

The flow of the document, what will be covered (UTLC, STLC, DTLC, Curry-Howard, LCCCs...). The aim of the document. Things will be formal but for key constructions that are deemed to be of practical use, a simple implementation will be given, usually in haskell, lean or similar. Implementations given will not necessarily be optimal since they are meant for educational purpose

Chapter 1

λ -Calculus

In the early 20th century, mathematicians such as Bertrand Russell, David Hilbert, and Kurt Gödel were trying to set complete, consistent, and decidable foundations for mathematics. Within this context, Alonzo Church, in his effort to formalize the notion of computability, introduced a minimal symbolic language based upon function abstraction and function application, this system is now called λ -calculus.

In 1936, Church used lambda calculus to address Hilbert's Entscheidungsproblem—whether a mechanical method could determine the truth of any first-order logic statement. He proved no such algorithm exists, establishing the undecidability of first-order logic.

One of the nuances formalized by the λ -calculus is the distinction between extensional and intensional equality. The extensional approach to equivalence states that two functions are equivalent if they share input output pairs for every possible input. Its intensional counterpart extends this notion of equality by requiring that the procedures that compute these pairs share complexity i.e. they take the same steps toward yielding a result.¹

Languages like LISP, Haskell, Erlang and others share the λ -calculus as their theoretical foundation. It has established itself as the backbone of functional programming. The introduction of type systems into the λ -calculus has allowed us to computer-verify mathematical proofs and develop programs that are correct by construction.²

¹Let p be a sufficiently large prime, and let $f, g : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ be defined by $f(x) = x^2$ and $g(x) = \log_a(a^{x+2})$, where $a \in \mathbb{Z}_p^\times$ is a fixed primitive root. Although f and g are extensionally equal, i.e., they yield the same output for all inputs in \mathbb{Z}_p , they are intensionally distinct. The function f performs a simple squaring operation, while g requires evaluating a discrete logarithm, intractable in general. ??[Pedro Bonilla]

²Principles of the λ -calculus and Type Theory underlie every computer assisted verification tool as well as proof assistants and kitchen table programming languages like C or Java.

1.1 Untyped λ -Calculus

In order to get acquainted with the λ -calculus, let us develop a simple example to familiarize ourselves before we begin with a more formal approach to this discipline. Consider the function $f(x) = x + 1$, the most straightforward way to express this using λ -calculus would be $(\lambda x.x + 1)$, where the lambda denotes that x is being captured and used as a parameter to perform some computation³. Evaluating $f(2)$ using this newly created lambda term would look like this: $(\lambda x.x + 1)(2) \rightsquigarrow_\beta (2 + 1) \rightsquigarrow_\beta 3$. This process of reducing a λ -expression is referred to as β -reduction, it will be covered formally later in the chapter.

Definition 1.1.1. The set of all lambda terms Λ is defined inductively as follows:

- (Variable) If $x \in \mathbf{V}$, then $x \in \Lambda$.
- (Abstraction) If $x \in \mathbf{V}$ and $M \in \Lambda$, then $(\lambda x.M) \in \Lambda$.
- (Application) If $M, N \in \Lambda$, then $(MN) \in \Lambda$.

Where $\mathbf{V} = \{x, y, z, \dots\}$ represents a countably infinite set of variable names.

The key takeaway of this definition is that abstraction and application together, when combined with β -reduction, enable computation—thus encapsulating the meaning of function in a way that is abstract yet useful.

Since we are dealing with a formal language, it is in our benefit to introduce a few other objects with the aim of defining a grammar to generate the set Λ :

- An alphabet $\Sigma = \{\lambda, ., (,), \dots\}$, is a finite set of symbols
- A string is a finite sequence of elements from Σ , the empty string is denoted by ε
- Σ^* denotes the set of all finite strings over Σ , $\varepsilon \in \Sigma^*$
- A language L over an alphabet Σ is a subset of Σ^*

Our aim now, to generate the set Λ , to do this, we will make use of a grammar. When dealing with grammars that define programming languages i.e. context-free grammars, Backus-Naur Form, BNF for short is the way to go:

- Nonterminals are enclosed in angle brackets (e.g. $\langle \text{expr} \rangle$)

³this is an e.g., $(\lambda x.x + 1)$ is not a valid lambda term, see Definition 1.1.2.

