Prove that $\int_{-\infty}^{\infty}\frac{e^{-tx^2}}{\cosh \pi x}\,dx \geq \frac{e^{t/2}}{t+1}$ for all $t \in [0,1]$

[+12] [8] TheSimpliFire
[2022-06-09 21:37:54]
[integration inequality]
[https://math.stackexchange.com/questions/4469245/prove-that-int-infty-infty-frace-tx2-cosh-pi-x-dx-ge-fracet-2]

This question is taken directly from Showing that $z^2e^{-z^2/2}\int\frac{\phi^2(x)}{\cosh(xz)}\,dx\geq\frac{1}{2\sqrt{\pi}}\frac{z^2}{z^2+1}$ [1] which unfortunately turned out to be untrue for all $z \geq 0$.

By plotting the functions, it appears that the inequality is true for $0 \leq z \leq 1$. That is, conjecturally,

$$\int_{-\infty}^{\infty}\frac{e^{-u^2}}{\cosh zu}\,du \geq \frac{e^{z^2/2}\sqrt{\pi}}{z^2+1},\qquad \forall z \in [0,1].$$

To simplify this, we can invoke the elegant identity [2]

$$\int_0^{\infty}\frac{e^{-u^2}}{\cosh \alpha u}\,du = \frac{\sqrt{\pi}}{\alpha}\int_0^{\infty}\frac{e^{-u^2}}{\cosh(\pi u/\alpha)}\,du$$

to obtain the equivalent

$$\int_{-\infty}^{\infty}\frac{e^{-u^2}}{\cosh(\pi u/z)}\,du \geq \frac{ze^{z^2/2}}{z^2+1}.$$

Note the identity can be used as the integrand is an even function.

Substituting $x = u/z$ and $t = z^2$ yields

$$\int_{-\infty}^{\infty}\frac{e^{-tx^2}}{\cosh \pi x}\,dx \geq \frac{e^{t/2}}{t+1}. \tag{1}$$

Can (1) be proven analytically for all $t \in [0,1]$?

[1] https://math.stackexchange.com/questions/4463692
[2] https://math.stackexchange.com/questions/18301/proving-that-the-given-two-integrals-are-equal?noredirect=1

Sort of. You can reduce it to the case $t = 1$ and one additional simple numeric inequality, but how are you going to verify those without computations? - **fedja**
@fedja If your whole proof simple? (I also have a proof assuming that it is not difficult to verify something like $\int_0^\infty \frac{\mathrm{e}^{-x^2-1/2}}{\cosh \pi x}\,\mathrm{d}x > \frac{1}{4}, \int_0^\infty \frac{x^2\mathrm{e}^{-x^2-1/2}}{\cosh \pi x}\,\mathrm{d}x > \frac{1}{30}$. - **River Li**
@RiverLi The reduction is rather simple (take the logarithmic derivative of both sides and go back from the inequality at $t = 1$), but the case $t = 1$ is rather delicate: we have a leeway of just about 0.01, so I don't know how to handle it properly yet. - **fedja**
@fedja Perhaps no easy way. So your proof is nice (remains to check $t = 1$). - **River Li**
The discussion between fedja and River Li seems quite close to the full solution (or already done) upon computation of $I(1)$ and $I'(1)$. It seems we can put together results around $t = 1$, by fedja and River Li. Around $t = 0$, by Jack. - **Sungjin Kim**
I have a solution assuming $|I'(1)|/I(1) > 1/8$ and $I(1) > 5/6$. - **Sungjin Kim**

[+9] [2022-06-10 13:50:20] Jack D'Aurizio

You may exploit the fact that

$$I(t) = 2\int_0^{+\infty}\frac{e^{-tx^2}}{\cosh(\pi x)}\,dx$$

like any moment, is a function with a convex logarithm. It follows that the graph of $J(t) = \log I(t)$ over $[0,1]$ lies above any tangent line. We have $J(0) = 0$ and

$$J'(0) = \frac{I'(0)}{I(0)} = I'(0) = 2\int_0^{+\infty}\frac{-x^2\,dx}{\cosh(\pi x)} = -\frac{1}{4}$$

so $J(t) \geq -\frac{t}{4}$ and $I(t) \geq \exp(-t/4)$ over $[0,1]$. By exploiting the log-convexity properties of $-I'(t)$ and $I''(t)$ and numerical approximations at $t = 1$ the inequality can be improved up to $J(t) \geq -\frac{t}{4} + \frac{t^2}{16}$, so

$$I(t) \geq \exp\left(-\frac{t}{4} + \frac{t^2}{16}\right)$$

which is sharper than $I(t) \geq \frac{e^{t/2}}{1+t}$.

