问题 1. (1). Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathscr{P}(\mathbb{N})$, $\mu = \nu =$ counting measure. Define $f(m,n) = 1$ if $m = n$, $f(m,n) = -1$ if $m = n+1$, and $f(m,n) = 0$ otherwise. Then $\int |f| \, \mathrm{d}(\mu \times \nu) = \infty$, and $\iint f \, \mathrm{d}\mu \, \mathrm{d}\nu$ and $\iint f \, \mathrm{d}\nu \, \mathrm{d}\mu$ exist and are equal.

(2). Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}([0,1])$, μ is Lebesgue measure, and ν is counting measure. If $D = \{(x,x) : x \in [0,1]\}$ is the diagonal in $X \times Y$, then $\iint \chi_D \, \mathrm{d}\mu \, \mathrm{d}\nu$, $\iint \chi_D \, \mathrm{d}\nu \, \mathrm{d}\mu$, and $\int \chi_D \, \mathrm{d}(\mu \times \nu)$ are all unequal.

问题 2. Let $(X, \mathcal{G})$ and $(Y, \mathcal{H})$ are measure spaces, and $f : X \times Y \to \mathbb{R}$ be measurable with respect to the product measure space $(X \times Y, \sigma(\mathcal{G} \times \mathcal{H}))$. Show that $x \mapsto f(x,y)$ is a Borel measurable function.

解答 3. Fix $y_0 \in Y$ and let $g(x) = f(x, y_0)$. To show that g is measurable, it suffices to show that if $a < b$, then $g^{-1}(a,b) \in \mathcal{G}$. We have

$$g^{-1}(a,b) = \{x \in X : a < g(x) < b\} = \{x \in X : a < f(x,y_0) < b\}.$$

Since f is $\sigma(\mathcal{G} \times \mathcal{H})$-measurable, we know that

$$f^{-1}(a,b) = \{(x,y) \in X \times Y : a < f(x,y) < b\} \in \sigma(\mathcal{G} \times \mathcal{H}).$$

Recall that if $(W, \mathcal{M})$ and $(Z, \mathcal{N})$ are measure spaces and $E \in \sigma(\mathcal{M} \times \mathcal{N})$, then for any $w \in W$,

$$E_w = \{z \in Z : (w,z) \in E\} \in \mathcal{N}$$

and for any $z \in Z$,

$$E^z = \{w \in W : (w,z) \in E\} \in \mathcal{M}.$$

It follows that

$$g^{-1}(a,b) = f^{-1}(a,b)^{y_0} = \{x \in X : (x,y_0) \in f^{-1}(a,b)\} \in \mathcal{G}.$$

问题 4. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. Suppose that $A \times B \subset X \times Y$ is measurable with respect to the product measure $\mu \times \nu$ and $B \in \mathcal{B}$. Is $A \in \mathcal{A}$?

解答 5. If $\mathcal{A} = \mathcal{B} =$ Borel sigma algebra of $\mathbb{R}$, $\mu = \nu =$ Lebesgue measure and A is a Lebesgue measurable set in $\mathbb{R}$ which is not a Borel set then $A \times \mathbb{R}$ is measurable with respect to the product measure but $A \notin \mathcal{A}$. If you really meant that $A \times B$ is $\mathcal{A} \times \mathcal{B}$ then it does follow that $A \in \mathcal{A}$. This is part of Fubini's Theorem.