连通紧度量空间 唯一的

Does every connected compact metric space have a unique always attainable average distance?

平均距离

[+8] [2] SmileyCraft

[2022-06-08 20:53:01]

[real-analysis probability-theory measure-theory metric-spaces average]

[https://math.stackexchange.com/questions/4468488/does-every-connected-compact-metric-space-have-a-unique-always-attainable-averag]

Problem statement

连通紧度量空间 存在唯一的$\alpha\geq 0$.

Let (X, d) be a connected compact metric space. Then does there always exist a unique value $\alpha \geq 0$ with the following property?

$\forall x_1,x_2,\cdots,x_k\in X$, 存在$x\in X$, s.t. $\sum_{i=1}^{k}\frac{d(x,x_i)}{k}=\alpha$.

For every $x_1, \ldots, x_k \in X$ there exists $x \in X$ such that the average distance from x to x_i for $i = 1, \ldots, k$ is exactly α.

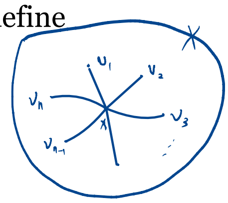
Proof of existence 存在性的证明

The following proves such a value always exists, so the main question is whether it is unique.

For $v \in X^n$, define

$\forall v\in X^n$, 定义

$$\begin{cases} f_v(x) := \frac{1}{n}\sum_{i=1}^{n} d(x,v_i) \\ a(v) := \min_{x\in X} f_v(x) \\ b(v) := \max_{x\in X} f_v(x) \end{cases}$$



$$f_v(x) := \frac{d(x, v_1) + \ldots + d(x, v_n)}{n},$$
$$a(v) := \min_{x \in X} f_v(x),$$

X紧, f_v连续 $\Rightarrow$ $\mathrm{Im} f_v$ 紧且有最值

$$b(v) := \max_{x \in X} f_v(x).$$

Because X is compact and f_v is continuous, the image of f_v is also compact and thus indeed has a minimum and a maximum.

设 $v\in X^n$, $w\in X^m$. 则 $a(w)\leq\frac{f_w(v_1)+\cdots+f_w(v_n)}{n}=\frac{f_v(w_1)+\cdots+f_v(w_m)}{m}\leq b(v)$

Let $v \in X^n$ and $w \in X^m$. We have

$\Rightarrow a(\cdot)$有上确界α, $b(\cdot)$有下确界β. 并有$\alpha\leq\beta$.

$$a(w) \leq \frac{f_w(v_1) + \ldots + f_w(v_n)}{n} = \frac{f_v(w_1) + \ldots + f_v(w_m)}{m} \leq b(v).$$

It follows that there exists a supremum α of a and an infimum β of b, and that we have $\alpha \leq \beta$.

Let $v \in X^n$, so we have $a(v) \leq \alpha \leq \beta \leq b(v)$. Since X is connected, the image of f_v is connected, so since it contains $a(v)$ and $b(v)$, it also contains $[a(v), b(v)]$. It follows that there exists $x \in X$ such that $f_v(x) = \alpha$.

于是 $\forall v\in X^n$, 有$a(v)\leq\alpha\leq\beta\leq b(v)$. 由于$X$连通, $\mathrm{Im} f_v$ 也是连通的. 且包含$a(v)$, $b(v)$. 故$[a(v),b(v)]\subseteq \mathrm{Im} f_v$

$\Rightarrow \exists x\in X$ s.t. $f_v(x)=\alpha$.

Note that uniqueness is thus equivalent with stating that $\alpha = \beta$ in the above proof.

唯一性等价于证明$\alpha=\beta$.

Looking for counterexamples

The most straight forward connected compact metric spaces that come to mind are $[0, 1]^n$. Let $v(t) := (t, \ldots, t)$. We have

最简单的连通紧度量空间是$[0,1]^n$. 设 $v(t):=(t,\cdots,t)$. 则 $a(v(0),v(1))=b(v(\frac{1}{2}))=\frac{\sqrt{n}}{2}=\alpha=\beta$

$$a(v(0), v(1)) = b(v(1/2)) = \sqrt{n}/2 = \alpha = \beta.$$

Consider a regular n-gon $P_1, \ldots, P_n$ with center O. We have

考虑中心为点O的正n边形$P_1P_2\cdots P_n$.

