Chapter 6

Consider Covariance Analysis

6.1 INTRODUCTION

The design and subsequent performance of the statistical estimation algorithms used for orbit determination, parameter identification, and navigation applications are dependent on the accuracy with which the dynamic system and the measurements used to observe the motion can be modeled. In particular, the design of orbit determination algorithms usually begins with the definition of the important error sources and a statistical description of these error sources. The effect of erroneous assumptions regarding (1) the mathematical description of the force model or the measurement model, (2) the statistical properties of the random errors, and (3) the accuracy of the numerical values assigned to the unestimated measurement and force model parameters, as well as the round-off and truncation characteristics that occur in the computation process, can lead to reduced estimation accuracy and, on occasion, to filter divergence. The general topic of covariance analysis treats the sensitivity of the estimation accuracy to these error sources. The first topic to be treated here is the effects of errors in the constant, but nonestimated, dynamic, and measurement model parameters. Errors of this nature lead to biased estimates. A second topic to be considered is the effect of errors in statistics such as the data noise covariance and the *a priori* state covariance.

On occasion it may be advantageous to ignore (i.e., not estimate) certain unknown or poorly known model parameters. *Consider covariance analysis* is a technique to assess the impact of neglecting to estimate these parameters on the accuracy of the state estimate.

The reasons for neglecting to estimate certain parameters are:

1. It is cost effective in computer cost, memory requirements, and execution time to use as small a parameter array as possible.
2. Large dimension parameter arrays may not be "totally" observable from an

observation set collected over a short time interval.

Consider covariance analysis is a "design tool" that can be used for sensitivity analysis to determine the optimal parameter array for a given estimation problem or to structure an estimation algorithm to achieve a more robust performance in the presence of erroneous force and/or measurement model parameters.

Covariance analysis is an outgrowth of the study of the effects of errors on an estimate of the state of a dynamical system. These errors manifest themselves as:

1. Large residuals in results obtained with a given estimation algorithm.
2. Divergence in the sequential estimation algorithms.
3. Incorrect navigation decisions based on an optimistic state error covariance.

As discussed in Chapter 4, the operational "fix" for the Kalman or sequential filter divergence problem caused by optimistic state error covariance estimates is the addition of process noise. In an application, the process noise model will cause the filter to place a higher weight on the most recent data. For the batch estimate, a short-arc solution, which reduces the time interval for collecting the batch of observations, can be used to achieve a similar result. Neither approach is very useful as a design tool.

The comments in the previous paragraphs can be summarized as follows. Consider covariance analysis, in the general case, attempts to quantify the effects of:

a. Nonestimated parameters, $\mathbf{C}$, whose uncertainty is neglected in the estimation procedure.
b. Incorrect *a priori covariance* for the *a priori* estimate of $\mathbf{X}$.
c. Incorrect *a priori covariance* for the measurement noise.

A *consider filter* will use actual data along with *a priori* information on certain consider parameters to improve the filter divergence characteristics due to errors in the dynamic and measurement models. The effects of bias in the unestimated model parameters is the most important of these effects and will be given primary emphasis in the following discussion.

6.2 BIAS IN LINEAR ESTIMATION PROBLEMS

Errors in the constant parameters that appear in the dynamic and/or measurement models may have a random distribution *a priori*. However, during any estimation procedure the values will be constant but unknown and, hence, must be treated as a bias. Bias errors in the estimation process are handled in one of three ways:

1. *Neglected.* The estimate of the state is determined, neglecting any errors in the nonestimated force model and measurement model parameters.

2. *Estimated.* The state vector is expanded to include dynamic and measurement model parameters that may be in error.

3. *Considered.* The state vector is estimated but the uncertainty in the nonestimated parameters is included in the estimation error covariance matrix. This assumes that the nonestimated parameters are constant and that their *a priori* estimate and associated covariance matrix is known.

In sequential filtering analysis, an alternate approach is to compensate for the effects of model errors through the addition of a process noise model, as discussed in Section 4.9 of Chapter 4.

6.3 FORMULATION OF THE CONSIDER COVARIANCE MATRIX

Consider the following partitioning of the generalized state vector $\mathbf{z}$, and observation-state mapping matrix H,

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{c} \end{bmatrix}, H = [H_x \vdots H_c] \tag{6.3.1}$$

where $\mathbf{x}$ is an $n \times 1$ vector of state variables whose values are to be estimated and $\mathbf{c}$ is a $q \times 1$ vector of measurement and force model variables whose values are uncertain but whose values will not be estimated. Note that we are considering a linearized system so that $\mathbf{c}$ represents a vector of deviations between the true and nominal values of the *consider parameters*, $\mathbf{C}$,

$$\mathbf{c} = \mathbf{C} - \mathbf{C}^* . \tag{6.3.2}$$

The measurement model for the i^{th} observation,

$$\mathbf{y}_i = H_i \mathbf{z}_i + \epsilon_i, i = 1, \ldots, l$$

can be expressed as

$$\mathbf{y}_i = H_{x_i} \mathbf{x} + H_{c_i} \mathbf{c} + \epsilon_i, i = 1, \ldots, l . \tag{6.3.3}$$

Assume that an *a priori* estimate of $\mathbf{x}$ and associated covariance (e.g., $(\overline{\mathbf{x}}, \overline{P}_x)$) is given along with $\overline{\mathbf{c}}$, an *a priori* estimate of $\mathbf{c}$. The filter equations can be derived by following the procedures used in Chapter 4. The relevant equations are

$$\mathbf{y} = H_x \mathbf{x} + H_c \mathbf{c} + \epsilon \tag{6.3.4}$$

where

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_l \end{bmatrix}, H_x = \begin{bmatrix} H_{x_1} \\ H_{x_2} \\ \vdots \\ H_{x_l} \end{bmatrix},$$

$$H_c = \begin{bmatrix} H_{c_1} \\ H_{c_2} \\ \vdots \\ H_{c_l} \end{bmatrix}, \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_l \end{bmatrix} \tag{6.3.5}$$

and

$\mathbf{y}_i$	$=$	$p \times 1$ vector of observations
$\mathbf{y}$	$=$	$lp \times 1$ vector of observations
H_{x_i}	$=$	$p \times n$ matrix
H_x	$=$	$lp \times n$ matrix
H_{c_i}	$=$	$p \times q$ matrix
H_c	$=$	$lp \times q$ matrix
$\mathbf{x}$	$=$	$n \times 1$ state vector
$\mathbf{c}$	$=$	$q \times 1$ vector of consider parameters
ϵ	$=$	$lp \times 1$ vector of observation errors
ϵ_i	$=$	$p \times 1$ vector of observation errors.

Recall that unless the observation-state relationship and the state propagation equations are linear, $\mathbf{y}$, $\mathbf{x}$, and $\mathbf{c}$ represent observation, state, and consider parameter deviation vectors, respectively. Also,

$$E[\epsilon_i] = 0, \qquad E[\epsilon_i \epsilon_j^T] = R_i \delta_{ij},$$
$$E[\epsilon \epsilon^T] = R = \begin{bmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_l \end{bmatrix}_{lp \times lp} \tag{6.3.6}$$

where δ_{ij} is the Kronecker delta.

A priori estimates for $\mathbf{x}$ and $\mathbf{c}$ are given by $\overline{\mathbf{x}}$ and $\overline{\mathbf{c}}$, where

$$\overline{\mathbf{x}} = \mathbf{x} + \boldsymbol{\eta}, \quad \overline{\mathbf{c}} = \mathbf{c} + \boldsymbol{\beta}. \tag{6.3.7}$$

The errors, $\boldsymbol{\eta}$ and $\boldsymbol{\beta}$, have the following statistical properties:

$$E[\boldsymbol{\eta}] = E[\boldsymbol{\beta}] = 0 \tag{6.3.8}$$
$$E[\boldsymbol{\eta}\boldsymbol{\eta}^T] = \overline{P}_x \tag{6.3.9}$$
$$E[\boldsymbol{\beta}\boldsymbol{\beta}^T] = \overline{P}_{cc} \tag{6.3.10}$$
$$E[\boldsymbol{\eta}\boldsymbol{\epsilon}^T] = E[\boldsymbol{\beta}\boldsymbol{\epsilon}^T] = 0 \tag{6.3.11}$$
$$E[\boldsymbol{\eta}\boldsymbol{\beta}^T] = \overline{P}_{xc}\,. \tag{6.3.12}$$

It is convenient to express this information in a more compact form, such as that of a data equation. From Eqs. (6.3.4) and (6.3.7)

$$\begin{aligned} \mathbf{y} &= H_x\mathbf{x} + H_c\mathbf{c} + \boldsymbol{\epsilon}; \boldsymbol{\epsilon} \sim (0, R) \\ \overline{\mathbf{x}} &= \mathbf{x} + \boldsymbol{\eta}; \qquad\qquad \boldsymbol{\eta} \sim (0, \overline{P}_x) \\ \overline{\mathbf{c}} &= \mathbf{c} + \boldsymbol{\beta}; \qquad\qquad \boldsymbol{\beta} \sim (0, \overline{P}_{cc})\,. \end{aligned} \tag{6.3.13}$$

Let

$$\begin{aligned} &\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{c} \end{bmatrix}; \quad \tilde{\mathbf{y}} = \begin{bmatrix} \mathbf{y} \\ \overline{\mathbf{x}} \\ \overline{\mathbf{c}} \end{bmatrix}; \\ &H_z = \begin{bmatrix} H_x & H_c \\ I & 0 \\ 0 & I \end{bmatrix}; \quad \tilde{\boldsymbol{\epsilon}} = \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \\ \boldsymbol{\beta} \end{bmatrix}. \end{aligned} \tag{6.3.14}$$

It follows that the observation equations can be expressed as

$$\tilde{\mathbf{y}} = H_z\mathbf{z} + \tilde{\boldsymbol{\epsilon}}; \quad \tilde{\boldsymbol{\epsilon}} \sim (0, \tilde{R}) \tag{6.3.15}$$

where

$$\tilde{R} = \begin{bmatrix} R & 0 & 0 \\ 0 & \overline{P}_x & \overline{P}_{xc} \\ 0 & \overline{P}_{cx} & \overline{P}_{cc} \end{bmatrix}. \tag{6.3.16}$$

We wish to determine the weighted least squares estimate of $\mathbf{z}$, obtained by choosing the value of $\mathbf{z}$, which minimizes the performance index

$$J = 1/2\, \tilde{\epsilon}\tilde{R}^{-1}\tilde{\epsilon}^T. \tag{6.3.17}$$

Note that the weight has been specified as the *a priori* data noise covariance matrix. As illustrated in Section 4.3.3, the best estimate of $\mathbf{z}$ is given by

$$\hat{\mathbf{z}} = \left(H_z^T \tilde{R}^{-1} H_z\right)^{-1} H_z^T \tilde{R}^{-1}\tilde{\mathbf{y}} \tag{6.3.18}$$

with the associated estimation error covariance matrix

$$P_z = E\left[(\hat{\mathbf{z}} - \mathbf{z})(\hat{\mathbf{z}} - \mathbf{z})^T\right] = \left(H_z^T \tilde{R}^{-1} H_z\right)^{-1}. \tag{6.3.19}$$

Equations (6.3.18) and (6.3.19) may be written in partitioned form in order to isolate the quantities of interest. First define

$$\tilde{R}^{-1} = \begin{bmatrix} R^{-1} & 0 & 0 \\ 0 & \overline{M}_{xx} & \overline{M}_{xc} \\ 0 & \overline{M}_{cx} & \overline{M}_{cc} \end{bmatrix}. \tag{6.3.20}$$

From

$$\tilde{R}\tilde{R}^{-1} = I, \tag{6.3.21}$$

it can be shown that

$$\begin{aligned}
\overline{M}_{xx} &= \overline{P}_x^{-1} + \overline{P}_x^{-1}\overline{P}_{xc}\overline{M}_{cc}\overline{P}_{cx}\overline{P}_x^{-1} \\
&= (\overline{P}_x - \overline{P}_{xc}\overline{P}_{cc}^{-1}\overline{P}_{cx})^{-1} && (6.3.22) \\
\overline{M}_{xc} &= -(\overline{P}_x - \overline{P}_{xc}\overline{P}_{cc}^{-1}\overline{P}_{cx})^{-1}\overline{P}_{xc}\overline{P}_{cc}^{-1} \\
&= -\overline{M}_{xx}\overline{P}_{xc}\overline{P}_{cc}^{-1} && (6.3.23) \\
\overline{M}_{cx} &= -(\overline{P}_{cc} - \overline{P}_{cx}\overline{P}_x^{-1}\overline{P}_{xc})^{-1}\overline{P}_{cx}\overline{P}_x^{-1} = \overline{M}_{xc}^T && (6.3.24) \\
\overline{M}_{cc} &= (\overline{P}_{cc} - \overline{P}_{cx}\overline{P}_x^{-1}\overline{P}_{xc})^{-1}. && (6.3.25)
\end{aligned}$$

From Eq. (6.3.18) it follows that

$$(H_z^T \tilde{R}^{-1} H_z)\hat{\mathbf{z}} = H_z^T \tilde{R}^{-1}\tilde{\mathbf{y}} \tag{6.3.26}$$

or

$$\begin{bmatrix} (H_x^T R^{-1} H_x + \overline{M}_{xx}) & (H_x^T R^{-1} H_c + \overline{M}_{xc}) \\ (H_c^T R^{-1} H_x + \overline{M}_{cx}) & (H_c^T R^{-1} H_c + \overline{M}_{cc}) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{c}} \end{bmatrix} \tag{6.3.27}$$

$$= \begin{bmatrix} H_x^T R^{-1} y + \overline{M}_{xx}\overline{x} + \overline{M}_{xc}\overline{c} \\ H_c^T R^{-1} y + \overline{M}_{cx}\overline{x} + \overline{M}_{cc}\overline{c} \end{bmatrix}.$$

Rewriting Eq. (6.3.27) as

$$\begin{bmatrix} M_{xx} & M_{xc} \\ M_{cx} & M_{cc} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{c}} \end{bmatrix} = \begin{bmatrix} \mathbf{N}_x \\ \mathbf{N}_c \end{bmatrix} \tag{6.3.28}$$

leads to the simultaneous equations

$$M_{xx}\hat{\mathbf{x}} + M_{xc}\hat{\mathbf{c}} = \mathbf{N}_x \tag{6.3.29}$$

$$M_{cx}\hat{\mathbf{x}} + M_{cc}\hat{\mathbf{c}} = \mathbf{N}_c \tag{6.3.30}$$

where

$$\begin{aligned} M_{xx} &= H_x^T R^{-1} H_x + \overline{M}_{xx} \\ M_{xc} &= H_x^T R^{-1} H_c + \overline{M}_{xc} \\ M_{cx} &= H_c^T R^{-1} H_x + \overline{M}_{cx} = M_{xc}^T \\ M_{cc} &= H_c^T R^{-1} H_c + \overline{M}_{cc} \\ \mathbf{N}_x &= H_x^T R^{-1} \mathbf{y} + \overline{M}_{xx}\overline{\mathbf{x}} + \overline{M}_{xc}\overline{\mathbf{c}} \\ \mathbf{N}_c &= H_c^T R^{-1} \mathbf{y} + \overline{M}_{cx}\overline{\mathbf{x}} + \overline{M}_{cc}\overline{\mathbf{c}}. \end{aligned} \tag{6.3.31}$$

From Eq. (6.3.29) it follows that

$$\hat{\mathbf{x}} = M_{xx}^{-1}\mathbf{N}_x - M_{xx}^{-1} M_{xc}\hat{\mathbf{c}}. \tag{6.3.32}$$

If Eq. (6.3.32) is substituted into Eq. (6.3.30), the following result is obtained:

$$M_{cx}(M_{xx}^{-1}\mathbf{N}_x - M_{xx}^{-1} M_{xc}\hat{\mathbf{c}}) + M_{cc}\hat{\mathbf{c}} = \mathbf{N}_c$$

or

$$(M_{cc} - M_{cx} M_{xx}^{-1} M_{xc})\hat{\mathbf{c}} = \mathbf{N}_c - M_{cx} M_{xx}^{-1}\mathbf{N}_x$$

and

$$\hat{\mathbf{c}} = (M_{cc} - M_{cx} M_{xx}^{-1} M_{xc})^{-1}(\mathbf{N}_c - M_{cx} M_{xx}^{-1}\mathbf{N}_x). \tag{6.3.33}$$

The value for $\hat{\mathbf{c}}$ determined by Eq. (6.3.33) can be substituted into Eq. (6.3.32) to obtain $\hat{\mathbf{x}}$.

