- Array
- Stack
- Queue
- Deque
- Linked List
    - Singly Linked List
    - Circularly Linked List
    - Doubly Linked List
    - Positional List
- Array-based vs. Linked-based
- Tree
    - General Tree
    - Binary Tree
    - Implementations
        - Linked Binary Tree
        - Array-based Binary Tree
        - Linked General Tree
    - Tree Traversal Algorithms
        - Preorder (General Tree)
        - Postorder (General Tree)
        - Breadth-first (General Tree)
        - Inorder (Binary Tree)
- Priority Queue
    - Unsorted Priority Queue
    - Sorted Priority Queue
    - Sorting with Priority Queue
    - Adaptable Priority Queue
- Heap
    - Heap-based Priority Queue
    - Heap-Sort
- Map
- Hash Table
    - Hash Functions
        - Hash codes
        - Compression functions
    - Collision-handling schemes
        - Separate Chaining
        - Open Addressing
            - Linear Probing
            - Quadratic Probing
            - Double Hashing
    - Rehashing
    - Sorted Search Table
    - Set
- Search Tree
    - Binary Search Tree

        - Efficiency
    - Balanced Search Tree
        - Rotation
        - AVL Tree
        - Splay Tree
        - (2, 4) Tree
        - Red-Black Tree
- Sorting
    - Insertion-Sort
    - Selection-Sort
    - Bubble-Sort
    - Heap-Sort
    - Merge-Sort
    - Quick-sort

Array

Referential array. Array of object references.

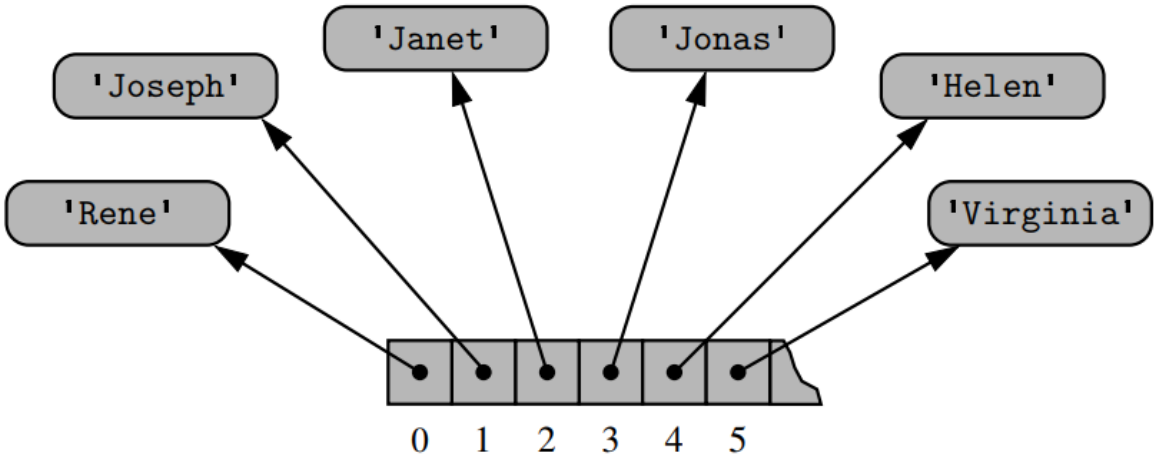
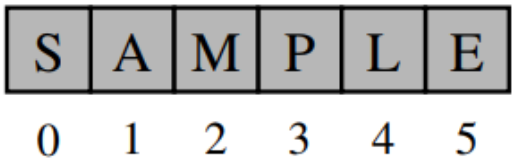


Figure 5.4: An array storing references to strings.

E.g., Python lists, tuples.

Compact array. Array that store bits representing primary data.



Dynamic array. Resizable array that grows or shrinks based on the number of items it contains, so that its operations can have amortized $O(1)$ runtime.

E.g., Python list is implemented using dynamic array.

Operation	Runtime
`data[j] = val`	$O(1)$
`data.append(value)`	$O(1)^*$
`data.insert(k, value)`	$O(n-k+1)^*$ (element shifts)
`data.pop()`	$O(1)^*$
`data.pop(k)` `del data[k]`	$O(n-k)^*$ (element shifts)
`data.remove(value)`	$O(n)^*$
`data1.extend(data2)` `data1 += data2`	$O(n_2)^*$

Operation	Runtime
`data.reverse()`	$O(n)$
`data.sort()`	$O(n \log n)$

- Amortized.

Stack

Method	Description	Runtime
`S.push(e)`	Add e to top of stack.	$O(1)$*
`S.pop()`	Remove and return item from top of stack.	$O(1)$*
`S.top()`	Return reference to item at top of stack.	$O(1)$
`S.is_empty()`	True if the stack is empty.	$O(1)$
`len(S)`	Return the number of items in the stack.	$O(1)$

*If implemented using a Python list, these operations are amortized.

Applications:

1. Reverse a list (push all items in and pop them one by one, first in last out).
2. Parenthesis matching.

Queue

Method	Description	Runtime
`Q.enqueue(e)`	Add `e` to end of queue.	$O(1)$*
`Q.dequeue()`	Remove and return item from front of queue.	$O(1)$*
`Q.first()`	Return item at front of queue.	$O(1)$
`Q.is_empty()`	True if the queue is empty.	$O(1)$
`len(Q)`	Return the number of items in the queue.	$O(1)$

*If implemented using a Python list (circular, wraps around when reaching end of list), these operations are amortized.

Deque

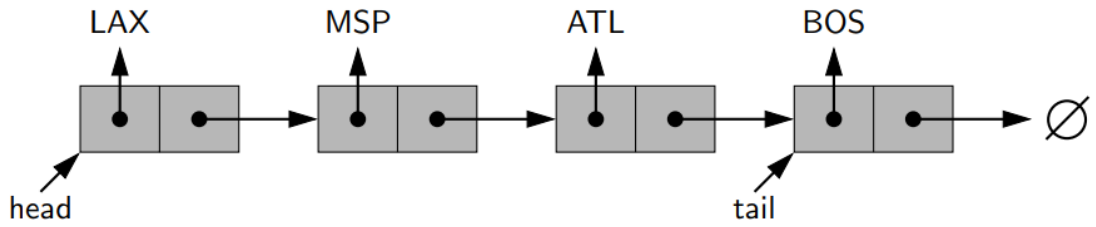
Double-ended queue. ADT that can add and remove elements from both ends of the queue.

Method	Description	Runtime
`D.add_first(e)`, `D.add_last(e)`	Add `e` to front/back of dequeue.	$O(1)^*$
`D.delete_first(e)`, `D.delete_last(e)`	Remove and return item from front/back of dequeue.	$O(1)^*$
`D.first()`, `D.last()`	Return and return item at the front/back of dequeue.	$O(1)$
`D.is_empty()`	True if the dequeue is empty.	$O(1)$
`len(D)`	Return the number of items in the dequeue.	$O(1)$

*If implemented using a Python list (circular, wraps around), these operations are amortized.

Linked List

Singly Linked List



Applications:

1. Implement the Stack ADT, all operations are worst-case $O(1)$.
2. Implement the Queue ADT, all operations are worst-case $O(1)$.

Circularly Linked List

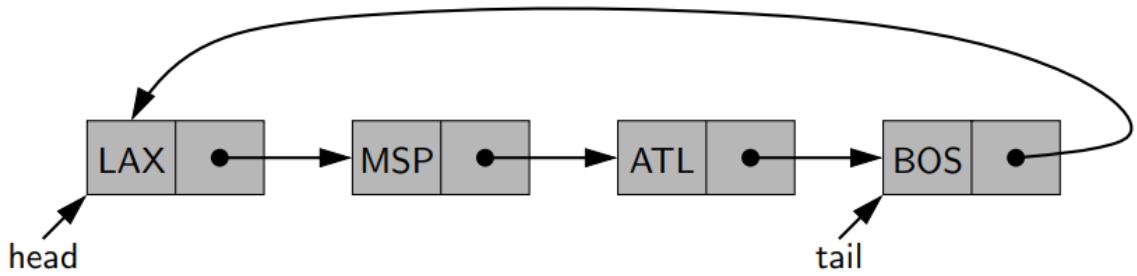


Figure 7.7: Example of a singly linked list with circular structure.

Applications:

1. Implement the Queue ADT, with more efficient method for wrapping around.

Doubly Linked List

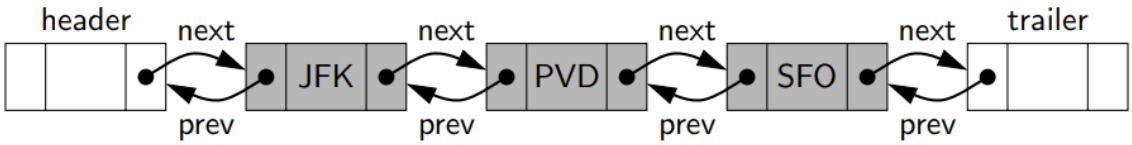


Figure 7.10: A doubly linked list representing the sequence { JFK, PVD, SFO }, using sentinels header and trailer to demarcate the ends of the list.

Applications:

1. Implement the Deque ADT.
2. Implement the Positional List ADT.

Positional List

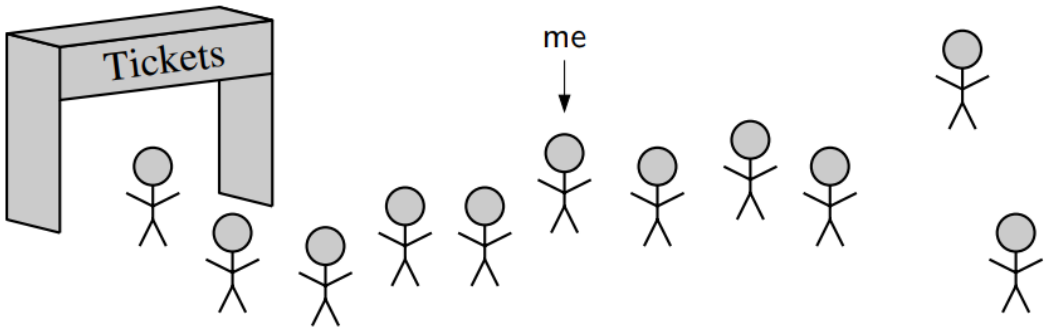


Figure 7.14: We wish to be able to identify the position of an element in a sequence without the use of an integer index.

