Appendix E

Analytical Theory for Near-Circular Orbits

E.1 DESCRIPTION

An analytical theory for the variation of the orbit elements under the influence of the zonal harmonics J_2 through J_5 has been developed by Brouwer (1959). This appendix presents a simplification of his equations for the time variation of the orbit elements for a near-circular orbit under the influence of J_2. The classical elements e and ω used by Brouwer have been replaced by $h = e\sin\omega$ and $k = e\cos\omega$ since the argument of perigee is not well defined for a near-circular orbit. The analytical solutions are given by (Born *et al.*, 2001):

$$
\begin{aligned}
a(t) &= \bar{a} + k_1 \cos 2\bar{\beta} \\
h(t) &= \bar{h}(t) + k_2 \sin\bar{\beta} + k_3 \sin 3\bar{\beta} \\
k(t) &= \bar{k}(t) + k_4 \cos\bar{\beta} + k_3 \cos 3\bar{\beta} \\
i(t) &= \bar{\imath} + k_5 \cos 2\bar{\beta} \\
\Omega(t) &= \overline{\Omega} + k_6 \sin 2\bar{\beta} \\
\beta(t) &= \bar{\beta} + k_7 \sin 2\bar{\beta} + k_8 (t - t_0)^2 \\
\bar{h}(t) &= \bar{e}\sin\bar{\omega} \\
\bar{k}(t) &= \bar{e}\cos\bar{\omega} \\
\overline{\Omega} &= \overline{\Omega}(t_0) + k_9 (t - t_0) \\
\bar{\omega} &= \bar{\omega}(t_0) + k_{10}(t - t_0) \\
\bar{\beta} &= \bar{\beta}(t_0) + k_{11}(t - t_0)
\end{aligned}
\tag{E.1.1}
$$

where $\beta = \omega + M$ and $(\bar{\ })$ represents the mean value of the element. Mean values are given by

$$
\begin{aligned}
\bar{a} &= a(t_0) - K_1 \cos 2\beta(t_0) \\
\bar{h}(t_0) &= \bar{e} \sin \bar{\omega}(t_0) = h(t_0) - K_2 \sin \beta(t_0) - K_3 \sin 3\beta(t_0) \\
\bar{k}(t_0) &= \bar{e} \cos \bar{\omega}(t_0) = k(t_0) - K_4 \cos \beta(t_0) - K_3 \cos 3\beta(t_0) \\
\bar{e} &= \sqrt{\bar{h}^2(t_0) + \bar{k}^2(t_0)} \\
\bar{\omega}(t_0) &= \operatorname{atan2}\left(\bar{h}(t_0), \bar{k}(t_0)\right) \\
\bar{\imath} &= i(t_0) - K_5 \cos 2\beta(t_0) \\
\overline{\Omega}(t_0) &= \Omega(t_0) - K_6 \sin 2\beta(t_0) \\
\bar{\beta}(t_0) &= \beta(t_0) - K_7 \sin 2\beta(t_0)\,.
\end{aligned}
\tag{E.1.2}
$$

Also,

$$
\begin{aligned}
K_1 &= 3\bar{a}\gamma_2\delta^2 \\
K_2 &= \frac{\gamma_2}{2}(6 - \frac{21}{2}\delta^2) \\
K_3 &= \frac{7}{4}\gamma_2\delta^2 \\
K_4 &= \frac{\gamma_2}{2}(6 - \frac{15}{2}\delta^2) \\
K_5 &= \frac{3}{2}\gamma_2\theta\delta \\
K_6 &= \frac{3}{2}\gamma_2\theta \\
K_7 &= \frac{3}{4}\gamma_2(3 - 5\theta^2) \\
K_8 &= \frac{3}{4}\frac{\rho v_{\mathrm{rel}}^2}{\bar{a}} C_D \frac{A}{m} \\
K_9 &= -3\bar{n}\theta\gamma_2 \\
K_{10} &= \frac{3}{2}\gamma_2\bar{n}(4 - 5\delta^2) \\
K_{11} &= \bar{n}\left(1 + 3\gamma_2(3 - 4\delta^2)\right)
\end{aligned}
\tag{E.1.3}
$$

and

$$
\begin{aligned}
\gamma_2 &= \frac{J_2}{2}\left(\frac{R}{\bar{a}}\right)^2 \\
\theta &= \cos \bar{\imath} \\
\delta &= \sin \bar{\imath}
\end{aligned}
\tag{E.1.4}
$$

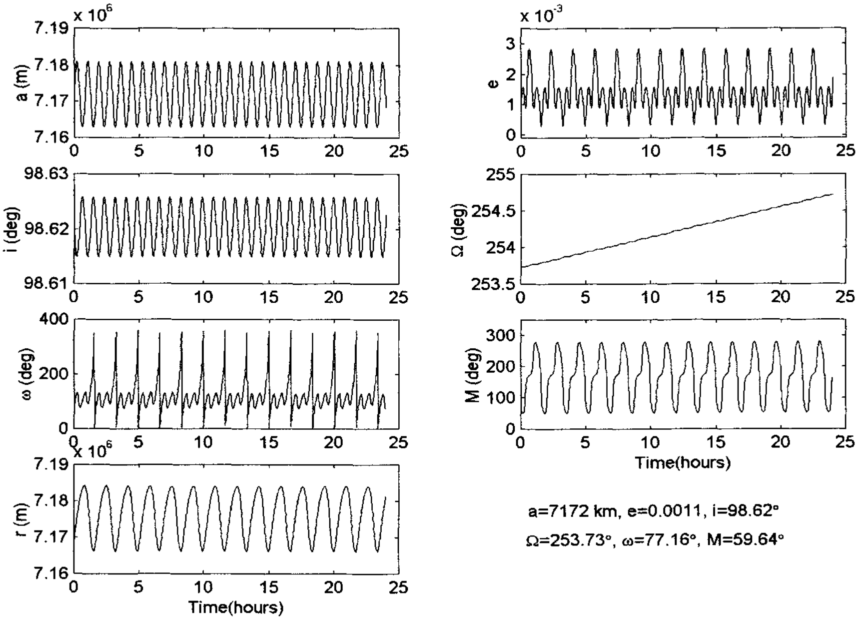

Figure E.1.1: Analytical solution results for the QUIKSCAT orbit using Eq. (E.1.1).

