Appendix H

# Transformation between ECI and ECF Coordinates

## H.1 INTRODUCTION

The transformation between ECI and ECF coordinates is defined by Eq. (2.4.12)

$$T_{XYZ}^{xyz} = T_{\text{ECI}}^{\text{ECF}} = WS'NP. \tag{H.1.1}$$

The transformation matrices W, S′, N, and P account for the following effects:

$W =$ the offset of the Earth's angular velocity vector with respect to the $z$ axis of the ECF (see Section 2.4.2)

$S' =$ the rotation of the ECF about the angular velocity vector (see Sections 2.3.3 and 2.4.2)

$N =$ the nutation of the ECF with respect to the ECI (see Section 2.4.1)

$P =$ the precession of the ECF with respect to the ECI (see Section 2.4.1).

The ECI and ECF systems have fundamental characteristics described in Chapter 2. The realization of these reference systems in modern space applications is usually through the *International Celestial Reference Frame* (ICRF) for the ECI and the *International Terrestrial Reference Frame* (ITRF) for the ECF. These realizations infer certain characteristics for highest accuracy and consistency. For example, the ICRF is defined using the coordinates of extragalactic radio sources, known as quasars, where the coordinates are derived from observations obtained on the surface of the Earth with a set of large antennas known as the *Very Long Baseline Interferometry* (VLBI) network. The coordinates of the quasars are then linked with visible sources to an optical star catalog and the coordinates of the celestial bodies in the solar system. In a similar manner, the ITRF is realized with ground networks of SLR, LLR, GPS, and DORIS (see Chapter 3), which establish the center of mass origin of the ITRF, as well as the reference meridian.

The following sections summarize the content of the matrices required for the transformation from an ECI system to an ECF system. This discussion follows the classical formulation and it will be assumed that the ECI is realized through J2000, which is not precisely the same as the actual ICRF adopted by the International Astronomical Union. If observed values of polar motion ($x_p$, $y_p$) and $\Delta$(UT1) are used, the transformation matrix has an accuracy of better than one arc-sec. The satellite motion is especially sensitive to polar motion and the rate of change in UT1, which is related to length of day. This sensitivity enables the estimation of these parameters from satellite observations.

The *International Earth Rotation Service* (IERS) distributes *Earth Orientation Parameters* (EOP) derived from VLBI and satellite observations. The main EOP are ($x_p$, $y_p$), $\Delta$(UT1) and corrections to the precession and nutation parameters. These parameters are empirical corrections derived from satellite and VLBI observations which account for changes resulting from tides and atmospheric effects, for example. The EOP are regularly distributed by the IERS as Bulletin B. Consult the IERS publications and web page for additional information. As the models adopted by the IERS are improved, they are documented in new releases of the IERS Standards (or "Conventions").

For the most part, the following discussion follows Seidelmann (1992) and McCarthy (1996). A comprehensive summary is given by Seeber (1993). A comparison of different implementations of the matrices, including a detailed review of the transformations, has been prepared by Webb (2002).

## H.2 MATRIX $P$

For this discussion, P represents the transformation from J2000 to a mean-of-date system. If is a function of the precession of the equinoxes (see Section 2.4.1). The transformation matrix is dependent on three angles and is given by (where C represents cosine and S denotes sine):

$$P = \qquad \text{(H.2.1)}$$
$$\begin{bmatrix} \mathrm{C}\zeta_A \mathrm{C}\theta_A \mathrm{C}z_A - \mathrm{S}\zeta_A \mathrm{S}z_A & -\mathrm{S}\zeta_A \mathrm{C}\theta_A \mathrm{C}z_A - \mathrm{C}\zeta_A \mathrm{S}z_A & -\mathrm{S}\theta_A \mathrm{C}z_A \\ \mathrm{C}\zeta_A \mathrm{C}\theta_A \mathrm{S}z_A + \mathrm{S}\zeta_A \mathrm{C}z_A & -\mathrm{S}\zeta_A \mathrm{C}\theta_A \mathrm{S}z_A + \mathrm{C}\zeta_A \mathrm{C}z_A & -\mathrm{S}\theta_A \mathrm{S}z_A \\ \mathrm{C}\zeta_A \mathrm{S}\theta_A & -\mathrm{S}\zeta_A \mathrm{S}\theta_A & \mathrm{C}\theta_A \end{bmatrix},$$

