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2

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Roger Temam

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NAVIER–STOKES EQUATIONS

THEORY AND NUMERICAL ANALYSIS

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FOREWORD

In the present work we derive a number of results concerned with the theory and numerical analysis of the Navier-Stokes equations for viscous incompressible fluids. We shall deal with the following problems: on the one hand, a description of the known results on the existence, the uniqueness and in a few cases the regularity of solutions in the linear and non-linear cases, the steady and time-dependent cases; on the other hand, the approximation of these problems by discretisation: finite difference and finite element methods for the space variables, finite differences and fractional steps for the time variable. The questions of stability and convergence of the numerical procedures are treated as fully as possible. We shall not restrict ourselves to these theoretical aspects: in particular, in the Appendix we give details of how to program one of the methods. All the methods we study have in fact been applied, but it has not been possible to present details of the effective implementation of all the methods. The theoretical results that we present (existence, uniqueness,...) are only very basic results and none of them is new; however we have tried as far as possible to give a simple and self-contained treatment. Energy and compactness methods lie at the very heart of the two types of problems we have gone into, and they form the natural link between them.

Let us give a more detailed description of the contents of this work: we consider first the linearized stationary case (Chapter 1), then the non-linear stationary case (Chapter 2), and finally the full non-linear time-dependent case (Chapter 3). At each stage we introduce new mathematical tools, useful both in themselves and in readiness for subsequent steps.

In Chapter 1, after a brief presentation of results on existence and uniqueness, we describe the approximation of the Stokes problem by various finite-difference and finite-element methods. This gives us an opportunity to introduce various methods of approximation of the divergence-free vector functions which are also vital for the numerical aspects of the problems studied in Chapters 2 and 3.

In Chapter 2 we introduce results on compactness in both the continuous and the discrete cases. We then extend the results obtained for the linear case in the preceding chapter to the non-linear case. The chapter ends with a proof of the non-uniqueness of solutions of the stationary Navier-Stokes equations, obtained by bifurcation and topological methods. The presentation is essentially self-contained.

Chapter 3 deals with the full non-linear time-dependent case. We first present a few results typical of the present state of the mathematical theory of the Navier-Stokes equations (existence and uniqueness theorems). We then present a brief introduction to the numerical aspects of the problem, combining the discretization of the space variables discussed in Chapter 1 with the usual methods of discretization for the time variable. The stability and convergence problems are treated by energy methods. We also consider the fractional step method and the method of artificial compressibility.

This brief description of the contents will suffice to show that this book is in no sense a systematic study of the subject. Many aspects of the Navier-Stokes equations are not touched on here. Several interesting approaches to the existence and uniqueness problems, such as semi-groups, singular integral operators and Riemannian manifolds methods, are omitted. As for the numerical aspects of the problem, we have not considered the particle approach nor the related methods developed by the Los Alamos Laboratory.

We have, moreover, restricted ourselves severely to the Navier-Stokes equations; a whole range of problems which can be treated by the same methods are not covered here. Nor are the difficult problems of turbulence and high Reynolds number flows.

The material covered by this book was taught at the University of Maryland in the first semester of 1972–3 as part of a special year on the Navier-Stokes equations and non-linear partial differential equations. The corresponding lecture notes published by the University of Maryland constitute the first version of this book.

I am extremely grateful to my colleagues in the Department of Mathematics and in the Institute of Fluid Dynamics and Applied Mathematics at the University of Maryland for the interest they showed in the elaboration of the notes. Direct contributions to the preparation of the manuscript were made by Arlett Williamson, and by Professors J. Osborn, J. Sather and P. Wolfe. I should like to thank them for correcting some of my mistakes in English and for their interesting comments and suggestions, all of which helped to improve the manuscript. Useful points were also made by Mrs Pelissier and by Messrs Fortin and Thomasset. Finally, I should like to express my thanks to the secretaries of the Mathematic Departments at Maryland and Orsay for all their assistance in the preparation of the manuscript.

CONTENTS

CHAPTER I

THE STEADY-STATE STOKES EQUATIONS

Introduction

In this chapter we study the stationary Stokes equations; that is, the stationary linearized form of the Navier-Stokes equations. The study of the Stokes equations is useful in itself, it also gives us an opportunity to introduce several tools necessary for a treatment of the full Navier-Stokes equations.