- Terminals are written literally (e.g. " λ ", ".", x)
- Productions define how nonterminals expand, written as $::=$
- The vertical bar $|$ denotes available expansions

Thus, the language for natural numbers in decimal notation is represented using:

```
<digit> ::= "0" | "1" | "2" | "3" | "4" | "5" | "6" | "7" | "8" | "9"
<number> ::= <digit> | <digit> <number>
```

Definition 1.1.2. Taking advantage of BNF notation, an alternative definition for Λ would be:

```
<term> ::= <variable>
        | "<math>\lambda</math>" <variable> "." <term>
        | "(" <term> <term> ")"
<variable> ::=  $\in V$ 
```

where $\Sigma = \{\lambda, ., (,)\} \cup V$ and $\Lambda \subset \Sigma^*$. Through use of algebraic types⁴, this grammar can be implemented in Haskell:

```
1 data Term = Var String
2           | App Term Term
3           | Abs String Term
```

Listing 1.1: Grammar implementation of the λ -calculus language using Haskell's algebraic types

Remark 1. From the definition above we extract that $\lambda x.x + 1 \notin \Lambda$, given this circumstance, the need to find an accurate representation for numbers within this definition arises. See Definition 1.1.15

Example 1.1.1. Some examples of valid λ terms generated using the grammar in 1.1.2: y , $(\lambda x.(xx))$, $(\lambda x.(\lambda y.x))$, $((\lambda x.(xy))(\lambda y.y))z$. The haskell counterpart for the second term is:

```
1 Abs "x" (App (Var "x") (Var "x"))
```

Listing 1.2: Haskell interpretation of the second term

⁴Algebraic Data Types are inductively defined types built from sums (either this or that) of products (this and that). They let you define structured data with multiple forms and support safe, exhaustive pattern matching.

As a convention to avoid notational cluttering, the outermost parenthesis can be omitted e.g. $(\lambda x.x)$ can be read as $\lambda x.x$. Also, application is left-associative and binds tighter than abstraction, so MNP is parsed as $(MN)P$ and $\lambda x.MN$ means $\lambda x.(MN)$, not $(\lambda x.M)N$. Likewise, abstraction is right-associative e.g. $\lambda x.\lambda y.M$ is $\lambda x.(\lambda y.M)$ and binds more weakly than application. All these are just to keep notation straightforward, the formal definition does not leave precedence under-specified thanks to parenthesis. Those interested in the Haskell implementation need not worry, as Haskell enforces an explicit order on constructors.

1.1.1 Equivalence of terms

Having defined the language of the λ -calculus, we move on to the mechanics of computation. To this end, several equivalence relations among terms will be defined. Let us begin by introducing the set that contains all variables that are not bound to the term via an abstraction.

Definition 1.1.3. Given $T \in \mathbf{\Lambda}$, $\mathbf{FV} : \mathbf{\Lambda} \rightarrow \mathcal{P}(\mathbf{V})$ outputs the set of free variables for T :

$$\begin{aligned}\mathbf{FV}(x) &= x \\ \mathbf{FV}(MN) &= \mathbf{FV}(M) \cup \mathbf{FV}(N) \\ \mathbf{FV}(\lambda x.M) &= \mathbf{FV}(M) \setminus \{x\}\end{aligned}$$

Where $x \in \mathbf{V}$ and $M, N \in \mathbf{\Lambda}$, and \mathcal{P} denotes the power set.

```

1 FV :: Term -> Set String
2 FV (Var x)      = Set.singleton x
3 FV (App t1 t2) = Set.union (FV t1) (FV t2)
4 FV (Abs x t)   = Set.delete x (FV t)
```

Listing 1.3: Implementation of FV in Haskell.

Example 1.1.2. Computing the free variables of $\lambda x.\lambda y.yxz$. The term well-formed, so we proceed:

$$\begin{aligned}\mathbf{FV}(\lambda x.\lambda y.yxz) &= \mathbf{FV}(\lambda y.yxz) \setminus \{x\} \\ &= \mathbf{FV}(yxz) \setminus \{x, y\} \\ &= \{x, y, z\} \setminus \{x, y\} \\ &= \{z\}\end{aligned}$$

Of course, z is the only free variable in the expression as it is the only variable that is not captured by some λ -abstraction.

