Robotic Motion Planning: Potential Functions

Robotics Institute 16-735

http://www.cs.cmu.edu/~motionplanning

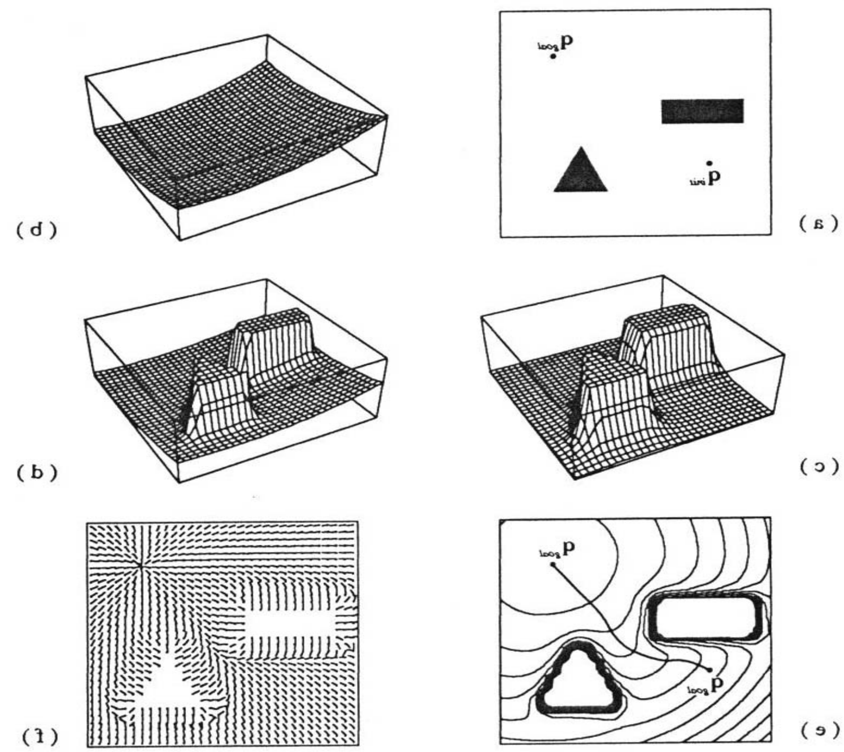
Howie Choset

http://www.cs.cmu.edu/~choset

The Basic Idea

$$U(q) = U_{\text{att}}(q) + U_{\text{rep}}(q)$$

$$\nabla U(q) = DU(q)^T = [\frac{\partial U}{\partial q_1}(q), \ldots, \frac{\partial U}{\partial q_m}(q)]^T$$

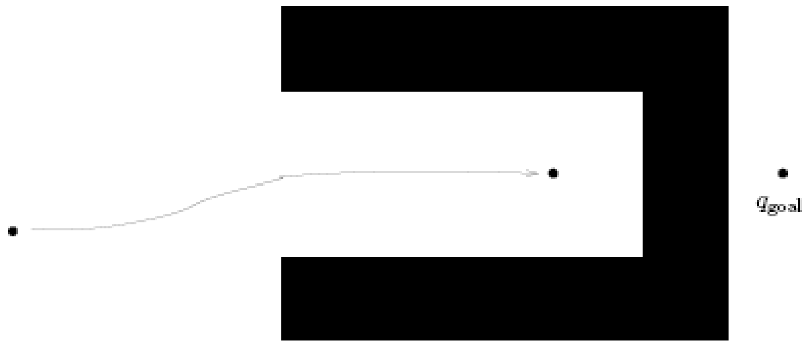


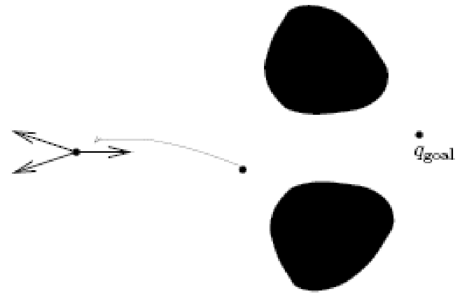
Compute Distance
- Polygon
- Sensor
- Grid

Local Minima
- Wavefront
- Navigation Function

Potential Functions Question

- How do we know that we have only a single (global) minimum





- We have two choices:
  - not guaranteed to be a global minimum: do something other than gradient descent (what?)
  - make sure only one global minimum (a navigation function, which we'll see later).

Remind reading grading sheet

Solutions??

- Wavefront
- Navigation Functions

The Wave-front Planner

- Apply the brushfire algorithm starting from the goal

- Label the goal pixel 2 and add all zero neighbors to L
  - While $L \neq \emptyset$
    - pop the top element of L, t
    - set d(t) to $1+\min_{t' \in N(t), d(t) > 1} d(t')$
    - Add all $t' \in N(t)$ with $d(t)=0$ to L (at the end)

- The result is now a distance for every cell
  - gradient descent is again a matter of moving to the neighbor with the lowest distance value

The Wavefront Planner: Setup

7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
3	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront in Action (Part 1)

- Starting with the goal, set all adjacent cells with “0” to the current cell + 1
  - 4-Point Connectivity or 8-Point Connectivity?
  - Your Choice. We’ll use 8-Point Connectivity in our example

7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
3	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	2
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront in Action (Part 2)

- Now repeat with the modified cells
  - This will be repeated until no 0’s are adjacent to cells with values >= 2
    - 0’s will only remain when regions are unreachable

7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	0	0	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront in Action (Part 3)

- Repeat again...

7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	0	0	5	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	0	5	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront in Action (Part 4)

- And again...