The covariance matrix for the errors in $\hat{\mathbf{x}}$ and $\hat{\mathbf{c}}$ given by Eq. (6.3.19) can be written in partitioned form as

$$\begin{bmatrix} P_{xx} & P_{xc} \\ P_{cx} & P_{cc} \end{bmatrix} = \begin{bmatrix} M_{xx} & M_{xc} \\ M_{cx} & M_{cc} \end{bmatrix}^{-1} \tag{6.3.34}$$

where M_{xx}, M_{cx}, M_{xc}, and M_{cc} are defined by Eq. (6.3.31). From the equation

$$\begin{bmatrix} M_{xx} & M_{xc} \\ M_{cx} & M_{cc} \end{bmatrix} \begin{bmatrix} P_{xx} & P_{xc} \\ P_{cx} & P_{cc} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{6.3.35}$$

It can be shown that

$$P_{cx} = -M_{cc}^{-1} M_{cx} P_{xx} \tag{6.3.36}$$

$$P_{xc} = -M_{xx}^{-1} M_{xc} P_{cc} \tag{6.3.37}$$

$$P_{xx} = (M_{xx} - M_{xc} M_{cc}^{-1} M_{cx})^{-1} \tag{6.3.38}$$

$$P_{cc} = (M_{cc} - M_{cx} M_{xx}^{-1} M_{xc})^{-1}. \tag{6.3.39}$$

Note that

$$P_{xc} = P_{cx}^T. \tag{6.3.40}$$

The expression for P_{xx} can be written in an alternate form. Using the Schur identity (Theorem 4 of Appendix B) and letting

$$\begin{aligned} A &= M_{xx} \\ B &= -M_{xc} M_{cc}^{-1} \\ C &= M_{cx}, \end{aligned}$$

it can be shown that

$$P_{xx} = M_{xx}^{-1} + M_{xx}^{-1} M_{xc} \left(M_{cc} - M_{cx} M_{xx}^{-1} M_{xc}\right)^{-1} M_{cx} M_{xx}^{-1}. \tag{6.3.41}$$

If we use the definition

$$P_x = M_{xx}^{-1} = (H_x^T R^{-1} H_x + \overline{M}_{xx})^{-1}, \tag{6.3.42}$$

and define S_{xc} as (in Section 6.4 we will show Eq. (6.3.43) to be the sensitivity matrix)

$$S_{xc} \equiv -M_{xx}^{-1} M_{xc} \tag{6.3.43}$$

and using Eq. (6.3.39) to define P_{cc}, we can write Eq. (6.3.41) as

$$P_{xx} = P_x + S_{xc} P_{cc} S_{xc}^T. \tag{6.3.44}$$

Also, from Eqs. (6.3.37) and (6.3.44) we have

$$P_{xc} = S_{xc} P_{cc} = P_{cx}^T . \tag{6.3.45}$$

Using Eqs. (6.3.39), (6.3.42), and (6.3.44) we may write Eqs. (6.3.32) and (6.3.33) as

$$\begin{aligned} \hat{\mathbf{x}} &= P_x \mathbf{N}_x + S_{xc} \hat{\mathbf{c}} & (6.3.46) \\ \hat{\mathbf{c}} &= P_{cc}(\mathbf{N}_c + S_{xc}^T \mathbf{N}_x) . & (6.3.47) \end{aligned}$$

The consider estimate is obtained by choosing not to compute $\hat{\mathbf{c}}$ and P_{cc} but to fix them at their *a priori* values $\overline{\mathbf{c}}$ and $\overline{P}_{cc}$, respectively. In this case, the equations that require modification are

$$\begin{aligned} P_{cc} &= \overline{P}_{cc} & (6.3.48) \\ P_{xx} &= P_x + S_{xc} \overline{P}_{cc} S_{xc}^T & (6.3.49) \\ P_{xc} &= S_{xc} \overline{P}_{cc} & (6.3.50) \\ \hat{\mathbf{x}} &= P_x \mathbf{N}_x + S_{xc} \overline{\mathbf{c}}. & (6.3.51) \end{aligned}$$

The *a priori* covariance of the state and consider parameters, $\overline{P}_{xc}$, generally is unknown and assumed to be the null matrix. For the case where $\overline{P}_{xc} = 0$, Eq. (6.3.31) reduces to

$$\begin{aligned} M_{xx} &= H_x^T R^{-1} H_x + \overline{P}_x^{-1} \equiv P_x^{-1} & (6.3.52) \\ M_{xc} &= H_x^T R^{-1} H_c . & (6.3.53) \end{aligned}$$

Hence,

$$S_{xc} = -M_{xx}^{-1} M_{xc} \tag{6.3.54}$$

reduces to

$$S_{xc} = -P H_x^T R^{-1} H_c . \tag{6.3.55}$$

The computational algorithm for the batch consider filter is

Given: $\overline{\mathbf{x}}$, $\overline{\mathbf{c}}$, $\overline{P}_x$, $\overline{P}_{xc}$, $\overline{P}_{cc}$, R_i, H_{x_i}, H_{c_i} and $\mathbf{y}_i$, $i = 1 \cdots l$

$\overline{M}_{xx}$, $\overline{M}_{xc}$, and $\overline{M}_{cc}$ are given by Eqs. (6.3.22), (6.3.23), and (6.3.25), respectively.

Compute:

$$\begin{aligned} M_{xx} &= H_x^T R^{-1} H_x + \overline{M}_{xx} & (6.3.56) \\ &= \sum_{i=1}^{l} H_{x_i}^T R_i^{-1} H_{x_i} + \overline{M}_{xx} = P_x^{-1} & (6.3.57) \end{aligned}$$

$$\begin{aligned}
M_{xc} &= H_x R^{-1} H_c + \overline{M}_{xc} && (6.3.58)\\
&= \sum_{i=1}^{l} H_{x_i}^T \, R_i^{-1} \, H_{c_i} + \overline{M}_{xc} && (6.3.59)\\
M_{cx} &= M_{xc}^T && (6.3.60)\\
M_{cc} &= \sum_{i=1}^{l} H_{c_i}^T R_i^{-1} H_{c_i} + \overline{M}_{cc} && (6.3.61)\\
P_x &= M_{xx}^{-1} && (6.3.62)\\
S_{xc} &= -P_x M_{xc} && (6.3.63)\\
\mathbf{N}_x &= \sum_{i=1}^{l} H_{x_i}^T R_i^{-1} y_i + \overline{P}_x^{\,-1} \overline{\mathbf{x}} && (6.3.64)\\
\hat{\mathbf{x}}_c &= M_{xx}^{-1} \mathbf{N}_x - M_{xx}^{-1} M_{xc} \overline{\mathbf{c}} && (6.3.65)\\
P_{xx} &= P_x + S_{xc} \overline{P}_{cc} S_{xc}^T && (6.3.66)\\
P_{xc} &= S_{xc} \overline{P}_{cc} = P_{cx}^T \,. && (6.3.67)
\end{aligned}$$

Equation (6.3.66) may also be written as

$$P_{xx} = P_x + P_{xc} \overline{P}_{cc}^{-1} P_{xc}^T \,. \tag{6.3.68}$$

The complete *consider covariance matrix* may be written as

$$\mathbf{P}_c = E\left\{ \begin{bmatrix} \hat{\mathbf{x}}_c - \mathbf{x} \\ \overline{\mathbf{c}} - \mathbf{c} \end{bmatrix} \left[(\hat{\mathbf{x}}_c - \mathbf{x})^T \;\; (\overline{\mathbf{c}} - \mathbf{c})^T \right] \right\} = \begin{bmatrix} P_{xx} & P_{xc} \\ P_{cx} & \overline{P}_{cc} \end{bmatrix} \tag{6.3.69}$$

where P_{xx} is the consider covariance associated with the state vector of estimated parameters, $\mathbf{x}$.

Note also that the expression for $\hat{\mathbf{x}}_c$ can be written

$$\begin{aligned}
\hat{\mathbf{x}}_c &= P_x \mathbf{N}_x - P_x M_{xc} \overline{\mathbf{c}} && (6.3.70)\\
\hat{\mathbf{x}}_c &= \hat{\mathbf{x}} - P_x M_{xc} \overline{\mathbf{c}} && (6.3.71)
\end{aligned}$$

or, from Eq. (6.3.53)

$$\hat{\mathbf{x}}_c = \hat{\mathbf{x}} - P_x H_x^T R^{-1} H_c \overline{\mathbf{c}} \tag{6.3.72}$$

where $\hat{\mathbf{x}}$ and P_x are the values of those parameters obtained from a batch processor, which assumes there are no errors in the consider parameters (i.e., $\mathbf{C}^* = \mathbf{C}$ and P_{xc}, $\overline{P}_{cc}$, $\mathbf{c}$, and $\overline{\mathbf{c}}$ are all zero).

6.4 THE SENSITIVITY AND PERTURBATION MATRICES

Several other matrices are often associated with the concept of consider analysis. The *sensitivity matrix* is defined to be

$$S_{xc} = \frac{\partial \hat{\mathbf{x}}_c}{\partial \hat{\mathbf{c}}} \tag{6.4.1}$$

which from Eq. (6.3.32) is

$$S_{xc} = -M_{xx}^{-1} M_{xc} \,. \tag{6.4.2}$$

From Eq. (6.3.52)

$$S_{xc} = -PM_{xc} \,, \tag{6.4.3}$$

and using Eq. (6.3.53) S_{xc} becomes

$$S_{xc} = -PH_x^T R^{-1} H_c \,. \tag{6.4.4}$$

Hence, Eq. (6.3.71) may be written as

$$\hat{\mathbf{x}}_c = \hat{\mathbf{x}} + S_{xc}\overline{\mathbf{c}} \,. \tag{6.4.5}$$

Recall from Eq. (6.3.66) that the consider covariance can be written in terms of the sensitivity matrix as

$$P_{xx} = P_x + S_{xc}\overline{P}_{cc}S_{xc}^T \,. \tag{6.4.6}$$

Also, using Eq. (6.3.67) the covariance, P_{xc}, can be written as

$$P_{xc} = S_{xc}\overline{P}_{cc} \,. \tag{6.4.7}$$

The sensitivity matrix describes how $\hat{\mathbf{x}}_c$ varies with respect to the consider parameters, $\mathbf{c}$. Another commonly used matrix is the *perturbation matrix* defined by

$$\Gamma = S_{xc} \cdot [\text{diagonal}\,(\sigma_c)] \tag{6.4.8}$$

where the elements of the diagonal matrix are the standard deviations of the consider parameters. Each element, Γ_{ij}, gives the error in the estimate of $\mathbf{x}_i$ due to a one-sigma error in the consider parameter c_j.

Additional discussion of consider covariance analysis can be found in Bierman (1977).

6.4.1 Example Application of a Sensitivity and Perturbation Matrix

Assume we have a state vector comprised of coordinates x and y and a parameter α, and consider parameters γ and δ,

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ \alpha \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}. \tag{6.4.9}$$

The perturbation matrix is defined by

$$\Gamma \equiv S_{xc} \cdot [\mathrm{diag}\, \sigma_c] = \begin{bmatrix} \frac{\partial \hat{x}}{\partial \gamma} & \frac{\partial \hat{x}}{\partial \delta} \\ \frac{\partial \hat{y}}{\partial \gamma} & \frac{\partial \hat{y}}{\partial \delta} \\ \frac{\partial \hat{\alpha}}{\partial \gamma} & \frac{\partial \hat{\alpha}}{\partial \delta} \end{bmatrix} \begin{bmatrix} \sigma_\gamma & 0 \\ 0 & \sigma_\delta \end{bmatrix}$$

or

$$\Gamma = \begin{bmatrix} \frac{\partial \hat{x}}{\partial \gamma}\sigma_\gamma & \frac{\partial \hat{x}}{\partial \delta}\sigma_\delta \\ \frac{\partial \hat{y}}{\partial \gamma}\sigma_\gamma & \frac{\partial \hat{y}}{\partial \delta}\sigma_\delta \\ \frac{\partial \hat{\alpha}}{\partial \gamma}\sigma_\gamma & \frac{\partial \hat{\alpha}}{\partial \delta}\sigma_\delta \end{bmatrix}. \tag{6.4.10}$$

Hence, the errors in the state estimate due to one-sigma errors in the consider parameters are given by

$$\begin{aligned} \Delta \hat{x} &= \frac{\partial \hat{x}}{\partial \gamma}\sigma_\gamma + \frac{\partial \hat{x}}{\partial \delta}\sigma_\delta \\ \Delta \hat{y} &= \frac{\partial \hat{y}}{\partial \gamma}\sigma_\gamma + \frac{\partial \hat{y}}{\partial \delta}\sigma_\delta \\ \Delta \hat{\alpha} &= \frac{\partial \hat{\alpha}}{\partial \gamma}\sigma_\gamma + \frac{\partial \hat{\alpha}}{\partial \delta}\sigma_\delta . \end{aligned} \tag{6.4.11}$$

Note that the information in Γ is meaningful only if $\overline{P}_{cc}$ (the covariance matrix of the errors in the consider parameters) is diagonal.

We may now form the consider covariance matrix. For simplicity, assume that

$\overline{P}_{cc}$ is diagonal. Using Eq. (6.4.6) we may write P_{xx} in terms of Γ,

$$P_{xx} = P_x + S_{xc}\overline{P}_{cc}S_{xc}^T = P_x + \Gamma\Gamma^T \,. \qquad (6.4.12)$$

The contribution of the consider parameters is given by

$$\begin{aligned} S_{xc}\,\overline{P}_{cc}\,S_{xc}^T &= P_{xc}\,S_{xc}^T \qquad (6.4.13)\\ &= \begin{bmatrix} \frac{\partial\hat{x}}{\partial\gamma}\sigma_\gamma^2 & \frac{\partial\hat{x}}{\partial\delta}\sigma_\delta^2 \\ \frac{\partial\hat{y}}{\partial\gamma}\sigma_\gamma^2 & \frac{\partial\hat{y}}{\partial\delta}\sigma_\delta^2 \\ \frac{\partial\hat{\alpha}}{\partial\gamma}\sigma_\gamma^2 & \frac{\partial\hat{\alpha}}{\partial\delta}\sigma_\delta^2 \end{bmatrix} \begin{bmatrix} \frac{\partial\hat{x}}{\partial\gamma} & \frac{\partial\hat{y}}{\partial\gamma} & \frac{\partial\hat{\alpha}}{\partial\gamma} \\ \frac{\partial\hat{x}}{\partial\delta} & \frac{\partial\hat{y}}{\partial\delta} & \frac{\partial\hat{\alpha}}{\partial\delta} \end{bmatrix} . \end{aligned}$$

The diagonal terms of this matrix are

$$(6.4.14)$$

$$= \begin{bmatrix} \left(\frac{\partial\hat{x}}{\partial\gamma}\right)^2\sigma_\gamma^2 + \left(\frac{\partial\hat{x}}{\partial\delta}\right)^2\sigma_\delta^2 & & \\ & \left(\frac{\partial\hat{y}}{\partial\gamma}\right)^2\sigma_\gamma^2 + \left(\frac{\partial\hat{y}}{\partial\delta}\right)^2\sigma_\delta^2 & \\ & & \left(\frac{\partial\hat{\alpha}}{\partial\gamma}\right)^2\sigma_\gamma^2 + \left(\frac{\partial\hat{\alpha}}{\partial\delta}\right)^2\sigma_\delta^2 \end{bmatrix}_{\text{consider}}$$

or in simplified notation, the upper triangular portion of this symmetric matrix is given by

$$S_{xc}\overline{P}_{cc}S_{xc}^T = \begin{bmatrix} \sigma_{\hat{x}}^2 & \sigma_{\hat{x}}\sigma_{\hat{y}}\rho_{xy} & \sigma_{\hat{x}}\sigma_{\hat{\alpha}}\rho_{x\alpha} \\ & \sigma_{\hat{y}}^2 & \sigma_{\hat{y}}\sigma_{\hat{\alpha}}\rho_{y\alpha} \\ & & \sigma_{\hat{\alpha}}^2 \end{bmatrix}_{\text{consider}} . \qquad (6.4.15)$$

For example, the total consider variance for the state variable, x, is given by

$$\begin{aligned} \sigma_{\hat{x}}^2 &= [\sigma_{\hat{x}}^2]_{\text{data noise}} + \left(\frac{\partial\hat{x}}{\partial\gamma}\right)^2\sigma_\gamma^2 + \left(\frac{\partial\hat{x}}{\partial\delta}\right)^2\sigma_\delta^2 \\ &= [\sigma_{\hat{x}}^2]_{\text{data noise}} + [\sigma_{\hat{x}}^2]_{\text{consider}} \,. \end{aligned} \qquad (6.4.16)$$

Comments:

1. If $\overline{P}_{cc}$ is diagonal, then the diagonal elements of P_{xx} consist of the variance due to data noise plus the sum of the squares of the perturbations due to each consider parameter.