In Section 1 we consider some function spaces (spaces of divergence-free vector functions with L^2-components). In Section 2 we give the variational formulation of the Stokes equations and prove existence and uniqueness of solutions by the projection theorem. In Sections 3 and 4 we recall a few definitions and results on the approximation of a normed space and of a variational linear equations (Section 3). We then propose several types of approximation of a certain fundamental space V of divergence-free vector functions; this includes an approximation by the finite-difference method (Section 3), and by conforming and non-conforming finite-element methods (Section 4). In Section 5 we discuss certain approximation algorithms for the Stokes equations and the corresponding discretized equations. The purpose of these algorithms is to overcome the difficulty caused by the condition div $u = 0$. As it will be shown, this difficulty, sometimes, is not merely solved by discretization.

Finally in Section 6 we study the linearized equations of slightly compressible fluids and their asymptotic convergence to the linear equations of incompressible fluids (i.e., Stokes' equations).

§1. Some function spaces

In this section we introduce and study certain fundamental function spaces. The results are important for what follows, but the methods used in this section will not reappear so that the reader can skim through the proofs and retain only the general notation described in Section 1.1 and the results summarized in Remark 1.6.

1.1. *Notation*

In Euclidean space $\mathbb{R}^n$ we write $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, \ldots, 0, 1)$, the canonical basis, and $x = (x_1, \ldots, x_n)$,

$y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n)$, ..., will denote points of the space.

The differential operator

$$\frac{\partial}{\partial x_i}\,(1 \leqslant i \leqslant n),$$

will be written D_i and if $j = (j_1, \ldots j_n)$ is a multi-index, D^j will be the differentiation operator

$$D^j = D_1^{j_1} \ldots D_n^{j_n} = \frac{\partial^{[j]}}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}} \tag{1.1}$$

where

$$[j] = j_1 + \ldots + j_n. \tag{1.2}$$

If $j_i = 0$ for some i, $D_i^{j_i}$ is the identity operator; in particular if $[j] = 0$, D^j is the identity.

The set Ω

Let Ω be an open set of $\mathbb{R}^n$ with boundary Γ. In general we shall need some kind of smoothness property fór Ω. Sometimes we shall assume that Ω is smooth in the following sense:

> The boundary Γ is a $(n-1)$-dimensional manifold of class $\mathscr{C}^r$ ($r \geqslant 1$ which must be specified) and Ω is locally located on one side of Γ. (1.3)

We will say that a set Ω satisfying (1.3) is of class $\mathscr{C}^r$. However this hypothesis is too strong for practical situations (such as a flow in a square) and all the main results will be proved under a weaker condition:

> The boundary of Ω is locally Lipschitz. (1.4)

This means that in a neighbourhood of any point $x \in \Gamma$, Γ admits a representation as a hypersurface $y_n = \theta(y_1, \ldots, y_{n-1})$ where θ is a Lipschitz function, and $(y_1, \ldots, y_n)$ are rectangular coordinates in $\mathbb{R}^n$ in a basis that may be different from the canonical basis $e_1, \ldots, e_n$.

Of course if Ω is of class $\mathscr{C}^1$, then Ω is locally Lipschitz.

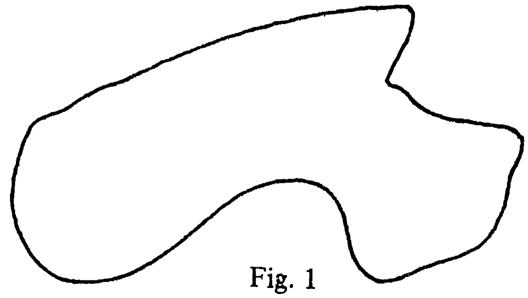

Fig. 1

It is useful for the sequel of this section to note that a set Ω satisfying (1.4) is "locally star-shaped". This means that each point $x_j \in \Gamma$, has an open neighbourhood $\mathcal{O}_j$, such that $\mathcal{O}'_j \doteq \Omega \cap \mathcal{O}_j$ is star-shaped with respect to one of its points. According to (1.4) we may, moreover, suppose that the boundary of $\mathcal{O}'_j$ is Lipschitz.

If Γ is bounded, it can be covered by a finite family of such sets $\mathcal{O}_j$, $j \in J$; if Γ is not bounded, the family $(\mathcal{O}_j)_{j \in J}$ can be chosen to be locally finite.