7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	6	6	6	6
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	0	6	5	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	6	5	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	6	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront in Action (Part 5)

- And again until...

7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	7	7	7	7	7
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	6	6	6	6
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	7	6	5	4	4	4
1	0	0	0	0	0	0	0	0	0	0	7	6	5	4	3	3
0	0	0	0	0	0	0	0	0	0	0	7	6	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront in Action (Done)

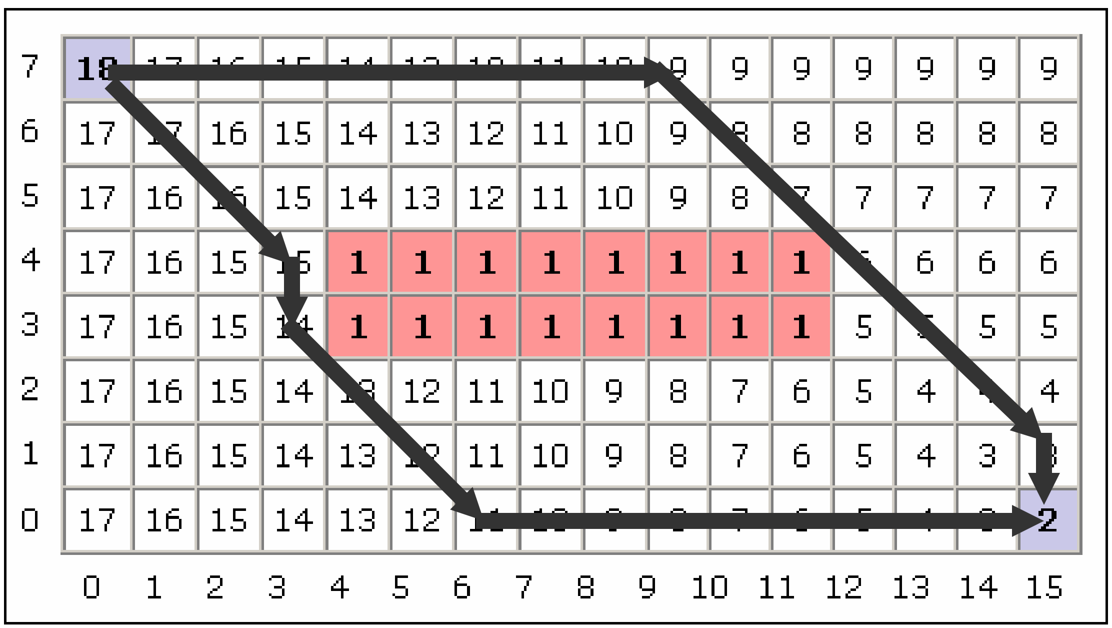
- You're done
  - Remember, 0's should only remain if unreachable regions exist

7	**18**	17	16	15	14	13	12	11	10	9	9	9	9	9	9	9
6	17	17	16	15	14	13	12	11	10	9	8	8	8	8	8	8
5	17	16	16	15	14	13	12	11	10	9	8	7	7	7	7	7
4	17	16	15	15	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	6	6	6	6
3	17	16	15	14	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	17	16	15	14	13	12	11	10	9	8	7	6	5	4	4	4
1	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	3
0	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront, Now What?

- To find the shortest path, according to your metric, simply always move toward a cell with a lower number
  - The numbers generated by the Wavefront planner are roughly proportional to their distance from the goal

Two possible shortest paths shown



Wavefront (Overview)

- Divide the space into a grid.
- Number the squares starting at the start in either 4 or 8 point connectivity starting at the goal, increasing till you reach the start.
- Your path is defined by any uninterrupted sequence of decreasing numbers that lead to the goal.

Navigation Functions

- A function ϕ: $Q_{free} \rightarrow [0,1]$ is called a *navigation function if it*
  - is smooth (or at least C^2)
  - has a unique minimum at q_{goal}
  - is uniformly maximal on the boundary of free space
  - is Morse

- A function is Morse if every critical point (a point where the gradient is zero) is isolated.

- The question: when can we construct such a function?

Sphere World

- Suppose that the world is a sphere of radius r_0 centered at q_0 containing n obstacles of radius r_i centered at q_i, i=1 .. n
  - $\beta_0(q) = -d^2(q,q_0) + r_0^2$
  - $\beta_i(q) = d^2(q,q_i) - r_i^2$

$$\mathcal{QO}_i = \{q \mid \beta_i(q) \leq 0\}$$

- Define $\beta(q) = \prod \beta_i(q)$ (Repulsive)
  - note this is zero on any obstacle boundary, positive in free space and negative inside an obstacle

- Define $\gamma_\kappa(q) = (d(q, q_{\text{goal}}))^{2\kappa}$ (Attractive)
  - note this will be zero at the goal, and increasing as we move away
  - κ controls the rate of growth

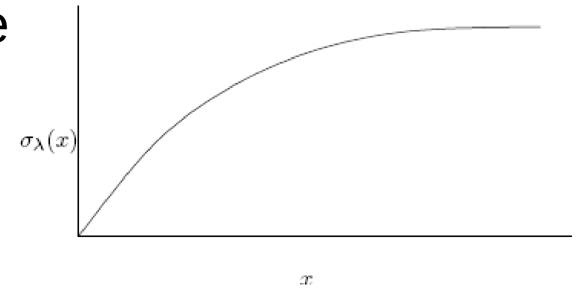
Sphere World

- Consider now $\frac{\gamma_\kappa}{\beta}(q)$
  - $\frac{\gamma_\kappa}{\beta}(q)$ is only zero at the goal
  - $\frac{\gamma_\kappa}{\beta}(q)$ goes to infinity at the boundary of any obstacle
  - By increasing κ, we can make the gradient at any direction point toward the goal
  - It is possible to show that the only stationary point is the goal, with positive definite Hessian because $\partial\gamma_\kappa/\partial q$ dominates $\partial\beta/\partial q$,
    - therefore no local minima

