问题 1. (Rudin) Suppose μ is a complex Borel measure on $[0, 2\pi)$ (or on the unit circle T), and define

$$\widehat{\mu}(n) = \int \mathrm{e}^{-\mathrm{i}nt} \,\mathrm{d}\mu(t) \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

Assume that $\widehat{\mu}(n) \to 0$ as $n \to +\infty$ and prove that then $\widehat{\mu}(n) \to 0$ as $n \to -\infty$. Hint: The assumption also holds with $f \,\mathrm{d}\mu$ in place of $\mathrm{d}\mu$ if f is any trigonometric polynomial, hence if f is continuous, hence if f is any bounded Borel function, hence if $\mathrm{d}\mu$ is replaced by $\mathrm{d}|\mu|$.

问题 2. (Rudin) Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f_n \in L^1(\mu)$ for $n = 1, 2, 3, \cdots$, $f_n(x) \to f(x)$ *a.e.*, and there exists $p > 1$ and $C < \infty$ such that $\int_X |f_n|^p \,\mathrm{d}\mu < C$ for all n. Prove that

$$\lim_{n\to\infty} \int_X |f_n - f| \,\mathrm{d}\mu = 0.$$

问题 3. (Folland) we have proved that:

Let ν be a finite signed measure and μ a positive measure on $(X, \mathscr{A})$, Then $\nu \ll \mu$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

Show that, the above theorem may fail when ν is not finite.

问题 4. (Folland) Suppose that ν is a signed measure on $(X, \mathscr{A})$ and $E \in \mathscr{A}$. Then

a. $\nu^+(E) = \sup\{\nu(F) : E \in \mathscr{A}, F \subset E\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathscr{A}, F \subset E\}$.

b. $|\nu|(E) = \sup\left\{\sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \cdots, E_n \text{ are disjoint, and } \bigcup_{j=1}^n E_j = E\right\}$.

问题 5. (Folland) A measure μ on $(X, \mathscr{A})$ is called decomposable if there is a family $\mathcal{F} \subset \mathscr{A}$ with the following properties:

1. $\mu(F) < \infty$ for all $F \in \mathcal{F}$;
2. the members of $\mathcal{F}$ are disjoint and their union is X;
3. if $\mu(E) < \infty$ then $\mu(E) = \sum_{F\in\mathcal{F}} \mu(E \cap F)$;
4. if $E \subset X$ and $E \cap F \in \mathscr{A}$ for all $F \in \mathcal{F}$ then $E \in \mathscr{A}$.

Show that every σ-finite measure is decomposable.