Method	Description
`L.first()`, `L.last()`	Return the position of the first/last item.
`L.before(p)`, `L.after(p)`	Return the position immediately before/after position `p`.
`L.is_empty()`	True if the positional list is empty.
`len(L)`	Return the number of items in the positional list.
`iter(L)`	Return a forward iterator of items in the positional list.
`L.add_first(e)`, `L.add_last(e)`	Add `e` to the front/back of the positional list.
`L.add_before(p, e)`, `L.add_after(p, e)`	Add `e` before/after position `p`.
`L.replace(p, e)`	Replace the item at position `p` with `e`.
`L.delete(p)`	Remove and return the item at position `p`.

Applications:

1. Maintain access frequencies.

Array-based vs. Linked-based

Metrics	**Array-based**	**Link-based**
access based on index	$O(1)$	$O(n)$
search	$O(\log n)$ if sorted (binary search)	$O(n)$
insertion, deletion	$O(n)$ worst case (need to shift elements)	$O(1)$ at arbitrary position
memory usage	$2n$ worst case (after resize)	$2n$ for singly-linked lists $3n$ for doubly-linked lists

Compromise between array-based and link-based structures: Skip lists achieve average $O(\log n)$ search and update operations via a probabilistic method.

Tree

General Tree

A tree T is set of nodes storing elements such that the nodes have a parent-child relationship that satisfies the following properties:

- If T is nonempty, it has a special node, called the root of T , that has no parent.
- Each node v of T different from the root has a unique parent node w; every node with parent w is a child of w.

Sibling. Two nodes are siblings if they have the same parent node.

External. A node is external if it has no children. A.k.a leaves.

Internal. A node is internal if it has >= 1 children.

Edge. An edge of tree T is a pair of nodes (u, v) such that u is the parent of v, or vice versa.

Path. A path of T is a sequence of nodes such that any two consecutive nodes in the sequence form an edge.

Ordered Tree. A tree is ordered if there is a meaningful linear order among the children of each node.

Depth of node. The depth of a node is the number of its ancestors, excluding itself.

Depth of node (recursive). If p is the root, then its depth is 0. Otherwise, the depth of p is $1+$ depth of p's parent.

Height of node (recursive). If p is a leaf, then its height is 0. Otherwise, the height of p is $1+$ the maximum of p's children's heights.

Height of tree. The height of a tree is the height of its root.

Method	Description
`T.root()`	Return the position of the tree's root.
`T.is_root(p)`	True if position `p` is the tree's root.
`T.parent(p)`	Return the position of `p`'s parent.
`T.num children(p)`	Return the number of `p`'s children.
`T.children(p)`	Generate an iteration of position `p`'s children.
`T.is leaf(p)`	True if position `p` does not have any children.
`len(T)`	Return the number of positions in the tree.
`T.is empty()`	True if the tree does not contain any position.
`T.positions()`	Generate an iteration of the positions in the tree.
`iter(T)`	Generate an iteration of the elements in the tree.

Method	Description
`T.depth(p)`	Return the depth of `p`.
`T.height(p)`	Return the height of `p`.

Proposition. The height of a nonempty tree is the maximum of its leaves' depths.

Proposition. In a tree with n nodes, the sum of the number of children of all nodes is $n-1$.

Proof. Every node except for the root is some other node's child.

Binary Tree

Binary tree. A binary tree is an ordered tree such that:

1. Every node has at most two children.
2. Each child node is either a left child or a right child.
3. A left child precedes a right child in the order of children of a node.

Binary tree (recursive). A binary tree is either empty or consists of:

- A node r, called the root of T, that stores an element
- A binary tree (possibly empty), called the left subtree of T
- A binary tree (possibly empty), called the right subtree of T

Method	Description
`T.left(p)`, `T.right(p)`	Return the position of `p`'s left/right child.
`T.sibling(p)`	Return the position of `p`'s sibling.

Proper/Full. A binary tree is proper or full if each node has either zero or two children. That is, all its internal nodes have two children.

Proposition. In a nonempty proper binary tree T, with n_E external nodes and n_I internal nodes, we have $n_E = n_I + 1$.

Proof. If h is T's height, then $n_E = 2^h$, $n_I = 2^0 + 2^1 + 2^2 + \cdots 2^h = 2^h - 1$.

Implementations

Linked Binary Tree

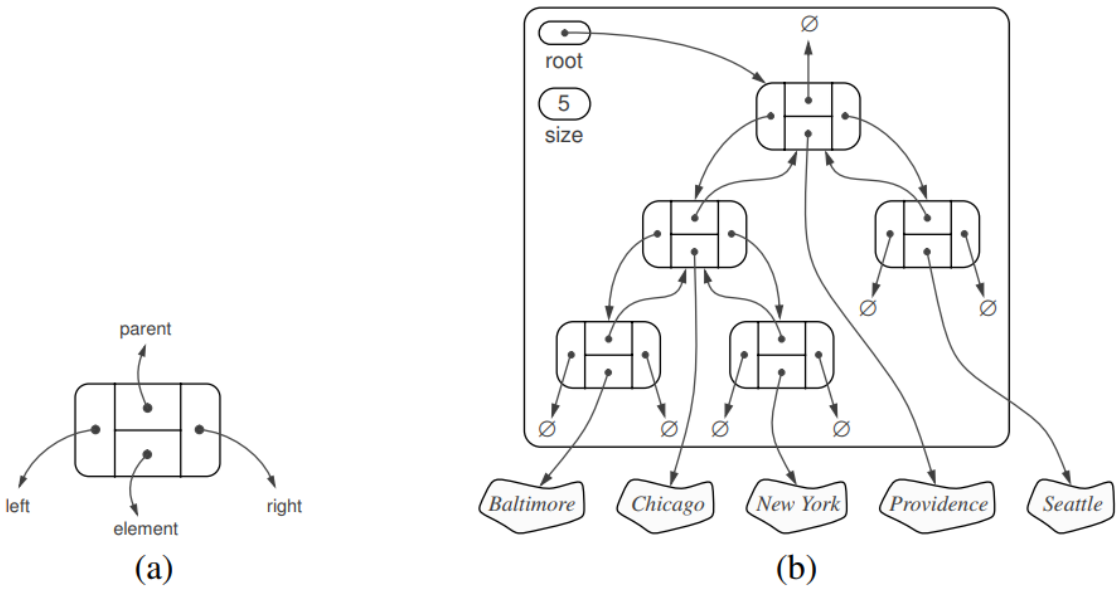


Figure 8.11: A linked structure for representing: (a) a single node; (b) a binary tree.

Method	Description
`T.add_root(e)`	Add root `e` to an empty tree.
`T.add_left(p, e)`, `T.add_right(p, e)`	Add `e` as left/right child to `p`.
`T.replace(p, e)`	Replace element at position `p` with `e`.
`T.delete(p)`	Remove the node at position `p` and replace it with its only child.
`T.attach(p, T1, T2)`	Attach `T1`, `T2` as left and right subtress of the leaf `p`.

Operation	Runtime
`len`, `is_empty`	$O(1)$
`root`, `parent`, `left`, `right`, `sibling`, `children`, `num_children`	$O(1)$
`is_root`, `is_leaf`	$O(1)$
`depth(p)`	$O(d_p + 1)$
`height`	$O(n)$
`add_root`, `add_left`, `add_right`, `replace`, `delete`, `attach`	$O(1)$

Array-based Binary Tree

For every position p of T, let $f(p)$ be the integer defined as follows. • If p is the root of T, then $f(p) = 0$. • If p is the left child of position q, then $f(p) = 2f(q) + 1$. • If p is the right child of position q, then $f(p) = 2f(q) + 2$.

Array-based binary tree. An array-based structure A (such as a Python list), with the element at position p of T stored at $A[f(p)]$.

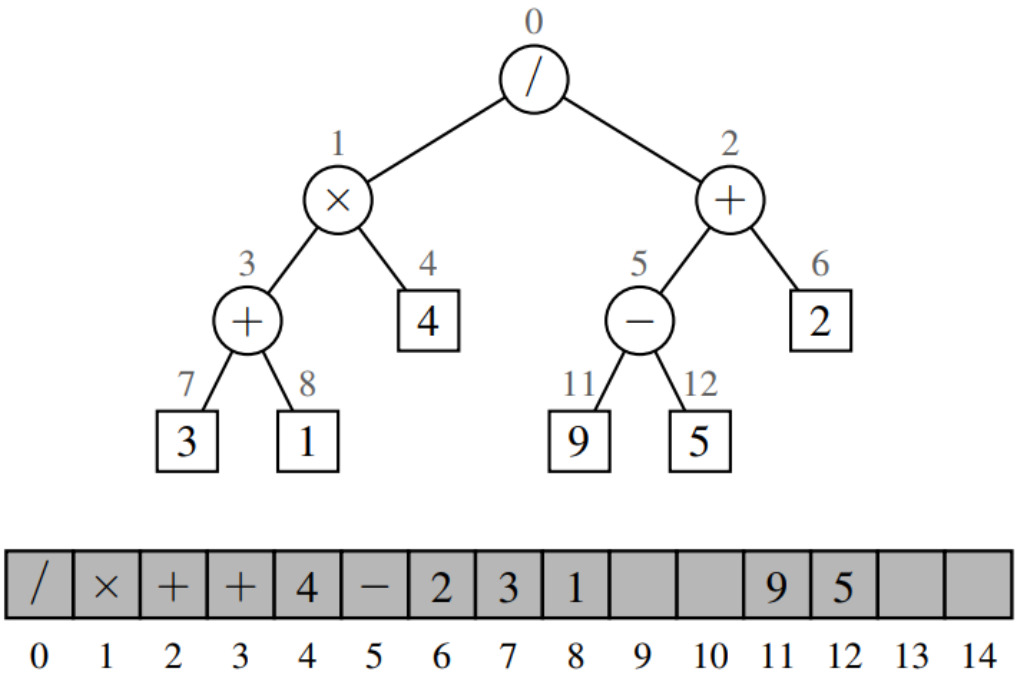


Figure 8.13: Representation of a binary tree by means of an array.

`delete` is $O(n)$ as all the node's descendents need to be shifted in the array.