Definition 1.1.4. A term $M \in \mathbf{\Lambda}$ is closed if and only if $\mathbf{FV}(M) = \emptyset$, closed lambda terms are often referred to as combinators. $\mathbf{\Lambda}^0$ is the set of all closed lambda terms.

Since every lambda term is made up of other smaller lambda terms, it is only natural to define the set that contains all subterms of a given term:

Definition 1.1.5. Given $T \in \mathbf{\Lambda}$, $\mathbf{Sub} : \mathbf{\Lambda} \rightarrow \mathcal{P}(\mathbf{V})$ maps a term to the set of it's subterms.

$$\begin{aligned}\mathbf{Sub}(x) &= \{x\} \\ \mathbf{Sub}(MN) &= \mathbf{Sub}(M) \cup \mathbf{Sub}(N) \cup \{MN\} \\ \mathbf{Sub}(\lambda x.M) &= \mathbf{Sub}(M) \cup \{\lambda x.M\}\end{aligned}$$

Where $x \in \mathbf{V}$ and $M, N \in \mathbf{\Lambda}$.

```

1 Sub :: Term -> Set Term
2 Sub v@(Var _)      = Set.singleton v
3 Sub a@(App t1 t2) = Set.insert a (Set.union (Sub t1)
4                                     (Sub t2))
5 Sub a@(Abs _ t)    = Set.insert a (Sub t)

```

Listing 1.4: Haskell implementation of Sub.

Remark 2. Let \preceq be the subterm relation on λ -terms, defined by

$$M \preceq N \iff M \in \mathbf{Sub}(N)$$

\preceq is a partial order relation. This statement follows intuitively from the definition, it can be proven using induction on the derivation tree.

Definition 1.1.6. A term $S \in \mathbf{Sub}(M)$, $M \in \mathbf{\Lambda}$ is said to be proper if $S \neq M$. $\mathbf{Sub}(M) \setminus \{M\}$ would be the set of proper subterms.

Definition 1.1.7. Let $M = \lambda x.N$ be a λ -term:

- The variable x is the binding variable of the abstraction $\lambda x.N$.
- An occurrence of a variable x in M is bound if $x \in \mathbf{Sub}(\lambda x.N)$.

Binding vs bound vs free variables. alpha equiv ultimately means replacing names of bound variables i.e. variables that are used as placeholders for terms so we dont care about renaming them

Intro to the notion of alpha equivalence

Definition 1.1.8. Capture-Avoiding Substitution Rules Let $M, N \in \mathbf{\Lambda}$ and $x, y, z \in \mathbf{V}$. The substitution $M[x := N]$ is defined inductively as:

$$\begin{aligned}
 x[x := N] &= N \\
 y[x := N] &= y & y \neq x \\
 (M_1 M_2)[x := N] &= (M_1[x := N]) (M_2[x := N]) \\
 (\lambda x. M)[x := N] &= \lambda x. M \\
 (\lambda y. M)[x := N] &= \lambda y. (M[x := N]) & y \neq x, y \notin \mathbf{FV}(N) \\
 (\lambda y. M)[x := N] &= \lambda z. (M[y := z][x := N]) & y \neq x, y \in \mathbf{FV}(N)
 \end{aligned}$$

where $z \notin \mathbf{FV}(M) \cup \mathbf{FV}(N)$.

Listing 1.5: asdasdasd

- De Bruijn
- Add insight and intuition upon the formal definition
- Add some haskell implementation details

Definition 1.1.9. Two lambda terms $M, N \in \mathbf{\Lambda}$ are α -equivalent, written $M =_\alpha N$, if they are structurally identical except for the names of bound variables. Formally:

$$\lambda x. M =_\alpha \lambda y. M[x := y]$$

Alpha-equivalence possesses the following properties:

$$\begin{aligned}
 M &=_\alpha M \\
 M &=_\alpha N \Rightarrow N =_\alpha M \\
 M &=_\alpha N \text{ and } N =_\alpha P \Rightarrow M =_\alpha P
 \end{aligned}$$

And thus, is an equivalence relation.

- Examples

Example 1.1.3.

