The Truncated Octahedral Conjecture

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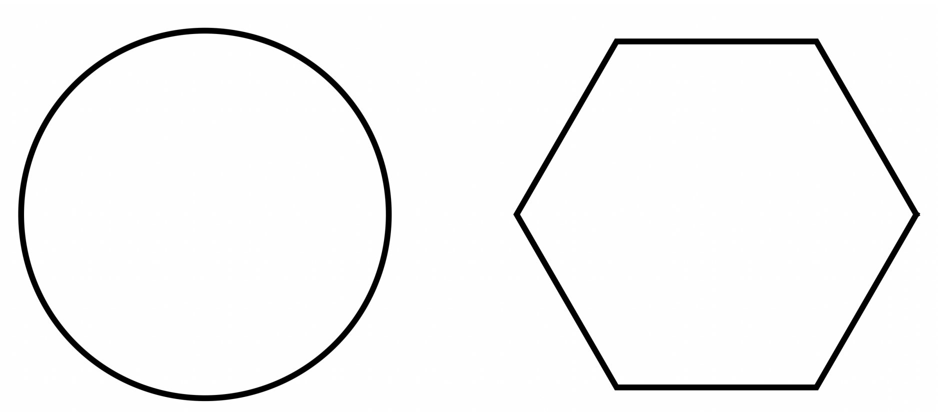
Isoperimetric Problems

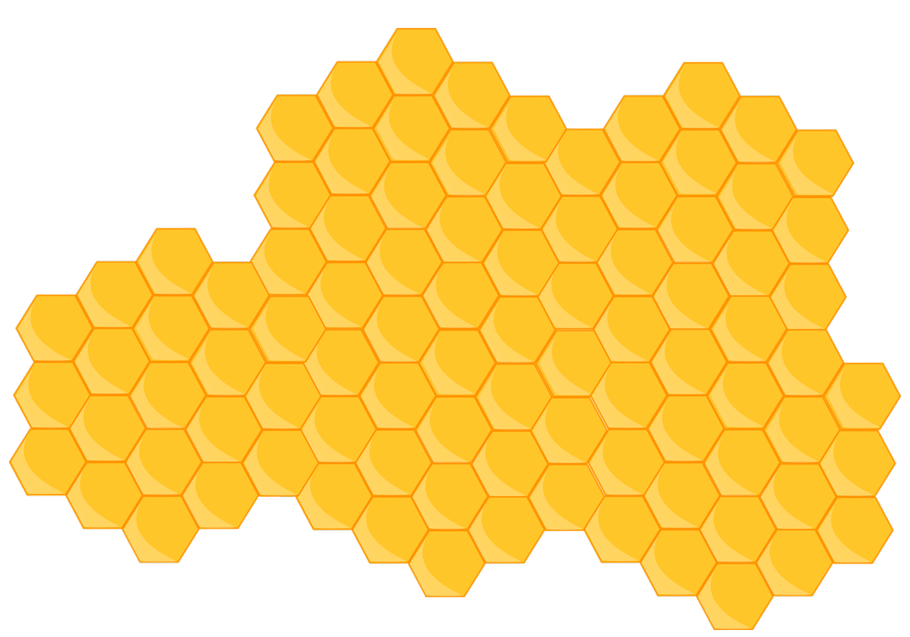
The **isoperimetric problem**, dating back to the ancient Greeks, is to determine among all planar shapes with fixed perimeter the one with the largest area.

The **circle** is the most isoperimetric shape in the plane.