Following your solution, I see that $J'(t) \geq -1/4 + t/4$. So, we might have $J(t) \geq -t/4 + t^2/8$. - **Sungjin Kim**
But, $J'(1) \geq 0$ at $t = 1$ is false. So, I guess it could not be improved to $t^2/8$. I am getting $J''(0) = 1/4$. Maybe I got incorrect $J''(0)$? - **Sungjin Kim**
Upon closer inspection, it appears that $J'(t)$ is actually concave so the direction of the inequality is the other way round, unfortunately. - **TheSimpliFire**
This is really interesting, as your bound $\exp\left(-\frac{t}{4}+\frac{t^2}{16}\right)$ graphically works for $0 \leq t \leq 1.268\cdots$ only. However, there is nothing currently to suggest why this wouldn't work for $t > 1.268\cdots$, so it may be a bit harder than it seems. Graphically, the upper bound of $\exp\left(-\frac{t}{4}+\frac{t^2}{8}\right)$ holds for all $t \geq 0$. - **TheSimpliFire**
Assuming J' is concave on $[0,1]$ and numerical result for $J'(1)$, the inequality $J'(t) \geq -1/4 + t/8$ becomes false at $t = 1$. So, $J(t) \geq -t/4 + t^2/16$ would not follow from the inequality for J'. - **Sungjin Kim**
Then there should be some other way to directly prove the inequality for J. It seems just not from J'. - **Sungjin Kim**
@JackD'Aurizio I've managed to show (using convexity of J'') that

$$I(t) \geq \exp\left(-\frac{t}{4} + \frac{t^2}{8} - \frac{t^3}{8}\right)$$

but this is weaker than the original $e^{t/2}/(t+1)$ when t is close to 1. - **TheSimpliFire**
Did you prove the convexity of J''? - **Sungjin Kim**
I used the log-convexity of $I'(t)$ and the tangent lines of $\log I'(t)$ at $t = 0$ and $t = 1$ - the latter involved some numerical approximation - now clarified. - **Jack D'Aurizio**
Then you will obtain an inequality of the form $I'(t) \geq \exp(\cdots)$. Does this prove your inequality for $I(t)$? - **Sungjin Kim**

Those exponentials can be integrated! - **Jack D'Aurizio**
If you use two tangent lines and log-convexity of $-I'(t)$, you will obtain $-I'(t) \geq \exp(\cdots)$. So, it will give upper estimates. - **Sungjin Kim**
Agreed, but $I''(t)$ still is log-convex and you can use it. - **Jack D'Aurizio**
(2) Okay. Firstly, integrating exponential will give possibly nonzero constant term. Secondly, it is still not clear how $t^2/16$ is obtained from repeated integration of the exponential. - **Sungjin Kim**

1

[+3] [2022-06-16 20:15:17] Sungjin Kim

1 Let $I(t) = \int_{\mathbb{R}} e^{-tx^2}\text{sech}(\pi x)\, dx$, $J(t) = \log I(t)$ be the functions defined in Jack D'Aurizio's solution. In his solution, the main idea was that the graph of convex functions lies above any tangent lines.

