The Reinhardt Conjecture as an Optimal Control Problem

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Abstract

Imagine you need to pay someone by covering a tray with a single layer of gold coins, all of the same shape, packed together as tightly as possible. What shape would you choose to hand over the least amount of gold? This is informally the Reinhardt conjecture, an open problem in discrete geometry looking for the worst-best packing of centrally symmetric, convex shapes in the plane.

Reinhardt conjectured in 1934 that this shape should be a smoothed octagon, constructed by taking a regular octagon and clipping the corners with hyperbolic arcs. Professor Hales along with his graduate student Koundinya Vajjha have recently made advancements in the proof by stating it as a problem in optimal control.

Background

The conjecture can be stated compactly as

$$\inf_{K \in \mathfrak{R}_{ccs}} \sup_{\mathcal{P}} \delta(K, \mathcal{P})$$

where $\mathfrak{R}_{ccs}$ is the set of convex, centrally symmetric discs and $\mathcal{P}$ is a packing with congruent copies of a disc.

A shape S is centrally symmetric if for every x in S, $-x$ is also in S. A shape is convex if for every x, y in S the line segment between x and y is in S.

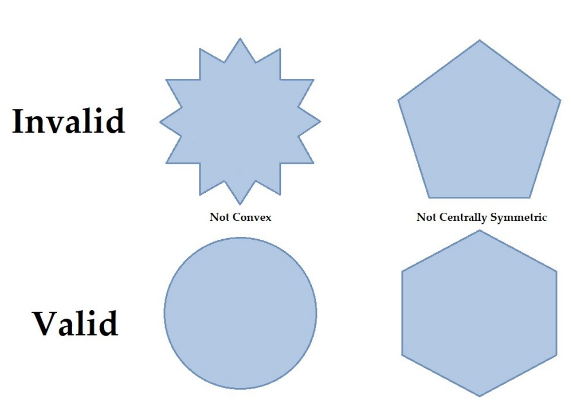

Figure 1: Elements of $\mathfrak{R}_{ccs}$

The smoothed octagon, constructed by hyperbolic corners and straight line edges, see Figure 2, is the conjectured solution.

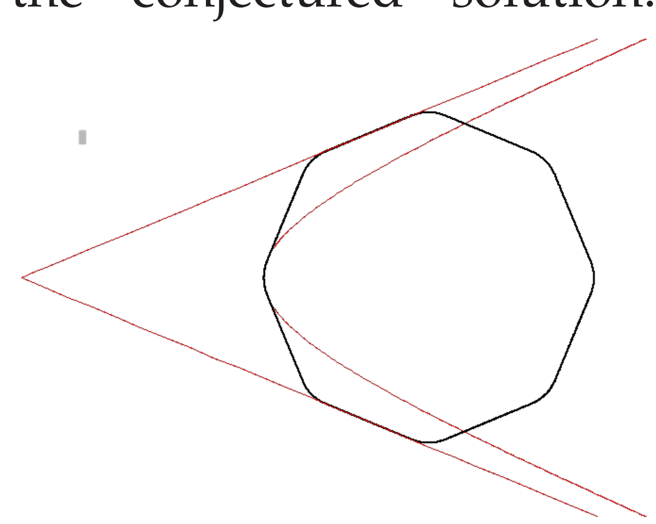

Figure 2: Smoothed Octagon

The abrupt switches from edge to corner are reminiscent of a well known problem in optimal control called the Dubin's Car Problem, which inspired the reformulation.

Previous Results

Professor Hales formulated Reinhardt as an optimal control problem by introducing the notion of multi-points and multi-curves. Multi-curves are used to trace the boundary of the smoothed octagon, which allows the geometric problem to be phrased in terms of control theory. Multi-points are defined by a function $E : \mathbb{Z}/6\mathbb{Z} \to \mathbb{R}^2$ s.t. $E_j + E_{j+2} + E_{j+4} = 0$, $E_j = E_{j+3}$, and $E_j \bigwedge E_{j+2} = \frac{\sqrt{3}}{2}$. A multi-curve is a function of multi-points

Here our multi-curve $\sigma_j(t) = g(y)\sigma_j(0)$ for $j = 1, ..., 5$ and $\forall\, t \in [0, t_f]$. Where $g : [0, t_f] \to SL_2(\mathbb{R})$, where $\mathfrak{sl}_2(\mathbb{R})$ is a set of 2×2 matrices with real entries and determinant $= 1$.

