问题. (Folland) Suppose $a > 0$. Use Poisson summation formula to show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = \frac{\pi}{a} \frac{1 + \mathrm{e}^{-2\pi a}}{1 - \mathrm{e}^{-2\pi a}}.$$

Then substract a^{-2} from both sides and let $a \to 0$ to show that $\zeta(2) = \pi^2/6$, where ζ is the Riemann zeta function. If you have spair time, try to show $\zeta(4) = \pi^4/90$ using Plancherel formula, or

$$\theta(x) := \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2/x}$$

using Poisson summation formula.

问题. prove the following statements:

1. (Rudin) Find
$$\lim_{A \to \infty} \int_{-A}^{A} \frac{\sin \lambda t}{t} \mathrm{e}^{\mathrm{i} t x} \, \mathrm{d}t, \qquad x \in \mathbb{R}, \lambda > 0.$$

2. Give eamples of $f \in L^2$ such that $f \notin L^1$, but $\widehat{f} \in L^1$.

问题. (Folland) Suppose $f \in L^2(\mathbb{R})$,

1. If the L^2 derivative f' exists, then
$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \leq 4 \int_{\mathbb{R}} |xf(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}} |f'(x)|^2 \, \mathrm{d}x.$$

2. (Heisenberg's Inequality) For any $b, \beta \in \mathbb{R}$,
$$\int_{\mathbb{R}} (x - b)^2 |f(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}} (\xi - \beta)^2 \left|\widehat{f}(\xi)\right|^2 \, \mathrm{d}\xi \geq \frac{\|f\|_2^4}{16\pi^2}.$$

问题. Show that $\widehat{\widehat{f}(\xi)}(t) = f(-t)$, then $\mathscr{F}^4[f] = f$, where $\mathscr{F}[f] = \widehat{f}$ for $f \in L^2$.

问题. Let $\mathscr{A} = \left\{\mathscr{F}f : f \in L^1(\mathbb{R})\right\}$ is the space of Fourier transform. Let $C_0(\mathbb{R})$ be the space of continuous functions that tends to 0 at ∞. Prove

1. $\mathscr{A} \subsetneq C_0(\mathbb{R})$;
2. $\mathscr{A}$ dense in $C_0(\mathbb{R})$.