Linked General Tree

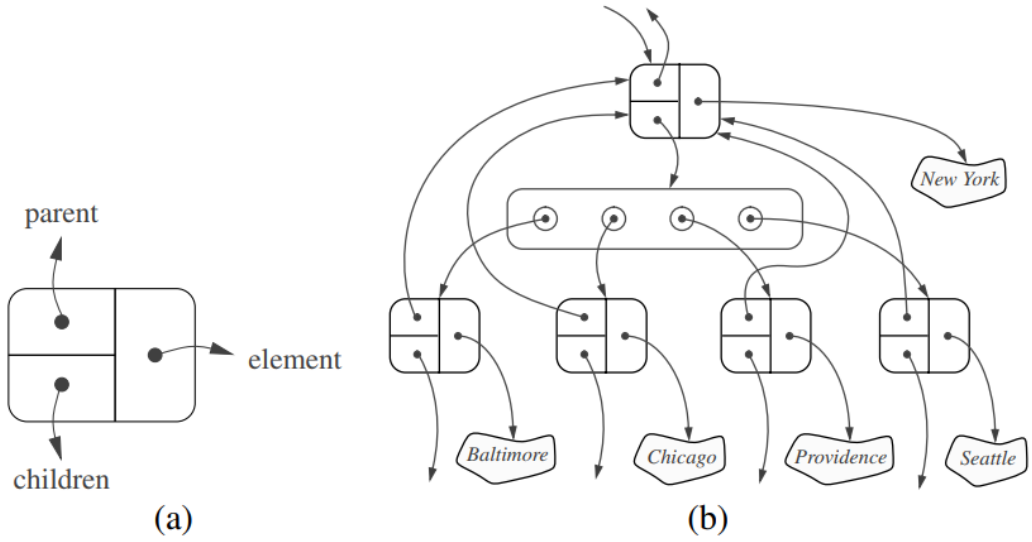


Figure 8.14: The linked structure for a general tree: (a) the structure of a node; (b) a larger portion of the data structure associated with a node and its children.

Operation	Runtime
`len`, `is_empty`	$O(1)$
`root`, `parent`, `is_root`, `is_leaf`	$O(1)$
`children(p)`	$O(c_p + 1)$
`depth(p)`	$O(d_p + 1)$

Operation	Runtime
`height`	$O(n)$

Tree Traversal Algorithms

Traversals are $O(n)$ as they must visit every node in the tree.

Binary search is $O(\log n)$ in a proper binary tree.

Preorder (General Tree)

Visit node, then visit node's children.

```
Algorithm preorder(T, p):
  perform the "visit" action for position p
  for each child c in T.children(p) do
    preorder(T, c)                    {recursively traverse the subtree rooted at c}
```

Code Fragment 8.12: Algorithm preorder for performing the preorder traversal of a subtree rooted at position p of a tree T.

Figure 8.15 portrays the order in which positions of a sample tree are visited during an application of the preorder traversal algorithm.

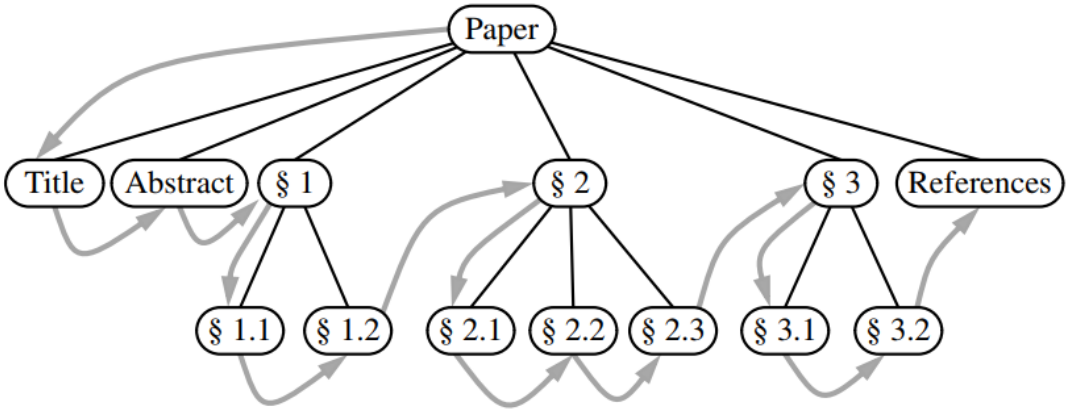


Figure 8.15: Preorder traversal of an ordered tree, where the children of each position are ordered from left to right.

Postorder (General Tree)

Visit node's children, then visit node.

```
Algorithm postorder(T, p):
  for each child c in T.children(p) do
    postorder(T, c)                    {recursively traverse the subtree rooted at c}
  perform the "visit" action for position p
```

Code Fragment 8.13: Algorithm postorder for performing the postorder traversal of a subtree rooted at position p of a tree T.

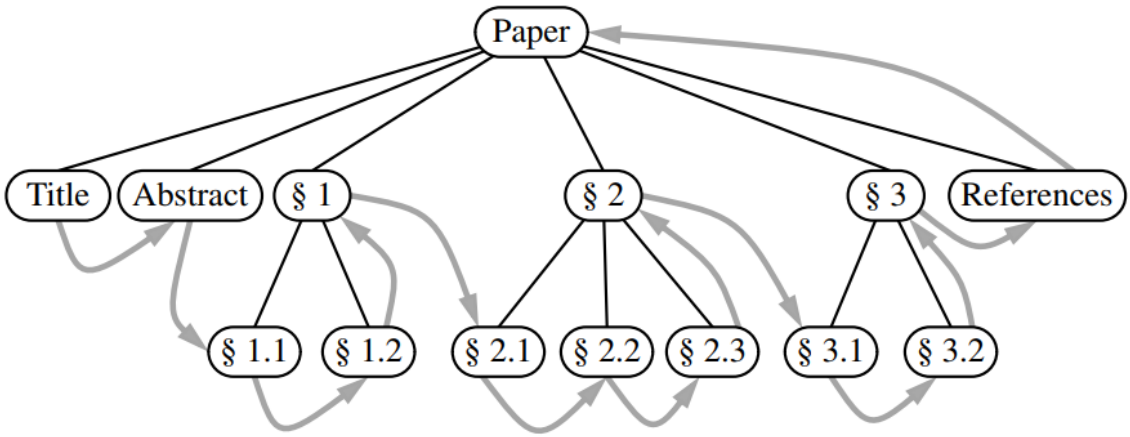


Figure 8.16: Postorder traversal of the ordered tree of Figure 8.15.

Breadth-first (General Tree)

Visit nodes level by level.

Not recursive.

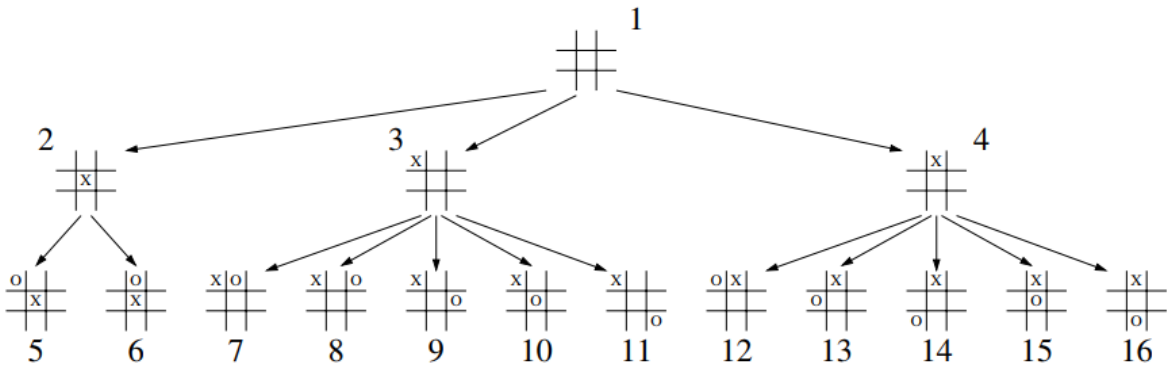


Figure 8.17: Partial game tree for Tic-Tac-Toe, with annotations displaying the order in which positions are visited in a breadth-first traversal.

Dequeue to get node. Visit node, then enqueue node's children.

```
Algorithm breadthfirst(T):
  Initialize queue Q to contain T.root( )
  while Q not empty do
    p = Q.dequeue( )                    {p is the oldest entry in the queue}
    perform the "visit" action for position p
    for each child c in T.children(p) do
      Q.enqueue(c)      {add p's children to the end of the queue for later visits}
```

Code Fragment 8.14: Algorithm for performing a breadth-first traversal of a tree.

Inorder (Binary Tree)

Visit left subtree. Visit right subtree. Visit node.

```
Algorithm inorder(p):
  if p has a left child lc then
    inorder(lc)                    {recursively traverse the left subtree of p}
  perform the "visit" action for position p
  if p has a right child rc then
    inorder(rc)                    {recursively traverse the right subtree of p}
```

Code Fragment 8.15: Algorithm inorder for performing an inorder traversal of a subtree rooted at position p of a binary tree.

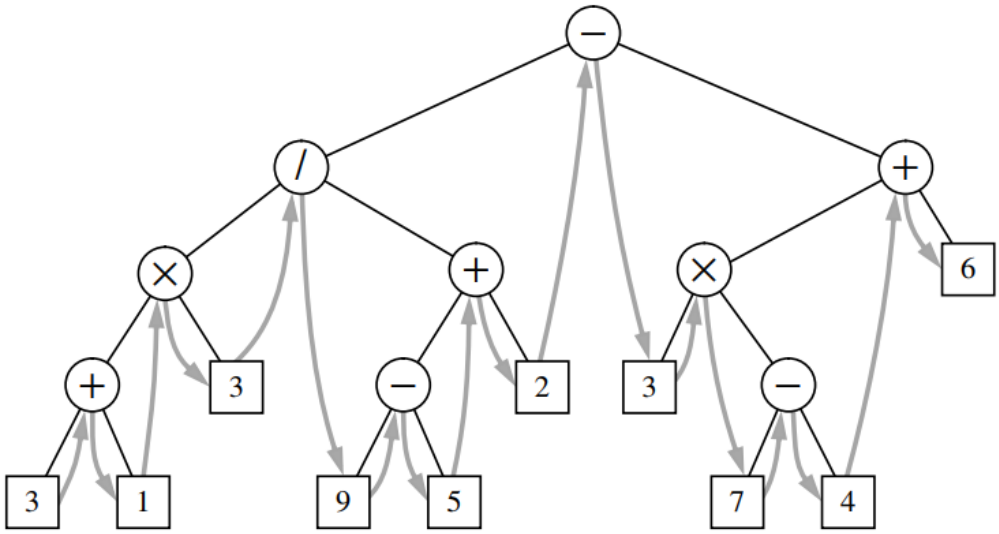


Figure 8.18: Inorder traversal of a binary tree.

Priority Queue

Method	Description
`P.add(k, v)`	Add item with key `k` and value `v` into the priority queue.
`P.min()`	Return item with the minimum key in the priority queue.
`P.remove_min()`	Remove and return an item with the minimum key in the priority queue.
`P.is_empty()`	True if the priority queue is empty.
`len(P)`	Return the number of items in the priority queue.

Unsorted Priority Queue

Add item to the end of the priority queue, find minimum to remove in $O(n)$.

$O(1)$ insertions, $O(n)$ removals (best-case, because it always takes $O(n)$ to find the minimum).

