Chapter 1

λ-Calculus

In the early 20th century, mathematicians such as Bertrand Russell, David Hilbert, and Kurt Gödel were trying to set complete, consistent, and decidable foundations for mathematics. Within this context, Alonzo Church, in his effort to formalize the notion of computability, introduced a minimal symbolic language based upon function abstraction and function application, this system is now called λ-calculus.

In 1936, Church used lambda calculus to address Hilbert's Entscheidungsproblem –whether a mechanical method could determine the truth of any first-order logic statement. He proved no such algorithm exists, establishing the undecidability of first-order logic.

One of the nuances formalized by the λ-calculus is the distinction between extensional and intensional equality. The extensional approach to equivalence states that two functions are equivalent if they share input output pairs for every possible input. Its intensional counterpart extends this notion of equality by requiring that the procedures that compute these pairs share complexity i.e. they take the same steps toward yielding a result.[1]

Languages like LISP, Haskell, Erlang and others share the λ-calculus as their theoretical foundation. The lambda calculus has established itself as the backbone of functional programming. The introduction of type systems into the λ-calculus has allowed us to computer-verify mathematical proofs and develop programs that are correct by construction.[2]

[1]Let p be a sufficiently large prime, and let $f, g : \mathbb{Z}_p \to \mathbb{Z}_p$ be defined by $f(x) = x^2$ and $g(x) = \log_a(a^{x+2})$, where $a \in \mathbb{Z}_p^\times$ is a fixed primitive root. Although f and g are extensionally equal, i.e., they yield the same output for all inputs in $\mathbb{Z}_p$, they are intensionally distinct. The function f performs a simple squaring operation, while g requires evaluating a discrete logarithm, intractable in general. [Pedro Bonilla]

[2]Principles of the λ-calculus and Type Theory underlie every computer assisted verification tool as well as proof assistants and kitchen table programming languages like C or Java.

1.1 Untyped λ-Calculus

In order to get acquainted with the λ-calculus, let us develop a simple example to familiarize ourselves before we begin with a more formal approach to this discipline. Consider the function $f(x) = x + 1$, the most straightforward way to express this using λ-calculus would be $(\lambda x.x + 1)$, where the lambda denotes that x is being captured and used as a parameter to perform some computation[3]. Evaluating $f(2)$ using this newly created lambda term would look like this: $(\lambda x.x + 1)(2) \rightsquigarrow_\beta (2 + 1) \rightsquigarrow_\beta 3$. This process of reducing a λ-expression is referred to as β-reduction, it will be covered formally later in the chapter.

Definition 1.1.1. The set of all lambda terms $\mathbf{\Lambda}$ is defined inductively as follows:

- (Variable) If $x \in \mathbf{V}$, then $x \in \mathbf{\Lambda}$.
- (Abstraction) If $x \in \mathbf{V}$ and $M \in \mathbf{\Lambda}$, then $(\lambda x.M) \in \mathbf{\Lambda}$.
- (Application) If $M, N \in \mathbf{\Lambda}$, then $(MN) \in \mathbf{\Lambda}$.

Where $\mathbf{V} = \{x, y, z, ...\}$ represents a countably infinite set of variable names.

The key takeaway of this definition is that abstraction and application together, when combined with β-reduction, enable computation—thus encapsulating the meaning of function in a way that is abstract yet useful.

Since we are dealing with a formal language, it is in our benefit to introduce a few other objects with the aim of defining a grammar to generate the set $\mathbf{\Lambda}$:

- An alphabet $\Sigma = \{\lambda, ., (,), \ldots\}$, is a finite set of symbols
- A string is a finite sequence of elements from Σ, the empty string is denoted by ε
- Σ^* denotes the set of all finite strings over Σ, $\varepsilon \in \Sigma^*$
- A language L over an alphabet Σ is a subset of Σ^*

Our aim now, to generate the set $\mathbf{\Lambda}$, to do this, we will make use of a grammar. When dealing with grammars that define programming languages i.e. context-free grammars, Backus-Naur Form, BNF for short is the way to go:

- Nonterminals are enclosed in angle brackets (e.g. `<expr>`)
- Terminals are written literally (e.g. `"λ"`, `"."`, x)
- Productions define how nonterminals expand, written as `::=`
- The vertical bar `|` denotes available expansions

Thus, the language for natural numbers in decimal notation is represented using:

```
 <digit> ::= "0" | "1" | "2" | "3" | "4" | "5" | "6" | "7" | "8" | "9"
<number> ::= <digit> | <digit> <number>
```

[3]this is an example, $(\lambda x.x + 1)$ is not a valid lambda term, see Definition 1.1.2.

Definition 1.1.2. Taking advantage of BNF notation, an alternative definition for $\mathbf{\Lambda}$ would be:

```
     <term> ::= <variable>
              | "λ" <variable> "." <term>
              | "(" <term> <term> ")"
 <variable> :: ∈ V
```

where $\Sigma = \{\lambda, ., (,)\} \cup \mathbf{V}$ and $\mathbf{\Lambda} \subset \Sigma^*$. Through use of inductive types types[4], this grammar can be implemented in Lean4:

Remark 1. From the definition above we extract that $\lambda x.x + 1 \notin \mathbf{\Lambda}$, given this circumstance, the need to find an accurate representation for numbers within this definition arises. See Definition 1.1.18

Notation 1. We use uppercase letters, e.g. M, N, L, P, Q to denote elements of $\mathbf{\Lambda}$ and lowercase e.g. x, y, z to denote elements of $\mathbf{V}$.

Example 1.1.1. Some examples of valid λ terms generated using the grammar in 1.1.2: y, $(\lambda x.(xx))$, $(\lambda x.(\lambda y.x))$, $(((\lambda x.(xy))(\lambda y.y))z)$. The Lean4 counterpart for the second term is:

As a convention to avoid notational cluttering, the outermost parenthesis can be omitted e.g. $\lambda x.x$ is read as $(\lambda x.x)$. Also, application is left-associative and binds tighter than abstraction, so MNP is parsed as $(MN)P$ and $\lambda x.MN$ means $\lambda x.(MN)$, not $(\lambda x.M)N$. Likewise, abstraction is right-associative e.g. $\lambda x.\lambda y.M$ is $\lambda x.(\lambda y.M)$ and binds more weakly than application. All these are just to keep notation straight-forward, the formal definition does not leave precedence under-specified thanks to parenthesis. Those interested in the implementation need not worry, as Lean4 enforces an explicit order on constructors. Below is a summary of common notational conventions used throughout the text:

- $\lambda x.\,x \iff (\lambda x.\,x)$
- $M\,N\,P \iff (M\,N)\,P$
- $\lambda x.M\,N \iff \lambda x.(M\,N)$
- $\lambda x.\lambda y.f(x,y) \iff \lambda xy.f(x,y)$ (Currying)

1.1.1 Equivalence of terms

Having defined the language of the λ-calculus, we move on to the mechanics of computation. To this end, several equivalences among terms will be defined, namely α-equivalence, β-equivalence, and η-equivalence. Firstly, and since every lambda term is made up of other smaller lambda terms, we define the set that contains all subterms of a given term:

[4]Inductive Data Types are inductively defined types built from sums (either this or that) of products (this and that). They let you define structured data with multiple forms and support safe, exhaustive pattern matching.

