The Smoothed Decagon Problem

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Conjecture

In 1934, Reinhardt conjectured that the shape of the centrally symmetric convex body in the plane whose densest packing has the smallest density is a smoothed octagon.

Building on Hales's reformulation of the Reinhardt conjecture as an optimal control problem, we propose an analogous conjecture.

Conjecture. The smoothed decagon has the highest packing density among all balanced disks in the plane.

Preliminaries

Definition. A set $S \subseteq \mathbb{R}^2$ is **convex** if for all $p_1, p_2 \in S$, the segment from p_1 to p_2 is also in S.

Definition. A convex body in $\mathbb{R}^2$ is called a **convex disk**.

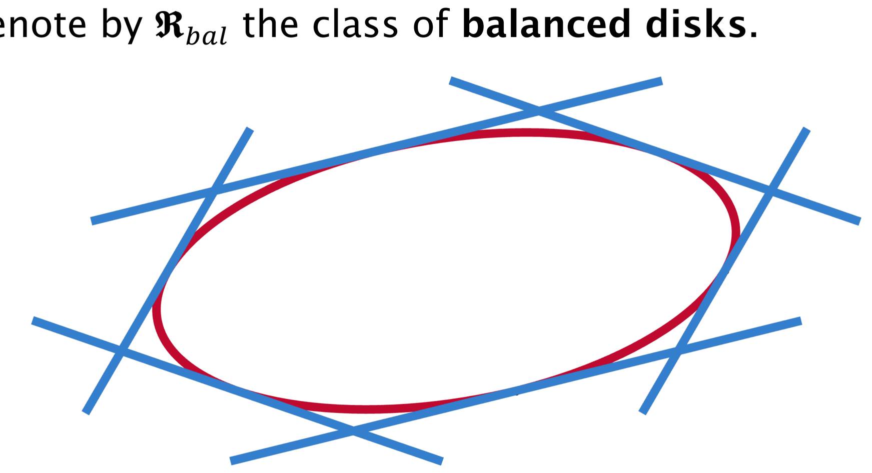
Definition. The **packing density** is the maximum proportion of space filled with non-overlapping congruent copies of a convex disk K.

Definition. A convex disk K is **centrally symmetric** such that if $v \in K$ then $-v \in K$.

Definition. A centrally symmetric convex disk K is **balanced** if the following conditions hold.

1. The boundary of K is parametrized counterclockwise by six regular C^1 curves $\sigma_j : \mathbb{R} \to \mathbb{R}^2$.
2. The derivatives σ_j' are Lipschitz.
3. For each t, $j \mapsto \sigma_j(t)$ is a multi-point with normalization convention.
4. For almost all t, we have the curvature constraint $\det\left(\sigma_j'(t), \sigma_j''(t)\right) \geq 0$.
5. For each t, the vector $\sigma_j'(t)$ points into the open cone with apex $\sigma_j(t)$ and rays passing through $\sigma_{j+1}(t)$ and $\sigma_j(t) + \sigma_{j+1}(t)$.
6. Each curve $\sigma_j : \mathbb{R}/(6t_f\mathbb{Z}) \to \mathbb{R}^2$ is a simple closed curve in $\mathbb{R}^2$, where $6t_f$ is the common period of the functions σ_j.

We denote by $\mathfrak{R}_{bal}$ the class of **balanced disks**.



Red: a balanced disk; Blue: the smallest hexagon containing the disk

Definition. A function $\mathbf{s} : \mathbb{Z}/6\mathbb{Z} \to \mathbb{R}^2$ is called a **multi-point** if it satisfies the following conditions

$$\mathbf{s}_j + \mathbf{s}_{j+2} + \mathbf{s}_{j+4} = 0, \qquad \mathbf{s}_{j+3} = -\mathbf{s}_j, \qquad \det(\mathbf{s}_j, \mathbf{s}_{j+1}) = \sqrt{3}/2$$

Definition. An indexed set of C^k curves $\sigma : \mathbb{Z}/6\mathbb{Z} \times [t_0, t_f] \to \mathbb{R}^2$ is a C^k **multi-curve** if $\forall\, t \in [t_0, t_f]$, $j \mapsto \sigma_j(t)$ is a multi-point.

Note: if the differentiability class C^k is not specified, then C^1 is assumed.

Remark. By Fejes Tóth, a centrally symmetric hexagon guarantees maximum density, making it the optimal packing choice.

Proof Outline

We pin down our proof in four steps:

1. formulate a well-defined control problem,
2. describe special solutions of this system,
3. show that for each positive integer k, the $(6k-2)$-gon given by a bang-bang control is a Pontryagin extremal,
4. discuss the existence of a counterexample.

An Optimal Control Problem

Definition. For a C^1 curve $g : [0, t_f] \to \mathrm{SL}_2(\mathbb{R})$, and for every t, define $X(t) \in \mathfrak{gl}_2(\mathbb{R})$ to be such that $g'(t) = g(t)X(t)$,

where $\mathrm{SL}_2(\mathbb{R})$ is the Lie group consisting of all 2×2 matrices with real entries and determinant 1; $\mathfrak{gl}_2(\mathbb{R})$ is the Lie algebra of 2×2 general matrices with real entries.

