Appendix F

Example of State Noise and Dynamic Model Compensation

F.1 INTRODUCTION

Consider a target particle that moves in one dimension along the x axis in the positive direction.* Nominally, the particle's velocity is a constant 10 m/sec. This constant velocity is perturbed by a sinusoidal acceleration in the x direction, which is unknown and is described by:

$$\eta(t) = \frac{2\pi}{10} \cos\left(\frac{2\pi}{10}t\right) \text{ m/sec.}$$

The perturbing acceleration, perturbed velocity, and position perturbation (perturbed position—nominal position) are shown in Fig. F.1.1.

A measurement sensor is located at the known position $x = -10$ m. This sensor takes simultaneous range and range-rate measurements at a frequency of 10 Hz. The range measurements are corrupted by uncorrelated Gaussian noise having a mean of zero and a standard deviation of 1 m. Likewise, the range-rate measurements are corrupted by uncorrelated Gaussian noise having a mean of zero and a standard deviation of 0.1 m/sec. We want to estimate the state of the particle, with primary emphasis on position accuracy, using these observations given the condition that the sinusoidal perturbing acceleration is unknown. The following estimation results were generated using the extended sequential algorithm discussed in Section 4.7.3.

A simple estimator for this problem incorporates a two-parameter state vector

*We thank David R. Cruickshank of Lockheed-Martin Corp., Colorado Springs, Colorado, for providing this example.

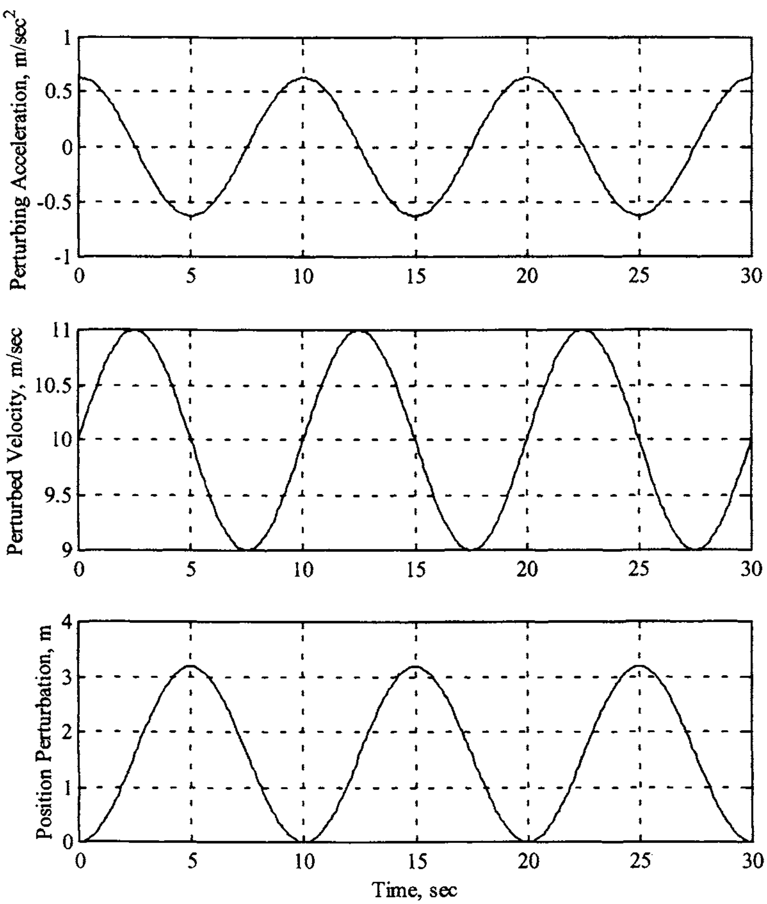

Figure F.1.1: Perturbed particle motion.

consisting of position and velocity:

$$\mathbf{X}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} x(t_0) + \dot{x}(t_0)(t - t_0) \\ \dot{x}(t_0) \end{bmatrix}.$$