- In short, following the gradient of $\frac{\gamma_\kappa}{\beta}(q)$ is guaranteed to get to the goal (for a large enough value of κ)

An Example: Sphere World

- One problem: the value $\frac{\gamma_\kappa}{\beta}(q)$(q) may be very large
- A solution: introduce a “switch” $\sigma_\lambda(x) = \frac{x}{\lambda + x}, \quad \lambda > 0.$



- Now, define $s(q,\lambda) = \left(\sigma_\lambda \circ \frac{\gamma_\kappa}{\beta}\right)(q) = \left(\frac{\gamma_\kappa}{\lambda\beta + \gamma_\kappa}\right)(q)$
  - this bounds the value of the function
  - however, $s(q,\lambda)$ may turn out not to be Morse
- A solution: introduce a “sharpening function” $\xi_\kappa(x) = x^{\frac{1}{\kappa}}$

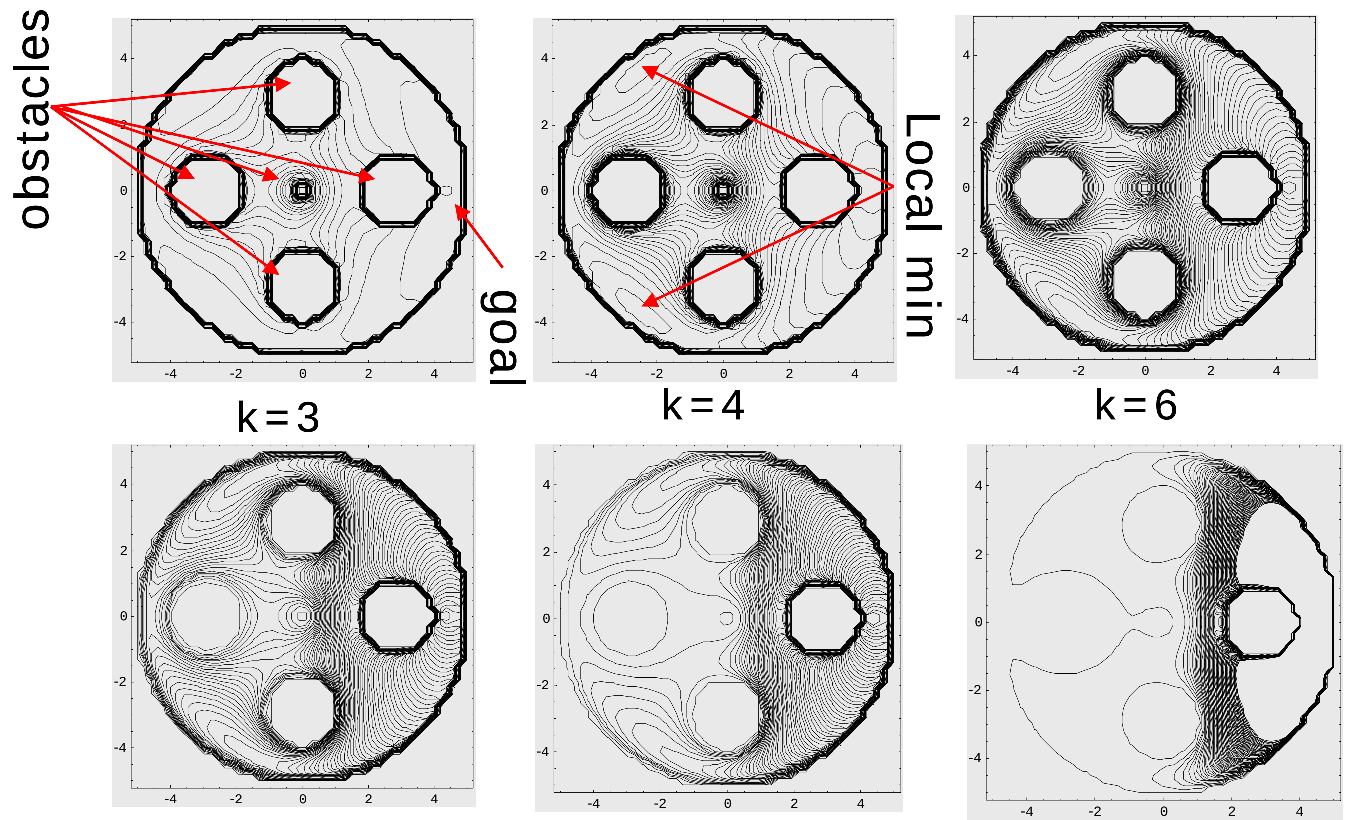
$$\varphi(q) = \left(\xi_\kappa \circ \sigma_1 \circ \frac{\gamma_\kappa}{\beta}\right)(q) = \frac{d^2(q, q_{\mathrm{goal}})}{\left[\left(d\left(q, q_{\mathrm{goal}}\right)\right)^{2\kappa} + \beta(q)\right]^{1/\kappa}}$$

For large enough κ, this is a navigation function on the sphere world!