Lemma. The area is given by the following **cost functional**

$$\operatorname{area}(K) = \frac{3}{2}\int_0^{t_f} \operatorname{trace}(JX)\,dt$$

where t_f is the arrival time, and J is the infinitesimal generator defined in the following problem.

Remark. In contrast, the Reinhardt conjecture seeks to minimize the negated cost functional.

Problem. The disks in $\mathfrak{R}_{bal}$ in circle representation arise via the following optimal control problem:

$$g' = gX, \qquad g : [0, t_f] \to \mathrm{SL}_2(\mathbb{R})$$

$$X' = \frac{[Z_u, X]}{\langle Z_u, X\rangle}, \qquad X : [0, t_f] \to \mathfrak{sl}_2(\mathbb{R})$$

$$\frac{3}{2}\int_0^{t_f} \langle J, X\rangle\,dt \to \max, \qquad \mathrm{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})$$

where $\mathfrak{sl}_2(\mathbb{R})$ is the Lie algebra of 2×2 matrices with real entries and trace 0; and where Z_u is the set of controls:

$$Z : U_T = \{(u_0, u_1, u_2) \mid \sum_i u_i = 1, u_i \geq 0\} \to \mathfrak{sl}_2(\mathbb{R})$$

$$Z_u = \begin{pmatrix} \dfrac{u_1 - u_2}{\sqrt{3}} & \dfrac{u_0 - 2u_1 - 2u_2}{3} \\ u_0 & \dfrac{u_2 - u_1}{\sqrt{3}} \end{pmatrix}$$

The problem has **initial conditions**:

$$g(0) = I_2 \in \mathrm{SL}_2(\mathbb{R}) \text{ and } X(0) = X_0 \in \mathfrak{sl}_2(\mathbb{R}),$$

and **terminal conditions**:

$$g(t_f) = R \text{ and } X(t_f) = RX_0R^{-1}$$

where $R := \exp(J\pi/3)$ is counterclockwise rotation by angle $\frac{\pi}{3}$.

Bang-Bang Control

Definition. A control function is said to be **bang-bang control** if its range is contained in the set of extreme points of the control set, with discontinuous switching.

Definition. Enumerate $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the three vertices of the control set U_T. Define $P_i = P(\mathbf{e}_i)$, where $P(u) = \frac{Z_u}{\langle X, Z_u\rangle}$. We define the **switching functions** as

$$\chi_{ij}(t) := \langle \Lambda_R(t), P_{k,j} - P_{k,i}\rangle$$

Hamiltonian and Pontryagin

- **Hamiltonian mechanics** is a useful approach in solving optimal control problems, wherein the Hamiltonian function incorporates the system dynamics and cost function.
- The **Pontryagin Maximum Principle (PMP)** is a powerful first-order necessary condition for optimality of solutions to an optimal control problem on a smooth manifold M with closed control set $U_T \subseteq \mathbb{R}^m$ and free-terminal time.
- The lifted curves which satisfy the conditions of the PMP are called **Pontryagin extremals** or simply **extremals**.

Lemma. The **full Hamiltonian** of the optimal control problem:

$$\mathcal{H}(\Lambda_1, \Lambda_R, X; Z_u) = \left\langle \Lambda_1 + \frac{3}{2}\lambda_{cost}J, X\right\rangle - \frac{\langle \Lambda_R, Z_u\rangle}{\langle X, Z_u\rangle}$$

Lemma. The PMP states that the extremals of the control problem are integral curves of the **maximized Hamiltonian**, which is the pointwise maximum of the control-dependent Hamiltonian over the control set U_T:

$$\mathcal{H}^+(\Lambda_1, X) = \left\langle \Lambda_1 + \frac{3}{2}\lambda_{cost}J, X\right\rangle + \max_{u \in U_T} \frac{-\langle \Lambda_R, Z_u\rangle}{\langle X, Z_u\rangle}$$

Theorem

Theorem. For each $k \in \mathbb{N}$, the smoothed $(6k-2)$-gon given by a bang-bang control is a **Pontryagin extremal**.

