Appendix G

Solution of the Linearized Equations of Motion

G.1 INTRODUCTION

The equations of motion for a satellite are given by:

$$\dot{\mathbf{X}} = F(\mathbf{X}, t),$$
where
$$\mathbf{X} = [X\; Y\; Z\; \dot{X}\; \dot{Y}\; \dot{Z}\; \boldsymbol{\beta}]^T. \tag{G.1.1}$$

$\mathbf{X}$ is the state vector containing six position and velocity elements and $\boldsymbol{\beta}$, an m vector, represents all constant parameters such as gravity and drag coefficients that are to be solved for. Hence, $\mathbf{X}$ is a vector of dimension $n = m + 6$.

Equation (G.1.1) can be linearized by expanding about a reference state vector denoted by $\mathbf{X}^*$,

$$\dot{\mathbf{X}}(t) = \dot{\mathbf{X}}^*(t) + \left[\frac{\partial \dot{\mathbf{X}}(t)}{\partial \mathbf{X}(t)}\right]^* (\mathbf{X}(t) - \mathbf{X}^*(t)) + \text{h.o.t.} \tag{G.1.2}$$

The $*$ indicates that the quantity is evaluated on the reference state. By ignoring higher-order terms (h.o.t.) and defining

$$\mathbf{x}(t) \equiv \mathbf{X}(t) - \mathbf{X}^*(t), \tag{G.1.3}$$

we can write Eq. (G.1.2) as

$$\dot{\mathbf{x}}(t) = \left[\frac{\partial \dot{\mathbf{X}}(t)}{\partial \mathbf{X}(t)}\right]^* \mathbf{x}(t)\,. \tag{G.1.4}$$

Define

$$A(t) \equiv \left[\frac{\partial \dot{\mathbf{X}}(t)}{\partial \mathbf{X}(t)}\right]^*,$$

then

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t). \tag{G.1.5}$$

Equation (G.1.5) is a linear system of first-order differential equations with $A(t)$ being an $n \times n$ time varying matrix evaluated on the known reference state $\mathbf{X}^*(t)$. Note that $\dot{\boldsymbol{\beta}} = 0$, so that

$$\frac{\partial \dot{\boldsymbol{\beta}}}{\partial \mathbf{X}(t)} = 0.$$

Because Eq. (G.1.4) is linear† and of the form

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t),$$

the solution can be written as

$$\mathbf{x}(t) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_0}\mathbf{x}_0.$$

It is also true that

$$\mathbf{x}(t) = \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}_0}\mathbf{x}_0. \tag{G.1.6}$$

This follows from the fact that the reference state does not vary in this operation,

$$\begin{aligned}\frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_0} &= \frac{\partial\left(\mathbf{X}(t) - \mathbf{X}^*(t)\right)}{\partial(\mathbf{X}_0 - \mathbf{X}_0^*)} \\ &= \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}_0}.\end{aligned}$$

The conditions under which Eq. (G.1.6) satisfies Eq. (G.1.4) are demonstrated as follows. First define

$$\Phi(t, t_0) \equiv \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}_0}. \tag{G.1.7}$$

Then Eq. (G.1.6) can be written as

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}_0. \tag{G.1.8}$$

† A differential equation of any order (order of highest-ordered derivative) and first degree (degree of its highest derivative) is said to be linear when it is linear in the dependent variable and its derivatives and their products; e.g., $\frac{d^2x}{dt^2} = \frac{x}{t} + 3t^2$ is a linear second-order equation of first degree, and $x\frac{dx}{dt} = 3t + 4$ is first order and degree but nonlinear.