Note that $I'(t) = \int_{\mathbb{R}} -x^2 e^{-tx^2}\text{sech}(\pi x)\, dx$ and $J'(t) = \frac{I'(t)}{I(t)}$. We have

$$J''(t) = \left(\frac{I'(t)}{I(t)}\right)' = \frac{I''(t)I(t) - (I'(t))^2}{I(t)^2}$$
$$= \frac{\iint_{\mathbb{R}^2}\left(\frac{x^4+y^4}{2} - x^2y^2\right)e^{-t(x^2+y^2)}\text{sech}(\pi x)\text{sech}(\pi y)dA}{\iint_{\mathbb{R}^2} e^{-t(x^2+y^2)}\text{sech}(\pi x)\text{sech}(\pi y)dA}$$
$$= \frac{\frac{1}{2}\iint_{\mathbb{R}^2}(x^2-y^2)^2 e^{-t(x^2+y^2)}\text{sech}(\pi x)\text{sech}(\pi y)dA}{\iint_{\mathbb{R}^2} e^{-t(x^2+y^2)}\text{sech}(\pi x)\text{sech}(\pi y)dA} > 0$$

Thus, J is convex.

Using convexity of I and the tangent line at $t=0$, we have by $I'(0) = -1/4$,

$$I(t) \geq 1 - \frac{t}{4}.$$

We use this for $t \in [0, 1/2]$. Then by $1 - \frac{t}{4} \geq \frac{e^{t/2}}{1+t}$ on $[0, 1/2]$, we have (1) on this range.

Using convexity of J and the tangent line at $t=1$, we have by $J'(1) = \frac{I'(1)}{I(1)}$ and $\frac{|I'(1)|}{I(1)} > \frac{1}{8}$, $I(1) > \frac{5}{6}$ (numerical results, need verification),

$$J(t) \geq J'(1)(t-1) + J(1) \geq \frac{1}{8}(1-t) + \log(\frac{5}{6}).$$

Then we have

$$I(t) \geq \frac{5}{6}\exp\left(\frac{1}{8}(1-t)\right).$$

We use this for $t \in [1/2, 1]$. Then by $\frac{5}{6}\exp\left(\frac{1}{8}(1-t)\right) \geq \frac{e^{t/2}}{1+t}$, we have (1) on this range.

A possible route to prove $I(t) \geq \exp\left(-\frac{t}{4} + \frac{t^2}{16}\right)$ is as follows.

If we prove that $J'''(t) < 0$ (I could not prove this), then the J' is concave. With a help of numerical result

$$J'(\frac{3}{4}) > -\frac{1}{4} + \frac{1}{8}\cdot\frac{3}{4},$$

we would have for $t \in [0, \frac{3}{4}]$,

$$J'(t) \geq -\frac{1}{4} + \frac{1}{8}t$$

Then for $t \in [0, \frac{3}{4}]$, we have

$$I(t) \geq \exp\left(-\frac{t}{4} + \frac{t^2}{16}\right).$$

For $t \in [\frac{3}{4}, 1]$, the previous bound

$$I(t) \geq \frac{5}{6}\exp\left(\frac{1}{8}(1-t)\right)$$

is stronger than $\exp\left(-\frac{t}{4} + \frac{t^2}{16}\right)$.

I'm busy for the rest of the month but I'll attempt to prove the numerical bounds at $t=1$. For $I(1)$ it is possible to write the integral as a series involving erf and truncate until the $5/6$ is reached. - **TheSimpliFire**

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[+2] [2022-06-20 05:44:16] River Li

Alternative proof for the original lower bound:

Problem 1: Let $0 \leq t \leq 1$. Prove that

$$2\int_0^\infty \frac{\mathrm{e}^{-tx^2}}{\cosh \pi x}\,\mathrm{d}x \geq \frac{\mathrm{e}^{t/2}}{t+1}.$$

Using $\mathrm{e}^{-u} \geq (1 - \frac{u}{5})^5$ for all $u \geq 0$ (equivalently $\mathrm{e}^{-u/5} \geq 1 - u/5$), we have

$$\begin{aligned}&2\int_0^\infty \frac{\mathrm{e}^{-tx^2}}{\cosh \pi x}\,\mathrm{d}x\\ &\ge 2\int_0^\infty \frac{(1-\frac{tx^2}{5})^5}{\cosh \pi x}\,\mathrm{d}x\\ &= 2\int_0^\infty \frac{1 - tx^2 + \frac{2}{5}\,t^2x^4 - \frac{2}{25}\,t^3x^6 + \frac{1}{125}\,t^4x^8 - \frac{1}{3125}\,t^5x^{10}}{\cosh \pi x}\,\mathrm{d}x\\ &= 1 - \frac{1}{4}\,t + \frac{1}{8}\,t^2 - \frac{61}{800}\,t^3 + \frac{277}{6400}\,t^4 - \frac{50521}{3200000}\,t^5\end{aligned}$$