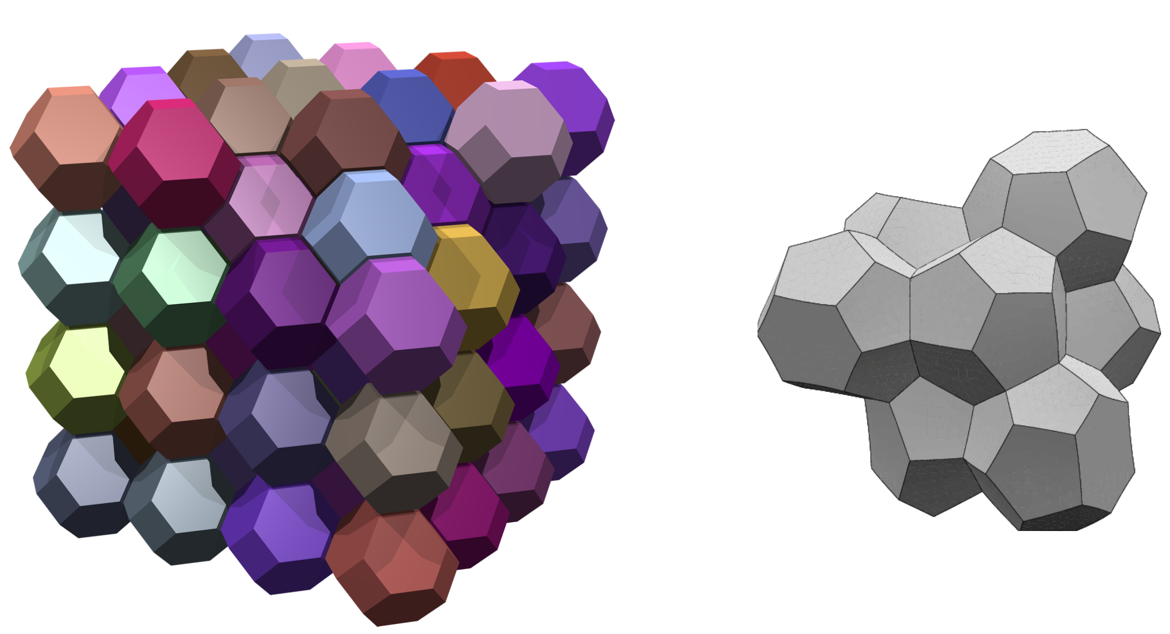


The **regular hexagon** is the most isoperimetric shape capable of tiling the plane without any gaps or overlaps.

► This is well known as the **Honeycomb Conjecture**, proved by Prof. Hales in 1999.



In 3 dimensions, the analogous version of the Honeycomb Conjecture is called the **Kelvin Problem**, which seeks the most efficient partition of space into cells of equal volume.



► The tessellation of the Kelvin structure and the Weaire-Phelan counterexample to Kelvin's conjecture.

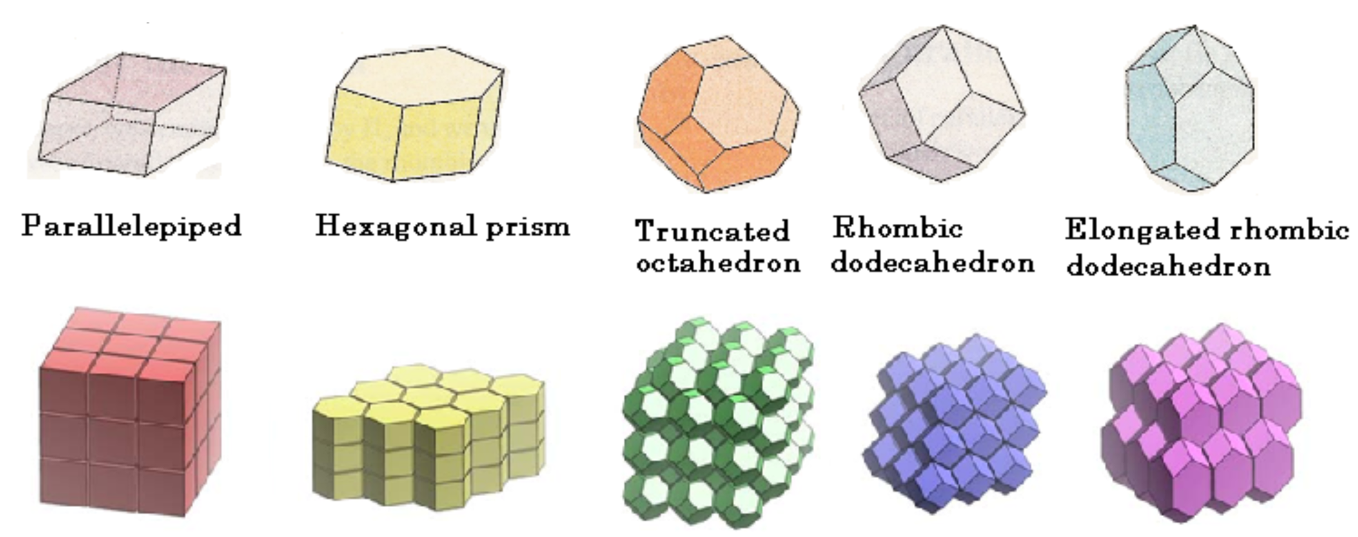
Our Problem

This counterexample makes the **Kelvin Problem** even harder than expected. However, Bezdek [1] noted that there remains an open, simpler yet fundamental problem:

Truncated Octahedral Conjecture. Among all parallelohedra with volume 1, the semi-regular truncated octahedron has the smallest surface area.