Definition 1.1.3. Given $T \in \mathbf{\Lambda}$, $\mathbf{Sub} : \mathbf{\Lambda} \to \mathcal{P}(\mathbf{V})$ maps a term to the set of it's subterms.

$$\begin{aligned}
&\mathbf{Sub}(x) = \{x\}\\
&\mathbf{Sub}(MN) = \mathbf{Sub}(M) \cup \mathbf{Sub}(N) \cup \{MN\}\\
&\mathbf{Sub}(\lambda x.M) = \mathbf{Sub}(M) \cup \{\lambda x.M\}
\end{aligned}$$

Where $x \in \mathbf{V}$ and $M, N \in \mathbf{\Lambda}$.

Remark 2. Let $\preceq$ be the subterm relation on λ-terms, defined by

$$M \preceq N \iff M \in \mathbf{Sub}(N)$$

$\preceq$ is a partial order relation. This statement is follows intuitively from the definition, it can be proven using induction on the derivation tree.

Definition 1.1.4. A term $S \in \mathbf{Sub}(M)$, $M \in \mathbf{\Lambda}$ is said to be proper if $S \neq M$. $\mathbf{Sub}(M) \setminus \{M\}$ would be the set of proper subterms.

Definition 1.1.5. Let $M = \lambda x.N$ be a λ-term:

- The variable x is the binding variable of the abstraction $\lambda x.N$.
- An occurrence of a variable x in M is bound if $x \in \mathbf{Sub}(\lambda x.N)$.
- Free variables are those that are not bound.

Example 1.1.2. Take $M \equiv \lambda x.\, x\, y$, in this example, the x in $\lambda x.$ is binding, y is free, and x is bound. This means that y is used literally, and that x, as denoted by $\lambda x.$ is abstracted and thus subject to being replaced within the context of said abtraction. A subexample might make this explicit: $(\lambda x.\, x\, y)\, M \rightsquigarrow_\beta M\, y$. The y remained while the x was used as a placeholder for M.

Remark 3. This taxonomy is key to understanding concepts such as α-equivalence intuitively.

The set of Free Variables **FV**, those that will not get replaced during the process of reduction, is computed using:

Definition 1.1.6. Given $T \in \mathbf{\Lambda}$, $\mathbf{FV} : \mathbf{\Lambda} \to \mathcal{P}(\mathbf{V})$ outputs the set of free variables for T:

$$\begin{aligned}
&\mathbf{FV}(x) = \{x\}\\
&\mathbf{FV}(MN) = \mathbf{FV}(M) \cup \mathbf{FV}(N)\\
&\mathbf{FV}(\lambda x.M) = \mathbf{FV}(M) \setminus \{x\}
\end{aligned}$$

Where $x \in \mathbf{V}$ and $M, N \in \mathbf{\Lambda}$, and $\mathcal{P}$ denotes the power set.

Example 1.1.3. Computing the free variables of $\lambda x.\lambda y.yxz$. The term well-formed, so we proceed:

$$\begin{aligned}\mathbf{FV}(\lambda x.\lambda y.yxz) &= \mathbf{FV}(\lambda y.yxz) \setminus \{x\}\\ &= \mathbf{FV}(yxz) \setminus \{x, y\}\\ &= \{x, y, z\} \setminus \{x, y\}\\ &= \{z\}\end{aligned}$$

Of course, z is the only free variable in the expression as it is the only variable that is not captured by some λ-abstraction.

Definition 1.1.7. A term $M \in \mathbf{\Lambda}$ is closed if and only if $\mathbf{FV}(M) = \emptyset$, closed lambda terms are often referred to as combinators. $\mathbf{\Lambda}^0$ is the set of all closed lambda terms.

Note 1. They are called combinators since having no free variables means that all variables are bound, and thus, a closed term just combines bound variables.

We approach the definition for α-equivalence to this end we have to introduce what it means to substitute a variable.

Definition 1.1.8. Capture-Avoiding Substitution Rules Let $M, N \in \mathbf{\Lambda}$ and $x, y, z \in \mathbf{V}$. The substitution $M[x := N]$ is defined inductively as:

$$\begin{aligned}x[x := N] &= N & &\\ y[x := N] &= y & & y \neq x\\ (M_1\, M_2)[x := N] &= (M_1[x := N])\,(M_2[x := N]) & &\\ (\lambda x.M)[x := N] &= \lambda x.M & &\\ (\lambda y.M)[x := N] &= \lambda y.(M[x := N]) & & y \neq x, y \notin \mathbf{FV}(N)\\ (\lambda y.M)[x := N] &= \lambda z.(M[y := z][x := N]) & & y \neq x, y \in \mathbf{FV}(N)\end{aligned}$$

where $z \notin \mathbf{FV}(M) \cup \mathbf{FV}(N)$.

Hopefully the meaning of the first three rules is clear, they ancapsulate renaming in the context of variables and applications. The subsequent rules are about abstractions, rule four states that renaming a binding variable has no effect, renaming only modifies a term whenever the renamed variable is different from the binding variable. Rule five is the general case, that is, when the substitution just moves in together with the abstracted term. Then there is the last case, when the binding variable is a bound occurrence in the term we are substituting, which poses a problem, to see why, let us work through an example:

Example 1.1.4. If we mistakenly substitute x by $(\lambda w.\, w\, y)$ in $(\lambda y.\, y\, x)$ using rule five, we get:

$$(\lambda y.\, yx)[x := (\lambda w.\, w\, y)] = (\lambda y.\, y\, (\lambda w.\, w\, y))$$

This is problematic since the free variable y became bound after the rename. Free and bound variables should mantain their freedom or their boundedness after the rename, hence the need for rule 6, where we rename the binding variable to avoid the name clash:

$$(\lambda y.\, yx)[x := (\lambda w.\, w\, y)] = (\lambda z.\, z\, (\lambda w.\, w\, y))$$

The purpuse of the binding variable is to serve as a placeholder, therefore renaming does not affect the integrity of the expression, the variable gets replaced anyway when we perform a reduction.