where we have used

$$\int_0^\infty \frac{1}{\cosh \pi x}\,\mathrm{d}x = \frac{1}{2}, \quad \int_0^\infty \frac{x^2}{\cosh \pi x}\,\mathrm{d}x = \frac{1}{8},$$
$$\int_0^\infty \frac{x^4}{\cosh \pi x}\,\mathrm{d}x = \frac{5}{32}, \quad \int_0^\infty \frac{x^6}{\cosh \pi x}\,\mathrm{d}x = \frac{61}{128},$$
$$\int_0^\infty \frac{x^8}{\cosh \pi x}\,\mathrm{d}x = \frac{1385}{512}, \quad \int_0^\infty \frac{x^{10}}{\cosh \pi x}\,\mathrm{d}x = \frac{50521}{2048}.$$

(See: The integral : $\frac{1}{2}\int_0^\infty x^n \operatorname{sech}(x)\mathrm{d}x$ [1])

It suffices to prove that

$$1 - \frac{1}{4}\,t + \frac{1}{8}\,t^2 - \frac{61}{800}\,t^3 + \frac{277}{6400}\,t^4 - \frac{50521}{3200000}\,t^5 \ge \frac{\mathrm{e}^{t/2}}{t+1}$$

or

$$\left(1 - \frac{1}{4}\,t + \frac{1}{8}\,t^2 - \frac{61}{800}\,t^3 + \frac{277}{6400}\,t^4 - \frac{50521}{3200000}\,t^5\right)(1+t) \ge \mathrm{e}^{t/2}.$$

Let $g(t) :=$ LHS and $h(t) := \mathrm{e}^{t/2}$. It is easy to prove that $g(t)$ is concave on $[0, 1]$. So $g(t) - h(t)$ is concave on $[0, 1]$. Also, $g(0) - h(0) = 0$ and $g(1) - h(1) > 0$. Thus, $g(t) \ge h(t)$ on $[0, 1]$.

We are done.

Remark 1: The bound is stronger than $\mathrm{e}^{-t/4 + t^2/17}$ on $[0, 1]$, i.e. for all $t \in [0, 1]$,

$$1 - \frac{1}{4}\,t + \frac{1}{8}\,t^2 - \frac{61}{800}\,t^3 + \frac{277}{6400}\,t^4 - \frac{50521}{3200000}\,t^5 \ge \mathrm{e}^{-t/4 + t^2/17}.$$

However, the bound is weaker than $\mathrm{e}^{-t/4+t^2/16}$ when $t > 0.887\ldots$

Remark 2: We can obtain a slightly better bound by using $\mathrm{e}^{-u} \ge (1 - \frac{u}{7})^7$.

[1] https://math.stackexchange.com/questions/4150325/the-integral-frac12-int-0-infty-xn-operatornamesechx-mathrm-dx

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[+2] [2022-06-23 13:30:14] River Li

Sketch of a proof for the lower bound $\mathrm{e}^{-t/4+t^2/16}$

Remarks: In my another answer, I proved a weaker lower bound by using a lower bound for e^{-tx^2}. Here, we used another lower bound for e^{-tx^2}.

Problem 2: Let $0 < t \le 1$. Prove that

$$2\int_0^\infty \frac{\mathrm{e}^{-tx^2}}{\cosh \pi x}\,\mathrm{d}x \ge \exp\left(-\frac{t}{4} + \frac{t^2}{16}\right).$$

Proof:

Fact 1: For all $u \ge 0$,

$$\mathrm{e}^{-u} \ge 1 - u + \frac{13}{30}u^2 - \frac{181}{1950}u^3 + \frac{9}{1000}u^4 - \frac{1}{3125}u^5.$$