Method	Runtime
`P.add(k, v)`	$O(1)$
`P.min()`	$O(n)$
`P.remove_min()`	$O(n)$
`P.is_empty()`	$O(1)$
`len(P)`	$O(1)$

Sorted Priority Queue

Maintain sortedness when inserting items, minimum is at front of the priority queue,

$O(n)$ insertions (best-case is $O(1)$, because the items may come in as sorted), $O(1)$ removals.

Method	Runtime
`P.add(k, v)`	$O(n)$
`P.min()`	$O(1)$
`P.remove_min()`	$O(1)$
`P.is_empty()`	$O(1)$
`len(P)`	$O(1)$

Sorting with Priority Queue

1. Add each item one by one to the priority queue.

2. Keep removing the minimum from the priority queue.

Implementation/Operation	`add`	`remove_min`
Unsorted List	Add to the end of list in $O(1)$.	Find minimum to remove in best-case $O(n)$.
Sorted List	Insert into sorted list in $O(n)$ (best-case $O(1)$).	Remove minimum from front of list in $O(1)$.

With unsorted list-based priority queue, this is selection-sort (best case $O(n^2)$). With sorted list-based priority queue, this is insertion-sort (best-case $O(n)$).

Adaptable Priority Queue

Additional operations:

1. Remove arbitrary entry.
2. Update the key (priority) of exiting entry.

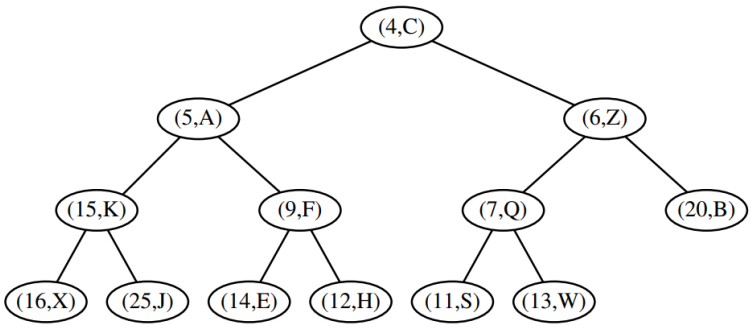
Heap

A heap is a binary tree T that stores a collection of items at its positions and that satisfies the following two properties:

Heap-Order Property. (relational): In a heap T, for every position p other than the root, the key stored at p >= the key stored at p's parent.

This implies that the minimum item is at the heap's root.

Complete Binary Tree Property. (structural): A heap T with height h is a complete binary tree if levels $0, 1, 2, \ldots, h-1$ of T have the maximum number of nodes possible and the remaining nodes at level h reside in the leftmost positions.

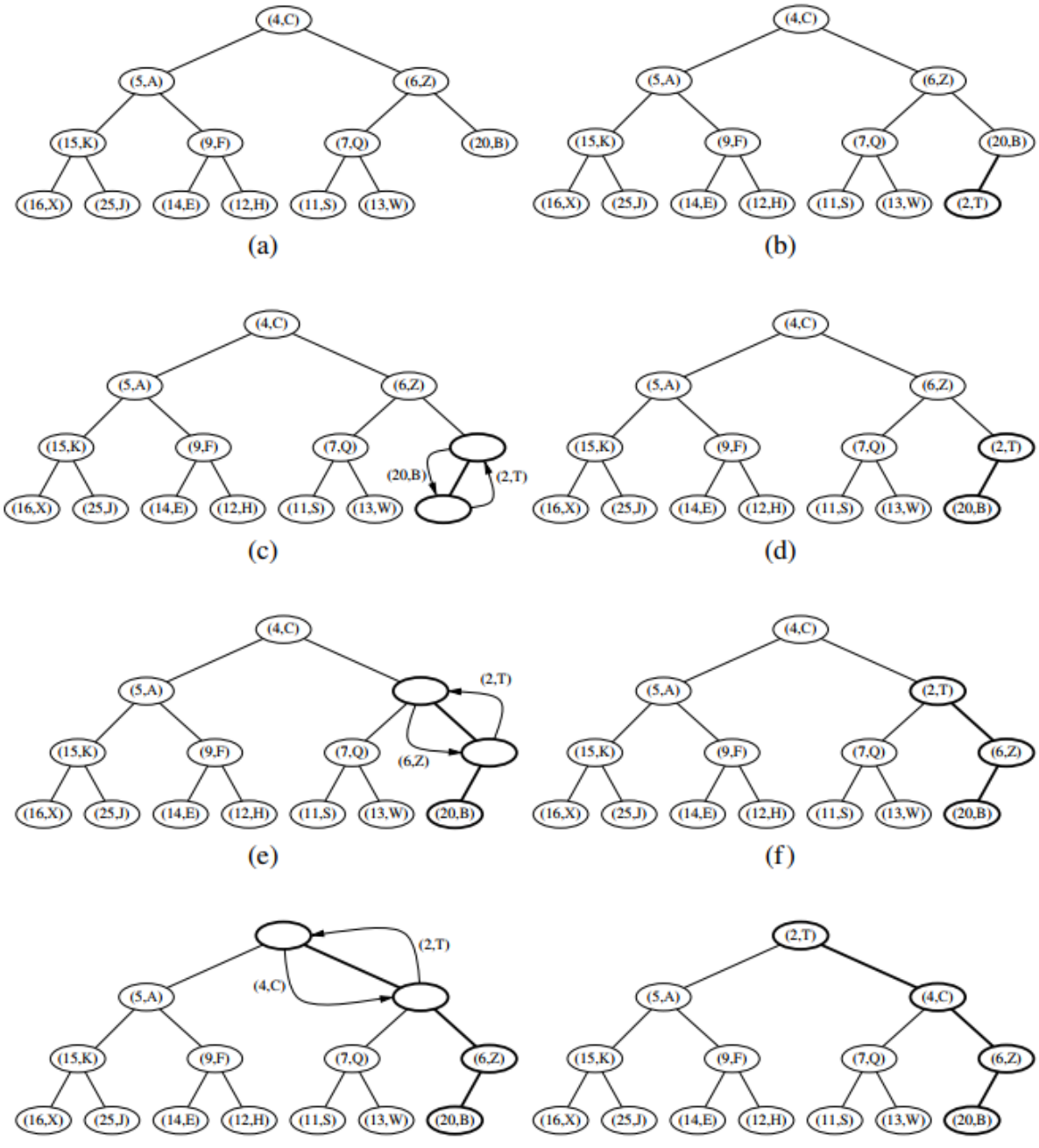


Proposition. A heap T storing n entries has height $h = \text{floor}(\log n)$.

Heap-based Priority Queue

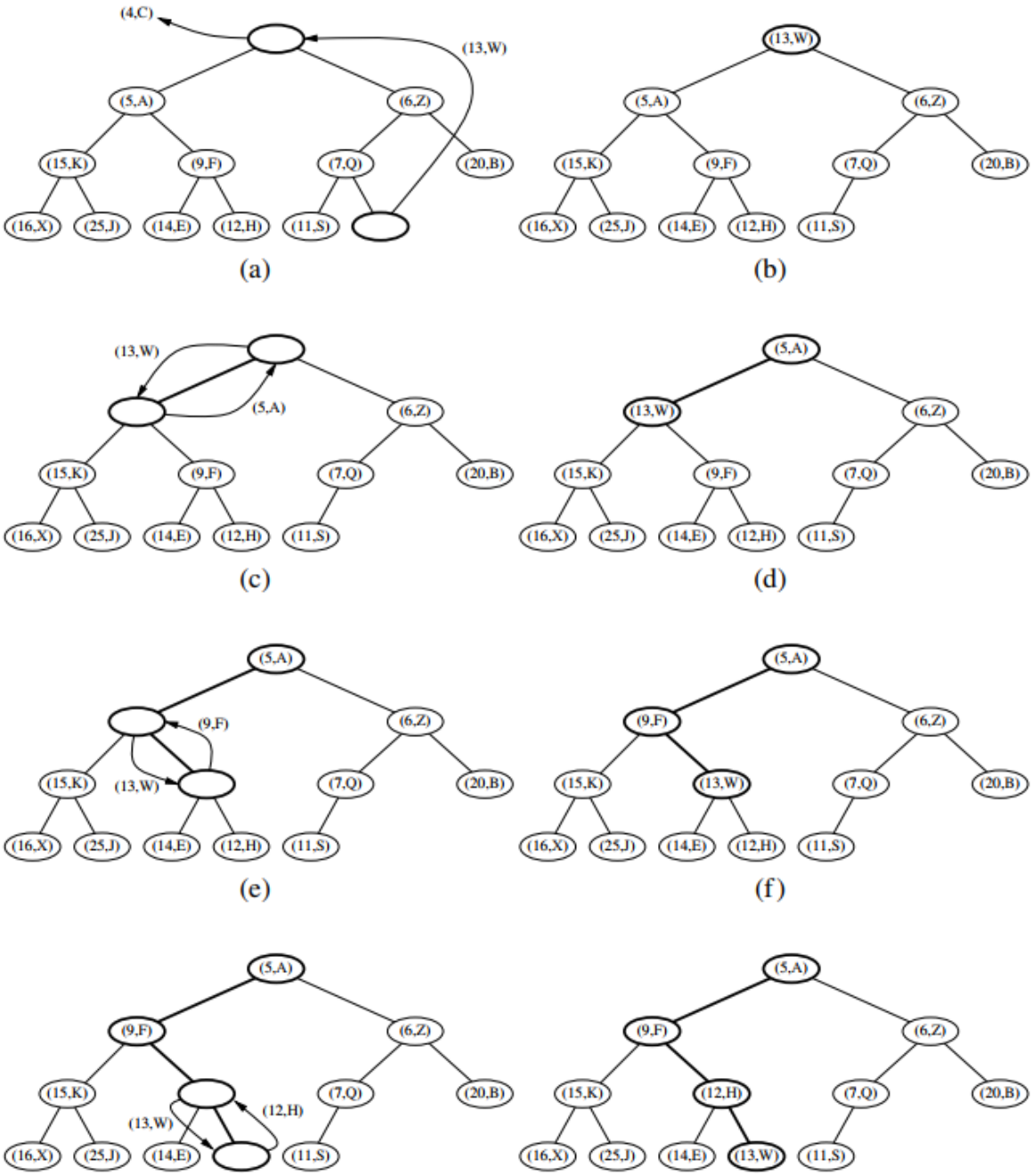
Add:

1. Add item to rightmost node at bottom level to maintain completeness.
2. Up-heap bubbling to swap the new node into correct position to maintain heap-order.



Remove:

1. Remove minimum item from top of heap (root).
2. Copy item at rightmost node at bottom level to root.
3. Down-heap bubbling to swap the node into correct position to maintain heap order.



