Chapter 4

Fundamentals of Orbit Determination

4.1 INTRODUCTION

During the decade of the 1960s, the accuracy of the radio frequency and optical measurement systems, the force models that govern the satellite motion, and the analysis techniques, combined with relatively primitive computing technology, restricted the positioning of Earth-orbiting satellites to accuracies of hundreds of meters. With the decade of the 1970s, improvements in all of these areas, particularly in mathematical force models and computing capability, facilitated orbit determination accuracy at the level of a few tens of meters by mid-decade and to a few meters by the end of the decade. During the 1980s, significant improvements were made in the models for the Earth's gravity field, including solid body and ocean tides, and in the models for the surface force effects. All these improvements, which were greatly advanced by the significant developments in computer technology, allowed orbit determination accuracies to increase to the tens-of-centimeter level by the end of the decade.

This improvement in orbit determination accuracy was motivated by the ever-increasing demands of scientists in the oceanographic and geodetic communities. In particular, the need for centimeter-level accuracy in global ocean topography obtained from altimetric satellites spurred extensive and unprecedented model improvements during the past two decades. These studies have led to even further improvements in the technology; today the orbit of the oceanographic satellite TOPEX/Poseidon, launched in 1992, is routinely computed with an accuracy approaching 2 cm RMS in the radial component and 8 cm RSS for all components (Tapley *et al.*,1994; Schutz *et al.*, 1994; Marshall *et al.*, 1995; Bertiger *et al.*, 1995). Orbits for the Jason-1 altimetric satellite, launched in 2001, are rou-

tinely computed with an accuracy of 1 cm in the radial component and 4 cm RSS in all components in a post-processing mode using GPS, SLR, and DORIS data (Lutchke *et al.*, 2003). In near-real time (3–5 hours) the Jason-1 radial component is computed to better than 2.5 cm RMS using GPS data (Desai and Haines, 2003).

The discussion in the following sections will focus on the techniques used to estimate the orbits. The approach adopted here will follow that given by Tapley (1973, 1989). The role of the estimation process in improving the force and measurement models as an integral step in the process will be illustrated.

4.2 LINEARIZATION OF THE ORBIT DETERMINATION PROCESS

In Chapter 1, the general formulation of the orbit determination problem was discussed using a dynamical system governed by a simple linear force model. The role of the measurement model in introducing nonlinearity to the process was described. In the general orbit determination problem, both the dynamics and the measurements involve significant nonlinear relationships. For the general case, the governing relations involve the nonlinear expression

$$\dot{\mathbf{X}} = F(\mathbf{X}, t), \qquad \mathbf{X}(t_k) \equiv \mathbf{X}_k \tag{4.2.1}$$

$$\mathbf{Y}_i = G(\mathbf{X}_i, t_i) + \boldsymbol{\epsilon}_i; \qquad i = 1, \ldots, \ell \tag{4.2.2}$$

where $\mathbf{X}_k$ is the unknown n-dimensional state vector at the time t_k, and $\mathbf{Y}_i$ for $i = 1, \ldots, \ell$, is a p-dimensional set of observations that are to be used to obtain a best estimate of the unknown value of $\mathbf{X}_k$ (i.e., $\hat{\mathbf{X}}_k$). In general, $p < n$ and $m = p \times \ell \gg n$. The formulation represented by Eqs. (4.2.1) and (4.2.2) is characterized by: (1) the inability to observe the state directly, (2) nonlinear relations between the observations and the state, (3) fewer observations at any time epoch than there are state vector components ($p < n$), and (4) errors in the observations represented by $\boldsymbol{\epsilon}_i$. The problem of determining the trajectory of a space vehicle in the presence of these effects is referred to as the nonlinear estimation (or orbit determination) problem. If the state vector and the observation vector can be related in a linear manner, then several powerful techniques from the field of linear estimation theory can be applied to the orbit determination problem.

If a reasonable reference trajectory is available and if $\mathbf{X}$, the true trajectory, and $\mathbf{X}^*$, the reference trajectory, remain sufficiently close throughout the time interval of interest, then the trajectory for the actual motion can be expanded in a Taylor's series about the reference trajectory at each point in time. If this expansion is truncated to eliminate higher order terms, then the deviation in the state from the reference trajectory can be described by a set of linear differential equations with time-dependent coefficients. A linear relation between the observation

deviation and the state deviation can be obtained by a similar expansion procedure. Then, the nonlinear orbit determination problem in which the complete state vector is to be estimated can be replaced by a linear orbit determination problem in which the deviation from some reference solution is to be determined.

To conduct this linearization procedure, let the $n \times 1$ state deviation vector, $\mathbf{x}$, and the $p \times 1$ observation deviation vector, $\mathbf{y}$, be defined as follows:

$$\mathbf{x}(t) = \mathbf{X}(t) - \mathbf{X}^*(t), \qquad \mathbf{y}(t) = \mathbf{Y}(t) - \mathbf{Y}^*(t). \tag{4.2.3}$$

It follows that

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{X}}(t) - \dot{\mathbf{X}}^*(t). \tag{4.2.4}$$

Expanding Eqs. (4.2.1) and (4.2.2) in a Taylor's series about the reference trajectory leads to

$$\begin{aligned}
\dot{\mathbf{X}}(t) &= F(\mathbf{X}, t) = F(\mathbf{X}^*, t) + \left[\frac{\partial F(t)}{\partial \mathbf{X}(t)}\right]^* [\mathbf{X}(t) - \mathbf{X}^*(t)] \\
&\quad + O_F [\mathbf{X}(t) - \mathbf{X}^*(t)] \\
\mathbf{Y}_i &= G(\mathbf{X}_i, t_i) + \boldsymbol{\epsilon}_i = G(\mathbf{X}_i^*, t_i) + \left[\frac{\partial G}{\partial \mathbf{X}}\right]_i^* [\mathbf{X}(t_i) - \mathbf{X}^*(t_i)]_i \\
&\quad + O_G [\mathbf{X}(t_i) - \mathbf{X}^*(t_i)] + \boldsymbol{\epsilon}_i
\end{aligned} \tag{4.2.5}$$

where $[\ ]^*$ indicates that the partial derivative matrix is evaluated on the reference solution, $\mathbf{X}^*(t)$, which is obtained by integrating Eq. (4.2.1) with the specified initial conditions, $\mathbf{X}^*(t_0)$. The symbols O_F and O_G indicate terms in the expansion containing products of the difference, $\mathbf{X}(t) - \mathbf{X}^*(t)$, higher than the first order. If the terms of order higher than the first in Eq. (4.2.5) are neglected, under the assumption that the higher order products are small compared to the first order terms, and if the condition $\dot{\mathbf{X}}^* = F(\mathbf{X}^*, t)$ and $\mathbf{Y}_i^* = G(\mathbf{X}_i^*, t_i)$ are used, Eq. (4.2.5) can be written as

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) \\
\mathbf{y}_i &= \widetilde{H}_i \mathbf{x}_i + \boldsymbol{\epsilon}_i \qquad (i = 1, \ldots, \ell)
\end{aligned} \tag{4.2.6}$$

where

$$A(t) = \left[\frac{\partial F(t)}{\partial \mathbf{X}(t)}\right]^* \qquad \widetilde{H}_i = \left[\frac{\partial G}{\partial \mathbf{X}}\right]_i^*.$$

Hence, the original nonlinear estimation problem is replaced by the linear estimation problem described by Eq. (4.2.6), where

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{X}(t) - \mathbf{X}^*(t), \\
\mathbf{x}_i &= \mathbf{X}(t_i) - \mathbf{X}^*(t_i)
\end{aligned}$$

and

$$\mathbf{y}_i = \mathbf{Y}_i - G(\mathbf{X}_i^*, t_i).$$

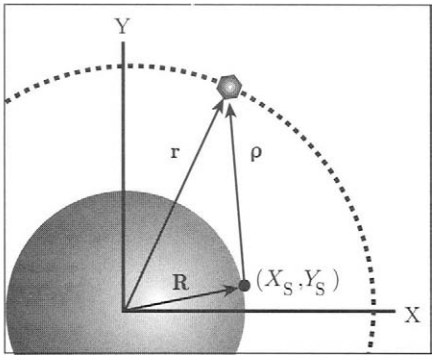
Notice that if the original system of differential equations $\dot{\mathbf{X}} = F(\mathbf{X}, t)$ is linear, the second and higher order partial derivatives of $F(\mathbf{X}, t)$ are zero (i.e., $\frac{\partial^i F}{\partial \mathbf{X}^i} = 0, i \geq 2$). The same statements apply to $G(\mathbf{X}_i, t_i)$ in Eq. (4.2.5). Hence, for a linear system there is no need to deal with a state or observational deviation vector or a reference solution. However, for the orbit determination problem, $F(\mathbf{X}, t)$ and $G(\mathbf{X}_i, t_i)$ will always be nonlinear in $\mathbf{X}(t)$, thus requiring that we deal with deviation vectors and a reference trajectory in order to linearize the system.

Generally in this text, uppercase $\mathbf{X}$ and $\mathbf{Y}$ will represent the state and the observation vectors and lowercase $\mathbf{x}$ and $\mathbf{y}$ will represent the state and observation deviation vectors as defined by Eq. (4.2.3). However, this notation will not always be adhered to and sometimes $\mathbf{x}$ and $\mathbf{y}$ will be referred to as the state and observation vectors, respectively.

Example 4.2.1

Compute the A matrix and the $\widetilde{H}$ matrix for a satellite in a plane under the influence of only a *central force*. Assume that the satellite is being tracked with range observations, ρ, from a single ground station. Assume that the station coordinates, (X_S, Y_S), and the gravitational parameter are unknown. Then, the state vector, $\mathbf{X}$, is given by

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ U \\ V \\ \mu \\ X_S \\ Y_S \end{bmatrix}$$



where U and V are velocity components and X_S and Y_S are coordinates of the tracking station. From Newton's Second Law and the law of gravitation,

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3}$$

or in component form,

$$\ddot{X} = -\frac{\mu X}{r^3}$$

$$\ddot{Y} = -\frac{\mu Y}{r^3}.$$

If these equations are expressed in first-order form, the following expression is obtained:

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{U} \\ \dot{V} \\ \dot{\mu} \\ \dot{X}_S \\ \dot{Y}_S \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} = \begin{bmatrix} U \\ V \\ -\frac{\mu X}{r^3} \\ -\frac{\mu Y}{r^3} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then,

$$A(t) = \frac{\partial F(\mathbf{X}^*, t)}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial F_1}{\partial X} & \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial U} & \frac{\partial F_1}{\partial V} & \frac{\partial F_1}{\partial \mu} & \frac{\partial F_1}{\partial X_S} & \frac{\partial F_1}{\partial Y_S} \\ \frac{\partial F_2}{\partial X} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F_2}{\partial Y_S} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_7}{\partial X} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F_7}{\partial Y_S} \end{bmatrix}^*$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\mu}{r^3}+\frac{3\mu X^2}{r^5} & \frac{3\mu XY}{r^5} & 0 & 0 & -\frac{X}{r^3} & 0 & 0 \\ \frac{3\mu XY}{r^5} & -\frac{\mu}{r^3}+\frac{3\mu Y^2}{r^5} & 0 & 0 & -\frac{Y}{r^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^* .$$

The $\widetilde{H}$ matrix is given by

$$\widetilde{H} = \frac{\partial \rho}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial \rho}{\partial X} & \frac{\partial \rho}{\partial Y} & \frac{\partial \rho}{\partial U} & \frac{\partial \rho}{\partial V} & \frac{\partial \rho}{\partial \mu} & \frac{\partial \rho}{\partial X_S} & \frac{\partial \rho}{\partial Y_S} \end{bmatrix}^*$$

where

$$\rho = \left[(X - X_S)^2 + (Y - Y_S)^2\right]^{1/2} .$$

It follows then that

$$\widetilde{H} = \begin{bmatrix} \frac{X - X_S}{\rho} & \frac{Y - Y_S}{\rho} & 0 & 0 & 0 & -\frac{(X - X_S)}{\rho} & -\frac{(Y - Y_S)}{\rho} \end{bmatrix}^* .$$

4.2.1 The State Transition Matrix

The first of Eq. (4.2.6) represents a system of linear differential equations with time-dependent coefficients. The symbol $[\,]^*$ indicates that the values of $\mathbf{X}$ are derived from a particular solution to the equations $\dot{\mathbf{X}} = F(\mathbf{X}, t)$ which is generated with the initial conditions $\mathbf{X}(t_0) = \mathbf{X}_0^*$. The general solution for this system, $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$, can be expressed as

$$\mathbf{x}(t) = \Phi(t, t_k)\mathbf{x}_k \tag{4.2.7}$$

where $\mathbf{x}_k$ is the value of $\mathbf{x}$ at t_k; that is, $\mathbf{x}_k = \mathbf{x}(t_k)$. The matrix $\Phi(t_i, t_k)$ is called the state transition matrix and was introduced in Chapter 1, Section 1.2.5. The state transition matrix for the two-body problem was discussed in Chapter 2, Section 2.2.6. $\Phi(t, t_k)$ has the following useful properties, which can be demonstrated from Eq. (4.2.7).

$$\begin{array}{lrcl} 1. & \Phi(t_k, t_k) & = & I \\ 2. & \Phi(t_i, t_k) & = & \Phi(t_i, t_j)\Phi(t_j, t_k) \\ 3. & \Phi(t_i, t_k) & = & \Phi^{-1}(t_k, t_i). \end{array} \tag{4.2.8}$$

The differential equation for $\Phi(t_i, t_k)$ can be obtained by differentiating Eq. (4.2.7) (noting that $\mathbf{x}_k$ is a constant). This yields

$$\dot{\mathbf{x}}(t) = \dot{\Phi}(t, t_k)\mathbf{x}_k. \tag{4.2.9}$$

Substituting Eq. (4.2.9) into the first of Eq. (4.2.6) and using Eq. (4.2.7) yields

$$\dot{\Phi}(t, t_k)\mathbf{x}_k = A(t)\Phi(t, t_k)\mathbf{x}_k.$$

Since this condition must be satisfied for all $\mathbf{x}_k$, it follows that

$$\dot{\Phi}(t, t_k) = A(t)\Phi(t, t_k) \tag{4.2.10}$$

with initial conditions

$$\Phi(t_k, t_k) = I.$$

By differentiating $\mathbf{x}_k = \Phi^{-1}(t, t_k)\mathbf{x}(t)$ from Eq. (4.2.7) and using the first of Eq. (4.2.6) it can be shown that

$$\dot{\Phi}^{-1}(t, t_k) = -\Phi^{-1}(t, t_k)\, A(t), \tag{4.2.11}$$

with initial conditions

$$\Phi^{-1}(t_k, t_k) = I.$$

Under certain conditions on $A(t)$, the state transition matrix may be inverted analytically (Battin, 1999).

If the matrix $A(t)$ can be partitioned in the form

$$A(t) = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \tag{4.2.12}$$

where the submatrices have the properties that

$$A_1^T = -A_4, \quad A_2^T = A_2 \quad \text{and} \quad A_3^T = A_3. \tag{4.2.13}$$

Then $\Phi(t, t_k)$ can be similarly partitioned as

$$\Phi(t, t_k) = \left[\begin{array}{c|c} \Phi_1 & \Phi_2 \\ \hline \Phi_3 & \Phi_4 \end{array}\right]$$

and $\Phi^{-1}(t, t_k)$ may be written as

$$\Phi^{-1}(t, t_k) = \left[\begin{array}{c|c} \Phi_4^T & -\Phi_2^T \\ \hline -\Phi_3^T & \Phi_1^T \end{array}\right]. \tag{4.2.14}$$

The proof follows: Define

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{4.2.15}$$

where I is the identity matrix and 0 is the null matrix. Then

$$\begin{aligned} \frac{d}{dt}(J\Phi(t, t_k)J) &= J\dot{\Phi}(t, t_k)J \\ &= -(JA(t)J)(J\Phi(t, t_k)J), \end{aligned} \tag{4.2.16}$$

where we have used the fact that $J^2 = -I$ and $\dot{\Phi}(t, t_k) = A(t)\Phi(t, t_k)$.

Define

$$V(t, t_k) \equiv -(J\Phi(t, t_k)J)^T. \tag{4.2.17}$$

Taking the transpose of Eq. (4.2.16) and using Eq. (4.2.17),

$$\begin{aligned} (J\dot{\Phi}(t, t_k)J)^T &= -\dot{V}(t, t_k) \\ &= -(J\Phi(t, t_k)J)^T(JA(t)J)^T \end{aligned}$$

or

$$\dot{V}(t, t_k) = -V(t, t_k)(JA(t)J)^T. \tag{4.2.18}$$

Using Eq. (4.2.12) for $A(t)$ yields

$$(JA(t)J)^T = \left[\begin{array}{c|c} -A_4^T & A_2^T \\ \hline A_3^T & -A_1^T \end{array}\right]. \tag{4.2.19}$$

Consequently, if $A(t)$ satisfies Eq. (4.2.13) and

$$(JA(t)J)^T = A(t), \tag{4.2.20}$$

then

$$\dot{V}(t, t_k) = -V(t, t_k)A(t) \tag{4.2.21}$$

and

$$V(t_0, t_0) = -(J\Phi(t_0, t_0)J)^T = I.$$

Hence, from Eq. (4.2.11) and Eq. (4.2.21),

$$V(t, t_k) = \Phi^{-1}(t, t_k)$$

or

$$\begin{aligned}\Phi^{-1}(t,t_k) &= -(J\Phi(t,t_k)J)^T \\ &= \left[\begin{array}{c|c} \Phi_4^T & -\Phi_2^T \\ \hline -\Phi_3^T & \Phi_1^T \end{array}\right] \end{aligned} \tag{4.2.22}$$

which is identical to Eq. (4.2.14).

An even dimensional matrix, B, which has the property that $B^T J B = J$ (where J is defined by Eq. (4.2.15)) is called *symplectic* (Battin, 1999). It easily is shown that $\Phi(t,t_k)$ has this property when A satisfies Eq. (4.2.13). An important case where $\Phi(t,t_k)$ is symplectic arises when the acceleration can be written as the gradient of a potential function; that is,

$$\ddot{\mathbf{r}} = \nabla U. \tag{4.2.23}$$

4.2.2 Solution for the State Transition Matrix

A linear differential equation of the type $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$ or $\dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0)$ has an infinite number of solutions in terms of arbitrary constants. However, when initial conditions, $\mathbf{x}(t_0)$ and $\Phi(t_0,t_0)$, are specified and the elements of $A(t)$ are continuous functions of time, the solution becomes unique. One could ask the question "Why bother to solve for the state transition matrix when the state deviation vector can be determined directly by solving $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$?" The answer is that the computational algorithms for determining the best estimate of $\mathbf{x}$ and for mapping the associated error covariance matrices are most easily formulated in terms of the state transition matrix. Since $\mathbf{x}_k$ in Eq. (4.2.7) is unknown in the orbit determination problem, the state transition matrix allows the solution, $\mathbf{x}(t)$, to be expressed in terms of the unknown initial state, $\mathbf{x}_k$. Hence, it is essential in relating observations made at different times.

The solution for $\Phi(t,t_0)$ is facilitated by noting that the individual columns of the differential equation for $\dot{\Phi}(t,t_0)$ are uncoupled and independent. To illustrate this, consider a one-dimensional case where the state vector consists of a single position and velocity coordinate. Equation (4.2.10) can be written in terms of the individual elements of the state transition matrix as follows:

$$\dot{\Phi}(t,t_0) = \begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} \\ \dot{\phi}_{21} & \dot{\phi}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \tag{4.2.24}$$

subject to the following initial conditions at t_0

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.2.25}$$

Equation (4.2.24) expands to

$$\begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} \\ \dot{\phi}_{21} & \dot{\phi}_{22} \end{bmatrix} = \begin{bmatrix} A_{11}\phi_{11} + A_{12}\phi_{21} & A_{11}\phi_{12} + A_{12}\phi_{22} \\ A_{21}\phi_{11} + A_{22}\phi_{21} & A_{21}\phi_{12} + A_{22}\phi_{22} \end{bmatrix}. \tag{4.2.26}$$

Recall that the A_{ij} are known quantities obtained by evaluating

$$A_{ij}(t) = \left[\frac{\partial F_i(t)}{\partial \mathbf{X}_j(t)}\right]^*$$

on the reference trajectory. From Eq. (4.2.26) we see that the columns of $\dot{\Phi}(t,\, t_0)$ are independent; for example, the first column of $\dot{\Phi}(t,\, t_0)$ does not contain elements of $\Phi(t,\, t_0)$ from the second column. Hence, we can solve for $\Phi(t,\, t_0)$ by integrating independently two 2×1 vector differential equations. For any practical orbit determination application, the solution for $\Phi(t,\, t_0)$ will be obtained via numerical integration. Hence, we can supply a vector of derivative values for the differential equation of the nominal state vector and $\dot{\Phi}(t,\, t_0)$ to the numerical integration routine. For this 1D case we would supply the integrator with the following vector at each time point:

$$\begin{bmatrix} \dot{X} \\ \dot{U} \\ \dot{\phi}_{11} \\ \dot{\phi}_{21} \\ \dot{\phi}_{12} \\ \dot{\phi}_{22} \end{bmatrix}. \tag{4.2.27}$$

The first two elements would provide the reference orbit, $\mathbf{X}^*(t)$, and the next four would yield the elements of $\Phi(t, t_0)$. The reference orbit is used to evaluate $A(t)$, which is needed to evaluate $\dot{\Phi}(t, t_0)$ in Eq. (4.2.26).

Notice that when $\dot{\Phi} = A\Phi$ is numerically integrated for the two-body case given in Example 4.2.1 we need to concern ourselves only with the upper 4×5 portion of $\dot{\Phi}$. The final three rows and two columns of Φ remain zero except for the values of unity in the last three diagonal elements. Hence, the numerical integration algorithm should be structured to take advantage of this fact.

Another approach to solving the linearized equations of motion is given in Appendix F. Additional clarification on the structure of Φ and associated matrices also is given.

Generally all orbit determination problems will result in $A(t)$ being a function of time. For example, the simple two-body Example 4.2.1 involves a time-varying matrix for $A(t)$. However, it is instructive to consider the case where A is a constant matrix because of the insight gained about the state transition matrix. Such a case is illustrated in the following example.

Example 4.2.2

Consider a system of linear first-order differential equations with constant coefficients

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= \beta x_1 \end{aligned} \tag{4.2.28}$$

where β is a constant. Initial conditions are $x_1(t_0) = x_{10}$ and $x_2(t_0) = x_{20}$.

a) Write Eq. (4.2.28) in state space form; that is, $\dot{\mathbf{X}} = A\mathbf{X}$,

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \beta & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{X}. \tag{4.2.29}$$

b) Solve Eq. (4.2.29) by determining the state transition matrix; that is, $\mathbf{X}(t) = \Phi(t, t_0)\mathbf{X}(t_0)$, where $\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0)$. Because $A(t)$ is a constant matrix there are a number of ways to solve for $\Phi(t, t_0)$. These include but are not limited to the following:

1. Because Eq. (4.2.28) is linear with constant coefficients it can be solved directly. From the first equation

$$\begin{aligned} \frac{dx_1}{x_1} &= dt \\ x_1(t) &= ce^t; \qquad \text{at } t = t_0,\ x_1 = x_{10} \\ c &= x_{10}e^{-t_0} \\ x_1(t) &= x_{10}e^{(t-t_0)}, \end{aligned} \tag{4.2.30}$$

from the second of Eq. (4.2.28)

$$\begin{aligned} dx_2 &= \beta x_{10}e^{(t-t_0)}dt \\ x_2(t) &= \beta x_{10}e^{(t-t_0)} + c; \qquad \text{at } t = t_0,\ x_2 = x_{20}. \end{aligned}$$

Hence,

$$x_2(t) = x_{20} + \beta x_{10}(e^{(t-t_0)} - 1). \tag{4.2.31}$$

The state transition matrix can be determined from a direct differentiation of the solution

$$\Phi(t, t_0) \equiv \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_0)} = \begin{bmatrix} \dfrac{\partial x_1(t)}{\partial x_1(t_0)} & \dfrac{\partial x_1(t)}{\partial x_2(t_0)} \\ \dfrac{\partial x_2(t)}{\partial x_1(t_0)} & \dfrac{\partial x_2(t)}{\partial x_2(t_0)} \end{bmatrix}. \tag{4.2.32}$$

The elements of Eq. (4.2.32) are obtained by differentiating Eqs. (4.2.30) and (4.2.31) to yield

$$\Phi(t,t_0) = \begin{bmatrix} e^{(t-t_0)} & 0 \\ \beta(e^{(t-t_0)}-1) & 1 \end{bmatrix}. \tag{4.2.33}$$

2. We may integrate $\dot{\Phi} = A\Phi$ directly:

$$\begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} \\ \dot{\phi}_{21} & \dot{\phi}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \beta & 0 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}; \quad \Phi(t_0,t_0) = I.$$

The equations to be integrated are

$$\begin{aligned} \dot{\phi}_{11} &= \phi_{11} \\ \dot{\phi}_{21} &= \beta\phi_{11} \\ \dot{\phi}_{12} &= \phi_{12} \\ \dot{\phi}_{22} &= \beta\phi_{12}. \end{aligned} \tag{4.2.34}$$

Solving Eq. (4.2.34) yields

$$\begin{aligned} \phi_{11} &= ce^t; \qquad \phi_{11}(t_0) = 1 \\ c &= e^{-t_0} \\ \phi_{11} &= e^{(t-t_0)}. \end{aligned}$$

Next,

$$\begin{aligned} \dot{\phi}_{21} &= \beta e^{(t-t_0)} \\ \phi_{21} &= \beta e^{(t-t_0)} + c; \qquad \phi_{21}(t_0) = 0. \end{aligned}$$

If this expression is evaluated at t_0, it follows that

$$c = -\beta$$

and

$$\phi_{21} = \beta(e^{(t-t_0)} - 1).$$

Finally, the second two of Eq. (4.2.34) along with the initial conditions lead to

$$\begin{aligned} \phi_{12} &= 0 \\ \phi_{22} &= 1. \end{aligned}$$

These solutions are identical to the elements given by Eq. (4.2.33).

3. We may use Laplace transforms to solve for $\Phi(t, t_0)$. In this case

$$\begin{aligned}\Phi(t,t_0) &= \mathcal{L}^{-1}\left[SI - A\right]^{-1} = \mathcal{L}^{-1}\begin{bmatrix} S-1 & 0 \\ -\beta & S \end{bmatrix}^{-1} \\ &= \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{S-1} & 0 \\ \frac{\beta}{S(S-1)} & \frac{1}{S} \end{bmatrix}.\end{aligned}$$

Using a table of Laplace transforms yields

$$\Phi(t,t_0) = \begin{bmatrix} e^{(t-t_0)} & 0 \\ \beta(e^{(t-t_0)} - 1) & 1 \end{bmatrix}.$$

4. Another solution, whenever A is a constant matrix, uses the eigenvalues and eigenvectors of A to yield the solution

$$\Phi(t,t_0) = Ve^{\lambda(t-t_0)}V^{-1}, \tag{4.2.35}$$

where V is the matrix of normalized eigenvectors of A. Also,

$$e^{\lambda(t,t_0)} = \begin{bmatrix} e^{\lambda_1(t-t_0)} & & & \\ & e^{\lambda_2(t-t_0)} & & 0 \\ & & \ddots & \\ 0 & & & e^{\lambda_n(t-t_0)} \end{bmatrix} \tag{4.2.36}$$

where $\lambda_1,\ \lambda_2 \ldots \lambda_n$ are the eigenvalues of the $n \times n$ matrix, A.

This method requires that A have a complete set of linearly independent eigenvectors. Otherwise there is no invertible matrix of eigenvectors, V, and the algorithm fails.

For the example we are considering, the matrix

$$A = \begin{bmatrix} 1 & 0 \\ \beta & 0 \end{bmatrix}$$

has eigenvalues (see Appendix B, Section B.6)

$$\begin{aligned}\lambda_1 &= 0 \\ \lambda_2 &= 1\end{aligned}$$

and normalized eigenvectors

$$V_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad V_2 = \begin{bmatrix} \dfrac{1}{\sqrt{1+\beta^2}} \\ \dfrac{\beta}{\sqrt{1+\beta^2}} \end{bmatrix}.$$

Hence,

$$V = \begin{bmatrix} 0 & \dfrac{1}{\sqrt{1+\beta^2}} \\ 1 & \dfrac{\beta}{\sqrt{1+\beta^2}} \end{bmatrix}, \qquad V^{-1} = \begin{bmatrix} -\beta & 1 \\ \sqrt{1+\beta^2} & 0 \end{bmatrix}$$

$$e^{\lambda(t,t_0)} = \begin{bmatrix} 1 & 0 \\ 0 & e^{(t-t_0)} \end{bmatrix}$$

and from Eq. (4.2.35)

$$\Phi(t,t_0) = \begin{bmatrix} e^{(t-t_0)} & 0 \\ \beta(e^{(t-t_0)}-1) & 1 \end{bmatrix}.$$

Moler and Van Loan (1978) describe 19 ways to compute the exponential of a matrix, including the methods presented here.

4.2.3 Relating the Observations to an Epoch State

Note from the second of Eq. (4.2.6) that there is an unknown state vector $\mathbf{x}_i$ corresponding to each observation $\mathbf{y}_i$. Hence, it is desirable to use the state transition matrix to express all observations in terms of the state at a single epoch in order to reduce the number of unknown state vectors from $\ell \times n$ to n. Using Eq. (4.2.7), the second of Eq. (4.2.6) may be written in terms of the state at t_k as

$$\begin{aligned} \mathbf{y}_1 &= \widetilde{H}_1 \Phi(t_1,t_k)\mathbf{x}_k + \boldsymbol{\epsilon}_1 \\ \mathbf{y}_2 &= \widetilde{H}_2 \Phi(t_2,t_k)\mathbf{x}_k + \boldsymbol{\epsilon}_2 \\ &\vdots \\ \mathbf{y}_\ell &= \widetilde{H}_\ell \Phi(t_\ell,t_k)\mathbf{x}_k + \boldsymbol{\epsilon}_\ell. \end{aligned} \tag{4.2.37}$$

Equation (4.2.37) now contains $m = p \times \ell$ observations and only n unknown components of the state. If $\boldsymbol{\epsilon}_i$, $i = 1, \ldots, \ell$ is zero, any linearly independent n of Eq. (4.2.37) can be used to determine $\mathbf{x}_k$.

If the following definitions are used

$$\mathbf{y} \equiv \begin{bmatrix} y_1 \\ \vdots \\ y_\ell \end{bmatrix}; \quad H \equiv \begin{bmatrix} \widetilde{H}_1\Phi(t_1, t_k) \\ \vdots \\ \widetilde{H}_\ell\Phi(t_\ell, t_k) \end{bmatrix}; \quad \boldsymbol{\epsilon} \equiv \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_\ell \end{bmatrix} \tag{4.2.38}$$

and if the subscript on $\mathbf{x}_k$ is dropped for convenience, then Eq. (4.2.37) can be expressed as follows:

$$\mathbf{y} = H\mathbf{x} + \boldsymbol{\epsilon} \tag{4.2.39}$$

where $\mathbf{y}$ is an $m \times 1$ vector, $\mathbf{x}$ is an $n \times 1$ vector, $\boldsymbol{\epsilon}$ is an $m \times 1$ vector, H is an $m \times n$ mapping matrix, where $m = p \times \ell$ is the total number of observations. If p or ℓ is sufficiently large, the essential condition $m > n$ is satisfied. However, we are still faced with m unknown observation errors resulting in $m + n$ total unknowns and only m equations. The least squares criterion provides us with conditions on the m observation errors that allow a solution for the n state variables, $\mathbf{x}$, at the epoch time t_k.

4.3 THE LEAST SQUARES SOLUTION

The least squares solution selects the estimate of $\mathbf{x}$ as that value that minimizes the sum of the squares of the calculated observation residuals. That is, $\mathbf{x}$ is selected to minimize the following *performance index* (Lawson and Hanson, 1974; Björck, 1997):

$$J(\mathbf{x}) = 1/2\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}. \tag{4.3.1}$$

The least squares criterion was first proposed by Gauss (1809) and is commonly used today. The sum of the squares of the calculated observation errors is a logical choice for the performance index. A criterion defined, for example, by the sum of the calculated observation errors could be identically zero with very large observation errors having plus and minus signs that cancel each other. Whether the observation error is positive or negative, its square will be positive and the performance index defined by Eq. (4.3.1) can vanish only if each of the observation errors is identically zero. If $\boldsymbol{\epsilon}$, as defined by Eq. (4.2.39), is substituted into Eq. (4.3.1), the following expression is obtained:

$$J(\mathbf{x}) = 1/2\boldsymbol{\epsilon}^T\boldsymbol{\epsilon} = \sum_{i=1}^{\ell} 1/2\boldsymbol{\epsilon}_i^T\boldsymbol{\epsilon}_i = 1/2(\mathbf{y} - H\mathbf{x})^T(\mathbf{y} - H\mathbf{x}). \tag{4.3.2}$$

Note that Eq. (4.3.2) is a quadratic function of $\mathbf{x}$, and as a consequence the expression will have a unique minima when (see Appendix B, Eq. (B.8.2))

$$\frac{\partial J}{\partial \mathbf{x}} = 0, \quad \text{and} \quad \delta\mathbf{x}^T \frac{\partial^2 J}{\partial \mathbf{x}^2} \delta\mathbf{x} > 0$$

for all $\delta\mathbf{x} \neq 0$. The second condition implies that the symmetric matrix

$$\frac{\partial^2 J}{\partial \mathbf{x}^2}$$

is positive definite.

Carrying out the first operation on Eq. (4.3.2) (see Appendix B, Eq. (B.7.3)) yields

$$\frac{\partial J}{\partial \mathbf{x}} = 0 = -(\mathbf{y} - H\mathbf{x})^T H = -H^T(\mathbf{y} - H\mathbf{x}). \tag{4.3.3}$$

The value of $\mathbf{x}$ that satisfies Eq. (4.3.3) will be the best estimate of $\mathbf{x}$, which we will call $\hat{\mathbf{x}}$. Hence,

$$(H^T H)\hat{\mathbf{x}} = H^T \mathbf{y}. \tag{4.3.4}$$

Also, from Eqs. (4.3.3) and (B.7.3) it follows that

$$\frac{\partial^2 J}{\partial \mathbf{x}^2} = H^T H \tag{4.3.5}$$

which will be positive definite if H is full rank.

Equation (4.3.4) is referred to as the normal equation, and $(H^T H)$ is referred to as the *normal matrix*. Note that the matrix $H^T H$ is an $n \times n$ symmetric matrix, and if this matrix is positive definite (H is rank n) then the solution for the best estimate of $\mathbf{x}$ is given by

$$\hat{\mathbf{x}} = (H^T H)^{-1} H^T \mathbf{y}. \tag{4.3.6}$$

Equation (4.3.6) is the well-known least squares solution for the best estimate of $\mathbf{x}$ given the linear observation state relationship expressed by Eq. (4.2.39). With the observations, $\mathbf{y}$, and a specified value of $\hat{\mathbf{x}}$, the value for the best estimate of the observation errors, $\hat{\epsilon}$, can be computed from Eq. (4.2.39) as

$$\hat{\epsilon} = \mathbf{y} - H\hat{\mathbf{x}}. \tag{4.3.7}$$

4.3.1 The Minimum Norm Solution

For the solution given by Eq. (4.3.6) to exist, it is required that $m \geq n$ and H have rank n. Consider the case where $m < n$; that is, H is of rank $< n$.

In other words, there are more unknowns than linearly independent observations. One could choose to specify any $n - m$ of the n components of $\mathbf{x}$ and solve for the remaining m components of $\mathbf{x}$ using the observation equations with $\epsilon = 0$. However, this leads to an infinite number of solutions for $\hat{\mathbf{x}}$. As an alternative, to obtain a unique solution, one can use the minimum norm criterion to determine $\hat{\mathbf{x}}$. Generally, a nominal or initial guess for $\mathbf{x}$ exists. Recall that the differential equations have been linearized and $\mathbf{x} = \mathbf{X} - \mathbf{X}^*$. The *minimum norm* criterion chooses $\mathbf{x}$ to minimize the sum of the squares of the difference between $\mathbf{X}$ and $\mathbf{X}^*$, subject to the constraint that $\epsilon = 0$; that is, $\mathbf{y} = H\mathbf{x}$. Hence, the performance index becomes

$$J(\mathbf{x}, \boldsymbol{\lambda}) = 1/2\mathbf{x}^T\mathbf{x} + \boldsymbol{\lambda}^T(\mathbf{y} - H\mathbf{x}) \tag{4.3.8}$$

where the constraint has been adjoined with an m-dimensional vector of Lagrange multipliers (see Appendix B, Section B.8). Since both $\mathbf{x}$ and $\boldsymbol{\lambda}$ are unknown, the necessary condition for a minimum of $J(\mathbf{x}, \boldsymbol{\lambda})$ is that its derivative with respect to $\mathbf{x}$ and $\boldsymbol{\lambda}$ vanish. This leads to

$$\frac{\partial J(\mathbf{x}, \boldsymbol{\lambda})}{\partial \mathbf{x}} = 0 = \mathbf{x} - H^T\boldsymbol{\lambda} \tag{4.3.9}$$

$$\frac{\partial J(\mathbf{x}, \boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}} = 0 = \mathbf{y} - H\mathbf{x}. \tag{4.3.10}$$

Substituting the expression for $\hat{\mathbf{x}}$ from Eq. (4.3.9) into Eq. (4.3.10) yields

$$\mathbf{y} = HH^T\boldsymbol{\lambda}, \tag{4.3.11}$$

and solving for $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda} = (HH^T)^{-1}\mathbf{y}. \tag{4.3.12}$$

Substituting Eq. (4.3.12) into Eq. (4.3.9) yields

$$\hat{\mathbf{x}} = H^T(HH^T)^{-1}\mathbf{y} \tag{4.3.13}$$

where HH^T is an $m \times m$ matrix of rank m. The quantities $H^T(HH^T)^{-1}$ of Eq. (4.3.13) and $(H^TH)^{-1}H^T$ of Eq. (4.3.6) are called the *pseudo inverses* of H in the equation $H\hat{\mathbf{x}} = \mathbf{y}$. They apply when there are more unknowns than equations or more equations than unknowns, respectively. Eq. (4.3.13) is the solution for $\mathbf{x}$ of minimum length. In summary,

$$\begin{aligned} \hat{\mathbf{x}} &= (H^TH)^{-1}H^T\mathbf{y}, && \text{if } m > n \\ \hat{\mathbf{x}} &= H^{-1}\mathbf{y}, && \text{if } m = n \\ \hat{\mathbf{x}} &= H^T(HH^T)^{-1}\mathbf{y}, && \text{if } m < n. \end{aligned} \tag{4.3.14}$$

4.3.2 SHORTCOMINGS OF THE LEAST SQUARES SOLUTION

Three major shortcomings of the simple least squares solution are:

1. Each observation error is weighted equally even though the accuracy of observations may differ.

2. The observation errors may be correlated (not independent), and the simple least squares solution makes no allowance for this.