Differentiating Eq. (G.1.8) yields

$$\dot{\mathbf{x}}(t) = \dot{\Phi}(t, t_0)\mathbf{x}_0. \tag{G.1.9}$$

Equating Eq. (G.1.4) and Eq. (G.1.9) yields

$$\frac{\partial \dot{\mathbf{X}}(t)}{\partial \mathbf{X}(t)}\mathbf{x}(t) = \dot{\Phi}(t, t_0)\mathbf{x}_0. \tag{G.1.10}$$

Substituting Eq. (G.1.8) for $\mathbf{x}(t)$ into Eq. (G.1.10) results in

$$\left[\frac{\partial \dot{\mathbf{X}}(t)}{\partial \mathbf{X}(t)}\right]^* \Phi(t, t_0)\mathbf{x}_0 = \dot{\Phi}(t, t_0)\mathbf{x}_0.$$

Equating the coefficients of $\mathbf{x}_0$ in this equation yields the differential equation for $\dot{\Phi}(t, t_0)$,

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \tag{G.1.11}$$

with initial conditions

$$\Phi(t_0, t_0) = I. \tag{G.1.12}$$

The matrix $\Phi(t, t_0)$ is referred to as the State Transition Matrix. Whenever Eqs. (G.1.11) and (G.1.12) are satisfied, the solution to $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$ is given by Eq. (G.1.8).

G.2 THE STATE TRANSITION MATRIX

Insight into the $n \times n$ state transition matrix can be obtained as follows. Let

$$\Phi(t, t_0) \equiv \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}_0} \equiv \begin{bmatrix} \phi_1(t, t_0) \\ \phi_2(t, t_0) \\ \phi_3(t, t_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{r}(t)}{\partial \mathbf{X}_0} \\ \frac{\partial \dot{\mathbf{r}}(t)}{\partial \mathbf{X}_0} \\ \frac{\partial \boldsymbol{\beta}(t)}{\partial \mathbf{X}_0} \end{bmatrix}. \tag{G.2.1}$$

Note that $\phi_3(t, t_0)$ is an $m \times n$ matrix of constants partitioned into an $m \times 6$ matrix of zeros on the left and an $m \times m$ identity matrix on the right, where m is the dimension of $\boldsymbol{\beta}$ and $\mathbf{X}$ is of dimension n. Because of this, it is only necessary to solve the upper $6 \times n$ portion of Eq. (G.1.11).

Equation (G.1.11) also can be written in terms of a second-order differential equation. This can be shown by differentiating Eq. (G.2.1):

$$\dot{\Phi}(t, t_0) = \frac{\partial \dot{\mathbf{X}}(t)}{\partial \mathbf{X}_0} = \begin{bmatrix} \dot{\phi}_1(t, t_0) \\ \dot{\phi}_2(t, t_0) \\ \dot{\phi}_3(t, t_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial \dot{\mathbf{r}}(t)}{\partial \mathbf{X}_0} \\ \frac{\partial \ddot{\mathbf{r}}(t)}{\partial \mathbf{X}_0} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \dot{\mathbf{r}}(t)}{\partial \mathbf{X}(t)} \\ \frac{\partial \ddot{\mathbf{r}}(t)}{\partial \mathbf{X}(t)} \\ 0 \end{bmatrix}_{n\times n} \left[\frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}_0}\right]_{n\times n}. \tag{G.2.2}$$

In these equations, 0 represents an $m \times n$ null matrix. Notice from the first of Eq. (G.2.2) that

$$\ddot{\phi}_1 = \frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{X}_0} = \dot{\phi}_2. \tag{G.2.3}$$