则有 $a(P_1,\cdots,P_n)=b(O)=|P_1O|=\alpha=\beta$.

$$a(P_1, \ldots, P_n) = b(O) = |P_1 O| = \alpha = \beta.$$

If we have the boundary of a unit square $ABCD$, let E, F, G, H be the centers of the edges. We have

若考虑单位正方形ABCD的边界为条件中的连通紧度量空间. 设E,F,G,H为边的中点. 则有

$a(A,B,C,D)=b(E,F,G,H)=1+\sqrt{5}=\alpha=\beta$

$$a(A, B, C, D) = b(E, F, G, H) = \frac{1}{4}(1 + \sqrt{5}) = \alpha = \beta.$$

If we have the unit circle, let v_n denote n evenly spaced points. We have

考虑单位圆上均匀分布的n个点v_n,有

$\lim_{n\to\infty} a(v_n)=\lim_{n\to\infty} b(v_n)=\frac{1}{2\pi}\int_0^{2\pi}|1-e^{i\theta}|d\theta = \frac{4}{\pi}=\alpha=\beta$

$$\lim_{n\to\infty} a(v_n) = \lim_{n\to\infty} b(v_n) = \frac{1}{2\pi}\int_0^{2\pi} |1-e^{i\theta}|\,\mathrm{d}\theta = \frac{4}{\pi} = \alpha = \beta,$$

so we can essentially approximate any measure, in this case giving us the average distance between two points on a circle. 考虑任意的锐角△ABC,外心为O,则$b(O)=|AO|$,因为可以逼近任何测度,引入权重w_A, w_B, w_C,固定w_A,则有唯一的w_B, w_C使得$f_{w_AA, w_BB, w_CC}(P)$在P=O处的梯度为0.

Consider an arbitrary acute triangle ABC with circumcenter O. Note that $b(O)=|AO|$. Since we can approximate any measure, we can introduce weights w_A, w_B, w_C. Fixing w_A, we find unique values of w_B, w_C such that $f_{w_AA,w_BB,w_CC}(P)$ has gradient 0 at $P=O$, which gives

从而给出$a(w_AA, w_BB, w_CC)=b(O)=|AO|=\alpha=\beta$

$$a(w_AA, w_BB, w_CC) = b(O) = |AO| = \alpha = \beta.$$

从上例子似乎都能说明α应该等于β.如何证明?

So it seems like $\alpha=\beta$ always holds, but I can not come up with a proof. I even tried some really ugly non-symmetric non-convex sets, and some metric spaces which are not subsets of $\mathbb{R}^n$, and while it becomes more work each time, I always end up proving $\alpha=\beta$. All proofs rely on constructing measures μ, ν such that $a(\mu)=b(\nu)$, but I can not figure out the bigger picture of what these measures are in general. I would not even know how to define measures in arbitrary metric spaces to begin with.

所有证明都依赖于构造μ,ν使得$a(\mu)=b(\nu)$

Edit: A measure theory approach

We can phrase the question in terms of probability measures as follows.

Let (X,d) be a connected compact metric space with Borel σ-algebra $B(X)$. Then does there always exist a unique value $\alpha\geq 0$ with the following property?

For every probability measure P on $(X, B(X))$, there exists $x\in X$, such that $\mathbb{E}_{y\sim P}(d(x,y))=\alpha$.

用概率测度的语言来说:设(X,d)是Borel代数$B(X)$上的连通紧度量空间,问是否存在唯一的$\alpha\geq 0$满足:

The existence proof is completely analogous if we use $f_P(x) := \mathbb{E}_{y\sim P}(d(x,y))$, for P a probability measure on $(X, B(X))$, because f_P is still continuous. Indeed, f_P has Lipschitz constant 1.