3. The method does not consider that the errors are samples from a random process and makes no attempt to utilize any statistical information.

The first of these objections is overcome through the use of the weighted least squares approach.

4.3.3 WEIGHTED LEAST SQUARES SOLUTION

Equation (4.3.14) has no means of preferentially ordering one observation with respect to another. A more general expression can be obtained by considering the following formulation. Given a vector sequence of observations $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_\ell$ related through the state transition matrix to the state at some epoch time, $\mathbf{x}_k$, and an associated weighting matrix, w_i, for each of the observation vectors, one can write

$$\begin{aligned} \mathbf{y}_1 &= H_1\mathbf{x}_k + \boldsymbol{\epsilon}_1; \qquad w_1 \\ \mathbf{y}_2 &= H_2\mathbf{x}_k + \boldsymbol{\epsilon}_2; \qquad w_2 \\ &\vdots \quad \vdots \quad \vdots \\ \mathbf{y}_\ell &= H_\ell\mathbf{x}_k + \boldsymbol{\epsilon}_\ell; \qquad w_\ell \end{aligned} \tag{4.3.15}$$

where

$$H_i = \widetilde{H}_i\Phi(t_i, t_k).$$

In Eq. (4.3.15) the weighting matrices, w_i, are assumed to be diagonal with their elements normalized to a range between zero and one. Observations weighted with a one would be given the highest possible weight and those weighted with zero would be neglected. To reduce Eq. (4.3.15) to an expression similar to

(4.2.14), the following definitions can be used:

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_\ell \end{bmatrix}; \quad H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_\ell \end{bmatrix};$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \boldsymbol{\epsilon}_1 \\ \boldsymbol{\epsilon}_2 \\ \vdots \\ \boldsymbol{\epsilon}_\ell \end{bmatrix}; \quad W = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & w_\ell \end{bmatrix}. \tag{4.3.16}$$

Each observation $\mathbf{y}_i$ is assumed to be a p-vector and $\mathbf{x}_k$ is an n-vector. Equation (4.3.15) now can be expressed as

$$\mathbf{y} = H\mathbf{x}_k + \boldsymbol{\epsilon}; \quad W. \tag{4.3.17}$$

One can then pose the weighted least squares problem as follows. Given the linear observation state relationship expressed by (4.3.17), find the estimate of $\mathbf{x}_k$ to minimize the weighted sum of the squares of the calculated observation errors. The performance index is

$$J(\mathbf{x}_k) = 1/2\boldsymbol{\epsilon}^T W \boldsymbol{\epsilon} = \sum_{i=1}^{\ell} 1/2\boldsymbol{\epsilon}_i^T w_i \boldsymbol{\epsilon}_i. \tag{4.3.18}$$

Using Eq. (4.3.17), $J(\mathbf{x}_k)$ can be expressed as

$$J(\mathbf{x}_k) = 1/2(\mathbf{y} - H\mathbf{x}_k)^T W(\mathbf{y} - H\mathbf{x}_k). \tag{4.3.19}$$

A necessary condition for a minimum of $J(\mathbf{x}_k)$ is that its first derivative with respect to $\mathbf{x}_k$ vanishes (see Eq. B.7.4),

$$\frac{\partial J}{\partial \mathbf{x}_k} = 0 = -(y - H\mathbf{x}_k)^T WH = -H^T W(\mathbf{y} - H\mathbf{x}_k). \tag{4.3.20}$$

This expression can be rearranged to obtain the normal equations analogous to Eq. (4.3.6) in the least squares formulation as

$$(H^T WH)\mathbf{x}_k = H^T W\mathbf{y}. \tag{4.3.21}$$

If the normal matrix $H^T WH$ is positive definite, it will have an inverse and the solution to (4.3.21) is

$$\hat{\mathbf{x}}_k = (H^T WH)^{-1} H^T W\mathbf{y}. \tag{4.3.22}$$

The value of $\hat{\mathbf{x}}_k$ given by Eq. (4.3.22) is the weighted least squares estimate and is the estimate that minimizes the sum of squares of the weighted observation errors. Note that Eq. (4.3.22) can be expressed as

$$\hat{\mathbf{x}}_k = P_k H^T W \mathbf{y},$$

where

$$P_k = (H^T W H)^{-1}. \tag{4.3.23}$$

The $n \times n$ matrix P_k is symmetric, as can be seen from the definition. Furthermore, if it exists, it must be positive definite, since it is computed as the inverse of the positive definite matrix, $H^T W H$. The parameter observability is related to the rank of this matrix. If all the parameters in $\mathbf{x}_k$ are observable (i.e., can be uniquely determined with the observation set $\mathbf{y}$), then P_k will be full rank and P_k will have an inverse. The number of independent observations must be greater than or equal to the number of parameters being estimated if P_k is to be invertible. Furthermore, P_k is related to the accuracy of the estimate, $\hat{\mathbf{x}}_k$. In general, the larger the magnitude of the elements of the matrix, P_k, the less accurate the estimate. Since the weighting matrix, W, usually results from an initial judgment on the accuracy of the observations followed by a normalization procedure to scale the weights to values between zero and one, this interpretation is not strictly valid in the statistical sense. Hence, some caution should be used when attempting to infer the accuracy of an estimate from the magnitude of P_k as obtained in the weighted least squares estimate. In Section 4.4, it will be shown that, with proper selection of W, P_k is the variance-covariance matrix of the estimation error associated with $\hat{\mathbf{x}}_k$.

If an *a priori* value, $\bar{\mathbf{x}}_k$, is available for $\mathbf{x}_k$ and an associated weighting matrix, $\overline{W}_k$, is given, the weighted least squares estimate for $\mathbf{x}_k$ can be obtained by choosing for $\hat{\mathbf{x}}_k$ the value of $\mathbf{x}_k$, which minimizes the performance index

$$J(\mathbf{x}_k) = 1/2(\mathbf{y} - H\mathbf{x}_k)^T W (\mathbf{y} - H\mathbf{x}_k) + 1/2(\bar{\mathbf{x}}_k - \mathbf{x}_k)^T \overline{W}_k (\bar{\mathbf{x}}_k - \mathbf{x}_k). \tag{4.3.24}$$

This results in

$$\hat{\mathbf{x}}_k = (H^T W H + \overline{W}_k)^{-1} (H^T W \mathbf{y} + \overline{W}_k \bar{\mathbf{x}}_k). \tag{4.3.25}$$

Here $\bar{\mathbf{x}}_k$ represents an *a priori* estimate of $\mathbf{x}_k$ and $\overline{W}_k$ represents a weighting matrix for the *a priori* estimate of $\mathbf{x}_k$. In Section 4.4 these terms will be introduced in terms of their statistical significance.

4.3.4 An Alternate Least Squares Approach

A somewhat classic approach to the problem of least squares that was introduced in Section 1.2.3 is described in this section. Assume that we have ℓ scalar

observations. The following least squares performance index, or cost function, can be defined:

$$J = \sum_{j=1}^{\ell} (O_j - C_j)^2, \tag{4.3.26}$$

where O_j is the observation, such as range, provided by an instrument; C_j is the *computed observation* using the reference value of $\mathbf{X}_j^*$; and ℓ is the number of observations. C_j is computed from the appropriate observation-state model for the measurement, and is based on a reference set of coordinates for the instrument and for the satellite. In classic terminology, $(O_j - C_j)$ is referred to as *O minus C* and represents an *observation residual.* The coordinates of the satellite are available as the solution to the equations of motion; for example, Eq. (2.3.31) with a specified set of initial conditions, represented by the position and velocity of the satellite at time t_0. The state vector at t_0, $\mathbf{X}(t_0)$, represents the initial position and velocity, specifically

$$\mathbf{X}(t_0) = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ \dot{X}_0 \\ \dot{Y}_0 \\ \dot{Z}_0 \end{bmatrix}. \tag{4.3.27}$$

As in the traditional least squares problem, the goal is to determine the initial conditions that minimize J in Eq. (4.3.26). Thus, the partial derivatives with respect to $\mathbf{X}(t_0)$ must be zero:

$$\begin{aligned} \frac{\partial J}{\partial X_0} &= 0 = \sum_{j=1}^{\ell} 2(O_j - C_j)\left(-\frac{\partial C_j}{\partial X_0}\right) \\ &\vdots \\ \frac{\partial J}{\partial \dot{Z}_0} &= 0 = \sum_{j=1}^{\ell} 2(O_j - C_j)\left(-\frac{\partial C_j}{\partial \dot{Z}_0}\right). \end{aligned} \tag{4.3.28}$$

These nonlinear algebraic equations can be written as

$$\begin{aligned} F_1 &= \sum_{j=1}^{\ell} (O_j - C_j)\left(\frac{\partial C_j}{\partial X_0}\right) \\ &\vdots \end{aligned} \tag{4.3.29}$$

$$F_6 = \sum_{j=1}^{\ell} (O_j - C_j) \left(\frac{\partial C_j}{\partial \dot{Z}_0} \right).$$

From Eqs. (4.3.28) and (4.3.29), the following vector equation is defined:

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = 0. \tag{4.3.30}$$

The problem is to solve these equations for the value of $\mathbf{X}(t_0)$, which produces the condition that $\mathbf{F} = 0$. Eq. (4.3.30) represents a set of nonlinear algebraic equations in the n unknown components of $\mathbf{X}(t_0)$.

The solution to a set of nonlinear algebraic equations can be accomplished with the well-known Newton–Raphson method, which usually is derived for a single equation. As shown by Dahlquist and Björck (1974), a nonlinear set of equations can be solved in a recursive manner by

$$\mathbf{X}(t_0)_{n+1} = \mathbf{X}(t_0)_n - \left[\frac{\partial \mathbf{F}}{\partial \mathbf{X}} \right]_n^{-1} [\mathbf{F}]_n, \tag{4.3.31}$$

where

$$\frac{\partial \mathbf{F}}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial F_1}{\partial X_0} & \frac{\partial F_1}{\partial Y_0} & \frac{\partial F_1}{\partial Z_0} & \frac{\partial F_1}{\partial \dot{X}_0} & \frac{\partial F_1}{\partial \dot{Y}_0} & \frac{\partial F_1}{\partial \dot{Z}_0} \\ \vdots & & & & & \vdots \\ \frac{\partial F_6}{\partial X_0} & \frac{\partial F_6}{\partial Y_0} & \frac{\partial F_6}{\partial Z_0} & \frac{\partial F_6}{\partial \dot{X}_0} & \frac{\partial F_6}{\partial \dot{Y}_0} & \frac{\partial F_6}{\partial \dot{Z}_0} \end{bmatrix} \tag{4.3.32}$$

and where n represents the iteration number. It is evident that the method requires an initial guess corresponding to $n = 0$. Consider, for example,

$$\frac{\partial F_1}{\partial X_0} = \sum_{j=1}^{\ell} \left[\left(-\frac{\partial C_j}{\partial X_0} \right) \left(\frac{\partial C_j}{\partial X_0} \right) + (O_j - C_j) \left(\frac{\partial^2 C_j}{\partial X_0^2} \right) \right]$$

and

$$\frac{\partial F_1}{\partial Y_0} = \sum_{j=1}^{\ell} \left[\left(-\frac{\partial C_j}{\partial Y_0} \right) \left(\frac{\partial C_j}{\partial X_0} \right) + (O_j - C_j) \left(\frac{\partial^2 C_j}{\partial X_0 \partial Y_0} \right) \right]. \tag{4.3.33}$$

In a converged solution, or nearly converged case, the $(O_j - C_j)$ can be expected to be small, thereby providing a reasonable justification for ignoring the terms involving the second partial derivative of C_j.

The solution to these nonlinear equations, $\mathbf{X}(t_0)$, can then be written as

$$\mathbf{X}(t_0)_{n+1} = \tag{4.3.34}$$

$$\mathbf{X}(t_0)_n + \begin{bmatrix} \sum_j \left(\frac{\partial C_j}{\partial X_0}\right)_n^2 & \cdots & \sum_j \left(\frac{\partial C_j}{\partial X_0}\right)_n \left(\frac{\partial C_j}{\partial \dot{Z}_0}\right)_n \\ \vdots & & \\ \sum_j \left(\frac{\partial C_j}{\partial X_0}\right)_n \left(\frac{\partial C_j}{\partial \dot{Z}_0}\right)_n & \cdots & \sum_j \left(\frac{\partial C_j}{\partial \dot{Z}_0}\right)_n^2 \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \sum_j \left(\frac{\partial C_j}{\partial X_0}\right)_n (O_j - C_j)_n \\ \vdots \\ \sum_j \left(\frac{\partial C_j}{\partial \dot{Z}_0}\right)_n (O_j - C_j)_n \end{bmatrix}$$

where the terms involving the second partial derivatives have been ignored. Note that, for example,

$$\frac{\partial C_j}{\partial X_0} = \frac{\partial C_j}{\partial X}\frac{\partial X}{\partial X_0} + \frac{\partial C_j}{\partial Y}\frac{\partial Y}{\partial X_0} + \cdots + \frac{\partial C_j}{\partial \dot{Z}}\frac{\partial \dot{Z}}{\partial X_0}$$

and it can be readily shown from

$$H = \widetilde{H}\Phi$$

that Eq. (4.3.31) is

$$\mathbf{X}(t_0)_{n+1} = \mathbf{X}(t_0)_n + (H_n^T H_n)^{-1} H_n^T \mathbf{y}_n \tag{4.3.35}$$

where $\mathbf{y}_n$ represents a vector of residuals $(\mathbf{O} - \mathbf{C})_n$ where $\mathbf{O}$ and $\mathbf{C}$ correspond to $\mathbf{Y}$ and $\mathbf{Y}^*$ in the notation used in Section 4.2. Furthermore, if

$$\hat{\mathbf{x}}_{n+1} = \mathbf{X}(t_0)_{n+1} - \mathbf{X}(t_0)_n, \tag{4.3.36}$$

it follows that

$$\hat{\mathbf{x}}_{n+1} = (H_n^T H_n)^{-1} H_n^T \mathbf{y}_n \tag{4.3.37}$$

corresponding to Eq. (4.3.6).

Some further conclusions can be drawn using this approach:

(1) For the orbit determination application, the least squares solution should be iterated, especially when the reference trajectory has significant deviations from the true trajectory.

(2) Ignoring the second partial derivatives in Eq. (4.3.33) may influence the convergence process if the deviations are large.

(3) Since the process is based on the Newton–Raphson method, it will exhibit *quadratic convergence* when near to the solution, namely

$$||\hat{\mathbf{x}}_n|| < ||\hat{\mathbf{x}}_{n-1}||^2.$$

(4) The least squares formulation allows *accumulation* in a sequential manner as illustrated by the summations in Eq. (4.3.33). That is, the method can be formulated to accumulate the measurements sequentially, followed by a matrix inversion or linear system solution.

(5) The iterative process for a specified set of observations can be repeated until, for example, $||\hat{\mathbf{x}}_{n+1}||$ is smaller than some convergence criteria.

Example 4.3.1

Section 3.6 illustrated the range residuals that would be obtained from two stations (Easter Island and Fortaleza, Brazil) for a specified error in the initial conditions of the two-body problem. The orbit error is illustrated in Fig. 2.2.9. For this case, the satellite is observed twice by Easter Island and once by Fortaleza in the time interval shown in Fig. 2.2.9, ignoring one pass from Fortaleza that rises less than one degree in elevation above the horizon. The true (and presumably unknown) initial conditions are given by Example 2.2.6.2 and the nominal or reference initial conditions (presumably known) are given by Example 2.2.4.1.

Using the algorithm described in this section for the three passes, assuming that error-free geometric ranges are collected every 20 seconds when the satellite is above the station's horizon, the results for the state corrections are shown in Table 4.3.1. To obtain these results, the Goodyear state transition matrix (Section 2.2.6) was used, but a numerically integrated state transition matrix, as given by Eqs. (4.2.1) and (4.2.10), could have been used. Note that the initial state being determined is expressed in the nonrotating system, so the partial derivatives comprising H must be found in that system. To form these partial derivatives, the station coordinates must be transformed from Earth-fixed to nonrotating. It is apparent from Table 4.3.1 that the method converges rapidly, since the error that exists after the first iteration is small (see Fig. 2.2.9).

Table 4.3.1:

Estimated Corrections[1]

	Iteration Number			Converged State[2]	
$\hat{\mathbf{x}}$	1	2	3	$(\mathbf{X}(t_0))$	
$\hat{x}_0$	0.808885	0.000566	0.000000	X_0	5492001.14945 m
$\hat{y}_0$	0.586653	0.000536	0.000000	Y_0	3984001.98719 m
$\hat{z}_0$	0.000015	0.000425	0.000000	Z_0	2955.81044 m
$\hat{\dot{x}}_0$	0.000000	0.000000	0.000000	$\dot{X}_0$	−3931.046491 m/sec
$\hat{\dot{y}}_0$	0.000000	0.000000	0.000000	$\dot{Y}_0$	5498.676921 m/sec
$\hat{\dot{z}}_0$	0.000000	0.000000	0.000000	$\dot{Z}_0$	3665.980697 m/sec

[1] Data: three passes described in Section 3.6

[2] Compare to Example 2.2.6.2

If only the first pass from Easter Island is used instead of three passes, a dramatically different result is obtained. With 33 observations, the correction on the first iteration is at the 100-meter level in position and the 0.1-m/sec level in velocity. The subsequent iteration produces corrections at a comparable level, but numerical instabilities are encountered on the third iteration. Close examination shows that $(H^T H)$ in Eq. (4.3.37) is ill conditioned. Explanation of this behavior is discussed in Section 4.12.

4.4 THE MINIMUM VARIANCE ESTIMATE

As noted, the least squares and weighted least squares methods do not include any information on the statistical characteristics of the measurement errors or the *a priori* errors in the values of the parameters to be estimated. The minimum variance approach is one method for removing this limitation. The minimum variance criterion is used widely in developing solutions to estimation problems because of the simplicity in its use. It has the advantage that the complete statistical description of the random errors in the problem is not required. Rather, only the first and second moments of the probability density function of the observation errors are required. This information is expressed in the mean and covariance matrix associated with the random error.

If it is assumed that the observation error ϵ_i is random with zero mean and specified covariance, the state estimation problem can be formulated as follows:

Given: The system of state-propagation equations and observation state equations

$$\mathbf{x}_i = \Phi(t_i, t_k)\mathbf{x}_k \tag{4.4.1}$$

$$\mathbf{y}_i = \widetilde{H}_i\mathbf{x}_i + \boldsymbol{\epsilon}_i \quad i = 1, \ldots, \ell. \tag{4.4.2}$$

Find: The linear, unbiased, minimum variance estimate, $\hat{\mathbf{x}}_k$, of the state $\mathbf{x}_k$.

The solution to this problem proceeds as follows. Using the state transition matrix and the definitions of Eq. (4.3.16), reduce Eq. (4.4.2) to the following form

$$\mathbf{y} = H\mathbf{x}_k + \boldsymbol{\epsilon} \tag{4.4.3}$$

where

$$E[\boldsymbol{\epsilon}] = \begin{bmatrix} E[\epsilon_1] \\ E[\epsilon_2] \\ \vdots \\ E[\epsilon_\ell] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad E[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T] = \begin{bmatrix} R_{11}R_{12} & \cdots & R_{1\ell} \\ R_{12}^T R_{22} & \cdots & R_{2\ell} \\ \vdots & \ddots & \vdots \\ R_{1\ell}^T & \cdots & R_{\ell\ell} \end{bmatrix} = R. \tag{4.4.4}$$

Generally, $R_{11} = R_{22} = \ldots = R_{\ell\ell}$ and $R_{ij} = 0$ $(i \neq j)$, but this is not a necessary restriction in the following argument. $R_{ij} \neq 0$ $(i \neq j)$ corresponds to the more general case of time-correlated observation errors.

From the problem statement, the estimate is to be the best linear, unbiased, minimum variance estimate. The consequences of each of these requirements are addressed in the following steps.

(1) *Linear:* The requirement of a linear estimate implies that the estimate is to be made up of a linear combination of the observations:

$$\hat{\mathbf{x}}_k = M\mathbf{y}. \tag{4.4.5}$$

The $(n \times m)$ matrix M is unspecified and is to be selected to obtain the best estimate.

(2) *Unbiased:* If the estimate is unbiased, then by definition

$$E[\hat{\mathbf{x}}] = \mathbf{x}. \tag{4.4.6}$$

Substituting Eqs. (4.4.5) and (4.4.3) into Eq. (4.4.6) leads to the following requirement:

$$E[\hat{\mathbf{x}}_k] = E[M\mathbf{y}] = E[MH\mathbf{x}_k + M\boldsymbol{\epsilon}] = \mathbf{x}_k.$$

But, since $E[\epsilon] = 0$, this reduces to

$$MH\mathbf{x}_k = \mathbf{x}_k$$

from which the following constraint on M is obtained

$$MH = I. \tag{4.4.7}$$

That is, if the estimate is to be unbiased, the linear mapping matrix M must satisfy Eq. (4.4.7). This condition requires the rows of M to be orthogonal to the columns of H.

(3) *Minimum Variance:* If the estimate is unbiased, then the *estimation error covariance matrix* can be expressed as (see Appendix A)

$$\begin{aligned} P_k &= E\left\{[(\hat{\mathbf{x}}_k - \mathbf{x}_k) - E(\hat{\mathbf{x}}_k - \mathbf{x}_k)][(\hat{\mathbf{x}}_k - \mathbf{x}_k) - E(\hat{\mathbf{x}}_k - \mathbf{x}_k)]^T\right\} \\ &= E[(\hat{\mathbf{x}}_k - \mathbf{x}_k)(\hat{\mathbf{x}}_k - \mathbf{x}_k)^T]. \end{aligned} \tag{4.4.8}$$

Hence, the problem statement requires that $\hat{\mathbf{x}}_k$ be selected to minimize P_k while satisfying Eqs. (4.4.6) and (4.4.7). By minimizing P_k, we mean that $P_k^* - P_k$ is nonnegative definite for any P_k^* that results from an M that satisfies Eq. (4.4.7) (Deutsch, 1965). Substituting Eqs. (4.4.5) and (4.4.3) into Eq. (4.4.8) leads to the following result:

$$\begin{aligned} P_k &= E[(M\mathbf{y} - \mathbf{x}_k)(M\mathbf{y} - \mathbf{x}_k)^T] \\ &= E[\{M(H\mathbf{x}_k + \epsilon) - \mathbf{x}_k\}\{M(H\mathbf{x}_k + \epsilon) - \mathbf{x}_k\}^T] \\ &= E[M\epsilon\epsilon^T M^T] \end{aligned}$$

where we have used $MH = I$. It follows from Eq. (4.4.4) that the covariance matrix can be written as

$$P_k = MRM^T \tag{4.4.9}$$

where M is to be selected to satisfy Eq. (4.4.7). To involve the constraint imposed by Eq. (4.4.7) and to keep the constrained relation for P_k symmetric, Eq. (4.4.7) is adjoined to Eq. (4.4.9) in the following form

$$P_k = MRM^T + \Lambda^T(I - MH)^T + (I - MH)\Lambda \tag{4.4.10}$$

where Λ is a $n \times n$ matrix of unspecified Lagrange multipliers. The final term is added to ensure that P_k remains symmetric. For a minimum of P_k, it is necessary that its first variation with respect to M vanish, and that $I - MH = 0$. Accordingly,

$$\delta P_k = 0 = (MR - \Lambda^T H^T)\delta M^T + \delta M(RM^T - H\Lambda). \tag{4.4.11}$$

Now, if δP_k is to vanish for an arbitrary δM, one of the following conditions must be met:

1. $RM^T - H\Lambda = 0$.
2. δM and/or $RM^T - H\Lambda$ must not be of full rank.

We will impose condition 1 and show that this yields a minimum value of P. Hence, it is required that

$$MR - \Lambda^T H^T = 0, \quad I - MH = 0. \tag{4.4.12}$$

From the first of these conditions

$$M = \Lambda^T H^T R^{-1} \tag{4.4.13}$$

since R is assumed to be positive definite. Substituting Eq. (4.4.13) into the second of Eqs. (4.4.12) leads to the following result

$$\Lambda^T (H^T R^{-1} H) = I. \tag{4.4.14}$$

Now, if the matrix $H^T R^{-1} H$ is full rank, which requires that $m \geq n$, then the inverse matrix will exist and

$$\Lambda^T = (H^T R^{-1} H)^{-1}. \tag{4.4.15}$$

Then, in view of Eq. (4.4.13),

$$M = (H^T R^{-1} H)^{-1} H^T R^{-1}. \tag{4.4.16}$$

This is the value of M that satisfies the unbiased and minimum variance requirements. Substitution of Eq. (4.4.16) into Eq. (4.4.9) leads to the following expression for the covariance matrix:

$$P_k = (H^T R^{-1} H)^{-1}. \tag{4.4.17}$$

With Eqs. (4.4.16) and (4.4.5), the linear unbiased minimum variance estimate of $\mathbf{x}_k$ is given as

$$\hat{\mathbf{x}}_k = (H^T R^{-1} H)^{-1} H^T R^{-1} \mathbf{y}. \tag{4.4.18}$$

It is not obvious that requiring $RM^T - H\Lambda = 0$ yields the minimum estimation error covariance matrix. We now demonstrate that $P_k^* - P_k$ is nonnegative definite, where P_k^* is the covariance matrix associated with any other linear unbiased estimator, $\tilde{\mathbf{x}}$. Without loss of generality let

$$\tilde{\mathbf{x}} = \hat{\mathbf{x}} + B\mathbf{y}.$$

Then

$$\begin{aligned} E[\tilde{\mathbf{x}}] &= E[\hat{\mathbf{x}}] + BE[\mathbf{y}] \\ &= \mathbf{x} + BH\mathbf{x}. \end{aligned}$$

Hence, $BH = 0$ in order for $\tilde{\mathbf{x}}$ to be unbiased. Since H is full rank, B cannot be full rank. We ignore the trivial solution $B = 0$.

Computing the estimation error covariance matrix associated with $\tilde{\mathbf{x}}$ yields

$$\begin{aligned} P_{\tilde{\mathbf{x}}} &= E\left[\left(\tilde{\mathbf{x}} - E(\tilde{\mathbf{x}})\right)\left(\tilde{\mathbf{x}} - E(\tilde{\mathbf{x}})\right)^T\right] \\ &= E\left[\left(\hat{\mathbf{x}} + B\mathbf{y} - \mathbf{x} - BH\mathbf{x}\right)\left(\hat{\mathbf{x}} + B\mathbf{y} - \mathbf{x} - BH\mathbf{x}\right)^T\right] \\ &= E\left[\left((\hat{\mathbf{x}} - \mathbf{x}) + B\boldsymbol{\epsilon}\right)\left((\hat{\mathbf{x}} - \mathbf{x}) + B\boldsymbol{\epsilon}\right)^T\right] \\ &= P + BRB^T + E\left[(\hat{\mathbf{x}} - \mathbf{x})\boldsymbol{\epsilon}^T\right]B^T + BE\left[\boldsymbol{\epsilon}(\hat{\mathbf{x}} - \mathbf{x})^T\right]. \end{aligned}$$

Also,

$$\begin{aligned} BE\left[\boldsymbol{\epsilon}(\hat{\mathbf{x}} - \mathbf{x})^T\right] &= BE\left[\boldsymbol{\epsilon}(PH^TR^{-1}\mathbf{y} - \mathbf{x})^T\right] \\ &= BE\left[\boldsymbol{\epsilon}(PH^TR^{-1}(H\mathbf{x} + \boldsymbol{\epsilon}) - \mathbf{x})^T\right] \\ &= BE(\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T)R^{-1}HP \\ &= BHP = 0, \end{aligned}$$

since $BH = 0$.

Hence,

$$P_{\tilde{\mathbf{x}}} - P = BRB^T$$

Because B is not of full rank, BRB^T must be positive semidefinite (i.e., its diagonal elements are each ≥ 0). Hence, the requirement that the difference of the two covariance matrices be nonnegative definite has been met. Consequently, all variances associated with $P_{\tilde{\mathbf{x}}}$ must be greater than or equal to those of P and the trace of $P_{\tilde{\mathbf{x}}}$ must be greater than that of P.

Note that computation of the estimate, $\hat{\mathbf{x}}_k$, requires inverting the $n \times n$ normal matrix, $H^TR^{-1}H$. For a large dimension system the computation of this inverse may be difficult. The solution given by Eq. (4.4.18) will agree with the weighted least squares solution if the weighting matrix, W, used in the least squares approach is equal to the inverse of the observation noise covariance matrix; that is, if $W = R^{-1}$.

4.4.1 Propagation of the Estimate and Covariance Matrix

If the estimate at a time t_j is obtained by using Eq. (4.4.18), the estimate may be mapped to any later time by using Eq. (4.4.2):

$$\overline{\mathbf{x}}_k = \Phi(t_k, t_j)\hat{\mathbf{x}}_j. \tag{4.4.19}$$

The expression for propagating the covariance matrix can be obtained as follows:

$$\overline{P}_k \equiv E[(\overline{\mathbf{x}}_k - \mathbf{x}_k)(\overline{\mathbf{x}}_k - \mathbf{x}_k)^T]. \tag{4.4.20}$$

In view of Eq. (4.4.19), Eq. (4.4.20) becomes

$$\overline{P}_k = E[\Phi(t_k, t_j)(\hat{\mathbf{x}}_j - \mathbf{x}_j)(\hat{\mathbf{x}}_j - \mathbf{x}_j)^T\Phi^T(t_k, t_j)]. \tag{4.4.21}$$

Since the state transition matrix is deterministic, it follows from Eq. (4.4.8) that

$$\overline{P}_k = \Phi(t_k, t_j)P_j\Phi^T(t_k, t_j). \tag{4.4.22}$$

Equations (4.4.19) and (4.4.22) can be used to map the estimate of the state and its associated covariance matrix from t_j to t_k.

4.4.2 Minimum Variance Estimate with *A Priori* Information

If an estimate and the associated covariance matrix are obtained at a time t_j, and an additional observation or observation sequence is obtained at a time t_k, the estimate and the observation can be combined in a straightforward manner to obtain the new estimate $\hat{\mathbf{x}}_k$. The estimate, $\hat{\mathbf{x}}_j$, and associated covariance, P_j, are propagated forward to t_k using Eqs. (4.4.19) and (4.4.22) and are given by

$$\overline{\mathbf{x}}_k = \Phi(t_k, t_j)\hat{\mathbf{x}}_j, \ \overline{P}_k = \Phi(t_k, t_j)P_j\Phi^T(t_k, t_j). \tag{4.4.23}$$

The problem to be considered can be stated as follows:

> *Given*: $\overline{\mathbf{x}}_k$, $\overline{P}_k$ and $\mathbf{y}_k = \widetilde{H}_k\mathbf{x}_k + \boldsymbol{\epsilon}_k$, where $E[\boldsymbol{\epsilon}_k] = 0$, $E[\boldsymbol{\epsilon}_k\boldsymbol{\epsilon}_j^T] = R_k\delta_{kj}$, and $E[(\bar{\mathbf{x}}_j - \mathbf{x}_j)\boldsymbol{\epsilon}_k^T] = 0$, find the linear, minimum variance, unbiased estimate of $\mathbf{x}_k$.

The solution to the problem can be obtained by reducing it to the previously solved problem. To this end, note that if $\hat{\mathbf{x}}_j$ is unbiased, $\overline{\mathbf{x}}_k$ will be unbiased since

$E[\overline{\mathbf{x}}_k] = \Phi(t_k, t_j)E[\hat{\mathbf{x}}_j] = \mathbf{x}_k$. Hence, $\overline{\mathbf{x}}_k$ can be interpreted as an unbiased observation of $\mathbf{x}_k$ and we may treat it as an additional data equation at t_k,

$$\begin{aligned} \mathbf{y}_k &= \widetilde{H}_k \mathbf{x}_k + \boldsymbol{\epsilon}_k \\ \overline{\mathbf{x}}_k &= \mathbf{x}_k + \boldsymbol{\eta}_k \end{aligned} \tag{4.4.24}$$

where

$$\begin{aligned} &E[\boldsymbol{\epsilon}_k] = 0, \quad E[\boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_k^T] = R_k, \quad E[\boldsymbol{\eta}_k] = 0, \\ &E[\boldsymbol{\eta}_k \boldsymbol{\epsilon}_k^T] = 0, \text{ and } \; E[\boldsymbol{\eta}_k \boldsymbol{\eta}_k^T] = \overline{P}_k. \end{aligned} \tag{4.4.25}$$

It is assumed that the errors in the observations, $\boldsymbol{\epsilon}_k$, are not correlated with the errors in the *a priori* estimate, $\boldsymbol{\eta}_k$. That is, $E[\boldsymbol{\eta}_k \boldsymbol{\epsilon}_k^T] = 0$. Now, if the following definitions are used

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} \mathbf{y}_k \\ \cdots \\ \overline{\mathbf{x}}_k \end{bmatrix}; H = \begin{bmatrix} \widetilde{H}_k \\ \cdots \\ I \end{bmatrix}; \\ \boldsymbol{\epsilon} &= \begin{bmatrix} \boldsymbol{\epsilon}_k \\ \cdots \\ \boldsymbol{\eta}_k \end{bmatrix}; R = \begin{bmatrix} R_k & 0 \\ & \cdots \\ 0 & \overline{P}_k \end{bmatrix}; \end{aligned} \tag{4.4.26}$$

Eq. (4.4.24) can be expressed as $\mathbf{y} = H\mathbf{x}_k + \boldsymbol{\epsilon}$ as in Eq. (4.4.3), and the solution for $\hat{\mathbf{x}}_k$ is given by Eq. (4.4.18),

$$\hat{\mathbf{x}}_k = (H^T R^{-1} H)^{-1} H^T R^{-1} \mathbf{y}. \tag{4.4.27}$$

In view of the definitions in Eq. (4.4.26),

$$\begin{aligned} \hat{\mathbf{x}}_k = &\left\{ [\widetilde{H}_k^T : I] \begin{bmatrix} R_k^{-1} & 0 \\ \cdots & \cdots \\ 0 & \overline{P}_k{}^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{H}_k \\ \cdots \\ I \end{bmatrix} \right\}^{-1} \\ &\left\{ [\widetilde{H}_k^T : I] \begin{bmatrix} R_k^{-1} & 0 \\ \cdots & \cdots \\ 0 & \overline{P}_k{}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_k \\ \cdots \\ \overline{\mathbf{x}}_k \end{bmatrix} \right\} \end{aligned} \tag{4.4.28}$$

or

$$\hat{\mathbf{x}}_k = (\widetilde{H}_k^T R_k^{-1} \widetilde{H}_k + \overline{P}_k{}^{-1})^{-1} (\widetilde{H}_k^T R_k^{-1} \mathbf{y}_k + \overline{P}_k{}^{-1} \overline{\mathbf{x}}_k). \tag{4.4.29}$$

Using Eq. (4.4.17) the covariance matrix associated with the estimation error in $\hat{\mathbf{x}}$ easily is shown to be

$$\begin{aligned} P_k &= E[(\hat{\mathbf{x}}_k - \mathbf{x}_k)(\hat{\mathbf{x}}_k - \mathbf{x}_k)^T] \\ &= (\widetilde{H}_k^T R_k^{-1} \widetilde{H}_k + \overline{P}_k{}^{-1})^{-1}. \end{aligned} \tag{4.4.30}$$

The inverse of the covariance matrix is called the *information matrix*,

$$\Lambda_k = P_k^{-1}. \tag{4.4.31}$$

Equation (4.4.29) often is seen written in normal equation form as

$$\Lambda_k \hat{\mathbf{x}}_k = \tilde{H}_k^T R_k^{-1} \mathbf{y}_k + \overline{P}_k^{-1} \bar{\mathbf{x}}_k. \tag{4.4.32}$$

The following remarks relate to Eq. (4.4.29):

1. The vector $\mathbf{y}_k$ may be only a single observation or it may include a batch of observations mapped to t_k.

2. The *a priori estimate*, $\bar{\mathbf{x}}_k$, may represent the estimate based on *a priori* initial conditions or the estimate based on the reduction of a previous batch of data.

3. The $n \times n$ normal matrix of Eq. (4.4.29) must be inverted and if the dimension n is large, this inversion can lead to computational problems. However, alternate solution techniques that avoid the accumulation and inversion of the normal matrix have been developed and are discussed in Chapter 5.

4. The algorithm for using Eq. (4.4.29) is referred to as the *batch processor*. The name derives from the fact that all data generally are accumulated prior to solving for $\hat{\mathbf{x}}_k$; that is, the data are processed in a single batch.

5. Note that Eq. (4.4.29) could also be implemented as a *sequential processor*; that is, after each observation the state estimate and covariance matrix could be mapped to the time of the next observation, where it would become the *a priori* information. This could then be combined with the observation at that time to yield the estimate for $\hat{\mathbf{x}}$ using Eq. (4.4.29).

4.5 MAXIMUM LIKELIHOOD AND BAYESIAN ESTIMATION

The method of *Maximum Likelihood Estimation* for determining the best estimate of a variable is due to Fisher (1912). The Maximum Likelihood Estimate (MLE) of a parameter Θ—given observations $y_1,\ y_2,\ \ldots y_k$ and the joint probability density function

$$f\,(y_1,\ y_2, \ldots\ y_k;\ \Theta) \tag{4.5.1}$$

is defined to be that value of Θ that maximizes the probability density function (Walpole and Myers, 1989). However, if Θ is a random variable and we have knowledge of its probability density function, the MLE of Θ is defined to be the

value of Θ, which maximizes the probability density function of Θ conditioned on knowledge of the observations $y_1, y_2, \ldots y_k$:

$$f(\Theta/y_1, y_2, \ldots y_k). \tag{4.5.2}$$

The *Bayes estimate* for Θ is defined to be the mean of the conditional density function given by Eq. (4.5.2) (Walpole and Myers, 1989). The joint density function, Eq. (4.5.1), and the conditional density function, Eq. (4.5.2), are referred to as the *likelihood function*, L. The logic behind maximizing L is that of all the possible values of Θ we should choose the one that maximizes the probability of obtaining the observations that actually were observed. If Θ is a random variable, this corresponds to the mode, or peak, of the conditional density function. In the case of a symmetric, unimodal, density function such as a Gaussian function, this will correspond to the mean of the conditional density function. Hence, the MLE and the Bayes estimate for a Gaussian density function are identical.

Since the logarithm of the density function is a monotonically increasing function of the density function, it is often simpler to determine the value of Θ that maximizes $\ln(L)$.