... α -equivalence relations

- This relation is the most horizontal of the ones that show up in the text. By this I mean it just relates terms with equivalent terms at the same level in the β -reduction tree. (find a way to say this without having introduced β -reduction yet)

Definition 1.1.10. Single-step β -reduction \rightsquigarrow_β is defined using the following inference rules:

$$\begin{array}{c} \frac{}{(\lambda x. M) N \rightsquigarrow_\beta M[x := N]} \text{BETA} \quad \frac{M \rightsquigarrow_\beta M'}{\lambda x. M \rightsquigarrow_\beta \lambda x. M'} \text{ABS} \\[10pt] \frac{M_1 \rightsquigarrow_\beta M'_1}{M_1 M_2 \rightsquigarrow_\beta M'_1 M_2} \text{APPL} \quad \frac{M_2 \rightsquigarrow_\beta M'_2}{M_1 M_2 \rightsquigarrow_\beta M_1 M'_2} \text{APPR} \end{array}$$

where $M, N \in \mathbf{\Lambda}$. For those not familiar with inference rules, here is an alternative definition where $L \in \mathbf{\Lambda}$

- $(\lambda x. M) N \rightsquigarrow_\beta M[x := N]$
- If $M \rightsquigarrow_\beta N$, then: $M L \rightsquigarrow_\beta N L$; $L M \rightsquigarrow_\beta L N$; $\lambda x. M \rightsquigarrow_\beta \lambda x. N$

Here, The Beta rule abstracts the concept of reduction, that encapsulates the idea of computation within the λ -calculus. The reason for this is that Beta reduces an application of an abstraction and removes the abstraction, hence simplifying the term.

Single β -reductions can be applied successively, which induces the following definition

Definition 1.1.11. Zero or more step β -reduction, also called reflexive transitive closure. We write $M \rightsquigarrow_\beta^* N$ if and only if there exists a sequence of one-step β -reductions starting from M and ending at N .⁵

$$M \equiv M_0 \rightsquigarrow_\beta M_1 \rightsquigarrow_\beta M_2 \rightsquigarrow_\beta \cdots \rightsquigarrow_\beta M_{n-1} \rightsquigarrow_\beta M_n \equiv N$$

That is, there exists an integer $n \geq 0$ and a sequence of terms M_0, M_1, \dots, M_n such that: $M_0 \equiv M$ and $M_n \equiv N$ for all i with $0 \leq i < n$, we have $M_i \rightsquigarrow_\beta M_{i+1}$.

The process of successively β -reducing a term leads raises the question of whether all terms can be reduced indefinitely. We know the answer to this, since for instance, variables in \mathbf{V} cannot be reduced. The subsequent question is then, Do all all terms have some kind of normal form that cannot be reduced further? This is answered in Section 1.1.3.

⁵The convention in the literature is to use \rightarrow_β for β -reductions \rightarrow_β for multi-step β -reductions. I prefer to use \rightsquigarrow_β and \rightsquigarrow_β^n since this allows me to use the superscript to note the number of steps in the process β -reduction e.g. $(\lambda f. \lambda x. x) M N \rightsquigarrow_\beta^2 N$. Instead of \rightsquigarrow_β^n I use \rightsquigarrow_β^* whenever the number of steps is undefined.

Definition 1.1.12. β -equivalence relation, equivalence of λ -terms through β -reductions.

$$M =_{\beta} N \iff M \rightsquigarrow_{\beta}^* N \vee N \rightsquigarrow_{\beta}^* M$$

Using definition 1.1.11:

$$M =_{\beta} N \iff \begin{cases} \exists M_0, \dots, M_n \text{ s.t. } M \equiv M_0 \rightsquigarrow_{\beta} M_1 \rightsquigarrow_{\beta} \dots \rightsquigarrow_{\beta} M_n \equiv N \\ \exists N_0, \dots, N_n \text{ s.t. } N \equiv N_0 \rightsquigarrow_{\beta} N_1 \rightsquigarrow_{\beta} \dots \rightsquigarrow_{\beta} N_n \equiv M \end{cases}$$

Remark 3. In the context of equivalence of programs, β -equivalence can be understood as means for intensional equivalence, the kind of equivalence that requires programs to be equivalent in a step-by-step fashion.