对于任意$(X, B(X))$上的概率测度P,存在$x\in X$ s.t. $\mathbb{E}_{y\sim P}(d(x,y))=\alpha$.

Equivalence with the finite average approach follows from discrete probability measures are dense in separable metric spaces[1]. Indeed, since X is compact, it is separable.

存在性证明如前类似证明,这只需使用$f_P(x):=\mathbb{E}_{y\sim P}(d(x,y))$,其中P是$(X, B(X))$上的概率测度,并注意$f_P$仍然连续,事实上,$f_P$有Lipschitz常数1. 由于X是紧的,从而是可分的.

Optimal probability measures

The generalization to arbitrary probability measures allows us to do the following.

测度状态(?)与有限平均方法等价是因为:离散概率测度在可分度量空间中稠密.

Since the set of probability measures P on $(X, B(X))$ is sequentially compact, see Theorem 10.2[2], there exist probabability measures P_a and P_b on $(X, B(X))$ such that $a(P_a)=\alpha$ and $b(P_b)=\beta$.

最优概率测度问题:由于$(X,B(X))$上的概率测度集是序列紧的,故在$(X,B(X))$上存在概率测度P_a和P_b使$a(P_a)=\alpha$, $b(P_b)=\beta$.

Claim: We have $\alpha=\beta$ if, and only if, we have $f_{P_a}(x)=\alpha$ for all $x\in\mathrm{supp}(P_b)$, and vice versa.

命题:$\alpha=\beta$, iff. $\forall x\in\mathrm{supp}(P_b)$有$f_{P_a}(x)=\alpha$,反之亦然.

Proof: For the left implication, since X is strongly Lindelöf, we have

证明:⇐:因X是强Lindelöf的,故

$\alpha=\mathbb{E}_{x\sim P_b}(f_{P_a}(x))=\mathbb{E}_{x\sim P_a}(f_{P_b}(x))=\beta$

$$\alpha = \mathbb{E}_{x\sim P_b}(f_{P_a}(x)) = \mathbb{E}_{x\sim P_a}(f_{P_b}(x)) = \beta.$$

For the right implication, note that, if $f_{P_a}(x)>\alpha$ for some $x\in\mathrm{supp}(P_b)$, then

⇒:若有$x\in\mathrm{supp}(P_b)$使$f_{P_a}(x)>\alpha$,则

$\alpha<\mathbb{E}_{x\sim P_b}(f_{P_a}(x))=\mathbb{E}_{x\sim P_a}(f_{P_b}(x))\leq\beta$

$$\alpha < \mathbb{E}_{x\sim P_b}(f_{P_a}(x)) = \mathbb{E}_{x\sim P_a}(f_{P_b}(x)) \leq \beta.$$

[1] https://math.stackexchange.com/questions/1310735/in-the-space-of-probability-distributions-is-the-set-of-discrete-distributions
[2] https://personalpages.manchester.ac.uk/staff/Charles.Walkden/ergodic-theory/lecture10.pdf

I find this problem incredibly fascinating. It would strike me as a surprise if uniqueness were true. - **Del**

If $\alpha<\beta$, not only does uniqueness fail, but it fails very badly with a continuum of counterexamples. This order of magnitude prove useful, using set theoretic trickery, to prove such a continuum of counter examples cannot exist. Just a thought. +1 for the intriguing question and the existence proof - **FShrike**

设$\epsilon>0$,则$\exists x_1,\dots,x_s\in X$, s.t. $\min_{1\leq i\leq s} d(x_i,\cdot)\leq\frac{\epsilon}{4}$,则$\forall v\in X^n$,$f_v(x)$的值与某$f_v(x_i)$不超过$\frac{\epsilon}{4}$.所以$b(v)-a(v)\leq\frac{\epsilon}{2}+\max_{1\leq i,j\leq s}(f_v(x_j)-f_v(x_i))$

Isn't β just radius of our space (minimal r s.t. our space fits into a ball of radius r)? - **mihaild**