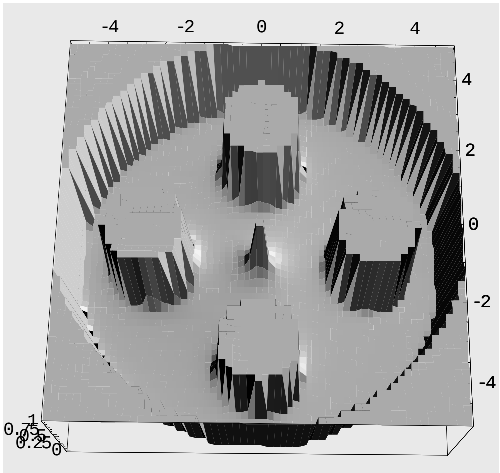
Navigation Function for Sphere World

$$\varphi(q) = \left(\xi_\kappa \circ \sigma_1 \circ \frac{\gamma_\kappa}{\beta}\right)(q) = \frac{d^2(q, q_{\mathrm{goal}})}{\left[\left(d\left(q, q_{\mathrm{goal}}\right)\right)^{2\kappa} + \beta(q)\right]^{1/\kappa}}$$

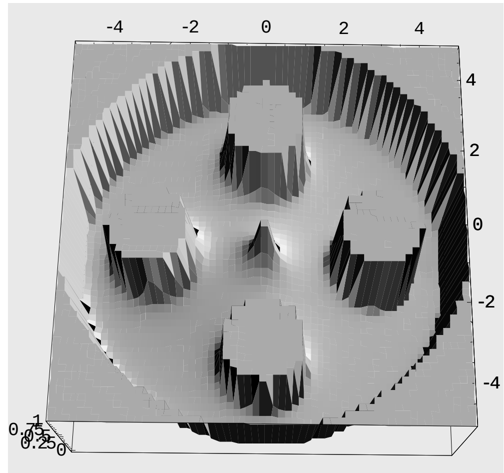
- For sufficiently large k, $\varphi(q)$ is a navigation function



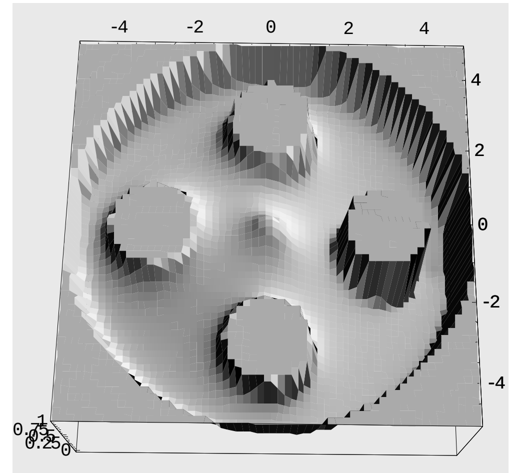
Navigation Function : $\varphi(q)$, varying k



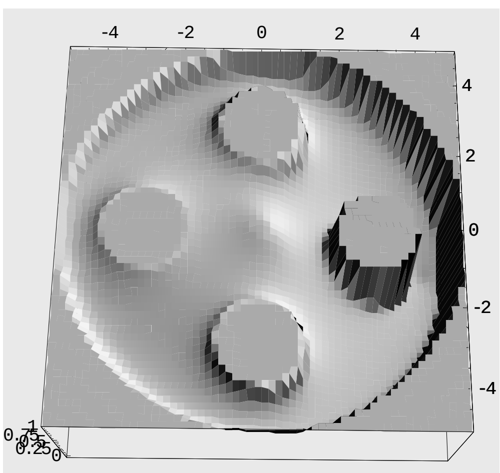
k = 3



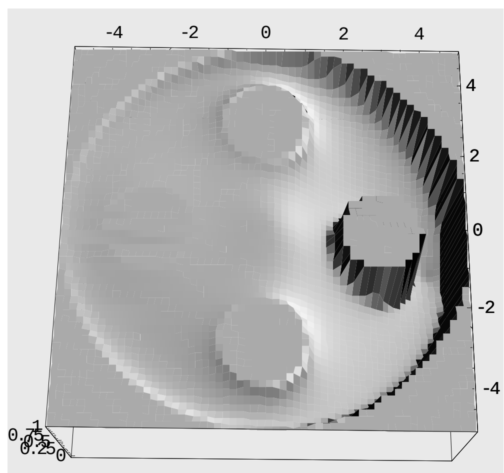
k = 4



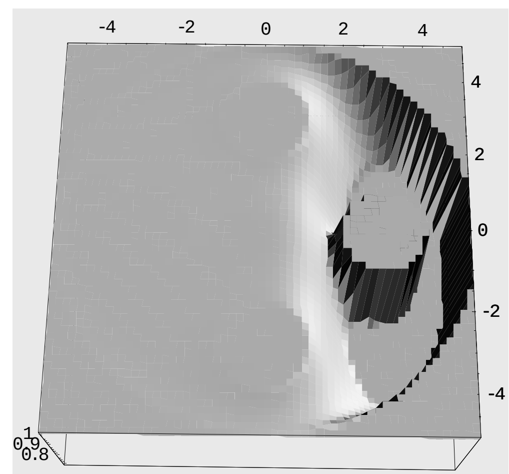
k = 6



k = 7



k = 8



k = 10

From Spheres to Stars and Beyond

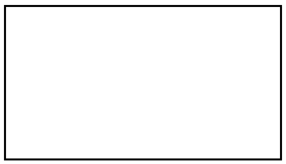
- While it may not seem like it, we have solved a very general problem
- Suppose we have a **diffeomorphism** δ from some world W to a sphere world S
  - if O''_κ is a navigation function on S then
  - $O'''_\kappa(q) = O''_\kappa(\delta(q))$ is a navigation function on W!
    - note we also need t[illegible] into account for distances
    - [illegible] Jacobian is full rank
    - [illegible] Jacobian is full rank, the gradient map cannot have new zeros introduced (which could only happen if the gradient was in the null space of the Jacobian)

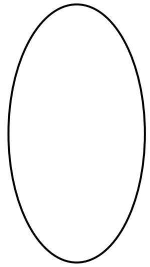
Explain diffeomorphism first!