$$\bar{n} = \frac{\mu^{1/2}}{\bar{a}^{3/2}}, \qquad \mu = GM$$
$$\rho = \text{average atmospheric density at satellite altitude}$$
$$v_{\rm rel} = \text{average velocity of the satellite relative to the atmosphere.}$$

This representation is based on the solution developed by Brouwer (1959), except for the last term in the $\beta(t)$ expression, $k_8(t - t_0)^2$. This quadratic term is introduced to compensate for drag.

Note that it is not necessary to iterate to determine mean values because this is a first-order theory in J_2. It is necessary to use $\bar{a}$ to compute $\bar{n}$ in order to avoid an error of $O(J_2)$ in computing K_{11}.

Other useful equations are

$$r(t) = \frac{a(t)\left[1 - \left(h(t)^2 + k^2(t)\right)\right]}{1 + k(t)\cos\beta(t) + h(t)\sin\beta(t)} \tag{E.1.5}$$

$$e(t) = \sqrt{h^2(t) + k^2(t)} \tag{E.1.6}$$

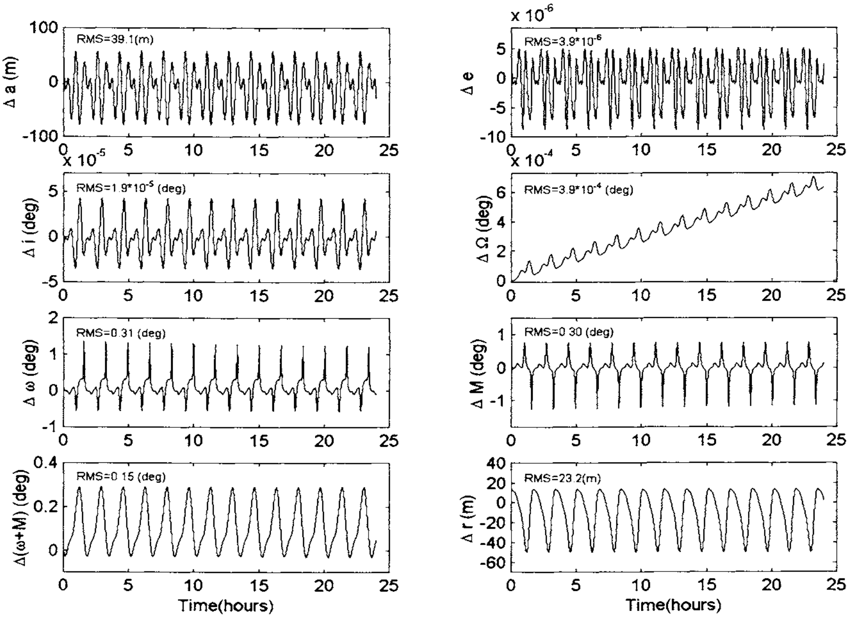

Figure E.1.2: Difference between QUIKSCAT analytical solutions and numerical integration.

$$
\begin{aligned}
\omega(t) &= \operatorname{atan} 2\,(h(t), k(t)) \\
M(t) &= \beta(t) - \omega(t).
\end{aligned}
\tag{E.1.7}
$$

The equation for $r(t)$ also can be written in terms of the mean elements as follows

$$
\begin{aligned}
r(t) = {} & \bar{a}[1 - \bar{e}\cos\overline{M}(t)] + \frac{3}{4}\frac{R^2}{\bar{a}}J_2(3\sin^2\bar{\imath} - 2) \\
& + \frac{1}{4}\frac{R^2}{a}J_2\sin^2\bar{\imath}\cos 2\bar{\beta}
\end{aligned}
\tag{E.1.8}
$$

where

$$
\overline{M}(t) = \bar{\beta}(t) - \bar{\omega}(t)\,. \tag{E.1.9}
$$

E.2 EXAMPLE

Figure E.2.1 presents the classical orbit elements and radius time history for the QUIKSCAT satellite over one day computed by using Eq. (E.1.1). Initial conditions are given on the figure. Fig. E.2.2 presents the differences between the analytical solutions and numerical integration.

We have ignored terms of $O(eJ_2)$ and $O(J_2^2)$ in developing this theory. Here the values of e and J_2 are the same order of magnitude. However, errors of $O(eJ_2)$ are dominant since reducing the initial eccentricity to 10^{-4} reduces the periodic errors in the analytical solutions by a factor of two. Secular errors of $O(J_2^2)$ are apparent in Ω. Including secular rates of $O(J_2^2)$, given by Brouwer (1959), reduces the RMS error in Ω by an order of magnitude.

E.3 REFERENCES

Born, G. H., D. B. Goldstein and B. Thompson, An Analytical Theory for Orbit Determination, *J. Astronaut. Sci.*, Vol. 49, No. 2, pp. 345–361, April–June 2001.

Brouwer, D., "Solutions of the problem of artificial satellite theory without drag," *Astron. J.*, Vol. 64, No. 9, pp. 378–397, November 1959.