Array

Referential array. Array of object references.

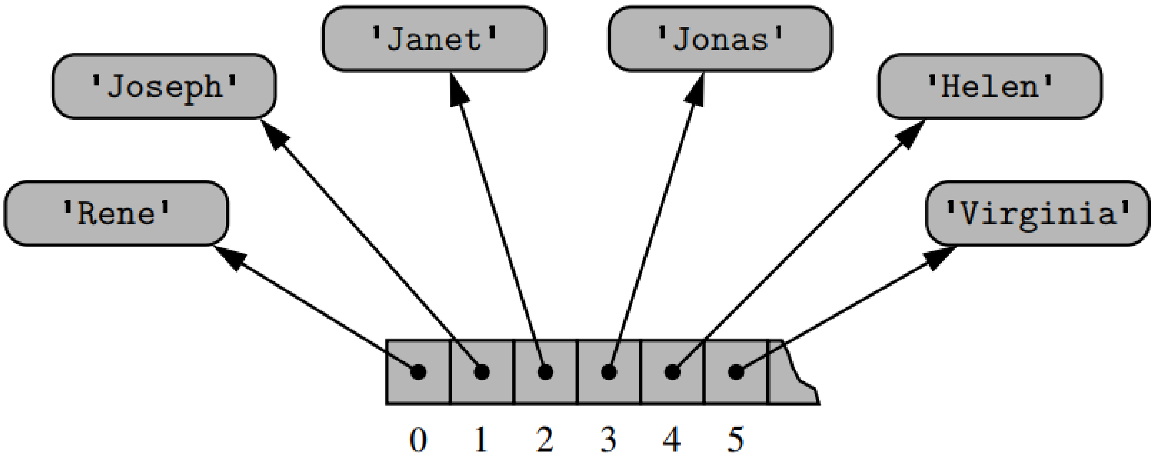
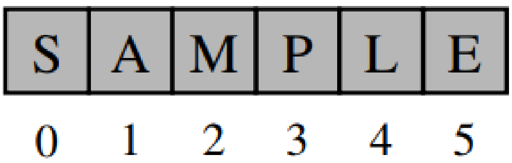


Figure 5.4: An array storing references to strings.

E.g., Python lists, tuples.

Compact array. Array that store bits representing primary data.



Dynamic array. Resizable array that grows or shrinks based on the number of items it contains, so that its operations can have amortized $O(1)$ runtime.

E.g., Python list is implemented using dynamic array.

Stack

Method	Description	Runtime
`S.push(e)`	Add e to top of stack.	$O(1)$*
`S.pop()`	Remove and return item from top of stack.	$O(1)$*
`S.top()`	Return reference to item at top of stack.	$O(1)$
`S.is_empty()`	True if the stack is empty.	$O(1)$
`len(S)`	Return the number of items in the stack.	$O(1)$

*If implemented using a Python list, these operations are amortized.

Applications:

1. Reverse a list (push all items in and pop them one by one, first in last out).
2. Parenthesis matching.

Queue

Method	Description	Runtime
`Q.enqueue(e)`	Add `e` to end of queue.	$O(1)$*
`Q.dequeue()`	Remove and return item from front of queue.	$O(1)$*
`Q.first()`	Return item at front of queue.	$O(1)$
`Q.is_empty()`	True if the queue is empty.	$O(1)$
`len(Q)`	Return the number of items in the queue.	$O(1)$

*If implemented using a Python list (circular, wraps around when reaching end of list), these operations are amortized.

Deque

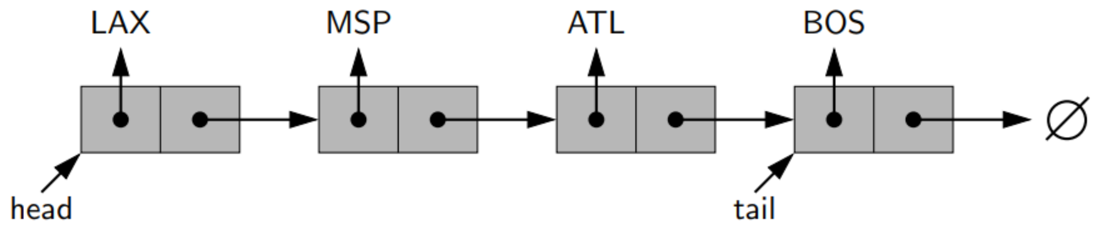
Double-ended queue. ADT that can add and remove elements from both ends of the queue.

Method	**Description**	**Runtime**
`D.add_first(e)`, `D.add_last(e)`	Add `e` to front/back of dequeue.	$O(1)$*
`D.delete_first(e)`, `D.delete_last(e)`	Remove and return item from front/back of dequeue.	$O(1)$*
`D.first()`, `D.last()`	Return and return item at the front/back of dequeue.	$O(1)$
`D.is_empty()`	True if the dequeue is empty.	$O(1)$
`len(D)`	Return the number of items in the dequeue.	$O(1)$

*If implemented using a Python list (circular, wraps around), these operations are amortized.

