The truncated octahedral conjecture

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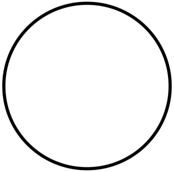
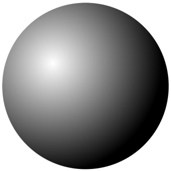
Outline

Appetizer

The isoperimetric problem is an old and important topic in mathematics. Ancient Greeks first identified that in the plane the **circle** has the maximum area for its perimeter, and in 3 dimensions the **sphere** has the minimum surface area for a given volume. The isoperimetric problem was not solved rigorously until the 19th century.





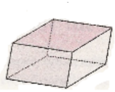
We are working on an isoperimetric problem in Euclidean space, specifically focusing on a family of shapes called the **parallelohedra**.

Definitions

- A **parallelohedron** is a polyhedron whose translates, without rotations, tile space ($\mathbb{R}^3$).
- In $\mathbb{R}^3$, the **isoperimetric problem for parallelohedra** asks which one minimizes surface area for a given volume (or maximizes volume for a given surface area).

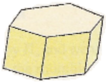
The five types of parallelohedra

determined by Fedorov (1885)



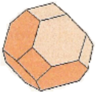
Parallelepiped

6 quadrilaterals

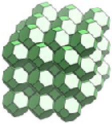


Hexagonal prism

6 quadrilaterals + 2 hexagons

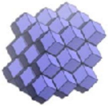


Truncated octahedron



6 quadrilaterals + 8 hexagons

Rhombic dodecahedron



12 quadrilaterals

Elongated rhombic dodecahedron

8 quadrilaterals + 4 hexagons

The problem

In his 2005 paper *Sphere packings revisited*, Bezdek noted that, while solving the Kelvin problem is harder than expected, there remains an open, simpler yet fundamental conjecture:

The truncated octahedral conjecture

Among all parallelohedra in $\mathbb{R}^3$ with unit volume, the semiregular truncated octahedron has the minimal surface area.

(*The Kelvin problem asks what is the most efficient way to divide space into cells of equal volume with the least total surface area.)

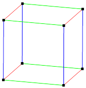
Proof sketch

1. We attack the problem by individually analyzing the five types of parallelohedra, structuring the potential proof into five propositions, each supported by relevant lemmas.

2. Our goal is to identify the local minima for each of these five types.

3. Ultimately, we aim to show that the local minimum for truncated octahedra is in fact the global minimum for parallelohedra in $\mathbb{R}^3$, thus proving the conjecture.

Parallelepiped



Proposition 1

The surface area of a parallelepiped in $\mathbb{R}^3$ with unit volume is minimized at 6 when it is a **cube**.

The following lemmas are used in the proof of Proposition 1.

Lemma 1.1

The lateral surface area of a parallelepiped with fixed parallelogram bases and height, only side faces variable, is minimized when it is a parallelogram-based right prism.

Lemma 1.2

The surface area of a parallelogram-based right prism with volume 1 is minimized at 6 when it is a cube.

Parallelepiped

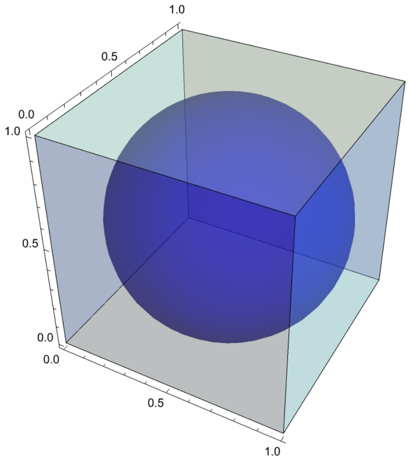
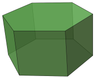


Figure: A cube that circumscribes a sphere, touching each of its faces, is the most isoperimetric parallelepiped in $\mathbb{R}^3$.

Hexagonal prism



Proposition 2

The surface area of a hexagonal prism in $\mathbb{R}^3$ of volume 1 is minimized at $2^{2/3} \cdot 3^{7/6} \approx 5.719$ when it is a **sphere-inscribing hexagonal prism of height** $\frac{2^{1/3}}{3^{1/6}}$.

The following two lemmas are used in the proof of Proposition 2.