It will be assumed that Ω *will always satisfy* (1.4), unless we explicitly state that Ω is any open set in $\mathbb{R}^n$ or that some other smoothness property is required.

L^p and Sobolev Spaces

Let Ω be any open set in $\mathbb{R}^n$. We denote by $L^p(\Omega)$, $1 < p < +\infty$ (or $L^\infty(\Omega)$) the space of real functions defined on Ω with the p-th power absolutely integrable (or essentially bounded real functions) for the Lebesgue measure $dx = dx_1 \ldots dx_n$. This is a Banach space with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p} \tag{1.5}$$

or

$$\|u\|_{L^\infty(\Omega)} = \text{ess.}\sup_\Omega |u(x)|).$$

For $p = 2$, $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(u, v) = \int_\Omega u(x) v(x) \, dx. \tag{1.6}$$

The Sobolev space $W^{m,p}(\Omega)$ is the space of functions in $L^p(\Omega)$ with derivatives of order less than or equal to m in $L^p(\Omega)$ (m an integer, $1 \leqslant p \leqslant +\infty$). This is a Banach space with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{[j] \leqslant m} \|D^j u\|^p_{L^p(\Omega)} \right)^{1/p} \tag{1.7}$$

When $p = 2$, $W^{m,2}(\Omega) = H^m(\Omega)$ is a Hilbert space with the scalar product

$$((u, v))_{H^m(\Omega)} = \sum_{[j] \leqslant m} (D^j u, D^j v) \tag{1.8}$$

Let $\mathscr{D}(\Omega)$ (or $\mathscr{D}(\bar{\Omega})$) be the space of $\mathscr{C}^\infty$ functions with compact support contained in Ω (or $\bar{\Omega}$). The closure of $\mathscr{D}(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$ ($H_0^m(\Omega)$ when $p = 2$).

We recall, when needed, the classical properties of these spaces such as the density or trace theorems (assuming regularity properties for Ω).

We shall often be concerned with n-dimensional vector functions with components in one of these spaces. We shall use the notation

$$\boldsymbol{L}^p(\Omega) = \{L^p(\Omega)\}^n, \qquad \boldsymbol{W}^{m,p}(\Omega) = \{W^{m,p}(\Omega)\}^n$$

$$\boldsymbol{H}^m(\Omega) = \{H^m(\Omega)\}^n, \quad \boldsymbol{\mathscr{D}}(\Omega) = \{\mathscr{D}(\Omega)\}^n,$$

and we suppose that these product spaces are equipped with the usual product norm or an equivalent norm (except $\mathscr{D}(\Omega)$ and $\mathscr{D}(\bar{\Omega})$, which are not normed spaces).

The following spaces will appear very frequently

$$L^2(\Omega), \boldsymbol{L}^2(\Omega), H_0^1(\Omega), \boldsymbol{H}_0^1(\Omega).$$

The scalar product and the norm are denoted by $(\cdot,\cdot)$ and $|\cdot|$ on $L^2(\Omega)$ or $\boldsymbol{L}^2(\Omega)$ (or $[\cdot,\cdot]$ and $[\cdot]$ on $H_0^1(\Omega)$ or $\boldsymbol{H}_0^1(\Omega)$).

We recall that if Ω is *bounded in some direction*[1] then the Poincaré inequality holds:

$$\|u\|_{L^2(\Omega)} \leqslant c(\Omega)\|Du\|_{L^2(\Omega)}, \ \forall\, u \in H_0^1(\Omega), \tag{1.9}$$

---

[1] *i.e.,* Ω lies within a slab whose boundary is two hyperplanes which are orthogonal to this direction. The minimal distance of such pair of hyperplanes is called the *thickness* of Ω in the corresponding direction.

where D is the derivative in that direction and $c(\Omega)$ is a constant depending only on Ω which is bounded by $2l$, l = the diameter of Ω or the thickness of Ω in any direction. In this case the norm $[\cdot]$ on $H_0^1(\Omega)$ (or $\boldsymbol{H}_0^1(\Omega)$) is equivalent to the norm:

$$\|\boldsymbol{u}\| = \left(\sum_{i=1}^{n} |D_i \boldsymbol{u}|^2 \right)^{1/2} \tag{1.10}$$

The space $H_0^1(\Omega)$ (or $\boldsymbol{H}_0^1(\Omega)$) is also a Hilbert space with the associated scalar product

$$((\boldsymbol{u}, \boldsymbol{v})) = \sum_{i=1}^{n} (D_i \boldsymbol{u}, D_i \boldsymbol{v}). \tag{1.11}$$

This scalar product and this norm are denoted by $((\cdot,\cdot))$ and $\|\cdot\|$ on $H_0^1(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$ (Ω bounded in some direction).