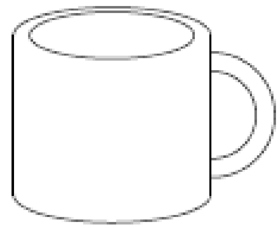
- A star world is one example where a diffeomorphism is known to exist
  - a star-shaped set is one in which all boundary points can be "seen" from some single point in the space.

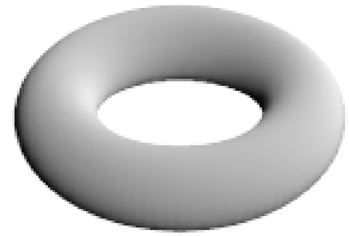
$$\exists x \text{ such that } \forall y \in S, \quad tx + (1-t)y \in S \quad \forall t \in [0,1]$$

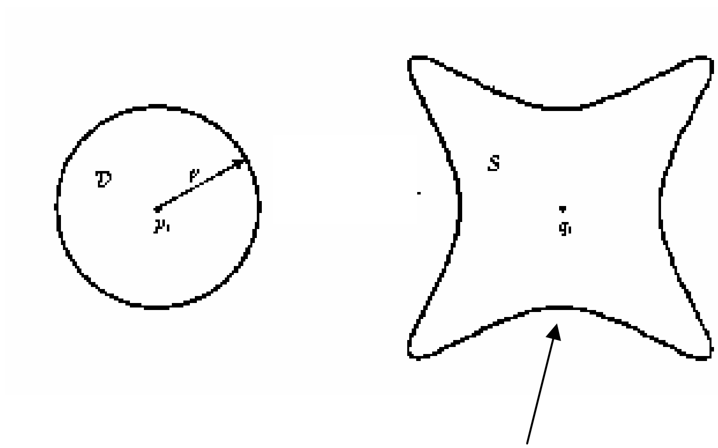
Which of the following are the same?

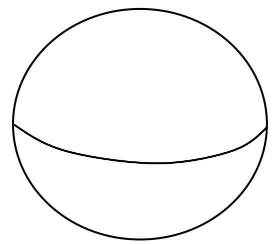






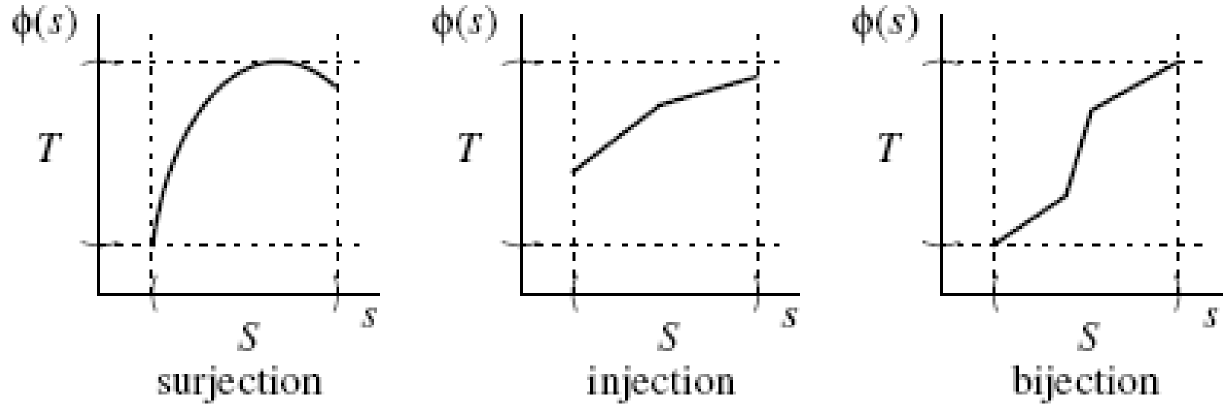






$\exists x$ such that $\forall y \in S, \quad tx + (1-t)y \in S \quad \forall t \in [0,1]$

______jections

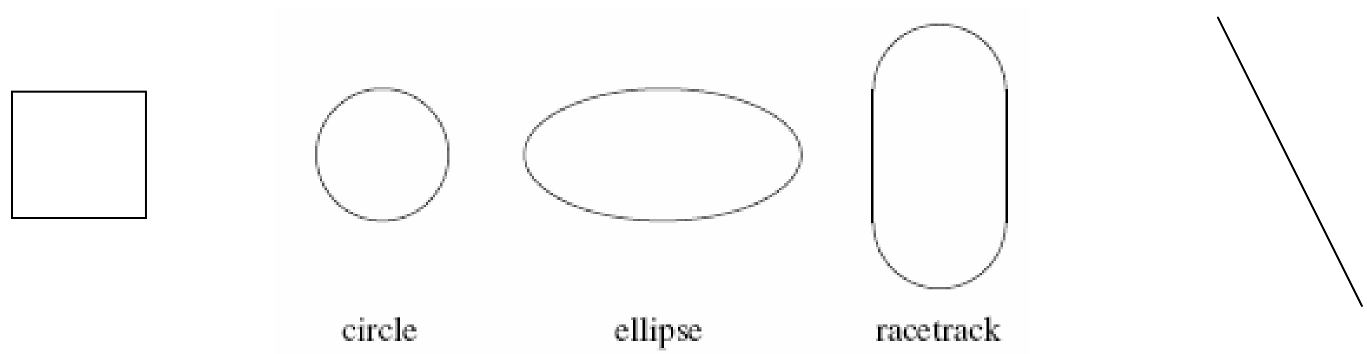


Diffeomorphism vs. Homeomorphism

HOMEOMORPHISM *If $\phi: S \to T$ is a bijection, and both ϕ and ϕ^{-1} are continuous, then ϕ is a* homeomorphism. *When such a ϕ exists, S and T are said to be* homeomorphic.