Linked List

Singly Linked List



Applications:

1. Implement the Stack ADT, all operations are worst-case $O(1)$.
2. Implement the Queue ADT, all operations are worst-case $O(1)$.

Circularly Linked List

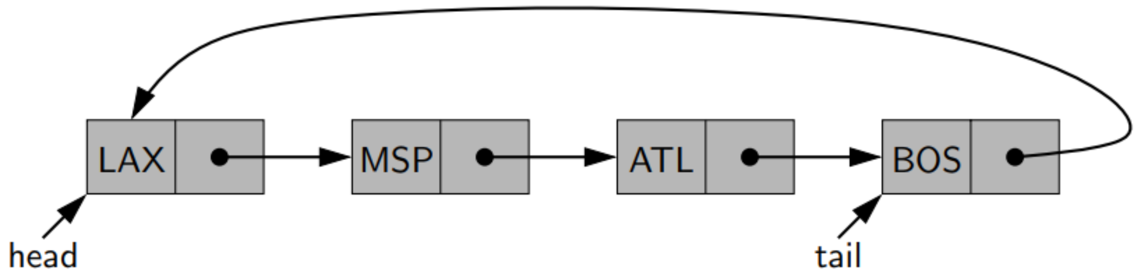


Figure 7.7: Example of a singly linked list with circular structure.

Applications:

1. Implement the Queue ADT, with more efficient method for wrapping around.

Doubly Linked List

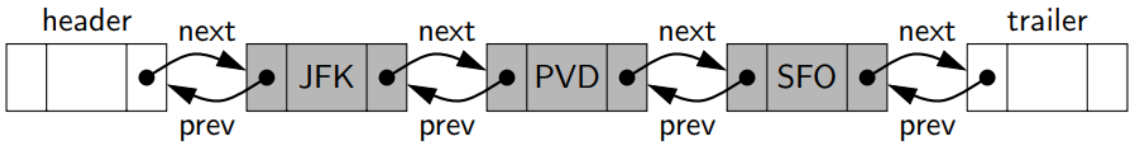


Figure 7.10: A doubly linked list representing the sequence { JFK, PVD, SFO }, using sentinels header and trailer to demarcate the ends of the list.

Applications:

1. Implement the Deque ADT.
2. Implement the Positional List ADT.

Positional List

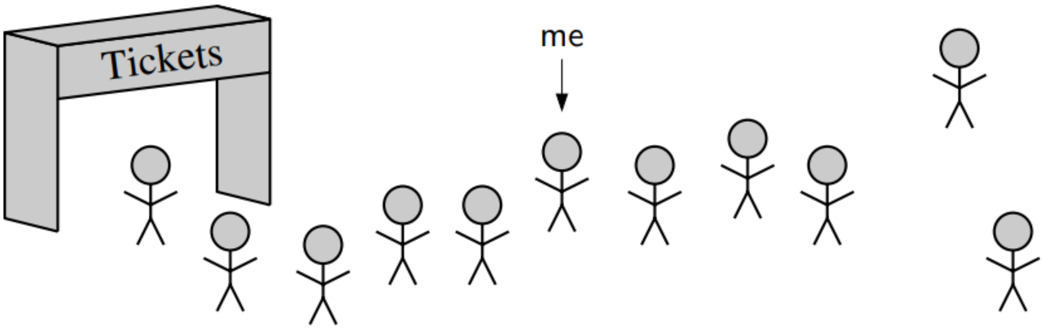


Figure 7.14: We wish to be able to identify the position of an element in a sequence without the use of an integer index.

Method	**Description**
`L.first()`, `L.last()`	Return the position of the first/last item.
`L.before(p)`, `L.after(p)`	Return the position immediately before/after position `p`.
`L.is_empty()`	True if the positional list is empty.
`len(L)`	Return the number of items in the positional list.
`iter(L)`	Return a forward iterator of items in the positional list.
`L.add_first(e)`, `L.add_last(e)`	Add `e` to the front/back of the positional list.
`L.add_before(p, e)`, `L.add_after(p, e)`	Add `e` before/after position `p`.
`L.replace(p, e)`	Replace the item at position `p` with `e`.
`L.delete(p)`	Remove and return the item at position `p`.

Applications:

1. Maintain access frequencies.

Array-based vs. Linked-based

Metrics	**Array-based**	**Linked-based**
access based on index	$O(1)$	$O(n)$
insertion, deletion	$O(n)$ worst case	$O(1)$ at arbitrary position
memory usage	$2n$ worst case (after resize)	$2n$ for singly-linked lists $3n$ for doubly-linked lists

Tree

General Tree

A tree T is set of nodes storing elements such that the nodes have a parent-child relationship that satisfies the following properties:

- If T is nonempty, it has a special node, called the root of T , that has no parent.
- Each node v of T different from the root has a unique parent node w; every node with parent w is a child of w.