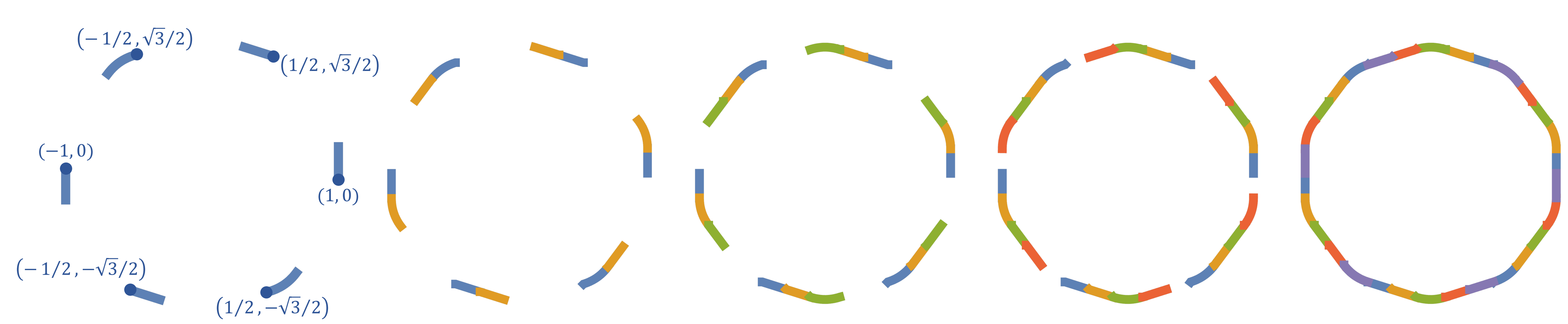
Note: this is a discrete set of extremals, but not on all balanced disks.

Method. The proof is based on the non-negativity of the switching functions, which are highly non-linear functions of the two variables $k \in \mathbb{N}$ and $t \in [0, t_f]$.

Example. When $k = 2$, the **smoothed decagon** K_{dec} is a Pontryagin extremal, and the area of its trajectory is

$$\operatorname{area}(K_{dec}) = \frac{3}{2}(3 \cdot 2 - 1)\int_0^{t_f} \operatorname{trace}(JX)\,dt \approx 3.22606$$

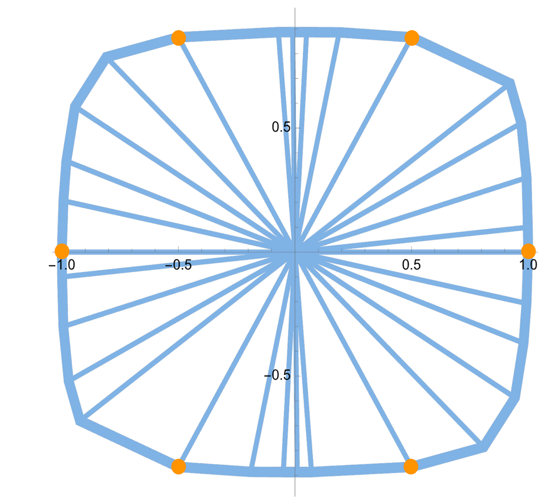
and its packing density is $\operatorname{area}(K_{dec})/\sqrt{12} \approx 0.931284$.



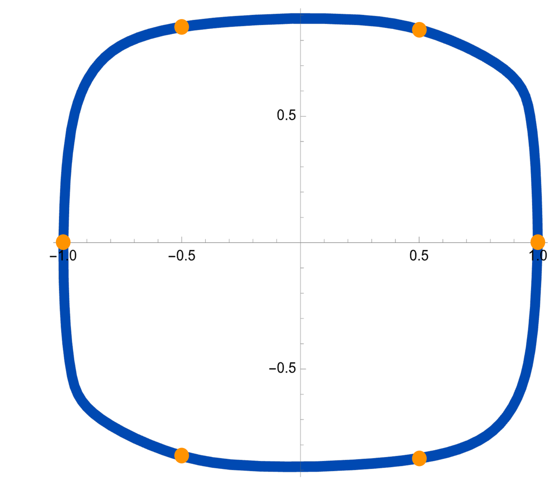
Multi-curves generating the smoothed decagon, starting from the sixth roots of unity.

Potential Counterexample

Example. Formed by line segments that meet both convexity and balanced conditions, an approximate counterexample can be divided into triangles. The total area of these triangles results in a packing density of 0.932067, slightly higher than the smoothed decagon ($\approx$ 0.931284).



Approximate counterexample



Proposed counterexample

Remark. The area of the approximate counterexample is an underestimated representation of a proposed counterexample, which follows a continuous trajectory (while the former is a polygon).

- Final verification is being made to confirm the existence of the proposed counterexample.

~~**Theorem.** There exists a counterexample that has a higher packing density than the smoothed decagon.~~

Future Directions

The Reinhardt conjecture is an active research area with many unsolved questions. These include:

1. The problem in 3 dimensions.
2. The possible existence of chaotic trajectories.
3. The question of whether balanced disks can achieve densities arbitrarily close to 1.

References

1. L. Fejes Tóth, G. Fejes Tóth, and W. Kuperberg. Lagerungen: Arrangements in the Plane, on the Sphere, and in Space. Springer, 2023.
2. T. C. Hales. On the Reinhardt conjecture. Vietnam Journal of Mathematics, 39(3):287–307, 2011.
3. T. C. Hales. The Reinhardt conjecture as an optimal control problem. ArXiv, 2017.
4. T. C. Hales and K. Vajjha. The Reinhardt Conjecture as an Optimal Control Problem. In Preparation.

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