Method	Runtime
`P.add(k, v)`	$O(\log n)$*
`P.min()`	$O(1)$
`P.remove_min()`	$O(\log n)$*
`P.is_empty()`	$O(1)$
`len(P)`	$O(1)$

*For array-based-tree-based heap, this is amortized, but array-based heap can locate last position in $O(1)$ via index access.

Implementation/Operation	`add`	`remove_min`

Implementation/Operation	`add`	`remove_min`
Unsorted List	Add to the end of list in $O(1)$.	Insert into sorted list in $O(n)$.
Sorted List	Find minimum in $O(n)$ to remove.	Remove minimum from front of list in $O(1)$.
Array-based Heap	1. Find last position in $O(1)$. 2. Up-heap bubbling in $O(\log n)$.	1. Remove minimum at root in $O(1)$. 2. Find last position and copy to root in $O(1)$. 3. Down-heap bubbling in $O(\log n)$.
Linked Heap	1. Find last position in $O(\log n)$. 2. Up-heap bubbling in $O(\log n)$.	1. Remove minimum at root in $O(1)$. 2. Find last position and copy to root in $O(\log n)$. 3. Down-heap bubbling in $O(\log n)$.

Heap-Sort

Since `add` and `remove_min` are both $O(\log n)$ for heap-based priority queue, heap-sort is $O(n \log n)$.

In-place heap-sort (can sort in-place because heap is complete binary tree which doesn't have gaps in array-based representation):

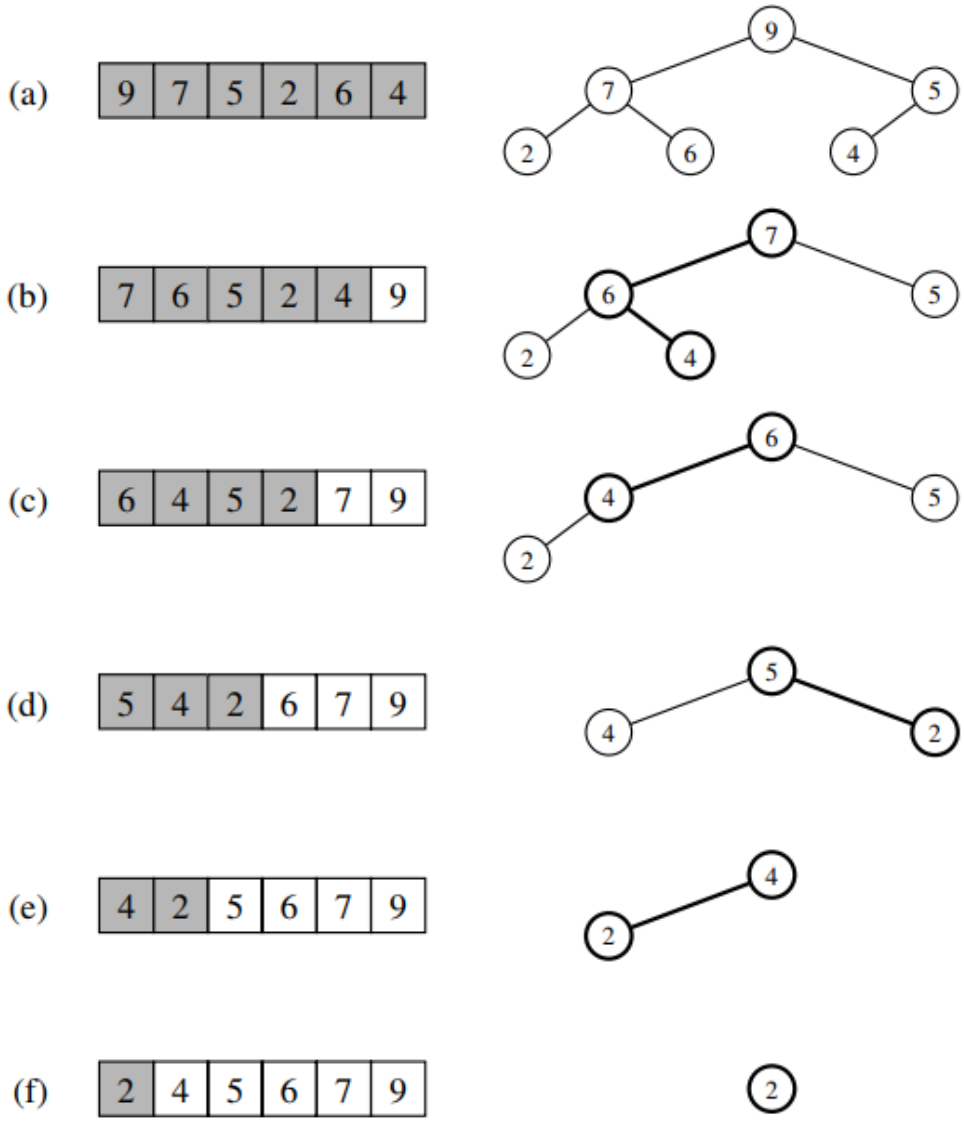


Figure 9.9: Phase 2 of an in-place heap-sort. The heap portion of each sequence representation is highlighted. The binary tree that each sequence (implicitly) represents is diagrammed with the most recent path of down-heap bubbling highlighted.

Map

Python uses dictionary to represent namespace. Can assume $O(1)$ lookup time.

Hash Table

Hash table is a lookup table structure that supports $O(1)$ lookup when implementing maps. The fast lookup is made possible by directly computing the hash table index from the map's key k via a hash function $h(k)$.

Hash Functions

We store the item (k, v) in the bucket $A[h(k)]$.

A hash function has two parts:

1. Hash code: keys $\to \mathbb{Z}$.
2. Compression function: $\mathbb{Z} \to [0, N-1]$ (N is the number of entries in the hash table).

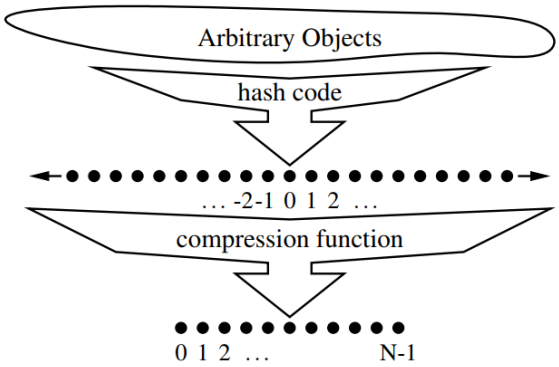


Figure 10.5: Two parts of a hash function: a hash code and a compression function.

Hash codes

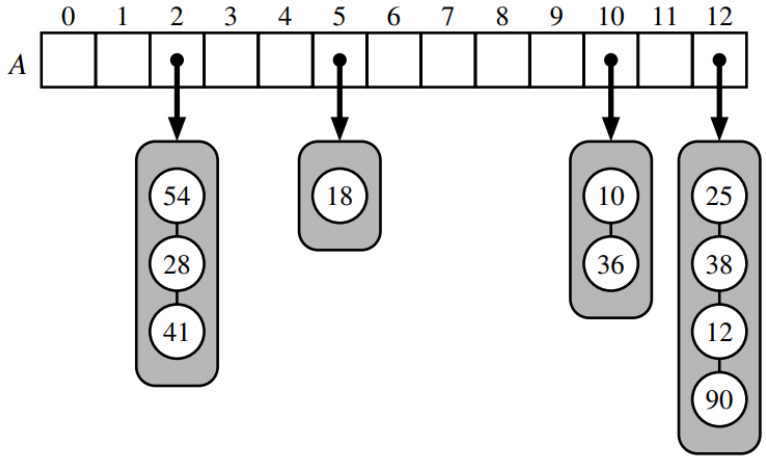
1. Interpret keys are integers. If overflow, sum the upper and lower 32-bits, or take bitwise exclusive-or. Does not preserve meaningful order in the key's characters, if any.
2. Polynomial in nonzero constant a. $x_0 a^{n-1} + x_1 a^{n-2} + \cdots + x_{n-2} a + x_{n-1}$
3. Cyclic shift. Sum each character in the key and shift the partial sum's bits cyclically in between.

Compression functions

1. "Division Method": $x \to x \mod N$.
2. "MAD Method": $x \to [(ax + b) \mod p] \mod N$.

Collision-handling schemes

Separate Chaining



Open Addressing

Linear Probing

Iteratively tries the buckets $A[(h(k)+i) \mod N]$, for $i = 0, 1, 2, \ldots$, until finding an empty bucket.

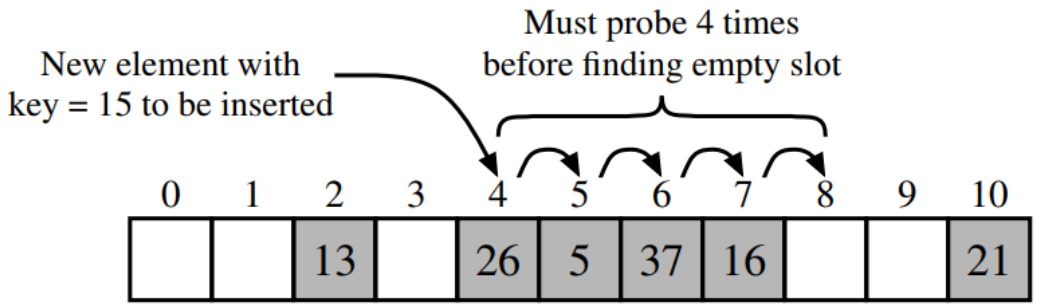


Figure 10.7: Insertion into a hash table with integer keys using linear probing. The hash function is $h(k) = k \mod 11$. Values associated with keys are not shown.

Quadratic Probing

Iteratively tries the buckets $A[(h(k)+f(i)) \mod N]$, for $i = 0, 1, 2, \ldots$, where $f(i) = i^2$, until finding an empty bucket.

Double Hashing

For secondary hash function $h'(i)$, iteratively tries the buckets $A[(h(k)+f(i)) \mod N]$, for $i = 0, 1, 2, \ldots$, where $f(i) = i \cdot h'(i)$, until finding an empty bucket.

Rehashing

Load factor. If n is the number of entries in a bucket array of capacity N, then the hash table's load factor is $\lambda = n/N$.

For each collision-handling scheme, there's a load factor threshold. If the load factor exceeds the threshold, the lookup efficiency will start degrading. For separate chaining, this is 0.9, linear probing 0.5, and Python dictionary's opening addressing $2/3$.

After the threshold is exceeded, the hash table is usually resized to twice the capacity (and all entries rehashed) to restore efficiency.

Sorted Search Table

A hash table where keys are sorted. The sortedness of the keys support inexact search such as searching for a range of keys.