In the previous example a bound variable is renamed to avoid a naming collision and avoid binding a previously free variable, α-equivalence is the generalization of this technique, it defines equivalence up to renaming of binding variables. Also, in many of the upcoming definitions inference rules are used, here is a simple introduction for those unfamiliar with the concept:

$$\frac{P_1 \quad P_2 \quad \cdots \quad P_n}{C}\ \textsc{Name}$$

where P_i are judgments assumed to hold, C is the judgment inferred from the premises and Name is an optional identification label.

Definition 1.1.9. Two lambda terms $M, N \in \mathbf{\Lambda}$ are α-equivalent, written $M =_\alpha N$, if they are structurally identical except for the names of bound variables. Formally:

$$\frac{}{\lambda x.\, L =_\alpha \lambda y.\, L[x := y]}\ \textsc{Alpha} \qquad \frac{L =_\alpha L'}{\lambda x.\, L =_\alpha \lambda x.\, L'}\ \textsc{Abs}$$

$$\frac{L_1 =_\alpha L_1'}{L_1\, L_2 =_\alpha L_1'\, L_2}\ \textsc{AppL} \qquad \frac{L_2 =_\alpha L_2'}{L_1\, L_2 =_\alpha L_1\, L_2'}\ \textsc{AppR}$$

α-equivalence is an equivalence relation:

(Reflexivity) $M =_\alpha M$
(Transitivity) $M =_\alpha N$ and $N =_\alpha P \Rightarrow M =_\alpha P$
(Symetry) $M =_\alpha N \Rightarrow N =_\alpha M$

Example 1.1.5. A simple equivalence class and two samples of not α-equivalent pairs.

$$\lambda x.\lambda y.\, y =_\alpha \lambda a.\lambda b.\, b =_\alpha \lambda z.\lambda x.\, x$$

$$\lambda x.\lambda y.\, x \neq_\alpha \lambda x.\lambda y.\, y \quad \lambda x.\, x\, y \neq_\alpha \lambda x.\, x\, z$$

Remember terms are α-equivalent only when bounded names are properly substituted.

Remark 4. α-equivalence comes naturally once substitution is defined, since the names of bound variables are just placeholders whose only purpuse is to get replaced.

Naming variables can be a nightmare, which is why there exists a way of methodically assigning numbers to variables in a way that eliminates the need for α-equivalence. These naming system works by using numeric indices that indicate how many binders away a variable's binder is. These indices are called De Bruijn indeces.

$$\lambda x.\, x \Rightarrow \lambda.\, 0 \quad \lambda x.\lambda y.\, x \Rightarrow \lambda.\lambda.\, 1 \quad \lambda x.\lambda y.\, y \Rightarrow \lambda.\lambda.\, 0$$

It turns out we are better at naming variables than at working with numbers, which is why we will stick to symbols and leave De Bruijn indices to computers. Programs like automated theorem provers do use this naming strategy to simplify their work.

Definition 1.1.10. Single-step β-reduction $\rightsquigarrow_\beta$ is defined using the following inference rules:

$$\frac{}{(\lambda x.\, M)\, N \rightsquigarrow_\beta M[x := N]}\ \textsc{Beta} \qquad \frac{M \rightsquigarrow_\beta M'}{\lambda x.\, M \rightsquigarrow_\beta \lambda x.\, M'}\ \textsc{Abs}$$

$$\frac{M_1 \rightsquigarrow_\beta M_1'}{M_1\, M_2 \rightsquigarrow_\beta M_1'\, M_2}\ \textsc{AppL} \qquad \frac{M_2 \rightsquigarrow_\beta M_2'}{M_1\, M_2 \rightsquigarrow_\beta M_1\, M_2'}\ \textsc{AppR}$$

where $M, N \in \mathbf{\Lambda}$.

Single β-reductions can be applied succesively, which induces the following definition

Definition 1.1.11. Zero or more step β-reduction, the reflexive transitive closure of β-reduction. We write $M \rightsquigarrow_\beta^* N$ if and only if there exists a sequence of one-step β-reductions starting from M and ending at N.[5]

$$M \equiv M_0 \rightsquigarrow_\beta M_1 \rightsquigarrow_\beta M_2 \rightsquigarrow_\beta \cdots \rightsquigarrow_\beta M_{n-1} \rightsquigarrow_\beta M_n \equiv N$$

That is, there exists an integer $n \geq 0$ and a sequence of terms $M_0, M_1, \ldots, M_n$ such that: $M_0 \equiv M$ and $M_n \equiv N$ for all i with $0 \leq i < n$, we have $M_i \rightsquigarrow_\beta M_{i+1}$.

The process of succesively β-reducing a term leads raises the question of whether all terms can be reduced indefinitly. We know the answer to this, since for instance, variables in $\mathbf{V}$ cannot be reduced. The subsequent question is then, Do all all terms have some kind of normal form that cannot be reduced further? This is answered in Section 1.1.3. To this end we introduce a formal definition of normal form that will allow us to gain insight into the normalization properties of the λ-calculus.

Definition 1.1.12. A λ-term M is in normal form if there is no term N such that $M \rightsquigarrow_\beta N$. Equivalently, M contains no subterm of the form $(\lambda x.\, P)\, Q$ that can be reduced by beta-reduction. In other words, M is irreducible under beta-reduction.

Definition 1.1.13. A reducible expression, or redex for short, is any subterm of the form $(\lambda x.\, P)\, Q$. Such a term can be reduced by a single beta-reduction step, resulting in $P[x := Q]$.

[5] The convention in the literature is to use $\rightarrow_\beta$ for β-reductions $\twoheadrightarrow_\beta$ for multi-step β-reductions. I prefer to use $\rightsquigarrow_\beta$ and $\rightsquigarrow_\beta^n$ since this allows me to use the superscript to note the number of steps in the process β-reduction e.g. $(\lambda f.\lambda x.\, x)\, M\, N \rightsquigarrow_\beta^2$ N. Instead of $\rightsquigarrow_\beta^n$ I use $\rightsquigarrow_\beta^*$ whenever the number of steps is undefined.