The dynamic model assumes constant velocity for the particle motion and does not incorporate process noise. The state transition matrix for this estimator is:

$$\Phi(t, t_0) = \begin{bmatrix} 1 & t - t_0 \\ 0 & 1 \end{bmatrix}$$

and the observation/state mapping matrix is a two-by-two identity matrix:

$$\widetilde{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since the sinusoidal acceleration is not included in the dynamic model and the filter has no means of compensation for the dynamic model deficiency, the filter quickly saturates and its estimation performance is poor. Fig. F.1.2 shows the actual position errors along with the one-sigma position standard deviation bounds estimated by the filter. The position error RMS is 0.9653 m, and only 18.60 percent of the actual position errors are within the estimated one-sigma standard deviation bounds. Fig. F.1.3 shows the corresponding velocity errors and one-sigma velocity standard deviation bounds. As with position, the velocity errors are generally well outside the standard deviation bounds.

F.2 STATE NOISE COMPENSATION

The *State Noise Compensation* (SNC) algorithm (see Section 4.9) provides a means to improve estimation performance through partial compensation for the unknown sinusoidal acceleration. SNC allows for the possibility that the state dynamics are influenced by a random acceleration. A simple SNC model uses a two-state filter but assumes that particle dynamics are perturbed by an acceleration that is characterized as simple white noise:

$$\ddot{x}(t) = \eta(t) = u(t)$$

where $u(t)$ is a stationary Gaussian process with a mean of zero and a variance of $\sigma_u^2 \delta(t - \tau)$, and $\delta(t - \tau)$ is the Dirac delta function. The Dirac delta function is not an ordinary function, and to be mathematically rigorous, white noise is a fictitious process. However, in linear stochastic models, it can be treated formally as an integrable function. Application of Eq. (4.9.1) to this case results in

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

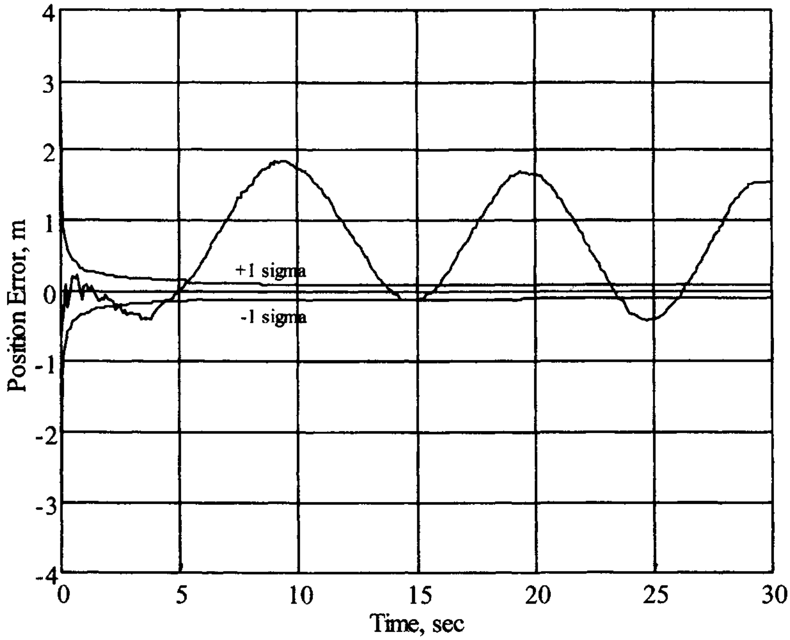

Figure F.1.2: Position errors and estimated standard deviation bounds from the two-state filter without process noise.

where the state propagation matrix A is identified as:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and the process noise mapping matrix is:

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The state transition matrix is the same:

$$\Phi(t, t_0) = \begin{bmatrix} 1 & t - t_0 \\ 0 & 1 \end{bmatrix}.$$

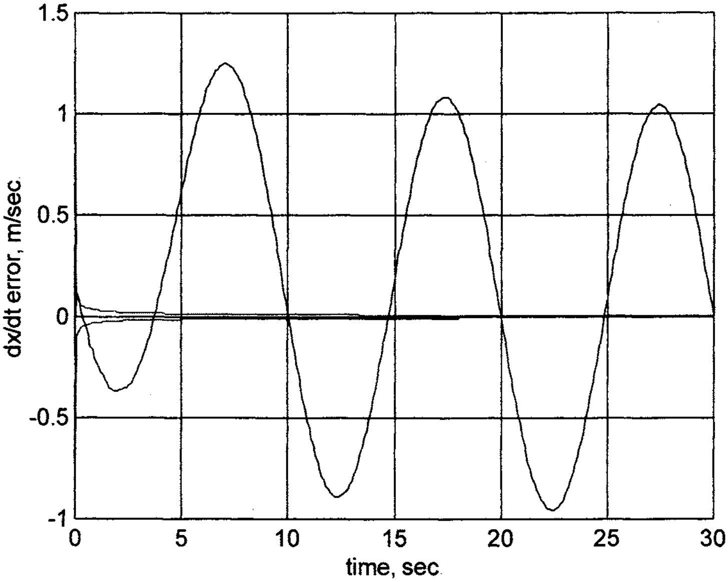

Figure F.1.3: Velocity errors and estimated standard deviation bounds from the two-state filter without process noise.

The process noise covariance integral (see Eq. (4.9.44)) needed for the time update of the estimation error covariance matrix at time t is expressed as:

$$Q_\eta(t) = \sigma_u^2 \int_{t_0}^{t} \Phi(t,\tau) B B^T \Phi^T(t,\tau) d\tau$$

where

$$\Phi(t,\tau) = \begin{bmatrix} 1 & (t-\tau) \\ 0 & 1 \end{bmatrix}.$$

Substituting for B and $\Phi(t,\tau)$ and evaluating results in the process noise covariance matrix:

$$Q_\eta(t) = \sigma_u^2 \begin{bmatrix} \frac{1}{3}(t-t_0)^3 & \frac{1}{2}(t-t_0)^2 \\ \frac{1}{2}(t-t_0)^2 & t-t_0 \end{bmatrix}$$

where $t - t_0$ is the measurement update interval; that is, $t_k - t_{k-1} = 0.1$ sec. The implication of this is that the original deterministic constant velocity model of particle motion is modified to include a random component that is a constant-diffusion Brownian motion process, σ_u^2 being known as the diffusion coefficient (Maybeck, 1979).

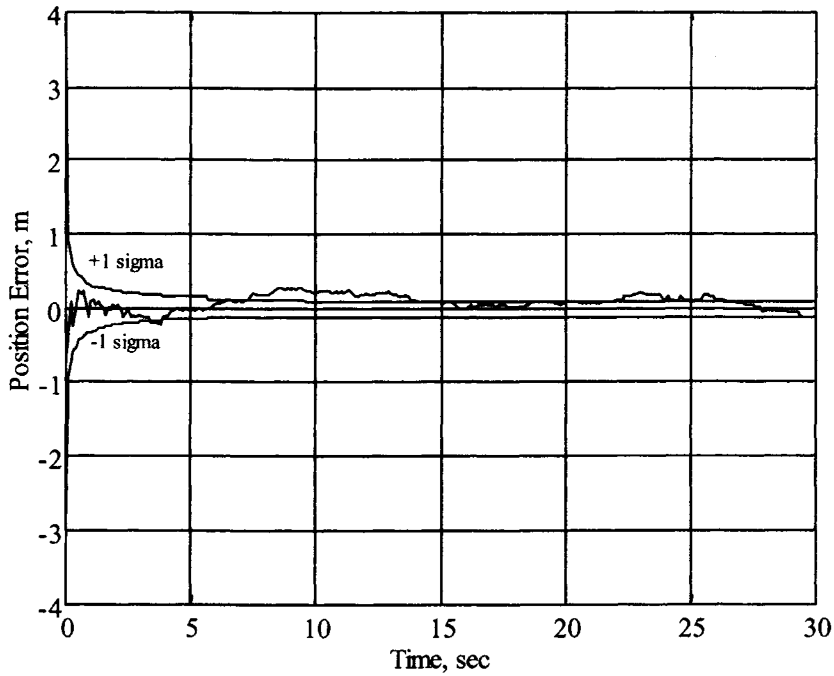

Figure F.2.1: Position errors and estimated standard deviation bounds from the two-state SNC filter.