This is the idea I had, but the key claim doesn't sound easy to show. Let $\epsilon>0$. There are $x_1,\dots,x_s\in X$ such that $\min_{1\leq i\leq s} d(x_i,\cdot)\leq\epsilon/4$. Then, for any $v\in X^n$, $f_v(x)$ is at distance at most $\epsilon/4$ from one of the $f_v(x_i)$. Thus, $b(v)-a(v)\leq\epsilon/2+\max_{1\leq i,j\leq s} f_v(x_j)-f_v(x_i)$. Now, "morally", if we draw the v_i iid according to some probability measure μ, SLLN bounds the RHS with probability one, as long as $\int d(x_i, y) d\mu(y)$ doesn't depend on i. I'm not sure whether μ exists in general. - **Aphelli** 由于 $\int d(x_i,y)d\mu(y)$ 不依赖于 i，所以不确定 μ 的存在性.

If you consider the average *quadratic* distance instead, I think I can show that for convex sets X of $\mathbb{R}^n$ we have $\tilde{\alpha} = \tilde{\beta} = r$, where r is the radius of X as suggested by @mihaild and $\tilde{\alpha}, \tilde{\beta}$ are the analogues for the quadratic distance. This is based on the fact that the barycenter of some points is the point that infimizes the average quadratic distance from them. (More in general, this should work whenever the center of a minimal ball containing X is also contained in X). However this is purely Euclidean - **Del**

(1) @mihaild I think this is correct for convex subsets of $\mathbb{R}^n$. A proof similar to that I gave for arbitrary acute triangles should apply in general. However, this is not true for the boundary of a square or a circle, as shown in the examples I gave. - **SmileyCraft**

I made an edit formalizing the ideas about measure, confirming my intuition that there always exist probability measures P_a and P_b such that $a(P_a) = \alpha$ and $b(P_b) = \beta$. The claim about supports also formalizes the idea that P_b has to be near the "middle" and P_a has to be as far as possible from the "middle". - **SmileyCraft**

After the reformulation with measures, I think that equality follows simply from a minimax theorem. I'll try to write something up. - **Del**

[+3] [2022-06-15 09:55:33] Del [✔ACCEPTED]

It seems that indeed $\alpha = \beta$.

First let us reformulate the problem making also the optimal point x a measure. By definition,

看上去好像确实有 $\alpha=\beta$.

首先，改写公式如下，并将最优点 x 看成一个测度，则

$a(\mu)=\inf_{x\in X}\int_X d(x,y)d\mu(y)$.

$$a(\mu) = \inf_{x \in X} \int_X d(x, y)\, \mathrm{d}\mu(y).$$

定义 $A(\mu):=\inf_{\nu\in P(X)}\int_X\int_X d(x,y)d\mu(y)d\nu(x)$

Let us introduce

$$A(\mu) := \inf_{\nu \in \mathcal{P}(X)} \int_X \int_X d(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\nu(x).$$

Claim. $A(\mu) = a(\mu)$ for every $\mu \in \mathcal{P}(X)$.

命题. $A(\mu)=a(\mu)$, $\forall\mu\in P(X)$.

Proof. On one hand it is clear that $A(\mu) \leq a(\mu)$ by choosing $\nu = \delta_{x_0}$, with x_0 optimal point for $a(\mu)$.

证明：显然当取 $\nu=\delta_{x_0}$ 时，$A(\mu)\le a(\mu)$，其中 x_0 为 $a(\mu)$ 的最优点.

On the other hand, let $\mu \in \mathcal{P}(X)$ be fixed and let ν be optimal for $A(\mu)$. First, for every $E \subseteq X$ we must have

另一方面，设 $\mu\in P(X)$ 固定，ν 是 $A(\mu)$ 的最优点.