and the precession angles are given by

$$
\begin{aligned}
\zeta_A &= 2306''.2181t + 0''.30188t^2 + 0''.017998t^3 \\
\theta_A &= 2004''.3109t - 0''.42665t^2 - 0''.041833t^3 \\
z_A &= 2306''.2181t + 1''.09468t^2 + 0''.018203t^3
\end{aligned}
\tag{H.2.2}
$$

with $t$ defined as

$$
t = \frac{[TT - J2000.0](\text{days})}{36525}. \tag{H.2.3}
$$

In Eq. H.2.3, J2000.0 is January 1, 2000, 12:00 in TT, or JD 2451545.0. These equations are based on the work of Lieske, *et al.* (1977). Terrestrial Time (TT) is defined in Section 2.4.2.

## H.3 MATRIX $N$

The transformation from a mean-of-date system to a true-of-date system is dependent on the nutations (see Section 2.4.1). Four angles are used: the mean obliquity of the ecliptic ($\epsilon_m$), the true obliquity of the ecliptic ($\epsilon_t$), the nutation in longitude ($\Delta\psi$), and the nutation in obliquity ($\Delta\epsilon$). The matrix N is given by

$$
N = \tag{H.3.1}
$$

$$
\begin{bmatrix}
C\Delta\psi & -C\epsilon_m S\Delta\psi & -S\epsilon_m S\Delta\psi \\
C\epsilon_t S\Delta\psi & C\epsilon_m C\epsilon_t C\Delta\psi + S\epsilon_m S\epsilon_t & S\epsilon_m C\epsilon_t C\Delta\psi - C\epsilon_m S\epsilon_t \\
S\epsilon_t S\Delta\psi & C\epsilon_m S\epsilon_t C\Delta\psi - S\epsilon_m C\epsilon_t & S\epsilon_m S\epsilon_t C\Delta\psi + C\epsilon_m C\epsilon_t
\end{bmatrix},
$$

where the mean obliquity is

$$
\epsilon_m = 84381''.448 - 46''.8150t - 0''.00059t^2 + 0''.001813t^3 \tag{H.3.2}
$$

and the true obliquity is

$$
\epsilon_t = \epsilon_m + \Delta\epsilon. \tag{H.3.3}
$$

The expression for the nutation in longitude and in obliquity is given by McCarthy (1996), for example. The IAU 1980 theory of nutation is readily available in a convenient Chebychev polynomial approximation with the planetary ephemerides (Standish, *et al.*, 1997). Corrections to the adopted series, based on observations, are provided by the IERS in Bulletin B.

## H.4 MATRIX $S'$

The true-of-date rotation to a pseudo-body-fixed system accounts for the 'spin' of the Earth. This diurnal rotation depends on a single angle $\alpha_G$, which is referred to as the *Greenwich Mean Sidereal Time* (GMST). The matrix S′ is identical to Eq. 2.3.20; namely,

$$S' = \begin{bmatrix} C\alpha_G & S\alpha_G & 0 \\ -S\alpha_G & C\alpha_G & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{H.4.1}$$

In some cases, especially when a satellite velocity vector with respect to the rotating ECF is required, the time derivative of the time-dependent elements of each of the transformation matrices will be needed. However, the time derivative of S′ is particularly significant since the angle $\alpha_G$ is the fastest changing variable in the matrices P, N, S′ and W. The time derivative of S′ is

$$\dot{S}' = \dot{\alpha}_G \begin{bmatrix} -S\alpha_G & C\alpha_G & 0 \\ -C\alpha_G & -S\alpha_G & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{H.4.2}$$

where $\dot{\alpha}_G$ is the rotation rate of the Earth. For some applications, this rate can be taken to be constant, but for high-accuracy applications the variability in this rate must be accounted for. According to the IERS, the average rate is $7.2921151467064 \times 10^{-5}$ rad/s, but this value should be adjusted using IERS Bulletin B to account for variations. The GMST for a given UT1 (UTC + Δ(UT1)) can be found from Kaplan, *et al.* (1981), modified to provide the result in radians:

$$\begin{aligned} \mathrm{GMST(UT1)} = {} & 4.894961212823058751375704430 \\ & + \Delta T\Big\{6.300388098984893552276513720 \\ & + \Delta T\Big(5.075209994113591478053805523 \times 10^{-15} \\ & -9.253097568194335640067190688 \times 10^{-24}\Delta T\Big)\Big\} \end{aligned} \tag{H.4.3}$$

where $\Delta T = \mathrm{UT1} - \mathrm{J2000.0}$, and where $\Delta T$ is in days, including the fractional part of a day.

An additional correction known as the equation of the equinoxes must be applied. For highest accuracy, additional terms amounting to a few milli-arc-sec amplitude should be applied (see McCarthy, 1996; or Simon, *et al.*, 1994) . With the equation of the equinoxes, the angle $\alpha_G$ becomes

$$\alpha_G = \mathrm{GMST(UT1)} + \Delta\psi \cos\epsilon_m. \tag{H.4.4}$$

## H.5 MATRIX $W$

The pseudo-body-fixed system is aligned with the instantaneous pole, equator and reference meridian of the Earth. As noted in Section 2.4.2, the Earth's spin axis is not fixed in the Earth and two angles ($x_p$, $y_p$) are used to describe the spin axis location with respect to an adopted ECF, such as an ITRF. Because of the small magnitude of the polar motion angles, the transformation matrix W can be represented by

$$T_{\mathrm{PBF}}^{\mathrm{BF}} = W = \begin{bmatrix} 1 & 0 & x_p \\ 0 & 1 & -y_p \\ -x_p & y_p & 1 \end{bmatrix}, \tag{H.5.1}$$

where the polar motion angles are expressed in radians. The angles are available in IERS Bulletin B, for example.

## H.6 REFERENCES


Aoki, S., and H. Kinoshita, "Note on the relation between the equinox and Guinot's non-rotating origin," *Celest. Mech.*, Vol. 29, No. 4, pp. 335–360, 1983.

Kaplan, G. H. (ed.), *The IAU Resolutions on astronomical constants, time scales, and the fundamental reference frame,* USNO Circular No. 163, U.S. Naval Observatory, 1981.

Lambeck, K., *Geophysical Geodesy*, Clarendon Press, Oxford, 1988.

Lieske, J. H., T. Lederle, W. Fricke, and B. Morando, "Expressions for the precession quantities based upon the IAU (1976) System of Astronomical Constants," *Astronomy and Astrophysics*, Vol. 58, pp. 1–16, 1977.

McCarthy, D. (ed.), *IERS Conventions (1996)*, IERS Technical Note 21, International Earth Rotation Service, Observatoire de Paris, July 1996.

Seeber, G., *Satellite Geodesy: Foundations, Methods & Applications*, Walter de Gruyter, New York, 1993.

Seidelmann, P. K., "1980 IAU Nutation: The Final Report of the IAU Working Group on Nutation," *Celest. Mech.*, Vol. 27, No. 1, pp. 79–106, 1982.

Seidelmann, P. K., *Explanatory Supplement to the Astronautical Almanac*, University Science Books, Mill Valley, CA, 1992.

Simon, J. L., P. Bretagnon, J. Chapront, M. Chapront-Touzé, G. Francou, J. Laskar, "Numerical expressions for precession formulae and mean elements for the Moon and planets," *Astronomy and Astrophysics*, Vol. 282, pp. 663–683, 1994.

Standish, E. M., X. Newhall, J. Williams, and W. Folkner, *JPL Planetary and Lunar Ephemerides* (CD-ROM), Willmann-Bell, Inc. Richmond, 1997.

Webb, C.,*The ICRF-ITRF Transformation: A Comparison of Fundamental Earth Orientation Models found in MSODP and CALC,* The University of Texas at Austin, Center for Space Research Report CSR-02-01, Austin, TX, 2002.