sliding hump method

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命题 1. l^1弱收敛等价于强收敛.

Proof. 反证, $f_n \rightharpoonup 0$ in l^1, 但$f_n \not\to 0$ in l^1.

则$\exists \varepsilon > 0$, $\forall N$, $\exists n > N$, s.t. $\|f_n\|_{l^1} > \varepsilon$.

取$N_1 = N$, $n_1 = n$, s.t. $\|f_{n_1}\|_{l^1} > \varepsilon$.

因为$f_{n_1} \in l^1$, 所以存在k_1, s.t.

$$\sum_{r=k_1+1}^{\infty} \left| f_{n_1}^{(r)} \right| < \frac{\varepsilon}{5},$$

定义$g^{(r)} = \mathrm{sgn} f_{n_1}^{(r)}$, $1 \le r \le k_1$.

由l^1中f_n弱收敛于0, 有对于任意的$r \in \mathbb{N}_+$, $\lim_{n\to\infty} f_n^{(r)} = 0$. 所以当$k_1$取定, 便$\exists N_2 > N_1$, s.t. $\forall n_2 > N_2$, 有

$$\sum_{r=1}^{k_1} \left| f_{n_2}^{(r)} \right| < \frac{\varepsilon}{5},$$

由于$f_n \not\to 0$ in l^1, 所以对于任意的$N_2 > N_1$, 存在$n_2 > N_2$使$\|f_{n_2}\|_{l^1} > \varepsilon$. 由$f_{n_2} \in l^1$, 故存在$k_2 > k_1$, s.t.

$$\sum_{r=k_2+1}^{\infty} \left| f_{n_2}^{(r)} \right| < \frac{\varepsilon}{5},$$

定义$g^{(r)} = \mathrm{sgn} f_{n_2}^{(r)}$, $k_1 < r \le k_2$.

如此继续下去, 则由$n_i > N_i$, $k_i > k_{i-1}$, s.t. $f_{n_i} \in l^1$且$\|f_{n_i}\|_{l^1} > \varepsilon$,

$$\sum_{r=1}^{k_{i-1}} \left| f_{n_i}^{(r)} \right| < \frac{\varepsilon}{5}, \qquad \sum_{r=k_i+1}^{\infty} \left| f_{n_i}^{(r)} \right| < \frac{\varepsilon}{5},$$

定义$g^{(r)} = \mathrm{sgn} f_{n_i}^{(r)}$, $k_{i-1} < r \le k_i$.

则$g \in l^\infty$, 于是$\langle f_{n_i}, g \rangle \to 0$, $i \to \infty$, (因为$f_n \rightharpoonup 0$ in l^1).

但

$$\begin{aligned}
\langle f_{n_i}, g \rangle &= \sum_r f_{n_i}^{(r)} g^{(r)} \\
&= \sum_{r=k_{i-1}+1}^{k_i} f_{n_i}^{(r)} g^{(r)} + \sum_{r=k_i+1}^{\infty} f_{n_i}^{(r)} g^{(r)} + \sum_{r=1}^{k_{i-1}} f_{n_i}^{(r)} g^{(r)} \\
&\ge \sum_{r=k_{i-1}+1}^{k_i} \left| f_{n_i}^{(r)} \right| - \sum_{r=k_i+1}^{\infty} \left| f_{n_i}^{(r)} \right| - \sum_{r=1}^{k_{i-1}} \left| f_{n_i}^{(r)} \right| \\
&\ge \|f_{n_i}\|_{l^1} - 2 \left(\sum_{r=k_i+1}^{\infty} \left| f_{n_i}^{(r)} \right| + \sum_{r=1}^{k_{i-1}} \left| f_{n_i}^{(r)} \right| \right) \\
&\ge \varepsilon - 2 \cdot \frac{2\varepsilon}{5} = \frac{\varepsilon}{5}.
\end{aligned}$$

矛盾. □

命题 2. $\alpha \in \mathcal{A}$, $\mathcal{A}$是指标集, 可数或不可数, $T_\alpha : E \to E_1$, E为$Banach$空间, E_1为赋范线性空间, $\forall x \in E$,

$$\sup_{\alpha \in \mathcal{A}} \{\|T_\alpha x\|\} < \infty,$$

则$\|T_\alpha\|$($\alpha \in \mathcal{A}$)有界.

Proof. 用反证法, 设$\{T_\alpha\}$为无界可数子列, 不妨设

$$\|T_n\| > 10^n,$$

则存在$x_n \in E$使得$\|x_n\|_E = 1$且

$$\|T_n x_n\|_{E_1} > \frac{1}{2} \|T_n\|,$$

取$x := \sum_{n=1}^{\infty} \frac{\varepsilon_n x_n}{9^n}$,其中$\varepsilon_n = 0$或1定义如下, 则$\|x\| \le \sum \frac{\|x_n\|_E}{9^n} < \infty$.
此段定义ε_n: □

- 取$\varepsilon_n = 1$, 如果

$$\left\| T_n \sum_{k=1}^{n-1} \frac{\varepsilon_k x_k}{9^k} \right\| \le \frac{\|T_n\|}{4 \cdot 9^n}.$$

- 取$\varepsilon_n = 0$, 如果

$$\left\| T_n \sum_{k=1}^{n-1} \frac{\varepsilon_k x_k}{9^k} \right\| > \frac{\|T_n\|}{4 \cdot 9^n}.$$

最后只需证明$\limsup_{n\to\infty} \|T_n x\| = \infty$.

- 当$\varepsilon_n = 0$时,

$$\begin{aligned}
\|T_n x\| &\ge \left\| T_n \sum_{k=1}^{n-1} \frac{\varepsilon_k x_k}{9^k} \right\| - \left\| T_n \sum_{k=n+1}^{\infty} \frac{\varepsilon_k x_k}{9^k} \right\| \\
&> \frac{\|T_n\|}{4 \cdot 9^n} - \sum_{k=n+1}^{\infty} \frac{\|T_n\|}{9^k} \\
&= \frac{\|T_n\|}{8 \cdot 9^n} > \frac{1}{8} \left(\frac{10}{9}\right)^n.
\end{aligned}$$

- 当$\varepsilon_n = 1$时,

$$\begin{aligned}
\|T_n x\| &\ge \left\| T_n \frac{x_n}{9^n} \right\| - \left\| T_n \sum_{k=1}^{n-1} \frac{\varepsilon_k x_k}{9^k} \right\| - \left\| T_n \sum_{k=n+1}^{\infty} \frac{\varepsilon_k x_k}{9^k} \right\| \\
&> \frac{\|T_n\|}{2 \cdot 9^n} - \frac{\|T_n\|}{4 \cdot 9^n} - \frac{\|T_n\|}{8 \cdot 9^n} \\
&= \frac{\|T_n\|}{8 \cdot 9^n} > \frac{1}{8} \left(\frac{10}{9}\right)^n.
\end{aligned}$$

导致矛盾, 即存在$x \in E$使$T_n x$无界.