For example, assume we are given the following joint probability density function of the independent random variables $y_i, i = 1, 2 \ldots k$ with common mean, α, and common standard deviation, σ, and we wish to determine the MLE of the parameter α:

$$f(y_1, \ , y_2 \cdots y_k;\ \alpha, \sigma) = f(y_1; \alpha, \sigma) f(y_2; \alpha, \sigma) \cdots f(y_k; \alpha, \sigma). \tag{4.5.3}$$

We are able to factor the joint density function into the product of the marginal density functions because the random variables y_i are independent. If the joint density function is Gaussian, we may write (see Eq. (A.19.1))

$$L = f(y_1, y_2 \cdots y_k;\ \alpha, \sigma) = \frac{1}{(2\pi)^{k/2}\sigma^k} \exp\left\{ \frac{-1}{2\sigma^2} \sum_{i=1}^{k} (y_i - \alpha)^2 \right\} \tag{4.5.4}$$

then

$$\ln L = -\frac{k}{2} \ln 2\pi - k \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{k} (y_i - \alpha)^2 \tag{4.5.5}$$

and for a maximum

$$\frac{\partial \ln L}{\partial \alpha} = \frac{1}{\sigma^2} \sum_{i=1}^{k} (y_i - \alpha) = 0 \tag{4.5.6}$$

and

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{k}{\sigma^2} < 0.$$

Hence,

$$\sum_{i=1}^{k}(y_i - \hat{\alpha}) = 0 \tag{4.5.7}$$

and the MLE of α is

$$\hat{\alpha} = \frac{1}{k}\sum_{i=1}^{k} y_i. \tag{4.5.8}$$

In terms of the orbit determination problem, we are given observations $\mathbf{y}_1$, $\mathbf{y}_2 \ldots \mathbf{y}_k$ and we wish to determine the MLE of the state, $\hat{\mathbf{x}}$. Hence, we wish to find the value of the state vector, $\mathbf{x}_k$, which maximizes the conditional density function

$$f(\mathbf{x}_k/\mathbf{y}_1, \mathbf{y}_2 \ldots \mathbf{y}_k). \tag{4.5.9}$$

We will assume that all density functions for this derivation are Gaussian. Using the first of Eq. (A.21.1) of Appendix A, we may write

$$f(\mathbf{x}_k/\mathbf{y}_1, \mathbf{y}_2 \ldots \mathbf{y}_k) = \frac{f(\mathbf{x}_k, \mathbf{y}_1, \mathbf{y}_2 \ldots \mathbf{y}_k)}{f(\mathbf{y}_1, \mathbf{y}_2 \ldots \mathbf{y}_k)}. \tag{4.5.10}$$

Using the second of Eq. (A.21.1) yields

$$f(\mathbf{x}_k, \mathbf{y}_1, \mathbf{y}_2 \ldots \mathbf{y}_k) = f(\mathbf{y}_k/\mathbf{x}_k, \mathbf{y}_1, \ldots \mathbf{y}_{k-1}) f(\mathbf{x}_k, \mathbf{y}_1, \ldots \mathbf{y}_{k-1}). \tag{4.5.11}$$

Assuming independent observations results in

$$f(\mathbf{y}_k/\mathbf{x}_k, \mathbf{y}_1, \ldots \mathbf{y}_{k-1}) = f(\mathbf{y}_k/\mathbf{x}_k), \tag{4.5.12}$$

and again using the first of Eq. (A.21.1) we have

$$f(\mathbf{x}_k, \mathbf{y}_1, \ldots \mathbf{y}_{k-1}) = f(\mathbf{x}_k/\mathbf{y}_1, \ldots \mathbf{y}_{k-1}) f(\mathbf{y}_1, \ldots \mathbf{y}_{k-1}). \tag{4.5.13}$$

Hence, Eq. (4.5.11) may be written as

$$f(\mathbf{x}_k, \mathbf{y}_1, \mathbf{y}_2 \ldots \mathbf{y}_k) = f(\mathbf{y}_k/\mathbf{x}_k) f(\mathbf{x}_k/\mathbf{y}_1, \ldots \mathbf{y}_{k-1}) f(\mathbf{y}_1, \ldots \mathbf{y}_{k-1}). \tag{4.5.14}$$

Substituting Eq. (4.5.14) into Eq. (4.5.10) yields

$$f(\mathbf{x}_k/\mathbf{y}_1, \mathbf{y}_2 \ldots \mathbf{y}_k) = \frac{f(\mathbf{y}_k/\mathbf{x}_k) f(\mathbf{x}_k/\mathbf{y}_1, \ldots \mathbf{y}_{k-1})}{f(\mathbf{y}_k)}, \tag{4.5.15}$$

where we have used the fact that the observations, $\mathbf{y}_i$, are independent so we can write

$$f(\mathbf{y}_1, \mathbf{y}_2 \ldots \mathbf{y}_k) = f(\mathbf{y}_1) f(\mathbf{y}_2) \ldots f(\mathbf{y}_k). \tag{4.5.16}$$

For our system,

$$\mathbf{x}_k = \Phi(t_k,\ t_i)\mathbf{x}_i \tag{4.5.17}$$

$$\mathbf{y}_i = \widetilde{H}_i\,\mathbf{x}_i\ +\ \boldsymbol{\epsilon}_i, \qquad i = 1 \ldots k \tag{4.5.18}$$

where $\boldsymbol{\epsilon}_i \sim N(0, R_i)$; that is, $\boldsymbol{\epsilon}_i$ has a normal distribution with zero mean and covariance R_i. We are assuming independent observations; hence,

$$E\left[\boldsymbol{\epsilon}_j \boldsymbol{\epsilon}_k^T\right] = \delta_{jk}\, R_k. \tag{4.5.19}$$

We seek the MLE estimate of $\mathbf{x}_k$, the value of $\mathbf{x}_k$ that maximizes the conditional density function of Eq. (4.5.15). Note that $f(\mathbf{y}_k)$, the marginal density function of $\mathbf{y}_k$, is by definition independent of $\mathbf{x}_k$. Hence, only the numerator of Eq. (4.5.15) is dependent on $\mathbf{x}_k$. $\mathbf{y}_k/\mathbf{x}_k$ has mean

$$\begin{aligned} E(\mathbf{y}_k/\mathbf{x}_k) &= E\left[(\widetilde{H}_k\mathbf{x}_k + \boldsymbol{\epsilon}_k)/\mathbf{x}_k\right] \\ &= \widetilde{H}_k\mathbf{x}_k \end{aligned} \tag{4.5.20}$$

and covariance

$$\begin{aligned} &E\left[(\mathbf{y}_k - \widetilde{H}_k\mathbf{x}_k)(\mathbf{y}_k - \widetilde{H}_k\mathbf{x}_k)^T/\mathbf{x}_k\right] \\ &= E\left[\boldsymbol{\epsilon}_k\boldsymbol{\epsilon}_k^T\right] = R_k. \end{aligned} \tag{4.5.21}$$

Hence,

$$\mathbf{y}_k/\mathbf{x}_k \sim N(\widetilde{H}_k\mathbf{x}_k, R_k). \tag{4.5.22}$$

Also,

$$E\left[\mathbf{x}_k/\mathbf{y}_1,\ \mathbf{y}_2 \ldots \mathbf{y}_{k-1}\right] \equiv \overline{\mathbf{x}}_k. \tag{4.5.23}$$

The associated covariance is

$$E[(\mathbf{x}_k - \bar{\mathbf{x}}_k)(\mathbf{x}_k - \bar{\mathbf{x}}_k)^T] \equiv \overline{P}_k. \tag{4.5.24}$$

Hence,

$$\mathbf{x}_k/\mathbf{y}_1, \ldots \mathbf{y}_{k-1} \sim N(\overline{\mathbf{x}}_k, \overline{P}_k). \tag{4.5.25}$$

The *likelihood function* defined by Eq. (4.5.15) is given by

$$\begin{aligned} L = f(\mathbf{x}_k/\mathbf{y}_1,\ \mathbf{y}_2 \ldots \mathbf{y}_k) &= \frac{1}{(2\pi)^{p/2}|R_k|^{1/2}} \\ &\times \exp -1/2\left\{(\mathbf{y}_k - \widetilde{H}_k\mathbf{x}_k)^T R_k^{-1}(\mathbf{y}_k - \widetilde{H}_k\mathbf{x}_k)\right\} \\ &\times \frac{1}{(2\pi)^{n/2}|\overline{P}_k|^{1/2}} \exp -1/2\left\{(\mathbf{x}_k - \bar{\mathbf{x}}_k)^T \overline{P}_k^{-1}(\mathbf{x}_k - \bar{\mathbf{x}}_k)\right\} \frac{1}{f(\mathbf{y}_k)}. \end{aligned} \tag{4.5.26}$$

Because this is a Gaussian density function, the mean, median (having equal probability weight on either side), and mode will be identical. Hence, any criterion that chooses one of these values for $\hat{\mathbf{x}}_k$ will yield the same estimator.

Accumulating the logarithms of all terms in $f(\mathbf{x}_k/\mathbf{y}_1,\ \mathbf{y}_2 \ldots \mathbf{y}_k)$ that are a function of $\mathbf{x}_k$ and calling this function $\ln L'$ yields

$$\begin{aligned}\ln L' = -1/2[&(\mathbf{y}_k - \widetilde{H}_k\mathbf{x}_k)^T R_k^{-1}(\mathbf{y}_k - \widetilde{H}_k\mathbf{x}_k) \\ &+(\mathbf{x}_k - \overline{\mathbf{x}}_k)^T \overline{P}_k^{-1}(\mathbf{x}_k - \overline{\mathbf{x}}_k)].\end{aligned} \tag{4.5.27}$$

Differentiating $\ln L'$ with respect to $\mathbf{x}_k$,

$$\frac{\partial \ln L'}{\partial \mathbf{x}_k} = -\left[-(\widetilde{H}_k^T R_k^{-1})(\mathbf{y}_k - \widetilde{H}_k\mathbf{x}_k) + \overline{P}_k^{-1}(\mathbf{x}_k - \overline{\mathbf{x}}_k)\right] = 0. \tag{4.5.28}$$

The value of $\mathbf{x}_k$ that satisfies Eq. (4.5.28) is $\hat{\mathbf{x}}_k$:

$$\left(\widetilde{H}_k^T R_k^{-1}\widetilde{H}_k + \overline{P}_k^{-1}\right)\hat{\mathbf{x}}_k = H_k^T R_k^{-1}\mathbf{y}_k + \overline{P}^{-1}\overline{\mathbf{x}}_k \tag{4.5.29}$$

$$\hat{\mathbf{x}}_k = \left(\widetilde{H}_k^T R_k^{-1}\widetilde{H}_k + \overline{P}_k^{-1}\right)^{-1}\left(\widetilde{H}_k^T R_k^{-1}\mathbf{y}_k + \overline{P}_k^{-1}\overline{\mathbf{x}}_k\right) \tag{4.5.30}$$

which is identical to the minimum variance estimate of $\hat{\mathbf{x}}_k$ as well as the weighted least squares estimate if $W_k = R_k^{-1}$.

Furthermore, from Eq. (4.5.28),

$$\frac{\partial^2 \ln L'}{\partial \mathbf{x}^2} = -\left[\widetilde{H}_k^T R_k^{-1}\widetilde{H}_k + \overline{P}_k^{-1}\right]. \tag{4.5.31}$$

Because $-\frac{\partial^2 \ln L'}{\partial \mathbf{x}^2}$ is positive definite, we have maximized the likelihood function (see Appendix B).

4.6 COMPUTATIONAL ALGORITHM FOR THE BATCH PROCESSOR

Assume that we wish to estimate the state deviation vector $\mathbf{x}_0$ at a reference time, t_0. Given a set of initial conditions $\mathbf{X}^*(t_0)$, an *a priori* estimate $\overline{\mathbf{x}}_0$ and the associated error covariance matrix, $\overline{P}_0$, the computational algorithm for the batch processor generally uses the normal equation form for $\hat{\mathbf{x}}_0$. Writing Eqs. (4.3.25) and (4.4.29) in normal equation form for a batch of observations and recognizing in Eq. (4.3.25) that $W = R^{-1}$ and $\overline{W} = \overline{P}_0^{-1}$ yields

$$(H^T R^{-1} H + \overline{P}_0^{-1})\hat{\mathbf{x}}_0 = H^T R^{-1}\mathbf{y} + \overline{P}_0^{-1}\overline{\mathbf{x}}_0. \tag{4.6.1}$$

Here t_0 is an arbitrary epoch and all quantities in Eq. (4.6.1) are assumed to have been mapped to this epoch using the appropriate state transition matrices as illustrated in Eqs. (4.3.15) and (4.3.16). Because we are dealing with a linearized system, Eq. (4.6.1) generally is iterated to convergence; that is, until $\hat{\mathbf{x}}_0$ no longer changes. Note that the two matrices in Eq. (4.6.1) that must be accumulated are $H^T R^{-1} H$ and $H^T R^{-1}\mathbf{y}$. If R is a block diagonal matrix—the observations are uncorrelated in time although correlations between the observations at any given time may exist—these matrices simply may be accumulated as follows:

$$H^T R^{-1} H = \sum_{i=1}^{\ell} \left[\widetilde{H}_i \Phi(t_i, t_0)\right]^T R_i^{-1} \widetilde{H}_i \Phi(t_i, t_0) \tag{4.6.2}$$

$$H^T R^{-1}\mathbf{y} = \sum_{i=1}^{\ell} [\widetilde{H}_i \Phi(t_i, t_0)]^T R_i^{-1} \mathbf{y}_i. \tag{4.6.3}$$

In general $\mathbf{X}^*(t_0)$ would be chosen so that $\bar{\mathbf{x}}_0 = 0$, and $\overline{P}_0$ would reflect the relative accuracy of the elements of the initial condition vector $\mathbf{X}^*(t_0)$. In theory $\bar{\mathbf{x}}_0$ and $\bar{P}_0$ represent information and should be treated as data that are merged with the observation data, as indicated by Eq. (4.6.1). Consequently, the value of $\mathbf{X}_0^* + \bar{\mathbf{x}}_0$ should be held constant for the beginning of each iteration. Since the initial condition vector $\mathbf{X}_0^*$ is augmented by the value of $\hat{\mathbf{x}}_0$ after each iteration, that is, $(\mathbf{X}_0^*)_n = (\mathbf{X}_0^*)_{n-1} + (\hat{\mathbf{x}}_0)_n$, holding $\mathbf{X}_0^* + \bar{\mathbf{x}}_0$ constant results in the following expression for $(\bar{\mathbf{x}}_0)_n$

$$\begin{aligned}
\mathbf{X}_0^* + \bar{\mathbf{x}}_0 &= (\mathbf{X}_0^*)_{n-1} + (\bar{\mathbf{x}}_0)_{n-1} \\
&= (\mathbf{X}_0^*)_n + (\bar{\mathbf{x}}_0)_n \\
&= (\mathbf{X}_0^*)_{n-1} + (\hat{\mathbf{x}}_0)_{n-1} + (\bar{\mathbf{x}}_0)_n
\end{aligned}$$

or

$$(\bar{\mathbf{x}}_0)_n = (\bar{\mathbf{x}}_0)_{n-1} - (\hat{\mathbf{x}}_0)_{n-1}. \tag{4.6.4}$$

Recall from Section 4.2 that the state transition matrix is obtained by integrating

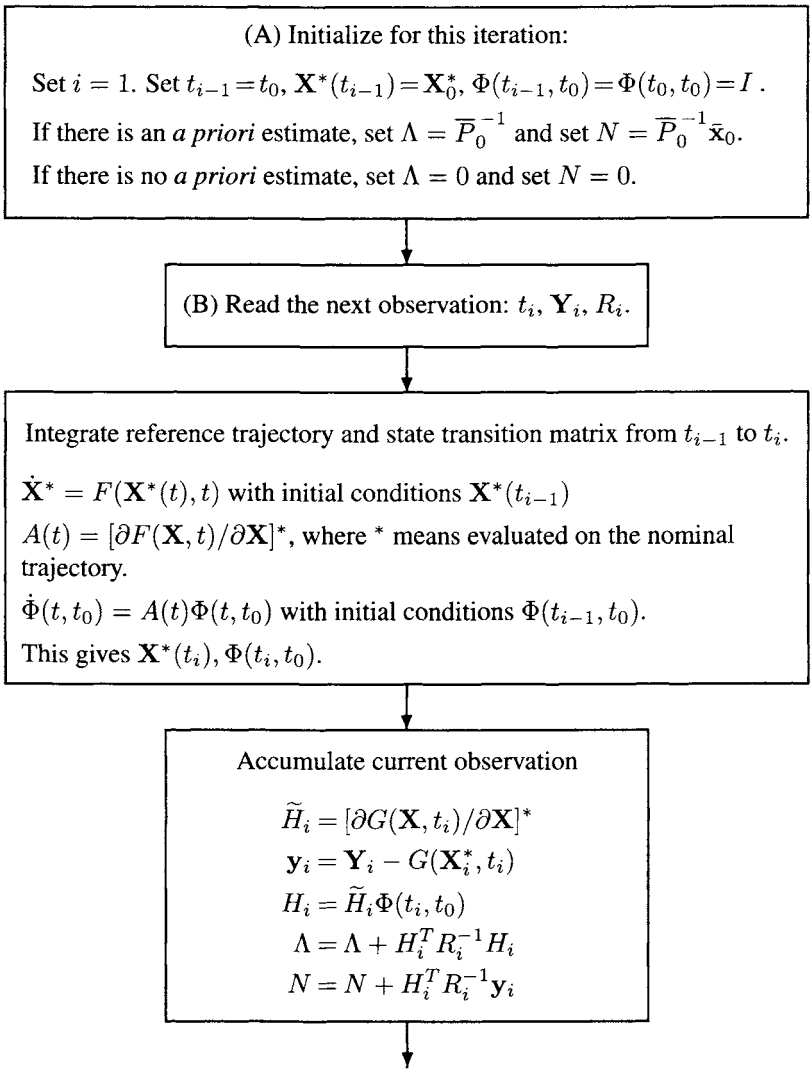
$$\dot{\Phi}(t, t_k) = A(t)\Phi(t, t_k)$$

subject to the initial conditions

$$\Phi(t_k, t_k) = I$$

along with the nonlinear equations, $\dot{\mathbf{X}}^* = F(\mathbf{X}^*, t)$, which define the nominal trajectory, $\mathbf{X}^*(t)$. The matrix $A(t)$ is evaluated on the reference trajectory,

$$A(t) = \frac{\partial F(\mathbf{X}^*, t)}{\partial \mathbf{X}} \tag{4.6.5}$$


(A) Initialize for this iteration:
Set $i = 1$. Set $t_{i-1} = t_0$, $\mathbf{X}^*(t_{i-1}) = \mathbf{X}_0^*$, $\Phi(t_{i-1}, t_0) = \Phi(t_0, t_0) = I$.
If there is an *a priori* estimate, set $\Lambda = \overline{P}_0^{-1}$ and set $N = \overline{P}_0^{-1}\bar{\mathbf{x}}_0$.
If there is no *a priori* estimate, set $\Lambda = 0$ and set $N = 0$.
(B) Read the next observation: t_i, $\mathbf{Y}_i$, R_i.
Integrate reference trajectory and state transition matrix from t_{i-1} to t_i.
$\dot{\mathbf{X}}^* = F(\mathbf{X}^*(t), t)$ with initial conditions $\mathbf{X}^*(t_{i-1})$
$A(t) = [\partial F(\mathbf{X}, t)/\partial \mathbf{X}]^*$, where * means evaluated on the nominal trajectory.
$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0)$ with initial conditions $\Phi(t_{i-1}, t_0)$.
This gives $\mathbf{X}^*(t_i), \Phi(t_i, t_0)$.
Accumulate current observation
$\widetilde{H}_i = [\partial G(\mathbf{X}, t_i)/\partial \mathbf{X}]^*$
$\mathbf{y}_i = \mathbf{Y}_i - G(\mathbf{X}_i^*, t_i)$
$H_i = \widetilde{H}_i \Phi(t_i, t_0)$
$\Lambda = \Lambda + H_i^T R_i^{-1} H_i$
$N = N + H_i^T R_i^{-1} \mathbf{y}_i$

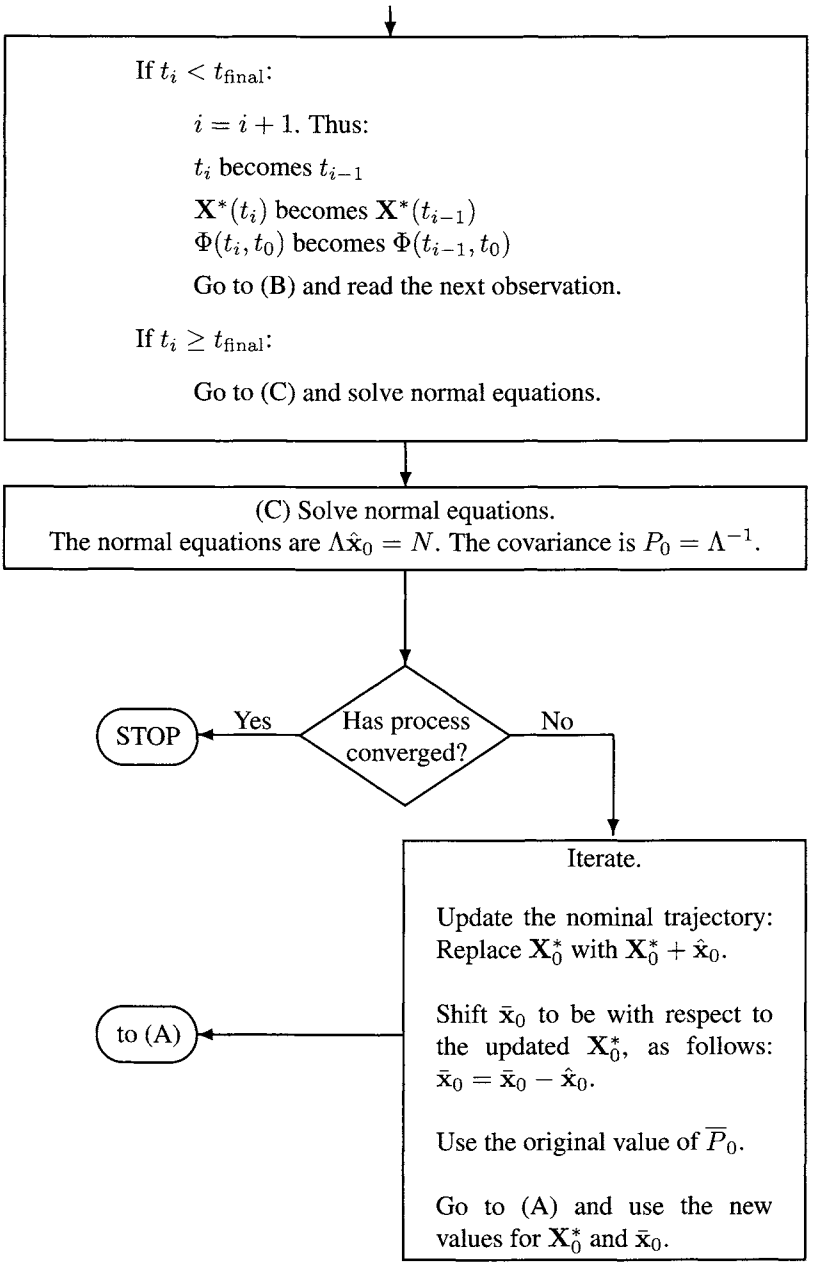


Figure 4.6.1: Batch processing algorithm flow chart.

where $F(\mathbf{X}^*, t)$ is the time derivative of the state vector in the differential equations governing the time evolution of the system. The observation-state mapping matrix is given by

$$\widetilde{H}_i = \frac{\partial G(\mathbf{X}_i^*, t_i)}{\partial \mathbf{X}} \tag{4.6.6}$$

where $G(\mathbf{X}_i^*, t_i)$ are the observation-state relationships evaluated on the nominal or reference trajectory.

Notice that the solution for $\hat{\mathbf{x}}_0$ involved inversion of the information matrix, Λ_0, where

$$\Lambda_0 = H^T R^{-1} H + \overline{P}_0^{-1}. \tag{4.6.7}$$

Generally the normal equation would not be solved by a direct inversion of Λ_0 but rather would be solved by an indirect but more accurate technique, such as the Cholesky decomposition described in Chapter 5, Section 5.2. The sequence of operations required to implement the batch estimation process is outlined in Fig. 4.6.1. Note that the algorithm in Fig. 4.6.1 assumes that there are no observations at t_0. If observations exist at t_0, set $\Lambda = \overline{P}_0^{-1} + H_0^T R_0^{-1} H_0$ and $N = H_0^T R_0^{-1} \mathbf{y}_0$ in the initialization box, A. As previously stated, the entire sequence of computations are repeated until the estimation process has converged. If there are observations at t_0, the state transition matrix for processing these observations is the identity matrix.

This procedure yields a minimum value of the performance index (see Eq. (4.3.24))

$$J(x) = (\hat{\mathbf{x}}_0 - \bar{\mathbf{x}}_0)^T \overline{P}_0^{-1} (\hat{\mathbf{x}}_0 - \bar{\mathbf{x}}_0) + \sum_{i=1}^{\ell} \hat{\boldsymbol{\epsilon}}_i^T R_i^{-1} \hat{\boldsymbol{\epsilon}}_i \tag{4.6.8}$$

where

$$\hat{\boldsymbol{\epsilon}}_i = \mathbf{y}_i - H_i \hat{\mathbf{x}}_0 \tag{4.6.9}$$

and $\hat{\boldsymbol{\epsilon}}_i$ is the best estimate of the observation error.

In practice, $\overline{P}_0$ is generally not a realistic representation of the accuracy of $\bar{\mathbf{x}}_0$ and it is used only to better condition the estimation error covariance matrix, P. In this case, $\bar{\mathbf{x}}_0$ usually is set to zero for each iteration and $\overline{P}_0$ is chosen to be a diagonal matrix with large diagonal values. Hence, the first term in Eq. (4.6.8) will be very small and the tracking data residuals will determine the value of $J(x)$. The RMS of the observation residuals generally is computed and may be used as a measure of convergence; when the RMS no longer changes the solution is assumed to be converged. The RMS is computed from

$$\text{RMS} = \left\{ \frac{\sum_{i=1}^{\ell} \hat{\boldsymbol{\epsilon}}_i^T R_i^{-1} \hat{\boldsymbol{\epsilon}}_i}{m} \right\}^{1/2}. \tag{4.6.10}$$

$\hat{\epsilon}_i$ is a p-vector and $m = \ell \times p$. Hence, m is the total number of observations. Eq. (4.6.10) is referred to as the weighted RMS. If the RMS is computed without including the weighting matrix, R_i^{-1}, it may be referred to as the unweighted RMS or just the RMS.

4.7 THE SEQUENTIAL ESTIMATION ALGORITHM

In this section, an alternate approach to the batch processor is discussed in which the observations are processed as soon as they are received. An advantage of this approach, referred to as the sequential processing algorithm, is that the matrix to be inverted will be of the same dimension as the observation vector. Hence, if the observations are processed individually, only scalar divisions will be required to obtain the estimate of $\mathbf{x}_k$. The algorithm was developed originally by Swerling (1959), but the treatment that received more popular acclaim is due to Kalman (1960). In fact, the sequential estimation algorithm discussed here often is referred to as the *Kalman filter*. A number of papers and textbooks have been written describing and providing variations of the Kalman filter. The collection of papers edited by Sorenson (1985) contains many of the pioneering papers in this area. Also, the treatments by Bierman (1977), Liebelt (1967), Tapley (1973), Gelb (1974), Maybeck (1979), Grewal and Andrews (1993), and Montenbruck and Gill (2000) are other references.

Recall that an estimate $\hat{\mathbf{x}}_j$ and a covariance matrix P_j can be propagated forward to a time t_k by the relations

$$\begin{aligned} \overline{\mathbf{x}}_k &= \Phi(t_k, t_j)\hat{\mathbf{x}}_j \\ \overline{P}_k &= \Phi(t_k, t_j)P_j\Phi^T(t_k, t_j). \end{aligned} \tag{4.7.1}$$

Assume that we have an additional observation at t_k (see Eq. (4.2.6)),

$$\mathbf{y}_k = \widetilde{H}_k\mathbf{x}_k + \boldsymbol{\epsilon}_k \tag{4.7.2}$$

where $E[\boldsymbol{\epsilon}_k] = 0$ and $E[\boldsymbol{\epsilon}_k\boldsymbol{\epsilon}_j^T] = R_k\delta_{kj}$, where δ_{kj} is the *Kronicker delta*. We wish to process $\mathbf{y}_k$ in order to determine $\hat{\mathbf{x}}_k$. The best estimate of $\mathbf{x}_k$ is obtained in Eq. (4.4.29) as

$$\hat{\mathbf{x}}_k = (\widetilde{H}_k^T R_k^{-1}\widetilde{H}_k + \overline{P}_k^{-1})^{-1}(\widetilde{H}_k^T R_k^{-1}\mathbf{y}_k + \overline{P}_k^{-1}\overline{\mathbf{x}}_k). \tag{4.7.3}$$

The primary computational problems are associated with computing the $(n \times n)$ matrix inverse in Eq. (4.7.3). Recall that in the original derivation it was shown that the quantity to be inverted is the information matrix Λ_k, which yields the covariance matrix P_k associated with estimate $\hat{\mathbf{x}}_k$ (see Eq. (4.4.30)),

$$P_k = \Lambda_k^{-1} = (\widetilde{H}_k^T R_k^{-1}\widetilde{H}_k + \overline{P}_k^{-1})^{-1}. \tag{4.7.4}$$

From Eq. (4.7.4), it follows that

$$P_k^{-1} = \widetilde{H}_k^T R_k^{-1} \widetilde{H}_k + \overline{P}_k^{-1}. \tag{4.7.5}$$

Premultiplying each side of Eq. (4.7.5) by P_k and then postmultiplying by $\overline{P}_k$ leads to the following expressions:

$$\overline{P}_k = P_k \widetilde{H}_k^T R_k^{-1} \widetilde{H}_k \overline{P}_k + P_k \tag{4.7.6}$$

or

$$P_k = \overline{P}_k - P_k \widetilde{H}_k^T R_k^{-1} \widetilde{H}_k \overline{P}_k. \tag{4.7.7}$$

Now if Eq. (4.7.6) is postmultiplied by the quantity $H_k^T R_k^{-1}$, the following expression is obtained:

$$\begin{aligned} \overline{P}_k \widetilde{H}_k^T R_k^{-1} &= P_k \widetilde{H}_k^T R_k^{-1} [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T R_k^{-1} + I] \\ &= P_k \widetilde{H}_k^T R_k^{-1} [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k] R_k^{-1}. \end{aligned} \tag{4.7.8}$$

Solving for the quantity $P_k \widetilde{H}_k^T R_k^{-1}$ leads to

$$P_k \widetilde{H}_k^T R_k^{-1} = \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}. \tag{4.7.9}$$

This relates the *a priori* covariance matrix $\overline{P}_k$ to the *a posteriori* covariance matrix P_k. If Eq. (4.7.9) is used to eliminate $P_k \widetilde{H}_k^T R_k^{-1}$ in Eq. (4.7.7), the following result is obtained:

$$P_k = \overline{P}_k - \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1} \widetilde{H}_k \overline{P}_k. \tag{4.7.10}$$

Equation (4.7.10) also can be derived by using the *Schur identity* (Theorem 4 of Appendix B). Note that Eq. (4.7.10) is an alternate way of computing the inverse in Eq. (4.7.4). In Eq. (4.7.10), the matrix to be inverted is of dimension $p \times p$, the same dimensions as the observation error covariance matrix. If the observations are processed as scalars (i.e., one at a time), only a scalar division is required. If the weighting matrix, K_k (sometimes referred to as the *Kalman gain* or simply gain matrix), is defined as

$$K_k \equiv \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}, \tag{4.7.11}$$

then Eq. (4.7.10) can be expressed in the compact form

$$P_k = [I - K_k \widetilde{H}_k] \overline{P}_k. \tag{4.7.12}$$

If Eq. (4.7.4) is substituted into Eq. (4.7.3), the sequential form for computing $\hat{\mathbf{x}}_k$ can be written as

$$\hat{\mathbf{x}}_k = P_k [\widetilde{H}_k^T R_k^{-1} \mathbf{y}_k + \overline{P}_k^{-1} \overline{\mathbf{x}}_k] \tag{4.7.13}$$

$$= P_k \widetilde{H}_k^T R_k^{-1} \mathbf{y}_k + P_k \overline{P}_k^{-1} \overline{\mathbf{x}}_k.$$

But from Eqs. (4.7.9) and (4.7.11)

$$K_k = P_k \widetilde{H}_k^T R_k^{-1}. \tag{4.7.14}$$

Using the preceding and Eq. (4.7.12), Eq. (4.7.13) becomes

$$\hat{\mathbf{x}}_k = K_k \mathbf{y}_k + [I - K_k \widetilde{H}_k] \overline{P}_k \overline{P}_k^{-1} \overline{\mathbf{x}}_k. \tag{4.7.15}$$

Collecting terms yields

$$\hat{\mathbf{x}}_k = \overline{\mathbf{x}}_k + K_k [\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k]. \tag{4.7.16}$$

Equation (4.7.16), with Eqs. (4.7.1), (4.7.11), and (4.7.12), can be used in a recursive fashion to compute the estimate of $\hat{\mathbf{x}}_k$, incorporating the observation $\mathbf{y}_k$.

4.7.1 THE SEQUENTIAL COMPUTATIONAL ALGORITHM

The algorithm for computing the estimate sequentially is summarized as

Given: $\hat{\mathbf{x}}_{k-1}$, P_{k-1}, $\mathbf{X}^*_{k-1}$, and R_k, and the observation $\mathbf{Y}_k$, at t_k (at the initial time t_0, these would be $\mathbf{X}^*_0$, $\hat{\mathbf{x}}_0$, and P_0).

(1) Integrate from t_{k-1} to t_k,

$$\begin{aligned} \dot{\mathbf{X}}^* &= F(\mathbf{X}^*, t), & \mathbf{X}^*(t_{k-1}) &= \mathbf{X}^*_{k-1} \\ \dot{\Phi}(t, t_{k-1}) &= A(t)\Phi(t, t_{k-1}), & \Phi(t_{k-1}, t_{k-1}) &= I. \end{aligned} \tag{4.7.17}$$

(2) Compute

$$\overline{\mathbf{x}}_k = \Phi(t_k, t_{k-1}) \hat{\mathbf{x}}_{k-1} \qquad \overline{P}_k = \Phi(t_k, t_{k-1}) P_{k-1} \Phi^T(t_k, t_{k-1}).$$

(3) Compute

$$\mathbf{y}_k = \mathbf{Y}_k - G(\mathbf{X}^*_k, t_k) \qquad \widetilde{H}_k = \frac{\partial G(\mathbf{X}^*_k, t_k)}{\partial \mathbf{X}}.$$

(4) Compute the measurement update

$$\begin{aligned} K_k &= \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1} \\ \hat{\mathbf{x}}_k &= \overline{\mathbf{x}}_k + K_k [\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k] \\ P_k &= [I - K_k \widetilde{H}_k] \overline{P}_k. \end{aligned} \tag{4.7.18}$$

(5) Replace k with $k+1$ and return to (1).

Equation (4.7.1) is known as the *time update* and Eq. (4.7.18) is called the *measurement update* equation. The flow chart for the sequential computational algorithm is given in Fig. 4.7.1. If there is an observation at t_0, a time update is not performed but a measurement update is performed.

Note that we do not multiply $\widetilde{H}_k$ by the state transition matrix, since the observation at $\mathbf{y}_k$ is not accumulated at another time epoch, as is the case for the batch processor. Also, note that the differential equations for the state transition matrix are reinitialized at each observation epoch. Therefore, the state transition matrix is reinitialized at each observation epoch. If there is more than one observation at each epoch and we are processing them as scalars, we would set $\Phi(t_i, t_i) = I$ after processing the first observation at each epoch; P and $\hat{\mathbf{x}}$ are not time updated until we move to the next observation epoch. The estimate of the state of the nonlinear system at t_k is given by $\hat{\mathbf{X}}_k = \mathbf{X}_k^* + \hat{\mathbf{x}}_k$.

One disadvantage of both the batch and sequential algorithm lies in the fact that if the true state and the reference state are not close together then the linearization assumption leading to Eq. (4.2.6) may not be valid and the estimation process may diverge. The extended sequential filter algorithm (see Section 4.7.3) is often used to overcome problems with the linearity assumption.

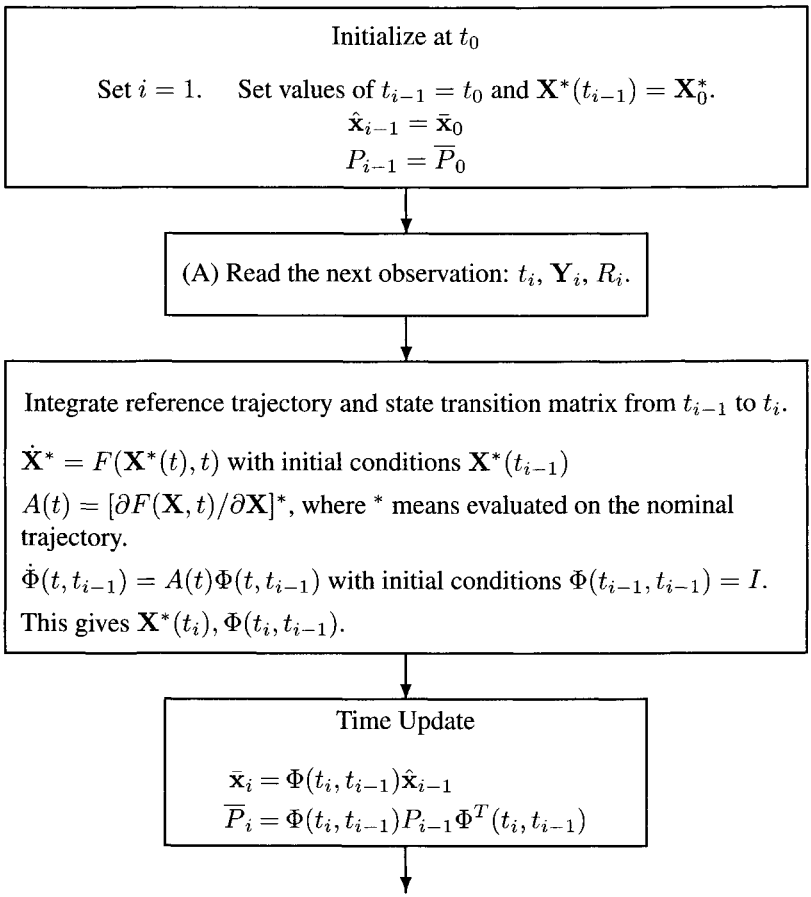
A second unfavorable characteristic of the sequential estimation algorithm is that the state estimation error covariance matrix may approach zero as the number of observations becomes large. The sketch in Fig. 4.7.2 illustrates the behavior of the trace of the state estimation error covariance matrix as discrete observations are processed. As illustrated in this sketch, the trace grows between observations and is reduced by the amount, trace $(K\widetilde{H}\overline{P})$, after each observation. Hence, the magnitude of the covariance matrix elements will decrease depending on the density, information content, and accuracy of the observations.

Examination of the estimation algorithm shows that as $P_k \rightarrow 0$ the gain approaches zero, and the estimation procedure will become insensitive to the observations. Consequently, the estimate will diverge due to either errors introduced in the linearization procedure, computational errors, or errors due to an incomplete mathematical model. To overcome this problem, process noise often is added to the state propagation equations (see Section 4.9).

