Modeling ODE Systems - A Pine Beetle Dispersion Analysis

Lucas Wilson

Math 451: Numerical Analysis II - Modeling ODE Systems - A Pine Beetle Dispersion Analysis

Introduction

- The goal of this project is to create a basic mathematical model of the dispersion of Mountain Pine Beetles using a system of ODEs.
- The model is calculated numerically using a fourth order Runge-Kutta method. I analyze the cost of calculating the model and the accuracy. I also look at the error of using the same time step with a lower order Runge-Kutta method.
- I vary carrying capacity, emigration/immigration rates, and the different starting conditions to analyze different effects they have.

Motivation

Mountain Pine Beetles infesting our trees is an important problem today. We depend heavily on our already limited trees, and the beetles are destroying that resource. Although this model doesn't take it into account, trees do die when infected with enough beetles. Learning how they spread can help us know the most effective way to eradicate or servery mitigate them.

Mathematics

I assume trees in a forest are evenly distributed in a grid (to reduce computational complexity). I assume there are four methods for beetles to spread and emerge: birth rate, death rate, immigration rate and emigration rate.

- Birth and death rate are modeled with a logistic model: $P(x) = px(1 - \frac{x}{C})$ where $C =$ carrying capacity, and p is the population rate proportion.
- Emigration is proportional to the population: $E(x) = ex$ where $e =$ the emigration proportion.
- Immigration is proportional to the sum of all other tree's emigration decayed with distance. $M(x_i) = m \sum_{i \neq j}^{n} \frac{E(x_j)}{dist(i,j)}$ where $dist(i,j)$ is the euclidean distance between tree i and j, and m is the immigration proportion (since not all that reach the tree infect the tree).

The rate of change in population of Mountain Pine Beetles in a tree is proportional to each of these.
$\frac{dx}{dt} = P(x) + E(x) + M(x)$ where $x = (x_1, x_2, ..., x_n)^T$ and x_i is the number of beetles in tree i.
Runge-Kutta is an explicit method of calculating x after Δt seconds derived from a taylor expansion:
$x^{(k)} = f(t^{(k-1)}, x^{(k-1)})$ or $x(t + \Delta t) = f(t, x(t))$
A fourth order Runge-Kutta uses the fourth order terms of a Taylor Series, so the err is of 5th order: $O(\Delta t^5)$.