Sibling. Two nodes are siblings if they have the same parent node.

External. A node is external if it has no children. A.k.a leaves.

Internal. A node is internal if it has >= 1 children.

Edge. An edge of tree T is a pair of nodes (u, v) such that u is the parent of v, or vice versa.

Path. A path of T is a sequence of nodes such that any two consecutive nodes in the sequence form an edge.

Ordered Tree. A tree is ordered if there is a meaningful linear order among the children of each node.

Depth of node. The depth of a node is the number of its ancestors, excluding itself.

Depth of node (recursive). If p is the root, then its depth is 0. Otherwise, the depth of p is $1+$ depth of p's parent.

Height of node (recursive). If p is a leaf, then its height is 0. Otherwise, the height of p is $1+$ the maximum of p's children's heights.

Height of tree. The height of a tree is the height of its root.

Method	Description
`T.root()`	Return the position of the tree's root.
`T.is_root(p)`	True if position `p` is the tree's root.
`T.parent(p)`	Return the position of `p`'s parent.
`T.num children(p)`	Return the number of `p`'s children.
`T.children(p)`	Generate an iteration of position `p`'s children.
`T.is leaf(p)`	True if position `p` does not have any children.
`len(T)`	Return the number of positions in the tree.
`T.is empty()`	True if the tree does not contain any position.
`T.positions()`	Generate an iteration of the positions in the tree.
`iter(T)`	Generate an iteration of the elements in the tree.

Method	Description
`T.depth(p)`	Return the depth of `p`.
`T.height(p)`	Return the height of `p`.

Proposition. The height of a nonempty tree is the maximum of its leaves' depths.

Proposition. In a tree with n nodes, the sum of the number of children of all nodes is $n-1$.

Proof. Every node except for the root is some other node's child.

Binary Tree

Binary tree. A binary tree is an ordered tree such that:

1. Every node has at most two children.
2. Each child node is either a left child or a right child.
3. A left child precedes a right child in the order of children of a node.

Binary tree (recursive). A binary tree is either empty or consists of:

- A node r, called the root of T, that stores an element
- A binary tree (possibly empty), called the left subtree of T
- A binary tree (possibly empty), called the right subtree of T

Method	Description
`T.left(p)`, `T.right(p)`	Return the position of `p`'s left/right child.
`T.sibling(p)`	Return the position of `p`'s sibling.

Proper/Full. A binary tree is proper or full if each node has either zero or two children. That is, all its internal nodes have two children.

Proposition. In a nonempty proper binary tree T, with n_E external nodes and n_I internal nodes, we have $n_E = n_I + 1$.

Proof. If h is T's height, then $n_E = 2^h$, $n_I = 2^0 + 2^1 + 2^2 + \cdots 2^h = 2^h - 1$.

Implementations

Linked Binary Tree

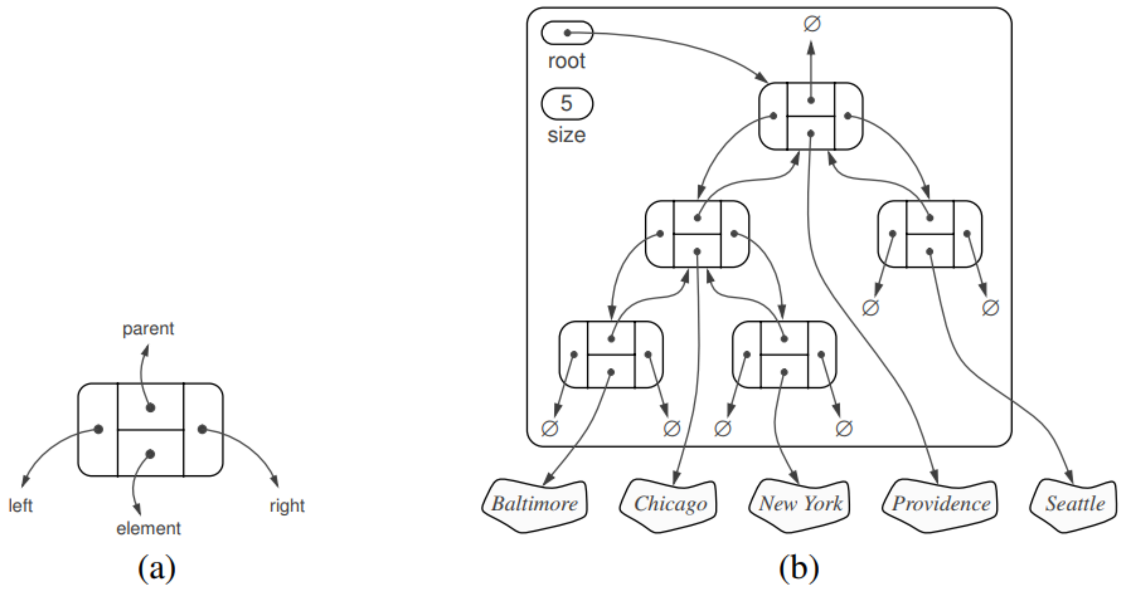


Figure 8.11: A linked structure for representing: (a) a single node; (b) a binary tree.