Definition 1.1.14. β-equivalence relation, equivalence of λ-terms through β-reductions.

$$M =_\beta N \iff M \rightsquigarrow_\beta^* N \lor N \rightsquigarrow_\beta^* M$$

Using definition 1.1.11:

$$M =_\beta N \iff \begin{cases} \exists M_0, \dots, M_n \text{ s.t. } M \equiv M_0 \rightsquigarrow_\beta M_1 \rightsquigarrow_\beta \cdots \rightsquigarrow_\beta M_n \equiv N \\ \exists N_0, \dots, N_n \text{ s.t. } N \equiv N_0 \rightsquigarrow_\beta N_1 \rightsquigarrow_\beta \cdots \rightsquigarrow_\beta N_n \equiv M \end{cases}$$

Remark 5. In the context of equivalence of programs, β-equivalence can be understood as means for intensional equivalece, the kind of equivalence that requires programs to be equivalent in a step-by-step fashion.

We now explore a key aspect of the λ-calculus, in the jargon referred to as first-class citizenship, or high-order, a property of terms that restrains them from being exclusively classified as either function or argument. The example below sheds some light on the matter:

Example 1.1.6. High Order by example. Let us β-reduce the λ-term $A\,B\,C$ and inspect the consequences:

$$\begin{aligned} \overbrace{(\lambda f.\lambda y.f\,f\,y)}^{A}\,\overbrace{(\lambda x.x+1)}^{B}\,\overbrace{(2)}^{C} &\rightsquigarrow_\beta (\lambda y.(\lambda x.x+1)(\lambda x.x+1)y)(2) \\ &\rightsquigarrow_\beta (\lambda x.x+1)(\lambda x.x+1)(2) \\ &\rightsquigarrow_\beta^* (\lambda x.x+1)(3) \\ &\rightsquigarrow_\beta^* (4) \end{aligned}$$

Taking a closer look, A, applies f twice to x, as a consequence, B is being applied twice to C. And so, very easily, we have implemented the $x + 2$ function via a sequential application of $x + 1$ to C.

Remark 6. As seen above, we can pass what we understand as conventional functions as arguments since they are treated alike. One can intuitively apreciate the computational expresivenes this brings with it, and how this syntactic-semantic homogeneity sets the λ-calculus apart from the classical set-theoretic approach to functions.

Theorem 1.1.1. The Church-Rosser Property states that $\forall M, N_1, N_2 \in \mathbf{\Lambda}$:

$$M \rightsquigarrow_\beta^* N_1 \quad \text{and} \quad M \rightsquigarrow_\beta^* N_2 \quad \Rightarrow \quad \exists N_3 \mid N_1 \rightsquigarrow_\beta^* N_3 \land N_2 \rightsquigarrow_\beta^* N_3$$

If a lambda term M has a normal form, then every reduction sequence starting from M eventually reduces to that same normal form, up to alpha-equivalence. The diagram below is the reason this theorem is often referred to as the diamond property.

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ N_1 \arrow[dr, dashed] & & N_2 \arrow[dl, dashed] \\ & N_3 & \end{tikzcd}$$

The proof to this theorem is a quite lengthy, we would be miss the point of this document if we were to introduce it [ChurchRosser].

Corollary 1.1.2. Normalization: If a λ-term M reduces to two normal forms N_1 and N_2, then they are alpha-equivalent:

$$M \to_\beta^* N_1 \text{ and } M \to_\beta^* N_2 \text{ and } N_1, N_2 \text{ are normal} \quad \Rightarrow \quad N_1 =_\alpha N_2$$

The last and indeed the least important notion of equivalence:

Definition 1.1.15. Two lambda terms M and N are said to be η-equivalent, written $M =_\eta N$, if:

$$\frac{x \notin \mathbf{FV}(M)}{(\lambda x.\, M) =_\eta M}\ \textsc{Eta} \qquad \frac{M =_\eta M'}{\lambda x.\, M =_\eta \lambda x.\, M'}\ \textsc{Abs}$$

$$\frac{M_1 =_\eta M_1'}{M_1\, M_2 =_\eta M_1'\, M_2}\ \textsc{AppL} \qquad \frac{M_2 =_\eta M_2'}{M_1\, M_2 =_\eta M_1\, M_2'}\ \textsc{AppR}$$

The abstraction $\lambda x.\,(M\ \ x)$ performs the same operation as M, so if x is not used freely within M, the abstraction is redundant. This is shown in Eta, the rest of the inference rules propagate this equivalence onto the whole term. Put into programming the idea becomes trivial: pseudo code example. given f of type , def g (n) := f n is equivalent to just calling g.

η-equivalence represents weak extensionality, since it does not encode observational equivalence. Two terms are observationally equivalent if they cannot be distinguished by any context that can be papplied to them.

1.1.2 Some Important Constructions

Within the λ-calculus there exist a number of conventional constructions that represent key ideas in programming (numbers, booleans, branching and lists among others) In the following we define a few of the most important ones. These constructions can be used to prove relevant statements concerning the expressiveness of the λ-calculus.

We begin defining $\mathbb{N}$ for no particular reason. The notion of natural numbers in the λ-calculus follows from the Peano axioms:

Definition 1.1.16. Peano Axioms, the set of natural numbers $\mathbb{N}$

- (Zero) $0 \in \mathbb{N}$
- (Successor) $\forall n \in \mathbb{N}, S(n) \in \mathbb{N}$
- (Initial) $\forall n \in \mathbb{N}, S(n) \neq 0$
- (Injectivity) $\forall n, m \in \mathbb{N}, S(n) = S(m) \Rightarrow n = m$
- (Induction) $P(0) \wedge \forall (P(n) \Rightarrow P(S(n))) \Rightarrow \forall n, P(n)$

The first four axioms can be summarized using a graph:

$$\begin{tikzcd}
\mathbf{0} \arrow[r, "s"] & 1 \arrow[r, "s"] & 2 \arrow[r, "s"] & 3 \arrow[r, "s"] & \cdots \arrow[r, "s"] & n
\end{tikzcd}$$

The last axiom is concerned with proving properties for all naturals given a precedent that satisfies the property and proof that if the property is satisfied for n then it is satisfied for $n + 1$, then the property is true of all naturals.

Example 1.1.7. Say for instance we mean to represent 3 using Peano notation, using the graph, we would come up with $s(s(s(0)))$, since it is 3 steps away from 0.

Definition 1.1.17. To add and multiply natural numbers one can just use following the sets of rules:

$$n + m = \begin{cases} n & \text{if } m = 0 \\ s(n + m') & \text{if } m = s(m') \end{cases} \qquad a \cdot b = \begin{cases} 0 & \text{if } b = 0 \\ a \cdot b' + a & \text{if } b = s(b') \end{cases}$$

- $\underline{a + b}$: A recursive rule that performs the addition, and a base case to stop the recursion for n + 0, since there is no such number that has 0 as succesor.
- $\underline{a \cdot b}$: Using addition we define multiplication in a similar way, adding a as many times as the function s is applied to 0 in b.