A mapping $\phi: U \to V$ is said to be *smooth* if all partial derivatives of ϕ, of all orders, are well defined (i.e., ϕ is of class C^∞). With the notion of smoothness, we define a second type of bijection.

DIFFEOMORPHISM *A smooth map $\phi: U \to V$ is a* diffeomorphism *if ϕ is bijective and ϕ^{-1} is smooth. When such a ϕ exists, U and V are said to be* diffeomorphic.



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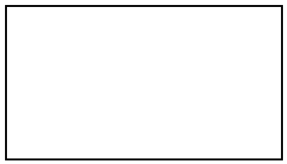

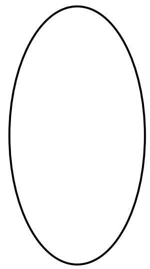

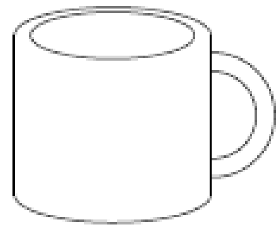

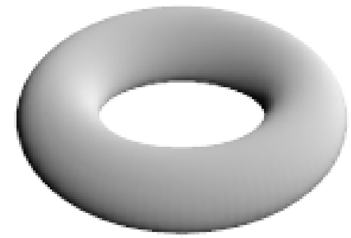

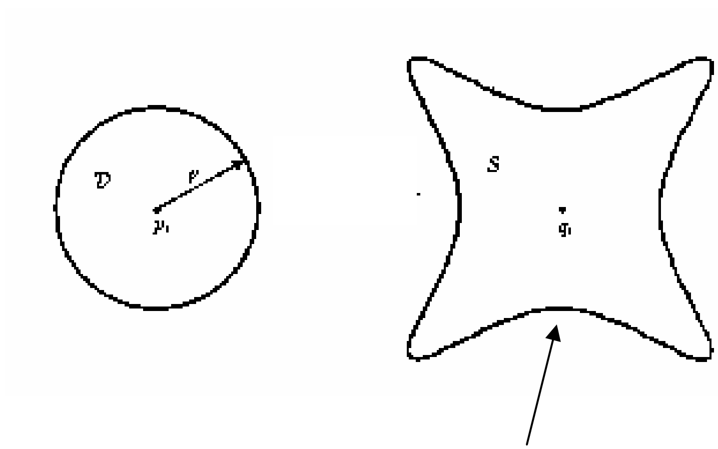

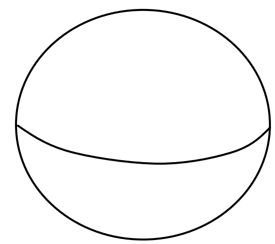

$$\exists x \text{ such that } \forall y \in S, \quad tx + (1-t)y \in S \ \ \forall t \in [0,1]$$

From Spheres to Stars and Beyond

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- A star world is one example where a diffeomorphism is known to exist
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$$\exists x \text{ such that } \forall y \in S, \quad tx + (1-t)y \in S \;\; \forall t \in [0,1]$$

Construct the Mapping

Radius of
Circle shaped set

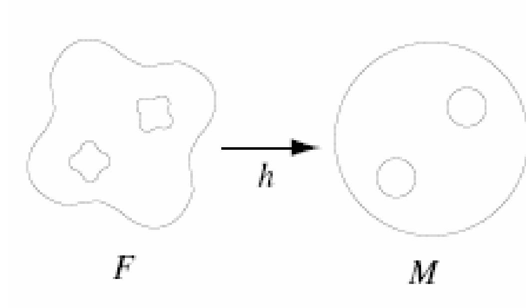
$$T_i(q) = \nu_i(q)(q - q_i) + p_i \qquad \nu_i(q) = (1 + \beta_i(q))^{1/2}\frac{r_i}{\|q - q_i\|}$$

Star shaped set

Center of
Star shaped set

Center of
Circle shaped set

Maps stars to spheres



For points on boundary of star shaped set $(1 + \beta_i(q)) = 1$

$$T_i(q) = r_i\frac{q - q_i}{d(q, q_i)} + p_i.$$

For the star-shaped obstacle $\mathcal{QO}_i$,

$$s_i(q, \lambda) = \left(\sigma_\lambda \circ \frac{\gamma_\kappa \bar{\beta}_i}{\beta_i}\right)(q) = \left(\frac{\gamma_\kappa \bar{\beta}_i}{\gamma_\kappa \bar{\beta}_i + \lambda \beta_i}\right)(q)$$

$$\bar{\beta}_i = \prod_{j=0, j\neq i}^{n} \beta_j$$

Zero on boundary of obstacles
except the “current” one

$$s_{q_{\text{goal}}}(q, \lambda) = 1 - \sum_{i=0}^{M} s_i$$

$s_i(q, \lambda)$ One on the boundary of $\mathcal{QO}_i$ and
Zero on the goal and other obstacle boundaries

$h_\lambda(q)$ is exactly $T_i(q)$ on the boundary of the $\mathcal{QO}_i$

$$h_\lambda(q) = s_{q_{\text{goal}}}(q, \lambda) T_{q_{\text{goal}}}(q) + \sum^{M} s_i(q, \lambda) T_i(q). \qquad T_{q_{\text{goal}}}(q) = q$$

for a suitable λ, $h_\lambda(q)$ is smooth, bijective, and has a smooth inverse.