Array-based implementation of the sorted search table allows $O(\log n)$ search via binary search, though update operations are $O(n)$ because elements need to be shifted.

Set

Set. A set is an unordered collection of elements, without duplicates, that typically supports efficient membership tests (e.g., using hash tables).

Sets are implemented using hash tables in Python. In fact, they are like maps with keys without values.

Multiset. A multiset is like a set but allows duplicates.

Mutlimap. A multimap is like a map but allows the same key to map to multiple values.

Operation	Description
`S.add(e)`	Add `e` to the set if it's not yet in the set.
`S.discard(e)`	Remove `e` from the set if it's in the set.
`e in S`	True if `e` is in the set.
`len(S)`	Return the number of elements in the set.
`iter(S)`	Return an iteration of the elements in the set.

Search Tree

Binary Search Tree

Binary search tree. A binary tree T with each position p storing a key-value pair (k, v) such that: • Keys in p's left subtree are $< k$. • Keys in p's right subtree are $> k$.

Proposition. An inorder traversal of a binary search tree visits positions in increasing order of their keys.

Successor of node. The successor of the node at position p is the node with the smallest value that is $>= p$ (the next node to visit right after p in an inorder traversal). If p has a right subtree, this is the leftmost node in its right subtree. Else, this is the nearest ancestor such that p is in its left subtree.

Predecessor of node. The predecessor of the node at position p is the node with the largest value that is $<= p$ (the node visited right before p in an inorder traversal). If p has a left subtree, this is the rightmost node in its left subtree. Else, this is the nearest ancestor such that p is in its right subtree.

Deletion from binary search tree:

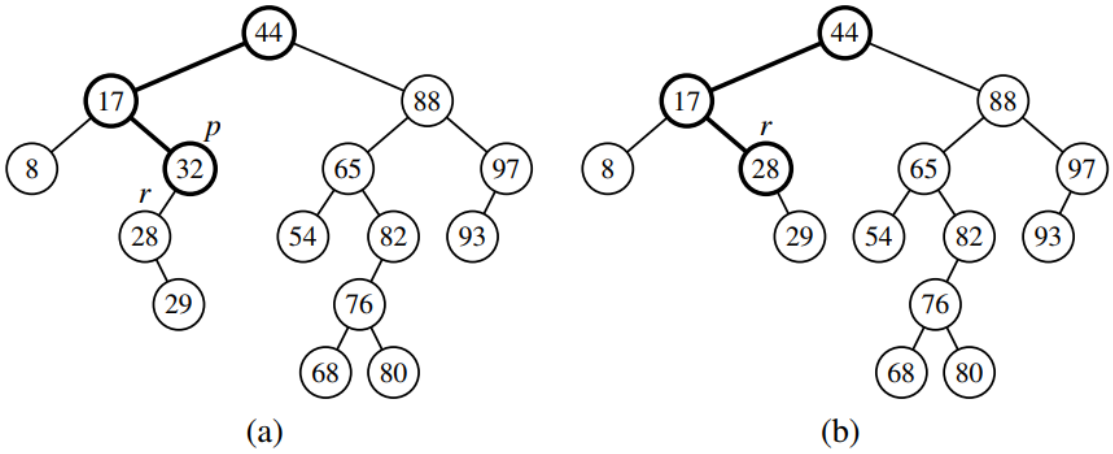


Figure 11.5: Deletion from the binary search tree of Figure 11.4b, where the item to delete (with key 32) is stored at a position p with one child r: (a) before the deletion; (b) after the deletion.

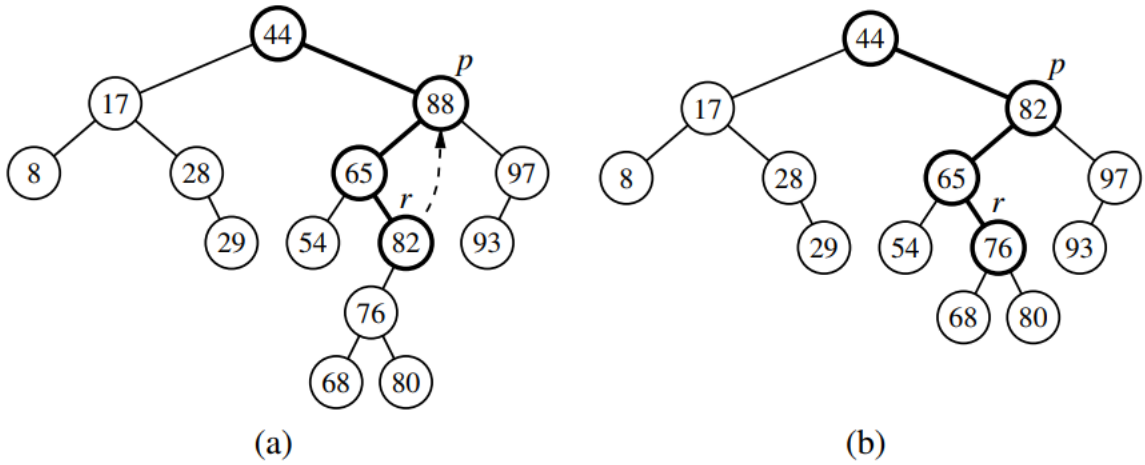


Figure 11.6: Deletion from the binary search tree of Figure 11.5b, where the item to delete (with key 88) is stored at a position p with two children, and replaced by its predecessor r: (a) before the deletion; (b) after the deletion.

Efficiency

Efficiency of binary search in a binary search tree depends on its height. So we need methods to balance a binary tree.

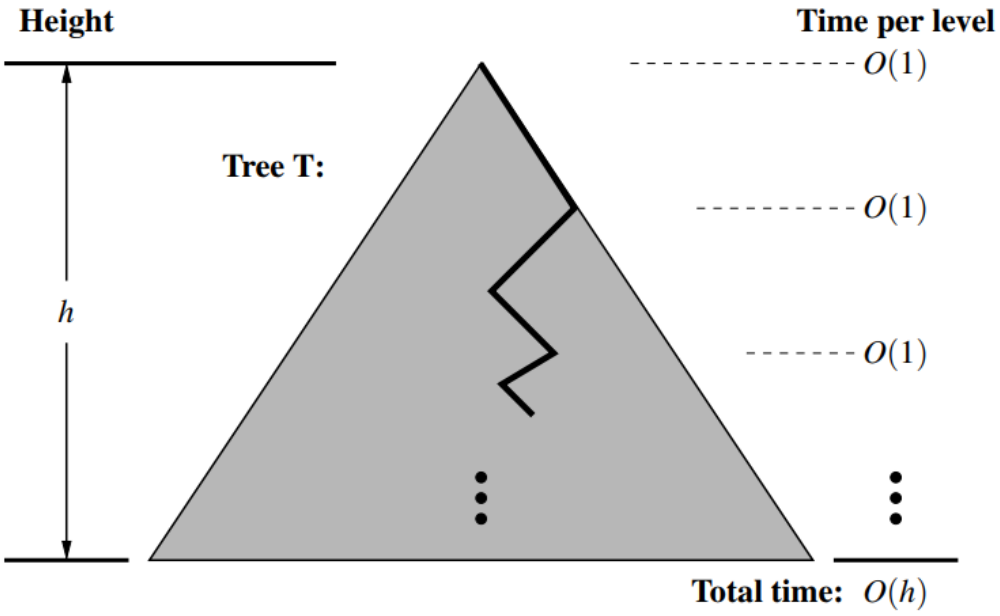
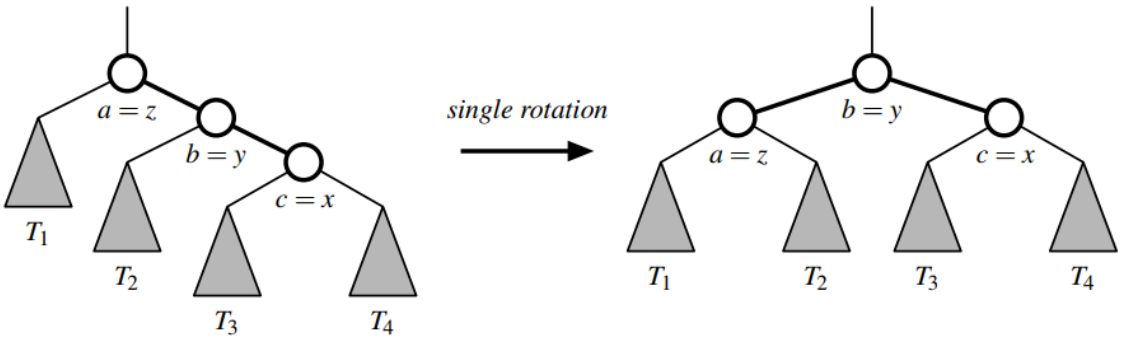


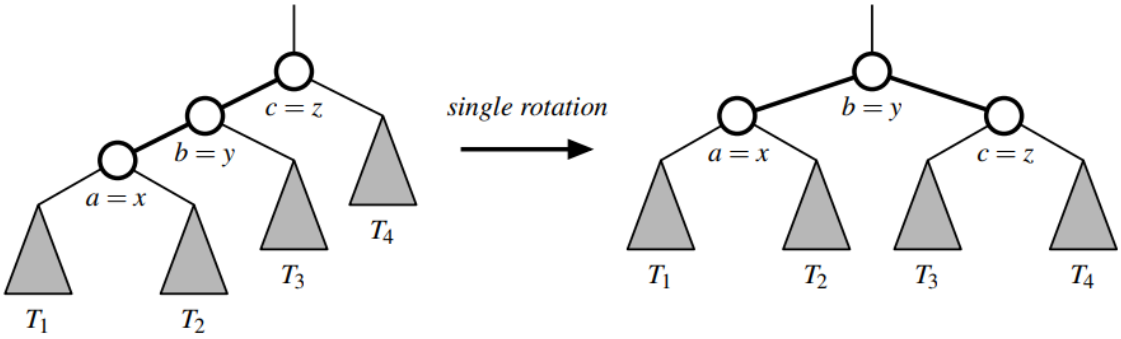
Figure 11.3: Illustrating the running time of searching in a binary search tree. The figure uses standard caricature of a binary search tree as a big triangle and a path from the root as a zig-zag line.