Method	Description
`T.add_root(e)`	Add root `e` to an empty tree.
`T.add_left(p, e)`, `T.add_right(p, e)`	Add `e` as left/right child to `p`.
`T.replace(p, e)`	Replace element at position `p` with `e`.
`T.delete(p)`	Remove the node at position `p` and replace it with its only child.
`T.attach(p, T1, T2)`	Attach `T1`, `T2` as left and right subtress of the leaf `p`.

Operation	Runtime
`len`, `is_empty`	$O(1)$
`root`, `parent`, `left`, `right`, `sibling`, `children`, `num_children`	$O(1)$
`is_root`, `is_leaf`	$O(1)$
`depth(p)`	$O(d_p+1)$
`height`	$O(n)$
`add_root`, `add_left`, `add_right`, `replace`, `delete`, `attach`	$O(1)$

Array-based Binary Tree

For every position p of T , let $f(p)$ be the integer defined as follows. • If p is the root of T, then $f(p)=0$. • If p is the left child of position q, then $f(p)=2f(q)+1$. • If p is the right child of position q, then $f(p)=2f(q)+2$.

Array-based binary tree. An array-based structure A (such as a Python list), with the element at position p of T stored at $A[f(p)]$.

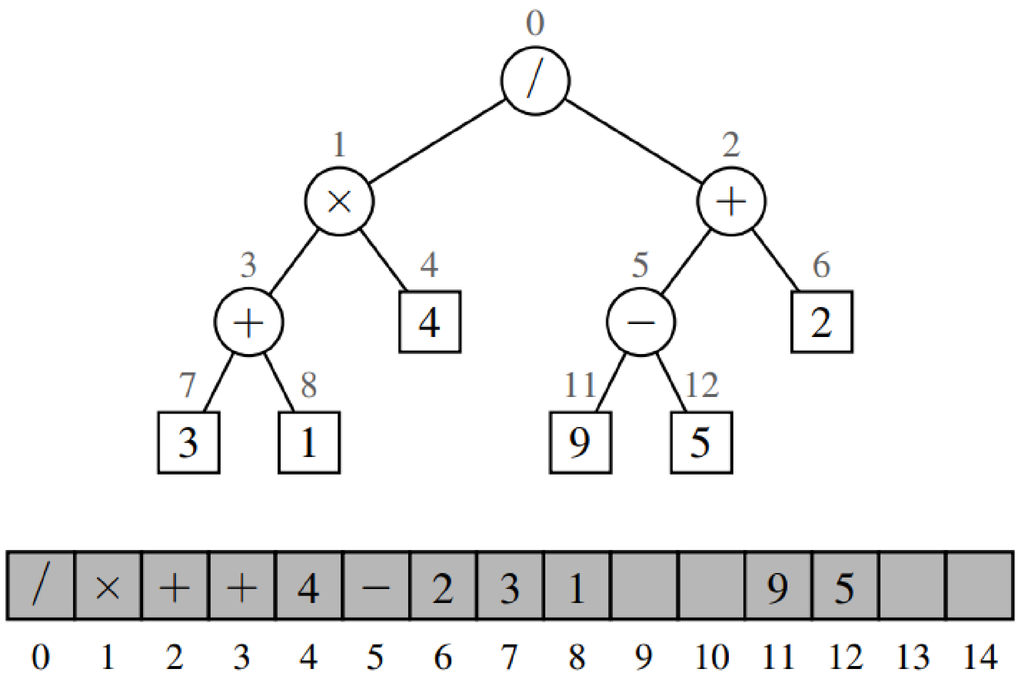


Figure 8.13: Representation of a binary tree by means of an array.

`delete` is $O(n)$ as all the node's descendents need to be shifted in the array.