In addition to these two problems, the Kalman filter may diverge because of numerical difficulties associated with the covariance measurement update, given by Eq. (4.7.12). This problem is described in the next section.

Numerical Considerations

The conventional Kalman filter, which uses the covariance update, (4.7.12), can sometimes fail due to numerical problems with this equation. The covariance


Initialize at t_0
Set $i = 1$. Set values of $t_{i-1} = t_0$ and $\mathbf{X}^*(t_{i-1}) = \mathbf{X}_0^*$.
$\hat{\mathbf{x}}_{i-1} = \bar{\mathbf{x}}_0$
$P_{i-1} = \overline{P}_0$
(A) Read the next observation: t_i, $\mathbf{Y}_i$, R_i.
Integrate reference trajectory and state transition matrix from t_{i-1} to t_i.
$\dot{\mathbf{X}}^* = F(\mathbf{X}^*(t), t)$ with initial conditions $\mathbf{X}^*(t_{i-1})$
$A(t) = [\partial F(\mathbf{X}, t)/\partial \mathbf{X}]^*$, where * means evaluated on the nominal trajectory.
$\dot{\Phi}(t, t_{i-1}) = A(t)\Phi(t, t_{i-1})$ with initial conditions $\Phi(t_{i-1}, t_{i-1}) = I$.
This gives $\mathbf{X}^*(t_i), \Phi(t_i, t_{i-1})$.
Time Update
$\bar{\mathbf{x}}_i = \Phi(t_i, t_{i-1})\hat{\mathbf{x}}_{i-1}$
$\overline{P}_i = \Phi(t_i, t_{i-1})P_{i-1}\Phi^T(t_i, t_{i-1})$

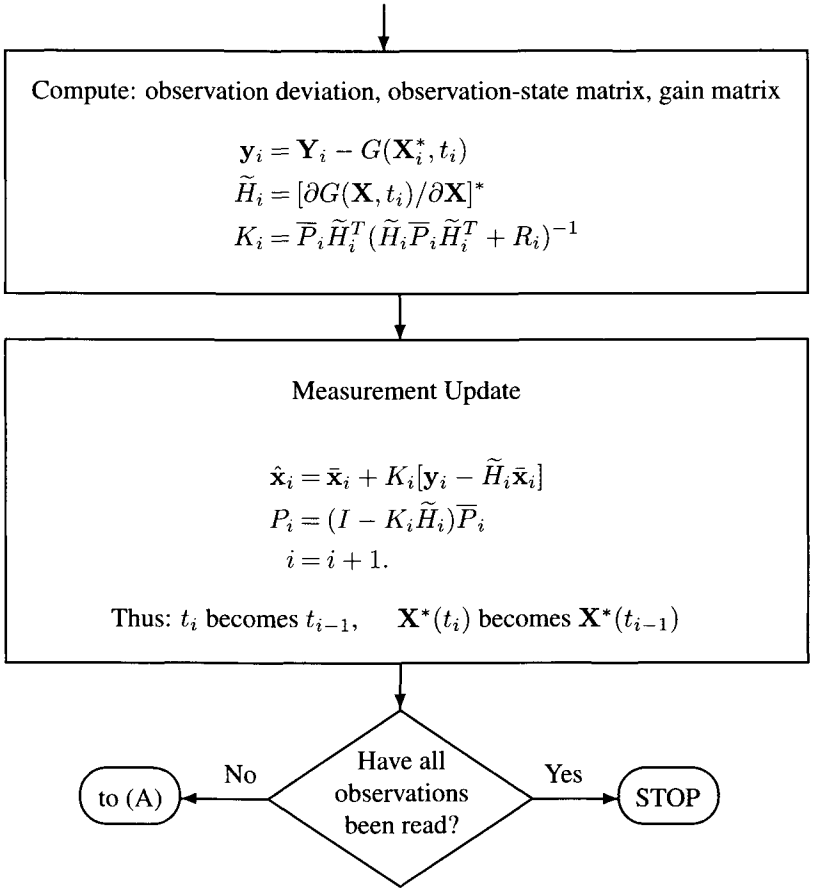


Figure 4.7.1: Sequential processing algorithm flow chart.

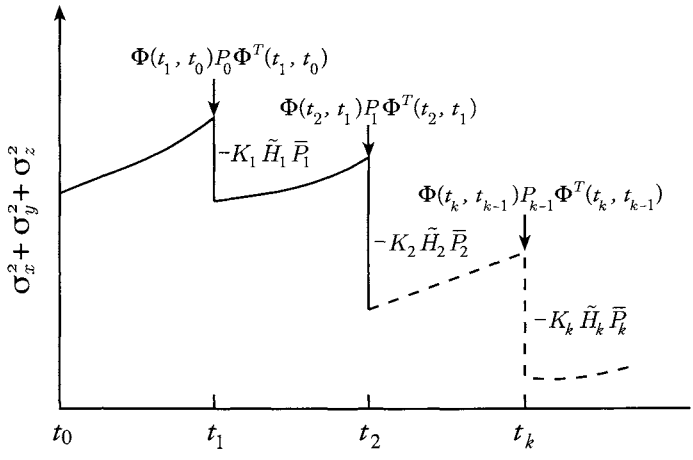


Figure 4.7.2: Illustration of the behavior of the trace of the state estimation error covariance matrix.

matrix may lose its properties of symmetry and become nonpositive definite when the computations are carried out with the finite digit arithmetic of the computer. In particular, this equation can fail to yield a symmetric positive definite result when a large *a priori* covariance is reduced by the incorporation of very accurate observation data.

In order to prevent these problems, several alternate algorithms have been suggested. A common alternative is to use the form of the equation given by Bucy and Joseph (1968):

$$P_k = (I - K_k\widetilde{H}_k)\overline{P}_k(I - K_k\widetilde{H}_k)^T + K_kR_kK_k^T. \tag{4.7.19}$$

Note that this formulation will always yield a symmetric result for P_k, although it may lose its positive definite quality for a poorly observed system.

The most common solution to numerical problems with the covariance update is to use a square root formulation to update the covariance matrix. The square root algorithms are discussed in Chapter 5.

An example of a poorly conditioned system taken from Bierman (1977, p. 96, Example V.2) illustrates the numerical instability problem. Consider the problem of estimating x_1 and x_2 from scalar measurements z_1 and z_2, where

$$z_1 = x_1 + \epsilon x_2 + v_1$$
$$z_2 = x_1 + x_2 + v_2.$$

We assume that the observation noise v_1 and v_2 are uncorrelated, zero mean ran-

dom variables with unit variance. If v_1 and v_2 do not have unit variance or are correlated, we can perform a simple whitening transformation so that their covariance matrix is the identity matrix (Bierman, 1977, or see Section 5.7.1). In matrix form,

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & \epsilon \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{4.7.20}$$

Assume that the *a priori* covariance associated with our knowledge of $[x_1 \; x_2]$ is given by

$$\overline{P}_1 = \sigma^2 I$$

where $\sigma = 1/\epsilon$ and $0 < \epsilon \ll 1$. The quantity ϵ is assumed to be small enough such that computer round-off produces

$$\begin{gathered} 1 + \epsilon^2 = 1, \\ \text{and} \\ 1 + \epsilon \neq 1; \end{gathered} \tag{4.7.21}$$

that is, $1 + \epsilon^2$ rounds to 1 and $1 + \epsilon$ does not. Note that $\sigma = 1/\epsilon$ is large; hence, this is an illustration of the problem where accurate data is being combined with a large *a priori* covariance.

This estimation problem is well posed. The observation z_1 provides an estimate of x_1 which, when combined with z_2, should determine x_2. However, the conventional Kalman update to P given by Eq. (4.7.12) will result in failure of the filter.

The observations will be processed one at a time. Let the gain and estimation error covariance associated with z_1 be denoted as K_1 and P_1, respectively. Similar definitions apply for z_2.

The exact solution yields

$$K_1 = \frac{1}{1 + 2\epsilon^2} \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}. \tag{4.7.22}$$

The estimation error covariance associated with processing z_1 is

$$P_1 = \frac{1}{1 + 2\epsilon^2} \begin{bmatrix} 2 & -\sigma \\ -\sigma & \sigma^2 + 1 \end{bmatrix}. \tag{4.7.23}$$

After processing z_2 it can be shown that the exact solution for the estimation error covariance is

$$P_2 = \frac{1}{\beta} \begin{bmatrix} 1 + 2\epsilon^2 & -(1 + \epsilon) \\ -(1 + \epsilon) & 2 + \epsilon^2 \end{bmatrix}, \tag{4.7.24}$$

where

$$\beta = 1 - 2\epsilon + 2\epsilon^2(2 + \epsilon^2)$$

and we see that both P_1 and P_2 are symmetric and positive definite.

The conventional Kalman filter, subject to Eq. (4.7.21), yields

$$K_1 = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$$

and

$$P_1 = \begin{bmatrix} 0 & -\sigma \\ -\sigma & \sigma^2 \end{bmatrix}. \tag{4.7.25}$$

Note that P_1 is no longer positive definite and the filter will not produce correct results. If z_2 is processed, the conventional Kalman filter yields

$$P_2 = \frac{1}{1-2\epsilon} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{4.7.26}$$

Note that P_2 does not have positive terms on the diagonal and $|P_2| = 0$. Hence, the conventional Kalman filter has failed.

By way of comparison, the Joseph formulation yields

$$P_1 = \begin{bmatrix} 2 & -\sigma \\ -\sigma & \sigma^2 \end{bmatrix} \tag{4.7.27}$$

$$P_2 = \begin{bmatrix} 1+2\epsilon & -(1+3\epsilon) \\ -(1+3\epsilon) & (2+\epsilon) \end{bmatrix}$$

which are symmetric and positive definite and agree with the exact solution to $O(\epsilon)$.

It should be pointed out that the batch processor for this problem, under the same assumption that $1 + \epsilon^2 = 1$, yields

$$P_2 = \begin{bmatrix} 1+2\epsilon & -(1+3\epsilon) \\ -(1+3\epsilon) & 2(1+2\epsilon) \end{bmatrix}.$$

This agrees to $O(\epsilon)$ with the exact solution for P_2. The batch solution is not as sensitive as the conventional Kalman filter to computer round-off errors. An intuitive understanding of this can be had by examining the equations for P.

For the batch processor

$$P = (\overline{P}_0{}^{-1} + \sum_{i=1}^{\ell} H_i^T R^{-1} H_i)^{-1},$$

and for the sequential processor

$$P = (I - \overline{P}\widetilde{H}^T(\widetilde{H}\overline{P}\widetilde{H}^T + R)^{-1}\widetilde{H})\overline{P}.$$

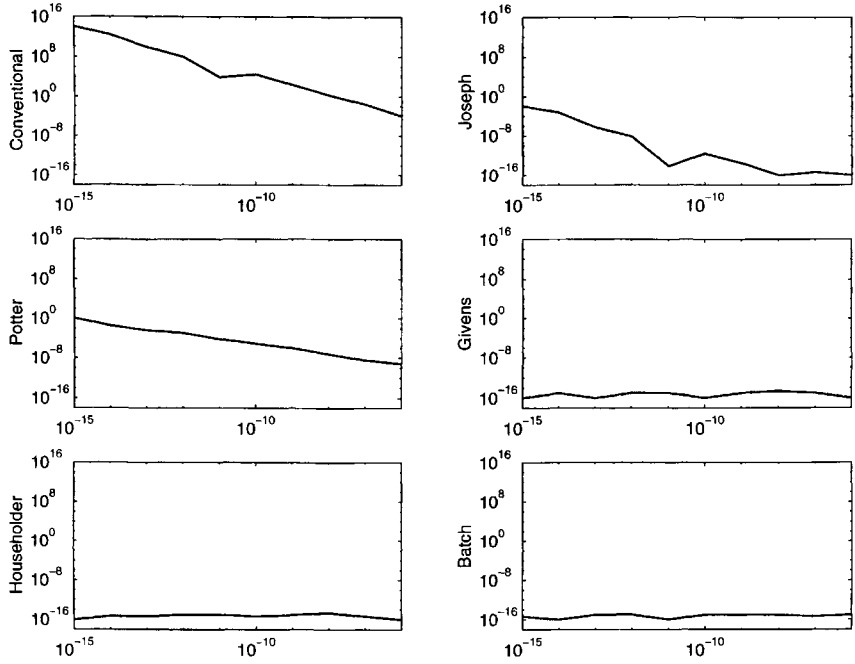


Figure 4.7.3: Difference of the exact trace of P_2 and that for various algorithms versus the value of ϵ on the abscissa.

Hence, if very accurate tracking data (small values for the elements of R) are combined with a large *a priori* (large values for the elements of $\overline{P}$), the sequential algorithm will ignore the data contribution, R, when it adds $\widetilde{H}\overline{P}\widetilde{H}^T + R$; whereas the batch processor will tend to ignore $\overline{P}_0^{-1}$ and base its solution on the strength of the data when it computes $(\Sigma H^T R^{-1} H + \overline{P}_0^{-1})$. It is emphasized that for sequential processing of the data the preferred solution to this problem is to use one of the square root formulations for propagating the covariance or information matrix discussed in Chapter 5.

Figure 4.7.3 shows the difference between the exact value of the trace of P_2 and the trace computed using various algorithms on a 16-digit machine. Notice that the conventional Kalman update yields an error of O(1) for $\epsilon \simeq 10^{-8}$, whereas the Joseph and Potter algorithms approach this value for $\epsilon = 10^{-15}$. Also, the Givens and Householder algorithms, which are introduced in Chapter 5, and the batch algorithm are all highly accurate.

4.7.2 THE EXTENDED SEQUENTIAL ESTIMATION ALGORITHM

To minimize the effects of errors due to the neglect of higher order terms in the linearization procedure leading to Eq. (4.2.6), the extended form of the sequential estimation algorithm is sometimes used. This algorithm is often referred to as the *Extended Kalman Filter* (EKF). The primary difference between the sequential and the extended sequential algorithm is that the reference trajectory for the extended sequential algorithm is updated after each observation to reflect the best estimate of the true trajectory. For example, after processing the k^{th} observation, the best estimate of the state vector at t_k is used to provide new initial conditions for the reference trajectory,

$$(\mathbf{X}_k^*)_{\text{new}} = \hat{\mathbf{X}}_k = \mathbf{X}_k^* + \hat{\mathbf{x}}_k. \tag{4.7.28}$$

Using $\hat{\mathbf{X}}_k$ for the reference trajectory leads to $\hat{\mathbf{x}}_k = 0$, which will result in $\bar{\mathbf{x}}_{k+1} = 0$. The integration for the reference trajectory and the state transition matrix is reinitialized at each observation epoch, and the equations are integrated forward from t_k to t_{k+1}. The estimate for $\hat{\mathbf{x}}_{k+1}$ is then computed from

$$\hat{\mathbf{x}}_{k+1} = K_{k+1}\mathbf{y}_{k+1} \tag{4.7.29}$$

where K_{k+1} and $\mathbf{y}_{k+1}$ are computed based on the new reference orbit. Then, the reference orbit is updated at time t_{k+1} by incorporating $\hat{\mathbf{x}}_{k+1}$ and the process proceeds to t_{k+2}. The process of incorporating the estimate at each observation point into the reference trajectory for propagating to the next observation epoch leads to the reference trajectory being the prediction of the estimate of the nonlinear state; for example, $\mathbf{X}^*(t) = \hat{\mathbf{X}}(t)$.

In actual practice, it is not a good idea to update the reference trajectory using the first observations. This is particularly true if the observations contain significant noise. After a few observations have been processed, the estimates of $\hat{\mathbf{x}}$ will stabilize, and the trajectory update can be initiated.

The advantage of the extended sequential algorithm is that convergence to the best estimate will be more rapid because errors introduced in the linearization process are reduced. In addition, because the state estimate deviation, $\hat{\mathbf{x}}(t)$, need not be mapped between observations, it is not necessary to compute the state transition matrix. The estimation error covariance matrix, $P(t)$, can be mapped by integrating the matrix differential equation (4.9.35) discussed in Section 4.9.

The major disadvantage of the extended sequential algorithm is that the differential equations for the reference trajectory must be reinitialized after each observation is processed.

4.7.3 THE EXTENDED SEQUENTIAL COMPUTATIONAL ALGORITHM

The algorithm for computing the extended sequential estimate can be summarized as follows:

Given: P_{k-1}, $\hat{\mathbf{X}}_{k-1}$ and $\mathbf{Y}_k$, R_k.

(1) Integrate from t_{k-1} to t_k,

$$\begin{aligned} \dot{\mathbf{X}}^* &= F(\mathbf{X}^*, t), & \mathbf{X}^*(t_{k-1}) &= \hat{\mathbf{X}}_{k-1} \\ \dot{\Phi}(t, t_{k-1}) &= A(t)\Phi(t, t_{k-1}), & \Phi(t_{k-1}, t_{k-1}) &= I. \end{aligned} \tag{4.7.30}$$

(2) Compute

$$\begin{aligned} \overline{P}_k &= \Phi(t_k, t_{k-1}) P_{k-1} \Phi^T(t_k, t_{k-1}) \\ \mathbf{y}_k &= \mathbf{Y}_k - G(\mathbf{X}_k^*, t_k) \\ \widetilde{H}_k &= \partial G(\mathbf{X}_k^*, t_k)/\partial \mathbf{X}_k. \end{aligned} \tag{4.7.31}$$

(3) Compute

$$\begin{aligned} K_k &= \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1} \\ \hat{\mathbf{X}}_k &= \mathbf{X}_k^* + K_k \mathbf{y}_k \\ P_k &= [I - K_k \widetilde{H}_k] \overline{P}_k. \end{aligned} \tag{4.7.32}$$

(4) Replace k with $k + 1$ and return to (1).

The flow chart for the extended sequential computational algorithm is given in Fig. 4.7.4.

4.7.4 THE PREDICTION RESIDUAL

It is of interest to examine the variance of the predicted residuals, which are sometimes referred to as the *innovation*, or new information, which comes from each measurement. The *predicted residual*, or innovation, is the observation residual based on the *a priori* or predicted state , $\bar{\mathbf{x}}_k$, at the observation time, t_k, and is defined as

$$\boldsymbol{\beta}_k = \mathbf{y}_k - \widetilde{H}_k \bar{\mathbf{x}}_k. \tag{4.7.33}$$

As noted previously,

$$\begin{aligned} \bar{\mathbf{x}}_k &= \mathbf{x}_k + \boldsymbol{\eta}_k \\ \mathbf{y}_k &= \widetilde{H}_k \mathbf{x}_k + \boldsymbol{\epsilon}_k \end{aligned}$$

where $\mathbf{x}_k$ is the true value of the state deviation vector and $\boldsymbol{\eta}_k$ is the error in $\bar{\mathbf{x}}_k$. Also,

$$E[\boldsymbol{\eta}_k] = 0,\ E[\boldsymbol{\eta}_k \boldsymbol{\eta}_k^T] = \overline{P}_k$$

and

$$\begin{aligned} E[\boldsymbol{\epsilon}_k] = 0,\ E[\boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_k^T] = R_k \\ E[\boldsymbol{\epsilon}_k \boldsymbol{\eta}_k^T] = 0. \end{aligned}$$

From these conditions it follows that $\boldsymbol{\beta}_k$ has mean

$$\begin{aligned} E[\boldsymbol{\beta}_k] \equiv \overline{\boldsymbol{\beta}}_k &= E(\widetilde{H}_k \mathbf{x}_k + \boldsymbol{\epsilon}_k - \widetilde{H}_k \bar{\mathbf{x}}_k) \\ &= E(\boldsymbol{\epsilon}_k - \widetilde{H} \boldsymbol{\eta}_k) = 0 \end{aligned}$$

and variance-covariance

$$\begin{aligned} P_{\beta_k} &= E[(\boldsymbol{\beta}_k - \overline{\boldsymbol{\beta}}_k)(\boldsymbol{\beta}_k - \overline{\boldsymbol{\beta}}_k)^T] = E[\boldsymbol{\beta}_k \boldsymbol{\beta}_k^T] \\ &= E[(\mathbf{y}_k - \widetilde{H}_k \bar{\mathbf{x}}_k)(\mathbf{y}_k - \widetilde{H}_k \bar{\mathbf{x}}_k)^T] \\ &= E[(\boldsymbol{\epsilon}_k - \widetilde{H}_k \boldsymbol{\eta}_k)(\boldsymbol{\epsilon}_k - \widetilde{H}_k \boldsymbol{\eta}_k)^T] \\ P_{\beta_k} &= R_k + \widetilde{H}_k \overline{P}_k \widetilde{H}_k^T. \end{aligned} \tag{4.7.34}$$

Hence, for a large prediction residual variance-covariance, the Kalman gain

$$K_k = \overline{P}_k \widetilde{H}_k^T P_{\beta_k}^{-1} \tag{4.7.35}$$

will be small, and the observation will have little influence on the estimate of the state. Also, large values of the prediction residual relative to the prediction residual standard deviation may be an indication of bad tracking data and hence may be used to edit data from the solution.

4.8 EXAMPLE PROBLEMS

This section provides several examples that involve processing observations with the batch and sequential processors.

4.8.1 LINEAR SYSTEM

Given: A system described by

$$\mathbf{x}(t_{i+1}) = \begin{bmatrix} x_1(t_{i+1}) \\ x_2(t_{i+1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix}$$

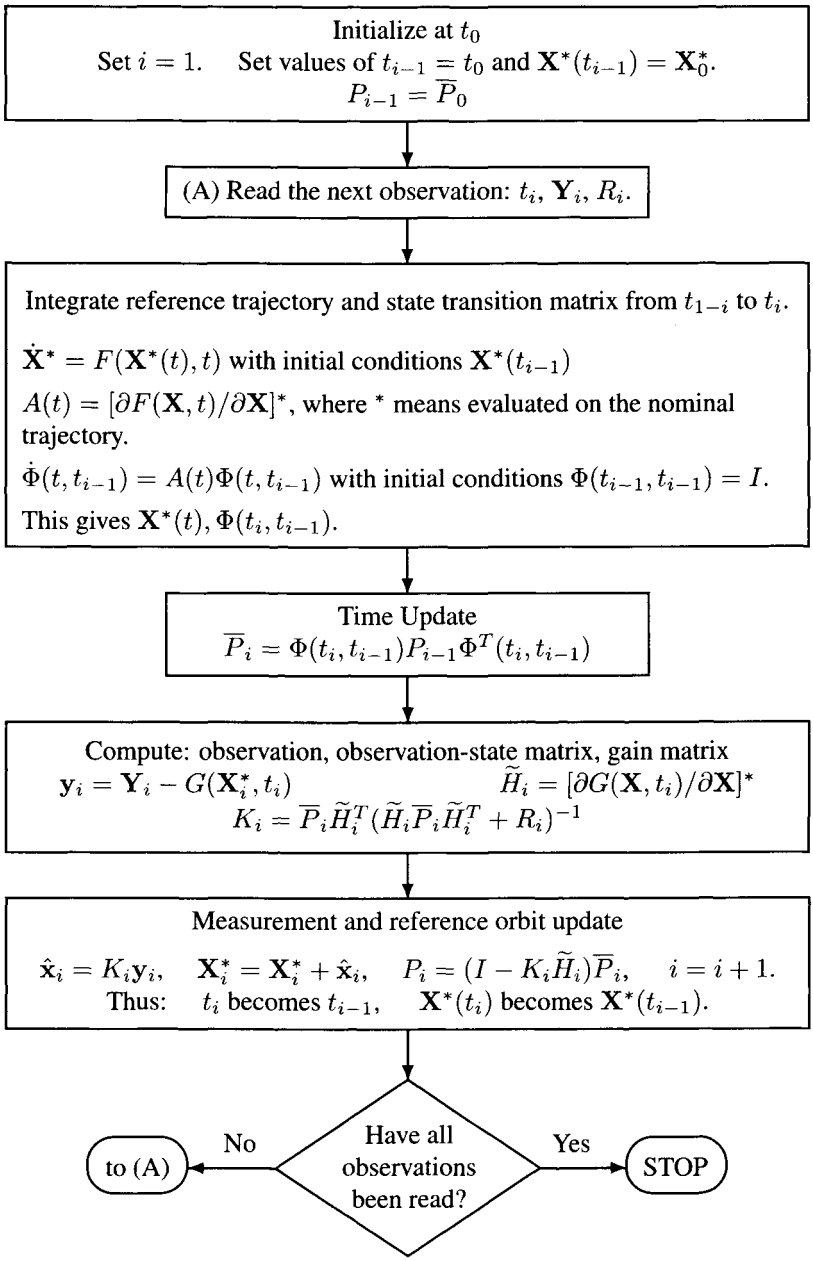


Figure 4.8.1: Extended sequential processing algorithm flow chart.

and observation-state relationship

$$\begin{bmatrix} y_1(t_i) \\ y_2(t_i) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

with *a priori* information

$$\overline{\mathbf{x}}_0 = \overline{\mathbf{x}}(t_0) = \begin{bmatrix} \overline{x}_1(t_0) \\ \overline{x}_2(t_0) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \overline{P}_0 = \overline{P}(t_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$W^{-1} = R = E\left\{ \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} [\epsilon_1 \epsilon_2] \right\} = \begin{bmatrix} 2 & 0 \\ 0 & 3/4 \end{bmatrix}$$

and observations at t_1 given by

$$\begin{bmatrix} y_1(t_1) \\ y_2(t_1) \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

(1) Using the batch processor algorithm, determine the best estimate of $\mathbf{x}(t_0)$ based on observations $y_1(t_1)$ and $y_2(t_1)$, $\overline{\mathbf{x}}(t_0)$, $\overline{P}(t_0)$, and R.

(2) Using the sequential algorithm and the information given, determine $\hat{\mathbf{x}}(t_1)$, the best estimate of $\mathbf{x}(t_1)$.

(3) Show that $\hat{\mathbf{x}}(t_0)$ obtained in (1), when mapped forward to t_1 is equal to $\hat{\mathbf{x}}(t_1)$ obtained in (2). Show that P_0 from the batch processor mapped forward to t_1 agrees with P_1 from the sequential processor.

(1) *Batch Processor*

$$\Phi(t_{i+1}, t_i) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\widetilde{H} = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}, \quad \overline{P}_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overline{\mathbf{x}}_o = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$W^{-1} = R = \begin{bmatrix} 2 & 0 \\ 0 & 3/4 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1(t_1) \\ y_2(t_1) \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

The computational algorithm for the batch processor is given in Section 4.6.

$$\hat{\mathbf{x}}_0 = (\overline{P}_0^{-1} + H_1^T W H_1)^{-1} (H_1^T W \mathbf{y} + \overline{P}_0^{-1} \overline{\mathbf{x}}_0)$$

$$H_1 = \widetilde{H}_1 \Phi(t_1, t_0) = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/2 & 1 \end{bmatrix}$$

$$\hat{\mathbf{x}}_0 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 4/3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1 \end{bmatrix} \right\}^{-1}$$
$$\left\{ \begin{bmatrix} 0 & 1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 4/3 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$$
$$\hat{\mathbf{x}}_o = \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 17/6 \end{bmatrix}^{-1} \begin{bmatrix} 17/3 \\ 31/3 \end{bmatrix}$$
$$\hat{\mathbf{x}}_o = \begin{bmatrix} 17/20 & -1/5 \\ -1/5 & 2/5 \end{bmatrix} \begin{bmatrix} 17/3 \\ 31/3 \end{bmatrix}$$
$$\hat{\mathbf{x}}_0 = \begin{bmatrix} 2\frac{3}{4} \\ 3 \end{bmatrix}$$

Note that

$$P_0 = \begin{bmatrix} 17/20 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}.$$

The correlation matrix, computed from P_0, has standard deviations on the diagonal and correlation coefficients for the lower off-diagonal terms,

$$\begin{bmatrix} \sigma_{x_1} \\ \rho_{x_1 x_2} \sigma_{x_2} \end{bmatrix} = \begin{bmatrix} .922 & \\ -.343 & .632 \end{bmatrix}.$$

(2) *Sequential Processor*

$$\begin{aligned} \hat{\mathbf{x}}_1 &= \overline{\mathbf{x}}_1 + \overline{P}_1 \widetilde{H}_1^T (W_1^{-1} + \widetilde{H}_1 \overline{P}_1 \widetilde{H}_1^T)^{-1} (\mathbf{y}_1 - \widetilde{H}_1 \overline{\mathbf{x}}_1) \\ &= \overline{\mathbf{x}}_1 + K_1 (\mathbf{y}_1 - \widetilde{H}_1 \overline{\mathbf{x}}_1) \end{aligned}$$

From the problem definition

$$\begin{aligned} \overline{\mathbf{x}}_1 &= \Phi(t_1, t_0) \overline{\mathbf{x}}_o \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \overline{P}_1 &= \Phi(t_1, t_0) \overline{P}_o \Phi^T(t_1, t_0) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

$$\overline{P}_1 \widetilde{H}_1^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 3/2 \\ 1 & 1 \end{bmatrix}$$

$$\widetilde{H}_1\overline{P}_1\widetilde{H}_1^T = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 3/2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 5/4 \end{bmatrix}$$

$$(W_1^{-1} + \widetilde{H}_1\overline{P}_1\widetilde{H}_1^T)^{-1} = \left[\begin{pmatrix} 2 & 0 \\ 0 & 3/4 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 5/4 \end{pmatrix}\right]^{-1}$$
$$= \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{bmatrix}$$

$$K_1 = \overline{P}_1\widetilde{H}_1^T(W_1^{-1} + \widetilde{H}_1\overline{P}_1\widetilde{H}_1^T)^{-1}$$
$$= \begin{bmatrix} 1 & 3/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 1/10 & 7/10 \\ 1/5 & 2/5 \end{bmatrix}.$$

With these results,

$$\hat{\mathbf{x}}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 1/10 & 7/10 \\ 1/5 & 2/5 \end{bmatrix}\left\{\begin{bmatrix} 6 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right\}$$
$$= \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 1/10 & 7/10 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 4 \\ 1/2 \end{bmatrix}$$
$$\hat{\mathbf{x}}_1 = \begin{bmatrix} 5\frac{3}{4} \\ 3 \end{bmatrix}$$
$$P_1 = (I - K_1\widetilde{H}_1)\overline{P}_1$$
$$= \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 7/20 & 9/20 \\ 1/5 & 2/5 \end{bmatrix}\right\}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$P_1 = \begin{bmatrix} 17/20 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}.$$

(3) Map $\hat{\mathbf{x}}_o$ obtained with the batch processor to t_1.

$$\hat{\mathbf{x}}_1 = \Phi(t_1, t_0)\hat{\mathbf{x}}_0$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2\frac{3}{4} \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 5\frac{3}{4} \\ 3 \end{bmatrix}$$

Hence, the batch and sequential estimates and covariances at t_1 agree. Note that P_0 from the batch processor may be mapped to t_1 by

$$P_1 = \Phi(t_1, t_0)P_0\Phi^T(t_1, t_0)$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 17/20 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

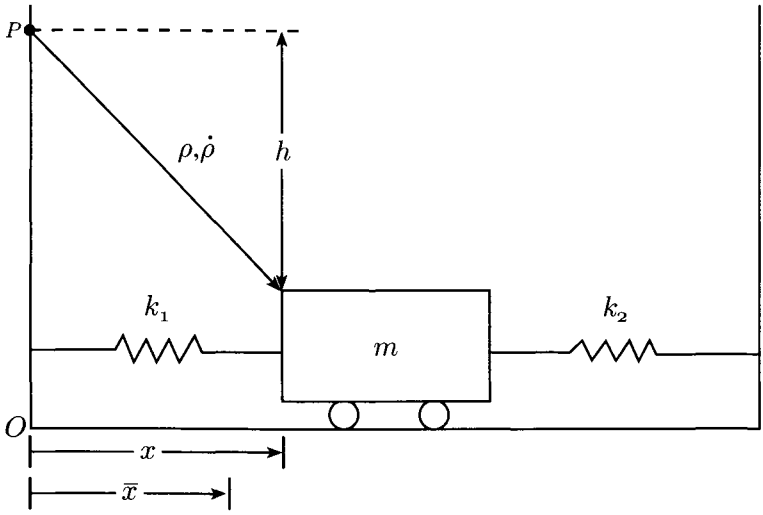


Figure 4.8.2: Spring-mass system.

$$P_1 = \begin{bmatrix} 17/20 & 1/5 \\ 1/5 & 2/5 \end{bmatrix},$$

which agrees with the sequential processor results.

4.8.2 Spring-Mass Problem

A block of mass m is attached to two parallel vertical walls by two springs as shown in Fig. 4.8.2. k_1 and k_2 are the spring constants. h is the height of the position P on one of the walls, from which the distance, ρ, and the rate of change of distance of the block from P, $\dot{\rho}$, can be observed.

Let the horizontal distances be measured with respect to the point O where the line OP, the lengths of the springs, and the center of mass of the block are all assumed to be in the same vertical plane. Then, if $\bar{x}$ denotes the position of the block at static equilibrium, the equation of motion of the block is given by

$$\ddot{x} \equiv \dot{v} = -(k_1 + k_2)(x - \bar{x})/m. \tag{4.8.1}$$

Let

$$\omega^2 = (k_1 + k_2)/m.$$

Since $\bar{x}$ is an arbitrary constant we have set it to zero so that Eq. (4.8.1) can be

written as

$$\dot{v} + \omega^2 x = 0. \tag{4.8.2}$$

Consider the problem of estimating the position and the velocity of the block with respect to the reference point O, by using the range and range-rate measurements of the block from the point, P. In order to formulate this problem mathematically, the estimation state vector is taken as $\mathbf{X}^T = [x \; v]$. Then the system dynamics are represented by

$$\dot{\mathbf{X}} = F(\mathbf{X}, t) = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -\omega^2 x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}. \tag{4.8.3}$$

The observation vector is

$$\mathbf{Y}(t) = \begin{bmatrix} \rho \\ \dot{\rho} \end{bmatrix}.$$

Also,

$$G(\mathbf{X}^*, t) = \begin{bmatrix} \rho \\ \dot{\rho} \end{bmatrix}^* = \begin{bmatrix} \sqrt{x^2 + h^2} \\ xv/\rho \end{bmatrix}^*. \tag{4.8.4}$$

The system parameters are m, k_1, k_2 (from dynamics), and h (from geometry).

Note that the state propagation equation, Eq. (4.8.3), is linear in the state variables $\mathbf{x}$ and v. However, the observation-state equation, Eq. (4.8.4), is nonlinear. Hence, the system must be linearized about a reference solution. Accordingly, let

$$\delta\mathbf{X}(t) = \begin{bmatrix} x(t) - x^*(t) \\ v(t) - v^*(t) \end{bmatrix}$$

and we will solve for $\delta\hat{\mathbf{X}}(t_0) \equiv \delta\hat{\mathbf{X}}_0$ and

$$\delta\dot{\hat{\mathbf{X}}}(t) = A(t)\delta\mathbf{X}(t)$$
$$\mathbf{y}(t) = \widetilde{H}\delta\mathbf{X}(t)$$

where

$$\mathbf{y}(t) = \mathbf{Y}(t) - G(\mathbf{X}^*, t)$$

$$A(t) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

$$\widetilde{H} = \begin{bmatrix} \frac{x}{\rho} & 0 \\ \left(\frac{v}{\rho} - \frac{x^2 v}{\rho^3}\right) & \frac{x}{\rho} \end{bmatrix}.$$

The solution to Eq. (4.8.3) is given by (assuming $t_0 = 0$)

$$\begin{aligned} x(t) &= x_0 \cos\omega t + \tfrac{v_0}{\omega} \sin\omega t \\ v(t) &= v_0 \cos\omega t - x_0 \omega \sin\omega t. \end{aligned} \tag{4.8.5}$$

Hence, we could determine the state transition matrix by differentiating Eq. (4.8.5),

$$\Phi(t,0) = \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_0)}.$$

Alternatively, the solution to the variational equation

$$\dot{\Phi}(t,0) = A(t)\Phi(t,0), \quad \Phi(0,0) = I$$

is given by

$$\Phi(t,0) = \begin{bmatrix} \cos\omega t & \frac{1}{\omega}\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{bmatrix}$$

hence,

$$\delta\mathbf{X}(t) = \Phi(t,0)\delta\mathbf{X}(t_0).$$

Also,

$$H_i = \widetilde{H}_i\Phi(t_i,0).$$

Perfect observations were simulated for a period of 10 seconds at one-second intervals (see Table 4.8.1), assuming the following values for the system parameters and the initial condition for the state parameters:

Table 4.8.1:

Observation Data

Time (s)	$\rho(m)$	$\dot{\rho}\ (m/s)$
0.00	6.1773780845922	0
1.00	5.56327661282686	1.31285863495514
2.00	5.69420161397342	−1.54488114381612
3.00	6.15294262127432	0.534923988815733
4.00	5.46251322092491	0.884698415328368
5.00	5.83638064328625	−1.56123248918054
6.00	6.08236452736002	1.00979943157547
7.00	5.40737619817037	0.31705117039215
8.00	5.97065615746125	−1.37453070975606
9.00	5.97369258835895	1.36768169443236
10.00	5.40669060248179	−0.302111588503166

$$\begin{aligned} k_1 &= 2.5 \text{ N/m} \\ k_2 &= 3.7 \text{ N/m} \\ m &= 1.5 \text{ kg} \\ h &= 5.4 \text{ m} \\ x_0 &= 3.0 \text{ m} \\ v_0 &= 0.0 \text{ m/s}. \end{aligned} \tag{4.8.6}$$

The 11 perfect observations were processed using the batch processor with the following *a priori* values:

$$\mathbf{X}_0^* = \begin{bmatrix} 4.0 \\ 0.2 \end{bmatrix}, \overline{P}_0 = \begin{bmatrix} 1000 & 0 \\ 0 & 100 \end{bmatrix}, \text{ and } \delta\overline{\mathbf{X}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The computed observations were calculated with the exact values of k_1, k_2, m, and h from Eq. (4.8.6); $\mathbf{X}_0^*$ was used for the values of x_0^* and v_0^*.

The least squares solution for $\delta\mathbf{X}_0$ is given by (assuming that $W_i^{-1} = R_i = I$)

$$\delta\hat{\mathbf{X}}_0 = \left(\sum_{i=0}^{10} H_i^T H_i + \overline{P}_0^{-1}\right)^{-1} \left(\sum_{i=0}^{10} H_i^T y_i + \overline{P}_0^{-1}\delta\overline{\mathbf{X}}_0\right)$$

and the best estimate of the initial state is

$$\hat{\mathbf{X}}_0 = \mathbf{X}_0^* + \delta\hat{\mathbf{X}}_0.$$

After each iteration shift the *a priori* state deviation vector so that

$$(\delta\bar{\mathbf{X}}_0)_n = (\delta\bar{\mathbf{X}}_0)_{n-1} - (\delta\hat{\mathbf{X}}_0)_{n-1}$$

following Eq. (4.6.4) and Fig. 4.6.1.