We now explore a key aspect of the λ -calculus, in the jargon referred to as first-class citizenship, or high-order, a property of terms that restrains them from being exclusively classified as either function or argument. The example below sheds some light on the matter:

Example 1.1.4. High Order by example. Let us β -reduce the λ -term ABC and inspect the consequences:

$$\begin{aligned} \overbrace{(\lambda f. \lambda y. f f y)}^A \overbrace{(\lambda x. x + 1)}^B \overbrace{(2)}^C &\rightsquigarrow_{\beta} (\lambda y. (\lambda x. x + 1)(\lambda x. x + 1)y)(2) \\ &\rightsquigarrow_{\beta} (\lambda x. x + 1)(\lambda x. x + 1)(2) \\ &\rightsquigarrow_{\beta}^* (\lambda x. x + 1)(3) \\ &\rightsquigarrow_{\beta}^* (4) \end{aligned}$$

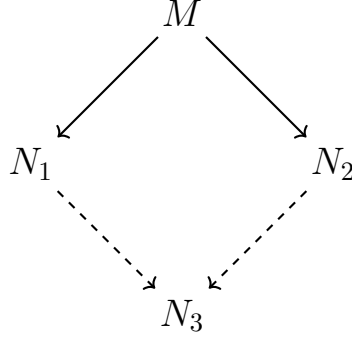
Taking a closer look, A , applies f twice to x , as a consequence, B is being applied twice to C . And so, very easily, we have implemented the $x + 2$ function via a sequential application of $x + 1$ to C .

Remark 4. As seen above, we can pass what we understand as conventional functions as arguments since they are treated alike. One can intuitively appreciate the computational expressiveness this brings with it, and how the syntactic-semantic homogeneity sets the λ -calculus apart from the classical set-theoretic approach to functions.

Theorem 1.1.1. The Church-Rosser Property states that $\forall M, N_1, N_2 \in \Lambda$:

$$M \rightsquigarrow_{\beta}^* N_1 \quad \text{and} \quad M \rightsquigarrow_{\beta}^* N_2 \quad \Rightarrow \quad \exists N_3 \mid N_1 \rightsquigarrow_{\beta}^* N_3 \wedge N_2 \rightsquigarrow_{\beta}^* N_3$$

If a lambda term M has a normal form, then every reduction sequence starting from M eventually reduces to that same normal form, up to alpha-equivalence. The diagram below is the reason this theorem is often referred to as the diamond property.



Note 1. In simple terms, if the solid arrows exist, then the dashed arrows must exist too.

1.1.2 Some Important Constructions

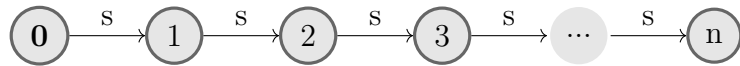
Within the λ -calculus there exist a number of conventional constructions that represent key ideas in programming (numbers, booleans, branching and lists among others) In the following we define a few of the most important ones. These constructions can be used to prove relevant statements concerning the expressiveness of the λ -calculus.

We begin defining \mathbb{N} for no particular reason. The notion of natural numbers in the λ -calculus follows from the Peano axioms:

Definition 1.1.13. Peano Axioms, the set of natural numbers \mathbb{N}

- (Zero) $0 \in \mathbb{N}$
- (Successor) $\forall n \in \mathbb{N}, S(n) \in \mathbb{N}$
- (Initial) $\forall n \in \mathbb{N}, S(n) \neq 0$
- (Injectivity) $\forall n, m \in \mathbb{N}, S(n) = S(m) \Rightarrow n = m$
- (Induction) $P(0) \wedge \forall (P(n) \Rightarrow P(S(n))) \Rightarrow \forall n, P(n)$

The first four axioms can be summarized using a graph:



The last axiom is concerned with proving properties for all naturals given a precedent that satisfies the property and proof that if the property is satisfied for n then it is satisfied for $n + 1$, then the property is true of all naturals.

Example 1.1.5. Say for instance we mean to represent 3 using Peano notation, using the graph, we would come up with $s(s(s(0)))$, since it is 3 steps away from 0.

Definition 1.1.14. To add and multiply natural numbers one can just use following the sets of rules:

$$n + m = \begin{cases} n & \text{if } m = 0 \\ s(n + m') & \text{if } m = s(m') \end{cases} \quad a \cdot b = \begin{cases} 0 & \text{if } b = 0 \\ a \cdot b' + a & \text{if } b = s(b') \end{cases}$$

- $a + b$: A recursive rule that performs the addition, and a base case to stop the recursion for $n + 0$, since there is no such number that has 0 as sucesor.
- $a \cdot b$: Using addition we define multiplication in a similar way, adding a as many times as the function s is applied to 0 in b .