Definition 1.1.18. Representation of the set of all naturals in the realm of λ-calculus, usually referred to as Church numerals:

$$\mathtt{0} \equiv \lambda f.\lambda x.x$$
$$\mathtt{SUCC} \equiv \lambda n.\lambda f.\lambda x.f(nfx)$$

Example 1.1.8. The church numerals have been defined similarly to the peano axioms, that is, by bringing into existance an initial element and using the succesor function to define the rest of them. Let us apply `SUCC` a few times to illustrate:

$$\begin{aligned}
\mathtt{0} &\equiv \lambda f.\lambda x.x \\
\mathtt{1} &\equiv \lambda f.\lambda x.fx \\
\mathtt{2} &\equiv \lambda f.\lambda x.ffx \\
\mathtt{3} &\equiv \lambda f.\lambda x.fffx \\
\mathtt{n} &\equiv \lambda f.\lambda x.f^n x
\end{aligned}$$

where n denotes the number of times we append f.

Remark 7. Since λ-terms are first-class citizens, and f and x are abstracted away, the reality is that numbers are a quite general construct that has many distinct applications. Recall Example 1.1.6 for instance.

Definition 1.1.19. Now that we have a construction for the naturals, we can define some arithmetic operations:

$$\begin{aligned}
\mathtt{ADD} &\equiv \lambda m.\lambda n.\lambda f.\lambda x.mf(nfx) \\
\mathtt{MUL} &\equiv \lambda f.\lambda x.\lambda f.\lambda x.m(nf)x \\
\mathtt{ISZERO} &\equiv \lambda n.n\,(\lambda x.\mathtt{FALSE})\,\mathtt{TRUE}
\end{aligned}$$

Example 1.1.9.

$$\begin{aligned}
\mathtt{ADD}\ \mathtt{2}\ \mathtt{3} &= (\lambda m.\lambda n.\lambda f.\lambda x.m\,f\,(n\,f\,x))\ (\lambda f.\lambda x.f\,f\,x)\ (\lambda f.\lambda x.f\,f\,f\,x) \\
&\rightsquigarrow_\beta^* \lambda f.\lambda x.\ (\lambda f.\lambda x.f\,f\,x)\ f\ ((\lambda f.\lambda x.f\,f\,f\,x)\ f\ x) \\
&\rightsquigarrow_\beta^* \lambda f.\lambda x.\ (\lambda x.f\,f\,x)\ ((\lambda x.f\,f\,f\,x)\ x) \\
&\rightsquigarrow_\beta (\lambda f.\lambda x.\ f\,f\,f\,f\,f\,x) \\
&\equiv \mathtt{5} \quad \text{Can I use alpha equiv o is it confusing?}
\end{aligned}$$

Multiplication works similarly to addition. As for `ISZERO`, the intuition behind it is that if the number is zero, then $\lambda x.$`FALSE` is dropped, and the result is `TRUE`; otherwise, it is applied at least once, which immediately yields `FALSE`.

Remark 8. This proves that the λ-calculus is capable of arithmetic, we leave them as a sidequest since we will not be extending or using them or in the rest of the text.

The next constructions in line are the truth values of boolean logic, "*true*" and "*false*" for short.

Definition 1.1.20. Boolean truth values:

$$\begin{aligned}
\mathtt{TRUE} &\equiv \lambda t.\lambda f.t \\
\mathtt{FALSE} &\equiv \lambda t.\lambda f.f
\end{aligned}$$

Remark 9. Note how `TRUE` $M\ N \rightsquigarrow_\beta M$ and `FALSE` $M\ N \rightsquigarrow_\beta N$

Definition 1.1.21. Conditional, in other words *"if statement"*.

$$\texttt{IF} \equiv \lambda b.\lambda t.\lambda e.b\,t\,e$$

The reality of the IF is that it is just a combinator that happens to behave similarly to a branching statement whenever we pass a church encoded boolean as a first argument.

Example 1.1.10. On the correct behabiour of IF i.e. check that it implements branching. $M, N \in \mathbf{\Lambda}$ and $\texttt{IF}, \texttt{TRUE}, \texttt{FALSE}$ are as usual:

$$\begin{aligned} & \texttt{IF}\ \texttt{TRUE}\ M\ N \\ \equiv\ & (\lambda b.\lambda t.\lambda e.\, b\,t\,e)\ \texttt{TRUE}\ M\ N \\ \rightsquigarrow_\beta\ & (\lambda t.\lambda e.\, \texttt{TRUE}\ t\,e)\ M\ N \\ \rightsquigarrow_\beta\ & (\lambda e.\, \texttt{TRUE}\ M\ e)\ N \\ \rightsquigarrow_\beta\ & \texttt{TRUE}\ M\,N \rightsquigarrow_\beta^* M \end{aligned} \qquad \begin{aligned} & \texttt{IF}\ \texttt{FALSE}\ M\ N \\ \rightsquigarrow_\beta^*\ & \texttt{FALSE}\ M\,N \rightsquigarrow_\beta^* N \end{aligned}$$

So the IF just combines b, t, e in such a way that whenever b is a church encoded boolean either t or e is dropped depending on the truth value of b. See Remark 9. From this example we extract that $\lambda x.x$ is equivalent to IF it terms of behaviour, and thus, we could use $\texttt{IF} \equiv \lambda x.x$ interchangeably.

Remark 10. Using the IF we can implement an universal set of logic gates:

$$\begin{aligned} \texttt{NOT} &\equiv \texttt{IF}\,b\,\texttt{FALSE}\,\texttt{TRUE} \\ \texttt{AND} &\equiv \texttt{IF}\,a\,(\texttt{IF}\,b\,\texttt{TRUE}\,\texttt{FALSE})\,\texttt{FALSE} \end{aligned}$$

This allows us to simulate a nand gate, known to be universal ?? universality, conjunctive normal form. Thus, the lambda calculus is at least equivalent to propositional logic ?? Something on turing completeness.