After four iterations the estimate for the state at $t = 0$ is

$$\left[\hat{\mathbf{X}}_0\right] = \begin{bmatrix} \hat{x}_0 \\ \hat{v}_0 \end{bmatrix} = \begin{bmatrix} 3.00019 \text{ m} \\ 1.18181 \times 10^{-3} \text{ m/s} \end{bmatrix}.$$

The computed standard deviations and correlation coefficient are

$$\sigma_{x_0} = 0.411 \text{ m}, \ \sigma_{v_0} = 0.765 \text{ m/s}, \quad \text{and} \quad \rho_{x_0 v_0} = 0.0406.$$

The mean of the residuals is $\rho_{\text{mean}} = -4.30 \times 10^{-5}$m, $\dot{\rho}_{\text{mean}} = -1.76 \times 10^{-6}$ m/s. The RMS of residuals is $\rho_{RMS} = 1.16 \times 10^{-4}$m, $\dot{\rho}_{\text{RMS}} = 4.66 \times 10^{-4}$ m/s.

4.9 STATE NOISE COMPENSATION ALGORITHM

In addition to the effects of the nonlinearities, the effects of errors in the dynamical model can lead to divergence in the estimate. See, for example, the discussion in Schlee *et al.*, (1967) . As pointed out previously, for a sufficiently large number of observations the elements of the covariance matrix P_k will asymptotically approach zero and the estimation algorithm will be insensitive to any further observations. This condition can lead to filter divergence. One approach to preventing this divergence is to recognize that the linearized equations for propagating the estimate of the state are in error and to compensate for this by assuming that the error in the linearized dynamics can be approximated by process noise.

The state dynamics of a linear system under the influence of process noise are described by

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) \tag{4.9.1}$$

where $A(t)$ and $B(t)$ are known functions of time. The vector $\mathbf{u}(t)$ is dimension $m \times 1$, and $B(t)$ is an $n \times m$ matrix. The functional form of $\mathbf{u}(t)$ can include a number of processes, including constant, piecewise constant, correlated, or white noise. In the following discussion, the function $\mathbf{u}(t)$ (called *state or process noise*) is assumed to be a white noise process with

$$\begin{aligned} E[\mathbf{u}(t)] &= 0 \\ E[\mathbf{u}(t)\mathbf{u}^T(\tau)] &= Q(t)\delta(t-\tau) \end{aligned} \tag{4.9.2}$$

where $\delta(t - \tau)$ is the Dirac Delta, and Q is called the *process noise covariance matrix.*

The algorithm that results from the assumption that $\mathbf{u}(t)$ is white noise with known covariance is known as *State Noise Compensation (SNC).* The use of more sophisticated models such as the process to compensate for state and/or measurement model errors generally is referred to as *Dynamic Model Compensation* (DMC). In the case of DMC, process noise parameters are often included in the state vector to be estimated. Appendix F provides more details on SNC and DMC.

The solution of Eq. (4.9.1) can be obtained by the method of variation of parameters. The homogeneous equation is given by

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) \tag{4.9.3}$$

which, as previously noted, has a solution of the form

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{C}_0. \tag{4.9.4}$$

The method of variation of parameters selects $\mathbf{C}_0$ as a function of time so that

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{C}(t). \tag{4.9.5}$$

It follows then that

$$\dot{\mathbf{x}}(t) = \dot{\Phi}(t, t_0)\mathbf{C}(t) + \Phi(t, t_0)\dot{\mathbf{C}}(t). \tag{4.9.6}$$

Substituting Eq. (4.9.6) into Eq. (4.9.1) yields

$$\dot{\Phi}(t, t_0)\mathbf{C}(t) + \Phi(t, t_0)\dot{\mathbf{C}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t). \tag{4.9.7}$$

Recall that

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0). \tag{4.9.8}$$

Substituting Eq. (4.9.8) into Eq. (4.9.7) and using Eq. (4.9.5) reduces Eq. (4.9.7) to

$$\Phi(t, t_0)\dot{\mathbf{C}}(t) = B(t)\mathbf{u}(t). \tag{4.9.9}$$

Hence,

$$\dot{\mathbf{C}}(t) = \Phi^{-1}(t, t_0)B(t)\mathbf{u}(t). \tag{4.9.10}$$

Integrating Eq. (4.9.10) yields

$$\mathbf{C}(t) = \mathbf{C}_0 + \int_{t_0}^{t} \Phi^{-1}(\tau, t_0)B(\tau)\mathbf{u}(\tau)d\tau. \tag{4.9.11}$$

Substituting Eq. (4.9.11) into Eq. (4.9.5) results in

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{C}_0 + \int_{t_0}^{t} \Phi(t, t_0)\Phi^{-1}(\tau, t_0)B(\tau)\mathbf{u}(\tau)d\tau. \tag{4.9.12}$$

Using the properties of the transition matrix, we may write

$$\Phi(t, t_0)\Phi^{-1}(\tau, t_0) = \Phi(t, t_0)\Phi(t_0, \tau) = \Phi(t, \tau). \tag{4.9.13}$$

At $t = t_0$, $\mathbf{x}(t_0) = \mathbf{x}_0$; hence, Eq. (4.9.12) can be used to determine that $\mathbf{C}_0 = \mathbf{x}_0$. With these results, Eq. (4.9.12) can be written as

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\mathbf{u}(\tau)d\tau. \tag{4.9.14}$$

Equation (4.9.14) is the general solution for the inhomogeneous Eq. (4.9.1) and indicates how the true state propagates under the influence of process noise.

The equations for propagating the state estimate $\hat{\mathbf{x}}(t_{k-1})$ to the next observation time, t_k, are obtained by recalling that

$$\overline{\mathbf{x}}(t) = E[\mathbf{x}(t)|\mathbf{y}_{k-1}] \quad \text{for} \quad t \geq t_{k-1}; \tag{4.9.15}$$

that is, the expected value of $\mathbf{x}(t)$ given observations through t_{k-1}. Also, because we have observations through t_{k-1}

$$\overline{\mathbf{x}}(t_{k-1}) = \hat{\mathbf{x}}(t_{k-1}). \tag{4.9.16}$$

Differentiating Eq. (4.9.15) and using Eq. (4.9.1) gives

$$\begin{aligned}\dot{\overline{\mathbf{x}}}(t) &= E\left[\dot{\mathbf{x}}(t)|\mathbf{y}_{k-1}\right] \\ &= E\left[\{A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)\}|\mathbf{y}_{k-1}\right] \\ &= A(t)\, E\left[\mathbf{x}(t)|\mathbf{y}_{k-1}\right].\end{aligned}$$

Since $E[\mathbf{u}(t)] = 0$, it follows then that

$$\dot{\overline{\mathbf{x}}}(t) = A(t)\overline{\mathbf{x}}(t) \tag{4.9.17}$$

with initial conditions $\overline{\mathbf{x}}(t_{k-1}) = \hat{\mathbf{x}}(t_{k-1})$. In Eq. (4.9.17), the assumption has been made that the state noise $\mathbf{u}(t)$ has zero mean and is independent of the observations,

$$E[\mathbf{u}(t)|\mathbf{y}_{k-1}] = E[\mathbf{u}(t)] = 0. \tag{4.9.18}$$

Hence, if the mean of the process noise is zero, ($E[\mathbf{u}(t)] = 0$), then the equation for propagating the state estimate is the same as without process noise,

$$\overline{\mathbf{x}}(t) = \Phi(t, t_{k-1})\hat{\mathbf{x}}_{k-1}. \tag{4.9.19}$$

One could derive a solution for the case where the mean is nonzero. In the case where $E[\mathbf{u}(t)] = \overline{\mathbf{u}}$, the solution would be obtained by applying the expectation operator to Eq. (4.9.14) to yield

$$\overline{\mathbf{x}}(t) = \Phi(t, t_{k-1})\hat{\mathbf{x}}_{k-1} + \Gamma(t_k, t_{k-1})\overline{\mathbf{u}} \tag{4.9.20}$$

where $\Gamma(t_k, t_{k-1})$ is defined by Eq. (4.9.47).

The equation for propagation of the estimation error covariance matrix is obtained by using the definition for $\overline{P}(t)$, given by

$$\overline{P}(t) = E[(\bar{\mathbf{x}}(t) - \mathbf{x}(t))(\overline{\mathbf{x}}(t) - \mathbf{x}(t))^T|\mathbf{y}_{k-1}] \;\; t \geq t_{k-1}. \tag{4.9.21}$$

Let

$$\Delta\mathbf{x}(t) \equiv \overline{\mathbf{x}}(t) - \mathbf{x}(t).$$

Then

$$\overline{P}(t) = E[\Delta\mathbf{x}\Delta\mathbf{x}^T|\mathbf{y}_{k-1}]$$

and differentiating $\overline{P}(t)$ yields

$$\begin{aligned}\dot{\overline{P}}(t) &= E\left[\frac{d}{dt}\,\Delta\mathbf{x}\Delta\mathbf{x}^T|\mathbf{y}_{k-1}\right] \\ &= E\left[\Delta\dot{\mathbf{x}}\Delta\mathbf{x}^T + \Delta\mathbf{x}\Delta\dot{\mathbf{x}}^T|\mathbf{y}_{k-1}\right]. \end{aligned} \tag{4.9.22}$$

From Eqs. (4.9.1) and (4.9.17)

$$\begin{aligned}\Delta\dot{\mathbf{x}}(t) &= \dot{\bar{\mathbf{x}}}(t) - \dot{\mathbf{x}}(t) \\ &= A(t)\bar{\mathbf{x}}(t) - A(t)\mathbf{x}(t) - B(t)\mathbf{u}(t).\end{aligned}$$

Hence,

$$\Delta\dot{\mathbf{x}}(t) = A(t)\Delta\mathbf{x}(t) - B(t)\mathbf{u}(t). \tag{4.9.23}$$

Substituting Eq. (4.9.23) into Eq. (4.9.22) yields

$$\begin{aligned}\dot{\overline{P}}(t) &= E\left[\left\{(A(t)\Delta\mathbf{x} - B(t)\mathbf{u}(t))\Delta\mathbf{x}^T + \Delta\mathbf{x}(A(t)\Delta\mathbf{x} - B(t)\mathbf{u}(t))^T\right\}|\mathbf{y}_{k-1}\right] \\ &= A(t)E\left[\Delta\mathbf{x}\Delta\mathbf{x}^T|\mathbf{y}_{k-1}\right] - B(t)E\left[\mathbf{u}(t)\Delta\mathbf{x}^T|\mathbf{y}_{k-1}\right] \\ &\quad + E\left[\Delta\mathbf{x}\Delta\mathbf{x}^T|\mathbf{y}_{k-1}\right]A^T(t) - E\left[\Delta\mathbf{x}\mathbf{u}^T(t)|\mathbf{y}_{k-1}\right]B^T(t)\end{aligned}$$

and using Eq. (4.9.21)

$$\begin{aligned}\dot{\overline{P}}(t) &= A(t)\overline{P}(t) + \overline{P}(t)A^T(t) - B(t)E\left[\mathbf{u}(t)\Delta\mathbf{x}^T|\mathbf{y}_{k-1}\right] \\ &\quad - E\left[\Delta\mathbf{x}\mathbf{u}^T(t)|\mathbf{y}_{k-1}\right]B^T(t).\end{aligned} \tag{4.9.24}$$

The solution for $\Delta\mathbf{x}(t)$ now is needed to substitute into Eq. (4.9.24). Note that the solution for Eq. (4.9.23) is identical in form to the solution of Eq. (4.9.1) given by Eq. (4.9.14). Substituting Eq. (4.9.14) with appropriate subscripts into the last term of Eq. (4.9.24) yields

$$\begin{aligned}E\left[\Delta\mathbf{x}(t)\mathbf{u}^T(t)|\mathbf{y}_{k-1}\right] &= E\left[\Phi(t,t_{k-1})\Delta\mathbf{x}_{k-1}\mathbf{u}^T(t)|\mathbf{y}_{k-1}\right] \\ &\quad - E\left[\int_{t_{k-1}}^{t}\Phi(t,\tau)B(\tau)\mathbf{u}(\tau)\mathbf{u}^T(t)d\tau|\mathbf{y}_{k-1}\right]\end{aligned} \tag{4.9.25}$$

where $t \geq t_{k-1}$. However,

$$E\left[\Delta\mathbf{x}_{k-1}\mathbf{u}^T(t)|\mathbf{y}_{k-1}\right] = 0$$

since the forcing function $\mathbf{u}(t)$ cannot influence the state at a time t_{k-1} for $t \geq$

t_{k-1}. Hence,

$$E\left[\Delta\mathbf{x}(t)\mathbf{u}^T(t)|\mathbf{y}_{k-1}\right] = -\int_{t_{k-1}}^{t} \Phi(t,\tau)B(\tau)E\left[\mathbf{u}(\tau)\mathbf{u}^T(t)d\tau|\mathbf{y}_{k-1}\right]. \tag{4.9.26}$$

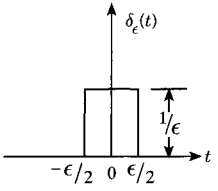
From Eq. (4.9.2)

$$E\left[\mathbf{u}(\tau)\mathbf{u}^T(t)\right] = Q(t)\delta(t-\tau) \tag{4.9.27}$$

where $\delta(t-\tau)$ is the *Dirac delta function*, which is defined to be zero everywhere except at $\tau = t$, where it is infinite in such a way that the integral of $\delta(t-\tau)$ across the singularity is unity; the Dirac delta function is defined by

$$\delta(t) = \lim_{\epsilon\to 0} \delta_\epsilon(t)$$

where $$\delta_\epsilon(t) = \begin{cases} 0 & |t| > \frac{\epsilon}{2} \\ \frac{1}{\epsilon} & |t| < \frac{\epsilon}{2} \end{cases}.$$



This definition implies that

$$\int_{t_1}^{t_2} \delta(t-\tau)d\tau = \int_{t-\epsilon/2}^{t+\epsilon/2} \delta(t-\tau)d\tau = 1.$$

Substituting Eq. (4.9.27) into Eq. (4.9.26) gives

$$E\left[\Delta\mathbf{x}\mathbf{u}^T(t)|\mathbf{y}_{k-1}\right] = -\int_{t_{k-1}}^{t} \Phi(t,\tau)B(\tau)Q(t)\delta(t-\tau)d\tau. \tag{4.9.28}$$

Now

$$\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{t-\epsilon/2} 0d\tau + \int_{t-\epsilon/2}^{t+\epsilon/2} \frac{1}{\epsilon}d\tau + \int_{t+\epsilon/2}^{\infty} 0d\tau\right].$$

Hence, Eq. (4.9.28) becomes

$$E\left[\Delta\mathbf{x}(t)\mathbf{u}^T(t)|\mathbf{y}_{k-1}\right] =$$

$$-\lim_{\epsilon\to 0}\left[\int_{t_{k-1}}^{t-\epsilon/2} 0d\tau + \int_{t-\epsilon/2}^{t} \Phi(t,\tau)B(\tau)Q(t)(\frac{1}{\epsilon})d\tau\right]. \tag{4.9.29}$$

Since ϵ is small, $\Phi(t,\tau)$ and $B(\tau)$ can be expanded in a Taylor series about $\Phi(t,t)$ and $B(t)$, respectively. To this end, let

$$\Phi(t,\tau) = \Phi(t,t) - \dot{\Phi}(t,t)(t-\tau) + O(t-\tau)^2. \tag{4.9.30}$$

The second term of this equation is negative, since $t \geq \tau$. Using the fact that $\Phi(t,t) = I$ and $\dot{\Phi}(t,t) = A(t)\Phi(t,t)$, we can write

$$\Phi(t,\tau) = I - A(t)(t-\tau) + O(t-\tau)^2. \tag{4.9.31}$$

Similarly we can expand $B(\tau)$:

$$B(\tau) = B(t) - \frac{d}{dt}B(t)(t-\tau) + O(t-\tau)^2. \tag{4.9.32}$$

Substituting Eqs. (4.9.31) and (4.9.32) into Eq. (4.9.29) yields

$$\begin{aligned} E\left[\Delta\mathbf{x}\mathbf{u}^T(t)|\mathbf{y}_{k-1}\right] = &- \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t-\epsilon/2}^{t}\Big[[I - A(t)(t-\tau)]\,[B(t)Q(t) \\ &- \frac{d}{dt}\,B(t)Q(t)(t-\tau)] + O(t-\tau)^2\Big]d\tau \\ = &- \lim_{\epsilon\to 0}\int_{t-\epsilon/2}^{t}\Big[B(t)Q(t) - \frac{d}{dt}B(t)Q(t)(t-\tau) \\ &- A(t)B(t)Q(t)(t-\tau) \\ &+ A(t)\frac{d}{dt}B(t)Q(t)(t-\tau)^2 \\ &+ O(t-\tau)^2\Big]\frac{d\tau}{\epsilon}. \end{aligned} \tag{4.9.33}$$

Ignoring higher order terms $(O(t-\tau)^2)$ in Eq. (4.9.33) results in

$$\begin{aligned} E\left[\Delta\mathbf{x}(t)\mathbf{u}^T(t)|\mathbf{y}_{k-1}\right] = &- \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t-\epsilon/2}^{t}\Big\{B(t)Q(t) - \Big[A(t)B(t)Q(t) \\ &+ \frac{d}{dt}\,B(t)Q(t)\Big](t-\tau)\Big\}d\tau. \end{aligned}$$

Defining $K(t) = A(t)B(t)Q(t) + \frac{d}{dt}B(t)Q(t)$ and carrying out the integration

$$\begin{aligned} E\left[\Delta\mathbf{x}\mathbf{u}^T(t)|\mathbf{y}_{k-1}\right] &= - \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[B(t)Q(t)\tau - K(t)(t\tau - \frac{\tau^2}{2})\right]_{t-\frac{\epsilon}{2}}^{t} \\ &= - \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[B(t)Q(t)\frac{\epsilon}{2} - K(t)\left(\frac{\epsilon^2}{8}\right)\right] \end{aligned}$$

$$= -\lim_{\epsilon \to 0} \left[\frac{B(t)Q(t)}{2} - \frac{K(t)}{8}\epsilon \right]$$
$$= -\frac{B(t)Q(t)}{2}. \tag{4.9.34}$$

Substituting (4.9.34) and its transpose into (4.9.24) leads to

$$\dot{\overline{P}}(t) = A(t)\overline{P}(t) + \overline{P}(t)A^T(t) + B(t)Q(t)B^T(t). \tag{4.9.35}$$

Equation (4.9.35) is an $n \times n$ matrix differential equation whose solution may be obtained by integrating with the initial conditions $\overline{P}(t_k) = P_k$; that is, the measurement update of the estimation error covariance matrix at t_k.

Equation (4.9.35) also can be expressed in integral form by using the method of variation of parameters. The homogeneous differential equation is given by

$$\dot{\overline{P}}(t) = A(t)\overline{P}(t) + \overline{P}(t)A^T(t), \tag{4.9.36}$$

which has the solution

$$\overline{P}(t) = \Phi(t, t_0)P_0\Phi^T(t, t_0), \tag{4.9.37}$$

where for convenience, t_{k-1} has been replaced by t_0. Letting P_0 become a function of time, Eq. (4.9.37) becomes

$$\dot{\overline{P}}(t) = \dot{\Phi}(t, t_0)P_0\Phi^T(t, t_0) + \Phi(t, t_0)P_0\dot{\Phi}^T(t, t_0) + \Phi(t, t_0)\dot{P}_0\Phi^T(t, t_0). \tag{4.9.38}$$

Equating Eqs. (4.9.35) and (4.9.38)

$$A(t)\overline{P}(t) + \overline{P}(t)A^T(t) + B(t)Q(t)B^T(t) = \dot{\Phi}(t, t_0)P_0\Phi^T(t, t_0) + \Phi(t, t_0)P_0\dot{\Phi}^T(t, t_0) + \Phi(t, t_0)\dot{P}_0\Phi^T(t, t_0). \tag{4.9.39}$$

However, from Eqs. (4.9.36) and (4.9.37)

$$A(t)\overline{P}(t) + \overline{P}(t)A^T(t) = \dot{\Phi}(t, t_0)P_0\Phi^T(t, t_0) + \Phi(t, t_0)P_0\dot{\Phi}^T(t, t_0). \tag{4.9.40}$$

Using Eq. (4.9.40), Eq. (4.9.39) reduces to

$$\Phi(t, t_0)\dot{P}_0\Phi^T(t, t_0) = B(t)Q(t)B^T(t),$$

or

$$\dot{P}_o = \Phi^{-1}(t, t_0)B(t)Q(t)B^T(t)\Phi^{-T}(t, t_0). \tag{4.9.41}$$

Integrating Eq. (4.9.41) results in

$$P_0(t) - P_0 = \int_{t_0}^{t} \Phi(t_0, \tau) B(\tau) Q(\tau) B^T(\tau) \Phi^T(t_0, \tau) d\tau. \tag{4.9.42}$$

Substituting Eq. (4.9.42) into (4.9.37)

$$\begin{aligned} \overline{P}(t) &= \Phi(t, t_0)[P_0 + \int_{t_0}^{t} \Phi(t_0\, \tau) B(\tau) Q(\tau) B^T(\tau) \Phi^T(t_0, \tau) d\tau] \Phi^T(t, t_0) \\ &= \Phi(t, t_0) P_0 \Phi^T(t, t_0) + \int_{t_0}^{t} \Phi(t, t_0) \Phi(t_0, \tau) B(\tau) Q(\tau) B^T(\tau) \\ &\times \Phi^T(t_0, \tau) \Phi^T(t, t_0) d\tau \,. \end{aligned} \tag{4.9.43}$$

If we use

$$\Phi(t, t_0)\Phi(t_0\, \tau) = \Phi(t, \tau),$$

and

$$\Phi^T(t_0, \tau)\Phi^T(t, t_0) = [\Phi(t, t_0)\Phi(t_0, \tau)]^T = \Phi^T(t, \tau)\,,$$

then after letting $t_0 = t_{k-1}$, Eq. (4.9.43) becomes

$$\begin{aligned} \overline{P}(t) &= \Phi(t, t_{k-1}) P_{k-1} \Phi^T(t, t_{k-1}) \\ &+ \int_{t_{k-1}}^{t} \Phi(t, \tau) B(\tau) Q(\tau) B^T(\tau) \Phi^T(t, \tau) d\tau. \end{aligned} \tag{4.9.44}$$

Equations (4.9.19) and (4.9.44) are the equations for propagating the estimate of the state and the covariance for a *continuous system*. Since the orbit determination problem generally consists of a continuous system (the trajectory) subjected to discrete observations, it is convenient to use Eq. (4.9.19) to propagate the state estimate and to discretize Eq. (4.9.44). This can be accomplished by replacing t with t_{k+1} and assuming that $\mathbf{u}(\tau)$ is a white *random sequence* rather than a process. Thus, as indicated in Fig. 4.9.1, $\mathbf{u}(t)$ is considered to be a piecewise constant function with covariance

$$E[\mathbf{u}(t_i)\mathbf{u}^T(t_j)] = Q_i \delta_{ij} \qquad \delta_{ij} = \begin{cases} 1\, i = j \\ 0\, i \neq j \end{cases} \tag{4.9.45}$$

where the Dirac delta function has been replaced by its analog for the discrete case, the Kroneker delta function. In the discrete case, Eq. (4.9.14) becomes

$$\mathbf{x}(t_{k+1}) \equiv \mathbf{x}_{k+1} = \Phi(t_{k+1}, t_k)\mathbf{x}_k \; + \Gamma(t_{k+1}, t_k)\mathbf{u}_k \tag{4.9.46}$$

where

$$\Gamma(t_{k+1}, t_k) = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau) B(\tau) d\tau. \tag{4.9.47}$$

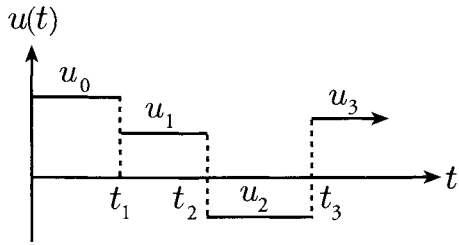


Figure 4.9.1: A white random sequence.

$\Gamma(t_{k+1}, t_k)$ is referred to as the *process noise transition matrix*, and Eq. (4.9.47) is an $n \times m$ quadrature since $\Phi(t_{k+1}, \tau)$ and $B(\tau)$ are known functions. Using the definition of the estimation error covariance matrix

$$\overline{P}_{k+1} = E[(\overline{\mathbf{x}}_{k+1} - \mathbf{x}_{k+1})(\overline{\mathbf{x}}_{k+1} - \mathbf{x}_{k+1})^T] \tag{4.9.48}$$

and substituting Eqs. (4.9.46) and (4.9.19) into (4.9.48) leads to

$$\overline{P}_{k+1} = E\Big\{ \big[\Phi(t_{k+1}, t_k)\hat{\mathbf{x}}_k - \Phi(t_{k+1}, t_k)\mathbf{x}_k - \Gamma(t_{k+1}, t_k)\mathbf{u}_k\big] \times \big[\Phi(t_{k+1}, t_k)\hat{\mathbf{x}}_k - \Phi(t_{k+1}, t_k)\mathbf{x}_k - \Gamma(t_{k+1}, t_k)\mathbf{u}_k\big]^T \Big\}. \tag{4.9.49}$$

Note that $E[(\hat{\mathbf{x}}_k - \mathbf{x}_k)\mathbf{u}_k^T] = 0$; that is, $\mathbf{u}_k$ cannot affect the estimation error at t_k since a finite time must evolve for $\mathbf{u}_k$ to affect the state. Finally, carrying out the expectation operation in Eq. (4.9.49) yields

$$\overline{P}_{k+1} = \Phi(t_{k+1}, t_k)P_k\Phi^T(t_{k+1}, t_k) + \Gamma(t_{k+1}, t_k)Q_k\Gamma^T(t_{k+1}, t_k). \tag{4.9.50}$$

The estimation error covariance matrix $\overline{P}_{k+1}$ can be obtained by integrating the differential equation for $\dot{\overline{P}}$ given by Eq. (4.9.35), or $\overline{P}_{k+1}$ may be obtained by using the state and process noise transition matrices as indicated in Eq. (4.9.50). A comparison of Eq. (4.9.35) and Eq. (4.9.50) indicates the following:

1. Since $P(t)$ is symmetric, only $n(n+1)/2$ of the $n \times n$ system of equations represented by Eq. (4.9.35) must be integrated. However, the $n(n+1)/2$ equations are coupled and must be integrated as a single first-order system of dimension $n(n+1)/2$.

2. The $n \times n$ system represented by Eq. (4.9.50) can be separated into an $n \times n$ system of differential equations for Φ and an $n \times m$ quadrature for

Γ. Furthermore, the $n \times n$ system of equations represented by the solution for $\Phi(t_{k+1}, t_k)$ can be integrated as a sequence of $n \times 1$ column vectors.

The comparison between the two methods indicates that integration of fewer equations is required to obtain the solution for $\overline{P}(t)$ with Eq. (4.9.35). However, the integration of these equations may be more difficult than the integration associated with the larger system represented by Eq. (4.9.50) since they are coupled.

The equations for determining $\hat{\mathbf{x}}$ using the sequential processing algorithm are unchanged whenever a zero-mean process noise is included. However, as has been shown, the equations that propagate the estimation error covariance matrix do change; that is, the first of Eq. (4.7.31) is replaced by Eq. (4.9.50). Generally, the batch processing algorithm is not used with process noise because mapping of the process noise effects from the observation times to the epoch time is cumbersome. It further results in a normal matrix with an observation weighting matrix that will be nondiagonal and whose dimension is equal to $m \times m$, where m is the total number of observations. Computation of the inverse of the normal matrix will be so cumbersome that the normal equation solution involving process noise in the data accumulation interval is impractical. For example, a one-day tracking period for the TOPEX/Poseidon satellite by the French tracking system, DORIS, typically yields 5000–7000 doppler observations.

The advantage of using the process noise compensated sequential estimation algorithm lies in the fact that the asymptotic value of $\overline{P}(t)$ will approach a nonzero value determined by the magnitude of $Q(t)$. That is, for certain values of $Q(t)$, the increase in the state error covariance matrix $\overline{P}(t)$ during the interval between observations will balance the decrease in the covariance matrix that occurs at the observation point. In this situation, the estimation procedure will always be sensitive to new observations.

The question of how to choose the process noise covariance matrix, $Q(t)$, is complex. In practice, it is often chosen as a simple diagonal matrix and its elements are determined by trial and error. Although this method can be effective for a particular estimation scenario, such a process noise matrix is not generally applicable to other scenarios. The dynamic evolution of the true states of parameters estimated in a filter typically is affected by stochastic processes that are not modeled in the filter's deterministic dynamic model. The process noise model is a characterization of these stochastic processes, and the process noise covariance matrix should be determined by this process noise model. Development of the process noise model will not be presented in depth here; however, extensive discussions are given by Cruickshank (1998), Ingram (1970), and Lichten (1990).

The Gauss-Markov process is used as a process noise model and will be introduced here. It is computationally well suited for describing unmodeled forces since it obeys Gaussian probability laws and is exponentially correlated in time.

4.9.1 THE GAUSS-MARKOV PROCESS

A first-order Gauss-Markov process is often used for dynamic model compensation in orbit determination problems to account for unmodeled or inaccurately modeled accelerations acting on a spacecraft. A Gauss-Markov process is one that obeys a Gaussian probability law and displays the Markov property. The Markov property means that the probability density function at t_n given its past history at t_{n-1}, $t_{n-2}, \ldots$ is equal to its probability density function at t_n given its value at t_{n-1}.

A Gauss-Markov process obeys a differential equation (often referred to as a Langevin equation) of the form

$$\dot{\eta}(t) = -\beta\eta(t) + u(t) \tag{4.9.51}$$

where $u(t)$ is white Gaussian noise with

$$\begin{aligned} E(u) &= 0 \\ E\left[u(t)u(\tau)\right] &= \sigma^2\delta(t-\tau) \end{aligned} \tag{4.9.52}$$

and

$$\beta = \frac{1}{\tau},$$

where τ is the time constant or correlation time (not the same as τ in Eq. (4.9.52)).

Equation (4.9.51) can be solved by the method of variation of parameters to yield

$$\eta(t) = \eta(t_0)e^{-\beta(t-t_0)} + \int_{t_0}^{t} e^{-\beta(t-\tau)}u(\tau)d\tau. \tag{4.9.53}$$

Hence, $\eta(t)$ consists of a deterministic part and a random part. The autocorrelation function is (Maybeck, 1979)

$$\begin{aligned} E\left[\eta(t_j)\eta(t_i)\right] &= e^{-\beta(t_j-t_i)}E\left[\eta(t_i)\eta(t_i)\right] \\ &\quad + E\left[\left(\int_{t_i}^{t_j} e^{-\beta(t_j-\tau)}u(\tau)d\tau\right)\eta(t_i)\right] \\ &= e^{-\beta(t_j-t_i)}E\left[\eta(t_i)\eta(t_i)\right] \end{aligned} \tag{4.9.54}$$

since the stochastic process represented by the integral consists of independent increments. The remaining expectation is the autocorrelation of $\eta(t)$ at t_i:

$$\begin{aligned} E\left[\eta(t_i)\eta(t_i)\right] &\equiv \Psi(t_i, t_i) \\ &= \eta^2(t_0)e^{-2\beta(t_i-t_0)} + \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta(t_i-t_0)}\right) \end{aligned} \tag{4.9.55}$$

where σ^2 is the variance (strength) of the driving noise in Eq. (4.9.51). Using Eq. (4.9.55), equation (4.9.54) can then be written as

$$E\left[\eta(t_j)\eta(t_i)\right] = \Psi(t_i, t_i)e^{-\beta(t_j - t_i)}. \tag{4.9.56}$$

This is important because it points out one of the salient characteristics of the first-order Gauss-Markov process; that is, its autocorrelation fades exponentially with the rate of the fade governed by the time constant, $\tau = 1/\beta$.

Equation (4.9.53) contains a stochastic integral that cannot, in general, be evaluated except in a statistical sense. The mean of the stochastic integral is zero since $E[\mathbf{u}(t)] = 0$. Its variance can be shown to be (Myers, 1973)

$$\frac{\sigma^2}{2\beta}(1 - e^{-2\beta(t_j - t_i)}). \tag{4.9.57}$$

Because the stochastic integral is a Gaussian process it is uniquely defined by its mean and variance. Hence, if a function can be found with the same mean and variance it will be an equivalent process. Such a discrete process is given by

$$L_k \equiv u_k \sqrt{\frac{\sigma^2}{2\beta}(1 - e^{-2\beta(t_j - t_i)})} \tag{4.9.58}$$

where u_k is a discrete, Gaussian random sequence with mean and variance

$$E[u_k] = 0, \;\; E[u_{k_i} u_{k_j}] = \delta_{ij}. \tag{4.9.59}$$

It is evident that L_k has the same mean and variance as the stochastic integral in Eq. (4.9.53); hence, the solution for $\eta(t)$ is given by

$$\begin{aligned} \eta(t_j) = {} & e^{-\beta(t_j - t_i)}\eta(t_i) \\ & + u_k(t_i)\sqrt{\frac{\sigma^2}{2\beta}(1 - e^{-2\beta(t_j - t_i)})} \end{aligned} \tag{4.9.60}$$

where $u_k(t_i)$ is a random number chosen by sampling from a Gaussian density function with a mean of zero and variance of 1.

The degree of correlation of the random process $\eta(t)$ is determined by the choice of σ and β. For a finite value of β and $\sigma^2 \simeq 0$, Eq. (4.9.60) yields

$$\eta(t_j) = e^{-\beta(t_j - t_i)}\eta(t_i) \tag{4.9.61}$$

and as β becomes small, $\eta(t_j) \rightarrow \eta(t_i)$, a constant. For a finite value of σ and $\beta \simeq 0$, Eq. (4.9.60) yields

$$\eta(t_j) = \eta(t_i) + u_k(t_i)\sigma\sqrt{t_j - t_i}; \tag{4.9.62}$$

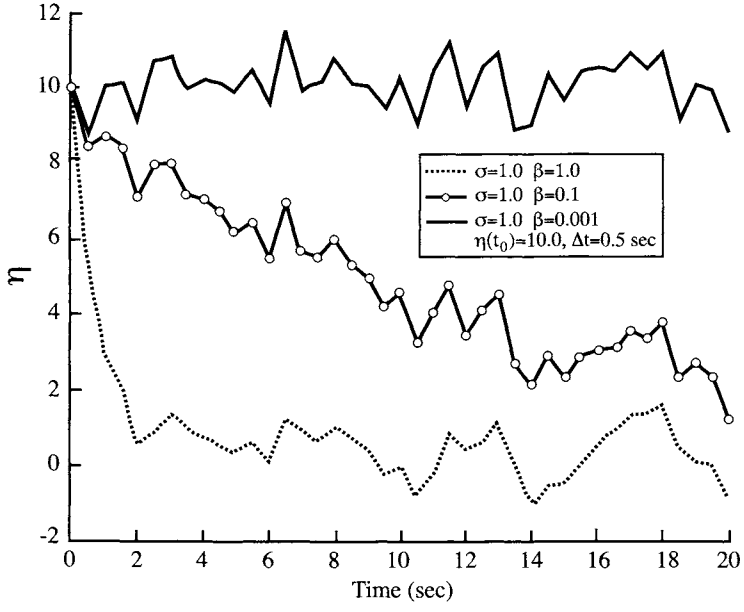


Figure 4.9.2: Gauss-Markov process for three time constants.

that is, a random walk process.

The term $\frac{\sigma^2}{2\beta}$ represents the steady-state variance of $\eta(t)$, the variance after a large enough time so that any transients in $\eta(t)$ have died out and it is in steady state.

With the proper choice of σ and β any number of time-correlated random functions can be generated. Figure 4.9.2 illustrates three histories of $\eta(t)$ for $\beta = 0.001$, 0.1, 1.0, and $\sigma = 1.0$. Note that for $\beta = 10^{-3}$, $\eta(t)$ is essentially constant. The shape of this curve can be varied by changing σ or β or both.

To use the Gauss-Markov process to account for unmodeled accelerations in the orbit determination procedure we may formulate the problem as follows. The equations of motion are

$$\begin{aligned} \dot{\mathbf{r}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{\mu \mathbf{r}}{r^3} + \mathbf{f}(t) + \boldsymbol{\eta}(t) \\ \dot{\boldsymbol{\eta}}(t) &= -\beta \boldsymbol{\eta}(t) + \mathbf{u}(t), \end{aligned} \tag{4.9.63}$$

where $-\frac{\mu \mathbf{r}}{r^3} + \mathbf{f}(t)$ represents the known accelerations and $\boldsymbol{\eta}(t)$ is a 3×1 vector of unknown accelerations. The procedure is to estimate the deterministic portion

of $\eta(t)$ and perhaps the associated time constants as part of the state vector. The random portion of $\eta(t)$ contributes to the process noise covariance matrix Q. For details on implementing this algorithm see Tapley and Ingram (1973), Ingram and Tapley (1974), Cruickshank (1998), Lichten (1990), or Goldstein *et al.* (2001).

An example of the use of process noise for SNC and DMC for a simple problem is given in Appendix F.