Using Fact 1, we have

$$\begin{aligned}&2\int_0^\infty \frac{\mathrm{e}^{-tx^2}}{\cosh \pi x}\,\mathrm{d}x\\ &\ge 2\int_0^\infty \frac{1 - tx^2 + \frac{13}{30}t^2x^4 - \frac{181}{1950}t^3x^6 + \frac{9}{1000}t^4x^8 - \frac{1}{3125}t^5x^{10}}{\cosh \pi x}\,\mathrm{d}x\\ &= 1 - \frac{1}{4}\,t + \frac{13}{96}\,t^2 - \frac{11041}{124800}\,t^3 + \frac{2493}{51200}\,t^4 - \frac{50521}{3200000}\,t^5\end{aligned}$$

where we have used

$$\int_0^\infty \frac{1}{\cosh \pi x}\,\mathrm{d}x = \frac{1}{2}, \quad \int_0^\infty \frac{x^2}{\cosh \pi x}\,\mathrm{d}x = \frac{1}{8},$$
$$\int_0^\infty \frac{x^4}{\cosh \pi x}\,\mathrm{d}x = \frac{5}{32}, \quad \int_0^\infty \frac{x^6}{\cosh \pi x}\,\mathrm{d}x = \frac{61}{128},$$
$$\int_0^\infty \frac{x^8}{\cosh \pi x}\,\mathrm{d}x = \frac{1385}{512}, \quad \int_0^\infty \frac{x^{10}}{\cosh \pi x}\,\mathrm{d}x = \frac{50521}{2048}.$$

(See: The integral : $\frac{1}{2}\int_0^\infty x^n \operatorname{sech}(x)\mathrm{d}x$ [1])

It suffices to prove that

$$1 - \frac{1}{4}\,t + \frac{13}{96}\,t^2 - \frac{11041}{124800}\,t^3 + \frac{2493}{51200}\,t^4 - \frac{50521}{3200000}\,t^5 \ge \mathrm{e}^{-t/4 + t^2/16}.$$

Let $f(t) :=$ LHS. Let

$$h(t) := 1 - \frac{1}{4}\,t + \frac{3}{32}\,t^2 - \frac{7}{384}\,t^3 + \frac{25}{6144}\,t^4 - \frac{27}{40960}\,t^5 + \frac{331}{2949120}\,t^6.$$

(*Note*: $h(t)$ is the 6-th order Taylor approximation of $\mathrm{e}^{-t/4+t^2/16}$ around $t = 0$.)

It suffices to prove that $f(t) \ge h(t)$ and $h(t) \ge \mathrm{e}^{-t/4+t^2/16}$ for all $t \in [0, 1]$. Omitted.

Proof of Fact 1:

Let

$$F(u) := 1 - u + \frac{13}{30}u^2 - \frac{181}{1950}u^3 + \frac{9}{1000}u^4 - \frac{1}{3125}u^5,$$

and

$$G(u) := -\frac{u^3 - 12u^2 + 60u - 120}{u^3 + 12u^2 + 60u + 120}.$$

(*Note*: $G(u)$ is $(3, 3)$-Pade approximant of e^{-u} at $u = 0$.)

We have

$$\begin{aligned} &G(u) - F(u)\\ &= \frac{u^2(312u^6 - 5031u^5 + 3920u^4 + 174440u^3 + 282000u^2 - 4740000u + 7800000)}{975000(u^3 + 12u^2 + 60u + 120)}\\ &\ge 0. \end{aligned}$$

It suffices to prove that $\mathrm{e}^{-u} \ge G(u)$.

Let

$$u_0 = (4 + 4\sqrt{5})^{1/3} - 4(4 + 4\sqrt{5})^{-1/3} + 4.$$

Then $G(u_0) = 0$, and $G(u) > 0$ on $[0, u_0)$, and $G(u) < 0$ on (u_0, ∞).

Let $H(u) := -u - \ln G(u)$. We have, for all $u \in [0, u_0)$,

$$H'(u) = G(u)\frac{u^6}{(u^3 - 12u^2 + 60u - 120)^2} \ge 0.$$

Also, we have $H(0) = 0$. Thus, we have $H(u) \ge 0$ on $(0, u_0)$.

We are done.