Linked General Tree

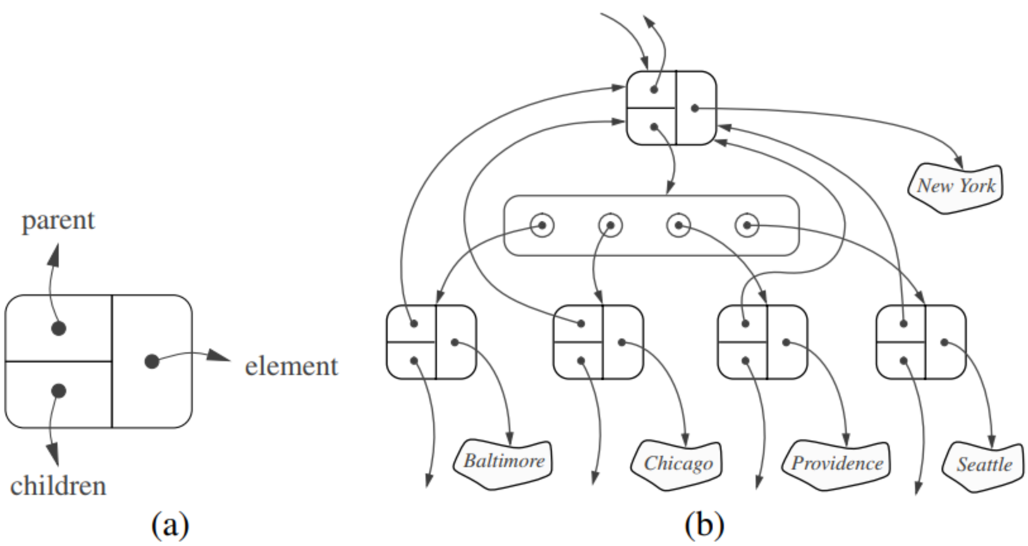


Figure 8.14: The linked structure for a general tree: (a) the structure of a node; (b) a larger portion of the data structure associated with a node and its children.

Operation	**Runtime**
`len`, `is_empty`	$O(1)$
`root`, `parent`, `is_root`, `is_leaf`	$O(1)$
`children(p)`	$O(c_p + 1)$
`depth(p)`	$O(d_p + 1)$

Operation	Runtime
`height`	$O(n)$

Tree Traversal Algorithms

Traversals are $O(n)$ as they must visit every node in the tree.

Binary search is $O(\log n)$ in a proper binary tree.

Preorder (General Tree)

Visit node, then visit node's children.

```
Algorithm preorder(T, p):
  perform the "visit" action for position p
  for each child c in T.children(p) do
    preorder(T, c)            {recursively traverse the subtree rooted at c}
```

Code Fragment 8.12: Algorithm preorder for performing the preorder traversal of a subtree rooted at position p of a tree T.

Figure 8.15 portrays the order in which positions of a sample tree are visited during an application of the preorder traversal algorithm.

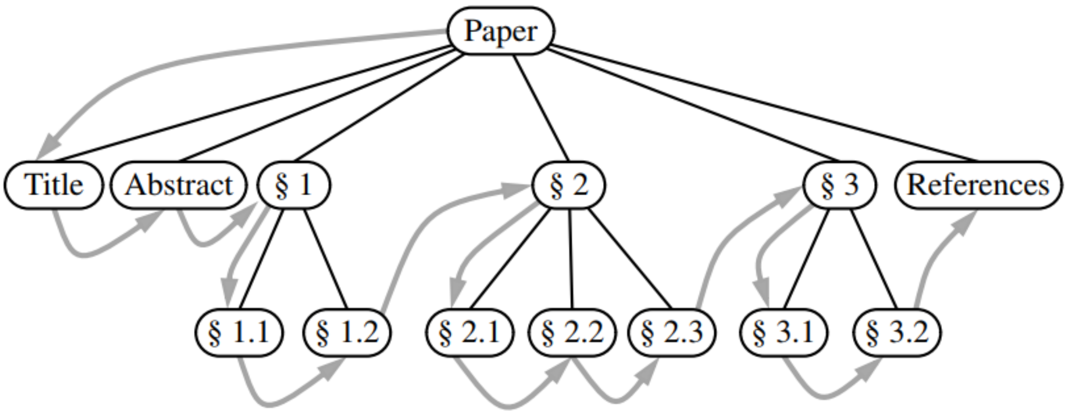


Figure 8.15: Preorder traversal of an ordered tree, where the children of each position are ordered from left to right.

Postorder (General Tree)

Visit node's children, then visit node.

```
Algorithm postorder(T, p):
  for each child c in T.children(p) do
    postorder(T, c)            {recursively traverse the subtree rooted at c}
  perform the "visit" action for position p
```

Code Fragment 8.13: Algorithm postorder for performing the postorder traversal of a subtree rooted at position p of a tree T.