► A **parallelohedron** is a polyhedron that can tile space through translations alone, without the need for rotations.

► There are 5 types of parallelohedra (Fedorov, 1885):



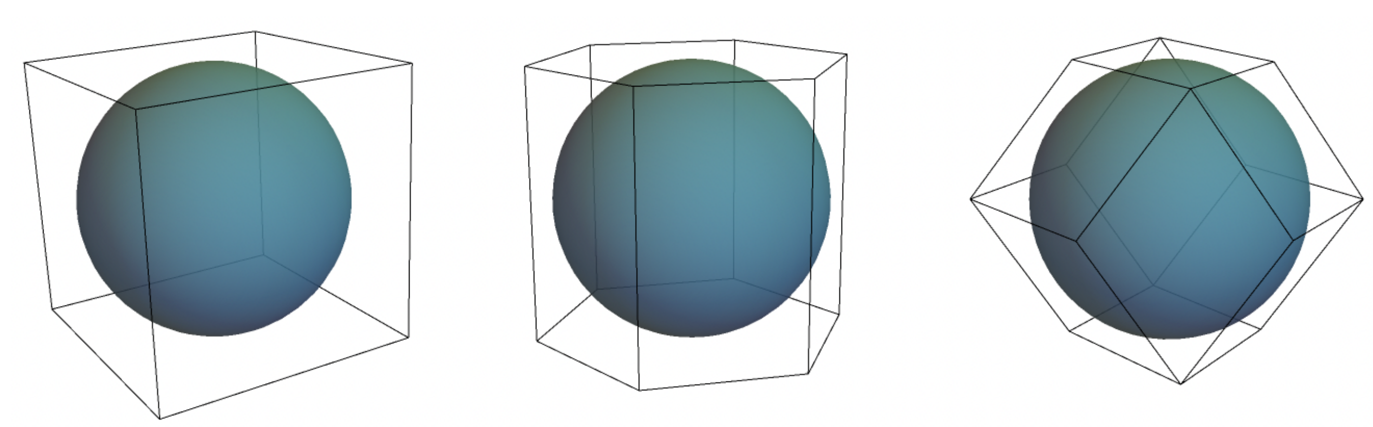
Significance. Successfully proving this conjecture would bring inspiration to disciplines including particle physics, crystallography, architecture, and materials science.

Results So Far

We have rigorously determined the local minima for parallelepipeds and hexagonal prisms.

► The cube is the most isoperimetric parallelepiped.

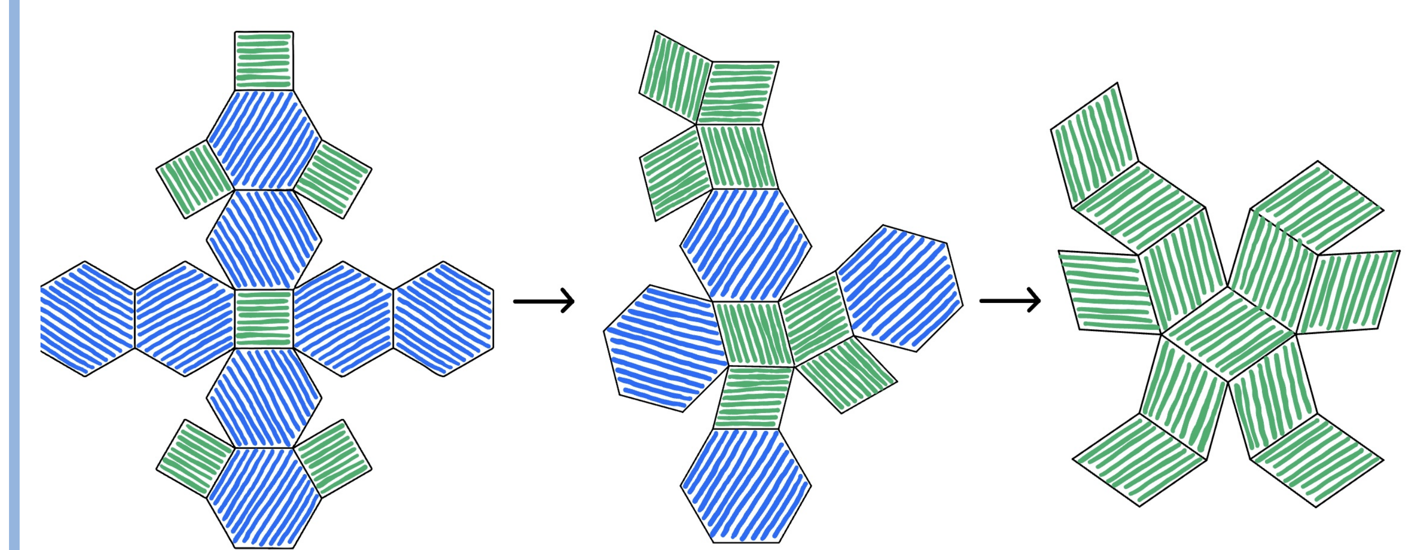
► The hexagonal prism circumscribing a sphere, touching each of its faces, is the most isoperimetric hexagonal prism.