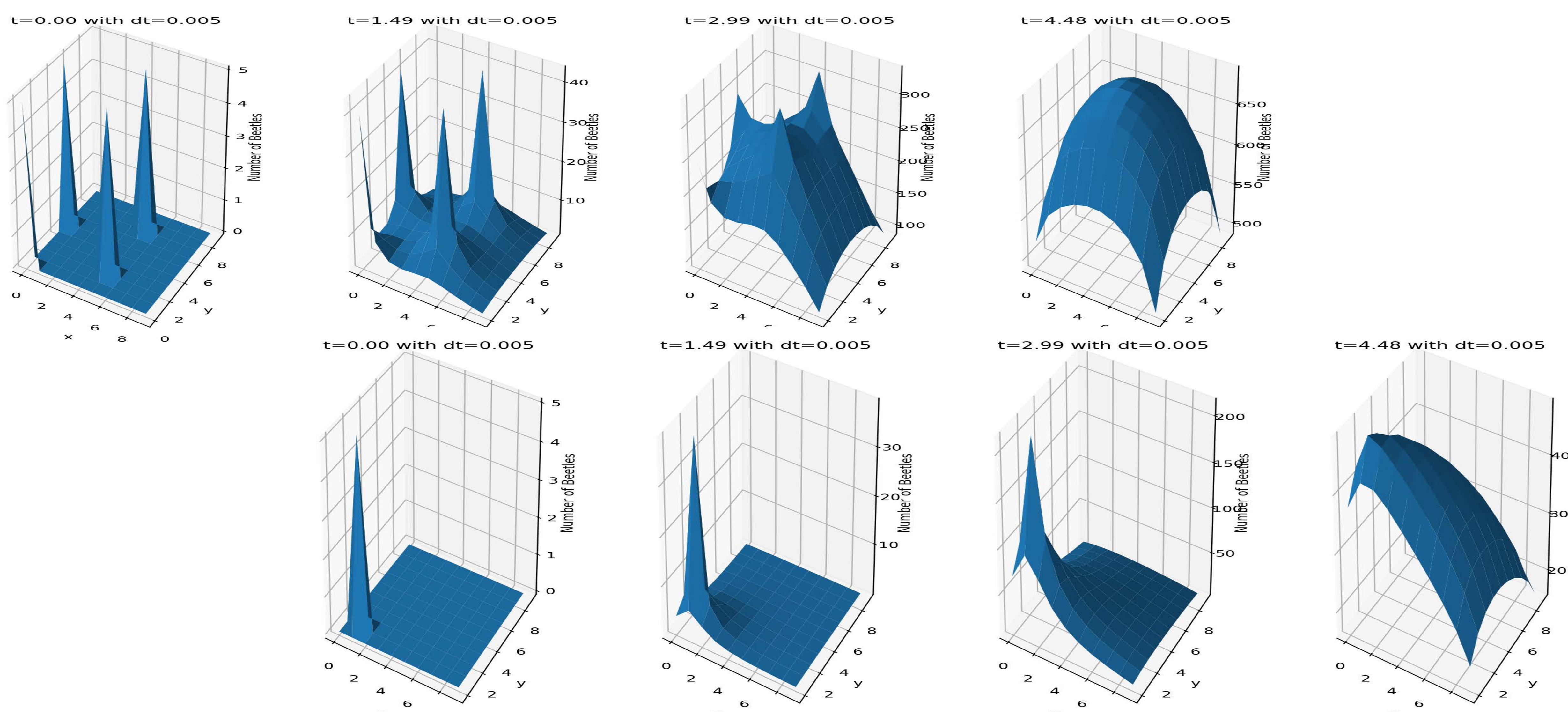
Evaluation of Solution

For $\Delta t, x^{(k)} = u + Ch^5$, and for $\frac{\Delta t}{2}, x^{(k)} = v + 2C\frac{h}{2}^5$
Solving for Ch^5 shows that error is proportional to $x^{(k)}_{\Delta t} - x^{(k)}_{\frac{\Delta t}{2}}$. Thus, if $x^{(k)}_{\frac{\Delta t}{n}}$ converges, then the error converges.
Using $\Delta t_i = \frac{1}{100*2^i}$ and finding $||x^{(k)}_{\Delta t_i} - x^{(k)}_{\Delta t_{i-1}}||_{\inf}$ yields $error = 0.124, 0.0618, 0.0308, 0.0153$.
Notice that the Ch^5 approximations are approaching zero (proportionally to Δt). Therefore, Runge-Kutta converges and is a valid method for numerically calculating my beetle model.

Vary Starting Conditions

Changing the starting conditions has no effect on the long term distribution of the beetles. The beetles migrate towards the center with time. The reason for this is because the emigration and immigration rate reach an equilibrium. Trees in the center of the forest are closest to all trees, thus more beetles emigrating from other trees can reach it more easily while edge trees are less exposed to migrating beetles.