$\forall E\subseteq X$，有 $\int_X\int_X d(x,y)d\mu(y)d(\nu\llcorner E)(x)=A(\mu)\nu(E)$

否则 $\frac{1}{\nu(E)}\nu\llcorner E$ 或 $\frac{1}{\nu(E^c)}\nu\llcorner E^c$ 是最优点. (这里 $\nu\llcorner E(B):=\nu(E\cap B)$)

$$\int_X \int_X d(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}(\nu \llcorner E)(x) = A(\mu)\nu(E),$$

otherwise either $\frac{1}{\nu(E)} \nu \llcorner E$ or $\frac{1}{\nu(E^c)} \nu \llcorner E^c$ would be a better competitor (here $\nu \llcorner E(B) := \nu(E \cap B)$ denotes the restriction of ν to E). It follows that for every $E \subseteq X$ with $\nu(E) > 0$

于是 $\forall E\subseteq X$, $\nu(E)>0$ 有

$\frac{1}{\nu(E)}\int_{x\in E}\int_{y\in X}d(x,y)d\mu(y)d\nu(x)=A(\mu)$

于是函数 $x\mapsto\int d(x,y)d\mu(y)$, ν-a.e. 是常数 $A(\mu)$.

故对于 ν-几乎处处的点 x_0 有 $\tilde{\nu}:=\delta_{x_0}$ 为 $A(\mu)$ 的最优点，从而 x_0 是 $a(\mu)$ 的最优点并且两者的值相同.

$$\frac{1}{\nu(E)} \int_{x \in E} \int_{y \in X} d(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\nu(x) = A(\mu)$$

and by standard arguments it follows that the function $x \mapsto \int d(x, y)\, \mathrm{d}\mu(y)$ is constant ν-a.e., equal to $A(\mu)$. Then, for ν-a.e. point x_0, $\tilde{\nu} := \delta_{x_0}$ would still be an optimal competitor for $A(\mu)$, and therefore x_0 would be optimal for $a(\mu)$ (with the same value). ■

With a similar argument, $b(\mu)$ equals

同理，$b(\mu)$ 等于

$B(\mu):=\sup_{\nu\in P(X)}\int_X\int_X d(x,y)d\mu(y)d\nu(x)$.

要证 $\alpha=\beta$，只需交换 μ 和 ν 的位置：

$\sup_{\mu\in P(X)}\inf_{\nu\in P(X)}\int_X\int_X d(x,y)d\mu(y)d\nu(y)=\inf_{\nu\in P(X)}\sup_{\mu\in P(X)}\int_X\int_X d(x,y)d\mu(y)d\nu(x)$

$$B(\mu) := \sup_{\nu \in \mathcal{P}(X)} \int_X \int_X d(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\nu(x).$$

To prove $\alpha = \beta$ we are thus reduced to prove (I exchanged μ and ν by symmetry)

由于

$P(X)\otimes P(X)\ni(\mu,\nu)\mapsto\int_X\int_X d(x,y)d\mu(y)d\nu(x)$

是双线性函数，关于一个变量凸，关于另一变量凹，

由极小极大定理得到所要等式.

$$\sup_{\mu \in \mathcal{P}(X)} \inf_{\nu \in \mathcal{P}(X)} \int_X \int_X d(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\nu(x) = \inf_{\nu \in \mathcal{P}(X)} \sup_{\mu \in \mathcal{P}(X)} \int_X \int_X d(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\nu(x).$$

This equality follows from a version of the minimax theorem [1] *, after observing that the function

$$\mathcal{P}(X) \times \mathcal{P}(X) \ni (\mu, \nu) \mapsto \int_X \int_X d(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\nu(x)$$

is bilinear (and thus convex in one variable and concave in the other).

*For instance, look at: Ky Fan, Minimax Theorems. *Proceedings of the National Academy of Sciences of the United States of America* Vol. 39, No. 1 (Jan. 15, 1953), pp. 42-47. There is probably a simpler reference but this is the first I could find that works for infinite-dimensional spaces.

[1] https://en.wikipedia.org/wiki/Minimax_theorem

@SmileyCraft Let me add again that I find it fascinating that equality holds (unless someone can spot a mistake above). This gives a "typical length" value to any compact metric space in a way that I haven't encountered anywhere else. Very interesting problem, how did you come up with it? - **Del** 因X和$\mathcal{P}(X)$是紧的，上下确界就是其最大最小值.