[1] https://math.stackexchange.com/questions/4150325/the-integral-frac12-int-0-infty-xn-operatornamesechx-mathrm-dx

Nice! Can you outline how to prove Fact1? - **Sungjin Kim**
@SungjinKim Thanks. It is not nice. I will edit. - **River Li**

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[+1] [2022-06-15 09:51:58] Erik Satie

Too long for a comment :

We have the obvious inequalities for $x \ge 0$ and $t \in [0, 0.25]$:

$$2\int_0^\infty \frac{e^{-tx^2}}{\cosh(\pi x)}dx \ge 2\int_0^\infty \frac{e^{-tx}}{\cosh(\pi x)}dx \ge 2\int_0^\infty \frac{1 - xt}{\cosh(\pi x)}dx$$

So we need to show for $x \ge 0$ and $t \in [0, 0.25]$:

$$\frac{e^{\frac{t}{2}}}{t+1} \le 2\int_0^\infty \frac{1 - xt}{\cosh(\pi x)}dx$$

We have :

$$2\int_0^\infty \frac{1}{\cosh(\pi x)}dx = 1$$

And :

$$2\int_0^\infty \frac{-xt}{\cosh(\pi x)}dx = -2tC/\pi^2$$

Where C is the Catalan's constant .

So we need to show :

$$\frac{e^{\frac{t}{2}}}{t+1} \le 1 - 2tC/\pi^2$$

Wich is easier and true .

Edit we have numerically the inequality for $t\in[0.25,0.75]$:

$$2\left(\int_{0}^{0.6}-\left(\frac{-1+x^2t-(3-((1-c)+2+ac(2-c)-ac(1-c)\ln a))}{\cosh(\pi\cdot x)}\right)dx+\int_{0.6}^{1}\frac{(1-x^2t)}{\cosh(\pi\cdot x)}dx\right)-\frac{e^{\frac{t}{2}}}{1+t}>0$$

Where $c=x^2t, a=e^{-1}$

In fact I have used this link [1] lemma 5.1

On the other hand we have the inequalities for $x\in[0,0.6]$ and $t\in[0.25,0.75]$:

$$e^{-tx^2}-1+x^2t-(3-((1-c)+2+ac(2-c)-ac(1-c)\ln a))\geq 0$$

And :

$$e^{-tx^2}\geq 1-x^2t$$

Last edit :

It seems we have the inequality for $x\in[0,1.252+5(t-1)]$ and $t\in[0.95,1]$:

$$-\left(2-\left(2(1-c)+a^bc(2-c)-ac(1-c)\ln a\right)-x^4t-1\right)\leq e^{-tx^2}$$

Where $b=\frac{\pi}{e}$ and $a=e^{-1}$, $c=x^2t$

Next it seems we have the inequality for $x\in[0,1.252+5(t-1)]$ and $t\in[0.95,1]$:

$$2\int_{0}^{1.252+5(t-1)}\frac{-\left(2-\left(2(1-c)+a^bc(2-c)-ac(1-c)\ln a\right)-x^4t-1\right)}{\cosh(\pi x)}dx+2\int_{1.252+5(t-1)}^{\infty}\frac{1}{\cosh(\pi x)^2}dx>\frac{e^{\frac{t}{2}}}{t+1}$$

[1] https://www.isr-publications.com/jnsa/articles-1563-proofs-of-three-open-inequalities-with-power-exponential-functions

5

[+1] [2022-06-19 09:22:59] rrv

Some thoughts. Let $\sigma^2=\frac{1}{2t}$. Then

$$\int_{-\infty}^{\infty}\frac{e^{-tx^2}}{\cosh\pi x}dx=\mathbf{E}\left(\frac{\sqrt{2\pi}\sigma}{\cosh\pi X}\right),$$

where X is normally distributed with zero mean and variance σ^2. Since $\cosh x\leq e^{\frac{x^2}{2}}$ then $\mathbf{E}\left(\frac{\sqrt{2\pi}\sigma}{\cosh\pi X}\right)\geq\mathbf{E}\left(\sqrt{2\pi}\sigma e^{-\frac{\pi^2X^2}{2}}\right)$. Now Jensen's inequality can be used.