Potential Fields on Non-Euclidean Spaces

- Thus far, we've dealt with points in R^n --- what about real manipulators
- Recall we can think of the gradient vectors as forces -- the basic idea is to define forces in the workspace (which is $\Re^2$ or $\Re^3$)

force f acting at a point $x = \phi(q)$

force u acting in the robot's configuration

$\dot{x} = J\dot{q}$, where $J = \partial\phi/\partial q$

$u^T\dot{q}$ Power in configuration space

$f^T\dot{x}$ Power in work space

<u>Power is conserved!</u>

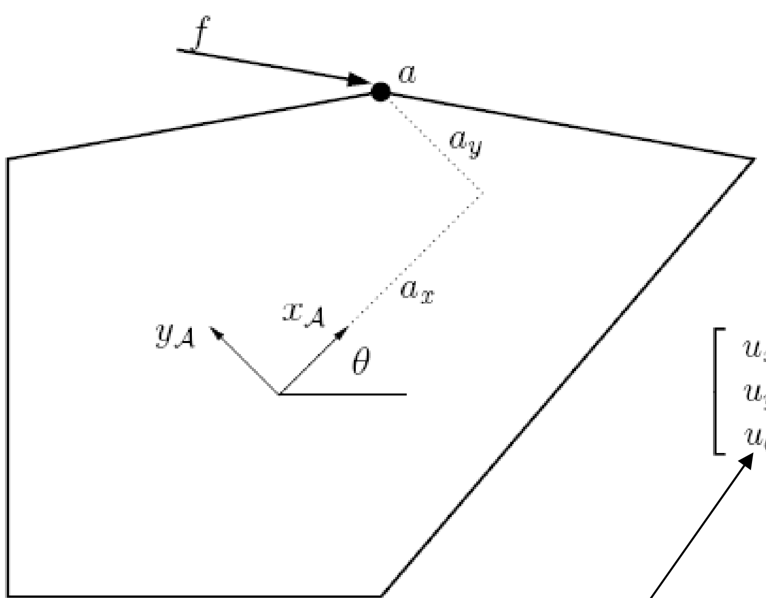
$$\begin{aligned} f^T J\dot{q} &= u^T\dot{q} \\ f^T J &= u^T \\ J^T f &= u. \end{aligned}$$

Force on an Object

$$q = [x, y, \theta]^T$$

$$[a_x, a_y]^T$$

robot's local coordinate frame.



$$\phi(q) = \begin{bmatrix} x + a_x \cos\theta - a_y \sin\theta \\ y + a_x \sin\theta + a_y \cos\theta \end{bmatrix}$$

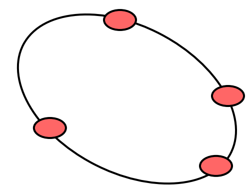
$$J(q) = \frac{\partial \phi}{\partial q}(q) = \begin{bmatrix} 1 & 0 & -a_x \sin\theta - a_y \cos\theta \\ 0 & 1 & a_x \cos\theta - a_y \sin\theta \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} u_x \\ u_y \\ u_\theta \end{bmatrix} &= J^T(q) \begin{bmatrix} f_x \\ f_y \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -a_x \sin\theta - a_y \cos\theta & a_x \cos\theta - a_y \sin\theta \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} \\ &= \begin{bmatrix} f_x \\ f_y \\ -f_x(a_x \sin\theta + a_y \cos\theta) + f_y(a_x \cos\theta - a_y \sin\theta) \end{bmatrix} \end{aligned}$$

torque

Potential Function on Rigid Body

pick control points $\{r_i\}$ on the robot



Pick enough points to “pin down” robot (2 in plane)

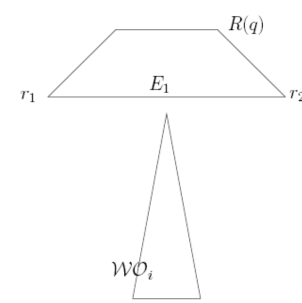
$$U_{\text{att},j}(q) = \begin{cases} \frac{1}{2}\zeta_i d^2(r_j(q), r_j(q_{\text{goal}})), & d(r_j(q), r_j(q_{\text{goal}})) \le d_0 \\ d\zeta_i d(r_i(q), r_i(q_{\text{goal}})) - \frac{1}{2}\zeta_i d^2, & d(r_i(q), r_i(q_{\text{goal}})) > d_0. \end{cases}$$

$$\nabla U_{\text{att},j}((q) = \begin{cases} \zeta_i(r_j(q) - r_j(q_{\text{goal}})), & d(r_i(q), r_i(q_{\text{goal}})) \le d_0, \\ \dfrac{d\zeta_j(r_j(q) - r_j(q_{\text{goal}}))}{d(r_j(q), r_j(q_{\text{goal}}))}, & d(r_j(q), r_j(q_{\text{goal}})) > d_0. \end{cases}$$

$$U_{\text{rep}i,j}(q) = \begin{cases} \frac{1}{2}\eta_j \left(\frac{1}{d_i(r_j(q))} - \frac{1}{Q_i^*}\right)^2, & d_i(r_j(q)) \le Q_i^* \\ 0, & d_i(r_j(q)) > Q_i^* \end{cases}$$

$$\nabla U_{\text{rep}i,j}(q) = \begin{cases} \eta_j \left(\frac{1}{Q_i^*} - \frac{1}{d_i(r_j(q))}\right) \frac{1}{d_i^2(r_j(q))} \nabla d_i(r_j(q)), & d_i(r_j(q)) \le Q_i^*, \\ 0, & d_i(r_j(q)) > Q_i^*. \end{cases}$$

i & j should switched?



