

The truncated
octahedral
conjecture

Lark Song

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The truncated octahedral conjecture

Lark Song

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Undergraduate Research Colloquium

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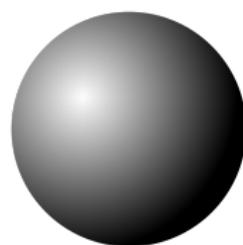
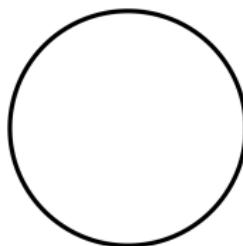
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Appetizer

The isoperimetric problem is an old and important topic in mathematics. Ancient Greeks first identified that in the plane the **circle** has the maximum area for its perimeter, and in 3 dimensions the **sphere** has the minimum surface area for a given volume. The isoperimetric problem was not solved rigorously until the 19th century.



We are working on an isoperimetric problem in Euclidean space, specifically focusing on a family of shapes called the **parallelohedra**.

Definitions

- A **parallelohedron** is a polyhedron whose translates, without rotations, tile space (\mathbb{R}^3).
- In \mathbb{R}^3 , the **isoperimetric problem for parallelohedra** asks which one minimizes surface area for a given volume (or maximizes volume for a given surface area).

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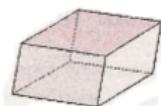
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The five types of parallelohedra

determined by Fedorov (1885)



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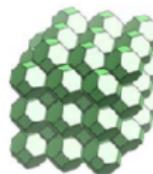
Elongated rhombic dodecahedron



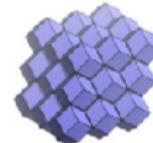
6 quadrilaterals



6 quadrilaterals + 2 hexagons



6 quadrilaterals + 8 hexagons



12 quadrilaterals



8 quadrilaterals + 4 hexagons

The problem

In his 2005 paper *Sphere packings revisited*, Bezdek noted that, while solving the Kelvin problem is harder than expected, there remains an open, simpler yet fundamental conjecture:

The truncated octahedral conjecture

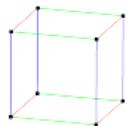
Among all parallelohedra in \mathbb{R}^3 with unit volume, the semiregular truncated octahedron has the minimal surface area.

(*The Kelvin problem asks what is the most efficient way to divide space into cells of equal volume with the least total surface area.)

Proof sketch

- ① We attack the problem by individually analyzing the five types of parallelohedra, structuring the potential proof into five propositions, each supported by relevant lemmas.
- ② Our goal is to identify the local minima for each of these five types.
- ③ Ultimately, we aim to show that the local minimum for truncated octahedra is in fact the global minimum for parallelohedra in \mathbb{R}^3 , thus proving the conjecture.

Parallelepiped



Proposition 1

The surface area of a parallelepiped in \mathbb{R}^3 with unit volume is minimized at 6 when it is a **cube**.

The following lemmas are used in the proof of Proposition 1.

Lemma 1.1

The lateral surface area of a parallelepiped with fixed parallelogram bases and height, only side faces variable, is minimized when it is a parallelogram-based right prism.

Lemma 1.2

The surface area of a parallelogram-based right prism with volume 1 is minimized at 6 when it is a cube.