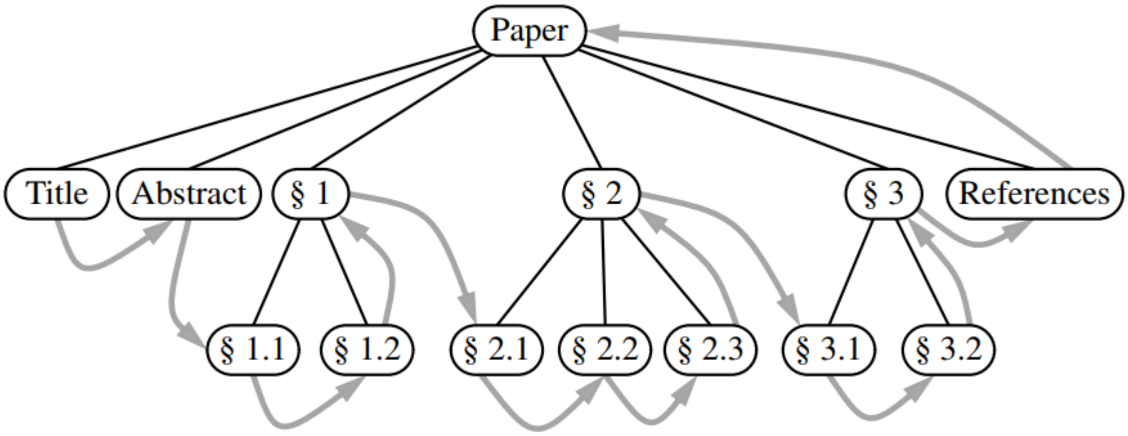


Figure 8.16: Postorder traversal of the ordered tree of Figure 8.15.

Breadth-first (General Tree)

Visit nodes level by level.

Not recursive.

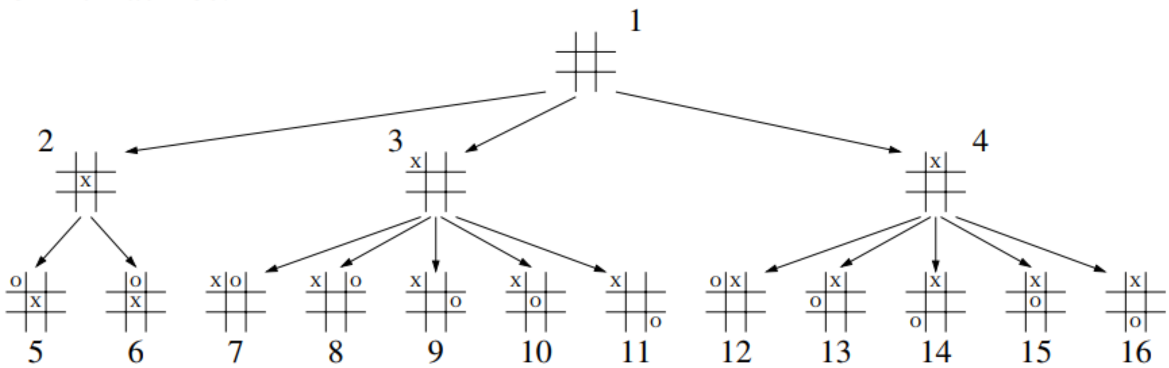


Figure 8.17: Partial game tree for Tic-Tac-Toe, with annotations displaying the order in which positions are visited in a breadth-first traversal.

Dequeue to get node. Visit node, then enqueue node's children.

Algorithm breadthfirst(T):
  Initialize queue Q to contain T.root()
  while Q not empty **do**
    p = Q.dequeue()   {p is the oldest entry in the queue}
    perform the "visit" action for position p
    for each child c in T.children(p) **do**
      Q.enqueue(c)   {add p's children to the end of the queue for later visits}

Code Fragment 8.14: Algorithm for performing a breadth-first traversal of a tree.

Inorder (Binary Tree)

Visit left subtree. Visit right subtree. Visit node.

Algorithm inorder(p):
  if p has a left child lc **then**
    inorder(lc)   {recursively traverse the left subtree of p}
  perform the "visit" action for position p
  if p has a right child rc **then**
    inorder(rc)   {recursively traverse the right subtree of p}

Code Fragment 8.15: Algorithm inorder for performing an inorder traversal of a subtree rooted at position p of a binary tree.