(2) Amazing! So just some clarifications. Since X and $\mathcal{P}(X)$ are compact, all infima and suprema can be replaced with minima and maxima, correct? In your proof of the claim, it seems you already use this fact anyways. Furthermore, do I understand correctly that the connectedness of X is not used in the proof that $\alpha = \beta$? For example, for $X = [0,1] \cup [2,3]$, we have $a(\{0,3\}) = b(\{0,3\}) = \frac{3}{2} = \alpha = \beta$, but there exists no $x \in X$ with $f_{\{0\}}(x) = \frac{3}{2}$. So that is pretty interesting how $\alpha = \beta$ still holds. - **SmileyCraft**

@SmileyCraft Yes, all infima and suprema are attained by compactness (I wrote them as $\inf$ and $\sup$ only out of habit). You are correct that this does not use connectedness of the space, and I find your simple example enlightening. - **Del**

I was trying to come up with an example of a non-compact space X such that $\alpha < \beta$, but even in this generality I am not able to find any examples. So I made this into a follow up question :) math.stackexchange.com/questions/4473618/... - **SmileyCraft**

(1) Looking at the answer of the other question it appears that this is a case where knowing the magic word "rendevouz number" can open up lots of search hits. I even found a previous question on mathoverflow: link. - **Del**

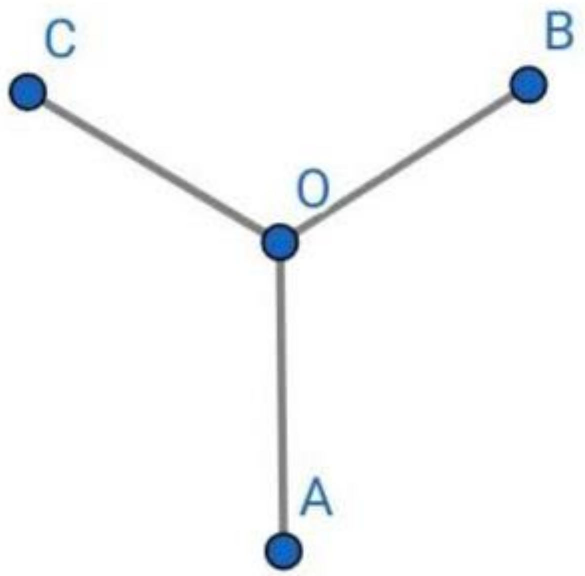
1

[+1] [2022-06-10 14:43:42] Antoine Labelle

Nice problem! Here are some thoughts which are too long for a comment.

One natural approach is to try to construct measures μ such that $b(\mu) - a(\mu)$ is arbitrarily small, as suggested by Aphelli in the comments. Unfortunately this is not always possible; consider for example the following tripod graph, endowed with the graph metric such that each edge has length 1.

构造μ使$b(\mu)-a(\mu)$任意小.
但这是不可能的.
设μ是此空间上的概率测度,
WLOG. 假设$[AO)$有测度$\leq \frac{1}{3}$.
则 $f_\mu(A) \geq f_\mu(O) + \frac{1}{3}$
$\Rightarrow b(\mu) - a(\mu) \geq \frac{1}{3}$



Let μ be a probability measure on this space. Without loss of generality, we can assume that the segment AO (with A included but O excluded) has measure $\leq \frac{1}{3}$. Then $f_\mu(A) \geq f_\mu(O) + \frac{1}{3}$, so $b(\mu) - a(\mu) \geq \frac{1}{3}$.

This is not a counterexample to the OP's question, however, since we have $a(A, B, C) = b(O) = 1$.

I had the same initial idea, because the first spaces I tried were $[0,1]$ and a circle, for which this was true. However, these spaces appear to be quite exceptional with this property, as none of the other examples i have tried share this property. Nice proof, though.

Although, I think the measure theory edit I just made allows for a more straight forward proof. Since $P_b = O$ suffices, we have $\operatorname{supp}(P_a) \subset \{A, B, C\}$. - **SmileyCraft**

2