Base Model and with Varying Initial Conditions

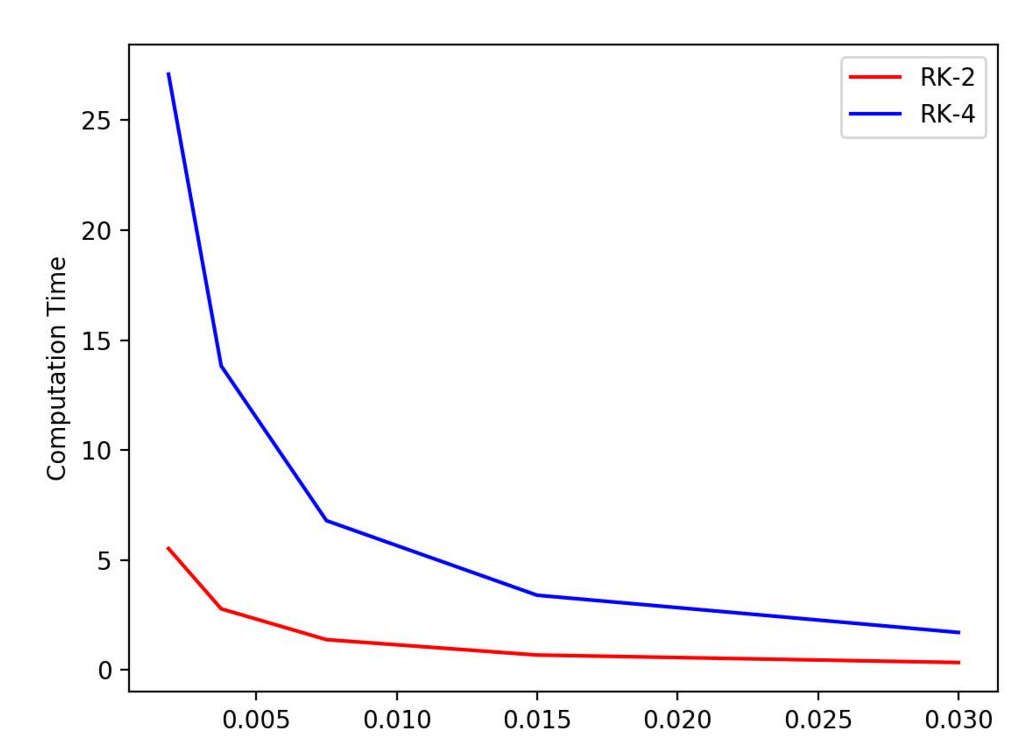


Lower Order Runge-Kutta Methods

In order to analyze the effectiveness of lower order Runge-Kutta methods, we will use fourth order (RK-4) as a baseline and look at new error and the performance increase of second order (RK-2). My analysis shows that RK-2 can be used as a performance increase with small error.

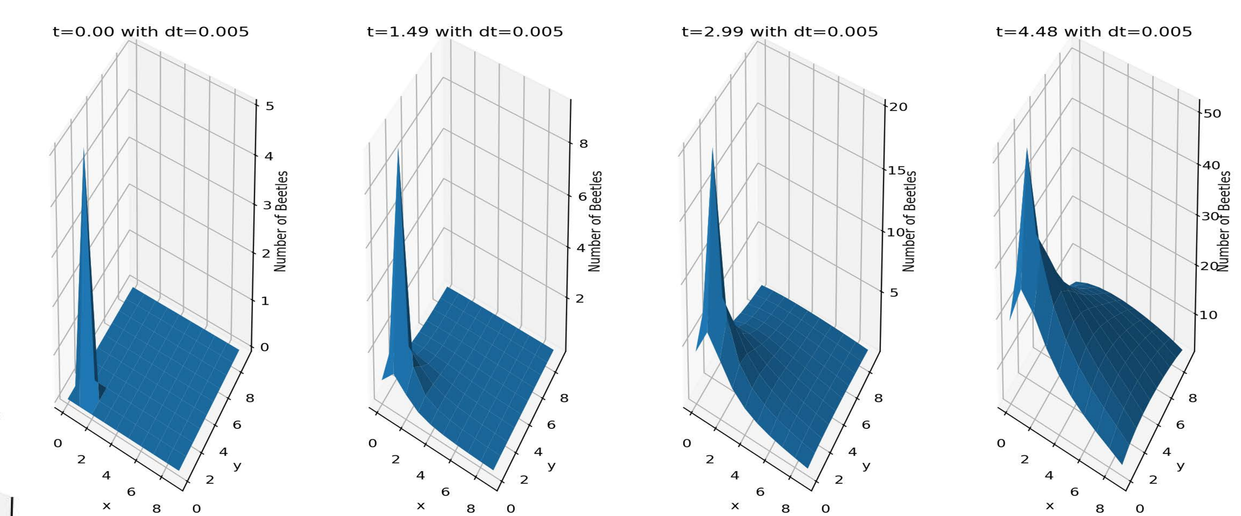
- A time analysis shows that computational time cost of RK-4 and RK-2 are proportional to Δt and RK-2 is about 6 times faster.
- Performing the same error analysis as above shows that RK-2 also converges.
- Using $\Delta t = 3.125e-4, x^{(k)}_{RK-4} - x^{(k)}_{RK-2} = 6.43e-6$.

Time Complexity vs Δt



Mitigating Reproduction Rate

Mitigating the reproduction rate causes the beetles to spread much more slowly and limits the ability for the beetles' population to grow within a specific tree. From the graph, we can see that the total number of beetles in even the starting tree is much lower than the base model. Since there are fewer beetles in total, there are fewer spreading to other trees.
This can be good or bad. It does mitigate the spread of beetles, and it also keeps the trees alive. Although the trees are alive, they are still spreading the beetles. Thus, beetles can spread to the entire forest. If at some point, they are able to reproduce to the point of killing a tree, the entire forest will die since every tree is infected.
Overall, mitigating reproduction ability reduces the total number of beetles within each tree.



Mitigating Immigration/Emigration Rate

Mitigating the beetles' ability to migrate from tree to tree has the effect of keeping the infestation very localized. The beetles spread very slowly throughout the forest. This is very similar to the situation of mitigating reproduction rate with the exception of the beetle population within each tree. As the beetle population increases, the number migrating also increases finally allowing beetles to spread to other trees.
However, if we consider the fact that a large amount of beetles will kill the tree and thus the beetles, then beetles will die before being capable of spreading to other trees. This causes the beetle infestation to be very localized.
Overall, the mitigation of immigration/emigration rates is very effective in fighting the beetles assuming the tree dies before beetles can successfully spread.