2. Off diagonal terms of the consider portion of P_{xx} contain correlations between errors in the estimated parameters caused by the consider parameters.

3. If $\overline{P}_{cc}$ is a full matrix, $S_{xc}\overline{P}_{cc}S_{xc}^T$ becomes much more complex, and the diagonal terms become a function of the covariances (correlations) between consider parameters.

For example, consider the case where $\overline{P}_{cc}$ is a full matrix and $\mathbf{X} = [x\, y]^T$ and $\mathbf{c} = [c_1\, c_2]^T$ are 2×1 vectors. Then

$$S_{xc}\overline{P}_{cc}S_{xc}^T = \begin{bmatrix} \frac{\partial \hat{x}}{\partial c_1} & \frac{\partial \hat{x}}{\partial c_2} \\ \frac{\partial \hat{y}}{\partial c_1} & \frac{\partial \hat{y}}{\partial c_2} \end{bmatrix} \begin{bmatrix} \sigma_{c_1}^2 & \mu_{12} \\ \mu_{12} & \sigma_{c_2}^2 \end{bmatrix} S_{xc}^T. \tag{6.4.17}$$

If this is expanded, the 1,1 element becomes

$$(P_c)_{11} = \left(\frac{\partial \hat{x}}{\partial c_1}\right)^2 \sigma_{c_1}^2 + 2\frac{\partial \hat{x}}{\partial c_1}\frac{\partial \hat{x}}{\partial c_2}\mu_{12} + \left(\frac{\partial \hat{x}}{\partial c_2}\right)^2 \sigma_{c_2}^2. \tag{6.4.18}$$

Depending on the sign of the components of the second term on the right-hand side, the consider uncertainty of x could be greater or less than that for the case where $\overline{P}_{cc}$ is diagonal. In general, inclusion of the correlations between the consider parameters (e.g., μ_{12}) results in a reduction in the consider variances.

6.5 INCLUSION OF TIME-DEPENDENT EFFECTS

In this section we will discuss consider covariance analysis under the assumption that the state vector is time dependent. However, in this chapter we will only "consider" the effect on the state vector of errors in constant measurement or model parameters.

The dynamical equations associated with the consider covariance model can be derived as follows. The differential equations for state propagation are given by (see Chapter 4):

$$\dot{\mathbf{X}} = F(\mathbf{X}, \mathbf{C}, t). \tag{6.5.1}$$

Expanding Eq. (6.5.1) in a Taylor series to first order about a nominal trajectory yields

$$\dot{\mathbf{X}} = F(\mathbf{X}^*, \mathbf{C}^*, t) + \left[\frac{\partial F}{\partial \mathbf{X}}\right]^* (\mathbf{X} - \mathbf{X}^*) + \left[\frac{\partial F}{\partial \mathbf{C}}\right]^* (\mathbf{C} - \mathbf{C}^*) + \ldots \tag{6.5.2}$$

Define

$$\dot{\mathbf{x}} \equiv \dot{\mathbf{X}} - F(\mathbf{X}^*, \mathbf{C}^*, t) = \left[\frac{\partial F}{\partial \mathbf{X}}\right]^* (\mathbf{X} - \mathbf{X}^*) + \left[\frac{\partial F}{\partial \mathbf{C}}\right]^* (\mathbf{C} - \mathbf{C}^*). \quad (6.5.3)$$

This may be written as

$$\dot{\mathbf{x}} = A(t)\mathbf{x}(t) + B(t)\mathbf{c} \quad (6.5.4)$$

where

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{X}(t) - \mathbf{X}^*(t) & (6.5.5)\\ \mathbf{c} &= \mathbf{C} - \mathbf{C}^* & (6.5.6)\\ A(t) &= \left[\frac{\partial F}{\partial \mathbf{X}}\right]^* \\ B(t) &= \left[\frac{\partial F}{\partial \mathbf{C}}\right]^* . \end{aligned}$$

For the conventional filter model $\mathbf{c} = 0$, and the solution for Eq. (6.5.4) is

$$\mathbf{x}(t) = \Phi(t, t_k)\mathbf{x}(t_k). \quad (6.5.7)$$

The general solution of Eq. (6.5.4) can be obtained by the method of variation of parameters to yield

$$\mathbf{x}(t) = \Phi(t, t_k)\mathbf{x}_k + \theta(t, t_k)\mathbf{c}. \quad (6.5.8)$$

In Eq. (6.5.8), the $n \times n$ mapping matrix, $\Phi(t, t_k)$, and the $n \times q$ mapping matrix, $\theta(t, t_k)$, are defined as

$$\Phi(t, t_k) = \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_k)} \quad (6.5.9)$$

$$\theta(t, t_k) = \frac{\partial \mathbf{X}(t)}{\partial \mathbf{C}(t_k)}. \quad (6.5.10)$$

The corresponding differential equations for $\Phi(t, t_k)$ and $\theta(t, t_k)$ are obtained by differentiating Eq. (6.5.1) and recognizing that differentiation with respect to time and $\mathbf{X}(t_k)$ and $\mathbf{C}$ are interchangeable for functions whose derivatives are continuous. From Eq. (6.5.1)

$$\frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}(t_k)} = \frac{\partial F}{\partial \mathbf{X}(t_k)}, \quad (6.5.11)$$

which may be written as

$$\frac{d}{dt}\frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_k)} = \left[\frac{\partial F}{\partial \mathbf{X}(t)}\right]^* \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_k)} \quad (6.5.12)$$

or

$$\dot{\Phi}(t, t_k) = A(t)\Phi(t, t_k) \tag{6.5.13}$$

with initial conditions $\Phi(t_0, t_0) = I$. Likewise,

$$\frac{d}{dt}\frac{\partial \mathbf{X}(t)}{\partial \mathbf{C}(t_k)} = \left[\frac{\partial F}{\partial \mathbf{X}(t)}\right]^* \frac{\partial \mathbf{X}(t)}{\partial \mathbf{C}(t_k)} + \left[\frac{\partial F}{\partial \mathbf{C}}\right]^*$$

becomes

$$\dot{\theta}(t, t_k) = A(t)\theta(t, t_k) + B(t) \tag{6.5.14}$$

with initial conditions $\theta(t_0, t_0) = 0$. Note that if C is partitioned as

$$\mathbf{C}^T = [\mathbf{C}_d^T \vdots \mathbf{C}_m^T]$$

where $\mathbf{C}_d$ are dynamic model parameters and $\mathbf{C}_m$ are measurement model parameters, then θ can be written as

$$\theta = [\theta_d \vdots \theta_m]$$

where the solution for θ_m will be $\theta_m(t, t_k) = 0$, the null matrix. This follows since measurement model parameters do not appear in the dynamic equations.

The estimation errors for the conventional filter model (Eq. 6.5.7) and the consider model (Eq. 6.5.8), labeled here as Filter and Consider, are mapped according to

Filter: $$\tilde{\mathbf{x}}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t) = \Phi(t, t_k)\,\tilde{\mathbf{x}}_k \tag{6.5.15}$$

where $\tilde{\mathbf{x}}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k$ and all quantities are computed assuming there are no consider parameters.

Consider:
$$\begin{aligned}\tilde{\mathbf{x}}_c(t) &= \hat{\mathbf{x}}_c(t) - \mathbf{x}(t)\\ &= [\Phi(t, t_k)\hat{\mathbf{x}}_{c_k} + \theta(t, t_k)\bar{\mathbf{c}}] - [\Phi(t, t_k)\mathbf{x}_k + \theta(t, t_k)\mathbf{c}]\\ &= \Phi(t, t_k)(\hat{\mathbf{x}}_{c_k} - \mathbf{x}_k) + \theta(t, t_k)(\bar{\mathbf{c}} - \mathbf{c}) \qquad (6.5.16)\\ &= \Phi(t, t_k)\tilde{\mathbf{x}}_{c_k} + \theta(t, t_k)\boldsymbol{\beta}\end{aligned}$$

where $\tilde{\mathbf{x}}_{c_k} = \hat{\mathbf{x}}_{c_k} - \mathbf{x}_k$ and $\boldsymbol{\beta} = \bar{\mathbf{c}} - \mathbf{c}$.

We have included the conventional filter results in this section to illustrate differences with the consider filter. Remember that the conventional filter results assume that there are no errors in the consider parameters ($\mathbf{C}^* = \mathbf{C}$). Hence, the true value, $\mathbf{x}_k$, for the filter results is different from the true value for the consider results.

The respective observation models are

Filter: $$\mathbf{y}_j = \tilde{H}_{x_j}\mathbf{x}_j + \epsilon_j \quad j = 1, \ldots, l\,. \tag{6.5.17}$$

Consider: $$\mathbf{y}_j = \tilde{H}_{x_j}\mathbf{x}_j + \tilde{H}_{c_j}\mathbf{c} + \epsilon_j \quad j = 1, \ldots, l\,. \tag{6.5.18}$$

The state vector, $\mathbf{x}_j$, is replaced by its value at the estimation epoch t_k, by using Eq. (6.5.7),

Filter: $$\mathbf{y}_j = \tilde{H}_{x_j}\Phi(t_j, t_k)\mathbf{x}_k + \epsilon_j\,. \tag{6.5.19}$$

The consider expression for $\mathbf{y}_j$ is obtained by recognizing that

$$\mathbf{y}_j = \tilde{H}_{x_j}\left[\Phi(t_j, t_k)\mathbf{x}_k + \theta(t_j, t_k)\mathbf{c}\right] + \tilde{H}_{c_j}\mathbf{c} + \epsilon_j\,. \tag{6.5.20}$$

This equation may be written as

$$\mathbf{y}_j = \left[\tilde{H}_{x_j}\tilde{H}_{c_j}\right]\begin{bmatrix} \Phi(t_j, t_k) & \theta(t_j, t_k) \\ 0 & I \end{bmatrix}\begin{bmatrix} \mathbf{x}_k \\ \mathbf{c} \end{bmatrix} + \epsilon_j \tag{6.5.21}$$

or

$$\mathbf{y}_j = \left[\tilde{H}_{x_j}\Phi(t_j, t_k) \vdots \tilde{H}_{x_j}\theta(t_j, t_k) + \tilde{H}_{c_j}\right]\begin{bmatrix} \mathbf{x}_k \\ \mathbf{c} \end{bmatrix} + \epsilon_j\,. \tag{6.5.22}$$

By using the definitions

$$H_{x_j} \equiv \tilde{H}_{x_j}\Phi(t_j, t_k) \quad j = 1, \ldots, l \tag{6.5.23}$$

$$H_{c_j} \equiv \tilde{H}_{x_j}\theta(t_j, t_k) + \tilde{H}_{c_j} \quad j = 1, \ldots, l \tag{6.5.24}$$

and

$$\mathbf{y} \equiv \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_l \end{bmatrix}, \qquad H_x \equiv \begin{bmatrix} H_{x_1} \\ \vdots \\ H_{x_l} \end{bmatrix}, \tag{6.5.25}$$

$$H_c \equiv \begin{bmatrix} H_{c_1} \\ \vdots \\ H_{c_l} \end{bmatrix}, \quad \boldsymbol{\epsilon} \equiv \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_l \end{bmatrix}$$

the two cumulative observation models used in Section 6.3 can then be obtained,

Filter: $$\mathbf{y} = H_x \mathbf{x}_k + \boldsymbol{\epsilon} \tag{6.5.26}$$

Consider: $$\mathbf{y} = H_x \mathbf{x}_k + H_c \mathbf{c} + \boldsymbol{\epsilon} \tag{6.5.27}$$

where $E[\boldsymbol{\epsilon}] = 0$ and $E[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T] = R$. The associated error covariance matrix for the filter model at the epoch, t_k, is

$$P_{xk} = E\big[(\hat{\mathbf{x}}_k - \mathbf{x}_k)(\hat{\mathbf{x}}_k - \mathbf{x}_k)^T\big] = \big[H_x^T R^{-1} H_x + \overline{P}_k^{-1}\big]^{-1}. \tag{6.5.28}$$

To simplify, notation P_{xx} is being replaced by P_c, S_{xc} is being replaced by S, and P_x is being replaced by P for the remainder of this chapter.

The consider covariance is given by

$$P_{xx_k} \equiv P_{c_k} = E\big[(\hat{\mathbf{x}}_{c_k} - \mathbf{x}_k)(\hat{\mathbf{x}}_{c_k} - \mathbf{x}_k)^T\big]\,.$$

From Eq. (6.4.5)

$$\hat{\mathbf{x}}_{c_k} = \hat{\mathbf{x}}_k + S_k \overline{\mathbf{c}}\,. \tag{6.5.29}$$

However, the true value of $\mathbf{x}_k$ for the consider covariance analysis is given by

$$\mathbf{x}_k = \mathbf{x}_k^* + S_k \mathbf{c} \tag{6.5.30}$$

where $\mathbf{x}_k^*$ is the true value of $\mathbf{x}_k$ for the conventional filter ($\mathbf{C}^* = \mathbf{C}$) and there are no errors in the consider parameters. $S_k\mathbf{c}$ is the contribution due to the true value of $\mathbf{c}$, the error in the consider parameters. Hence, $\Delta\hat{\mathbf{x}}_{c_k}$, the error in the consider estimate, $\hat{\mathbf{x}}_{c_k}$, is given by

$$\Delta\hat{\mathbf{x}}_{c_k} \equiv \hat{\mathbf{x}}_{c_k} - \mathbf{x}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k^* + S_k(\overline{\mathbf{c}} - \mathbf{c})\,. \tag{6.5.31}$$

Since $P_{c_k} = E[\Delta\hat{\mathbf{x}}_{c_k}\Delta\hat{\mathbf{x}}_{c_k}^T]$, it follows that

$$P_{c_k} = P_k + S_k\overline{P}_{cc}S_k^T \tag{6.5.32}$$

where

$$P_k = E\big[(\hat{\mathbf{x}}_k - \mathbf{x}_k^*)(\hat{\mathbf{x}}_k - \mathbf{x}_k^*)^T\big]\,.$$

P_k is the conventional filter data noise covariance given by Eq. (6.5.28). Also,

$$E\big[(\hat{\mathbf{x}}_k - \mathbf{x}_k^*)(\bar{\mathbf{c}} - \mathbf{c})^T\big] = 0, \tag{6.5.33}$$

because errors in the conventional filter estimate are by definition independent of errors in the consider parameters.