Definition 1.1.15. Representation of the set of all naturals in the realm of λ -calculus, usually referred to as Church numerals:

$$0 \equiv \lambda f. \lambda x. x \\ \text{SUCC} \equiv \lambda n. \lambda f. \lambda x. f(nfx)$$

Example 1.1.6. The church numerals have been defined similarly to the peano axioms, that is, by bringing into existance an initial element and using the sucesor function to define the rest of them. Let us apply **SUCC** a few times to illustrate:

$$\begin{aligned} 0 &\equiv \lambda f. \lambda x. x \\ 1 &\equiv \lambda f. \lambda x. fx \\ 2 &\equiv \lambda f. \lambda x. ffx \\ 3 &\equiv \lambda f. \lambda x. fffx \\ n &\equiv \lambda f. \lambda x. f^n x \end{aligned}$$

where n denotes the number of times we append f .

Definition 1.1.16. Now that we have a construction for the naturals, we can define some arithmetic operations:

$$\begin{aligned} \text{ADD} &\equiv \lambda m. \lambda n. \lambda f. \lambda x. mf(nfx) \\ \text{MUL} &\equiv \lambda f. \lambda x. \lambda f. \lambda x. m(nf)x \\ \text{ISZERO} &\equiv \lambda n. n (\lambda x. \text{FALSE}) \text{TRUE} \end{aligned}$$

Example 1.1.7. Beta reduction of a sample multiplication Notion behind **ISZERO**

Remark 5. This proves that the λ -calculus is capable of arithmetic, we leave them as a sidequest since we will not be extending or using them or in the rest of the text.

The next constructions in line are the truth values of boolean logic, “*true*” and “*false*” for short.

Definition 1.1.17. Boolean truth values:

$$\begin{aligned}\text{TRUE} &\equiv \lambda t.\lambda f.t \\ \text{FALSE} &\equiv \lambda t.\lambda f.f\end{aligned}$$

Remark 6. Note how $\text{TRUE } M N \rightsquigarrow_{\beta} M$ and $\text{FALSE } M N \rightsquigarrow_{\beta} N$

Definition 1.1.18. Conditional, in other words “*if statement*”.

$$\text{IF} \equiv \lambda b.\lambda t.\lambda e.b t e$$

The reality of the IF is that it is just a combinator that happens to behave similarly to a branching statement whenever we pass a church encoded boolean as a first argument.

Example 1.1.8. On the correct behaviour of IF i.e. check that it implements branching. $M, N \in \mathbf{\Lambda}$ and IF, TRUE, FALSE are as usual:

$$\begin{aligned}\text{IF TRUE } M N & \\ \equiv (\lambda b.\lambda t.\lambda e.b t e) \text{ TRUE } M N & \quad \text{IF FALSE } M N \\ \rightsquigarrow_{\beta} (\lambda t.\lambda e.\text{TRUE } t e) M N & \quad \rightsquigarrow_{\beta}^* \text{FALSE } M N \rightsquigarrow_{\beta}^* N \\ \rightsquigarrow_{\beta} (\lambda e.\text{TRUE } M e) N & \\ \rightsquigarrow_{\beta} \text{TRUE } M N \rightsquigarrow_{\beta}^* M & \end{aligned}$$

So the IF just combines b, t, e in such a way that whenever b is a church encoded boolean either t or e is dropped depending on the truth value of b . See Remark 6. From this example we extract that $\lambda x.x$ is equivalent to IF in terms of behaviour, and thus, we could use $\text{IF} \equiv \lambda x.x$ interchangeably.

Remark 7. Using the IF we can implement an universal set of logic gates:

$$\begin{aligned}\text{NOT} &\equiv \text{IF } b \text{ FALSE TRUE} \\ \text{AND} &\equiv \text{IF } a (\text{IF } b \text{ TRUE FALSE}) \text{ FALSE}\end{aligned}$$

This allows us to simulate a nand gate, known to be universal ?? **universality, conjunctive normal form**. Thus, the lambda calculus is at least equivalent to propositional logic ?? **Something on turing completeness**.