4.10 INFORMATION FILTER

A sequential estimation algorithm can be developed by propagating the *information matrix*, $\Lambda \equiv P^{-1}$ (Maybeck, 1979). This form of the filter offers some numerical properties with better characteristics than the covariance filter. Writing Eq. (4.7.3) in terms of the information matrix gives

$$\{\overline{\Lambda}_k + \widetilde{H}_k^T R_k^{-1} \widetilde{H}_k\}\hat{\mathbf{x}}_k = \overline{\Lambda}_k \overline{\mathbf{x}}_k + \widetilde{H}_k^T R_k^{-1} \mathbf{y}_k \tag{4.10.1}$$

or

$$\Lambda_k \hat{\mathbf{x}}_k = \overline{\Lambda}_k \overline{\mathbf{x}}_k + \widetilde{H}_k^T R_k^{-1} \mathbf{y}_k.$$

Recall that $\overline{P}_{k+1}$ is expressed in terms of P_k by Eq. (4.9.50) and $\overline{\mathbf{x}}_k$ is obtained by propagating $\hat{\mathbf{x}}_{k-1}$ according to Eq. (4.9.19). The Schur identity is given by (see Appendix B)

$$(A + BC)^{-1} = A^{-1} - A^{-1}B(I + CA^{-1}B)^{-1}CA^{-1}. \tag{4.10.2}$$

In Eq. (4.9.50) let

$$\begin{aligned} A &= \Phi\,(t_{k+1}, t_k) P_k \Phi^T\,(t_{k+1}, t_k) \\ B &= \Gamma(t_{k+1}, t_k) Q_k \\ C &= \Gamma^T(t_{k+1}, t_k). \end{aligned} \tag{4.10.3}$$

Then, for Q_k nonsingular, the Schur identity yields

$$\begin{aligned} \overline{\Lambda}_{k+1} = \overline{P}_{k+1}^{-1} &= M(t_{k+1}) - M(t_{k+1})\Gamma(t_{k+1}, t_k) \\ &\times [\Gamma^T(t_{k+1}, t_k) M(t_{k+1}) \Gamma(t_{k+1}, t_k) + Q_k^{-1}]^{-1} \\ &\times \Gamma^T(t_{k+1}, t_k) M(t_{k+1}) \end{aligned} \tag{4.10.4}$$

where

$$M(t_{k+1}) = A^{-1} = \Phi^T(t_k, t_{k+1}) P_k^{-1} \Phi(t_k, t_{k+1}). \tag{4.10.5}$$

If there is no process noise, Eq. (4.10.4) reduces to

$$\overline{\Lambda}_{k+1} = \overline{P}_{k+1}^{-1} = M(t_{k+1}).$$

Define

$$L_{k+1} \equiv M(t_{k+1})\Gamma(t_{k+1}, t_k) \times \left[\Gamma^T(t_{k+1}, t_k)M(t_{k+1})\Gamma(t_{k+1}, t_k) + Q_k^{-1}\right]^{-1}. \quad (4.10.6)$$

In terms of L_{k+1}, Eq. (4.10.4) becomes

$$\overline{\Lambda}_{k+1} = M(t_{k+1}) - L_{k+1}\Gamma^T(t_{k+1}, t_k)M(t_{k+1}). \quad (4.10.7)$$

The measurement update for Λ_{k+1} is obtained by inverting Eq. (4.7.4); that is,

$$\Lambda_{k+1} = \overline{\Lambda}_{k+1} + \widetilde{H}_{k+1}^T R_{k+1}^{-1} \widetilde{H}_{k+1}. \quad (4.10.8)$$

Define the following quantities:

$$\overline{\mathbf{D}}_k \equiv \overline{\Lambda}_k \overline{\mathbf{x}}_k \quad (4.10.9)$$

$$\hat{\mathbf{D}}_k \equiv \Lambda_k \hat{\mathbf{x}}_k. \quad (4.10.10)$$

The recursion relations for $\mathbf{D}$ are given by

$$\overline{\mathbf{D}}_{k+1} = \overline{\Lambda}_{k+1}\overline{\mathbf{x}}_{k+1}. \quad (4.10.11)$$

Using Eq. (4.10.7),

$$\overline{\mathbf{D}}_{k+1} = \left\{I - L_{k+1}\Gamma^T(t_{k+1}, t_k)\right\} \times \Phi^T(t_k, t_{k+1})P_k^{-1}\Phi(t_k, t_{k+1})\overline{\mathbf{x}}_{k+1} \quad (4.10.12)$$

but $\hat{\mathbf{x}}_k = \Phi(t_k, t_{k+1})\overline{\mathbf{x}}_{k+1}$ and $\hat{\mathbf{D}}_k = P_k^{-1}\hat{\mathbf{x}}_k$. Hence,

$$\overline{\mathbf{D}}_{k+1} = \left\{I - L_{k+1}\Gamma^T(t_{k+1}, t_k)\right\} \Phi^T(t_k, t_{k+1})\hat{\mathbf{D}}_k. \quad (4.10.13)$$

Also, from Eq. (4.10.9) and the second of Eq. (4.10.1),

$$\hat{\mathbf{D}}_{k+1} = \Lambda_{k+1}\hat{\mathbf{x}}_{k+1} = \overline{\mathbf{D}}_{k+1} + \widetilde{H}_{k+1}^T R_{k+1}^{-1}\mathbf{y}_{k+1}. \quad (4.10.14)$$

The procedure starts with *a priori* values $\overline{P}_0$ and $\overline{\mathbf{x}}_0$. From these compute

$$\overline{\Lambda}_0 = \overline{P}_0^{-1}$$

$$\overline{\mathbf{D}}_0 = \overline{\Lambda}_0\overline{\mathbf{x}}_0. \quad (4.10.15)$$

Next compute M_1, L_1, $\overline{\Lambda}_1$, $\overline{\mathbf{D}}_1$, $\hat{\mathbf{D}}_1$, and Λ_1 from Eqs. (4.10.5), (4.10.6), (4.10.7), (4.10.13), (4.10.14), and (4.10.8). Once Λ_k becomes nonsingular, its inverse can

be computed to obtain P_k and the optimal estimate of $\mathbf{x}$ can be computed from Eq. (4.10.10); that is,

$$\hat{\mathbf{x}}_k = P_k \hat{\mathbf{D}}_k. \tag{4.10.16}$$

Time updates are given by Eqs. (4.10.5), (4.10.6), (4.10.7), and (4.10.13), with initial conditions given by Eq. (4.10.15). The measurement update is accomplished using Eqs. (4.10.8) and (4.10.14).

In summary, the covariance filter and the information filter time and measurement updates are given by

Time Update—Covariance Filter

$$\overline{\mathbf{x}}_{k+1} = \Phi(t_{k+1}, t_k)\hat{\mathbf{x}}_k \tag{4.10.17}$$

$$\overline{P}_{k+1} = \Phi(t_{k+1}, t_k)P_k\Phi^T(t_{k+1}, t_k) + \Gamma(t_{k+1}, t_k)Q_k\Gamma^T(t_{k+1}, t_k). \tag{4.10.18}$$

Measurement Update—Covariance Filter

$$K_{k+1} = \overline{P}_{k+1}\widetilde{H}_{k+1}^T(R_{k+1} + \widetilde{H}_{k+1}\overline{P}_{k+1}\widetilde{H}_{k+1}^T)^{-1} \tag{4.10.19}$$

$$\hat{\mathbf{x}}_{k+1} = \overline{\mathbf{x}}_{k+1} + K_{k+1}(\mathbf{y}_{k+1} - \widetilde{H}_{k+1}\overline{\mathbf{x}}_{k+1}) \tag{4.10.20}$$

$$P_{k+1} = (I - K_{k+1}\widetilde{H}_{k+1})\overline{P}_{k+1}. \tag{4.10.21}$$

Time Update—Information Filter

$$M(t_{k+1}) = \Phi^T(t_k, t_{k+1})\Lambda_k\Phi(t_k, t_{k+1}) \tag{4.10.22}$$

$$L_{k+1} = M(t_{k+1})\Gamma(t_{k+1}, t_k) \times [\Gamma^T(t_{k+1}, t_k)M(t_{k+1})\Gamma(t_{k+1}, t_k) + Q_k^{-1}]^{-1} \tag{4.10.23}$$

$$\overline{\mathbf{D}}_{k+1} = \{I - L_{k+1}\Gamma^T(t_{k+1}, t_k)\} \times \Phi^T(t_k, t_{k+1})\hat{\mathbf{D}}_k \tag{4.10.24}$$

$$\overline{\Lambda}_{k+1} = \overline{P}_{k+1}^{-1} = [I - L_{k+1}\Gamma^T(t_{k+1}, t_k)] \times M(t_{k+1}). \tag{4.10.25}$$

Measurement Update—Information Filter

$$\hat{\mathbf{D}}_{k+1} = \overline{\mathbf{D}}_{k+1} + \widetilde{H}_{k+1}^T R_{k+1}^{-1}\mathbf{y}_{k+1} \tag{4.10.26}$$

$$P_{k+1}^{-1} = \Lambda_{k+1} = \overline{\Lambda}_{k+1} + \widetilde{H}_{k+1}^T R_{k+1}^{-1}\widetilde{H}_{k+1} \tag{4.10.27}$$

$$\hat{\mathbf{x}}_{k+1} = \Lambda_{k+1}^{-1}\hat{\mathbf{D}}_{k+1}. \tag{4.10.28}$$

We can initialize the information filter with $P_0 = \infty$ or with P_0 singular, and obtain valid results for $\hat{\mathbf{x}}$ once P_k becomes nonsingular. The Cholesky Algorithm of Chapter 5 may be used to solve for $\hat{\mathbf{x}}$. However, the solution for $\hat{\mathbf{x}}$ is not required by the algorithm and needs to be computed only when desired. The conventional sequential estimation algorithm fails in these cases. Also, as indicated in Section 4.7.1, the conventional sequential estimator can fail in the case where very accurate measurements are processed, which rapidly reduce the estimation error covariance matrix. This can be mitigated with the information filter.

Consider the example discussed in Section 4.7.1,

$$\overline{P}_0 = \sigma^2 I, \quad \sigma = 1/\epsilon, \quad \widetilde{H}_1 = \begin{bmatrix} 1 \vdots \epsilon \end{bmatrix}, \quad R = I$$

where $\epsilon \ll 1$ and we assume that our computer word length is such that

$$1 + \epsilon \neq 1$$
$$1 + \epsilon^2 = 1.$$

The objective is to find P_1; that is, the estimation error covariance matrix after processing one observation using the information filter. The information filter yields

$$\begin{aligned} M_1 &= \epsilon^2 I \\ \overline{\Lambda}_1 &= M_1 \\ \Lambda_1 &= \overline{\Lambda}_1 + \widetilde{H}_1^T R^{-1} \widetilde{H}_1 \\ &= \epsilon^2 I + \begin{bmatrix} 1 & \epsilon \\ \epsilon & \epsilon^2 \end{bmatrix} \\ \Lambda_1 &= P_1^{-1} = \begin{bmatrix} 1 & \epsilon \\ \epsilon & 2\epsilon^2 \end{bmatrix} \end{aligned}$$

and

$$P_1 = \begin{bmatrix} 2 & -\sigma \\ -\sigma & \sigma^2 \end{bmatrix}.$$

This is the same symmetric, positive definite result obtained from the batch processor and agrees with the exact solution to $O(\epsilon)$. Because we are accumulating the information matrix at each stage, the accuracy of the information filter should be comparable to that of the batch processor. Hence, the conventional covariance filter fails for this example, but the information filter does not.

4.11 BATCH AND SEQUENTIAL ESTIMATION

As described in previous sections, two general categories of estimators are used, the batch processor and the sequential processor, both with distinct advantages and disadvantages. The batch formulation provides an estimate of the state at some chosen epoch using an entire batch or set of data. This estimate and its associated covariance matrix can then be mapped to other times. The sequential processor, on the other hand, provides an estimate of the state at each observation time based on observations up to that time. This solution and its associated covariance also can be mapped to another time.

In the sequential formulation without process noise, a mathematical equivalence can be shown between the batch and sequential algorithms; given the same data set, both algorithms produce the same estimates when the estimates are mapped to the same times. In the extended form of the sequential algorithm, where the reference orbit is updated at each observation time, the algorithms are not equivalent, but numerical experiments have shown a very close agreement.

Normally, the batch and sequential algorithm will need to be iterated to convergence, whereas the extended sequential will obtain near convergence in a single iteration. The sequential algorithm, unlike the batch, requires restarting a numerical integrator at each observation time. In general, the sequential processor is used in real-time applications supporting control or other decision functions and it is appropriate to incorporate some representation of the state noise to ensure that divergence does not occur. This implementation provides a means of compensating for various error sources in the processing of ground-based or onboard data. As indicated previously, inclusion of process noise in the batch algorithm substantially complicates the solution of the normal equations by increasing the dimensions of the normal matrix from n (the number of state parameters) to m (the number of observations).

4.12 OBSERVABILITY

The property of *observability* refers to the ability to apply an estimator to a particular system and obtain a unique estimate for all components of the state vector. As applied to orbital mechanics, the observability property refers to the ability to obtain a unique estimate of the spacecraft state as opposed to whether the satellite can be physically observed. Cases of unobservability rarely occur for properly formulated problems. However, these cases illustrate clearly the caution that must be exercised in applying an estimation algorithm to a particular problem. An unobservable parameter can best be illustrated by the following example.

Consider a satellite moving in an orbit about a perfectly spherical planet with a homogeneous mass distribution. Assuming no forces other than the gravitational attraction of the planet, which can be represented by a point mass, the orbit

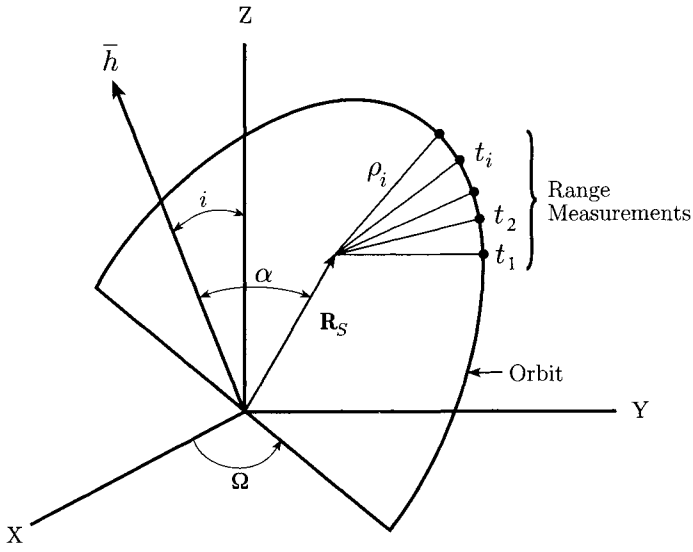


Figure 4.11.1: Single-station range geometry.

will be a conic section. Suppose a laser or radar range system exists on the surface of the planet capable of measuring to infinite precision the distance between the instrument and the satellite. Assuming the planet does not rotate and only a single range measuring system exists, the previously described estimators can be applied to the problem of estimating the state of the spacecraft at some appropriate time. Consider a single pass of the satellite over the station as shown in Fig. 4.11.1, where $\mathbf{R_S}$ is the station position vector, assumed known. Given a sequence of range measurements, ρ_i, as shown, an orbit could be determined from which the computed ranges match the measured values. However, a subtle problem exists because there is no change in the location of the tracking station relative to the orbit plane; hence, multiple solutions are possible—the same range history occurs for each orbit as we rotate the orbit's angular momentum vector about $\mathbf{R_S}$ keeping α constant. This results in an infinite number of solutions that could have the same range measurement sequence. This circumstance leads to an ambiguity in the inclination, i, and longitude of the ascending node, Ω. The difficulty is not alleviated by using multiple passes, since each pass would have the same geometry relative to the station. The possible multiple solutions will be manifested by a singular normal matrix $(H^T H)$ in the ordinary least squares formulation, regardless of the number of observations used.

A unique solution can be obtained if (1) i or Ω is known and not estimated, resulting in five elements being estimated, or (2) an *a priori* covariance matrix is assigned to the state indicative of knowledge of i or Ω. The problem can be avoided by other techniques as well; for example, using angular data such as azimuth and elevation or by using a second ranging station. However, this difficulty is not altered by varying the accuracy of the range measurements, or if range-rate is used instead of, or in addition to, range.

In practice, in a single pass the earth rotates slightly, resulting in a change in the station-orbit geometry. However, the resulting least squares system is frequently ill conditioned; that is, nearly singular. Numerous other examples of nonobservability exist, particularly in the recovery of spherical harmonic coefficients of the earth's gravity field from inadequate amounts and distributions of tracking data.

The mathematical conditions for parameter observability can be derived from the observability criterion for a linear dynamical system.

Theorem: For the linear system and linear observation set,

$$\begin{aligned} \mathbf{x}_i &= \Phi(t_i, t_k)\mathbf{x}_k \\ \mathbf{y}_i &= \widetilde{H}_i\mathbf{x}_i + \boldsymbol{\epsilon}_i; \qquad i = 1, \ldots, \ell \end{aligned} \tag{4.12.1}$$

complete observability of the $n \times 1$ state vector, at the general time, t_k, requires satisfying the condition

$$\delta\mathbf{x}_k^T \Lambda \delta\mathbf{x}_k > 0 \tag{4.12.2}$$

for all arbitrary real vectors, $\delta\mathbf{x}_k$, where the $n \times n$ information matrix, Λ, is defined as

$$\Lambda = \sum_{i=1}^{\ell} \Phi^T(t_i, t_k)\widetilde{H}_i^T R_i^{-1} \widetilde{H}_i \Phi(t_i, t_k) = H^T R^{-1} H \tag{4.12.3}$$

and H is defined by Eq. (4.2.38). Hence, complete observability of the state with the data in the batch accumulated from the observation set $\mathbf{y}_i; i = 1, \ldots, \ell$ requires that the symmetric information matrix, Λ, be positive definite. Note that in order for Λ to be positive definite, H must be full rank.

In addition to the constraints placed on the data as a necessary criterion for observability, care in the mathematical formulation of the problem must be exercised to ensure that only a minimal set of parameters is estimated. If spurious or unnecessary parameters are included in the problem formulation, the solution will be nonobservable regardless of the data characteristics. As an example, the expression for the acceleration on a satellite due to effects of atmospheric drag can be written as

$$\mathbf{D} = -\frac{1}{2}\frac{C_D A}{m}\rho(h)|\mathbf{V}_{rel}|\mathbf{V}_{rel} \tag{4.12.4}$$

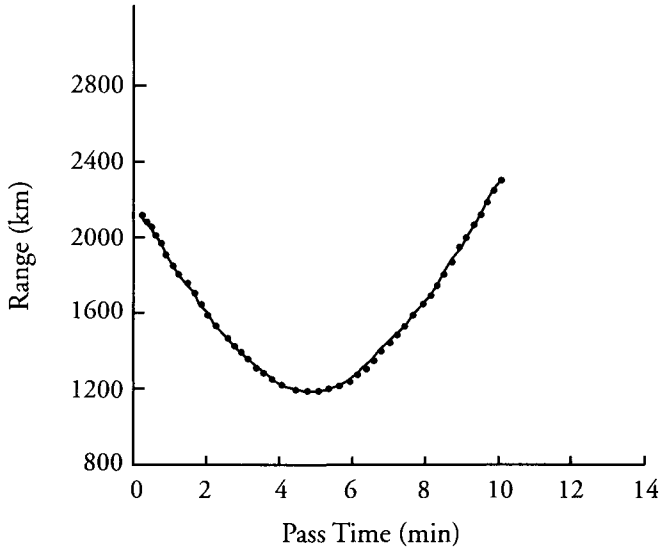


Figure 4.12.1: Range vs time from ground-based station.

where $\mathbf{V}_{rel} = \mathbf{V} - \boldsymbol{\omega}_e \times \mathbf{r}$, $\mathbf{V}$ is the inertial satellite velocity, $\boldsymbol{\omega}_e$ is the rotational velocity of the atmosphere (which is assumed to be the same as the earth's rotational velocity), C_D is the satellite drag coefficient, A is the satellite projected cross-sectional area, m is the satellite mass, and $\rho(h)$ is the atmospheric density at the satellite altitude, h. In many applications, C_D, A, and m will be sufficiently uncertain that the errors in their *a priori* values may degrade the estimate of the satellite state. However, if any two or all three of these parameters are added to the state vector, a singular or nonobservable state vector will result, no matter how dense or how complete the observation set is. In this case, the H matrix will have two rows that are identical except for a scalar mutiplication factor. This condition will cause H to be non-full-rank and $H^T H$ to be nonpositive definite. The single scalar parameter,

$$\beta = \frac{C_D A}{m} \tag{4.12.5}$$

can be included as a state variable, and a well-posed estimation problem will occur. Two constants, say C_1 and C_2, that do not appear separately in other force functions cannot be observed. Rather the product $C_3 = C_1 C_2$ may be observed. Any state variables that are linearly related to one another cannot be uniquely determined (see Exercise 2 of Chapter 1). In general, the observability criterion tests not only the completeness of the data set in both the spatial and temporal sense, but it also tests the validity of the problem formulation.

4.13 ERROR SOURCES

In the application of an estimation procedure to a satellite or trajectory problem, measurements are obtained by various ground-based or on-board instruments. For example, a ground-based ranging system may make the measurements shown in Fig. 4.13.1 with time measured in minutes since the first measurement. Based on a mathematical model of the dynamical system and the measurement system, a predicted or computed measurement could be generated and compared with the actual measurement. If, in fact, the models are quite accurate, the difference (or residual) between the actual and predicted (or computed) measurements (O-C) will exhibit the random component in the measurement system as in Fig. 4.13.3. On the other hand, as is usually the case, the model has some inaccuracies associated with it, and the residual pattern will exhibit the character shown in Fig. 4.13.2. By using a polynomial function of time, the systematic component in Fig. 4.13.2 can be removed to determine the noise-only components. The data editing functions to eliminate the spurious or erroneous measurements are applied to these residuals. Finally, the edited data in Fig. 4.13.2 are used by the estimators to improve the state and the force and measurement models.

In the ideal case, the nonzero difference between the actual measurement and the predicted value should be due to the noise and biasing that occur in making the measurement. In practice, however, the mathematical models that describe the satellite force environment and those that describe the instrument performing some measurement are not completely accurate, or certain approximations are made for the benefit of computer storage and/or computer execution time, which introduce some discrepancy or error in the data processing. It is frequently necessary to ascribe the source of an error to a phenomenon in the physical world or to an approximation made in the model of the real world. Knowledge of various parameters in the mathematical models, such as the mass of the Earth or the coefficients that describe the inhomogeneous mass distribution within the Earth, have been obtained through various experiments or through use of many measurements and are only approximately known.

Typical error sources are as follows.

- **SATELLITE FORCE MODEL:**

  Gravitation parameters

  - Mass of the earth (GM)
  - Geopotential coefficients, (C_{lm} and S_{lm})
  - Solid earth and ocean tide perturbations
  - Mass and position of the moon and planets
  - General relativistic perturbation

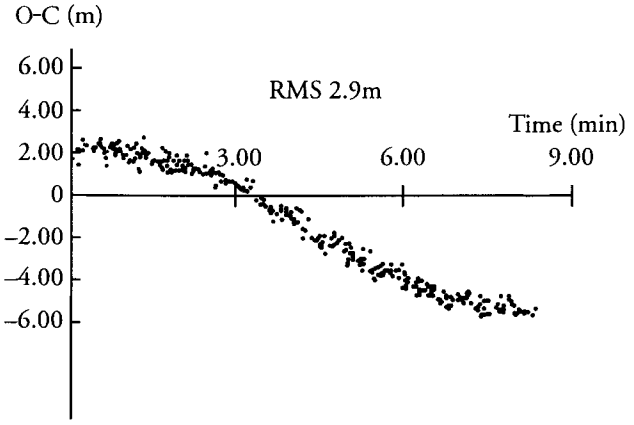


Figure 4.13.1: O-C, random and systematic component.

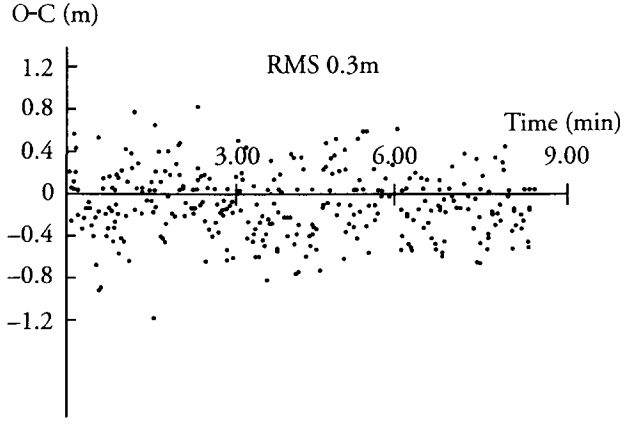


Figure 4.13.2: Random component.

Nongravitational parameters

- Drag (C_D, atmospheric density)
- Solar and earth radiation pressure
- Thrust (including venting and momentum dumping)
- Other (magnetic origin, etc.)

- **MEASUREMENT MODEL:**

Inertial and terrestrial coordinate systems

- Precession and nutation
- Polar motion

Ground-based measurements

- Coordinates of tracking station
- Atmospheric effects (tropospheric and ionospheric refraction)
- Instrument modeling
- Clock accuracy
- Tectonic plate motion

The error sources are dependent on the satellite under consideration, that is, satellite configuration, the orbit altitude and inclination, and measurement systems. Some of these error sources have distinct signatures in the data, and others may be very similar, thus producing aliasing between these components and making it difficult or impossible to separate their effects into individual components.

As a consequence, one constantly attempts to improve the model of the satellite environment. However, this improvement is normally done outside the operations aspect in which state estimates may be needed for an ongoing mission. To improve the gravitational model, for example, observations of various types such as range, range-rate, and angular data from different satellites would be used in a large parameter least squares solution. These large parameter solutions, in which the number of parameters may be 5000 or more, consume large amounts of computer time and, consequently, can be performed only infrequently. A family of such solutions has been generated since 1972 at the Goddard Space Flight Center and the Center for Space Research, and in several international centers. The models start with the Goddard Earth Model, GEM-1, and progress through the recent JGM-3 model (Tapley *et al.*, 1996) developed for the TOPEX/Poseidon mission. In the case of JGM-3, a full degree-70 and order-70 solution was produced. Other solutions have been obtained at the Smithsonian Astrophysical Observatory (SAO), the Department of Defense, and by foreign institutions, including GFZ in Germany and GRGS in France.

Implicit in all of these solutions is the time standard used in solving Newton's Equations, as well as time tagging the observations by various instruments. In the case of Newton's Laws, a uniform time scale is inherent, and we attempt to provide such a system in time tagging the measurements. Uniform time scales are provided by atomic clocks, aside from small relativistic effects, and an operational problem exists to ensure that all clocks used in making measurements are adequately synchronized. For ground-based systems, station clocks are usually synchronized to a national standard. Clearly, in the event that a station is out of synchronization with the other stations, the measurements made at that station will produce residuals that are somewhat larger than the others.

4.14 ORBIT ACCURACY

In general, the orbit accuracy is dependent on the following factors:

1. Truncation error in the application of an approximating algorithm to a mathematical process;
2. Round-off error in the application of a finite computational precision;
3. Mathematical model simplifications;
4. Errors in the parameters used in the mathematical model of the physical system or the instrument model;
5. Amount, type, and accuracy of tracking data.

For missions with high orbit-accuracy requirements, the limitation on the accuracy usually is imposed by the accuracy of the dynamical model, particularly the geopotential. But even the geopotential errors are dependent on the particular orbit; for example, an orbit at an altitude of one earth radius will be less affected by those errors than a satellite at an altitude of 800 km. The accuracy of the models is reflected in the current ability to determine an orbit for TOPEX/Poseidon (1334 km altitude) of ten days duration, which yields 1 cm root mean square (RMS) of the laser range residuals (Tapley *et al.*, 1994). This RMS reflects an overall orbit accuracy of about 8 cm.

Determining the true accuracy of an orbit determination solution based on actual tracking data is difficult because the true trajectory is never known. Furthermore, the estimation error covariance matrix for a given solution is generally overly optimistic depending on the weights assigned to the tracking data. If realistic process noise has been included, the estimation error covariance matrix may be a good indicator of orbit accuracy but it is difficult to determine the level of process noise needed to properly scale the covariance matrix. This is best done through simulation studies, but these require quantitative knowledge of the error

sources. Covariance analysis, described in Chapter 6, may also aid in accuracy assessment assuming statistical knowledge of the error sources is available.

A few quantitative indicators of accuracy can be examined depending on what tracking information is available. If solutions derived from different independent tracking systems are available they can be compared. The locations of most laser tracking sites are known to the centimeter level and the measurements themselves are accurate to this level. Therefore, if the satellite is tracked by lasers as it flies at high elevation over a site, the laser residuals will be a good indication of radial orbit accuracy. Furthermore, the estimation error covariance matrix can be mapped to these times, rotated to radial, along-track and cross-track directions and the radial standard deviation compared with the RMS of laser residuals. This comparison will provide a calibration of the radial component of the covariance matrix. The RMS of tracking data residuals is also an indicator of accuracy. However, small tracking residuals do not guarantee an accurate orbit because there may be a component of the satellite's position that is insensitive to the particular tracking data type (see Section 4.12).

Another measure of accuracy is the RMS of orbit solution overlaps. For example, five days of contiguous tracking data may be fit as two arcs of three days length with one day of overlap. The RMS of the three components of overlapping position is an indicator of orbit accuracy. However, any error common to a given coordinate during the overlap period will not be evident in the RMS. Finally, orbit solutions for satellites that make any kind of metric measurements (radar altimeters, laser altimeters, SAR, etc.) can be evaluated by examining the quality of parameters based on these measurements themselves. For example, for laser or radar altimeter satellites the RMS of crossover residuals discussed in Chapter 3 are an indicator of orbit accuracy.

The accuracy just described relates to the estimation accuracy. Another important accuracy consideration occurs in the problem of prediction. Given some estimate of the satellite state, how well can the state of the spacecraft be predicted at some future time? Such prediction accuracy is important for (1) predicting and scheduling events at ground-based stations, including antenna or telescope pointing, and (2) scheduling events for future orbital maneuvers. The prediction accuracy is influenced by the same effects that influence the estimation accuracy; however, it is also dependent on the estimation accuracy itself. If, for instance, a perfect physical model was known but the state estimate used to initiate the prediction was in error, this error would grow during the prediction interval. As an example of prediction accuracy, the position of Lageos can be predicted to about 200 meters after two months. For the TOPEX/Poseidon Mission, the orbit can be predicted to about 0.5 km after a week based on tracking with laser ranging or the Global Positioning System.

4.15 SMOOTHING

It is often desirable to perform a smoothing operation when using a sequential filter. In this case, we are searching for the best estimate of the state at some time t_k based on all observations through time t_ℓ where $\ell > k$. For the case where there is no random component to the dynamical equation of state—for example, the no-process noise case—the batch estimation algorithm along with the prediction equation, Eqs. (4.4.19) and (4.4.22), will give the smoothed solution. However, as noted, the batch estimation approach has difficulty including the effects of process noise. The smoothing algorithms have been developed to overcome this difficulty. Following Jazwinski (1970), the smoothing algorithm can be derived using a Bayesian approach of maximizing the density function of the state conditioned on knowledge of the observations through time, t_ℓ. Our system is described in Section 4.9 (see Eq. (4.9.46)).

$$\begin{aligned}\mathbf{x}_{k+1} &= \Phi(t_{k+1}, t_k)\mathbf{x}_k + \Gamma(t_{k+1}, t_k)\mathbf{u}_k \\ \mathbf{y}_k &= \widetilde{H}_k \mathbf{x}_k + \boldsymbol{\epsilon}_k.\end{aligned}$$

We will use the notation $\hat{\mathbf{x}}_k^\ell$ to indicate the best estimate of $\mathbf{x}$ at t_k based on observations through t_ℓ, where in general $\ell > k$. Following the Maximum Likelihood philosophy, we wish to find a recursive expression for $\hat{\mathbf{x}}_k^\ell$ in terms of $\hat{\mathbf{x}}_{k+1}^\ell$, which maximizes the conditional density function

$$p(\mathbf{x}_k,\ \mathbf{x}_{k+1}/\mathbf{Y}_\ell), \quad \text{where} \quad \mathbf{Y}_\ell = \mathbf{y}_1, \mathbf{y}_2 \cdots \mathbf{y}_k \cdots \mathbf{y}_\ell. \tag{4.15.1}$$

From Bayes Rule

$$\begin{aligned}p(\mathbf{x}_k,\ \mathbf{x}_{k+1}\ /\ \mathbf{Y}_\ell) &= \frac{p(\mathbf{x}_k,\ \mathbf{x}_{k+1},\ \mathbf{Y}_\ell)}{p(\mathbf{Y}_\ell)} = \frac{p(\mathbf{x}_k,\ \mathbf{x}_{k+1},\ \mathbf{Y}_k,\ \mathbf{y}_{k+1} \cdots \mathbf{y}_\ell)}{p(\mathbf{Y}_\ell)} \\ &= \frac{p(\mathbf{Y}_k)}{p(\mathbf{Y}_\ell)} p(\mathbf{x}_k,\ \mathbf{x}_{k+1},\ \mathbf{y}_{k+1} \cdots \mathbf{y}_\ell\ /\ \mathbf{Y}_k) \\ &= \frac{p(\mathbf{Y}_k)}{p(\mathbf{Y}_\ell)} p(\mathbf{y}_{k+1} \cdots \mathbf{y}_\ell\ /\ \mathbf{x}_k,\ \mathbf{x}_{k+1}, \mathbf{Y}_k) \\ &\times p(\mathbf{x}_k,\ \mathbf{x}_{k+1}\ /\ \mathbf{Y}_k).\end{aligned} \tag{4.15.2}$$

Notice that

$$p(\mathbf{y}_{k+1} \cdots \mathbf{y}_\ell\ /\ \mathbf{x}_k,\ \mathbf{x}_{k+1},\ \mathbf{Y}_k) = p(\mathbf{y}_{k+1} \cdots \mathbf{y}_\ell\ /\ \mathbf{x}_{k+1}), \tag{4.15.3}$$

and

$$\begin{aligned}p(\mathbf{x}_k,\ \mathbf{x}_{k+1}\ /\ \mathbf{Y}_k) &= p(\mathbf{x}_{k+1}\ /\ \mathbf{x}_k,\ \mathbf{Y}_k)\ p(\mathbf{x}_k\ /\ \mathbf{Y}_k) \\ &= p(\mathbf{x}_{k+1}\ /\ \mathbf{x}_k)\, p(\mathbf{x}_k\ /\ \mathbf{Y}_k),\end{aligned} \tag{4.15.4}$$

since knowledge of $\mathbf{x}_k$ and $\mathbf{Y}_k$ is redundant. Using Eqs. (4.15.3) and (4.15.4), Eq. (4.15.2) may be written

$$p(\mathbf{x}_k, \mathbf{x}_{k+1} / \mathbf{Y}_\ell) = \frac{p(\mathbf{Y}_k)}{p(\mathbf{Y}_\ell)} p(\mathbf{y}_{k+1} \cdots \mathbf{y}_\ell / \mathbf{x}_{k+1}) \times p(\mathbf{x}_{k+1} / \mathbf{x}_k)\, p(\mathbf{x}_k / \mathbf{Y}_k). \tag{4.15.5}$$

The first three density functions on the right-hand side of Eq. (4.15.5) are independent of $\mathbf{x}_k$; hence, we need to be concerned only with $p(\mathbf{x}_{k+1}/\mathbf{x}_k)$ and $p(\mathbf{x}_k/\mathbf{Y}_k)$. Assuming that these are Gaussian and that the process noise is zero mean, it is easily shown that

$$\begin{aligned} p(\mathbf{x}_{k+1}/\mathbf{x}_k) &\sim N\big(\Phi(t_{k+1}, t_k)\mathbf{x}_k, \Gamma(t_{k+1}, t_k) Q_k \Gamma^T(t_{k+1}, t_k)\big) \\ p(\mathbf{x}_k/\mathbf{Y}_k) &\sim N(\hat{\mathbf{x}}_k, P_k). \end{aligned} \tag{4.15.6}$$

It may seem like the covariance of $p(\mathbf{x}_{k+1}/\mathbf{x}_k)$ should be

$$\Phi(t_{k+1}, t_k) P_k \Phi^T(t_{k+1}, t_k) + \Gamma(t_{k+1}, t_k) Q_k \Gamma^T(t_{k+1}, t_k).$$

However, notice that this is the density function of $\mathbf{x}_{k+1}$ conditioned on knowledge of $\mathbf{x}_k$. Since P_k describes the error in $\hat{\mathbf{x}}_k$ and $\mathbf{x}_k$ has occurred and is known, P_k must be a null matrix.

In order to maximize the conditional probability density function given in Eq. (4.15.5), we may maximize the logarithm of $p(\mathbf{x}_{k+1} / \mathbf{x}_k)\, p(\mathbf{x}_k / \mathbf{Y}_k)$,

$$\begin{aligned} \ln L = &-\frac{1}{2}[\mathbf{x}_{k+1} - \Phi(t_{k+1}, t_k)\mathbf{x}_k]^T [\Gamma(t_{k+1}, t_k) Q_k \Gamma^T(t_{k+1}, t_k)]^{-1} \\ &\times [\mathbf{x}_{k+1} - \Phi(t_{k+1}, t_k)\mathbf{x}_k] - \frac{1}{2}[\mathbf{x}_k - \hat{\mathbf{x}}_k]^T P_k^{-1} [\mathbf{x}_k - \hat{\mathbf{x}}_k]. \end{aligned} \tag{4.15.7}$$

Suppose that $\hat{\mathbf{x}}_{k+1}^\ell$, the maximizing value of $\mathbf{x}_{k+1}$ based on observations through t_ℓ, is available. We wish to find the value of $\hat{\mathbf{x}}_k^\ell$ that maximizes $\ln L$. Differentiating $\ln L$ with respect to $\mathbf{x}_k$ and setting this to zero yields (for simplicity we have dropped time identifiers on $\Phi(t_{k+1}, t_k)$ and $\Gamma(t_{k+1}, t_k)$)

$$\begin{aligned} \hat{\mathbf{x}}_k^\ell = &[(P_k^k)^{-1} + \Phi^T(\Gamma Q \Gamma^T)^{-1}\Phi]^{-1} \\ &\times [(P_k^k)^{-1} \hat{\mathbf{x}}_k^k + \Phi^T(\Gamma Q \Gamma^T)^{-1} \hat{\mathbf{x}}_{k+1}^\ell], \end{aligned} \tag{4.15.8}$$

using our current notation,

$$\begin{aligned} P_k^k &\equiv P_k \\ \hat{\mathbf{x}}_k^k &\equiv \hat{\mathbf{x}}_k. \end{aligned}$$

Applying the Schur identity we can write this in a more conventional form

$$\hat{\mathbf{x}}_k^\ell = \hat{\mathbf{x}}_k^k + S_k \left(\hat{\mathbf{x}}_{k+1}^\ell - \Phi(t_{k+1}, t_k)\hat{\mathbf{x}}_k^k\right) \tag{4.15.9}$$

where

$$\begin{aligned} S_k &= P_k^k \Phi^T(t_{k+1}, t_k)[\Phi(t_{k+1}, t_k) P_k^k \Phi^T(t_{k+1}, t_k) \\ &\quad + \Gamma(t_{k+1}, t_k) Q_k \Gamma^T(t_{k+1}, t_k)]^{-1} \\ &= P_k^k \Phi^T(t_{k+1}, t_k)(P_{k+1}^k)^{-1}. \end{aligned} \tag{4.15.10}$$

Eq. (4.15.9) is the smoothing algorithm. Computation goes backward in index k, with $\hat{\mathbf{x}}_\ell^\ell$, the filter solution, as initial conditions. Note that the filter solutions for $\hat{\mathbf{x}}_k^k$, P_k^k, $\Phi(t_{k+1}, t_k)$, and $\Gamma(t_{k+1}, t_k)$ are required and should be stored in the filtering process. The time update of the covariance matrix, P_{k+1}^k, may be stored or recomputed.