The magnitude of the process noise covariance matrix and its effect on estimation performance are functions of this diffusion coefficient, hence σ_u is a tuning parameter whose value can be adjusted to optimize performance. Figures F.2.1 and F.2.2 show the result of the tuning process. The optimal value of σ_u is 0.42 m/sec$^{3/2}$ at which the position error RMS is 0.1378 m and 56.81 percent of the actual position errors fall within the estimated one-sigma standard deviation bounds. The large sinusoidal error signature in both position and velocity displayed by the uncompensated filter is eliminated by SNC.

Note that there is no noticable change in the position standard deviations; however, the velocity standard deviations show a significant increase when process noise is included. This increase in the velocity variances prevents the components of the Kalman gain matrix from going to zero with the attendant saturation of the filter.

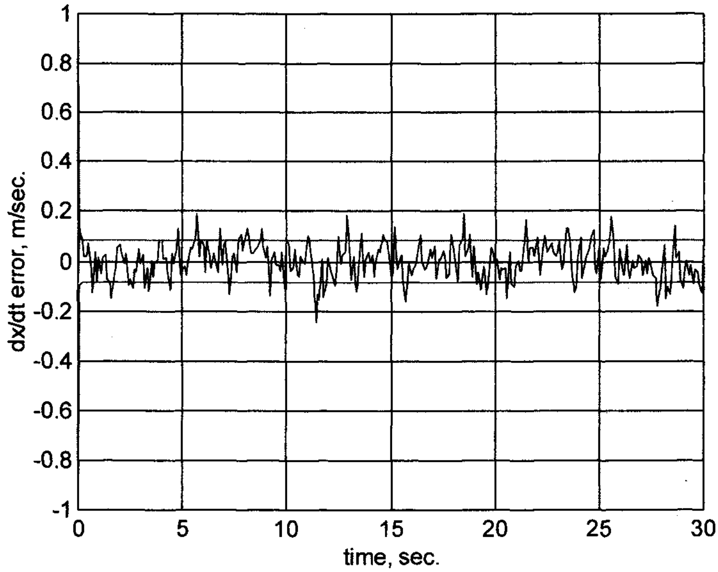

Figure F.2.2: Velocity errors and estimated standard deviation bounds from the two-state filter with SNC process noise.

F.3 DYNAMIC MODEL COMPENSATION

A more sophisticated process noise model is provided by the *Dynamic Model Compensation* (DMC) formulation. DMC assumes that the unknown acceleration can be characterized as a first-order linear stochastic differential equation, commonly known as a Langevin equation:

$$\dot{\eta}(t) + \beta\eta(t) = u(t),$$

where $u(t)$ is a white zero-mean Gaussian process as described earlier and β is the inverse of the correlation time:

$$\beta = \frac{1}{\tau}.$$

The solution to this Langevin equation is

$$\eta(t) = \eta_0 e^{-\beta(t-t_0)} + \int_{t_0}^{t} e^{-\beta(t-\tau)} u(\tau) d\tau.$$