On the same track, another must-have for programmers is lists. Since they are a bit more involved, we require tuples to define them. Note that the IF is a tuple that contains two elements, we query the first element using a **TRUE** i.e. the “*then*” branch, and the second using a **FALSE** i.e. the “*else*” branch. We will be using this alternative definition for convenience:

Definition 1.1.19. $\text{PAIR} \equiv \lambda x.\lambda y.\lambda f.f\ x\ y$

Remark 8. This is more convenient since $\text{PAIR}\ M\ N \rightsquigarrow_{\beta}^* \lambda f.f\ M\ N$, which allows us to query the first element using $(\lambda f.f\ M\ N)\ \text{TRUE}$ and the second by $(\lambda f.f\ M\ N)\ \text{FALSE}$, this motivates the definition below:

Definition 1.1.20.

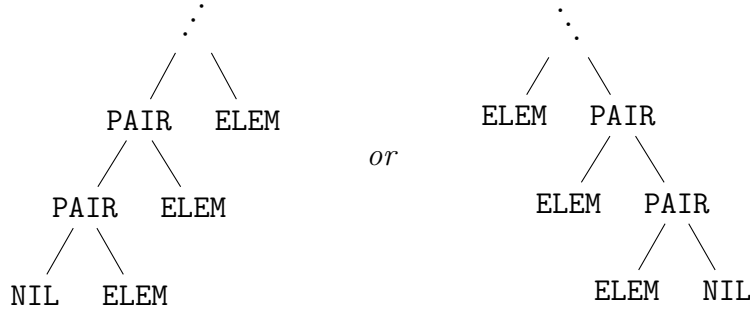
$$\text{FST} \equiv \lambda p.p\ (\lambda x.\lambda y.x)$$

$$\text{SND} \equiv \lambda p.p\ (\lambda x.\lambda y.y)$$

Proposition 1.1.2. A key property of **PAIR**’s is:

$$\text{PAIR}\ A\ B \equiv_{\beta} \text{PAIR}\ (\text{FST}\ A\ B)\ (\text{SND}\ A\ B)$$

Using pairs, we now can define linear lists, we can achieve this using either left linear trees or right linear trees:



Definition 1.1.21. We define **NIL** in the same way we defined **0** and **FALSE** before: $\text{NIL} \equiv \lambda f.\lambda z.z$; **NIL** represents the terminating element of the inductive definition for lists. To build lists, we cons lists together using:

$$\text{CONS} \equiv \lambda h.\lambda t.\lambda f.\lambda z.f\ h\ (t\ f\ z)$$

that binds a new element h to to an pre-existing list or to **NIL**.

Note 2. This notion of inductively defined lists is lays the foundations of the **LISP** family of programming languages, hence the name (**LI**St **PR**ocessing).

Definition 1.1.22. Relevant list operators:

$$\text{ISNIL} \equiv \lambda l.l (\lambda h.\lambda t.\text{FALSE}) \text{TRUE}$$

$$\text{HEAD} \equiv \lambda l.l (\lambda h.\lambda t.h) \text{undef}$$

1.1.3 Fixed Point Combinators and Recursion

Reintroduce the question from before

The most straight forward way to test this, is to check whether there exists some term that reduces to itself, effectively creating a loop in the β -reduction chain so to speak, that is $M \rightsquigarrow_{\beta}^n M$ with $n > 0$. It turns out $\Omega \equiv (\lambda x.x x)(\lambda x.x x)$ fullfills this requirement. Let us inspect what goes on as we keep β -reducing this term:

$$\Omega \equiv (\lambda x.x x)(\lambda x.x x) \rightsquigarrow_{\beta} (\lambda x.x x)(\lambda x.x x) \rightsquigarrow_{\beta}^* (\lambda x.x x)(\lambda x.x x) \equiv \Omega$$

This so called Ω -combinator, proves that not all terms have a normal form, and shines a beam of light onto how recursion can be achieved in the λ -calculus.

There is one caveat to all of this, namely that classically recursion is achieved through naming, take for instance this definition for the gcd:

$$\underline{\text{gcd}}(a, b) = \begin{cases} a & b = 0 \\ \underline{\text{gcd}}(b, a \bmod b) & b \neq 0 \end{cases}$$

Here, when $b \neq 0$ the name gcd is used to refer to the same definition again. In the λ -calculus such self-referencing definitions are not permitted since the language does not allow names. See 1.1.2. To bypass this setback the notion of fixed point is of particular interest.