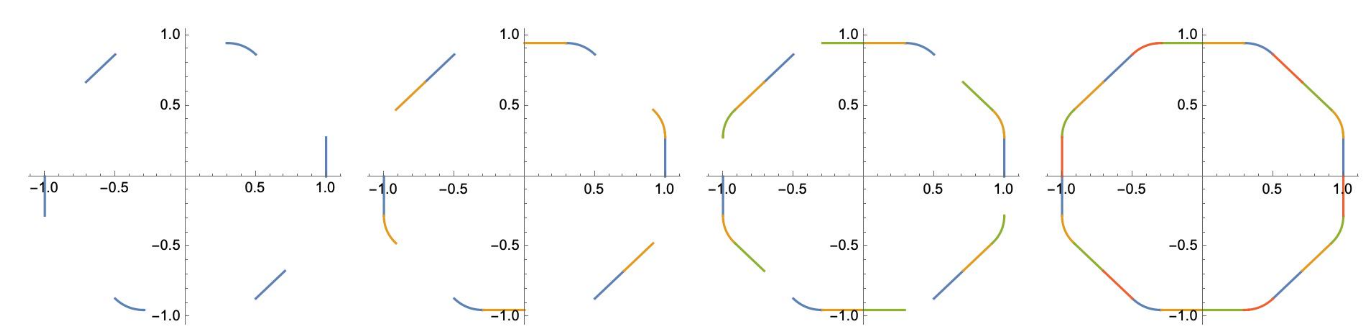

Figure 5: Construction of the Octagon using Multicurves $\sigma_j(t)$

This reformulation involves creating an optimal control problem and then proving the only valid solution is the smoothed octagon. The equations are thus:

$$g' = gX, \quad g : [0, t_f] \to SL_2\mathbb{R}$$

$$X' = \frac{1}{<X, Z_u>}[Z_u, X], \quad X : [0, t_f] \to \mathfrak{sl}_2(\mathbb{R})$$

$$-\frac{3}{2}\int_0^{t_f} < J, X > dt \to \min, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where J is an infinitesimal generator of the rotation group $SO_2(\mathbb{R})$, and $R := e^{J\theta}$ is counterclockwise rotation by some angle θ.

The general solutions in terms of matrix exponentials have been obtained for constant control, which we use during our own research.

The theory which has been developed by Professor Hales in order to prove the Reinhardt conjecture gives rise to numerous other areas of research.

Research Directions

As we familiarize ourselves with the Reinhardt problem we have developed two conjectures which we plan to explore further over the Summer.

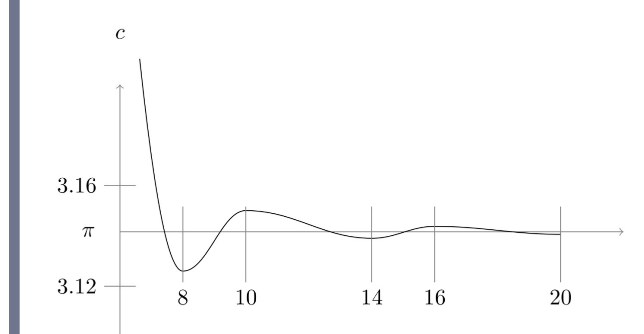

Figure 3: Area enclosed in $(6k \pm 2)$-gons for various values of k

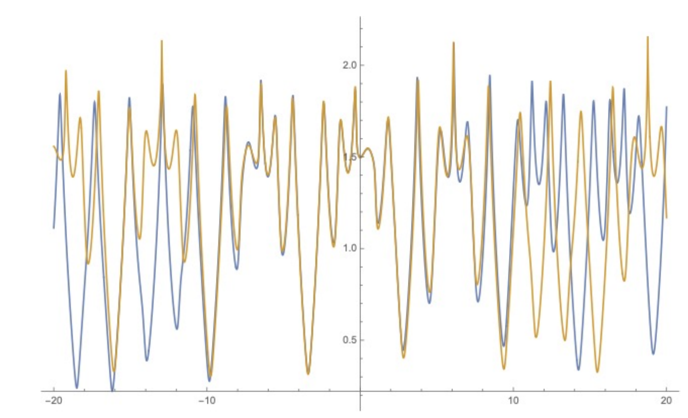

Figure 4: Chaotic solutions

The figure above shows that the smoothed octagon and smoothed decagon seem to be the minimizer and maximizer of area respectively. We want to find the sequence of balanced convex discs where the density is tending to 1.

Conjecture 1 *The smoothed decagon is the balanced convex disc with the highest packing density.*

Conjecture 2 *The solutions to the system of ODEs for Reinhardt optimal control problem exhibit chaotic behavior for certain initial conditions.*

References

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