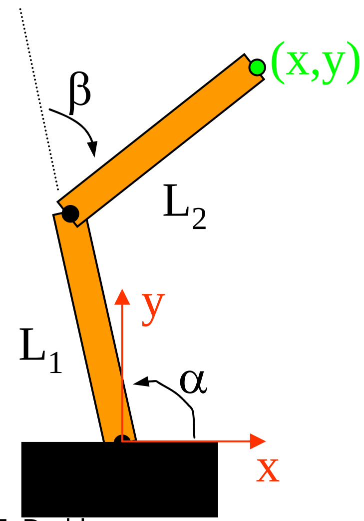
More points please

$$\begin{aligned} u(q) &= \sum_i u_{\text{att}i}(q) + \sum_j u_{\text{rep}j}(q) \\ &= \sum_i J_i^T(q) f_{\text{att}i}(q) + \sum_j J_j^T(q) f_{\text{rep}j}(q) \end{aligned}$$

Potential Fields for Multiple Bodies

- Recall we can think of the gradient vectors as forces -- the basic idea is to define forces in the workspace (which is $\Re^2$ or $\Re^3$)
  - We have $J^t f = u$ where f is in W and u is in Q
  - Thus, we can define forces in W and then map them to Q
  - Example: our two-link manipulator

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 c_\alpha \\ L_1 s_\alpha \end{pmatrix} + \begin{pmatrix} L_2 c_{\alpha+\beta} \\ L_2 s_{\alpha+\beta} \end{pmatrix}$$



Potential Fields on Non-Euclidean Spaces

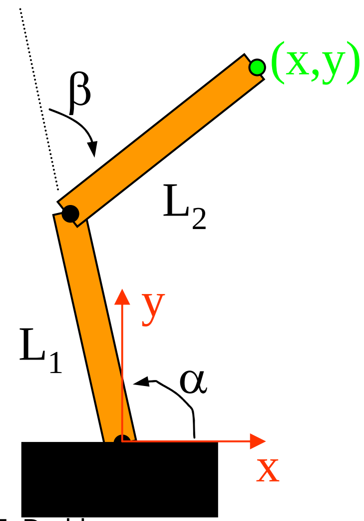
– Example: our two-link manipulator

– $J = \begin{matrix} -L_1 s_\alpha - L_2 s_{\alpha+\beta} & -L_2 s_{\alpha+\beta} \\ L_1 c_\alpha + L_2 c_{\alpha+\beta} & L_2 c_{\alpha+\beta} \end{matrix}$

Suppose $q_{goal} = (0,0)^t$, then $f_W = (x,y)$

$$f_q = \begin{matrix} x\,(-L_1 s_\alpha - L_2 s_{\alpha+\beta}) + y\,(L_1 c_\alpha + L_2 c_{\alpha+\beta}) \\ x\,(-L_2 s_{\alpha+\beta}) + y\,L_2 c_{\alpha+\beta} \end{matrix}$$

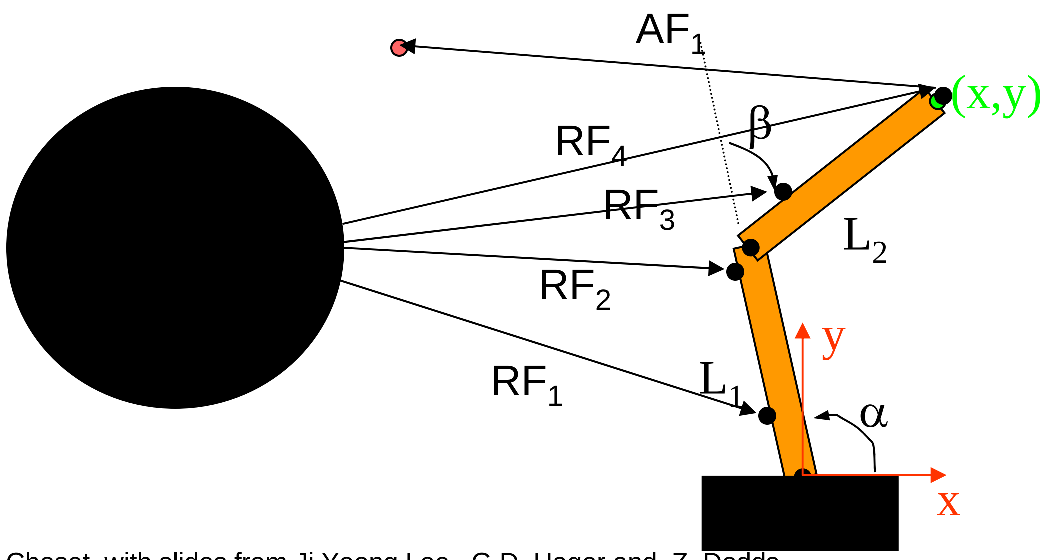
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 c_\alpha \\ L_1 s_\alpha \end{pmatrix} + \begin{pmatrix} L_2 c_{\alpha+\beta} \\ L_2 s_{\alpha+\beta} \end{pmatrix}$$



In General

- Pick several points on the manipulator
- Compute attractive and repulsive potentials for each
- Transform these into the configuration space and add
- Use the resulting force to move the robot (in its configuration space)

Be careful to use the correct Jacobian!



Summary

- Basic potential fields
  - attractive/repulsive forces
- Gradient following and Hessian
- Navigation functions
- Extensions to more complex manipulators