6.6 PROPAGATION OF THE ERROR COVARIANCE

The models for propagating the covariance matrix from time t_j to t_k can be obtained by noting that

Filter:
$$P_k = E\big[(\hat{\mathbf{x}}_k - \mathbf{x}_k)(\hat{\mathbf{x}}_k - \mathbf{x}_k)^T\big] \tag{6.6.1}$$

where
$$\hat{\mathbf{x}}_k - \mathbf{x}_k = \Phi(t_k,t_j)[\hat{\mathbf{x}}_j - \mathbf{x}_j] \tag{6.6.2}$$

and

Consider:
$$P_{c_k} = E\big[(\hat{\mathbf{x}}_{c_k} - \mathbf{x}_k)(\hat{\mathbf{x}}_{c_k} - \mathbf{x}_k)^T\big] \tag{6.6.3}$$

where

$$\begin{aligned} \hat{\mathbf{x}}_{c_k} - \mathbf{x}_k &= \big[\Phi(t_k,t_j)\hat{\mathbf{x}}_{c_j} + \theta(t_k,t_j)\bar{\mathbf{c}})\big] \\ &\quad -\big[\Phi(t_k,t_j)\mathbf{x}_j + \theta(t_k,t_j)\mathbf{c}\big]\,. \end{aligned} \tag{6.6.4}$$

Substituting Eqs. (6.6.2) and (6.6.4) into Eqs. (6.6.1) and (6.6.3), respectively, and performing the expected value operation leads to

Filter:
$$P_k = \Phi(t_k,t_j)P_j\Phi^T(t_k,t_j) \tag{6.6.5}$$

and

Consider:

$$\begin{aligned} P_{c_k} &= E\Big\{\big[\Phi(t_k,t_j)(\hat{\mathbf{x}}_{c_j} - \mathbf{x}_j) + \theta(t_k,t_j)(\bar{\mathbf{c}} - \mathbf{c})\big]\big[\;\big]^T\Big\} \\ &= \Phi(t_k,t_j)P_{c_j}\Phi^T(t_k,t_j) + \theta(t_k,t_j)\overline{P}_{cc}\theta^T(t_k,t_j) \\ &+ \Phi(t_k,t_j)S_j\overline{P}_{cc}\theta^T(t_k,t_j) + \theta(t_k,t_j)\overline{P}_{cc}S_j^T\Phi^T(t_k,t_j)\,. \end{aligned} \tag{6.6.6}$$

This result is obtained by noting from Eq. (6.4.7) that

$$P_{xc_j} = E\big[(\hat{\mathbf{x}}_{c_j} - \mathbf{x}_j)(\overline{\mathbf{c}} - \mathbf{c})^T\big] = S_j\overline{P}_{cc} \tag{6.6.7}$$

and

$$P_{cx_j} = \overline{P}_{cc}S_j^T \tag{6.6.8}$$

where we have used the notation

$$S_j \equiv S_{xc_j}\,.$$

By defining

$$S_k = \Phi(t_k, t_j)S_j + \theta(t_k, t_j) \tag{6.6.9}$$

and using

$$P_{cj} = P_j + S_j\overline{P}_{cc}S_j^T \tag{6.6.10}$$

from Eq. (6.5.32), we can write Eq. (6.6.6) for the consider covariance P_{ck} as

$$P_{c_k} = P_k + S_k\overline{P}_{cc}S_k^T\,. \tag{6.6.11}$$

The second term in the consider covariance will always be at least positive semidefinite and, as a consequence

$$\text{Trace}[P_{c_k}] > \text{Trace}[P_k]\,.$$

Hence, the effect of the uncertainty in the consider parameters will lead to a larger uncertainty in the value of the estimate at a given time.

P_{xc} is propagated as follows

$$\begin{aligned} P_{xc_k} &= E\big[(\hat{\mathbf{x}}_{c_k} - \mathbf{x}_k)\boldsymbol{\beta}^T\big] \\ &= E\big[\Phi(t_k, t_j)(\hat{\mathbf{x}}_{c_j} - \mathbf{x}_j) + \theta(t_k, t_j)(\overline{\mathbf{c}} - \mathbf{c})\boldsymbol{\beta}^T\big] \\ &= \Phi(t_k, t_j)P_{xc_j} + \theta(t_k, t_j)\overline{P}_{cc} \end{aligned} \tag{6.6.12}$$

or P_{xc_k} may be determined directly from S_k by using Eq. (6.6.7) and Eq. (6.6.9),

$$P_{xck} = S_k\overline{P}_{cc}\,. \tag{6.6.13}$$

Mapping of the complete consider covariance matrix may be written more compactly as follows. Define

$$\Psi(t_k, t_j) \equiv \begin{bmatrix} \Phi\ (t_k, t_j) & \theta(t_k, t_j) \\ 0 & I \end{bmatrix} \tag{6.6.14}$$

then

$$\mathbf{P}_{c_k} = \Psi(t_k, t_j)\mathbf{P}_{cj}\Psi^T(t_k, t_j)\,. \tag{6.6.15}$$

Recall that

$$\mathbf{P}_{c_k} \equiv \begin{bmatrix} P_{c_k} & P_{xc_k} \\ P_{cx_k} & \overline{P}_{cc} \end{bmatrix} \tag{6.6.16}$$

where the expressions for P_{c_k} and P_{xc_k} are given by Eqs. (6.6.11) and (6.6.13) respectively.

6.7 SEQUENTIAL CONSIDER COVARIANCE ANALYSIS

The algorithms for consider covariance analysis developed in batch form also can be developed in sequential form. The batch algorithm for $\hat{\mathbf{x}}$ can be modified to function in sequential form as follows.

Write Eq. (6.3.72) at the k^{th} stage

$$\hat{\mathbf{x}}_{c_k} = \hat{\mathbf{x}}_k - P_k H_{x_k}^T R_k^{-1} H_{c_k}\overline{\mathbf{c}} \tag{6.7.1}$$

where

$$P_k = \left(\overline{P}_k^{-1} + \sum_{i=1}^{k} H_{x_i}^T R_i^{-1} H_{x_i}\right)^{-1} \tag{6.7.2}$$

and

$$H_{x_k}^T R_k^{-1} H_{c_k} = \sum_{i=1}^{k} H_{x_i}^T R_i^{-1} H_{c_i} \tag{6.7.3}$$

that is, $\overline{P}_0$, H_{x_i}, and H_{c_i} all have been mapped to the appropriate time t_k (e.g., $\overline{P}_k = \Phi(t_k, t_0)\overline{P}_0, \Phi^T(t_k, t_0))$, and H_{x_i} and H_{c_i} are defined by Eq. (6.5.23) and (6.5.24), respectively,

$$H_{x_i} = \widetilde{H}_{x_i}\Phi(t_i, t_k).$$

The sensitivity matrix S_k is given by

$$S_k = \frac{\partial \hat{\mathbf{x}}_k}{\partial \overline{\mathbf{c}}} = -P_k H_{x_k}^T R_k^{-1} H_{c_k}\,. \tag{6.7.4}$$

Hence (see also Eq. (6.4.5)),

$$\hat{\mathbf{x}}_{c_k} = \hat{\mathbf{x}}_k + S_k\overline{\mathbf{c}}\,. \tag{6.7.5}$$

The *a priori* value of $\hat{\mathbf{x}}_{c_k}$ is given by

$$\overline{\mathbf{x}}_{c_k} = \Phi(t_k, t_{k-1})\hat{\mathbf{x}}_{c_{k-1}} + \theta(t_k, t_{k-1})\overline{\mathbf{c}}\,. \tag{6.7.6}$$

Substituting Eq. (6.7.5) at the $k-1^{\text{st}}$ stage into Eq. (6.7.6) yields

$$\overline{\mathbf{x}}_{c_k} = \Phi(t_k, t_{k-1})\hat{\mathbf{x}}_{k-1} + \big(\Phi(t_k, t_{k-1})S_{k-1} + \theta(t_k, t_{k-1})\big)\overline{\mathbf{c}}\,. \tag{6.7.7}$$

Define

$$\overline{S}_k = \Phi(t_k, t_{k-1})S_{k-1} + \theta(t_k, t_{k-1})\,, \tag{6.7.8}$$

then

$$\overline{\mathbf{x}}_{c_k} = \overline{\mathbf{x}}_k + \overline{S}_k\overline{\mathbf{c}}\,. \tag{6.7.9}$$

Recall that $\overline{\mathbf{x}}_k$ and $\hat{\mathbf{x}}_k$ are the values of these quantities assuming there are no errors in the consider parameters (i.e., $\mathbf{c} = \overline{\mathbf{c}} = 0$) and $\overline{P}_{cc}$ is the null matrix.

The deviation of $\overline{\mathbf{x}}_{ck}$ in Eq. (6.7.9) from the true value of $\mathbf{x}_c$ may be written in terms of the error in each component

$$\delta\overline{\mathbf{x}}_{c_k} = \delta\overline{\mathbf{x}}_k + \overline{S}_k\delta\overline{\mathbf{c}} \tag{6.7.10}$$

Multiplying Eq. (6.7.10) by its transpose and taking the expected value yields

$$E\left[\delta\overline{\mathbf{x}}_{c_k}\delta\overline{\mathbf{x}}_{c_k}^T\right] = \overline{P}_{c_k} = E\left[\delta\overline{\mathbf{x}}_k\delta\overline{\mathbf{x}}_k^T\right] + \overline{S}_k E\left[\boldsymbol{\beta}\boldsymbol{\beta}^T\right]\overline{S}_k^{\,T} \tag{6.7.11}$$

where $\delta\overline{\mathbf{c}} = \overline{\mathbf{c}} - \mathbf{c} = \boldsymbol{\beta}$, and all cross covariances are zero because $\delta\overline{\mathbf{x}}_k$ is by definition independent of the consider parameters. Hence,

$$\overline{P}_{c_k} = \overline{P}_k + \overline{S}_k\overline{P}_{cc}\overline{S}_k^T\,. \tag{6.7.12}$$

Note that this equation is equivalent to Eq. (6.6.11) since both these equations simply map the consider covariance matrix forward in time.

The measurement update for $\overline{P}_{c_k}$ is defined by

$$P_{c_k} = E\big[(\hat{\mathbf{x}}_{c_k} - \mathbf{x}_k)(\hat{\mathbf{x}}_{c_k} - \mathbf{x}_k)^T\big]\,. \tag{6.7.13}$$

Using Eq. (6.7.5), we may write the error in $\hat{\mathbf{x}}_{c_k}$ as

$$\delta\hat{\mathbf{x}}_{c_k} = \delta\hat{\mathbf{x}}_k + S_k\delta\overline{\mathbf{c}}\,. \tag{6.7.14}$$

Hence Eq. (6.7.13) may be written as

$$P_{c_k} = E\left[\delta\hat{\mathbf{x}}_{c_k}\delta\hat{\mathbf{x}}_{c_k}^T\right] = E\left[\delta\hat{\mathbf{x}}_k\delta\hat{\mathbf{x}}_k^T\right] + S_k E\left[\boldsymbol{\beta}\,\boldsymbol{\beta}^T\right]S_k^T \tag{6.7.15}$$

or

$$P_{c_k} = P_k + S_k\overline{P}_{cc}S_k^T\,. \tag{6.7.16}$$

The cross covariances $\overline{P}_{xc_k}$ and P_{xc_k} are obtained from Eqs. (6.7.10) and (6.7.14), respectively,

$$\begin{aligned}\overline{P}_{xc_k} &= E\left[\delta\overline{\mathbf{x}}_{c_k}\boldsymbol{\beta}^T\right] \\ &= \overline{S}_k\overline{P}_{cc} \end{aligned} \quad (6.7.17)$$

$$\begin{aligned}P_{xc_k} &= E\left[\delta\hat{\mathbf{x}}_{c_k}\boldsymbol{\beta}^T\right] \\ &= S_k\overline{P}_{cc}\,. \end{aligned} \quad (6.7.18)$$

To obtain an expression for the measurement update of $\overline{S}_k$, note that an analogy with the expression for $\hat{\mathbf{x}}_k$ of Eq. (4.7.16), $\hat{\mathbf{x}}_{c_k}$, may be written as

$$\hat{\mathbf{x}}_{c_k} = \overline{\mathbf{x}}_{c_k} + K_k\left(\mathbf{y}_k - \widetilde{H}_{x_k}\overline{\mathbf{x}}_{c_k} - \widetilde{H}_{c_k}\overline{\mathbf{c}}\right). \quad (6.7.19)$$

Using Eq. (6.7.9) this may be written as

$$\hat{\mathbf{x}}_{c_k} = \overline{\mathbf{x}}_k + \overline{S}_k\overline{\mathbf{c}} + K_k\left[\mathbf{y}_k - \widetilde{H}_{x_k}\left(\overline{\mathbf{x}}_k + \overline{S}_k\overline{\mathbf{c}}\right) - \widetilde{H}_{c_k}\overline{\mathbf{c}}\right] \quad (6.7.20)$$

and recognizing that $\hat{\mathbf{x}}_k = \bar{\mathbf{x}}_k + K_k(\mathbf{y}_k - \widetilde{H}_{x_k}\bar{\mathbf{x}}_k)$, Eq. (6.7.20) becomes

$$\hat{\mathbf{x}}_{c_k} = \hat{\mathbf{x}}_k + (\overline{S}_k - K_k\widetilde{H}_{x_k}\overline{S}_k - K_k\widetilde{H}_{c_k})\overline{\mathbf{c}}\,. \quad (6.7.21)$$

Comparing this with Eq. 6.7.5), it is seen that

$$S_k = (I - K_k\widetilde{H}_{x_k})\overline{S}_k - K_k\widetilde{H}_{c_k}\,. \quad (6.7.22)$$

We now have the equations needed to write the computational algorithm for the sequential consider covariance filter.