I think the estimate $\cosh x\leq e^{\frac{x^2}{2}}$ is not enough (too loose). You can check $t=1/2$. - **River Li**

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[+1] [2022-06-21 09:51:32] orangeskid

Attempts:

We have the sech distribution [1], so the moments of the function $\operatorname{sech}\pi x$ are known and can be expressed in terms of the Euler functions. We have the formula [2]:

$$\int_{-\infty}^{\infty}\frac{e^{-tx}}{\cosh\pi x}dx=\sec\frac{t}{2}$$

for $|\operatorname{Re}t|<\pi$

Now, to estimate the integrals

$$\int_{-\infty}^{\infty}\frac{e^{-tx^2}}{\cosh\pi x}dx$$

we'll use the Gaussian quadrature of order 3. The orhogonal polynomials for the distrubution $\frac{1}{\cosh\pi x}$ are $1, x, x^2-\frac{1}{4}, x^3-\frac{5}{4}x, \ldots$. The Gaussian quadrature of order 3 (exact for polynomials of degree ≤ 5) is

$$\int_{-\infty}^{\infty}\frac{f(x)}{\cosh\pi x}\simeq\frac{1}{10}f(-\frac{\sqrt{5}}{2})+\frac{4}{5}f(0)+\frac{1}{10}f(\frac{\sqrt{5}}{2})$$

For $f(x)=e^{-tx^2}$ we get

$$\int_{-\infty}^{\infty}\frac{e^{-tx^2}}{\cosh\pi x}\simeq\frac{1}{5}e^{-\frac{5}{4}t}+\frac{4}{5}$$

It turns out that the RHS in the above approximation is larger for all $t>0$. To get a lower estimate for $t\in[0,1]$, substitute RHS with

$$\frac{1}{4.5}e^{-\frac{4.5}{3.5}t}+\frac{3.5}{4.5}$$

This is larger than $\frac{e^{\frac{t}{2}}}{1+t}$ for $t\in[0,1]$.

This lower estimate is valid for $t\in[0,1]$. A lower estimate that works for all $t>0$ is given by another Gauss quadrature, $e^{-\frac{t}{4}}$.

[1] https://en.wikipedia.org/wiki/Hyperbolic_secant_distribution
[2] https://www.wolframalpha.com/input?i=Integrate%5B%20Exp%5B-%20Catalan%20x%5D%2F%20Cosh%5BPi%20x%5D%2C%20%7Bx%2C%20-Infinity%2C%20Infinity%7D%5D

7

[-1] [2022-06-18 16:08:18] Martin Gales

I use a systematic approach.

$$f(t)=\int_0^\infty \frac{e^{-tx^2}}{\cosh(\pi x)}\,dx$$

Apply the Laplace transform

$$F(s)=\int_0^\infty f(t)e^{-ts}\,dt$$

or

$$F(s)=\int_0^\infty \frac{dx}{(s+x^2)\cosh(\pi x)}=\frac{2}{\sqrt{s}}\int_0^1 \frac{x^{2\sqrt{s}}}{1+x^2}$$

The last integral is due to Hardy

Now, taylor-expand the integrand in the last integral and integrate term by term to get

$$F(s)=\sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{s}(\sqrt{s}+k+\frac{1}{2})}$$

To inverse the received expression we use a table of Laplace transform pairs.

Result

$$f(t)=\sum_{k=0}^\infty (-1)^k e^{t(k+\frac{1}{2})^2}\operatorname{erfc}\left[\sqrt{\mathrm{t}}(\mathrm{k}+\frac{1}{2})\right]$$

where $\mathrm{erfc}(\mathrm{x})$ is the Complementary Error Function.

From this result, for large t, the asymptotic behavior of $f(t)$ seems to be

$$\frac{2}{\sqrt{\pi t}}$$

But for moderate values of t as required i am not sure how to get suggested bounds.

But in any case, since the series is alternating, the error should be less than the absolute value of the first omitted term.

Just mentioning that you can write `\operatorname{erfc}` to make it look like an operator. - **Martin R**
@MartinR Good point! - **Martin Gales**

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