AI 512 Assignment #2

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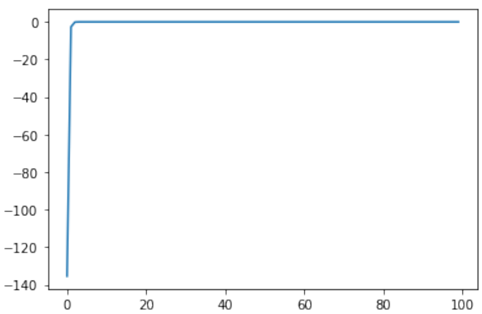
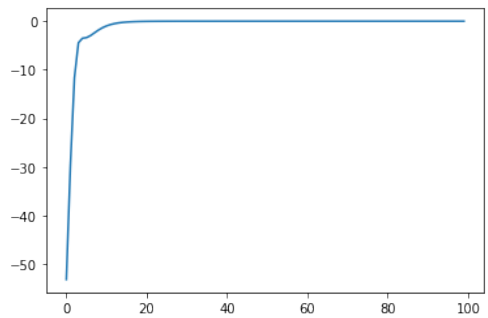
Solution 1

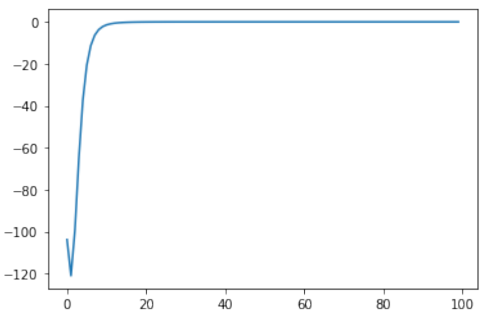
The attachments consists of .py and .ipynb files. The Python code is contained as is in the .py file. The .ipynb contains a literate program for better visualisation. The convergence plots for top 5 eigenvalues obtained are placed here, and can also be obtained by running the Python program.



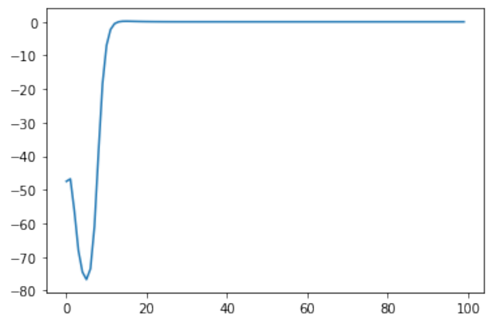
(a) first



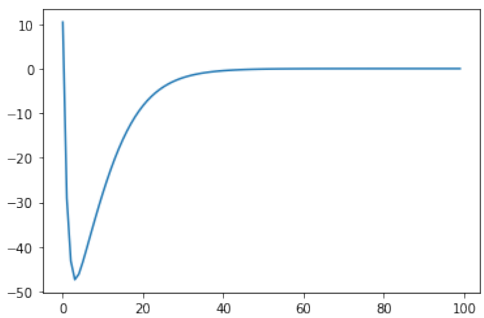
(b) second



(c) third



(d) fourth



(e) fifth

Figure 1: caption

Solution 2

(1) To show $\mathcal{D}(v) = v^T L v$.

$$\begin{aligned}
\mathcal{D}(v) &= ||A^T v||^2 \\
&= (A^T v)^T (A^T v) \\
&= v^T (A^T)^T A^T v \\
&= v^T A A^T v \\
&= v^T L v, \quad \because A A^T = L
\end{aligned}$$

(2) Let A represent the $m \times n$ incidence matrix described by the function:

$$A_{ij} = \begin{cases} 1 & \text{if edge j points to node i} \\ -1 & \text{if edge j points from node i} \\ 0 & \text{otherwise} \end{cases}$$

Let L_{ij} represent the generic entry within matrix L, where i, j are parametric row and column indices constrained by dimension bounds.

For $i = j$,

$$\begin{aligned}
L_{ij} &= L_{ii} \\
&= (AA^T)_{ii} \\
&= \sum_{k=1}^{n} A_{ik} A^T_{ki} \\
&= \sum_{k=1}^{n} A^2_{ik}
\end{aligned}$$

A_{ij} can take the values $\{-1, 0, 1\}$, thus A^2_{ij} is either 0 or 1, depending on whether or not edge j is connected to node i. In simpler terms, L_{ii} represents the degree of the node i.

For $i \neq j$,

$$\begin{aligned}
L_{ij} &= (AA^T)_{ij} \\
&= \sum_{k=1}^{n} A_{ik} A^T_{kj} \\
&= \sum_{k=1}^{n} A_{ik} A_{jk}
\end{aligned}$$

If edge k has same parity towards nodes i and j with respect to direction, L_{ij} receives a contribution of $+1$. If there is disparity, -1 is received. If edge k does not connect to either or both the nodes, there is no contribution. This last case may thus be ignored for calculation of the L_{ij} term.

A directed edge can only point to one node. Therefore, a contribution of $+1$ is only possible for node i and node j being coincident, but this case is restricted to the situation for $i \neq j$. Hence, a contribution of $+1$ is never received. If the edge k connects nodes i and j (the edge points *from* one node *toward* another), there is a contribution of -1. Note that oppositely directed edges between i and j will contribute the same.

Simply put, the magnitude of the term L_{ij}, for $i \neq j$, represents the number of edges connecting nodes i and j.

Solution 3

The question describes the function f as:

$$f = \frac{1}{2}x^T Sx$$

or alternatively,

$$2f = ax^2 + 2bxy + cy^2, \quad x = \begin{bmatrix} x \\ y \end{bmatrix}, \; S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Given that $x^T Sx = 2x_1x_2$, where x_1 and x_2 are represent the constituents of the matrix argument, it is evident that $a = 0, b = 1, c = 0$. Matrix S can be described as:

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues may be obtained from solving the characteristic equation for S.

$$\begin{aligned} |S - \lambda I| &= 0 \\ \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} &= 0 \\ (-\lambda)^2 - 1 &= 0 \\ \lambda &= \pm 1 \end{aligned}$$

Since S has both positive and negative eigenvalues, it has a saddle point.