Given: $P_{k-1}, \hat{\mathbf{x}}_{k-1}, S_{k-1}, \widetilde{H}_{x_k}, \widetilde{H}_{c_k}, \mathbf{y}_k$

1. Compute the time updates

$$\begin{aligned}
\overline{P}_k &= \Phi\left(t_k, t_{k-1}\right)P_{k-1}\Phi^T\left(t_k, t_{k-1}\right) && (6.7.23)\\
K_k &= \overline{P}_k\widetilde{H}_{x_k}^T(\widetilde{H}_{x_k}\overline{P}_k\widetilde{H}_{x_k}^T + R_k)^{-1} && (6.7.24)\\
\overline{S}_k &= \Phi(t_k, t_{k-1})S_{k-1} + \theta\left(t_k, t_{k-1}\right) && (6.7.25)\\
\overline{\mathbf{x}}_k &= \Phi(t_k, t_{k-1})\hat{\mathbf{x}}_{k-1} && (6.7.26)\\
\overline{\mathbf{x}}_{c_k} &= \overline{\mathbf{x}}_k + \overline{S}_k\overline{\mathbf{c}} && (6.7.27)\\
\overline{P}_{c_k} &= \overline{P}_k + \overline{S}_k\overline{P}_{cc}\overline{S}_k^{\,T} && (6.7.28)\\
\overline{P}_{xc_k} &= \Phi\left(t_k, t_{k-1}\right)P_{xc_{k-1}} + \theta(t_k, t_{k-1})\overline{P}_{cc} \\
&= \overline{S}_k\overline{P}_{cc}. && (6.7.29)
\end{aligned}$$

2. Compute the measurement update

$$
\begin{aligned}
P_k &= (I - K_k \widetilde{H}_{x_k})\overline{P}_k && (6.7.30)\\
S_k &= (I - K_k \widetilde{H}_{x_k})\overline{S}_k - K_k \widetilde{H}_{c_k} && (6.7.31)\\
\hat{\mathbf{x}}_k &= \overline{\mathbf{x}}_k + K_k(\mathbf{y}_k - \widetilde{H}_{x_k}\overline{\mathbf{x}}_k) && (6.7.32)\\
\hat{\mathbf{x}}_{c_k} &= \hat{\mathbf{x}}_k + S_k \overline{\mathbf{c}} && (6.7.33)\\
P_{c_k} &= P_k + S_k \overline{P}_{cc} S_k^T && (6.7.34)\\
P_{xc_k} &= S_k \overline{P}_{cc}\,. && (6.7.35)
\end{aligned}
$$

Note that $\bar{\mathbf{x}}_{c_0} = \bar{\mathbf{x}}_0$ (from Eq. 6.7.9); the consider parameters do not affect the *a priori* value of $\bar{\mathbf{x}}_{c_0}$, hence $\bar{S}_0 = 0$ and

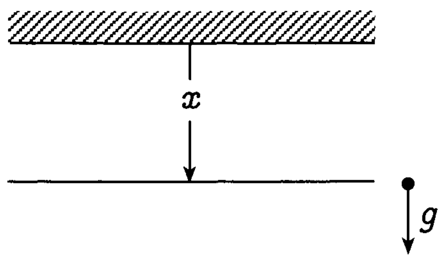
$$S_0 = -K_0 \widetilde{H}_{c_0}\,. \tag{6.7.36}$$

Normally in a covariance analysis one would be interested only in computing the consider covariances; hence, the estimates for $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}_c$ need not be computed. Generally process noise is not included in a consider covariance analysis. However, if process noise has been included, Eq. (6.7.23) is replaced by

$$\overline{P}_k = \Phi(t_k, t_{k-1})P_{k-1}\Phi^T(t_k, t_{k-1}) + \Gamma(t_k, t_{k-1})Q_{k-1}\Gamma^T(t_k, t_{k-1})\,. \tag{6.7.37}$$

6.8 EXAMPLE: FREELY FALLING POINT MASS

A point mass is in free fall only under the influence of gravity. Observations of range, x, are made from a fixed referent point. Set up the consider analysis equations assuming that x is estimated and g is a consider parameter, and that $\overline{P}_0 = I$, $R = I$, $\overline{P}_{cc} = \Pi$, and $\overline{P}_{xc} = 0$.



Equation of motion: $\ddot{x} = g$

State vector: $\mathbf{X} = \begin{bmatrix} x \\ v \end{bmatrix}$

System dynamics: $\dot{\mathbf{X}} = F(\mathbf{X}, t) = \begin{bmatrix} v \\ g \end{bmatrix}$

$$A = \frac{\partial F}{\partial \mathbf{X}} = \begin{bmatrix} \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial v} \\[2ex] \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The consider parameter, C, is g. Hence,

$$B = \frac{\partial F}{\partial g} = \begin{bmatrix} \dfrac{\partial v}{\partial g} \\[2ex] \dfrac{\partial g}{\partial g} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The observation equation is

$$\begin{aligned} Y(\mathbf{X}, t) &= G(\mathbf{X}, t) + \epsilon \\ &= x + \epsilon\,. \end{aligned}$$

Because, $G(\mathbf{X}, t) = x$

$$\begin{aligned} \tilde{H}_x &= \frac{\partial G}{\partial \mathbf{X}} = \begin{bmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial v} \end{bmatrix} = [1 \;\; 0] \\ \tilde{H}_c &= \frac{\partial G}{\partial g} = 0\,. \end{aligned}$$

The differential equation for the state transition matrix is

$$\dot{\Phi} = A\Phi, \quad \Phi(t_o, t_0) = I$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The solution for $\Phi(t, t_0)$ is

$$\Phi(t, t_0) = Ke^{A(t-t_0)}$$

where K is a constant matrix. Since $\Phi(t_0, t_0) = I$, we have $Ke^{A0} = I$ or $K = I$. Therefore,

$$\Phi(t, t_0) = e^{A(t-t_0)}$$

$$= \quad I + A(t - t_0) + \frac{1}{2!}A^2(t - t_0)^2 + \ldots$$

Now

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, A^2 and higher powers of A are all zero. Hence,

$$\begin{aligned}
\Phi(t, t_0) &= I + A(t - t_0) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(t - t_0) \\
&= \begin{bmatrix} 1 & (t - t_0) \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

The equation for the consider parameter mapping matrix is

$$\dot{\theta} = A\theta + B, \quad \theta(t_0, t_0) = 0$$

or

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \theta_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned}
\dot{\theta}_2 &= 1 \\
\theta_2(t, t_0) &= t + k_1. \quad At \;\; t = t_0, \quad \theta_2 = 0 \\
\theta_2(t, t_0) &= t - t_0\,.
\end{aligned}$$

Also $\dot{\theta}_1 = \theta_2$; therefore,

$$\theta_1(t, t_0) = \frac{(t - t_0)^2}{2} + k_2\,.$$

At $t = t_0$, $\theta_1 = 0$; hence,

$$\begin{aligned}
\theta_1 &= \frac{(t - t_0)^2}{2} \\
\theta(t, t_0) &= \begin{bmatrix} (t - t_0)^2/2 \\ (t - t_0) \end{bmatrix}.
\end{aligned}$$

The transformation of observation-state partials to epoch, t_0, is obtained by using Eqs. (6.5.23) and (6.5.24):

$$\begin{aligned}
H_x(t) &= \tilde{H}_x(t)\Phi(t,t_0) \\
&= [1\ \ 0]\begin{bmatrix} 1 & (t-t_0) \\ 0 & 1 \end{bmatrix} \\
&= [1\ \ (t-t_0)] \\
H_c(t) &= \tilde{H}_x(t)\theta(t,t_0) + \tilde{H}_c(t) \\
&= [1\ \ 0]\begin{bmatrix} (t-t_0)^2/2 \\ (t-t_0) \end{bmatrix} + 0 \\
&= (t-t_0)^2/2\,.
\end{aligned}$$

The (2×2) normal matrix of the state and the 2×1 normal matrix of the consider parameter at time t is given by

$$\begin{aligned}
(H_x^T H_x)_t &= \begin{bmatrix} 1 \\ (t-t_0) \end{bmatrix}[1\ (t-t_0)] = \begin{bmatrix} 1 & (t-t_0) \\ (t-t_0) & (t-t_0)^2 \end{bmatrix} \\
(H_x^T H_c)_t &= \begin{bmatrix} 1 \\ (t-t_0) \end{bmatrix}\frac{(t-t_0)^2}{2} = \begin{bmatrix} (t-t_0)^2/2 \\ (t-t_0)^3/2 \end{bmatrix}.
\end{aligned}$$

Given that $\overline{P}_0 = I$, $R = I$ and that three measurements are taken at $t = 0, 1,$ and 2, and let $t_0 = 0$. Then the accumulation of the normal matrices yields

$$H_x^T H_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

and

$$H_x^T H_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 9/2 \end{bmatrix}.$$

The computed covariance at epoch, t_0, is given by

$$P_0 = \left(H_x^T H_x + \overline{P}_0^{-1}\right)^{-1} = \begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 4/15 \end{bmatrix}.$$

Assuming the variance of g to be Π, the consider covariance P_{c_o} at the epoch is given by Eq. (6.5.32) with $k = 0$,

$$P_{c_0} = P_0 + S_0 \Pi S_0^T.$$

The sensitivity matrix, S_0, is given by Eq. (6.4.4), with $R = I$,

$$\begin{aligned} S_0 &= -P_0(H_x^T H_c) \\ &= -\begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 4/15 \end{bmatrix} \begin{bmatrix} 5/2 \\ 9/2 \end{bmatrix} = -\begin{bmatrix} 1/10 \\ 7/10 \end{bmatrix}. \end{aligned}$$

Therefore,

$$S_0 \Pi S_0^T = \frac{\Pi}{100} \begin{bmatrix} 1 & 7 \\ 7 & 49 \end{bmatrix}$$

and the value of P_{c_0} is given by

$$P_{c_0} = \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 4/15 \end{bmatrix} + \frac{\Pi}{100} \begin{bmatrix} 1 & 7 \\ 7 & 49 \end{bmatrix}.$$

Also,

$$P_{xc_0} = S_0 \Pi = -\Pi \begin{bmatrix} 1/10 \\ 7/10 \end{bmatrix}.$$

Finally from Eq. (6.6.16),

$$\begin{aligned} \mathbf{P}_{c_0} &= \begin{bmatrix} P_{c_0} & P_{xc_0} \\ P_{cx_0} & \overline{P}_{cc} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{40+\Pi}{100} & \dfrac{-20+7\Pi}{100} & \dfrac{-\Pi}{10} \\ & \dfrac{80+147\Pi}{300} & \dfrac{-7\Pi}{10} \\ & & \Pi \end{bmatrix}. \end{aligned}$$

Thus, the consider standard deviations of the position and velocity,

$$\sqrt{\frac{(40+\Pi)}{100}}$$

and

$$\sqrt{\frac{(80 + 147\Pi)}{300}}$$

are greater than the respective data noise standard deviations; namely, $\sqrt{2/5}$ and $\sqrt{4/15}$.

6.8.1 Propagation with Time

The computed covariance is mapped from $t = 0$ to $t = 2$ by Eq. (6.6.10),

$$\begin{aligned} P_2 &= \Phi(t_2, t_0) P_0 \Phi^T(t_2, t_0) \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 4/15 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 4/15 \end{bmatrix}. \end{aligned}$$

The contribution to the propagated covariance from the consider parameter uncertainty is given by the last term of Eq. (6.6.11),

$$S_2 \Pi S_2^T$$

where, from Eq. (6.6.9)

$$\begin{aligned} S_2 &= \Phi(t_2, t_0) S_0 + \theta(t_2, t_0) \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/10 \\ -7/10 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 \\ 13/10 \end{bmatrix}. \end{aligned}$$

Therefore,

$$S_2 \Pi S_2^T = \begin{bmatrix} 1/2 \\ 13/10 \end{bmatrix} \Pi \begin{bmatrix} 1/2 & 13/10 \end{bmatrix} = \Pi \begin{bmatrix} 1/4 & 13/20 \\ 13/20 & 169/100 \end{bmatrix}.$$

Thus, the consider covariance propagated to time $t = 2$ is

$$P_{c_2} = P_2 + S_2 \Pi S_2^T = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 4/15 \end{bmatrix} + \Pi \begin{bmatrix} 1/4 & 13/20 \\ 13/20 & 169/100 \end{bmatrix}.$$

Also,

$$P_{xc_2} = S_2\Pi = \frac{\Pi}{2}\begin{bmatrix} 1 \\ 13/5 \end{bmatrix}.$$

Hence,

$$\mathbf{P}_{c_2} = \begin{bmatrix} \frac{8+3\Pi}{12} & \frac{20+39\Pi}{60} & \frac{\Pi}{2} \\ & \frac{80+507\Pi}{300} & \frac{13\Pi}{10} \\ & & \Pi \end{bmatrix}$$

where again the consider standard deviation is greater than the computed or data noise standard deviation. Recall that in this development we have let $\sigma_g^2 = \Pi$. Also note that $P_{c_2} - P_2$ is positive definite as was the case at t_0.

6.8.2 The Sequential Consider Algorithm

As a continuation of this example, compute $\mathbf{P}_{c_2}$ using the sequential consider filter algorithm. We will compute P_{c_0}, P_{c_1}, and P_{c_2} based on observations at $t = 0$; $t = 0, 1$; $t = 0, 1$, and 2, respectively.

Beginning with $R = I$, $\overline{P}_0 = I$, $\overline{P}_{cc} = \Pi$, $P_{xc_0} = 0$, $\widetilde{H}_{x_0} = [1 \;\; 0]$, $\widetilde{H}_{c_0} = 0$, compute

$$\begin{aligned} K_0 &= \overline{P}_0\widetilde{H}_{x_0}^T(\widetilde{H}_{x_0}\overline{P}_0\widetilde{H}_{x_0}^T + R)^{-1} \\ K_0 &= \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \\ S_0 &= -K_0\widetilde{H}_{c_0} \\ &= 0 \end{aligned}$$

$$\begin{aligned} P_{c_0} &= P_0 + S_0\Pi S_0^T \\ &= P_0 \end{aligned}$$

where

$$\begin{aligned} P_0 &= (I - K_0\widetilde{H}_{x_0})\overline{P}_0 \\ &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Hence, as expected, an error in g cannot affect the state at t_0 and the data noise and consider covariance for the state are identical. Therefore,

$$\mathbf{P}_{c_0} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Pi \end{bmatrix}.$$

Now compute $\mathbf{P}_{c_1}$. The time update for P_0 is given by Eq. (6.7.23)

$$\begin{aligned}
\overline{P}_1 &= \Phi(t_1, t_0) P_0 \Phi^T(t_1, t_0) \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3/2 & 1 \\ 1 & 1 \end{bmatrix}.
\end{aligned}$$

Also, from Eq. (6.7.24)

$$\begin{aligned}
K_1 &= \overline{P}_1 \widetilde{H}_{x_1}^T (\widetilde{H}_{x_1} \overline{P}_1 \widetilde{H}_{x_1}^T + R)^{-1} \\
&= \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}.
\end{aligned}$$

The measurement update for $\overline{P}_1$ is given by Eq. (6.7.30)

$$\begin{aligned}
P_1 &= (I - K_1 \widetilde{H}_{x_1}) \overline{P}_1 \\
&= \left[I - \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix} [1 \;\; 0] \right] \begin{bmatrix} 3/2 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix}.
\end{aligned}$$

From Eq. (6.7.25), the time update for S_0 is

$$\bar{S}_1 = \Phi(t_1, t_0) S_0 + \theta(t_1, t_0)$$

and the measurement update is given by Eq. (6.7.31)

$$S_1 = (I - K_1 \widetilde{H}_{x_1}) \bar{S}_1 - K_1 \widetilde{H}_{c_1}.$$

However, $S_0 = 0$, so

$$\bar{S}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

and

$$S_1 = \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix}.$$

From Eq. (6.7.34)

$$\begin{aligned} P_{c_1} &= P_1 + S_1 \Pi S_1^T \\ &= \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix} + \Pi \begin{bmatrix} 1/25 & 4/25 \\ 4/25 & 16/25 \end{bmatrix} \end{aligned}$$

and from Eq. (6.7.35)

$$P_{xc_1} = S_1 \Pi = \begin{bmatrix} \dfrac{\Pi}{5} \\ \dfrac{4\Pi}{5} \end{bmatrix}.$$

Substituting these results into Eq. (6.6.16) yields the desired result

$$\mathbf{P}_{c_1} = \begin{bmatrix} \dfrac{15+\Pi}{25} & \dfrac{10+4\Pi}{25} & \dfrac{\Pi}{5} \\ & \dfrac{15+16\Pi}{25} & \dfrac{4\Pi}{5} \\ & & \Pi \end{bmatrix}.$$

Finally, it can be shown that

$$\begin{aligned} P_2 &= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 4/15 \end{bmatrix} \\ \bar{S}_2 &= \begin{bmatrix} 3/2 \\ 9/5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
S_2 &= \begin{bmatrix} 1/2 \\ 13/10 \end{bmatrix} \\
P_{c_2} &= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 4/15 \end{bmatrix} + \Pi \begin{bmatrix} 1/4 & 13/20 \\ 13/20 & 169/100 \end{bmatrix} \\
P_{xc_2} &= \Pi \begin{bmatrix} 1/2 \\ 13/10 \end{bmatrix}
\end{aligned}$$

and

$$\mathbf{P}_{c_2} = \begin{bmatrix} \dfrac{8+3\Pi}{12} & \dfrac{20+39\Pi}{60} & \dfrac{\Pi}{2} \\ & \dfrac{80+507\Pi}{300} & \dfrac{13\Pi}{10} \\ & & \Pi \end{bmatrix}$$

which is in agreement with results from the batch processor.