Hence, we could solve this second-order system of differential equations to obtain $\Phi(t, t_0)$,

$$\begin{aligned}\ddot{\phi}_1(t,t_0) &= \frac{\partial \ddot{\mathbf{r}}(t)}{\partial \mathbf{X}_0} = \frac{\partial \ddot{\mathbf{r}}(t)}{\partial \mathbf{X}(t)} \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}_0} \\ &= \left[\frac{\partial \ddot{\mathbf{r}}(t)}{\partial \mathbf{r}(t)}\ \frac{\partial \ddot{\mathbf{r}}(t)}{\partial \dot{\mathbf{r}}(t)}\ \frac{\partial \ddot{\mathbf{r}}(t)}{\partial \boldsymbol{\beta}}\right]_{3\times n} \begin{bmatrix} \frac{\partial \mathbf{r}(t)}{\partial \mathbf{X}_0} \\ \frac{\partial \dot{\mathbf{r}}(t)}{\partial \mathbf{X}_0} \\ \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{X}_0} \end{bmatrix}_{n\times n}\end{aligned} \tag{G.2.4}$$

or

$$\ddot{\phi}_1(t,t_0) = \frac{\partial \ddot{\mathbf{r}}(t)}{\partial \mathbf{r}(t)}\phi_1(t,t_0) + \frac{\partial \ddot{\mathbf{r}}(t)}{\partial \dot{\mathbf{r}}(t)}\dot{\phi}_1(t,t_0) + \frac{\partial \ddot{\mathbf{r}}(t)}{\partial \boldsymbol{\beta}}\phi_3(t,t_0). \tag{G.2.5}$$

With initial conditions

$$\begin{aligned}\phi_1(t_0,t_0) &= \left[[I]_{3\times 3}[0]_{3\times(n-3)}\right] \\ \dot{\phi}_1(t_0,t_0) &= \phi_2(t_0,t_0) = \left[[0]_{3\times 3}\ [I]_{3\times 3}\ [0]_{3\times m}\right].\end{aligned}$$

We could solve Eq. (G.2.5), a $3 \times n$ system of second-order differential equations, instead of the $6 \times n$ first-order system given by Eq. (G.2.2). Recall that the partial derivatives are evaluated on the reference state and that the solution of the $m \times n$ system represented by $\dot{\phi}_3(t, t_0) = 0$ is trivial,

$$\phi_3(t,t_0) = \left[[0]_{m\times 6}[I]_{m\times m}\right].$$

In solving Eq. (G.2.5) we could write it as a system of $n \times n$ first-order equations,

$$\begin{aligned}\dot{\phi}_1(t,t_0) &= \phi_2(t,t_0) \\ \dot{\phi}_2(t,t_0) &= \tfrac{\partial \ddot{\mathbf{r}}(t)}{\partial \mathbf{r}(t)}\phi_1(t,t_0) + \tfrac{\partial \ddot{\mathbf{r}}(t)}{\partial \dot{\mathbf{r}}(t)}\phi_2(t,t_0) + \tfrac{\partial \ddot{\mathbf{r}}(t)}{\partial \boldsymbol{\beta}}\phi_3(t,t_0) \\ \dot{\phi}_3(t,t_0) &= 0.\end{aligned} \tag{G.2.6}$$

It is easily shown that Eq. (G.2.6) is identical to Eq. (G.1.11),

$$\begin{bmatrix} \dot{\phi}_1(t,t_0) \\ \dot{\phi}_2(t,t_0) \\ \dot{\phi}_3(t,t_0) \end{bmatrix}_{n\times n} = \begin{bmatrix} [0]_{3\times 3} & [I]_{3\times 3} & [0]_{3\times m} \\ \left[\frac{\partial \ddot{\mathbf{r}}(t)}{\partial \mathbf{r}(t)}\right]_{3\times 3} & \left[\frac{\partial \ddot{\mathbf{r}}(t)}{\partial \dot{\mathbf{r}}(t)}\right]_{3\times 3} & \left[\frac{\partial \ddot{\mathbf{r}}(t)}{\partial \boldsymbol{\beta}}\right]_{3\times m} \\ [0]_{m\times 3} & [0]_{m\times 3} & [0]_{m\times m} \end{bmatrix}^*_{n\times n} \begin{bmatrix} \phi_1(t,t_0) \\ \phi_2(t,t_0) \\ \phi_3(t,t_0) \end{bmatrix}_{n\times n}$$

or

$$\dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0)\,.$$