The equation for propagating the smoothed covariance is derived next (Jazwinski, 1970; Rausch *et al.*, 1965) . It can easily be shown from Eq. (4.15.9) that $\hat{\mathbf{x}}_k^\ell$ is unbiased; hence, the smoothed covariance is defined by

$$P_k^\ell = E\left[(\hat{\mathbf{x}}_k^\ell - \mathbf{x}_k)(\hat{\mathbf{x}}_k^\ell - \mathbf{x}_k)^T\right]. \tag{4.15.11}$$

Subtracting $\mathbf{x}_k$ from both sides of Eq. (4.15.9) and moving all terms involving smoothed quantities to the LHS yields:

$$\tilde{\mathbf{x}}_k^\ell - S_k \hat{\mathbf{x}}_{k+1}^\ell = \tilde{\mathbf{x}}_k^k - S_k \Phi(t_{k+1}, t_k)\hat{\mathbf{x}}_k^k \tag{4.15.12}$$

where

$$\tilde{\mathbf{x}}_k^\ell \equiv \hat{\mathbf{x}}_k^\ell - \mathbf{x}_k, \quad \tilde{\mathbf{x}}_k^k \equiv \hat{\mathbf{x}}_k^k - \mathbf{x}_k.$$

Multiplying both sides of Eq. (4.15.12) by their respective transpose and taking the expected value yields

$$\begin{aligned} &E\left(\tilde{\mathbf{x}}_k^\ell \tilde{\mathbf{x}}_k^{\ell^T}\right) + S_k E\left(\hat{\mathbf{x}}_{k+1}^\ell \hat{\mathbf{x}}_{k+1}^{\ell^T}\right) S_k^T = \\ &\qquad E\left(\tilde{\mathbf{x}}_k^k \tilde{\mathbf{x}}_k^{k^T}\right) + S_k \Phi(t_{k+1}, t_k) E\left(\hat{\mathbf{x}}_k^k \hat{\mathbf{x}}_k^{k^T}\right) \Phi^T(t_{k+1}, t_k) S_k^T. \end{aligned} \tag{4.15.13}$$

By definition

$$P_k^\ell \equiv E\left[\tilde{\mathbf{x}}_k^\ell \tilde{\mathbf{x}}_k^{\ell^T}\right] \tag{4.15.14}$$

and

$$P_k^k \equiv E\left[\tilde{\mathbf{x}}_k^k \tilde{\mathbf{x}}_k^{k^T}\right]. \tag{4.15.15}$$

The cross product terms that have been dropped in Eq. (4.15.13) can be shown to be null matrices,

$$\begin{aligned} E\left(\tilde{\mathbf{x}}_k^{\ell}\hat{\mathbf{x}}_{k+1}^{\ell^T}\right) &= E\left(\hat{\mathbf{x}}_k^{\ell} - \mathbf{x}_k\right)\hat{\mathbf{x}}_{k+1}^{\ell^T} \\ &= E\left(\hat{\mathbf{x}}_k^{\ell}\hat{\mathbf{x}}_{k+1}^{\ell^T}\right) - E\left(\mathbf{x}_k\hat{\mathbf{x}}_{k+1}^{\ell^T}\right) \\ &= \hat{\mathbf{x}}_k^{\ell}\hat{\mathbf{x}}_{k+1}^{\ell^T} - E\left(\mathbf{x}_k/\mathbf{y}_{\ell}\right)\hat{\mathbf{x}}_{k+1}^{\ell^T} = 0. \end{aligned} \tag{4.15.16}$$

Here $\hat{\mathbf{x}}$ is the conditional mean of the appropriate conditional density function and is not a random variable; hence,

$$E\left[\mathbf{x}_k/\mathbf{y}_{\ell}\right] \equiv \hat{\mathbf{x}}_k^{\ell}. \tag{4.15.17}$$

Likewise, for the filter results,

$$\begin{aligned} E\left(\tilde{\mathbf{x}}_k^{k}\hat{\mathbf{x}}_k^{k^T}\right) &= E\left(\hat{\mathbf{x}}_k^{k} - \mathbf{x}_k\right)\hat{\mathbf{x}}_k^{k^T} \\ &= \hat{\mathbf{x}}_k^{k}\hat{\mathbf{x}}_k^{k^T} - E\left(\mathbf{x}_k/\mathbf{y}_k\right)\hat{\mathbf{x}}_k^{k^T} = 0. \end{aligned} \tag{4.15.18}$$

Hence, Eq. (4.15.13) becomes

$$P_k^{\ell} + S_k\hat{\mathbf{x}}_{k+1}^{\ell}\hat{\mathbf{x}}_{k+1}^{\ell^T}S_k^T = P_k^k + S_k\Phi(t_{k+1},t_k)\hat{\mathbf{x}}_k^k\hat{\mathbf{x}}_k^{k^T}\Phi^T(t_{k+1},t_k)S_k^T. \tag{4.15.19}$$

Now,

$$\begin{aligned} P_{k+1}^{\ell} &\equiv E\left\{\left(\hat{\mathbf{x}}_{k+1}^{\ell} - \mathbf{x}_{k+1}\right)\left(\hat{\mathbf{x}}_{k+1}^{\ell} - \mathbf{x}_{k+1}\right)^T/\mathbf{y}_{\ell}\right\} \\ &= -\hat{\mathbf{x}}_{k+1}^{\ell}\hat{\mathbf{x}}_{k+1}^{\ell^T} + E\left(\mathbf{x}_{k+1}\mathbf{x}_{k+1}^T\right) \end{aligned}$$

or

$$\hat{\mathbf{x}}_{k+1}^{\ell}\hat{\mathbf{x}}_{k+1}^{\ell^T} = E(\mathbf{x}_{k+1}\mathbf{x}_{k+1}^T) - P_{k+1}^{\ell}. \tag{4.15.20}$$

Also, in terms of $\mathbf{x}_k$ (let $G = \Phi(t_{k+1},t_k)\mathbf{x}_k + \Gamma(t_{k+1},t_k)\mathbf{u}_k$)

$$\begin{aligned} E(\mathbf{x}_{k+1}\mathbf{x}_{k+1}^T) &= E\left[GG^T\right] \\ &= \Phi(t_{k+1},t_k)E(x_k,x_k^T)\Phi^T(t_{k+1},t_k) \\ &\quad +\Gamma(t_{k+1},t_k)Q_k\Gamma^T(t_{k+1},t_k) \end{aligned} \tag{4.15.21}$$

and

$$\begin{aligned} P_k^k &= E\left\{\left(\hat{\mathbf{x}}_k^k - \mathbf{x}_k\right)\left(\hat{\mathbf{x}}_k^k - \mathbf{x}_k\right)^T/\mathbf{y}_k\right\} \\ &= -\hat{\mathbf{x}}_k^k\hat{\mathbf{x}}_k^{k^T} + E(\mathbf{x}_k\mathbf{x}_k^T) \end{aligned}$$

or

$$\hat{\mathbf{x}}_k^k \hat{\mathbf{x}}_k^{k^T} = E\left(\mathbf{x}_k \mathbf{x}_k^T\right) - P_k^k. \tag{4.15.22}$$

Substituting Eqs. (4.15.20), (4.15.21), and (4.15.22) into (4.15.19) yields

$$\begin{aligned} P_k^\ell + S_k \left[-P_{k+1}^\ell + \Gamma(t_{k+1}, t_k) Q_k \Gamma^T(t_{k+1}, t_k)\right] S_k^T \\ = P_k^k - S_k \Phi(t_{k+1}, t_k) P_k^k \Phi^T(t_{k+1}, t_k) S_k^T. \end{aligned} \tag{4.15.23}$$

Finally,

$$P_k^\ell = P_k^k + S_k \left(P_{k+1}^\ell - P_{k+1}^k\right) S_k^T. \tag{4.15.24}$$

Note that neither the smoothed covariance nor the observation data appear explicitly in the smoothing algorithm. The algorithm derived previously is identical to the Rauch, Tung, and Striebel smoother (1965).

Suppose there is no process noise (i.e., $Q = 0$), then the smoothing algorithm reduces to

$$S_k = \Phi^{-1}(t_{k+1},\, t_k), \tag{4.15.25}$$

and

$$\begin{aligned} \hat{\mathbf{x}}_k^\ell &= \Phi^{-1}(t_{k+1},\, t_k)\, \hat{\mathbf{x}}_{k+1}^\ell \\ &= \Phi\left(t_k, t_{k+1}\right) \hat{\mathbf{x}}_{k+1}^\ell \\ &= \Phi(t_k, t_\ell) \hat{\mathbf{x}}_\ell^\ell. \end{aligned} \tag{4.15.26}$$

Also,

$$\begin{aligned} P_k^\ell &= \Phi^{-1}(t_{k+1},\, t_k) P_{k+1}^\ell \Phi^{-T}(t_{k+1},\, t_k) \\ &= \Phi\left(t_k,\, t_{k+1}\right) P_{k+1}^\ell \Phi^T(t_k,\, t_{k+1}) \qquad (4.15.27) \\ &= \Phi(t_k, t_\ell) P_\ell^\ell \Phi^T(t_k, t_\ell). \qquad (4.15.28) \end{aligned}$$

Hence, with no process noise the smoothing algorithm simply maps the final filter state estimate and covariance matrix to earlier epochs.

4.15.1 Computational Algorithm for Smoother

Given (from the filtering algorithm)

$$\hat{\mathbf{x}}_\ell^\ell,\ \hat{\mathbf{x}}_{\ell-1}^{\ell-1},\ P_\ell^{\ell-1},\ P_{\ell-1}^{\ell-1},\ \Phi\left(t_\ell,\, t_{\ell-1}\right);$$

set $k = \ell - 1$

$$\begin{aligned} S_{\ell-1} &= P_{\ell-1}^{\ell-1}\, \Phi^T(t_\ell,\, t_{\ell-1})\, \left(P_\ell^{\ell-1}\right)^{-1} \\ \hat{\mathbf{x}}_{\ell-1}^\ell &= \hat{\mathbf{x}}_{\ell-1}^{\ell-1} + S_{\ell-1}(\hat{\mathbf{x}}_\ell^\ell - \Phi\left(t_\ell,\, t_{\ell-1}\right) \hat{\mathbf{x}}_{\ell-1}^{\ell-1}). \end{aligned} \tag{4.15.29}$$

Given (from the filtering algorithm and the previous step of the smoothing algorithm)

$$\hat{\mathbf{x}}_{\ell-2}^{\ell-2},\ P_{\ell-1}^{\ell-2},\ P_{\ell-2}^{\ell-2},\ \hat{\mathbf{x}}_{\ell-1}^{\ell},\ \Phi(t_{\ell-1}, t_{\ell-2});$$

set $k = \ell - 2$, and compute

$$\begin{aligned} S_{\ell-2} &= P_{\ell-2}^{\ell-2}\, \Phi^T\,(t_{\ell-1}, t_{\ell-2})\,(P_{\ell-1}^{\ell-2})^{-1} \\ \hat{\mathbf{x}}_{\ell-2}^{\ell} &= \hat{\mathbf{x}}_{\ell-2}^{\ell-2} + S_{\ell-2}\,(\hat{\mathbf{x}}_{\ell-1}^{\ell} - \Phi\,(t_{\ell-1}, t_{\ell-2})\,\hat{\mathbf{x}}_{\ell-2}^{\ell-2}) \end{aligned} \tag{4.15.30}$$

$$\vdots$$

and so on.

4.16 THE PROBABILITY ELLIPSOID

Given a normally distributed random vector, $\mathbf{x}$, with mean $\overline{\mathbf{x}}$, and variance-covariance P, the function

$$(\mathbf{x} - \overline{\mathbf{x}})^T P^{-1} (\mathbf{x} - \overline{\mathbf{x}}) = \ell^2 \tag{4.16.1}$$

is a positive definite quadratic form representing a family of hyperellipsoids of constant probability density (Mikhail, 1976; Bryson and Ho, 1975). The 3D case is important because we often are interested in the 3D ellipsoids associated with the position uncertainty of a satellite. For example, in the case of interplanetary missions, we are interested in the probability ellipsoid of the spacecraft as it impacts the plane that contains the center of gravity of the target planet and that is normal to the asymptote of the spacecraft trajectory relative to the planet. This plane, shown in Fig. 4.16.1 and referred to as the B-plane (Dorroh and Thornton, 1970), is the reference plane used in interplanetary navigation applications. The associated coordinate system has two orthogonal axes in this plane and one normal to it. The axes of the ellipsoid in the B-plane give information on the uncertainty of the nearest approach distance to the planet that is needed to give assurance that the spacecraft will not impact the planet. The out-of-plane axis relates to the accuracy of the encounter time. If the spacecraft is to be placed in orbit about the target planet, information from the covariance matrix used to generate the probability ellipsoid is used to design the capture maneuver, and to compute the *a priori* uncertainty of the resulting orbit parameters. Construction of the probability ellipsoid is most easily accomplished relative to the principal axes. To this end, we introduce the following theorem (Kreyszig, 1993).

Theorem: If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is an orthonormal system of eigenvectors associated, respectively, with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ symmetric positive definite matrix, P, and if

$$U = [\mathbf{u}_1,\ \mathbf{u}_2, \ldots,\ \mathbf{u}_n]_{n \times n},$$

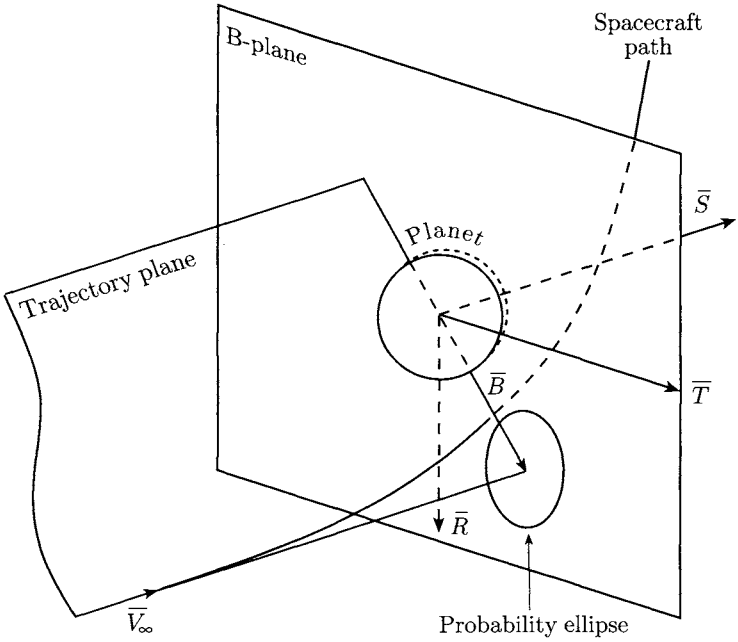


Figure 4.16.1: Probability ellipse on the B-plane.

then

$$U^T P U = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D\,[\lambda_1,\ \lambda_2, \ldots,\ \lambda_n]; \tag{4.16.2}$$

that is, $U^T PU$ is a diagonal matrix containing the eigenvalues of P. The normalized vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are called principal axes of P, and the transformation matrix U^T used to diagonalize P is called a *principal axes transformation.* The matrix U^T is the matrix of normalized eigenvectors and is orthogonal. For the random vector $\mathbf{x}$ with mean $\overline{\mathbf{x}}$ and variance-covariance, P, the principal axes, $\mathbf{x}'$, are given by

$$\mathbf{x}' = U^T \mathbf{x}. \tag{4.16.3}$$

The variance-covariance matrix, P', associated with the principal axes is given by

$$\begin{aligned} P' &\equiv E[(\mathbf{x}' - \overline{\mathbf{x}}')(\mathbf{x}' - \overline{\mathbf{x}}')^T] \\ &= U^T E[(\mathbf{x} - \overline{\mathbf{x}})(\mathbf{x} - \overline{\mathbf{x}})^T]\, U \\ &= U^T P U = D\,[\lambda_1 \ldots \lambda_n]. \end{aligned} \tag{4.16.4}$$

In our case, $\Delta\mathbf{x}$ represents the estimation error vector defined by

$$\Delta\mathbf{x} \equiv \hat{\mathbf{x}} - \mathbf{x} \equiv [\tilde{x}\ \tilde{y}\ \tilde{z}]^T,$$

with zero mean and variance-covariance given by

$$P = E[\Delta\mathbf{x}\Delta\mathbf{x}^T]. \tag{4.16.5}$$

Although this restriction is unnecessary, we will simplify matters and deal only with the three position coordinates of $\Delta\mathbf{x}$ and the associated 3×3 portion of the estimation error covariance matrix.

In this case, the equation for the probability ellipsoid is

$$[\tilde{x}\ \tilde{y}\ \tilde{z}]P^{-1}\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{bmatrix} = \ell^2. \tag{4.16.6}$$

The ellipsoids for $\ell = 1, 2$, and 3 are called the 1σ, 2σ, and 3σ error ellipsoids. The probability of the state estimate error falling inside these ellipsoids, assuming a trivariate Gaussian density function, is 0.200, 0.739, and 0.971, respectively.

To obtain the principal axes, we use the theorem just introduced and determine the matrix of normalized eigenvectors, U, and the eigenvalues λ_i, $i = 1, 2, 3$ of P. The principal axes are given by

$$\begin{bmatrix}\tilde{x}'\\ \tilde{y}'\\ \tilde{z}'\end{bmatrix} = U^T\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{bmatrix}, \tag{4.16.7}$$

and the associated covariance matrix is

$$P' = U^T P U.$$

The probability ellipsoids are given by

$$[\tilde{x}'\ \tilde{y}'\ \tilde{z}']\begin{bmatrix}1/\lambda_1 & & \\ & 1/\lambda_2 & \\ & & 1/\lambda_3\end{bmatrix}\begin{bmatrix}\tilde{x}'\\ \tilde{y}'\\ \tilde{z}'\end{bmatrix} = \ell^2, \tag{4.16.8}$$

or

$$\frac{\tilde{x}'^2}{\lambda_1} + \frac{\tilde{y}'^2}{\lambda_2} + \frac{\tilde{z}'^2}{\lambda_3} = \ell^2. \tag{4.16.9}$$

It is convenient to arrange the eigenvectors so that $\lambda_1 > \lambda_2 > \lambda_3$ (i.e., in order of descending values of λ_i). The axes of the 1σ ellipsoid are given by solving Eq.

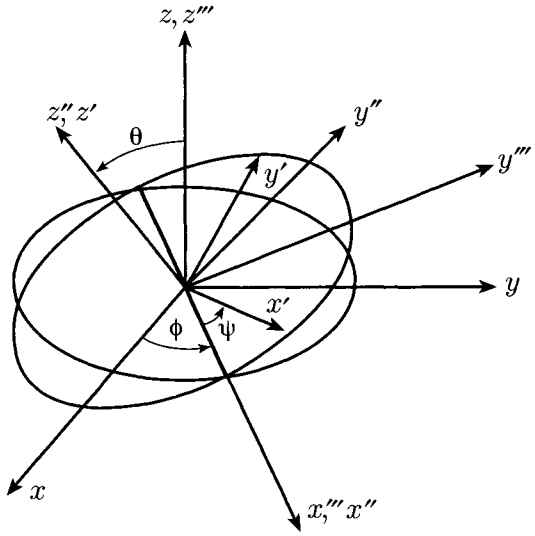


Figure 4.16.2: Euler angles defined.

(4.16.9) for $\ell = 1$ and sequentially setting two of the coordinate values to zero; that is, to obtain the semimajor axis, a, set $\tilde{x}' = a$ and $\tilde{y}' = \tilde{z}' = 0$. This yields

$$a^2 = \lambda_1, b^2 = \lambda_2, c^2 = \lambda_3. \tag{4.16.10}$$

The orientation of the ellipse relative to the original axis system is obtained by solving for the three Euler angles given by the transformation matrix, U. It is convenient to choose the sign of the normalized eigenvectors so that U defines a right-hand triad. This can be accomplished by requiring that $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3$.

The Euler angles are defined in Fig. 4.16.2. The first rotation is about the z axis through the angle ϕ and takes the unprimed frame into the ( )''' frame. The second rotation is about the x''' axis through the angle θ to the ( )'' frame, and the final rotation is about the z'' axis through the angle ψ to the ( )' or principal axes frame (Reddy and Rasmussen, 1990). This transformation is analogous to that of transforming Earth-centered, Earth-fixed coordinates into orbital plane coordinates with the x' axis through perigee, the z' axis along the angular momentum vector, and the y' axis completing the right-hand triad. The transpose of the transformation matrix that accomplishes this is the matrix we have identified as U; that is, $\mathbf{x}' = U^T\mathbf{x}$ where

$$U = \begin{bmatrix} C_\phi C_\psi - C_\theta S_\phi S_\psi & -C_\phi S_\psi - C_\theta S_\phi C_\psi & S_\theta S_\phi \\ S_\phi C_\psi + C_\theta C_\phi S_\psi & -S_\phi S_\psi + C_\theta C_\phi C_\psi & -S_\theta C_\phi \\ S_\theta S_\psi & S_\theta C_\psi & C_\theta \end{bmatrix}, \quad (4.16.11)$$

and where C and S represent cosine and sine, respectively. The Euler angles are given by

$$\phi = \text{atan2}\left[\frac{U_{13}}{-U_{23}}\right], \quad 0 \leq \phi \leq 2\pi \quad (4.16.12)$$

$$\theta = \text{acos}\left[U_{33}\right], \quad 0 \leq \theta \leq \pi \quad (4.16.13)$$

$$\psi = \text{atan2}\left[\frac{U_{31}}{U_{32}}\right], \quad 0 \leq \psi \leq 2\pi. \quad (4.16.14)$$

Example: Consider a normally distributed 2D random vector, $\mathbf{x}$, where

$$\mathbf{x} \sim N\,(0,\ P),$$

and

$$P = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}.$$

Sketch the 1-, 2-, and 3-σ probability ellipses.

The eigenvalues are given by the polynomial

$$|P - \lambda I| = 0,$$

or

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0;$$

hence,

$$\lambda^2 - 6\lambda + 4 = 0,$$

and

$$\lambda_1 = 5.236, \quad \lambda_2 = 0.764.$$

The corresponding eigenvectors are given by

$$[P - \lambda_i I] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0, \quad i = 1,\ 2.$$

The normalized eigenvectors are given by

$$U = \begin{bmatrix} .851 & -.526 \\ .526 & .851 \end{bmatrix}.$$

The angle between the principal and original axes system is obtained by recognizing that the coordinate transformation matrix is given by (this can be determined by setting $\theta = \psi = 0$ in Eq. (4.16.11))

$$U = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Hence,

$$\phi = \tan^{-1}\frac{\sin\phi}{\cos\phi} = 31.7^\circ$$

where ϕ is the angle between the x_1 and x_1' axes.

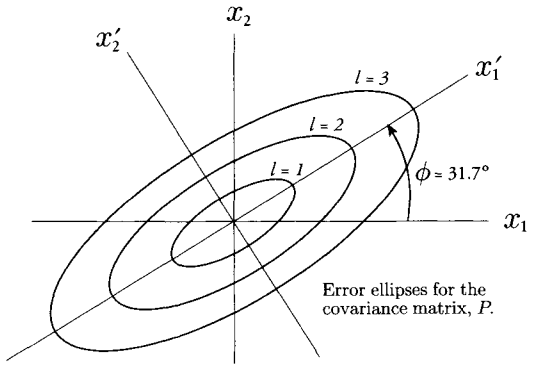
The semimajor axes a_i and minor axes b_i are given by

$$\begin{aligned} a_i &= \sqrt{\ell_i^2 \lambda\,(\text{Max})} \\ & \qquad i = 1,2,3;\ \ell_1 = 1,\ \ell_2 = 2,\ \ell_3 = 3 \\ b_i &= \sqrt{\ell_i^2 \lambda(\text{Min})} \end{aligned}$$

with numerical values,

$$\begin{aligned} a_1 &= 2.29, \quad a_2 = 4.58, \quad a_3 = 6.86 \\ b_1 &= 0.87, \quad b_2 = 1.75, \quad b_3 = 2.62 \end{aligned}$$

The error ellipses can now be constructed:



4.16.1 Transformation of the Covariance Matrix between Coordinate Systems

Sometimes it is desirable to transform the state vector and the estimation error covariance into alternate coordinate systems. For example, it may be of interest to view these quantities in a radial, transverse, and normal (RTN) system. Here the transverse direction is normal to the radius vector and the normal direction lies along the instantaneous angular momentum vector. RTN forms a right-hand triad.

The general transformation between any two coordinate frames (say prime to unprime) for a position vector is given by

$$\mathbf{r} = \gamma \mathbf{r}' + \mathbf{a}, \tag{4.16.15}$$

where γ is an orthogonal transformation matrix, $\mathbf{r}$ is the vector in the unprimed frame, and $\mathbf{a}$ is the vector offset of the origin of the two systems expressed in the unprimed frame. Generally $\mathbf{a}$ will be zero unless, for example, we are transforming from a geocentric to a topocentric frame.

The velocity transforms according to

$$\dot{\mathbf{r}} = \gamma \dot{\mathbf{r}}' + \dot{\gamma} \mathbf{r}'. \tag{4.16.16}$$

Generally $\dot{\gamma}$ will be zero unless we are transforming from a rotating to a nonrotating frame or vice versa; for example, Earth-centered-Earth-fixed to Earth-centered inertial (ECEF to ECI). Let the unprimed system be the inertial, nonrotating frame. It can be shown (Wiesel, 1997) that $\dot{\gamma}\mathbf{r}' = \boldsymbol{\omega} \times \mathbf{r}$, where $\boldsymbol{\omega}$ is the angular velocity vector of the rotating frame expressed in the nonrotating frame coordinate system.

The transformation we want is ECI to RTN. We assume that the RTN frame is fixed to the osculating orbit at each point in time; hence, $\dot{\gamma} = 0$ and

$$\begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix}_{\text{RTN}} = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix}_{\text{ECI}}. \tag{4.16.17}$$

The covariance of the estimation error is transformed as follows:

$$[\hat{\mathbf{x}} - \mathbf{x}]_{\text{RTN}} = \psi \, [\hat{\mathbf{x}} - \mathbf{x}]_{\text{ECI}},$$

where

$$\psi = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad [\hat{\mathbf{x}} - \mathbf{x}] = \begin{bmatrix} \hat{\mathbf{r}} - \mathbf{r} \\ \hat{\mathbf{v}} - \mathbf{v} \\ \hat{\boldsymbol{\beta}} - \boldsymbol{\beta} \end{bmatrix}, \tag{4.16.18}$$

and $\mathbf{r}$, $\mathbf{v}$, and β represent the true values of the position, velocity, and all other quantities in the state vector, respectively. It is assumed that none of the elements of β are affected by the coordinate transformation. The desired covariance is given by

$$\begin{aligned} P_{\mathrm{RTN}} &= E\left[(\hat{\mathbf{x}}-\mathbf{x})(\hat{\mathbf{x}}-\mathbf{x})^T\right]_{\mathrm{RTN}} \\ &= \psi E\left[(\hat{\mathbf{x}}-\mathbf{x})(\hat{\mathbf{x}}-\mathbf{x})^T\right]_{\mathrm{ECI}} \psi^T. \end{aligned} \tag{4.16.19}$$

The elements of γ for the ECI to RTN transformation are given by

$$\begin{aligned} \mathbf{u}_{\mathrm{R}} &= \frac{\mathbf{r}^*}{|\mathbf{r}^*|} = \epsilon_X \mathbf{i} + \epsilon_Y \mathbf{j} + \epsilon_Z \mathbf{k} \\ \mathbf{u}_{\mathrm{T}} &= \mathbf{u}_N \times \mathbf{u}_R = \delta_X \mathbf{i} + \delta_Y \mathbf{j} + \delta_Z \mathbf{k} \\ \mathbf{u}_N &= \frac{\mathbf{r}^* \times \mathbf{v}^*}{|\mathbf{r}^* \times \mathbf{v}^*|} = \alpha_X \mathbf{i} + \alpha_Y \mathbf{j} + \alpha_Z \mathbf{k}, \end{aligned} \tag{4.16.20}$$

where $\mathbf{u}_{\mathrm{R}}$, $\mathbf{u}_{\mathrm{T}}$, $\mathbf{u}_{\mathrm{N}}$ are unit vectors in the RTN frame, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the ECI frame, and $\mathbf{r}^*$ and $\mathbf{v}^*$ are the position and velocity vectors of the reference orbit.

Equation (4.16.20) may be written

$$\begin{bmatrix} \mathbf{u}_{\mathrm{R}} \\ \mathbf{u}_{\mathrm{T}} \\ \mathbf{u}_{\mathrm{N}} \end{bmatrix} = \begin{bmatrix} \epsilon_X & \epsilon_Y & \epsilon_Z \\ \delta_X & \delta_Y & \delta_Z \\ \alpha_X & \alpha_Y & \alpha_Z \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}. \tag{4.16.21}$$

Hence, the transformation matrix relating the RTN and ECI frame is

$$\gamma = \begin{bmatrix} \epsilon_X & \epsilon_Y & \epsilon_Z \\ \delta_X & \delta_Y & \delta_Z \\ \alpha_X & \alpha_Y & \alpha_Z \end{bmatrix}. \tag{4.16.22}$$

4.17 COMBINING ESTIMATES

Assume we are given two unbiased and uncorrelated estimates $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ for the n-vector $\mathbf{x}$. Assume that the associated estimation errors η_1 and η_2 are Gaussian with covariance matrices P_1 and P_2. Our objective is to establish a performance index and determine the combination of $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ that is an optimal, unbiased ($E[\hat{\mathbf{x}}] = \mathbf{x}$) estimate of $\mathbf{x}$.

Using the fact that the errors in $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are zero mean and uncorrelated (hence, being Gaussian, they are independent), the joint density function for the

estimation errors, η_1 and η_2, where

$$\begin{aligned} \eta_1 &= \hat{\mathbf{x}}_1 - \mathbf{x} \\ \eta_2 &= \hat{\mathbf{x}}_2 - \mathbf{x}, \end{aligned} \tag{4.17.1}$$

is given by

$$\begin{aligned} f(\eta_1, \eta_2) &= f(\eta_1) f(\eta_2) \\ &= \frac{1}{(2\pi)^{n/2}} \frac{1}{|P_1|^{\frac{1}{2}}} e^{-\frac{1}{2}(\eta_1^T P_1^{-1} \eta_1)} \\ &\times \frac{1}{(2\pi)^{n/2} |P_2|^{\frac{1}{2}}} e^{-\frac{1}{2}(\eta_2^T P_2^{-1} \eta_2)}. \end{aligned} \tag{4.17.2}$$

The method of Maximum Likelihood selects the value of $\mathbf{x}$ that maximizes the likelihood function $L = f(\eta_1, \eta_2)$. Because the density functions involved are Gaussian, all classical estimation techniques, Bayesian, Minimum Variance, or Maximum Likelihood, will yield the same results.

Maximizing the logarithm of L is equivalent to maximizing L. Hence, we wish to maximize

$$\begin{aligned} \ln L &= K - 1/2\left[\eta_1^T P_1^{-1} \eta_1 + \eta_2^T P_2^{-1} \eta\right] \\ &= K - 1/2\left[(\hat{\mathbf{x}}_1 - \mathbf{x})^T P_1^{-1} (\hat{\mathbf{x}}_1 - \mathbf{x})\right. \\ &\left. + (\hat{\mathbf{x}}_2 - \mathbf{x})^T P_2^{-1} (\hat{\mathbf{x}}_2 - \mathbf{x})\right] \end{aligned} \tag{4.17.3}$$

where

$$K = \ln\left(\frac{1}{(2\pi)^{\eta/2}} \frac{1}{|P_1|^{1/2}}\right) + \ln\left(\frac{1}{(2\pi)^{\eta/2}} \frac{1}{|P_2|^{1/2}}\right).$$

Using Eq. (B.7.3) of Appendix B, for a maximum of $\ln L$, it is necessary that

$$\frac{d \ln L}{d\mathbf{x}} = -\frac{1}{2}\left[-2P_1^{-1}(\hat{\mathbf{x}}_1 - \mathbf{x}) - 2P_2^{-1}(\hat{\mathbf{x}}_2 - \mathbf{x})\right] = 0.$$

Hence,

$$P_1^{-1}(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}) + P_2^{-1}(\hat{\mathbf{x}}_2 - \hat{\mathbf{x}}) = 0$$

and

$$\hat{\mathbf{x}} = (P_1^{-1} + P_2^{-1})^{-1}[P_1^{-1}\hat{\mathbf{x}}_1 + P_2^{-1}\hat{\mathbf{x}}_2]. \tag{4.17.4}$$

Also,

$$\frac{d^2 \ln L}{d\mathbf{x}^2} = -(P_1^{-1} + P_2^{-1}) \tag{4.17.5}$$

which is negative definite; therefore, this is a maximum of $\ln L$.

It is easily shown that $\hat{\mathbf{x}}$ is unbiased,

$$\begin{aligned} E[\hat{\mathbf{x}}] &= (P_1 + P_2)^{-1}[P_1^{-1}\, E(\hat{\mathbf{x}}_1) + P_2^{-1} E(\hat{\mathbf{x}}_2)] \\ &= \mathbf{x}, \end{aligned}$$

since $E(\hat{\mathbf{x}}_1) = E(\hat{\mathbf{x}}_2) = \mathbf{x}$.

The estimation error covariance, P, associated with $\hat{\mathbf{x}}$ is given by

$$P = E[(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T], \tag{4.17.6}$$

where

$$\hat{\mathbf{x}} = \gamma[P_1^{-1}\hat{\mathbf{x}}_1 + P_2^{-1}\hat{\mathbf{x}}_2],$$

and

$$\gamma \equiv (P_1^{-1} + P_2^{-1})^{-1}.$$

Using Eqs. (4.17.1) and the fact that

$$E[\boldsymbol{\eta}_1] = E[\boldsymbol{\eta}_2] = 0,$$

$$E[\boldsymbol{\eta}_1 \boldsymbol{\eta}_1^T] = P_1, E[\boldsymbol{\eta}_2 \boldsymbol{\eta}_2^T] = P_2, E[\boldsymbol{\eta}_1 \boldsymbol{\eta}_2^T] = 0, \tag{4.17.7}$$

we can write

$$\hat{\mathbf{x}} = \gamma[P_1^{-1}(\mathbf{x} + \boldsymbol{\eta}_1) + P_2^{-1}(\mathbf{x} + \boldsymbol{\eta}_2)].$$

Hence,

$$\hat{\mathbf{x}} - \mathbf{x} = \gamma[P_1^{-1}\boldsymbol{\eta}_1 + P_2^{-1}\boldsymbol{\eta}_2], \tag{4.17.8}$$

therefore,

$$\begin{aligned} P &= \gamma E\left\{[P_1^{-1}\boldsymbol{\eta}_1 + P_2^{-1}\boldsymbol{\eta}_2][P_1^{-1}\boldsymbol{\eta}_1 + P_2^{-1}\boldsymbol{\eta}_2]^T\right\}\gamma \\ &= \gamma[P_1^{-1}P_1P_1^{-1} + P_2^{-1}P_2P_2^{-1}]\gamma \\ &= \gamma \\ &= (P_1^{-1} + P_2^{-1})^{-1}. \end{aligned} \tag{4.17.9}$$

It is not necessary to assume that the errors in $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are Gaussian. Knowing that the two estimates $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are unbiased and uncorrelated, we could have simply chosen to minimize a performance index that yields a value of $\mathbf{x}$ that minimizes the weighted sum of squares of the estimation errors for $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$,

$$Q = \frac{1}{2}[(\hat{\mathbf{x}}_1 - \mathbf{x})^T P_1^{-1}(\hat{\mathbf{x}}_1 - \mathbf{x}) + (\hat{\mathbf{x}}_2 - \mathbf{x})^T P_2^{-1}(\hat{\mathbf{x}}_2 - \mathbf{x})]. \tag{4.17.10}$$

Choosing $\mathbf{x}$ to minimize Q will yield the result given by Eq. (4.17.4).

If there are n independent solutions to combine, it is easily shown that

$$\hat{\mathbf{x}} = \left(\sum_{i=1}^{n} P_i^{-1} \right)^{-1} \sum_{i=1}^{n} P_i^{-1} \hat{\mathbf{x}}_i. \tag{4.17.11}$$

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4.19 EXERCISES

(1) A dynamical system is described by

$$\dot{\mathbf{X}}(t) = A\mathbf{X}(t).$$

Given that the state transition matrix for this system is

$$\Phi = \begin{bmatrix} e^{-2at} & 0 \\ 0 & e^{bt} \end{bmatrix},$$

determine the matrix A.

(2) Given the solution to the differential equation

$$\dot{\mathbf{x}}(t_i) = A(t_i)\mathbf{x}(t_i)$$

is

$$\mathbf{x}(t_i) = \Phi(t_i, t_k)\mathbf{x}(t_k)$$

where

$$\Phi(t_k, t_k) = I,$$

show that

(a) $\dot{\Phi}(t_i, t_k) = A(t_i)\Phi(t_i, t_k)$

(b) $\Phi(t_i, t_j) = \Phi(t_i, t_k)\Phi(t_k, t_j)$

(c) $\Phi^{-1}(t_i, t_k) = \Phi(t_k, t_i)$

(d) $\dot{\Phi}^{-1}(t_i, t_k) = -\Phi^{-1}(t_i, t_k)A(t_i)$.

(3) Given a vector of observations, $\mathbf{y} = H\mathbf{x} + \boldsymbol{\epsilon}$ with weighting matrix W, and *a priori* information, $(\overline{\mathbf{x}}, \overline{W})$, determine the least squares estimate for $\hat{\mathbf{x}}$. (Note that W corresponds to R^{-1} and $\overline{W}$ to $\bar{P}_0^{-1}$ in the case where we have statistical information on $\boldsymbol{\epsilon}$ and $\bar{\mathbf{x}}$.) Let the performance index be

$$J(x) = 1/2\ \boldsymbol{\epsilon}^T W \boldsymbol{\epsilon} + 1/2\ \overline{\boldsymbol{\eta}}^T \overline{W} \overline{\boldsymbol{\eta}}$$

where $\overline{\boldsymbol{\eta}}$ is the error in the *a priori* estimate $\overline{\mathbf{x}}$,

$$\overline{\boldsymbol{\eta}} = \overline{\mathbf{x}} - \mathbf{x}.$$

Answer: $\hat{\mathbf{x}} = (H^T W H + \overline{W})^{-1}(H^T W \mathbf{y} + \overline{W}\overline{\mathbf{x}})$

(4) Determine the state transition matrix associated with the matrix

$$A = \begin{bmatrix} a & 0 \\ b & g \end{bmatrix}, \quad a \neq g, \text{ and } \dot{\Phi} = A\Phi,\ \Phi(t_0, t_0) = I.$$

(5) Express the linear system of equations

$$\ddot{x} = -abx$$

in the matrix form

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

Find the state transition matrix for this system.

(6) Show that the matrix

$$\Phi(t, t_0) = \begin{bmatrix} 3e^{at} & 0 \\ 0 & 2e^{-bt} \end{bmatrix}$$

satisfies the relation

$$\dot{\Phi} = A\Phi$$

but that $\Phi(t, t_0)$ is not a transition matrix. (Assume $t_0 = 0$.)

(7) Show that whenever $\Phi(t, t_k)$ satisfies Eq. (4.2.22), it is symplectic; that is, $\Phi(t, t_k)^T J \phi(t, t_k) = J$.

(8) The displacement of a particle, under the influence of a constant acceleration $\ddot{x}_0$, can be expressed as

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}\ddot{x}_0 t^2,$$

where x_0 is the initial displacement, $\dot{x}_0$ is the initial velocity, and $\ddot{x}_0$ is the acceleration at the initial time, $t_0 = 0$.