6.8.3 Perturbation in the State Estimate

Since the sensitivity matrix is by definition

$$S_0 = \frac{\partial \hat{\mathbf{X}}_0}{\partial g}.$$

It follows that

$$\Delta \hat{\mathbf{X}}_0 = \frac{\partial \hat{\mathbf{X}}_0}{\partial g} \sqrt{\Pi}.$$

In general, if σ_g is the 1σ uncertainty in the value of g, then

$$\Delta \hat{\mathbf{X}}_0 = \begin{bmatrix} \Delta \hat{x}_0 \\ \Delta \hat{\dot{x}}_0 \end{bmatrix} = S_0 \sigma_g = -\begin{bmatrix} 1/10 \\ 7/10 \end{bmatrix} \sigma_g$$

or

$$\begin{aligned}
\Delta \hat{x}_0 &= -\sigma_g/10 \\
\Delta \hat{\dot{x}}_0 &= -7\sigma_g/10
\end{aligned}$$

which illustrates how the estimate of the epoch state for the batch processor will be in error as a function of the uncertainty in the consider parameter g after processing the three observations of this example.

6.9 EXAMPLE: SPRING-MASS PROBLEM

A block of mass m is attached to two parallel vertical walls by two springs as shown in Fig. 4.8.2. k_1 and k_2 are the spring constants. h is the height of the position P on one of the walls, from which the distance, ρ, and the rate of change of distance of the block from P, $\dot{\rho}$ can be observed.

Let the horizontal distances be measured with respect to the point O where the line OP, the lengths of the springs, and the center of mass of the block are all assumed to be in the same vertical plane. Then, if $\overline{x}$ denotes the position of the block at static equilibrium, the equation of motion of the block is given by

$$\ddot{x} = -(k_1 + k_2)(x - \overline{x})/m\,. \tag{6.9.1}$$

Let

$$\omega^2 = (k_1 + k_2)/m, \text{ and } \bar{x} = 0$$

so that Eq. (6.9.1) can be written as

$$\ddot{x} + \omega^2 x = 0\,. \tag{6.9.2}$$

Consider the problem of estimating the position and the velocity of the block with respect to the reference point O, by using the range and range-rate measurements of the block from the point, P. To formulate this problem mathematically, the estimation state vector is taken as $\mathbf{X}^T = [x \quad v]$. Then the system dynamics are represented by

$$\dot{\mathbf{X}} = F(\mathbf{X}, t) = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -\omega^2 x \end{bmatrix}, \tag{6.9.3}$$

or in state space form

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}. \tag{6.9.4}$$

The observation vector is

$$G(\mathbf{X}, t) = \begin{bmatrix} \rho \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} \sqrt{x^2 + h^2} \\ x\dot{x}/\rho \end{bmatrix}. \tag{6.9.5}$$

Suppose that some parameters are not known exactly and that the effect of their errors on the state estimates are to be evaluated. For example, let m, k_2, and h be the set of "consider parameters." Then the consider vector $\mathbf{C}$ will be

$$\mathbf{C}^T = \left[m \vdots k_2 \vdots h \right]$$

and $\mathbf{c} = \mathbf{C} - \mathbf{C}^*$. The linearized state and observation equations, including the consider parameters, are given by

$$\delta\dot{\mathbf{X}} = A(t)\delta\mathbf{X} + B(t)\mathbf{c} \tag{6.9.6}$$

$$\mathbf{y} = \tilde{H}_x\delta\mathbf{X} + \tilde{H}_c\mathbf{c} \tag{6.9.7}$$

where

$$\delta\mathbf{X} = \begin{bmatrix} \delta x \\ \delta v \end{bmatrix}$$

$$A(t) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{\omega^2 x}{m} & \dfrac{-x}{m} & 0 \end{bmatrix} \tag{6.9.8}$$

and

$$\tilde{H}_x = \begin{bmatrix} \dfrac{x}{\rho} & 0 \\ \dfrac{\dot{x}}{\rho} - \dfrac{x^2\dot{x}}{\rho^3} & \dfrac{x}{\rho} \end{bmatrix}, \quad \tilde{H}_c = \begin{bmatrix} 0 & 0 & \dfrac{h}{\rho} \\ 0 & 0 & -\dfrac{x\dot{x}h}{\rho^3} \end{bmatrix}.$$

The solution to Eq. (6.9.4) is given by

$$\begin{aligned} x(t) &= x_o \cos\omega t + \tfrac{v_o}{\omega} \sin\omega t \\ v(t) &= v_o \cos\omega t - x_o\omega \sin\omega t. \end{aligned} \tag{6.9.9}$$

Note that the original differential equation for the state Eq. (6.9.2) is linear; hence the filter solution (i.e., $c = 0$) for

$$\delta\mathbf{X}(t) = \begin{bmatrix} \delta\mathbf{x}(t) \\ \delta v(t) \end{bmatrix}$$

is also given by Eq. (6.9.9) when x_0 and v_0 are replaced by δx_0 and δv_0, respectively.

The differential equations for the filter state transition matrix are

$$\dot{\Phi}(t,0) = A(t)\Phi(t,0), \quad \Phi(0,0) = I$$

and the consider parameter transition matrix equations are

$$\dot{\theta}(t,0) = A(t)\theta(t,0) + B(t), \quad \theta(0,0) = 0\,.$$

Solutions to these are given by

$$\Phi(t,0) = \begin{bmatrix} \cos\omega t & \frac{1}{\omega}\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{bmatrix} \tag{6.9.10}$$

and

$$\begin{aligned}
\theta_{11}(t,0) &= \frac{\beta_2}{\omega}\sin\omega t + t[-\beta_2\cos\omega t + \beta_1\omega\sin\omega t] \\
\theta_{12}(t,0) &= -\theta_{11}(t,0)\omega^2 \\
\theta_{13}(t,0) &= 0 \\
\theta_{21}(t,0) &= \beta_1\omega\sin\omega t + t[\beta_2\omega\sin\omega t + \beta_1\omega^2\cos\omega t] \\
\theta_{22}(t,0) &= -\theta_{21}/\omega^2 \\
\theta_{23}(t,0) &= 0
\end{aligned} \tag{6.9.11}$$

where

$$\beta_1 = \frac{x_0}{2m} \quad \text{and} \quad \beta_2 = \frac{\dot{x}_0}{2m}\,. \tag{6.9.12}$$

Eleven perfect observations of ρ and $\dot{\rho}$ were simulated over a period of 10 seconds at one-second intervals and are given in Chapter 4 (Table 4.8.1), assuming the following values for the system parameters and the initial condition:

$$\begin{aligned}
k_1 &= 2.5\ \text{N/m} \\
k_2 &= 3.7\ \text{N/m} \\
m &= 1.5\ \text{kg} \\
h &= 5.4\ \text{m} \\
x_0 &= 3.0\ \text{m} \\
\dot{x}_0 &= 0.0\ \text{m/s}\,.
\end{aligned} \tag{6.9.13}$$

The *a priori* values used are:

$$\bar{\mathbf{X}}_0 = \begin{bmatrix} 4.0 \\ 0.2 \end{bmatrix} \quad \text{and} \quad \overline{P}_o = \begin{bmatrix} 100.0 & 0 \\ 0 & 10.0 \end{bmatrix}. \tag{6.9.14}$$

When perfect values of the consider parameters, denoted by the vector $\mathbf{C}$, are used, the batch processor "recovers" the true estimate of the state, as shown in the example described in Section 4.8.2. However, when the consider parameters are in error, not only will the estimates of the state parameters deviate from their true values, but these errors will propagate with time. In the following cases the consider covariance is propagated with time under the influence of the indicated covariances on the consider parameters.

Twenty Percent Error in Consider Parameters

Assume that the values of m, k_2, and h actually used in the batch processor are

$$\mathbf{C}^* = \begin{bmatrix} m \\ k_2 \\ h \end{bmatrix} = \begin{bmatrix} 1.8 \\ 3.0 \\ 6.4 \end{bmatrix}; \text{ hence, } \mathbf{c} = \begin{bmatrix} -.3 \\ .7 \\ -1 \end{bmatrix}. \tag{6.9.15}$$

$\mathbf{C}^*$ is approximately 20 percent in error relative to the true values given in Eq. (6.9.13). The corresponding consider covariance is assumed to be:

$$\overline{P}_{cc} = \begin{bmatrix} 0.09 & 0 & 0 \\ 0 & 0.49 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6.9.16}$$

The solution for the epoch state using the observations of Table 4.8.1 and the values of the consider parameters given by Eq. (6.9.15) and *a priori* values given by Eq. (6.9.14) is

$$\begin{aligned} \hat{x}_0 &= 2.309 \pm 0.623 \text{ m} \\ \hat{\dot{x}}_0 &= -0.758 \pm 0.712 \text{ m/s}. \end{aligned}$$

The error covariance matrices are propagated with time using Eqs. (6.6.10) and (6.6.11) with $\overline{P}_{cc}$ given by Eq. (6.9.16), and the results are shown in Fig. 6.9.1. The upper panel shows the error in the position estimate, and the positive values of the computed standard deviation and the consider standard deviation. The time span includes the measurement period (0–10 sec) and a prediction interval (10–20 sec). The center panel shows the previous three quantities for the

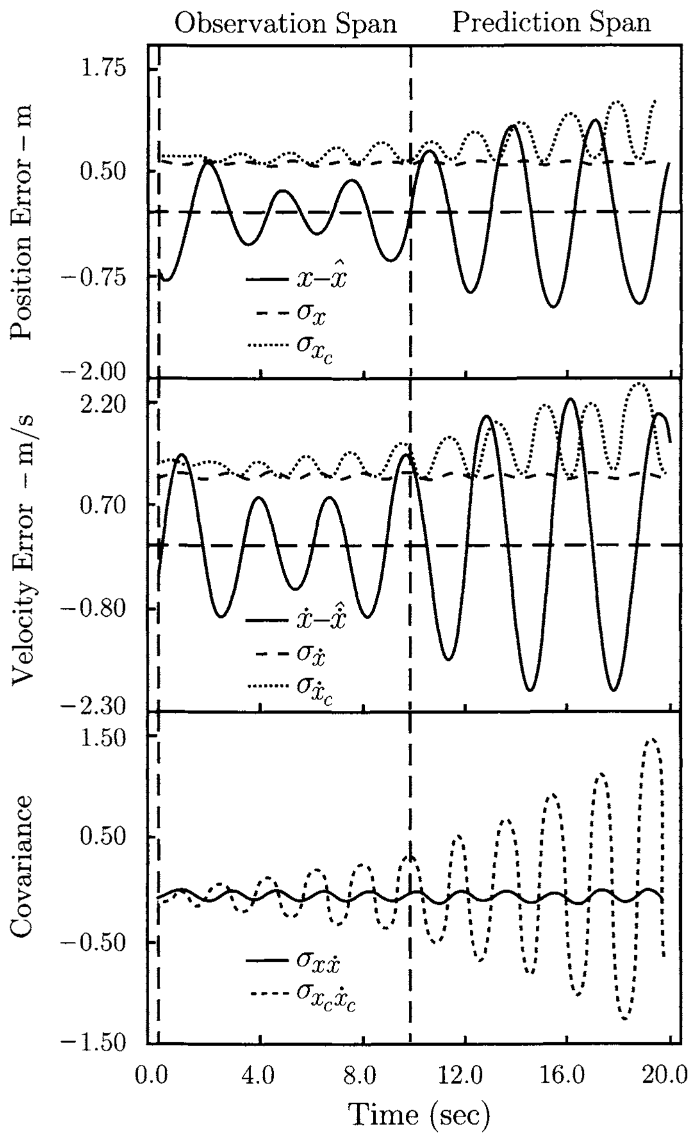


Figure 6.9.1: Consider analysis: Spring mass problem, 20% error in consider parameters.

velocity component. The computed and the consider covariance between the position and velocity estimates are shown in the lower panel. Note that the consider sigmas are always greater than the computed sigmas. The phase difference between the actual error and consider error is caused by the errors in m and k_2, which result in an error in frequency and phase. It is also interesting to note that the state errors are bounded above by the computed sigmas in the measurement interval but not in the prediction interval.

Five Percent Error in Consider Parameter

In this case, the only change is to $\mathbf{C}^*$ and $\overline{P}_{cc}$, with

$$\mathbf{C}^* = \begin{bmatrix} 1.6 \\ 3.5 \\ 5.7 \end{bmatrix}.$$

$\mathbf{C}^*$ is approximately 5 percent in error. The consider covariance is assumed to be:

$$\overline{P}_{cc} = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.09 \end{bmatrix}.$$

The batch solution for the epoch state is:

$$\begin{aligned} \hat{x}_0 &= 2.873 \pm 0.505 \\ \hat{\dot{x}}_0 &= -0.499 \pm 0.513. \end{aligned}$$

The propagation errors and the standard deviations corresponding to this case are shown in Fig. 6.9.2. The results are similar to the previous case, except that the magnitude of error is proportionally smaller.

6.10 ERRORS IN THE OBSERVATION NOISE AND *A PRIORI* STATE COVARIANCES

The effects of errors in the assumed values of the data noise and *a priori* state covariance matrices for the batch processor can be evaluated as follows. Assume that the values of these matrices used in the batch processor differ from the true values (denoted by $(\ )^*$) by the matrix δ,

$$\begin{aligned} R^* &= R + \delta R & (6.10.1) \\ \overline{P}_0^* &= \overline{P}_0 + \delta \overline{P}_0 . & (6.10.2) \end{aligned}$$

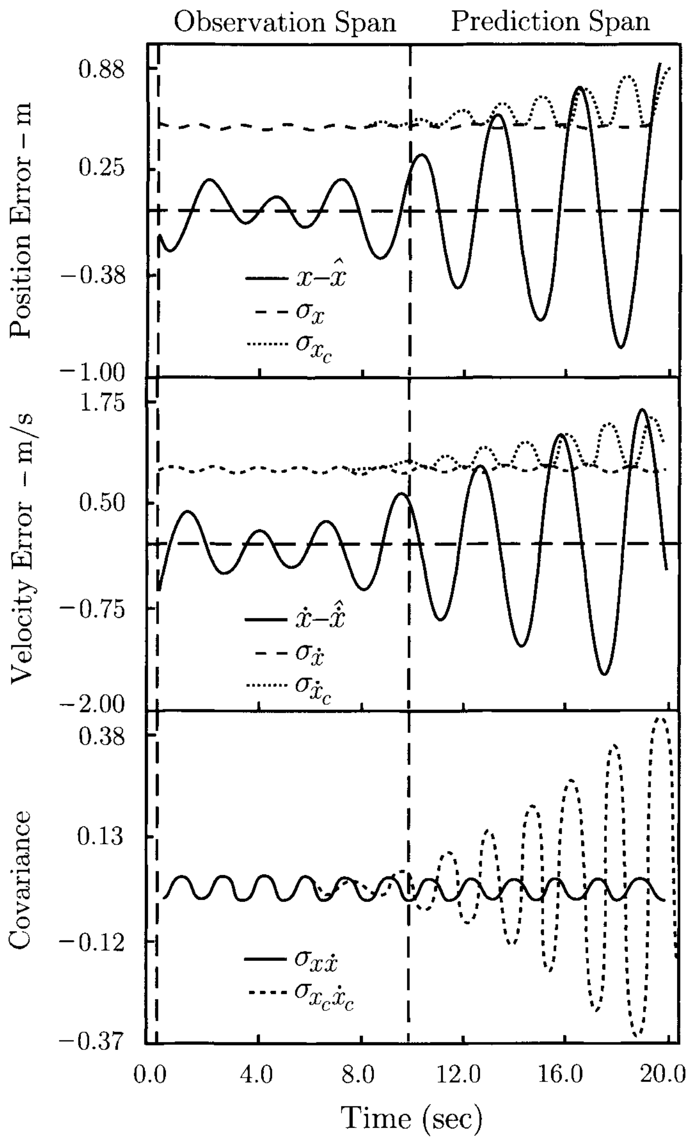


Figure 6.9.2: Consider analysis: spring mass problem, 5% error in consider parameters.