Balanced Search Tree

Rotation

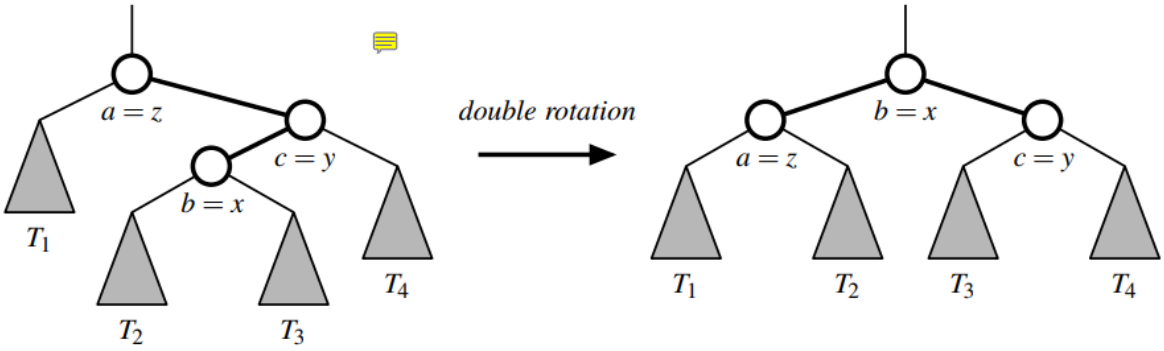


(a)

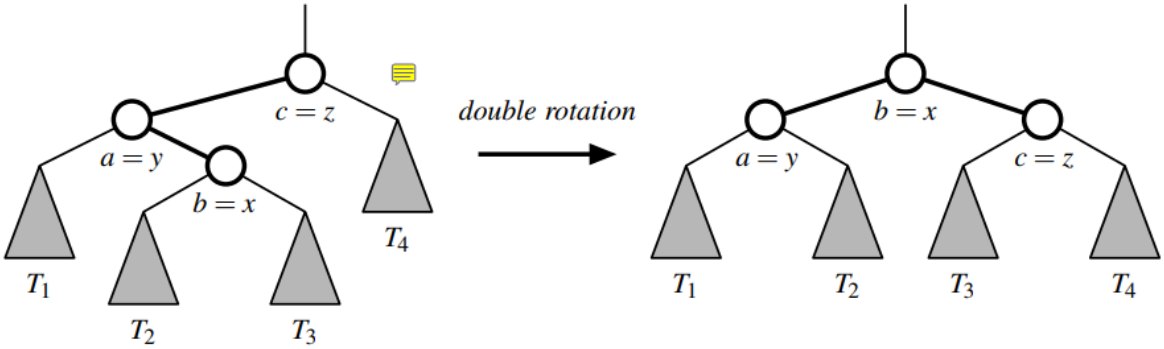


(b)

Rotating y above parent z, with x remaining as y's subtree.



(c)



(d)

Rotating x above parent y, then above grandparent z (with y remaining as x's subtree).

After the first rotation, the first tree becomes

```
a=z
    b=x
        c=y
```

And the second tree becomes

```
        c=z
    b=x
a=y
```

Rotating x above z gives the result (the relation between x and y remains unchanged in this rotation).

AVL Tree

An AVL tree is a binary search that satisfies the following:

Height balance property. For every position p of T, the heights* of the children of p differ by at most 1.

***Height of node (alternate definition).** Number of nodes (instead of edges) in a longest path from p to a leaf. E.g., height of a leaf is 1.

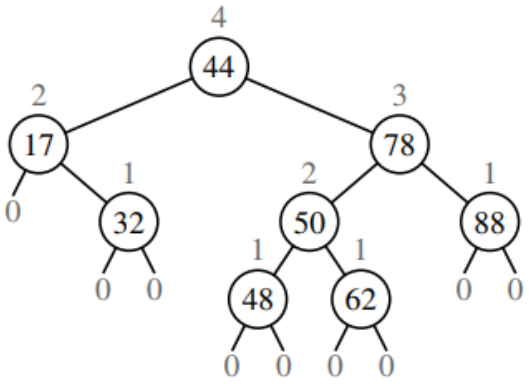


Figure 11.11: An example of an AVL tree. The keys of the items are shown inside the nodes, and the heights of the nodes are shown above the nodes (with empty subtrees having height 0).

Proposition. The height of an AVL tree storing n entries is $O(\log n)$.

Balanced node. In a binary search tree T, a position is balanced if the absolute value of the difference between the heights of its children is at most 1.

So the height balance property <=> every node is balanced.

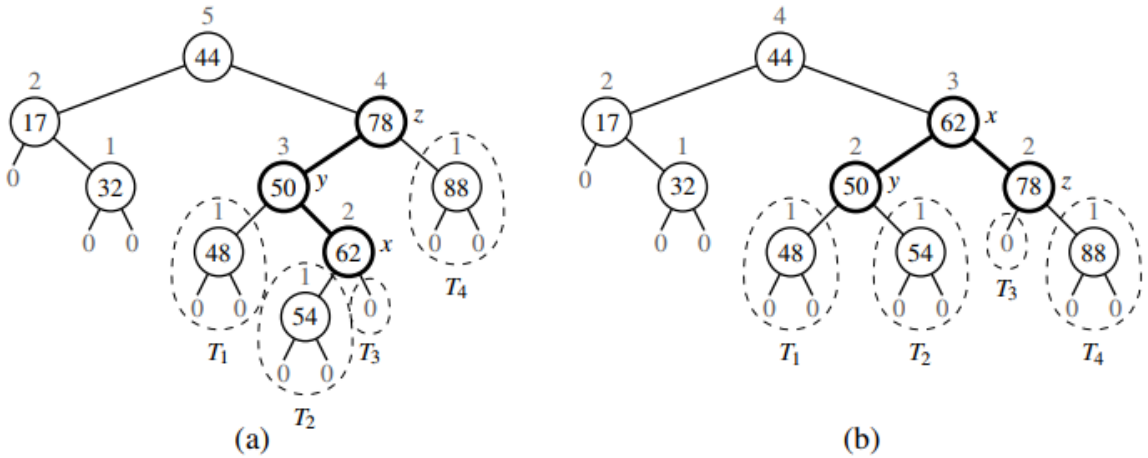


Figure 11.12: An example insertion of an item with key 54 in the AVL tree of Figure 11.11: (a) after adding a new node for key 54, the nodes storing keys 78 and 44 become unbalanced; (b) a trinode restructuring restores the height-balance property. We show the heights of nodes above them, and we identify the nodes x, y, and z and subtrees T_1, T_2, T_3, and T_4 participating in the trinode restructuring.

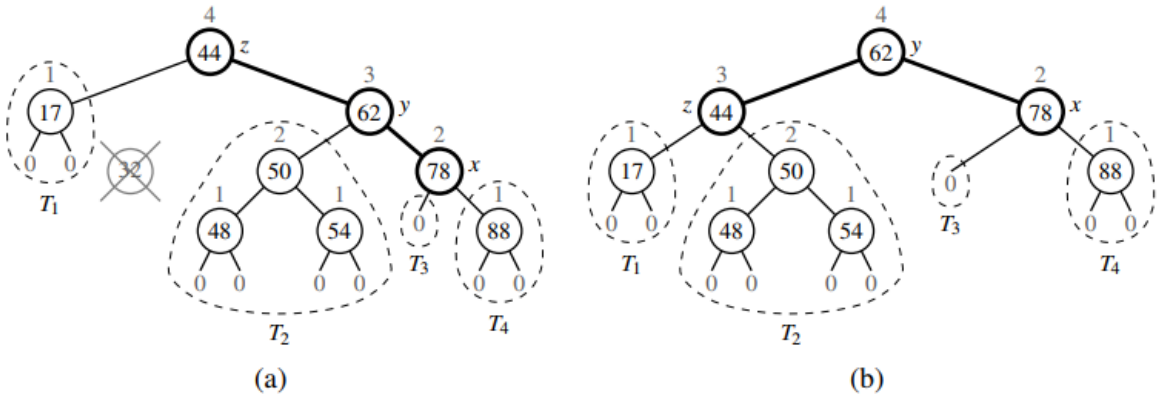


Figure 11.14: Deletion of the item with key 32 from the AVL tree of Figure 11.12b: (a) after removing the node storing key 32, the root becomes unbalanced; (b) a (single) rotation restores the height-balance property.

For deletion, need to continue walking upward to repair further unbalances, until reaching the root. The number of operations is bounded by tree height O(log n).

So AVL tree guarantees $O(\log n)$ bound for binary search tree operations.

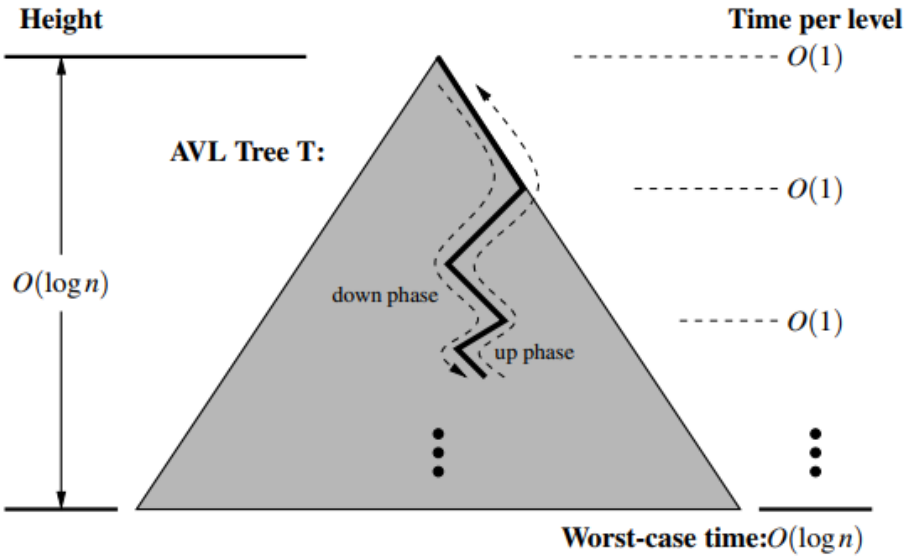


Figure 11.15: Illustrating the running time of searches and updates in an AVL tree. The time performance is $O(1)$ per level, broken into a down phase, which typically involves searching, and an up phase, which typically involves updating height values and performing local trinode restructurings (rotations).

Splay Tree

Moves more frequently accessed nodes closer to the root.

Amortized $O(\log n)$ for search, insertions, deletions.

(2, 4) Tree

Particular case of a multiway search tree, where each node may have multiple children. Each internal node has 2, 3, or 4 children.

Operations' runtime is the same as AVL tree.

Red-Black Tree

Only requires $O(1)$ structural operations for rebalancing (instead of $O(\log n)$ as in AVL tree).

A red-black tree is a binary search tree with nodes colored red and black such that:
Root Property. The root is black.
Red Property. The children of a red node (if any) are black.
Depth Property. All nodes with zero or one children have the same black depth, defined as the number of black ancestors. (Recall that a node is its own ancestor).

