Appendix C

Equations of Motion

C.1 LAGRANGE PLANETARY EQUATIONS

If the perturbing force $\mathbf{f}$ is conservative, it follows that $\mathbf{f}$ is derivable from a *disturbing function*, D, such that $\mathbf{f} = \nabla D$. The force $\mathbf{f}$ will produce temporal changes in the orbit elements that can be expressed by Lagrange's Planetary Equations (e.g., Kaula, 1966):

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{na} \frac{\partial D}{\partial M} \\
\frac{de}{dt} &= \frac{(1-e^2)^{1/2}}{na^2 e} \left((1-e^2)^{1/2} \frac{\partial D}{\partial M} - \frac{\partial D}{\partial \omega} \right) \\
\frac{di}{dt} &= \frac{1}{h \, \sin i} \left(\cos i \, \frac{\partial D}{\partial \omega} - \frac{\partial D}{\partial \Omega} \right) \\
\frac{d\Omega}{dt} &= \frac{1}{h \, \sin i} \frac{\partial D}{\partial i} \\
\frac{d\omega}{dt} &= -\frac{\cos i}{h \, \sin i} \frac{\partial D}{\partial i} + \frac{(1-e^2)^{1/2}}{na^2 e} \frac{\partial D}{\partial e} \\
\frac{dM}{dt} &= n - \frac{1-e^2}{na^2 e} \frac{\partial D}{\partial e} - \frac{2}{na} \frac{\partial D}{\partial a}.
\end{aligned}
$$

Note that $h = na^2[1 - e^2]^{1/2}$.

C.2 GAUSSIAN FORM

If the perturbing force $\mathbf{f}$ is expressed as

$$\mathbf{f} = \hat{R}\, \overline{u}_r + \hat{T}\, \overline{u}_T + \hat{N}\, \overline{u}_n$$

where the unit vectors are defined by the RTN directions (radial, along-track, and cross-track) and $\hat{R}$, $\hat{T}$, $\hat{N}$ represent force components, the temporal changes in orbit elements can be expressed in the Gaussian form of Lagrange's Planetary Equations (e.g., Pollard, 1966) as:

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2a^2e}{h}\sin f\hat{R} + \frac{2a^2h}{\mu r}\hat{T} \\
\frac{de}{dt} &= \frac{h}{\mu}\left[\sin f\hat{R} + \hat{T}(e + 2\cos f + e\cos^2 f)/(1 + e\cos f)\right] \\
\frac{di}{dt} &= \frac{r}{h}\cos(\omega + f)\hat{N} \\
\frac{d\Omega}{dt} &= \frac{r\sin(\omega + f)\hat{N}}{h\sin i} \\
\frac{d\omega}{dt} &= -\frac{h}{\mu e}\cos f\hat{R} - \frac{r}{h}\cot i\,\sin(\omega + f)\,\hat{N} \\
&\quad + \frac{(h^2 + r\mu)\sin f}{\mu e h}\hat{T} \\
\frac{dM}{dt} &= n - \frac{1}{na}\left(\frac{2r}{a} - \frac{1-e^2}{e}\cos f\right)\hat{R} \\
&\quad - \frac{1-e^2}{nae}\left(1 + \frac{r}{p}\right)\sin f\hat{T}.
\end{aligned}
$$

The Gaussian form applies to either conservative or nonconservative forces.

C.3 REFERENCES

Kaula, W. M., *Theory of Satellite Geodesy*, Blaisdell Publishing Co., Waltham, MA, 1966 (republished by Dover Publications, New York, 2000).

Pollard, H., *Mathematical Introduction to Celestial Mechanics*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1966.