This is the Gauss-Markov process (more precisely known as an Ornstein-Uhlenbeck process) described in Section 4.9. Note that, unlike SNC, the DMC process noise model yields a deterministic acceleration term as well as a purely random term. The deterministic acceleration can be added to the state vector and estimated along with the velocity and position; the augmented state vector becomes a three-state filter with

$$\mathbf{X}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \eta_D(t) \end{bmatrix}$$

where $\eta_D(t)$ is the deterministic part of $\eta(t)$; that is, $\eta_0 e^{-\beta(t-t_0)}$. Since the acceleration integrates into velocity and position increments, the dynamic model of the particle's motion becomes

$$\mathbf{X}(t) = \begin{bmatrix} x_0 + \dot{x}_0(t-t_0) + \frac{\eta_0}{\beta}(t-t_0) + \frac{\eta_0}{\beta^2}\left(e^{-\beta(t-t_0)} - 1\right) \\ \dot{x}_0 + \frac{\eta_0}{\beta}\left(1 - e^{-\beta(t-t_0)}\right) \\ \eta_0 e^{-\beta(t-t_0)} \end{bmatrix}.$$

The correlation time, τ, can also be added to the estimated state, resulting in a four-parameter state vector. However, a tuning strategy that works well for many DMC estimation problems is to set τ to a near-optimal value and hold it constant, or nearly constant, during the estimation span. For simplicity, τ is held constant in this case, allowing the use of the three-parameter state vector just noted. The observation/state mapping matrix is a simple extension of the two-state case:

$$\widetilde{H} = \begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \end{bmatrix}.$$

Using the methods described in Example 4.2.3, the state transition matrix is found to be

$$\Phi(t,t_0) = \frac{\partial X(t)}{\partial X(t_0)} = \begin{bmatrix} 1 & t-t_0 & \frac{1}{\beta}(t-t_0) + \frac{1}{\beta^2}\left(e^{-\beta(t-t_0)} - 1\right) \\ 0 & 1 & \frac{1}{\beta}\left(1 - e^{-\beta(t-t_0)}\right) \\ 0 & 0 & e^{-\beta(t-t_0)} \end{bmatrix}.$$

The state propagation matrix for this case is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\beta \end{bmatrix}$$

and the process noise mapping matrix is

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

With these, the process noise covariance integral becomes (in terms of the components of $\Phi(t,\tau)$)

$$Q_\eta(t) = \sigma_u^2 \times$$
$$\int_{t_0}^{t} \begin{bmatrix} \Phi_{1,3}^2(t,\tau) & \Phi_{1,3}(t,\tau)\Phi_{2,3}(t,\tau) & \Phi_{1,3}(t,\tau)\Phi_{3,3}(t,\tau) \\ \Phi_{2,3}(t,\tau)\Phi_{1,3}(t,\tau) & \Phi_{2,3}^2(t,\tau) & \Phi_{2,3}(t,\tau)\Phi_{3,3}(t,\tau) \\ \Phi_{3,3}(t,\tau)\Phi_{1,3}(t,\tau) & \Phi_{3,3}(t,\tau)\Phi_{2,3}(t,\tau) & \Phi_{3,3}^2(t,\tau) \end{bmatrix} d\tau\,.$$

Note that here τ designates the integration variable, not the correlation time. The expanded, integrated process noise covariance matrix terms follow:
position variance:

$$Q_{\eta 1,1}(t) = \sigma_u^2 \Bigg(\frac{1}{3\beta^2}(t-t_0)^3 - \frac{1}{\beta^3}(t-t_0)^2 + \frac{1}{\beta^4}(t-t_0)(1-2e^{-\beta(t-t_0)}) + \frac{1}{2\beta^5}(1-e^{-2\beta(t-t_0)}) \Bigg)$$

position/velocity covariance:

$$Q_{\eta 1,2}(t) = \sigma_u^2 \Bigg(\frac{1}{2\beta^2}(t-t_0)^2 - \frac{1}{\beta^3}(t-t_0)(1-e^{-\beta(t-t_0)}) + \frac{1}{\beta^4}(1-e^{-\beta(t-t_0)}) - \frac{1}{2\beta^4}(1-e^{-2\beta(t-t_0)}) \Bigg)$$

position/acceleration covariance:

$$Q_{\eta 1,3}(t) = \sigma_u^2 \left(\frac{1}{2\beta^3}(1-e^{-2\beta(t-t_0)}) - \frac{1}{\beta^2}(t-t_0)e^{-\beta(t-t_0)} \right)$$