Parallelepiped

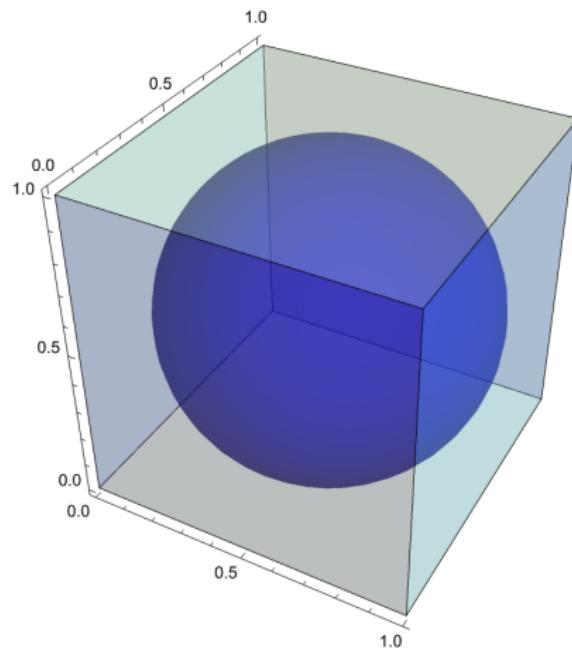


Figure: A cube that circumscribes a sphere, touching each of its faces, is the most isoperimetric parallelepiped in \mathbb{R}^3 .

Hexagonal prism



Proposition 2

The surface area of a hexagonal prism in \mathbb{R}^3 of volume 1 is minimized at $2^{2/3} \cdot 3^{7/6} \approx 5.719$ when it is a **sphere-inscribing hexagonal prism of height $\frac{2^{1/3}}{3^{1/6}}$** .

The following two lemmas are used in the proof of Proposition 2.

Lemma 2.1

A hexagonal prism with a fixed volume and height has the minimum surface area when its top and bottom faces are regular hexagons, assuming it is a right prism (w.l.o.g.)

Lemma 2.2

The surface area of a right prism with regular hexagonal top and bottom faces of volume 1 is minimized at $2^{2/3} \cdot 3^{7/6}$ when the prism is of height $\frac{2^{1/3}}{3^{1/6}}$.

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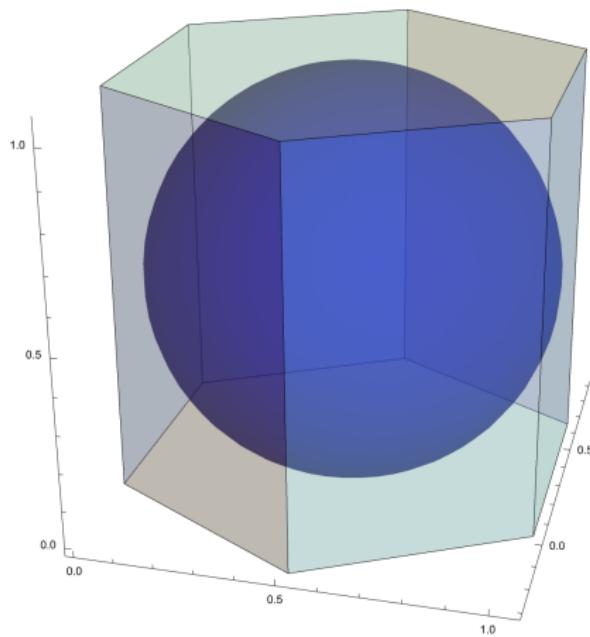


Figure: A hexagonal prism that circumscribes a sphere, touching each of its faces, is the most isoperimetric hexagonal prism in \mathbb{R}^3 .