From Eq. (4.6.1),

$$\hat{\mathbf{x}} = \left(H^T R^{-1} H + \overline{P}_0^{-1}\right)^{-1}\left(H^T R^{-1}\mathbf{y} + \overline{P}_0^{-1}\overline{\mathbf{x}}\right) \tag{6.10.3}$$

where

$$\begin{aligned} \overline{\mathbf{x}} &= \mathbf{x} + \boldsymbol{\eta} & (6.10.4)\\ \mathbf{y} &= H\mathbf{x} + \boldsymbol{\epsilon} & (6.10.5) \end{aligned}$$

and

$$\begin{aligned} E[\boldsymbol{\eta}\,\boldsymbol{\eta}^T] &= \overline{P}_0^* \\ E[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T] &= R^* . \end{aligned} \tag{6.10.6}$$

Evaluating the estimation error covariance using the assumed values of R and $\overline{P}_0$ results in

$$\hat{\mathbf{x}} = \left[H^T R^{-1} H + \overline{P}_0^{-1}\right]^{-1}\left[H^T R^{-1}(H_x + \boldsymbol{\epsilon}) + \overline{P}_0^{-1}(\mathbf{x} + \boldsymbol{\eta})\right] \tag{6.10.7}$$

or

$$\hat{\mathbf{x}} - \mathbf{x} = \left[H^T R^{-1} H + \overline{P}_0^{-1}\right]^{-1}\left[H^T R^{-1}\boldsymbol{\epsilon} + \overline{P}_0^{-1}\boldsymbol{\eta}\right]. \tag{6.10.8}$$

Assuming $E[\boldsymbol{\epsilon}\boldsymbol{\eta}^T] = 0$, we have

$$\begin{aligned} P_c &= E\left[(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T\right] \\ &= \left[H^T R^{-1} H + \overline{P}_0^{-1}\right]^{-1}\left[H^T R^{-1} E[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T] R^{-1} H\right] \\ &\quad + \left[\overline{P}_0^{-1} E[\boldsymbol{\eta}\,\boldsymbol{\eta}^T]\overline{P}_0^{-1}\right]\left[H^T R^{-1} H + \overline{P}_0^{-1}\right]^{-1} \\ &= P\left[H^T R^{-1} R^* R^{-1} H + \overline{P}_0^{-1}\overline{P}_0^*\overline{P}_0^{-1}\right]P . \end{aligned} \tag{6.10.9}$$

Substituting Eqs. (6.10.1) and (6.10.2) for R^* and $\overline{P}_0^*$ yields

$$P_c = P + P\left[H^T R^{-1}\delta R R^{-1} H + \overline{P}_0^{-1}\delta\overline{P}_0\overline{P}_0^{-1}\right]P . \tag{6.10.10}$$

The second term in Eq. (6.10.10) is the contribution to the consider covariance from errors in the data noise and *a priori* state covariance matrices.

6.11 ERRORS IN PROCESS NOISE, OBSERVATION NOISE, AND STATE COVARIANCE

The effects of errors in the process noise, observation noise, and *a priori* state covariance matrices can be evaluated using a sequential or Kalman filter algorithm. Following Heffes (1996), assume that the values of these quantities used in the filter are given by Q, R, and $\overline{P}_0$. Also,

$$\begin{aligned} Q^* &= Q + \delta Q \\ R^* &= R + \delta R \\ \overline{P}_0^* &= \overline{P}_0 + \delta \overline{P}_0 \end{aligned} \tag{6.11.1}$$

where $(\quad)^*$ represents the optimal or true value, and δ denotes the error in the assumed value.

We wish to find the covariance associated with the actual state error. The actual error in the estimated states at time t_k is given by

$$\delta \mathbf{x}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k \tag{6.11.2}$$

where

$$\hat{\mathbf{x}}_k = \overline{\mathbf{x}}_k + K_k(\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k) \tag{6.11.3}$$

$$K_k = \overline{P}_k \widetilde{H}_k^T (\widetilde{H}_k \overline{P}_k \widetilde{H}^T + R)^{-1} \tag{6.11.4}$$

and

$$\mathbf{y}_k = \widetilde{H}_k \mathbf{x}_k + \epsilon_k \,. \tag{6.11.5}$$

Here $\hat{\mathbf{x}}_k$ is the state estimate obtained using Q, R, and P_0 in Eq. (6.11.3), and $\mathbf{x}_k$ is the true value of the state at t_k.

Substituting Eqs. (6.11.3) and (6.11.5) into Eq. (6.11.2) yields

$$\begin{aligned} \delta \mathbf{x}_k &= \overline{\mathbf{x}}_k + K_k(\widetilde{H}_k \mathbf{x}_k + \epsilon_k - \widetilde{H}_k \overline{\mathbf{x}}_k) - \mathbf{x}_k \\ &= (I - K_k \widetilde{H}_k)(\overline{\mathbf{x}}_k - \mathbf{x}_k) + K_k \epsilon_k \,. \end{aligned} \tag{6.11.6}$$

The covariance of the actual estimation error is

$$\begin{aligned} P_k &\equiv E\big[\delta \mathbf{x}_k \delta \mathbf{x}_k^T\big] \\ &= \big(I - K_k \widetilde{H}_k\big) \overline{P}_k \big(I - K_k \widetilde{H}_k\big)^T + K_k R^* K_k^T \end{aligned} \tag{6.11.7}$$

where

$$\overline{P}_k \equiv E\big[\delta \overline{\mathbf{x}}_k \delta \overline{\mathbf{x}}_k^T\big] = E\big[(\overline{\mathbf{x}}_k - \mathbf{x}_k)(\overline{\mathbf{x}}_k - \mathbf{x}_k)^T\big] \,. \tag{6.11.8}$$

The equation relating $\overline{P}_k$ to P_{k-1} is found by using the expressions that relate the state at time t_{k-1} to the state at t_k,

$$\overline{\mathbf{x}}_k = \Phi(t_k, t_{k-1}) \hat{\mathbf{x}}_{k-1} \tag{6.11.9}$$

$$\mathbf{x}_k = \Phi(t_k, t_{k-1}) \mathbf{x}_{k-1} + \Gamma(t_k, t_{k-1}) \mathbf{u}_{k-1} \,. \tag{6.11.10}$$

Thus,

$$\begin{aligned}
\delta\overline{\mathbf{x}}_k &= \overline{\mathbf{x}}_k - \mathbf{x}_k \\
&= \Phi(t_k, t_{k-1})\hat{\mathbf{x}}_{k-1} - \Phi(t_k, t_{k-1})\mathbf{x}_{k-1} - \Gamma(t_k, t_{k-1})\mathbf{u}_{k-1} \\
&= \Phi(t_k, t_{k-1})(\hat{\mathbf{x}}_{k-1} - \mathbf{x}_{k-1}) - \Gamma(t_k, t_{k-1})\mathbf{u}_{k-1}\,. \qquad (6.11.11)
\end{aligned}$$

Substituting this into Eq. (6.11.8) yields

$$\begin{aligned}
\overline{P}_k &= \Phi(t_k, t_{k-1})P_{k-1}\Phi^T\,(t_k,\, t_{k-1}) \\
&+ \Gamma(t_k, t_{k-1})Q^*_{k-1}\Gamma^T(t_k, t_{k-1}) \qquad (6.11.12)
\end{aligned}$$

where $E[(\hat{\mathbf{x}}_{k-1} - \mathbf{x}_{k-1})\mathbf{u}^T_{k-1}\Gamma^T(t_k, t_{k-1})] = 0$ because $\mathbf{x}_{k-1}$ is dependent on $\mathbf{u}_{k-2}$ but not $\mathbf{u}_{k-1}$.

Hence, the covariance of the actual error given by Eqs. (6.11.12) and (6.11.7) is

$$\begin{aligned}
\overline{P}_{k_a} &= \Phi(t_k, t_{k-1})P_{k-1_a}\Phi^T(t_k, t_{k-1}) \qquad (6.11.13) \\
&\quad +\Gamma(t_k, t_{k-1})Q^*_{k-1}\Gamma^T(t_k, t_{k-1}) \\
P_{k_a} &= (I - K_k\widetilde{H}_k)\overline{P}_{k_a}(I - K_k\widetilde{H}_k)^T + K_kR^*K_k^T\,. \qquad (6.11.14)
\end{aligned}$$

The recursion is initiated with $\overline{P}_{0_a} = \overline{P}^*_0$.

The gain matrix, K_k, at each stage is computed by using the suboptimal $\overline{P}$ and R, and is initiated with $\overline{P}_0$: K_k is given by

$$K_k = \overline{P}_k\widetilde{H}_k^T(\widetilde{H}_k\overline{P}_k\widetilde{H}_k^T + R)^{-1}\,. \qquad (6.11.15)$$

The suboptimal values of $\overline{P}_k$ and P_k are given by

$$\begin{aligned}
\overline{P}_k &= \Phi(t_k, t_{k-1})P_{k-1}\Phi^T(t_k, t_{k-1}) \qquad (6.11.16) \\
&\quad +\Gamma(t_k, t_{k-1})Q_k\Gamma^T(t_k, t_{k-1}) \\
P_k &= (I - K_k\widetilde{H}_k)\overline{P}_k\,. \qquad (6.11.17)
\end{aligned}$$

Hence, there will be three covariance matrices one can compute using Eqs. (6.11.13) and (6.11.14):

1. The optimal value of P based on $\overline{P}^*_0$, Q^*, R^*.

2. The actual value of P determined by using $\overline{P}^*_0$, Q^*, R^* and $\overline{P}_0$, Q, and R.

3. The suboptimal value of P determined from $\overline{P}_0$, Q, and R. Note that the suboptimal value of $\overline{P}_k$ must be computed in order to determine the actual value of P_k.

6.12 COVARIANCE ANALYSIS AND ORTHOGONAL TRANSFORMATIONS

The consider covariance formulation can also be developed in square-root form by using orthogonal transformations. Consider the dynamic and measurement models given by Eqs. (6.5.8) and (6.5.18):

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t, t_k)\mathbf{x}_k + \theta(t, t_k)\mathbf{c}_k & (6.12.1)\\ \mathbf{y}_j &= \widetilde{H}_{x_j}\mathbf{x}_j + \widetilde{H}_{c_j}\mathbf{c} + \boldsymbol{\epsilon}_j, \quad j = 1, \ldots, \ell. & (6.12.2) \end{aligned}$$

By substituting Eq. (6.12.1) into Eq. (6.12.2) (see Eq. (6.5.22)), this set can be reduced to

$$\mathbf{y}_j = H_{x_j}\mathbf{x}_k + H_{c_j}\mathbf{c} + \boldsymbol{\epsilon}_j, \quad j = 1, \ldots, \ell \tag{6.12.3}$$

where

$$\begin{aligned} H_{x_j} &= \widetilde{H}_{x_j}\Phi(t_j, t_k)\\ H_{c_j} &= \widetilde{H}_{x_j}\theta(t_j, t_k) + \widetilde{H}_{c_j}, \quad j = 1, \ldots, \ell. \end{aligned}$$

Then the set of ℓ observations can be combined as

$$\mathbf{y} = H_x\mathbf{x}_k + H_c\mathbf{c} + \boldsymbol{\epsilon}. \tag{6.12.4}$$

where $\mathbf{y}$, H_x, H_c, and $\boldsymbol{\epsilon}$ are defined in Eq. (6.5.25).

We next need to write the *a priori* information on the n-vector, $\mathbf{x}$, and the q-vector, $\mathbf{c}$, in terms of a data equation. Given *a priori* values $\bar{\mathbf{x}}$ and $\bar{\mathbf{c}}$, we may write

$$\begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{c}} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{c} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\beta} \end{bmatrix} \tag{6.12.5}$$

where $\mathbf{x}$ and $\mathbf{c}$ are the true values. $\boldsymbol{\eta}$ and $\boldsymbol{\beta}$ have known mean and covariance,

$$\begin{aligned} E[\boldsymbol{\eta}] &= E[\boldsymbol{\beta}] = 0\\ \overline{P} &= E\left[\begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\beta} \end{bmatrix}\begin{bmatrix}\boldsymbol{\eta}^T \boldsymbol{\beta}^T\end{bmatrix}\right] = E\begin{bmatrix} \boldsymbol{\eta}\boldsymbol{\eta}^T & \boldsymbol{\eta}\boldsymbol{\beta}^T \\ \boldsymbol{\beta}^T\boldsymbol{\eta} & \boldsymbol{\beta}\boldsymbol{\beta}^T \end{bmatrix}\\ &= \begin{bmatrix} \overline{P}_x & \overline{P}_{xc} \\ \overline{P}_{cx} & \overline{P}_{cc} \end{bmatrix}. \end{aligned} \tag{6.12.6}$$

We may now determine the upper triangular square root of the inverse of $\overline{P}$ with a Cholesky decomposition

$$\text{Chol}\begin{bmatrix} \overline{P}_x & \overline{P}_{xc} \\ \overline{P}_{cx} & \overline{P}_{cc} \end{bmatrix}^{-1} \equiv \begin{bmatrix} \overline{R}_x & \overline{R}_{xc} \\ 0 & \overline{R}_c \end{bmatrix}. \tag{6.12.7}$$

Multiplying Eq. (6.12.5) by Eq. (6.12.7) in order to write it as a data equation yields

$$\begin{aligned} \begin{bmatrix} \overline{R}_x & \overline{R}_{xc} \\ 0 & \overline{R}_c \end{bmatrix} \begin{bmatrix} \overline{\mathbf{x}} \\ \overline{\mathbf{c}} \end{bmatrix} &= \begin{bmatrix} \overline{R}_x & \overline{R}_{xc} \\ 0 & \overline{R}_c \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{c} \end{bmatrix} \\ &+ \begin{bmatrix} \overline{R}_x & \overline{R}_{xc} \\ 0 & \overline{R}_c \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\beta} \end{bmatrix}. \end{aligned} \tag{6.12.8}$$

Because we do not intend to estimate $\mathbf{c}$, we need to deal with only the first row of Eq. (6.12.8). Define

$$\begin{aligned} \overline{\mathbf{b}}_c &\equiv \overline{R}_c\overline{\mathbf{c}} \\ \overline{\mathbf{b}} &\equiv \overline{R}_x\overline{\mathbf{x}} + \overline{R}_{xc}\overline{\mathbf{c}} \\ \overline{\boldsymbol{\eta}} &\equiv \overline{R}_x\boldsymbol{\eta} + \overline{R}_{xc}\boldsymbol{\beta}. \end{aligned} \tag{6.12.9}$$

We will not use the first of Eq. (6.12.9) unless we estimate $\mathbf{c}$.