velocity variance:

$$Q_{\eta 2,2}(t) = \sigma_u^2 \left(\frac{1}{\beta^2}(t-t_0) - \frac{2}{\beta^3}\left(1-e^{-\beta(t-t_0)}\right) + \frac{1}{2\beta^3}\left(1-e^{-2\beta(t-t_0)}\right) \right)$$

velocity/acceleration covariance:

$$Q_{\eta 2,3}(t) = \sigma_u^2 \left(\frac{1}{2\beta^2}\left(1+e^{-2\beta(t-t_0)}\right) - \frac{1}{\beta^2}e^{-\beta(t-t_0)} \right)$$

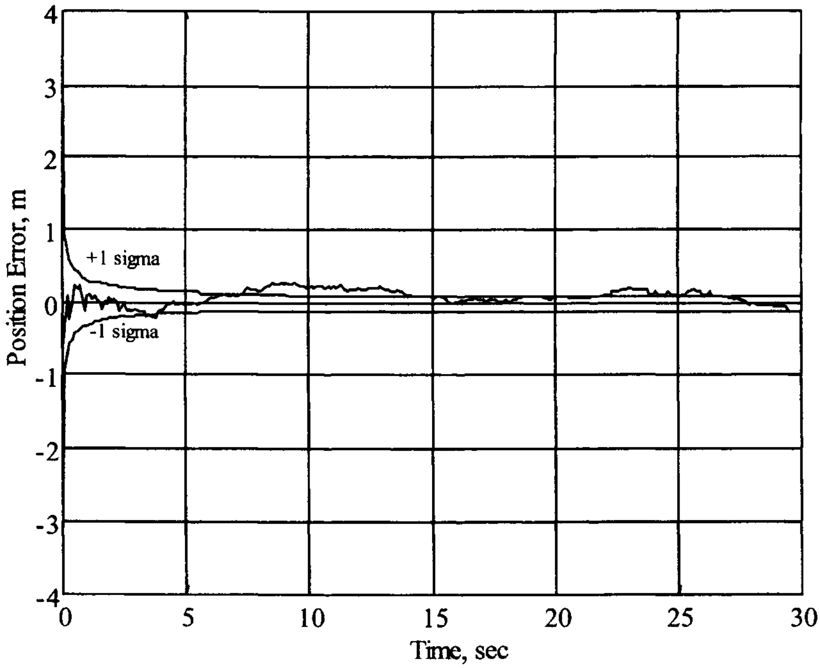

Figure F.3.1: Position errors and estimated standard deviation bounds from the three-state DMC filter.

acceleration variance:

$$Q_{\eta 3,3}(t) = \frac{\sigma_u^2}{2\beta}\left(1 - e^{-2\beta(t-t_0)}\right) .$$

With τ fixed at 200.0 sec (or $\beta = 0.005$ sec^{-1}), the optimal value of σ_u is 0.26 m/sec$^{5/2}$. This combination produces the position error results shown in Fig. F.3.1. The position error RMS is 0.1376 m and 56.81 percent of the actual position errors are within the estimated one-sigma standard deviation bounds. Although not shown here, the plot of velocity errors exhibits a behavior similar to the velocity errors for the SNC filter.

Figure F.3.2 is a plot of the RMS of the position error as a function of σ_u for both the two-state SNC filter and the three-state DMC filter. Although the optimally tuned SNC filter approaches the position error RMS performance of the DMC filter, it is much more sensitive to tuning. DMC achieves very good performance over a much broader, suboptimal range of σ_u values than does SNC.

