3 (Folland) we have proved that:

> Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{A})$, then $\nu \ll \mu$ iff. $\forall \varepsilon > 0$, $\exists \delta > 0$. s.t. $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

Show that, the above theorem may fail when ν is not finite.

Pf: $d\mu = dm$, $d\nu = e^x d\mu \Rightarrow \nu \ll \mu$

$\forall \varepsilon > 0$, $\forall \delta > 0$, $\mu(E) \le \delta$, 取 $E = (n, n+\delta)$, but $\int_E d\nu = \nu(E) \to \infty$, $n \to \infty$.

1. (Folland). $X = [0,1]$, $\mathcal{M} = \mathcal{B}[0,1]$, m = Lebesgue measure. μ = counting measure on $\mathcal{M}$.

① $m \ll \mu$ but $dm \neq f d\mu$. $\forall f$.

② μ has no Lebesgue decomposition with respect to m.

pf: ①. $f \equiv 0$.

②. 反证: $\mu = \mu_a + \mu_s$. $\mu_a \ll m$, $\mu_s \perp m$

$\exists N$. s.t. $m(N) = \mu_s(N^c) = 0$.

$\forall x \in N^c$, $\mu_a(x) = \mu(x) = 1$, 而 $m(x) = 0$. 矛盾.