The first row of Eq. (6.12.8) becomes

$$\overline{\mathbf{b}} = [\overline{R}_x \;\; \overline{R}_{xc}] \begin{bmatrix} \mathbf{x} \\ \mathbf{c} \end{bmatrix} + \overline{\boldsymbol{\eta}}. \tag{6.12.10}$$

Using Eqs. (6.12.10) and (6.12.4) we may write the array to be upper triangularized,

$$\begin{array}{c} \overbrace{\quad}^{n} \; \overbrace{\quad}^{q} \; \overbrace{\quad}^{1} \\ \begin{bmatrix} \overline{R}_x & \overline{R}_{xc} & \overline{\mathbf{b}} \\ H_x & H_c & \mathbf{y} \end{bmatrix} \begin{array}{l} \}n \\ \}\ell\,. \end{array} \end{array} \tag{6.12.11}$$

We need only partially triangularize Eq. (6.12.11) by nulling the terms below the diagonal of the first n columns (n is the dimension of $\mathbf{x}$). Hence

$$T \begin{array}{c} \begin{array}{ccc} \overbrace{\hphantom{\overline{R}_x}}^{n} & \overbrace{\hphantom{\overline{R}_{xc}}}^{q} & \overbrace{\hphantom{\overline{\mathbf{b}}}}^{1} \end{array} \\ \left[\begin{array}{ccc} \overline{R}_x & \overline{R}_{xc} & \overline{\mathbf{b}} \\ H_x & H_c & \mathbf{y} \end{array} \right] \begin{array}{l} \}n \\ \}\ell \end{array} \end{array} = \begin{array}{c} \begin{array}{ccc} \overbrace{\hphantom{\hat{R}_x}}^{n} & \overbrace{\hphantom{\hat{R}_{xc}}}^{q} & \overbrace{\hphantom{\hat{\mathbf{b}}}}^{1} \end{array} \\ \left[\begin{array}{ccc} \hat{R}_x & \hat{R}_{xc} & \hat{\mathbf{b}} \\ 0 & \widetilde{R}_c & \widetilde{\mathbf{b}}_c \end{array} \right] \begin{array}{l} \}n \\ \}q\,. \end{array} \end{array} \tag{6.12.12}$$

There are $\ell - q$ additional rows on the right-hand side of Eq. (6.12.12), which are not needed and are not shown here. Also, the terms $\widetilde{R}_c$ and $\widetilde{\mathbf{b}}_c$ are not needed to compute the consider covariance and are needed only if $\mathbf{c}$ is to be estimated. For consider covariance analysis $\overline{\mathbf{c}}$ is used wherever an estimate of $\mathbf{c}$ is needed. The first row of Eq. (6.12.12) yields

$$\hat{R}_x \mathbf{x} + \hat{R}_{xc}\overline{\mathbf{c}} = \hat{\mathbf{b}}\,. \tag{6.12.13}$$

In keeping with the notation of Section 6.3 and Eq. (6.3.70) we will refer to the estimate of $\mathbf{x}$ obtained from Eq. (6.12.13) as $\hat{\mathbf{x}}_c$; accordingly,

$$\hat{\mathbf{x}}_c = \hat{R}_x^{-1}\hat{\mathbf{b}} - \hat{R}_x^{-1}\hat{R}_{xc}\overline{\mathbf{c}}. \tag{6.12.14}$$

The estimate of $\mathbf{x}$ obtained by ignoring $\overline{\mathbf{c}}$ is called the computed estimate of $\mathbf{x}$ and is given by

$$\hat{\mathbf{x}} = \hat{R}_x^{-1}\hat{\mathbf{b}}. \tag{6.12.15}$$

Hence,

$$\hat{\mathbf{x}}_c = \hat{\mathbf{x}} - \hat{R}_x^{-1}\hat{R}_{xc}\overline{\mathbf{c}}. \tag{6.12.16}$$

Comparing Eq. (6.12.16) with Eq. (6.4.5) or by recalling that the sensitivity matrix is given by

$$S = \frac{\partial \hat{\mathbf{x}}_c}{\partial \overline{\mathbf{c}}},$$

we have

$$S = -\hat{R}_x^{-1}\hat{R}_{xc} \tag{6.12.17}$$

and Eq. (6.12.16) may be written as

$$\hat{\mathbf{x}}_c = \hat{\mathbf{x}} + S\overline{\mathbf{c}}. \tag{6.12.18}$$

From Eq. (6.12.15) the data noise covariance matrix is given by

$$P = (\hat{R}_x^T\hat{R}_x)^{-1} = \hat{R}_x^{-1}\hat{R}_x^{-T} \tag{6.12.19}$$

and the consider covariance for $\mathbf{x}$ is (see Eq. (6.3.64))

$$P_c = P + S\overline{P}_{cc}S^T . \tag{6.12.20}$$

The cross covariance is given by Eq. (6.3.65)

$$P_{xc} = S\overline{P}_{cc} \tag{6.12.21}$$

and the complete consider covariance matrix is given by Eq. (6.3.67)

$$\mathbf{P}_c = \begin{bmatrix} P_c & P_{xc} \\ P_{cx} & \overline{P}_{cc} \end{bmatrix} . \tag{6.12.22}$$

We could also determine the consider covariance directly by replacing $\widetilde{R}_c$ with $\overline{R}_c$ in Eq. (6.12.12). Then,

$$\begin{aligned}
\mathbf{P}_c &= \begin{bmatrix} \hat{R}_x & \hat{R}_{xc} \\ 0 & \overline{R}_c \end{bmatrix}^{-1} \begin{bmatrix} \hat{R}_x^T & 0 \\ \hat{R}_{xc}^T & \overline{R}_c^T \end{bmatrix}^{-1} \qquad\qquad (6.12.23) \\
&= \begin{bmatrix} \hat{R}_x^{-1} & -\hat{R}_x^{-1}\hat{R}_{xc}\overline{R}_c^{-1} \\ 0 & \overline{R}_c^{-1} \end{bmatrix} \begin{bmatrix} \hat{R}_x^{-T} & 0 \\ -\hat{R}_c^{-T}\hat{R}_{xc}^T\hat{R}_{xc}^{-T} & \overline{R}_c^{-T} \end{bmatrix} \\
&= \begin{bmatrix} \hat{R}_x^{-1}\hat{R}_x^{-T} + S\overline{R}_c^{-1}\overline{R}_c^{-T}S^T & S\overline{R}_c^{-1}\overline{R}_c^{-T} \\ \overline{R}_c^{-1}\overline{R}_c^{-T}S^T & \overline{R}_c^{-1}\overline{R}_c^{-T} \end{bmatrix} \\
&= \begin{bmatrix} P_c & P_{xc} \\ P_{cx} & \overline{P}_{cc} \end{bmatrix} .
\end{aligned}$$

In component form

$$P_c = \hat{R}_x^{-1}\hat{R}_x^{-T} + S\overline{P}_{cc}S^T \tag{6.12.24}$$

$$P_{xc} = S\overline{P}_{cc} \tag{6.12.25}$$

$$\overline{P}_{cc} = \overline{R}_c^{-1}\overline{R}_c^{-T} . \tag{6.12.26}$$

If we wish to estimate $\mathbf{c}$ we would combine $\widetilde{R}_c$ and $\widetilde{\mathbf{b}}_\mathbf{c}$ from Eq. (6.12.12) with the *a priori* information on $\mathbf{c}$ given by the first of Eq. (6.12.9). We then perform a series of orthogonal transformations on

$$T\begin{bmatrix} \overline{R}_c & \overline{\mathbf{b}}_\mathbf{c} \\ \hat{R}_c & \widetilde{\mathbf{b}}_\mathbf{c} \end{bmatrix} = \begin{bmatrix} \hat{R}_c & \hat{\mathbf{b}}_\mathbf{c} \\ 0 & \mathbf{e} \end{bmatrix} \tag{6.12.27}$$

and

$$\hat{\mathbf{c}} = \hat{R}_c^{-1}\hat{\mathbf{b}}_c \tag{6.12.28}$$

the associated estimation error covariance for $\hat{\mathbf{c}}$ is

$$P_{cc} = \hat{R}_c^{-1}\hat{R}_c^{-T}. \tag{6.12.29}$$

We could now replace $\overline{\mathbf{c}}$ with $\hat{\mathbf{c}}$ in Eq. (6.12.16) to obtain the proper estimate for $\mathbf{x}$ and replace $\overline{R}_c$ with $\hat{R}_c$ in the first of Eq. (6.12.23) to obtain the corresponding consider covariance matrices. This will result in $\overline{P}_{cc}$ being replaced by P_{cc} in Eqs. (6.12.24) and (6.12.25).

The computational algorithm for consider covariance analysis using orthogonal transformations is as follows.

Given *a priori* information $\overline{P}_x$, $\overline{P}_{xc}$, and $\overline{P}_{cc}$ at an epoch time, t_0, and observations $\mathbf{y}_j = \widetilde{H}_{x_j}\mathbf{x}_j + \widetilde{H}_{c_j}\mathbf{c} + \epsilon_j \quad j = 1, \ldots, \ell$.

1. Use a Cholesky decompostion to compute the upper triangular matrix of Eq. (6.12.7).

2. Using the procedure indicated in Eqs. (6.12.8) through (6.12.10), form the matrix indicated by Eq. (6.12.11). The observations may be processed one at a time, in small batches, or simultaneously (see Chapter 5).

3. Partially upper triangularize the matrix of Eq. (6.12.11) by nulling the terms below the diagonal of the first n columns to yield the results of Eq. (6.12.12).

4. After processing all observations replace $\widetilde{R}_c$ in Eq. (6.12.12) with the *a priori* value $\overline{R}_c$ and evaluate Eq. (6.12.23) to obtain $\mathbf{P}_c$.

Note that if only a consider covariance is needed, and no estimation is to be done, the last column of Eq. (6.12.12) containing $\overline{\mathbf{b}}$ and $\mathbf{y}$ may be eliminated since the *a priori* state and the actual values of the measurements do not affect the estimation error covariance.

More extensive treatment of the square-root formulations of the covariance analysis problem including the effects of system model errors, errors in the *a priori* statistics, and errors in correlated process noise can be found in Curkendall (1972), Bierman (1977), Thornton (1976), and Jazwinski (1970).

6.13 REFERENCES

Bierman, G. J., *Factorization Methods for Discrete Sequential Estimation*, Academic Press, New York, 1977.

Curkendall, D. W., *Problems in estimation theory with applications to orbit determination*, UCLA-ENG-7275, UCLA School Engr. and Appl. Sci., Los Angeles, CA, September 1972.

Heffes, H., "The effects of erroneous models on the Kalman filter response", *Trans. Auto. Cont.*, Vol. AC-11, pp. 541–543, July 1966.

Jazwinski, A. H., *Stochastic Processes and Filtering Theory*, Academic Press, New York, 1970.

Thornton, C. L., *Triangular covariance factorizations for Kalman filtering*, Technical Memorandum, 33–798, Jet Propulsion Laboratory, Pasadena, CA, October 15, 1976.

6.14 EXERCISES

1. Map the consider covariance, $\mathbf{P}_{c_0}$, for the batch algorithm from t_0 to t_1 for Example 6.8. Now map $\mathbf{P}_{c_2}$, the result at t_2 from the sequential algorithm to t_1 and show that it agrees with the batch result.

   Answer:

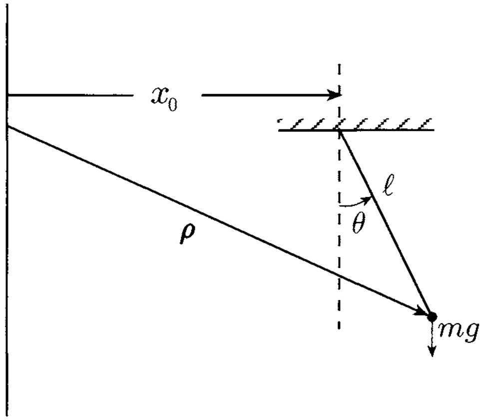
$$\mathbf{P}_{c_1} = \begin{bmatrix} \dfrac{80+27\Pi}{300} & \dfrac{20-27\Pi}{300} & -\dfrac{3\Pi}{10} \\ \dfrac{20-27\Pi}{300} & \dfrac{80+27\Pi}{300} & \dfrac{3\Pi}{10} \\ -\dfrac{3\Pi}{10} & \dfrac{3\Pi}{10} & \Pi \end{bmatrix}$$

2. Compute the consider covariance for Example 6.8 at $t = 0$ using the square root formulation of Section 6.12. Compare your results to those given in Section 6.8.

3. Beginning with Eq. (6.3.38), derive the alternate equation for P_{xx} given by Eq. (6.3.41).

4. Beginning with Eqs. (6.6.3) and (6.6.4), derive Eqs. (6.6.6) and (6.6.11).

5. Given the pendulum problem shown here. Assume that θ_0 and $\dot{\theta}_0$ are to be estimated, that the observation is range magnitude, ρ, and x_0 and g are to be considered.



Assume: $\sigma(x_0) = \sigma(g) = \sigma(\rho) = 1$, and small oscillations so that the equation of motion is linear.

(a) Define: $\Phi(t, t_0)$, $\theta(t, t_0)$, H_x, H_c.

(b) Assuming $\sqrt{g/l} = 10$ and observations are taken at $t = 0, 1, 2$, compute the data noise covariance, the sensitivity, perturbation, and consider covariance matrices. Discuss the interpretation of these matrices. Assume $x_0 = \ell = 1$, $\overline{P} = R = I$, $\theta_0 = 0.01$ rad, and $\dot{\theta}_0 = 0$.

Answer:

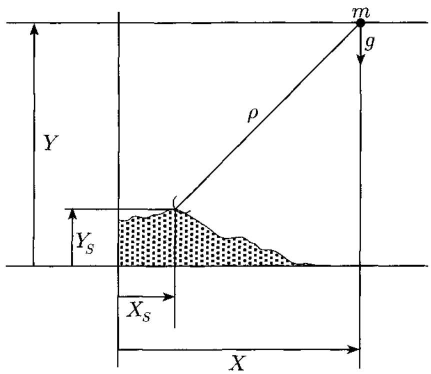
$$\mathbf{P}_{c_0} = \begin{bmatrix} 0.539 & -0.0196 & -0.147 & 0.000154 \\ -0.0196 & 0.995 & -0.0122 & 0.000042 \\ -0.147 & -0.0122 & 1.0 & 0.0 \\ 0.000154 & 0.000042 & 0.0 & 1.0 \end{bmatrix}$$

6. Examine the problem of estimating the position and velocity of a spacecraft in low Earth orbit from ground-based range measurements. The dynamic model is chosen as a simple "flat Earth." Assume range measurements are taken from a single station. Derive the expression for the consider covariance matrix for the estimated and consider parameters shown.

Estimate:
$X, Y, \dot{X}, \dot{Y}$

Consider:
g, X_s, Y_s

Observe: ρ



7. A vehicle travels in a straight line according to the following equation:

$$u(t) = u_0 + et + ft^2$$

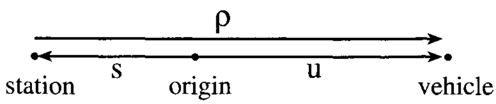
where

u = distance from origin

t = time

e, f = constants.

A tracking station location a distance s from the origin takes range measurements ρ.



The true range at time t is

$$\rho(t) = u(t) + s\,.$$

The observed range, $y(t)$, has some error $\epsilon(t)$:

$$\begin{aligned} y(t) &= \rho(t) + \epsilon(t) \\ &= u(t) + s + \epsilon(t)\,. \end{aligned}$$

Assume that we wish to estimate $u(t_0) = u_0$ and e, and consider s and f. The following observations with measurement noise of zero mean and unit variance are given.

Assume $\sigma_\epsilon = \sigma_s = \sigma_f = 1$, $\overline{P} = I$ and $\bar{\mathbf{c}} = \begin{bmatrix} s \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

t	$y(t)$
0	1.5
1	10.8
2	24.8

Compute $\hat{u}_0$, $\hat{e}$, the computed and consider covariance matrices, and the sensitivity and perturbation matrix.

Answer:

$$\begin{bmatrix} \hat{u}_o \\ \hat{e} \end{bmatrix} = \begin{bmatrix} 2.76 \\ 8.69 \end{bmatrix},$$

$$\mathbf{P}_{c_0} = \begin{bmatrix} 0.80 & 0.20 & -0.60 & -0.20 \\ 0.20 & 2.27 & -0.20 & -1.40 \\ -0.60 & -0.20 & 1.0 & 0.0 \\ -0.20 & -1.40 & 0.0 & 1.0 \end{bmatrix}$$

8. Work the spring-mass problem of Example 6.9. Assume k_1 and k_2 are known exactly; estimate x_0, and v_0 and consider a 20 percent error in h.

9. Work Exercise 8 using the orthogonal transformation procedure of Section 6.12.