On the same track, another must-have for programmers is lists. Sice they are a bit more involved, we require tuples to define them. Note that the IF is a tuple that contains two elements, we query the first element using a TRUE i.e. the *"then"* branch, and the sencond using a FALSE i.e. the *"else"* branch. We will be using this alternative definition for convenience:

Definition 1.1.22. $\texttt{PAIR} \equiv \lambda x.\lambda y.\lambda f.f\,x\,y$

Remark 11. This is more convenient since $\texttt{PAIR}\,M\,N \rightsquigarrow_\beta^* \lambda f.f\,M\,N$, which allows us to query the first element using $(\lambda f.f\ M\ N)\,\texttt{TRUE}$ and the second by $(\lambda f.f\ M\ N)\,\texttt{FALSE}$, this motivates the definition below:

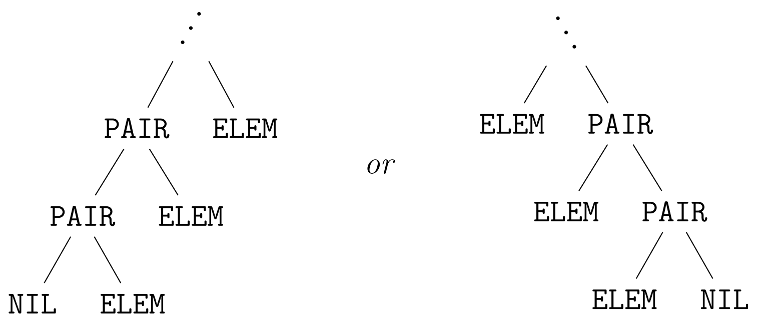
Definition 1.1.23.

$$\begin{aligned} \texttt{FST} &\equiv \lambda p.p\,(\lambda x.\lambda y.x) \\ \texttt{SND} &\equiv \lambda p.p\,(\lambda x.\lambda y.y) \end{aligned}$$

Proposition 1.1.3. A key property of PAIR's is:

$$\texttt{PAIR}\,A\,B \equiv_\beta \texttt{PAIR}\,(\texttt{FST}\,A\,B)\,(\texttt{SND}\,A\,B)$$

Using pairs, we now can define linear lists, we can achieve this using either left linear trees or right linear trees:



Definition 1.1.24. We define `NIL` in the same way we defined `0` and `FALSE` before: `NIL` $\equiv \lambda f.\lambda z.z$; `NIL` represents the terminating element of the inductive definition for lists. To build lists, we cons lists together using:

$$\texttt{CONS} \equiv \lambda h.\lambda t.\lambda f.\lambda z.f\,h\,(t\,f\,z)$$

that binds a new element h to to an pre-existing list or to `NIL`.

Note 2. This notion of inductively defined lists is lays the foundations of the **LISP** family of programming languages, hence the name (LISt Processing).

Definition 1.1.25. Relevant list operators:

$$\texttt{ISNIL} \equiv \lambda l.l\,(\lambda h.\lambda t.\texttt{FALSE})\,\texttt{TRUE}$$
$$\texttt{HEAD} \equiv \lambda l.l\,(\lambda h.\lambda t.h)\,\text{undef}$$

Turing completeness of the λ-calculus

$$\begin{aligned}
\texttt{CONS} &\equiv \lambda h.\lambda t.\lambda f.\lambda z.f\,h\,(t\,f\,z)\\
\texttt{IS\_NIL} &\equiv \lambda l.l\,(\lambda h.\lambda t.\texttt{FALSE})\,\texttt{TRUE}\\
\texttt{HEAD} &\equiv \lambda l.l\,(\lambda h.\lambda t.h)\,\text{undef}\\
\texttt{TAIL} &\equiv \lambda l.\,\texttt{FIRST}\,(l\,(\lambda p.\lambda h.\,\texttt{PAIR}\,(\texttt{SECOND}\,p)\,(\texttt{CONS}\,h\,(\texttt{SECOND}\,p)))\,(\texttt{PAIR}\,\texttt{NIL}\,\texttt{NIL}))
\end{aligned}$$

Recursion:

$$\texttt{Y} \equiv \lambda f.(\lambda x.f\,(x\,x))\,(\lambda x.f\,(x\,x))$$

1.1.3 Fixed Point Combinators and Recursion

The Church Rosser Property labels the untyped λ-calculus as a confluent system, therefore we know that whenever a term has a normal form it must be unique. what we did cover We still have not given an answer to whether all λ-terms have a normal form.

The most straight forward way to test this, is to check whether there exists some term that reduces to itself, effectively creating a loop in the β-reduction chain so to speak. In more formal terms this means there exists some M such that $M \rightsquigarrow_\beta^n M$ with $n > 0$. It turns out $\Omega \equiv (\lambda x.\, x\, x)(\lambda x.\, x\, x)$ fullfills this requirement. Let us inspect what goes on while β-reducing this term:

$$\Omega \equiv (\lambda x.\, x\, x)(\lambda x.\, x\, x) \rightsquigarrow_\beta (\lambda x.\, x\, x)(\lambda x.\, x\, x) \rightsquigarrow_\beta^* (\lambda x.\, x\, x)(\lambda x.\, x\, x) \equiv \Omega$$

This so called Ω-combinator, proves that not all terms have a normal form, and shines a beam of light onto how recursion can be achieved in the λ-calculus.

There is one caveat to this, that classically, recursion is achieved through naming, take for instance this definition for the gdc:

$$\underline{\mathrm{gcd}}(a,b) = \begin{cases} a & b = 0 \\ \underline{\mathrm{gcd}}(b,\, a \bmod b) & b \neq 0 \end{cases}$$

Here, when $b \neq 0$ the name gcd is used to refer to the same definition again. In the λ-calculus such self-referencing definitions are not permited since the language does not allow names. To bypass this setback the notion of fixed point is of particular interest.

Definition 1.1.26. Let $f : X \to X$ be a function. An element $x \in X$ is called a fixed point of f if $f(x) = x$.

Example 1.1.11. Some simple functions that illustrate what fixed points are:

Function	*Fixed Point(s)*
$f(x) = c$	$x = c$
$g(x) = x$	$\{x \mid x \in \mathbb{R}\}$
$h(x) = x^2$	$x = 0,\ x = 1$

This harmless idea satisfies an interesting property, namely that $x = f(x) = f(f(x)) = f^n(x)$ effectively creating a chain where every element $f^{n+1}(x) = f(f^n(x))$, thus attaining self-reference without naming. This idempotent behaviour under application is important, since it showcases the ability of fixed points to induce recursion. Our quest now, is to find a term that simulates this behavior. To do this, let us translate the definition for fixed point into the λ-world.

Definition 1.1.27. Given a lambda term $L \in \mathbf{\Lambda}$, a term $M \in \mathbf{\Lambda}$ is called a fixed point of L if $M =_\beta LM$.