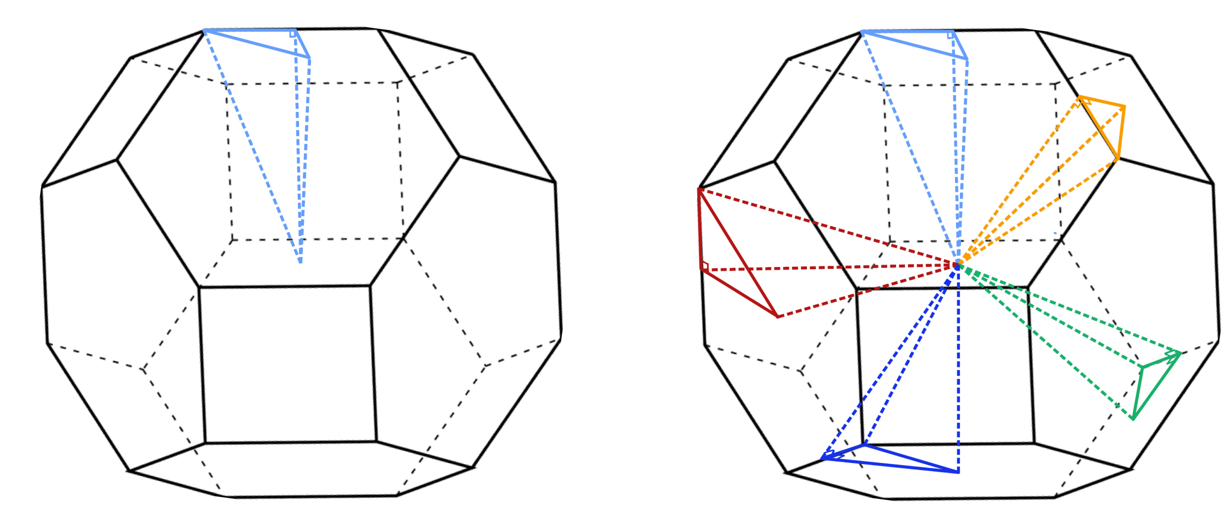
► Although not rigorously proved, we found that the rhombic dodecahedron circumscribing a sphere, touching each of its faces, is the most isoperimetric among its type.

Method & Next Step

The elongated dodecahedron and the rhombic dodecahedron can be understood as being derived from the truncated octahedron by reducing one and two vectors to zero, respectively. So we now focus on the truncated octahedron.



We employ a technique developed by Prof. Hales [2] in his proof of the **Kepler Conjecture** in 1998.



The technique splits a truncated octahedron into smaller pieces called **orthosimplices**. A total of 144 orthosimplices form a truncated octahedron. We are developing a linear programming for this optimization problem. **The goal** is to assemble the orthosimplices such that minimizes surface area while constraining the total volume to be 1.

References

[1] Karoly Bezdek. Sphere packings revisited. *European Journal of Combinatorics*, 27(6):864–883, 2006.

[2] Thomas C. Hales. *Dense Sphere Packings: a Blueprint for Formal Proofs*. Cambridge University Press, 2012.

Acknowledgements

This research is supported by a Chancellor's Undergraduate Research Fellowship.