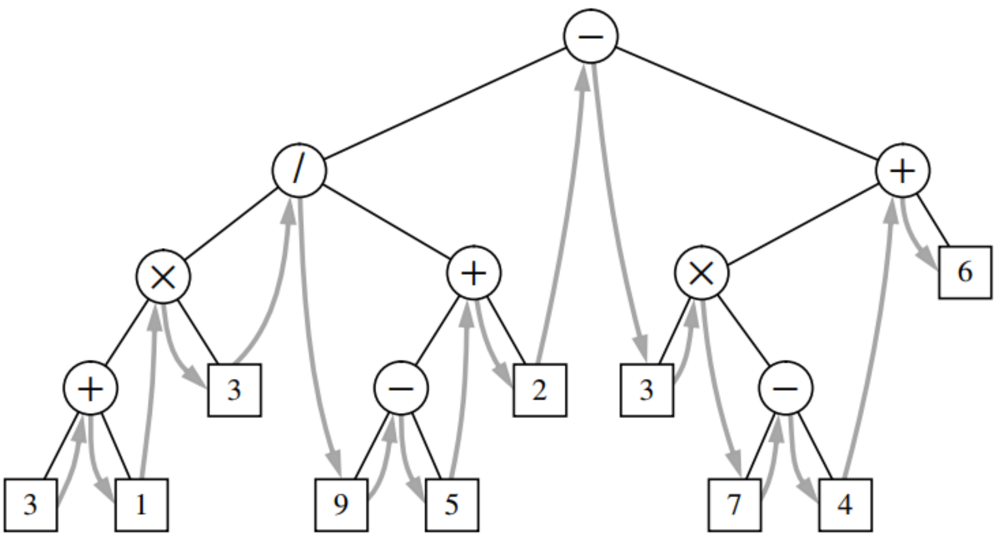


Figure 8.18: Inorder traversal of a binary tree.

Priority Queue

Method	Description
`P.add(k, v)`	Add item with key `k` and value `v` into the priority queue.
`P.min()`	Return item with the minimum key in the priority queue.
`P.remove_min()`	Remove and return an item with the minimum key in the priority queue.
`P.is_empty()`	True if the priority queue is empty.
`len(P)`	Return the number of items in the priority queue.

Unsorted Priority Queue

Add item to the end of the priority queue, find minimum to remove in $O(n)$.

$O(1)$ insertions, $O(n)$ removals (best-case, because it always takes $O(n)$ to find the minimum).

Method	Runtime
`P.add(k, v)`	$O(1)$
`P.min()`	$O(n)$
`P.remove_min()`	$O(n)$
`P.is_empty()`	$O(1)$
`len(P)`	$O(1)$

Sorted Priority Queue

Maintain sortedness when inserting items, minimum is at front of the priority queue,

$O(n)$ insertions (best-case is $O(1)$, because the items may come in as sorted), $O(1)$ removals.

Method	Runtime
`P.add(k, v)`	$O(n)$
`P.min()`	$O(1)$
`P.remove_min()`	$O(1)$
`P.is_empty()`	$O(1)$
`len(P)`	$O(1)$

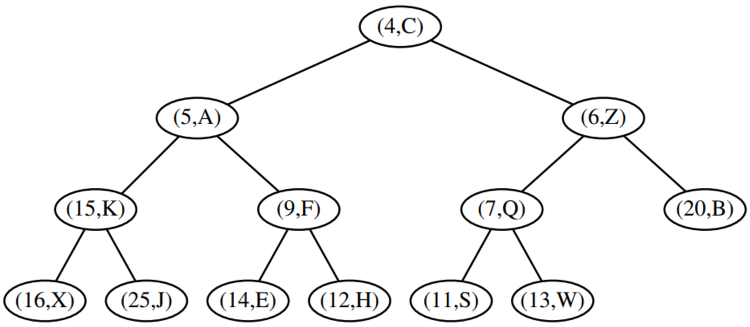
Heap

A heap is a binary tree T that stores a collection of items at its positions and that satisfies the following two properties:

Heap-Order Property. (relational): In a heap T, for every position p other than the root, the key stored at p >= the key stored at p's parent.

This implies that the minimum item is at the heap's root.

Complete Binary Tree Property. (structural): A heap T with height h is a complete binary tree if levels $0, 1, 2, \ldots, h-1$ of T have the maximum number of nodes possible and the remaining nodes at level h reside in the leftmost positions.