(a) By successive differentiation of this expression, show that the linear system

$$\dot{\mathbf{x}} = A\mathbf{x}$$

describes the motion, where

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}; A = \begin{bmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 0\,0\,0 \end{bmatrix}.$$

(b) Prove that the transition matrix $\Phi(t, t_0)$ of the system in (a) is

$$\Phi(t, t_0) = \begin{bmatrix} 1 & (t - t_0) & (t - t_0)^2/2 \\ 0 & 1 & (t - t_0) \\ 0 & 0 & 1 \end{bmatrix}$$

first by differentiating Φ and showing that $\dot{\Phi}(t, t_0) = A\Phi(t, t_0)$ where $\Phi(t_0, t_0) = I$, and then by integrating this system of differential equations.

(c) Show that $\Phi(t, t_0)$ can be represented as

$$\Phi(t, t_0) = e^{A(t-t_0)} = I + \sum_{n=1}^{\infty} \frac{1}{n!} A^n (t - t_0)^n.$$

(d) Calculate $\Phi(t_0, t)$ by direct inversion.

(e) Let $\Phi^{-1}(t, t_0) = \Theta(t, t_0)$ and show that

$$\dot{\Theta}(t, t_0) = -\Theta(t, t_0) A, \Theta(t_0, t_0) = I$$

by integration and comparison with the results of d.

(f) Calculate $\Phi(t_2, t_1)$ by finding the product $\Phi(t_2, t_0)\, \Phi(t_0, t_1)$.

(g) Compare this result with the result obtained by integrating the equation

$$\dot{\Phi}(t, t_1) = A\, \Phi(t, t_1),$$

with the condition $\Phi(t_1, t_1) = I$.

(h) Show that

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} = \Phi(t, t_0),$$

where $\mathbf{x}^T = (x \;\; \dot{x} \;\; \ddot{x})$ and $\mathbf{x}_0^T = (x_0 \;\; \dot{x}_0 \;\; \ddot{x}_0)$.

(9) The equations of motion for a satellite moving in the vicinity of a body with a homogeneous mass distribution can be expressed as

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3}$$

where $\mathbf{r}$ is the position vector, $\ddot{\mathbf{r}}$ is the acceleration vector, and where $r = |\mathbf{r}|$. Let $\mathbf{v} = \dot{\mathbf{r}}$ denote the velocity vector, and express the equations in first-order form as

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ -\dfrac{\mu}{r^3}\mathbf{r} \end{bmatrix}.$$

(a) The relations that define the deviations from a given reference orbit due to deviations in $\mathbf{r}$ and $\mathbf{v}$ and μ at a given time t_0 can be used to analyze the trajectory sensitivity. Show that

$$\frac{d}{dt} \begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \dfrac{\partial \mathbf{f}}{\partial \mathbf{r}} & 0 \end{bmatrix} \begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{v} \end{bmatrix} - \begin{bmatrix} 0 \\ \dfrac{\mathbf{r}}{r^3} \end{bmatrix} \delta\mu,$$

where

$$\mathbf{f} = -\frac{\mu \mathbf{r}}{r^3}$$

$$-\frac{\partial \mathbf{f}}{\partial \mathbf{r}} = \begin{bmatrix} \frac{\mu}{r^3} - 3\mu\frac{x_1^2}{r^5} & -3\mu\frac{x_1x_2}{r^5} & -3\mu\frac{x_1x_3}{r^5} \\ -3\mu\frac{x_1x_2}{r^5} & \frac{\mu}{r^3} - 3\mu\frac{x_2^2}{r^5} & -3\mu\frac{x_2x_3}{r^5} \\ -3\mu\frac{x_1x_3}{r^5} & -3\mu\frac{x_2x_3}{r^5} & \frac{\mu}{r^3} - 3\mu\frac{x_3^2}{r^5} \end{bmatrix}.$$

Note that $\partial \mathbf{f}/\partial \mathbf{r}$ is symmetric.

(b) The constant μ is a physical constant and $\delta\mu$ represents the error in the knowledge of μ. Show that the error in $\mathbf{r}(t)$ and $\mathbf{v}(t)$ can be related to the error in μ by the following expression (Hint: Use the solution for $x(t)$ given by Eq. (4.9.14).):

$$\begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{v}(t) \end{bmatrix} = \Phi(t, t_0) \begin{bmatrix} \delta\mathbf{r}_0 \\ \delta\mathbf{v}_0 \end{bmatrix} - \delta\mu \int_{t_0}^{t} \Phi(t, \tau) \begin{bmatrix} 0 \\ \frac{\mathbf{r}}{r^3} \end{bmatrix} d\tau.$$

(10) Assume an orbit plane coordinate system for Exercise 9 with $\mu = 1$.

(a) Generate a "true" solution by numerically integrating the resulting differential equations

$$\ddot{x} = -\frac{x}{r^3}$$
$$\ddot{y} = -\frac{y}{r^3}$$
$$r^2 = x^2 + y^2$$

for the initial conditions

$$\mathbf{X}(t_0) = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix}_{t=t_0} = \begin{pmatrix} 1.0 \\ 0.0 \\ 0.0 \\ 1.0 \end{pmatrix}.$$

Save the values of the state vector $\mathbf{X}(t_i)$ for $t_i = i*10.; i = 0, \ldots, 10$.

(b) Perturb the previous set of initial conditions by an amount

$$\mathbf{X}^*(t_0) = \mathbf{X}(t_0) - \delta\mathbf{X}(t_0)$$

where

$$\delta\mathbf{X}(t_0) = \begin{pmatrix} 1\times 10^{-6} \\ -1\times 10^{-6} \\ 1\times 10^{-6} \\ 1\times 10^{-6} \end{pmatrix}$$

and numerically integrate this "nominal" trajectory along with the associated state transition matrix to find $\mathbf{X}^*(t_i)$ and $\Phi(t_i, t_0)$ at $t_i = i * 10.; i = 0, \ldots, 10.$

(c) For this problem, $\Phi(t_i, t_0)$ is symplectic. Demonstrate this for $\Phi(t_{100}, t_0)$ by multiplying it by $\Phi^{-1}(t_{100}, t_0)$, given by Eq. (4.2.22), and showing that the result is the identity matrix.

(d) Calculate the perturbation vector, $\delta\mathbf{X}(t_i)$, by the following two methods:

(1) $\delta\mathbf{X}(t_i) = \mathbf{X}(t_i) - \mathbf{X}^*(t_i)$

(2) $\delta\mathbf{X}(t_i) = \Phi(t_i, t_0)\delta\mathbf{X}(t_0)$

and compare the results of (1) and (2). A program such as Matlab works well for this problem.

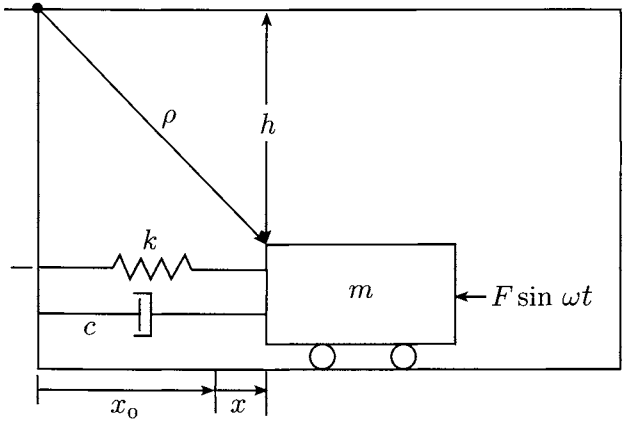
(11) Show that if the acceleration on a spacecraft is derivable from a potential function,

$$\ddot{\mathbf{r}} = \nabla U$$

then the state transition matrix is symplectic; that is, Eq. (4.2.13) is true under the assumption that the state vector is

$$\mathbf{x} = \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix}.$$

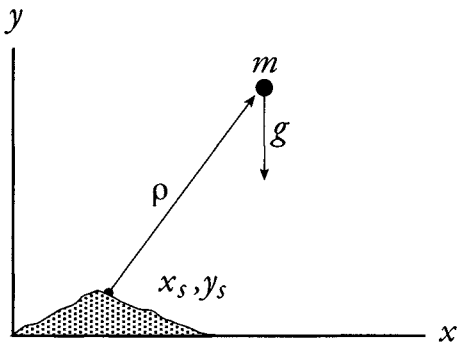
(12) Given the spring-mass system, as shown,



$$m\ddot{x} + c\dot{x} + kx + F\sin\omega t = 0.$$

Assume that we wish to estimate x, $\dot{x}$, F, c, and h using observations ρ. Derive the A and $\widetilde{H}$ matrix for this state vector.

(13) Given a point mass falling under the acceleration of gravity and being observed by range observations, ρ, as shown. Determine the state transition matrix, and the $\widetilde{H}$ and H matrices. Assume the state vector is to be estimated at $t_0 = 0$.



$$\text{let } \mathbf{X} = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \\ g \\ x_s \end{bmatrix}, \qquad \begin{array}{c} @ \ t_0 \\ x = x_0 \\ \dot{x} = \dot{x}_0 \\ y = y_0 \\ \dot{y} = \dot{y}_0 \end{array}$$

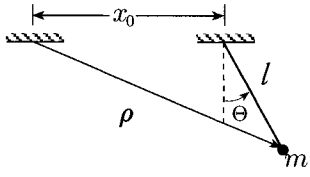
(14) Given a simple pendulum with range observations $|\rho|$, from a fixed point, as shown in the figure.

(a) Write the equations of motion and form the observation-state matrix ($\tilde{H}$) and the state propagation matrix (A). Assume the state vector is

$$\mathbf{X} = \begin{bmatrix} \Theta \\ \dot{\Theta} \\ x_0 \end{bmatrix}.$$

(b) Assume small oscillations; that is, $\sin\Theta \approx \Theta$, $\cos\Theta \approx 1$. Solve the equations of motion and derive expressions for the state transition matrix.

(c) How does the assumption that Θ is small differ from a linearized formulation of this problem?



at $t = t_0$, $\Theta = \Theta_0$, $\dot{\Theta} = \dot{\Theta}_0$

(d) Write a computer program to process range observations of the pendulum mass using both a batch processor and a sequential filter.

(e) Generate a set of observations and process these observations using both the batch and sequential estimation algorithms. Use both perfect observations and observations with random noise.

(15) Write a computer program to process range and range-rate data for the spring-mass system of Section 4.8.2. Use the following data set (or create your own), which has Gaussian noise with mean zero and $\sigma_\rho = 0.25$ m

and $\sigma_{\dot{\rho}} = 0.10$ m/s added. Use the same initial conditions given in Example 4.8.2. Use a weighting matrix, W, that reflects the noise on the data,

$$E[\epsilon\epsilon^T] = R = \begin{bmatrix} \sigma_\rho^2 & 0 \\ 0 & \sigma_{\dot{\rho}}^2 \end{bmatrix} = \begin{bmatrix} .0625 & 0 \\ 0 & .01 \end{bmatrix} = W^{-1}.$$

Observation Data

Time	$\rho(m)$	$\dot{\rho}(m/s)$
0.00	6.37687486186586	−0.00317546143535849
1.00	5.50318198665912	1.17587430814596
2.00	5.94513302809067	−1.47058865193489
3.00	6.30210798411686	0.489030779000695
4.00	5.19084347133671	0.993054430595876
5.00	6.31368240334678	−1.40470245576321
6.00	5.80399842220377	0.939807575607138
7.00	5.45115048359871	0.425908088320457
8.00	5.91089305965839	−1.47604467619908
9.00	5.6769731201352	1.42173765213734
10.00	5.25263404969825	−0.12082311844776

Answer after three iterations:

$$x_0 = 2.9571m, \quad v_0 = -0.1260m/s$$
$$\rho_{\text{RMS}} = 0.247m, \dot{\rho}_{\text{RMS}} = 0.0875m/s$$
$$\sigma_{x_0} = 0.0450m, \ \sigma_{v_0} = 0.0794m/s, \ \ \rho_{x_0 v_0} = 0.0426$$

(16) Repeat Exercise 15, except program the sequential processor algorithm. Use the same initial conditions. Solve for the state at each observation time. Map the state estimate and covariance matrix from the final time to the initial time and show that they agree with the batch results.

(17) Given: The equation of motion of a particle moving in a uniform gravity field influenced by a resistive drag force; for example,

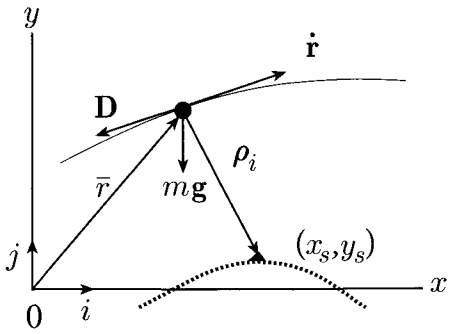
$$m\ddot{\mathbf{r}} = -mg\mathbf{j} - m\mathbf{D}$$

where

$$\mathbf{D} = \frac{1}{2}\rho\beta\ \dot{r}\ \dot{\mathbf{r}},$$

and the sequence of observations

$$\rho_i = \sqrt{(x - x_s)_i^2 + (y - y_s)_i^2}\, i = 1, \ldots, m.$$



Set up the necessary equations to estimate the initial position and velocity of the particle assuming that β and x_s, y_s are unknown but constant force-model and measurement-model parameters to be estimated. Discuss the conditions necessary to ensure complete parameter observability.

(18) Given the following dynamic system

$$\ddot{x} = -kx - \beta x^2 + c \sin \omega t$$
$$\rho_i = \sqrt{d^2 + (\ell + x_i)^2} + \epsilon_i; i = 1, \ldots, m.$$

$$E[\epsilon_i] = 0, \qquad E[\epsilon_i \epsilon_j] = \sigma_i^2 \delta_{ij}$$

Set up all necessary equations for sequentially estimating the position and velocity of the system from observations (ρ_i). Assume that [d, ℓ, k, β, c] are unknowns to be estimated. State clearly all information that must be given as input information beyond what is given in the problem statement.

(19) Given the observation-state relation

$$y(t) = \sum_{i=1}^{3} (t)^i x_i,$$

and the observation sequence at $t = 1$, $y(1) = 2$, and at $t = 2$, $y(2) = 1$.

Find the "best" estimate of $\mathbf{x}_i$, $i = 1, 2, 3$. (Hint: There are fewer observations than unknowns, so use the minimum norm solution.)

(20) Given the observation state relation $\mathbf{y} = H\,x + \boldsymbol{\epsilon}$, where x is a scalar and given that

$$\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \text{ and } E[\boldsymbol{\epsilon}] = 0,$$

$$R = E\left[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T\right] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

with *a priori* information, $\overline{x} = 2$ and $\sigma^2(\overline{x}) = \frac{1}{2}$.

(a) Using R^{-1} as the weighting matrix, find $\hat{x}$ using the batch processing algorithm.

(b) What is the standard deviation of the estimation error associated with $\hat{x}$?

(c) What is the best estimate of the observation error, $\hat{\boldsymbol{\epsilon}}$?

(21) Given the system

$$\begin{aligned} x(t) &= x_0 + \dot{x}_0 t + \frac{1}{2}\ddot{x}_0 t^2; \quad t_0 = 0 \\ \dot{x}(t) &= \dot{x}_0 + \ddot{x}_0 t \\ \ddot{x}(t) &= \ddot{x}_0 \end{aligned}$$

with

$$\mathbf{X}(t) = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}, \quad \overline{P}_0 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \overline{\mathbf{X}}(t_0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

At $t_1 = 1$, an observation of $x(t)$ is made; that is, $Y(t_1) = x(t_1) = 2$. The variance of the error in $Y(t_1)$ is $\sigma^2(\epsilon_Y) = 1$.

(a) Compute $\hat{\mathbf{X}}(t_0)$ using the batch processing algorithm.
Answer:

$$\hat{\mathbf{X}}(t_0) = \begin{bmatrix} 21/29 \\ 25/29 \\ 28/29 \end{bmatrix}$$

(b) Using the sequential processing algorithm, determine $\hat{\mathbf{X}}(t_0)$. Remember you are to solve for $\hat{\mathbf{X}}(t_0)$, not $\hat{\mathbf{X}}(t_1)$. You should get identical results to part a.

(c) Compute $\hat{\mathbf{X}}(t_1)$ using the sequential algorithm, then map $\hat{\mathbf{X}}(t_1)$ to t_0 using the state transition matrix. You should get the same value of $\hat{\mathbf{X}}(t_0)$ as you computed in parts a and b.

Answer:

$$\hat{\mathbf{X}}(t_1) = \begin{bmatrix} 60/29 \\ 53/29 \\ 28/29 \end{bmatrix}$$

(22) Assume a vehicle is moving along a rectilinear path with a constant acceleration. The generic equation of motion is

$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2.$$

At discrete constant intervals of time $t = t_i$, $i = 0, 1, 2, 3 \ldots 10$ the position and velocity of the vehicle are observed.

(a) Set up a flow chart for the batch and sequential estimation algorithm to estimate a state vector consisting of the position, velocity, and acceleration. For the batch algorithm estimate the state at t_0, and for the sequential estimate the state at t_i.

(b) Write a program (using Matlab or a similar software package) to solve for the state vector using the batch processor. Include an *a priori* covariance matrix and a measurement weighting matrix and the capability to map the state estimate and the estimation error covariance matrix from the epoch to any desired time. Generate a sequence of simulated observations to which random noise is added. Solve for the state vector and associated covariance matrix at t_0. Map these quantities to the final time for comparison with results from the sequential processor in part c.

(c) Repeat part b for the sequential processor and estimate the state at t_i, $i = 1 \ldots 10$. Compare results at the final measurement time with the mapped results of the batch solution of part b.

(d) Experiment with your program by changing the *a priori* covariance and weighting matrix. Write a brief report describing your results.

(23) For the problem posed in Section 4.7.1:

(a) Show that the exact value for P_2 is given by Eq. (4.7.24).

(b) Under the assumption that $1 + \epsilon^2 = 1$, show that the conventional Kalman filter yields

$$P_2 = \frac{1}{1-2\epsilon}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(c) Under the assumption that $1 + \epsilon^2 = 1$, show that the batch processor yields

$$P_2 = \frac{1}{1-2\epsilon}\begin{bmatrix} 1 & -(1+\epsilon) \\ -(1+\epsilon) & 2 \end{bmatrix} = \begin{bmatrix} 1+2\epsilon & -(1+3\epsilon) \\ -(1+3\epsilon) & 2(1+2\epsilon) \end{bmatrix},$$

which agrees with the exact result to $O(\epsilon)$, thus illustrating the improved numerical behavior of the batch over the sequential processor.

(24) Two observations are taken of the parameter x,

$$y_1 = x + \epsilon_1, y_2 = x + \epsilon_2,$$

where ϵ_1 and ϵ_2 are zero mean random noise. It is known that the variance of ϵ_2 is twice the variance of ϵ_1. Determine the weighted least squares estimate of x.

(25) Derive the following two equations for the estimation error covariance matrix:

(a) Beginning with Eq. (4.4.29), derive the expression for the estimation error covariance matrix given by Eq. (4.4.30). Assume that

$$E\left[(\bar{\mathbf{x}} - \mathbf{x})\boldsymbol{\epsilon}^T\right] = 0.$$

(b) Beginning with the equation for $\hat{\mathbf{x}}$ (Eq. 4.7.16), derive the Joseph formulation for the measurement update of the estimation error covariance matrix given by Eq. (4.7.19).

(c) Show that the Joseph formulation for the measurement update of P is equivalent to the conventional Kalman update given by Eq. (4.7.18),

$$P = (I - K\widetilde{H})\overline{P}.$$

(26) For the joint density function given by Eq. (4.5.4),

(a) Determine the MLE of σ.

(b) Show that the MLE of α given by Eq. (4.5.8) is unbiased.

(27) Let $\mathbf{X} = [x_1\, x_2\, \ldots\, x_n]^T$ have a multivariate Gaussian distribution with each component of $\mathbf{X}$ having the same (unknown) mean α,

$$f(\mathbf{X};\, \alpha) = \frac{1}{(2\pi)^{n/2}\, |P|^{\frac{1}{2}}} \exp[-\frac{1}{2}(\mathbf{X} - A\alpha)^T P^{-1}(\mathbf{X} - A\alpha)]$$

with

$$A = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

(a) Identify the likelihood function and show that the expression for the maximum likelihood estimate of α is $\hat{\alpha} = (A^T P^{-1} A)^{-1} A^T P^{-1} \mathbf{X}$.

(b) Show that $\hat{\alpha}$ is an unbiased estimator of α.

(28) Given the estimator $\hat{\mathbf{x}} = \overline{\mathbf{x}} + K(\mathbf{y} - H\overline{\mathbf{x}})$, state all necessary conditions on $\overline{\mathbf{x}}$ and $\mathbf{y}$ for $\hat{\mathbf{x}}$ to be an unbiased estimate of $\mathbf{x}$.

(29) Given: Random variables X and Y and

$$Y = a \pm bX$$

where a and b are constants. Show that the correlation coefficient $\rho(X, Y)$ has the value

$$\rho(X, Y) = \pm 1.$$

(30) Given that $\mathbf{x}$ is an $n \times 1$ vector of random variables with mean $\overline{\mathbf{x}}$ and covariance P. Let $\mathbf{y}$ be an $m \times 1$ vector of random variables related to $\mathbf{x}$ by

$$\mathbf{y} = H\mathbf{x} + \boldsymbol{\epsilon}$$

where $\boldsymbol{\epsilon}$ is zero mean with covariance R and is independent of $\mathbf{x}$.

(a) Find the mean of $\mathbf{y}$.

(b) Show that the variance-covariance of $\mathbf{y}$ is given by

$$\begin{aligned} P_y &= E\left[(\mathbf{y} - \overline{\mathbf{y}})(\mathbf{y} - \overline{\mathbf{y}})^T\right] \\ &= H\,P\,H^T + R. \end{aligned}$$

(c) Show that the covariance of $\mathbf{x}$ and $\mathbf{y}$ is

$$P_{\mathbf{xy}} \equiv E[(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{y}-\bar{\mathbf{y}})^T] = PH^T.$$

(31) An estimate, $\hat{\mathbf{x}}$ is made of $\mathbf{x}$ (an $n \times 1$ vector), based on m observations $\mathbf{y}$ $(m > n)$, where

$$\mathbf{y} = H\mathbf{x} + \boldsymbol{\epsilon}$$

and

$$\boldsymbol{\epsilon} \text{ is } N(\bar{\epsilon}, R).$$

An *a priori* value of $\mathbf{x}$ is given:

$$\bar{\mathbf{x}} = \mathbf{x} + \mathbf{e}$$

where $\mathbf{e}$ is $N(0, \overline{P})$. Assume $E[\mathbf{e}\,\boldsymbol{\epsilon}^T] = 0$.

The estimate is

$$\hat{\mathbf{x}} = \bar{\mathbf{x}} + K(\mathbf{y} - H\bar{\mathbf{x}}).$$

(a) What is the bias for this estimator?

(b) What is the variance-covariance associated with the estimation error for $\hat{\mathbf{x}}$? Note that $\hat{\mathbf{x}}$ is biased so use the definition of P given in the answer.

(c) Show that by redefining the noise vector to be $\boldsymbol{\epsilon} = \bar{\boldsymbol{\epsilon}} + \boldsymbol{\epsilon}'$, and by including $\bar{\boldsymbol{\epsilon}}$ in the state vector, an unbiased estimator may be formed. Assume that an *a priori* estimate of $\bar{\boldsymbol{\epsilon}}$ with covariance $P_{\bar{\boldsymbol{\epsilon}}}$ is available.

Answer:

(a) $E[\hat{\mathbf{x}}] = \mathbf{x} + K\bar{\boldsymbol{\epsilon}}$, bias $= K\bar{\boldsymbol{\epsilon}}$

(b) $P = (I - KH)\overline{P}(I - KH)^T + KRK^T$ where

$$R = E\left[(\boldsymbol{\epsilon} - \bar{\boldsymbol{\epsilon}})(\boldsymbol{\epsilon} - \bar{\boldsymbol{\epsilon}})^T\right]$$

and

$$P \equiv E\left[(\tilde{\mathbf{e}} - E(\tilde{\mathbf{e}}))(\tilde{\mathbf{e}} - E(\tilde{\mathbf{e}}))^T\right]$$
$$\tilde{\mathbf{e}} \equiv \hat{\mathbf{x}} - \mathbf{x}, \quad E(\tilde{\mathbf{e}}) = K\bar{\boldsymbol{\epsilon}}$$

(32) Assuming we are given no *a priori* information on $\hat{\mathbf{x}}$,

(a) show that there is no correlation between the residual observation error, $\hat{\mathbf{y}} - \mathbf{y} = \hat{\epsilon}$, and the estimation error, $\hat{\mathbf{x}} - \mathbf{x}$, that

$$E\left[\hat{\epsilon}(\hat{\mathbf{x}} - \mathbf{x})^T\right] = 0.$$

If this were not zero, it would mean that we had not extracted all of the available information from the observations.

Hint: Use

$$\begin{aligned}
\hat{\epsilon} &= \mathbf{y} - H\hat{\mathbf{x}} \\
\hat{\mathbf{x}} &= (H^T R^{-1} H)^{-1} H^T R^{-1} \mathbf{y} \\
\mathbf{y} &= H\mathbf{x} + \epsilon, \quad E[\epsilon] = 0, \quad E[\epsilon\epsilon^T] = R.
\end{aligned}$$

(b) Assuming that *a priori* information is given; that is, $\bar{\mathbf{x}} = \mathbf{x} + \eta$ where $E[\eta] = 0$, $E[\eta\eta^T] = \overline{P}$ and $E[\epsilon\eta^T] = 0$. Show that $E[\hat{\epsilon}(\hat{\mathbf{x}} - \mathbf{x})^T] = 0$ is still true.

(33) Occasionally *a priori* information, $\overline{\mathbf{x}}$ and $\overline{P}$, used to initiate a batch filter is obtained from processing an earlier batch of data so that the errors in $\overline{\mathbf{x}}$ and the observation error, ϵ, are correlated. Derive the expression for the covariance matrix of the estimation error, P, assuming

$$\begin{aligned}
P &= E\left[(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T\right] \\
\hat{\mathbf{x}} &= (\overline{P}^{-1} + H^T R^{-1} H)^{-1} (\overline{P}^{-1} \overline{\mathbf{x}} + H^T R^{-1} \mathbf{y}),
\end{aligned}$$

$$\begin{aligned}
\overline{\mathbf{x}} &= \mathbf{x} + \epsilon_{\overline{\mathbf{x}}}, \quad E(\epsilon_{\overline{\mathbf{x}}}) = 0, \qquad E(\epsilon_{\overline{\mathbf{x}}} \epsilon_{\overline{\mathbf{x}}}^T) = \overline{P}, \\
\mathbf{y} &= H\mathbf{x} + \epsilon, E\left[\epsilon\right] = 0, \qquad E\left[\epsilon\, \epsilon^T\right] = R, \\
&\qquad\qquad E\left[\epsilon\, \epsilon_{\overline{\mathbf{x}}}^T\right] = M.
\end{aligned}$$

Answer:

$$\begin{aligned}
P = &(\overline{P}^{-1} + H^T R^{-1} H)^{-1} + (\overline{P}^{-1} + H^T R^{-1} H)^{-1} \\
&\times \left[H^T R^{-1} M \overline{P}^{-1} + \overline{P}^{-1} M^T R^{-1} H\right] (\overline{P}^{-1} + H^T R^{-1} H)^{-1}.
\end{aligned}$$

(34) Given that

$$\begin{aligned}
\hat{\mathbf{x}}_k &= \mathbf{x}_k + \eta_k \\
\hat{\mathbf{x}}_{k+1} &= \mathbf{x}_{k+1} + \eta_{k+1}
\end{aligned}$$

and

$$\mathbf{x}_{k+1} = \Phi(t_{k+1}, t_k)\mathbf{x}_k + \Gamma(t_{k+1}, t_k)\mathbf{u}_k,$$

where

$$E[\boldsymbol{\eta}_k] = E[\boldsymbol{\eta}_{k+1}] = 0$$

and $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ are the true values, show that

$$P_{\boldsymbol{\eta}_k \boldsymbol{\eta}_{k+1}} = E[\boldsymbol{\eta}_k \boldsymbol{\eta}_{k+1}^T] = P_k \Phi^T(t_{k+1}, t_k)(I - K_{k+1} H_{k+1})^T.$$

(35) Consider the linear system defined in Exercise 21. Assume $t_0 = 0$; otherwise replace t with $(t - t_0)$.

$$\Phi(t, t_0) = \frac{\partial X(t)}{\partial X(t_0)} = \begin{bmatrix} 0 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\overline{P}_0 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Show that the time update for P at the first measurement time, t_1, obtained by integrating the differential equation for $\dot{\overline{P}}$ (Eq. 4.9.36) with initial conditions $\overline{P}_0$ is given by

$$\overline{P}(t_1) = \begin{bmatrix} 4 + 2t_1 + \frac{t_1^4}{4} & 2t_1 + \frac{t_1^3}{3} & \frac{t_1^2}{2} \\ 2t_1 + \frac{t_1^3}{3} & 2 + t_1^2 & t_1 \\ \frac{t_1^2}{2} & t_1 & 1 \end{bmatrix}.$$

(b) Show that this agrees with the conventional time update given by

$$\overline{P}(t_1) = \Phi(t_1, t_0)\overline{P}_0 \Phi^T(t_1, t_0).$$

(36) Given the information matrix

$$\Lambda = \begin{bmatrix} 13 & 2 & 6 \\ 2 & 2 & 6 \\ 6 & 6 & 18 \end{bmatrix}.$$

Are all three elements of the state vector observable? Why or why not?

(37) In Example 4.2.1 the state vector includes X, Y, X_s, and Y_s. Are all of these quantities observable? Why or why not?

(38) Given a 2D state vector, $\mathbf{X}$, with

$$E[\mathbf{X}] = 0, \quad E[\mathbf{X}\mathbf{X}^T] = P = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}.$$

Sketch the 2σ probability ellipse.

(39) For the transformation of coordinates

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x+2 \\ y+2 \end{bmatrix},$$

determine $\overline{x}'$, $\overline{y}'$, and P' where $\overline{x}' = E(x')$, $\overline{y}' = E(y')$ and

$$P' = \begin{bmatrix} \sigma^2(x') & \mu_{11}(x'y') \\ \mu_{11}(x'y') & \sigma^2(y') \end{bmatrix}.$$

(40) The differential equation for the Ornstein-Uhlenbeck process, a first-order Gauss-Markov process, is

$$\dot{\eta}(t) + \beta\eta(t) = u(t)$$

with initial conditions $\eta = \eta(t_0)$, at $t = t_0$ where $u(t)$ is a Gaussian white noise process with mean zero. Show that the solution is given by

$$\eta(t) = \eta(t_0)e^{-\beta(t-t_0)} + \int_{t_0}^{t} e^{-\beta(t-\tau)}u(\tau)d\tau.$$

(41) Generate 1000 equally spaced observations of one cycle of a sine wave with amplitude 1 and period 10. Add Gaussian random noise with zero mean and variance $= 0.25$. Set up a sequential estimation procedure to estimate the amplitude of the sine wave as a function of time using the noisy raw data. Model the sine wave as a Gauss-Markov process as given by Eq. (4.9.60),

$$\eta_{i+1} = m_{i+1}\eta_i + \Gamma_{i+1}u_i$$

where

$$u_i = N(0,1)$$

$$m_{i+1} = e^{-\beta(t_{i+1}-t_i)}$$

$$\Gamma_{i+1} = \sqrt{\frac{\sigma^2}{2\beta}(1 - m_{i+1}^2)}$$

$$\beta = \frac{1}{\tau}$$

and τ is the time constant. The sequential algorithm is given by

1. $\bar{\eta}_i = \Phi(t_i, t_{i-1})\hat{\eta}_{i-1} \quad (i = 1, 2 \ldots 1000)$
   $\Phi(t_i, t_{i-1}) = m_i = e^{-\beta(t_i - t_{i-1})}$
   $\overline{P}_i = \Phi(t_i, t_{i-1})P_{i-1}\Phi^T(t_i, t_{i-1}) + \Gamma_i Q_{i-1}\Gamma_i^T$
   Note that P, Φ, Q, and Γ are scalars
   $Y_i = \eta_i$, thus $\widetilde{H}_i = 1$, assume $R_i = 1$, $Q_i = 1$, $\bar{\eta}_0 = 0$, $\overline{P}_0 = 1$
   $K_i = \overline{P}_i\widetilde{H}_i^T(\widetilde{H}_i\overline{P}_i\widetilde{H}_i^T + R_i)^{-1} = \dfrac{\overline{P}_i}{\overline{P}_i + 1}$
   $\hat{\eta}_i = \bar{\eta}_i + K_i(Y_i - \widetilde{H}_i\bar{\eta}_i) = \bar{\eta}_i + K_i(Y_i - \bar{\eta}_i)$, ($Y_i$ is the observation data)
   $P_i = (I - K_i\widetilde{H}_i)\overline{P}_i = K_i$
   Next i

Plot your observations, the truth data, and $\hat{\eta}$ versus time. You will need to guess initial values for σ and β. Try several values to see which gives you the best results (i.e., the smallest RMS of estimation errors).

$$\text{RMS} = \left\{\sum_{i=1}^{N} \frac{(T_i - \hat{\eta}_i)^2}{N}\right\}^{1/2},$$

where T_i is the true amplitude of the sine wave and $N = 1000$. Fig. 4.19.1 illustrates the truth, the raw data, and one example of the filtered results (not necessarily the optimal) for the indicated values of σ and β. The truth is the smooth curve. You may also plot a histogram of the post-fit residuals to see if they conform to a Gaussian distribution.

(42) Using the optimum values of σ and τ determined in Exercise 41, solve for the smoothed history of $\hat{\eta}(t)$ using the algorithm of Section 4.15.1. Plot the true values of η, the filter values determined in Exercise 41, and the smoothed values. Compute the RMS of the smoothed fit. You should find the RMS from the smoothed solution somewhat smaller than the filter result. Fig. 4.19.2 is an example of the truth, the filter, and the smoothed results for the case illustrated in Fig. 4.19.1. Is the histogram of residuals

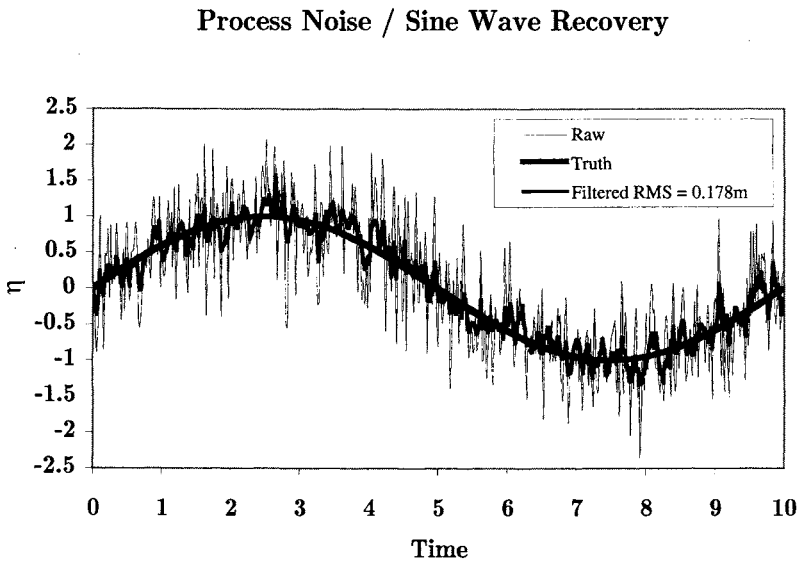


Figure 4.19.1: Process noise/sine wave recovery showing truth, raw data (truth plus noise) and the filtered solution. $\bar{\eta}_0 = 0$, $\sigma = 2.49$, $\beta = .045$.

for the smoothed solution more nearly Gaussian than those from the filter in Exercise 41?

(43) Given a system described by

$$\mathbf{X}(t_{i+1}) = \begin{bmatrix} x_1(t_{i+1}) \\ x_2(t_{i+1}) \end{bmatrix} = \begin{bmatrix} 1\,1 \\ 0\,1 \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix} + \begin{bmatrix} 1\,1 \\ 1\,0 \end{bmatrix} \begin{bmatrix} u_1(t_i) \\ u_2(t_i) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t_i) \\ y_2(t_i) \end{bmatrix} = \begin{bmatrix} 0\,1/2 \\ 1\,1/2 \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

$$\overline{\mathbf{X}}(t_0) = \begin{bmatrix} \overline{x}_1(t_0) \\ \overline{x}_2(t_0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \overline{P}(t_0) = \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}$$

$$R = E\left\{ \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} [\epsilon_1 \epsilon_2] \right\} = \begin{bmatrix} 1\,0 \\ 0\,2 \end{bmatrix},$$

$$E[u_1(t_i)] = E[u_2(t_i)] = 0$$

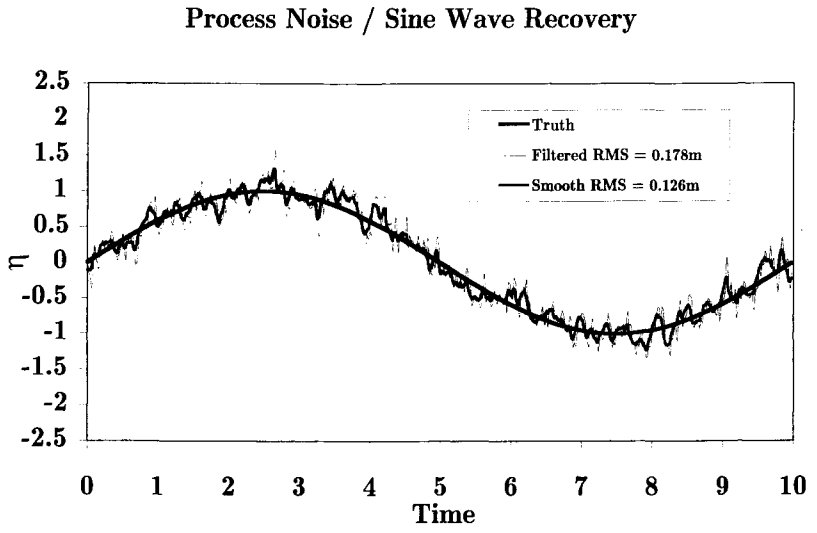


Figure 4.19.2: Process noise/sine wave recovery showing the truth, the filtered, and the smoothed solution. $\bar{\eta}_0 = 0$, $\sigma = 2.49$, $\beta = .045$.

$$E\left\{\begin{bmatrix} u_1(t_i) \\ u_2(t_i) \end{bmatrix} [u_1(t_j)u_2(t_j)]\right\} = I\delta_{ij}.$$

Observations at t_1 are given by

$$\begin{bmatrix} y_1(t_1) \\ y_2(t_1) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

(a) Using the sequential estimation algorithm, determine the optimal estimate of $\mathbf{X}(t_1)$ and $P(t_1)$.

(b) What is the smoothed estimate of $\mathbf{X}(t_0)$?

(c) What is the optimal estimate of $\mathbf{X}(t_1)$ if no process noise were present?

(44) Use the information filter to determine the optimal estimate of $\mathbf{X}(t_1)$ and $P(t_1)$ for Exercise 43a.