Lemma 2.1

A hexagonal prism with a fixed volume and height has the minimum surface area when its top and bottom faces are regular hexagons, assuming it is a right prism (w.l.o.g.)

Lemma 2.2

The surface area of a right prism with regular hexagonal top and bottom faces of volume 1 is minimized at $2^{2/3} \cdot 3^{7/6}$ when the prism is of height $\frac{2^{1/3}}{3^{1/6}}$.

Hexagonal prism

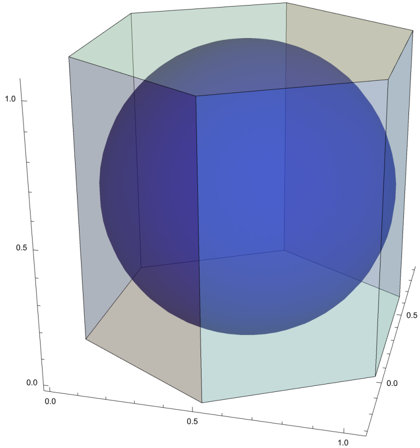
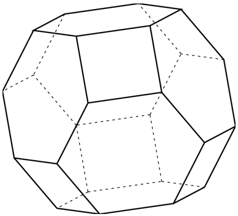


Figure: A hexagonal prism that circumscribes a sphere, touching each of its faces, is the most isoperimetric hexagonal prism in $\mathbb{R}^3$.

Truncated octahedron

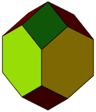
Remaining proof

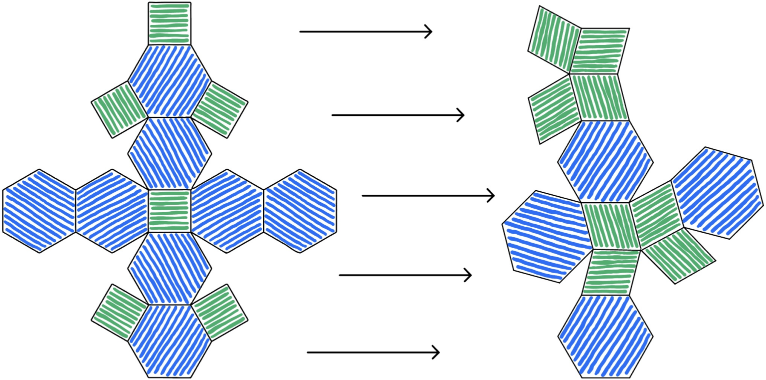
The surface area of a truncated octahedron in $\mathbb{R}^3$ of volume 1 is minimized when it is a **semiregular truncated octahedron**.



(8 congruent regular hexagonal faces and 6 congruent square faces)

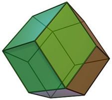
Elongated dodecahedron

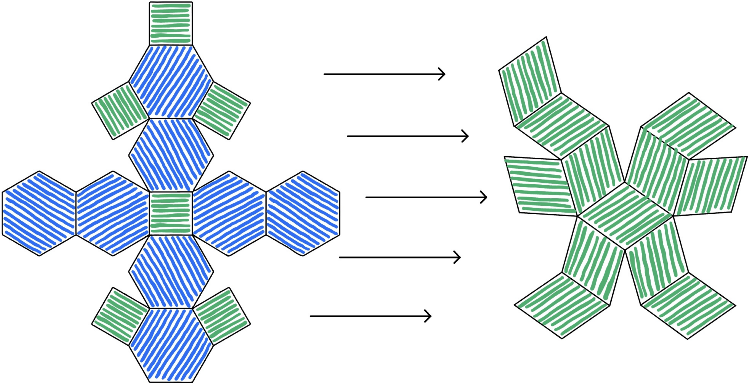




The elongated dodecahedron can be visualized as the result of **reducing a vector** in the truncated octahedron **to zero**.

Rhombic dodecahedron





The rhombic dodecahedron can be visualized as the result of **reducing two vectors** in the truncated octahedron **to zero**.

Computer estimates

Mathematica computes the minimal normalized surface area for the five types of parallelohedra, giving us the following approximate values:

1. Parallelepiped: 6
2. Hexagonal prism: 5.71911
3. Elongated dodecahedron: 5.34539
4. Rhombic dodecahedron: 5.34539
5. Truncated octahedron: 5.31474

Thank you for your attention!