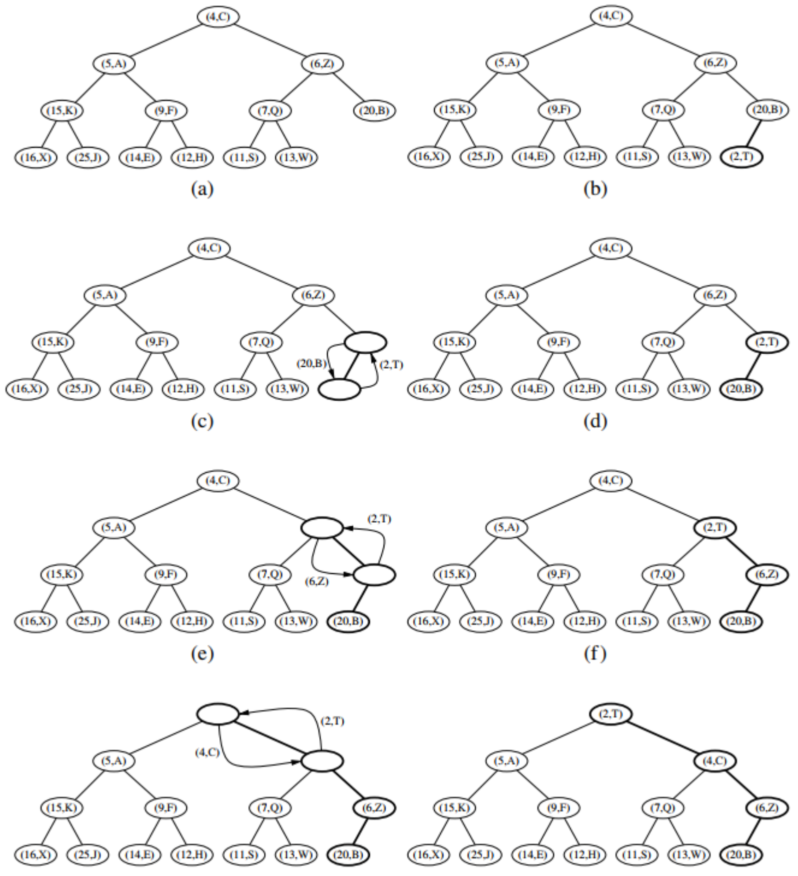


Proposition. A heap T storing n entries has height $h = \text{floor}(\log n)$.

Heap-based Priority Queue

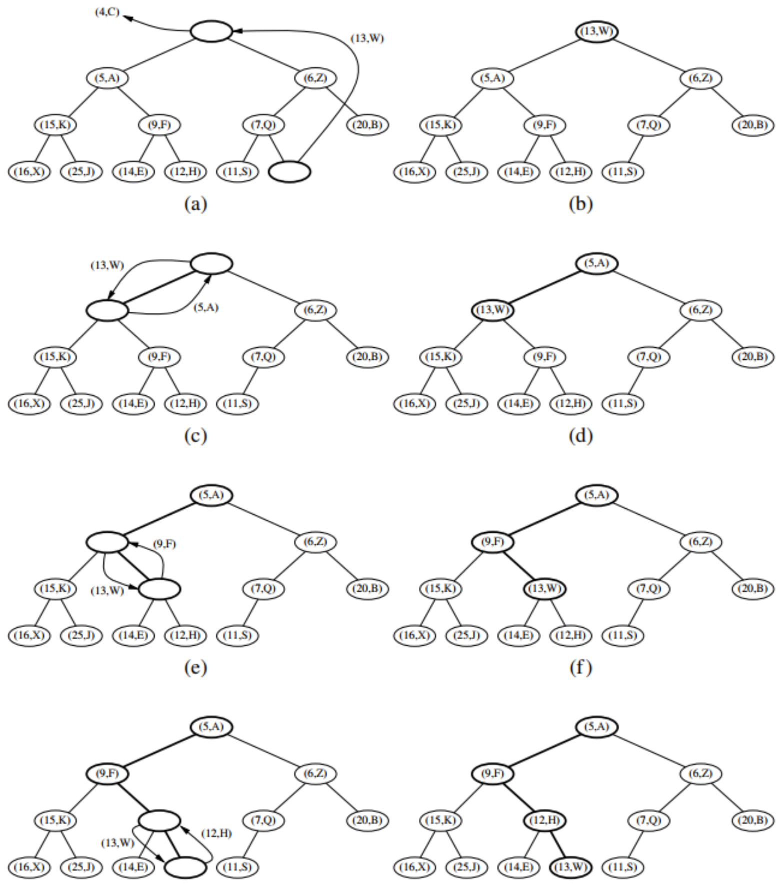
Add:

1. Add item to rightmost node at bottom level to maintain completeness.
2. Up-heap bubbling to swap the new node into correct position to maintain heap-order.



Remove:

1. Remove minimum item from top of heap (root).
2. Copy item at rightmost node at bottom level to root.
3. Down-heap bubbling to swap the node into correct position to maintain heap order.



Method	Runtime
`P.add(k, v)`	$O(\log n)$*
`P.min()`	$O(1)$
`P.remove_min()`	$O(\log n)$*
`P.is_empty()`	$O(1)$
`len(P)`	$O(1)$

*Amortized for array-based tree.

Implementation/Operation	`add`	`remove_min`
Unsorted List	Add to the end of list in $O(1)$.	Insert into sorted list in $O(n)$.
Sorted List	Find minimum in $O(n)$ to remove.	Remove minimum from front of list in $O(1)$.
Array-based Heap	1. Find last position in $O(1)$. 2. Up-heap bubbling in $O(\log n)$.	1. Remove minimum at root in $O(1)$. 2. Find last position and copy to root in $O(1)$. 3. Down-heap bubbling in $O(\log n)$.

Implementation/Operation	`add`	`remove_min`
Linked Heap	1. Find last position in $O(\log n)$. 2. Up-heap bubbling in $O(\log n)$.	1. Remove minimum at root in $O(1)$. 2. Find last position and copy to root in $O(\log n)$. 3. Down-heap bubbling in $O(\log n)$.