No surprises here, the definition is close to a one-to-one mapping of its functional counterpart seen previously. Interestingly enough, it turns out that every lambda term has a fixed point, as stated in the theorem proven below:

Theorem 1.1.4. For every $L \in \mathbf{\Lambda}$, there exists $M \in \mathbf{\Lambda}$ such that M is a fixed point for M i.e. $M =_\beta LM$. Referred to as the Fixed Point Theorem in the literature.

Proof. Given $L \in \mathbf{\Lambda}$, define M as below:

$$M := (\lambda x.\, L\,(x\,x))(\lambda x.\, L\,(x\,x))$$

M is a reducible expression, redex for short:

$$\begin{aligned} M &\equiv (\lambda x.\, L\,(x\,x))(\lambda x.\, L\,(x\,x)) \\ &\rightsquigarrow_\beta L\,((\lambda x.\, L\,(x\,x))(\lambda x.\, L\,(x\,x))) \\ &\equiv L\,M. \end{aligned}$$

Hence, $M =_\beta LM$, as required. □

Remark 12. See how LM is the application of M to L and that using AppR we can keep β-reducing M the same way we did for $x = f(x) = f(f(x))\ldots$ only this time using lambda terms:

$$M \rightsquigarrow_\beta L\,M \rightsquigarrow_\beta L\,L\,M \rightsquigarrow^*_\beta L^n M.$$

Now that nameless self-reference has been proven to exist within the the λ-calculus, we take a step forward and introduce a few of the most important fixed point combinators and explore their different natures.

Definition 1.1.28. Y Combinator. The term that given some other term returns its fixed point.

$$Y \triangleq \lambda f.\,(\lambda x.\, f\,(x\,x))\,(\lambda x.\, f\,(x\,x))$$

Note 3. $Y \equiv \lambda L.M$ M defined previously. Improve this, find some other way to put it? note it is an improved Ω

Remark 13. Even though the Y combinator is of theoretical interest since it allows us to find recursion within the λ-calculus, it is of little practical use since it always runs into non-termination e.g.

$$\begin{aligned} Y\,F &\equiv (\lambda f.\,(\lambda x.\, f\,(x\,x))(\lambda x.\, f\,(x\,x)))F \\ &\rightsquigarrow_\beta (\lambda x.\, F\,(x\,x))(\lambda x.\, F\,(x\,x)) \\ &\rightsquigarrow_\beta F((\lambda x.\, F\,(x\,x))(\lambda x.\, F\,(x\,x))) \\ &\rightsquigarrow_\beta F(F((\lambda x.\, F\,(x\,x))(\lambda x.\, F\,(x\,x)))) \\ &\rightsquigarrow_\beta \cdots \end{aligned}$$

This is due to its lazy nature, this means that the argument F is not evaluated until Y is done unravelling i.e. never.

To work around this issue, several other combinators exist, one of the most well-known, the Z combinator is defined next:

Definition 1.1.29. Z combinator

$$Z \triangleq \lambda f.\,(\lambda x.\, f\,(\lambda v.\, x\,x\,v))\,(\lambda x.\, f\,(\lambda v.\, x\,x\,v))$$

Example 1.1.12.

$$\begin{aligned}
Z\,F &\equiv (\lambda f.\,(\lambda x.\,f\,(\lambda v.\,x\,x\,v))\,(\lambda x.\,f\,(\lambda v.\,x\,x\,v)))\,F \\
&\rightsquigarrow_\beta (\lambda x.\,F\,(\lambda v.\,x\,x\,v))\,(\lambda x.\,F\,(\lambda v.\,x\,x\,v)) \\
&\rightsquigarrow_\beta F\,(\lambda v.\,(\lambda x.\,F\,(\lambda v.\,x\,x\,v))\,(\lambda x.\,F\,(\lambda v.\,x\,x\,v))\,v) \\
&\equiv F\,(\lambda v.\,Z\,F\,v)
\end{aligned}$$

1.2 Simply Typed λ-Calculus

We covered the untyped λ-calculus, highlighting its strengths and limitations. This section introduces the notion of typing, which restricts its expressiveness while improving the tractability of studying its properties. The idea of typing arises from the usual definition of a function, understood as a mapping between elements of sets. For example, consider a function $f : A \to B$ defined by $a \mapsto f(a)$, to develop the analogy.

Notice how we are able to recognize A as the domain and B as the codomain, and how the mapping is meaningful only within this context—otherwise f would lack sense and purpose. Hence, it is important to first restrict the environment in which f lives and then remain within it while defining the mapping. In this instance, that means using constructions and operations that exist within the realms of A and B.

In the context of the λ-calculus, types serve as a means of restriction, similar to how we use logical predicates to define sets. Just as $f : ? \to ?$ is refined to $f : A \to B$ by specifying requirements on the arguments and ensuring properties of the outputs, we use types to impose requirements on the arguments of an abstraction and to ensure properties of the reduction process of that term.

Since the λ-calculus treats all terms as first-class citizens, attempting to impose restrictions directly on arguments is not effective. Instead, we define restrictions on all terms by pairing each term with additional information called a *type*. This tells us about the nature of a given term, which may be of two kinds: a simple type or a function type (written as $\to$). Because Λ is infinite, we require a set of rules that automatically assigns types to terms.

Introducing a typing system into the λ-calculus splits Λ into two categories: Typable and Non-typable terms. These sets are disjoint, and the separation is strict and exhaustive. This simple type system classifies expressions such as $x(\lambda y.y)$ or Y as untypable, while expressions such as $\lambda x.x$ have type $\alpha \to \alpha$.

Let us now proceed by introducing some of the necessary concepts required to type λ-terms.

Definition 1.2.1. The set of all types, $\mathbf{T}$:

Let $\mathbf{B} = \{\alpha, \beta, \gamma, \dots\}$ be the set of all basic type, usually denoted using symbols from the greek alphabet.

...

```
<type> ::= <atom>
         | "(" <type> "→" <type> ")"
<atom> :: ∈ B
```

typing a la church, typing a la curry

Chapter 2

Intuitionistic Logic

2.1 Intuitionistic Logic

This section serves as a shallow introduction to intuitionistic logic...

2.2 Natural Deduction

2.3 The Curry-Howard Bijection

Chapter 3

Category Theory

Category theory abstracts the notion of internal structure and studies mathematical objects indirectly through connections to other objects. In practical terms, instead of studying objects in isolation, category theory shifts the focus onto morphisms among objects. This means that instead of studying mathematical entities from a definitional stand point, it emphasizes the importance of how constructions relate to one another.