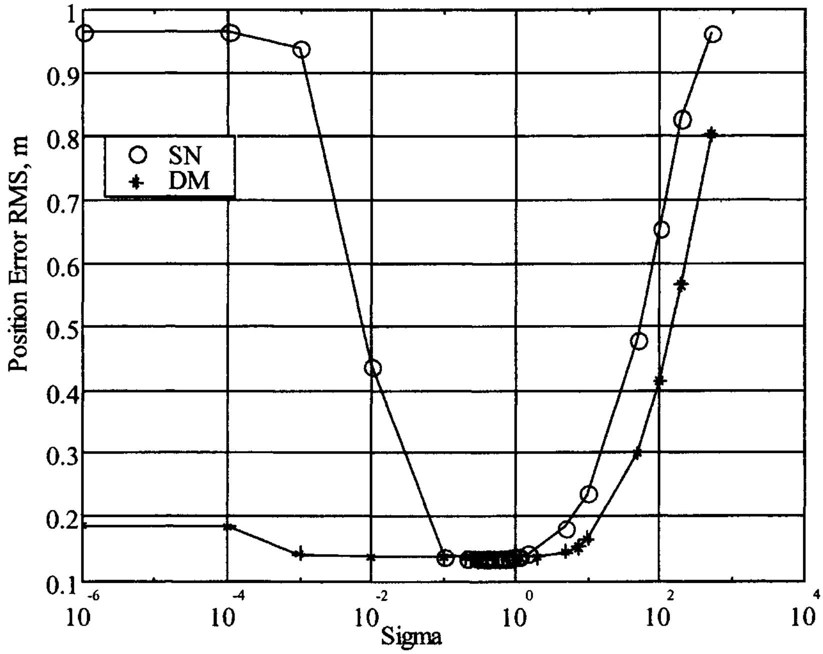

Figure F.3.2: RMS Position error as a function of σ_u for the two-state SNC and three-state DMC filters.

This is a significant advantage for DMC applications where the true state values are not available to guide the tuning effort, as they are in a simulation study, and the precisely optimal tuning parameter values are not known.

Aside from this, the DMC filter also provides a direct estimate of the unknown, unmodeled acceleration. This information could, in turn, be useful in improving the filter dynamic model. Figure F.3.3 is a plot of the estimated and true sinusoidal accelerations. This plot shows excellent agreement between the estimated and true accelerations; their correlation coefficient is 0.9306.

F.4 REFERENCE

Maybeck, P. S., *Stochastic Models, Estimation and Control*, Vol. 1, Academic Press, 1979.

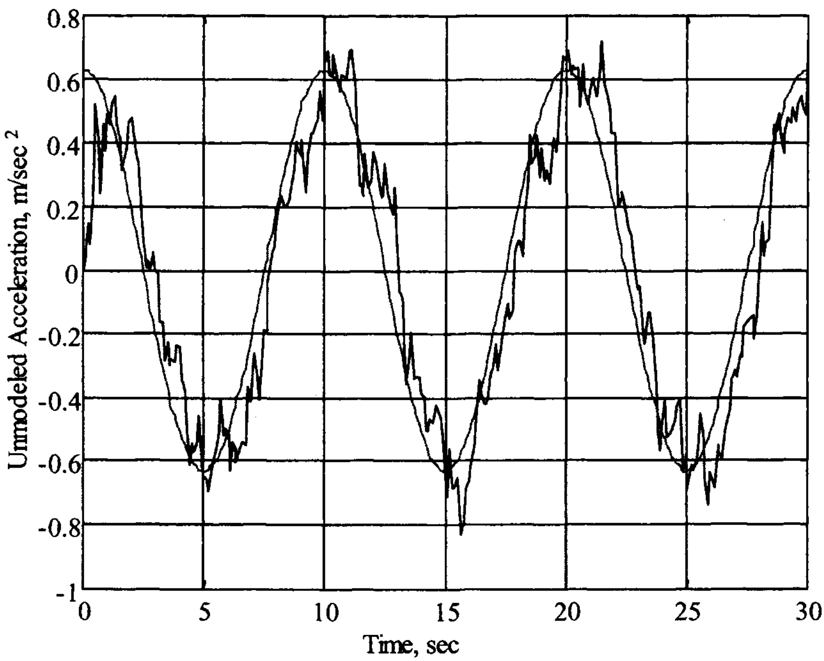

Figure F.3.3: Estimated and true sinusoidal accelerations.