Proposition. The height of a red-black tree storing n entries is $O(\log n)$.

Sorting

Runtime of sorting algorithms is bounded from above by $n!$, because sorting produces a permutation of the original sequence, and there are $n!$ permutations.

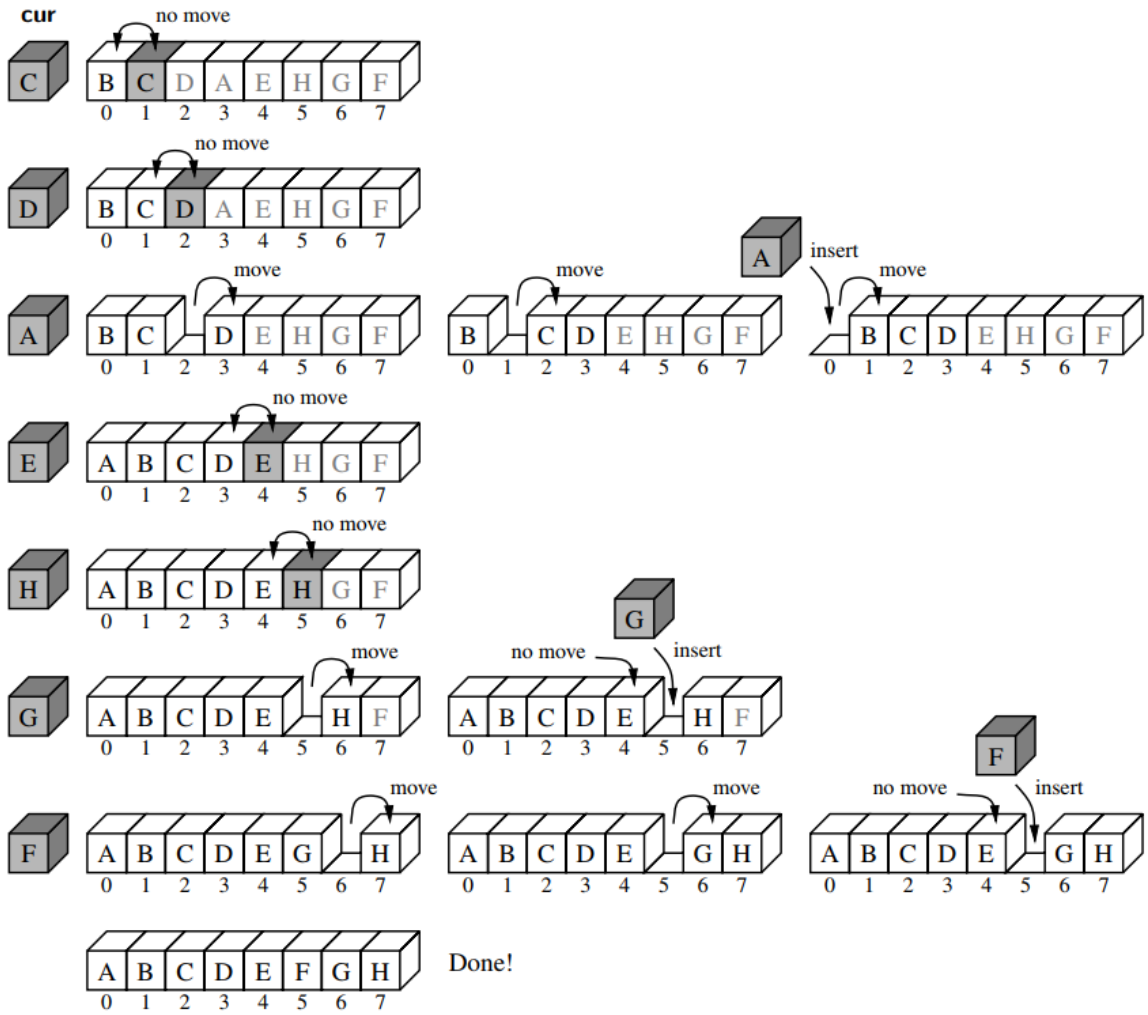
Insertion-Sort

Maintain a sorted sub-list at the front, keep inserting the next element from the unsorted sub-list into the sorted sub-list to make sure it stays sorted.

Scenario	Runtime	Description
Worst case	$O(n^2)$	When input is sorted in reverse.
Best case	$O(n)$	When input is sorted.
Average	$O(n^2)$	

Algorithm InsertionSort(A):
Input: An array A of n comparable elements
Output: The array A with elements rearranged in nondecreasing order
for k from 1 to n − 1 **do**
Insert A[k] at its proper location within A[0], A[1], . . ., A[k].

Code Fragment 5.9: High-level description of the insertion-sort algorithm.



Selection-Sort

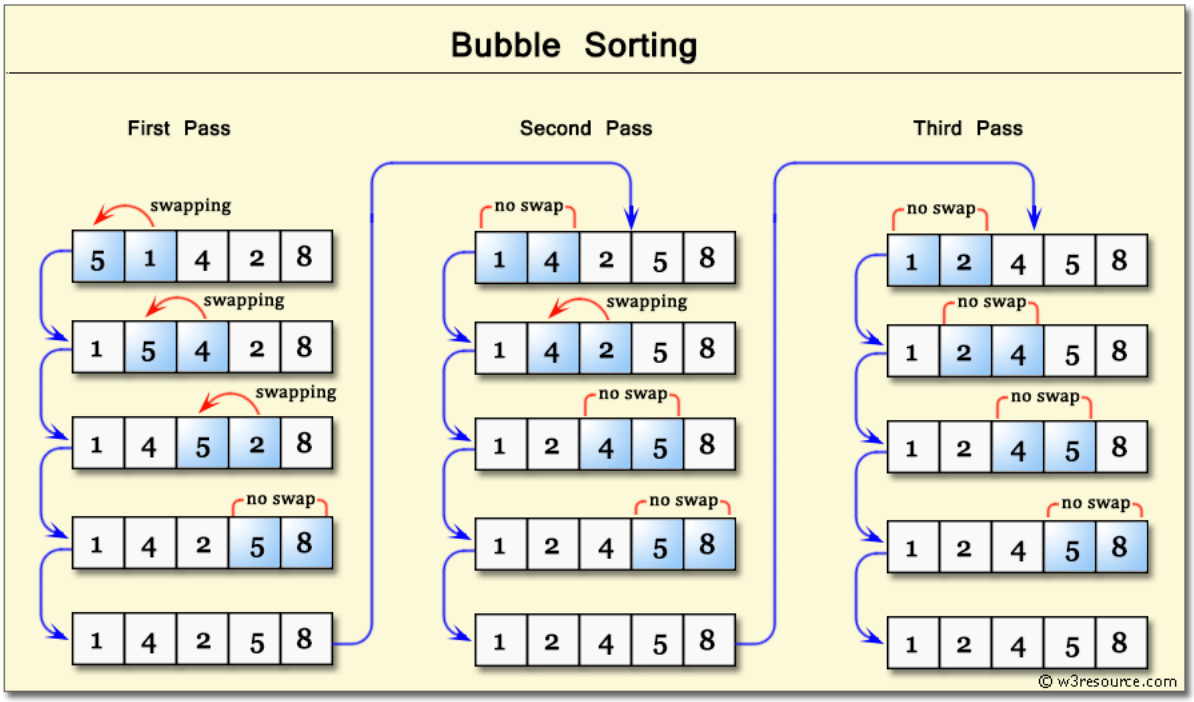
Maintain a sorted sub-list at the front, keep selecting the minimum from the unsorted sub-list and append to the sorted sub-list.

Scenario	Runtime	Description
Worst case	$O(n^2)$	
Best case	$O(n^2)$	Even when input is sorted.
Average	$O(n^2)$	

Bubble-Sort

Make n passes, swapping every two items into order in each pass.

Scenario	Runtime	Description
Worst case	$O(n^2)$	When input is sorted in reverse.
Best case	$O(n)$	When input is sorted.
Average	$O(n^2)$	



Heap-Sort

First, add each item to heap. Then, keep popping the minimum from the heap.

With unsorted/sorted list-based heaps, this is selection/insertion-sort.

With linked/array-based binary tree-based heaps, this is heap-sort.

Scenario	Runtime	Description
Worst case	$O(n \log n)$	

Scenario	Runtime	Description
Best case	$O(n \log n)$	If input is sorted, adding items to heap is $O(1)$ with array-based heaps, but down-heap bubbling when removing items is still $O(\log n)$.
Average	$O(n \log n)$	

Merge-Sort

Divide and conquer.

1. Divide into two halves.
2. Recursively sort on each half.
3. Merge the sorted halves into a sorted list.

There are $\log n$ levels, and the merge step at each level is $O(n)$ ($O(n/2 * 2)$, $O(n/4 * 4)$, etc). So merge-sort is $O(n \log n)$.

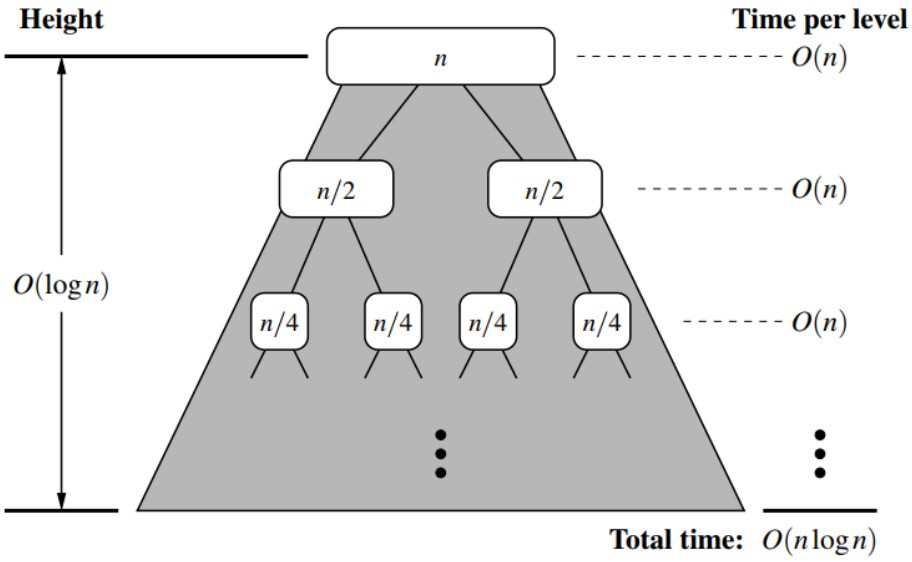


Figure 12.6: A visual analysis of the running time of merge-sort. Each node represents the time spent in a particular recursive call, labeled with the size of its subproblem.

Quick-sort

Divide and conquer.