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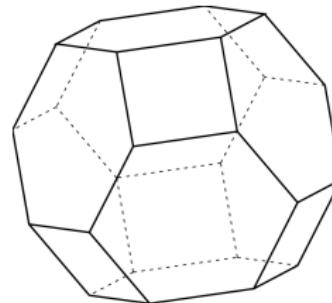
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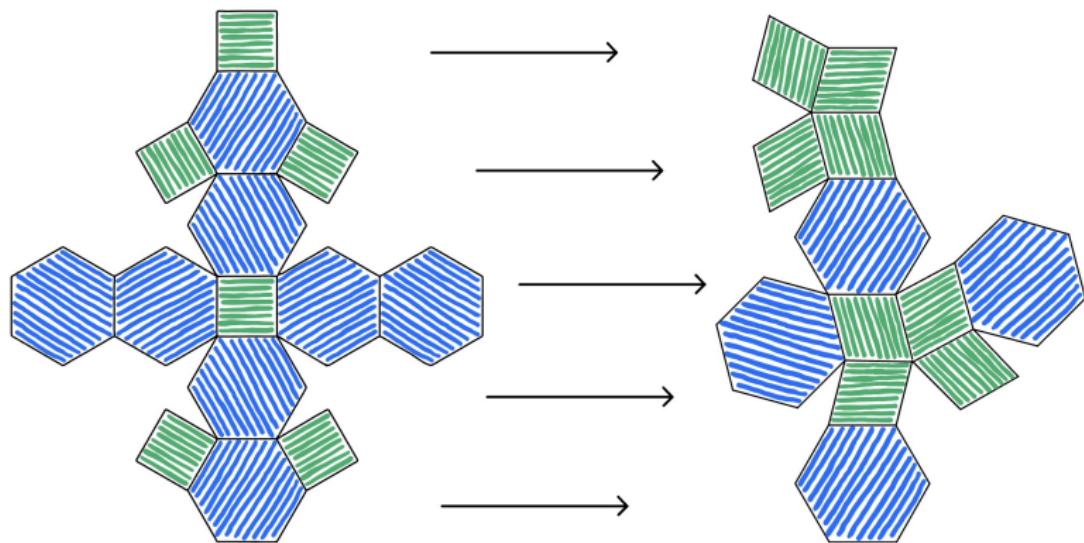
Remaining proof

The surface area of a truncated octahedron in \mathbb{R}^3 of volume 1 is minimized when it is a **semiregular truncated octahedron**.



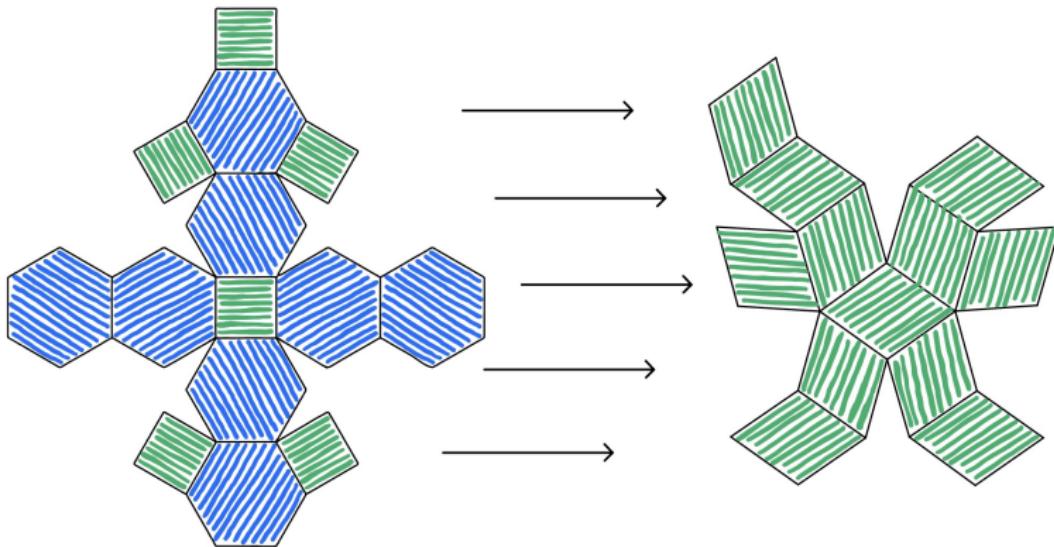
(8 congruent regular hexagonal faces and 6 congruent square faces)

Elongated dodecahedron



The elongated dodecahedron can be visualized as the result of **reducing a vector** in the truncated octahedron **to zero**.

Rhombic dodecahedron



The rhombic dodecahedron can be visualized as the result of **reducing two vectors** in the truncated octahedron **to zero**.

Computer estimates

Mathematica computes the minimal normalized surface area for the five types of parallelohedra, giving us the following approximate values:

- ① Parallelepiped: 6
- ② Hexagonal prism: 5.71911
- ③ Elongated dodecahedron: 5.34539
- ④ Rhombic dodecahedron: 5.34539
- ⑤ Truncated octahedron: 5.31474

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Thank you for your attention!