It is against the rules of categorical thinking to manipulate mathematical objets directly, a categorical lens must be used. This categorical lens abstracts away the details of mathematical constructions but enables a more general and abstract toolkit to tacke problems. This perspective unifies many areas of mathematics under a common language.

Several topics in computer science are deeply related to category theory, we only mention CCC's and LCCC's in the effort not to give away the purpuse of the chapter.

Definition 3.0.1. A category $\mathcal{C}$ consists of:

- A collection of objects
- A collection of arrows or morphisims where each arrow f has a domain object $dom(f)$ and a codomain object $cod(f)$. These can be defined using $f : A \to B$ to signify $dom(f) = A$ and $cod(f) = B$
- A composition operator assigning to each pair of arrows f and g, with $cod(f) = dom(g)$, a composite arrow $g \circ f : dom(f) \to cod(g)$ satisfying the following associative law:
$$h \circ (g \circ f) = (h \circ g) \circ f$$
where $f : A \to B, g : B \to C, h : C \to D$ are arrows of a category
- For each object A, an identity arrow $id_A : A \to A$ satisfying the following identity law:
$$id_B \circ f = f \quad f \circ id_A = f$$
for any $f : A \to B$

Remark 14. prev definition contemplates only small categories. usual categories are defined in terms of collections. that is a class of arrows and objects for example. we will just deal with small categories, sets of arrows and objects

Example 3.0.1. The most elemental category is the category that has no elements and no arrows, it is commonly represented with a 0. Category 1 contains one object and one identity arrow, as required by the definition. Category 2 contains three arrows, the identities and one arrow from 1 to 2. Category 3 builds up on this as is observed in the following diagram:

$$\begin{tikzcd} 1 \arrow[loop right, "id_1"] \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[loop left, "id_1"] \arrow[r] & 2 \arrow[loop right, "id_2"] \end{tikzcd}$$

$$\begin{tikzcd} & 2 \arrow[loop right, "id_2"] \arrow[d] \\ 1 \arrow[loop left, "id_1"] \arrow[ur] \arrow[r] & 3 \arrow[loop right, "id_3"] \end{tikzcd}$$

Example 3.0.2. In this example we show that **Set** is a category where objects correspond to sets $A, B, C, \ldots$ and arrows to functions. For $f : A \to B$ and $g : B \to C$, composition of functions is $(g \circ f)(a) = g(f(a))$, which is a function $g \circ f : A \to C$. Composition is associative since for $h : C \to D$ we have that: $h \circ (g \circ f)(a) = h(g(f(a))) = (h \circ g)(f(a)) = (h \circ g) \circ f(a)$. Lastly, each set A, the identity morphism is $\mathrm{id}_A(a) = a$, satisfying $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$. Therefore, **Set** is a category. □

Example 3.0.3. In this example we show that any partially ordered set $(P, \leq)$ can be viewed as a category **Pos** whose objects are elements of P and there is a unique arrow $p \to q$ if and only if $p \leq q$. Composition is defined by transitivity: if $p \leq q$ and $q \leq r$, then $p \leq r$, so the composite $p \to r$ exists and is unique. Associativity holds automatically since there is at most one arrow between any two objects. Lastly, each $p \in P$ has an identity arrow $p \to p$ that corresponds to the reflexivity property of partial orders. Therefore, every poset defines a category. □

There exist several well known categories tied to mathematics and compite. Category of categories... Monoid

Category	**Description**
Set	Sets as objects, functions as morphisms.
Grp	Groups and group homomorphisms.
Ring	Rings and ring homomorphisms.
$\mathbf{Mod}_R$	Modules over a ring R and R-linear maps.
$\mathbf{Vect}_k$	Vector spaces over field k and linear maps.
Top	Topological spaces and continuous functions.
Man	Smooth manifolds with smooth maps.
Hilb	Hilbert spaces and bounded linear maps.

Table 3.1: Key mathematical categories.

to build some intuition on why categories relate to programs que alguien que controle corrija el siguiente ejemplo q es mas inventado que que los gamusinos

Example 3.0.4. Let us consider a simple functional programming language with the following types:

$$T ::= \mathrm{Nat} \mid \mathrm{Bool} \mid T \times T \mid T \to T$$

Category	Description
Mon	Monoids and monoid homomorphisms.
CPO	Complete partial orders and continuous maps.
Dom	Domains as posets with Scott-continuous maps.
Rel	Sets and binary relations as morphisms.
Pfn	Sets and partial functions as morphisms.
Par	Sets and partial maps (generalized morphisms).
Kl(T)	Kleisli category of a monad T on a base category.
Endo(**Set**)	Endofunctors on the category of sets.

Table 3.2: Key categories in computer science.

The type system is closed under product and function type constructors. A valid compound type is $(\text{Nat} \times \text{Bool}) \to \text{Nat}$. A categorical interpretation of this scenario reveals types are objects and functions morphisms.

Identity $\text{id}_A : A \to A$ can be defined as $\text{id}_A = \lambda x : A.\, x$, which is well-typed. Identity laws are verified trivially: $\text{id}_B \circ f = f = f \circ \text{id}_A$.

Compositionality is defined as sequential application of functions that share domain and codomain through $f : A \to B$ and $g : B \to C$ like $g \circ f : A \to C$ by $g \circ f = \lambda x : A.\, g(f(x))$, which is well-typed by function application.

Composition is associative due to β-equivalence in lambda calculus:

$$h \circ (g \circ f) = \lambda x{:}A.h(g(f(x))) = (h \circ g) \circ f$$

Therefore, **FL** satisfies all axioms of a category.

Aside from defining categories from mathematical objects, categores can be build from other categories. One way to do this is via opposite categories.

Definition 3.0.2. For a category $\mathcal{C}$, the opposite category $\mathcal{C}^{op}$ is defined by:

- $\text{Ob}(\mathcal{C}^{op}) = \text{Ob}(\mathcal{C})$,
- For $A, B \in \text{Ob}(\mathcal{C})$, $\mathcal{C}^{op}(A, B) = \mathcal{C}(B, A)$,
- Composition is reversed: if $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$, then

$$f^{op} \circ g^{op} = (g \circ f)^{op}.$$

Definition 3.0.3. Let $g_1, g_2 : X \to A$ (resp. $h_1, h_2 : B \to Y$). An arrow $f : A \to B$ in $\mathcal{C}$ is a monomorphism (resp. epimorphism) if:

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2 \quad (\text{resp. } h_1 \circ f = h_2 \circ f \implies h_1 = h_2)$$

Note 4. Monomorphisims generalize the notion of injectivity, while epimorphisms are the categorical generalization of surjectivity.