Definition 1.1.23. Let $f : X \rightarrow X$ be a function. An element $x \in X$ is called a fixed point of f if $f(x) = x$.

Example 1.1.9. Some simple functions that illustrate what fixed points are about:

<i>Function</i>	<i>Fixed Point(s)</i>
$f(x) = c$	$x = c$
$g(x) = x$	$\{x \mid x \in \mathbb{R}\}$
$h(x) = x^2$	$x = 0, x = 1$

This harmless concept satisfies an interesting property, namely that $x = f(x) = f(f(x)) = f^n(x)$ effectively creating a chain every element $f^{n+1}(x) = f(f^n(x))$, thus attaining self-reference without naming. This idempotent behaviour under application is important, since it showcases

the ability of fixed points to induce recursion. Our quest now, is to find a term that simulates this behaviour. To do this, let us translate the definition for fixed point into the λ -world. **maybe use circ (i.e function composition) in the chain since it might have some meaning in the context of categories.**

Definition 1.1.24. Given a lambda term $L \in \Lambda$, a term $M \in \Lambda$ is called a fixed point of L if $M =_{\beta} LM$.

Theorem 1.1.3. For every $L \in \Lambda$, there exists $M \in \Lambda$ such that M is a fixed point for M i.e. $M =_{\beta} LM$. Referred to as the Fixed Point Theorem in the literature.

Proof. Given $L \in \Lambda$, define M as below:

$$M := (\lambda x. L(x x))(\lambda x. L(x x))$$

M is a reducible expression, redex for short:

$$\begin{aligned} M &\equiv (\lambda x. L(x x))(\lambda x. L(x x)) \\ &\rightsquigarrow_{\beta} L((\lambda x. L(x x))(\lambda x. L(x x))) \\ &\equiv LM. \end{aligned}$$

Hence, $M =_{\beta} LM$, as required. \square

See how LM is the application of M to L and that using APPR we can keep β -reducing M the same way we did for $x = f(x) = f(f(x)) \dots$ only this time using lambda terms:

$$M \rightsquigarrow_{\beta} LM \rightsquigarrow_{\beta} LLM \rightsquigarrow_{\beta}^* L^n M.$$

Now that nameless self-reference has been proven to exist within the realm of the λ -calculus, we take a step forward and introduce a few of the most important fixed point combinators and explore their different natures.

Definition 1.1.25. Y Combinator

$$Y \triangleq \lambda f. (\lambda x. f(x x)) (\lambda x. f(x x))$$

Note 3. $Y \equiv \lambda L.M$ **M defined previously. Improve this, find some other way to put it? note it is an improved Ω**

Remark 9. Even though the Y combinator is of theoretical interest since it allows us to find recursion within the λ -calculus, it is of little practical use since it runs into non-termination e.g.

$$\begin{aligned}
 Y F &\equiv (\lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))) F \\
 &\rightsquigarrow_{\beta} (\lambda x. F (x x)) (\lambda x. F (x x)) \\
 &\rightsquigarrow_{\beta} F ((\lambda x. F (x x)) (\lambda x. F (x x))) \\
 &\rightsquigarrow_{\beta} F (F ((\lambda x. F (x x)) (\lambda x. F (x x)))) \\
 &\rightsquigarrow_{\beta} \dots
 \end{aligned}$$

This is due to its lazy nature, this means that the argument F is not evaluated until Y is done unravelling i.e. never.

To work around this issue, several other combinators exist, one of the most notorious, the Z combinator is defined next:

Definition 1.1.26. Z combinator

$$Z \triangleq \lambda f. (\lambda x. f (\lambda v. x x v)) (\lambda x. f (\lambda v. x x v))$$

Example 1.1.10.

$$\begin{aligned}
 Z F &\equiv (\lambda f. (\lambda x. f (\lambda v. x x v)) (\lambda x. f (\lambda v. x x v))) F \\
 &\rightsquigarrow_{\beta} (\lambda x. F (\lambda v. x x v)) (\lambda x. F (\lambda v. x x v)) \\
 &\rightsquigarrow_{\beta} F (\lambda v. (\lambda x. F (\lambda v. x x v)) (\lambda x. F (\lambda v. x x v))) v \\
 &\equiv F (\lambda v. Z F v)
 \end{aligned}$$