Let $\mathscr{V}$ be the space (without topology)

$$\mathscr{V} = \{\boldsymbol{u} \in \mathscr{D}(\Omega), \operatorname{div} \boldsymbol{u} = 0\}. \tag{1.12}$$

The closures of $\mathscr{V}$ in $\boldsymbol{L}^2(\Omega)$ and in $\boldsymbol{H}_0^1(\Omega)$ are two basic spaces in the study of the Navier-Stokes equations; we denote them by H and V. The results of this section will allow us to give a characterization of H and V.

1.2. *A density theorem*

Let $E(\Omega)$ be the following auxiliary space:

$$E(\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \operatorname{div} \boldsymbol{u} \in L^2(\Omega)\}.$$

This is a Hilbert space when equipped with the scalar product

$$((\boldsymbol{u}, \boldsymbol{v}))_{E(\Omega)} = (\boldsymbol{u}, \boldsymbol{v}) + (\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{v}). \tag{1.13}$$

It is clear that (1.13) is a scalar product on $E(\Omega)$; it is easy to see that $E(\Omega)$ is complete for the associated norm [(1)]

$$\|\boldsymbol{u}\|_{E(\Omega)} = \{((\boldsymbol{u}, \boldsymbol{u}))_{E(\Omega)}\}^{1/2}$$

---

(1) For if $\boldsymbol{u}_m$ is a Cauchy sequence in $E(\Omega)$, then $\boldsymbol{u}_m$ is also a Cauchy sequence in $\boldsymbol{L}^2(\Omega)$; $\boldsymbol{u}_m$ converges to some limit $\boldsymbol{u}$ in $\boldsymbol{L}^2(\Omega)$ and $\operatorname{div} \boldsymbol{u}_m$ converges to some limit g in $L^2(\Omega)$, necessarily $g = \operatorname{div} \boldsymbol{u}$, and so $\boldsymbol{u} \in E(\Omega)$ and $\boldsymbol{u}_m$ converges to $\boldsymbol{u}$ in $E(\Omega)$.

Our goal is to prove a trace theorem: for $u \in E(\Omega)$ one can define the value on Γ of the normal component $u \cdot \nu$, ν = the unit vector normal to the boundary. The method we use is the classical Lions-Magenes [1] one. We begin by proving

Theorem 1.1. *Let Ω be a Lipschitz open set in $\mathbb{R}^n$. Then the set of vector functions belonging to $\mathcal{D}(\overline{\Omega})$ is dense in $E(\Omega)$.*

Proof. Let u be some element of $E(\Omega)$. We have to prove that u is a limit in $E(\Omega)$ of vector functions of $\mathcal{D}(\overline{\Omega})$.

(i) When Ω is not bounded we first approximate u by functions of $E(\Omega)$ with compact support in $\overline{\Omega}$ (i.e. functions with a bounded support).

Let $\phi \in \mathcal{D}(\mathbb{R}^n)$, $0 \leqslant \phi \leqslant 1$, $\phi = 1$ for $|x| \leqslant 1$, and $\phi = 0$ for $|x| \geqslant 2$. For $a > 0$ let ϕ_a be the restriction to Ω of the function $x \mapsto \phi(x/a)$. It is easy to check that $\phi_a u \in E(\Omega)$ and that $\phi_a u$ converges to u in this space as $a \to \infty$.

The functions with bounded support are a dense subspace of $E(\Omega)$ and we may assume that u has a bounded support.

(ii) Let us consider first the case $\Omega = \mathbb{R}^n$; hence $u \in E(\mathbb{R}^n)$ and u has a compact support.

The result is then proved by regularization. Let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a smooth $\mathscr{C}^\infty$ function with compact support, such that $\rho \geqslant 0$, $\int_{\mathbb{R}^n} \rho(x)\mathrm{d}x = 1$. For $\epsilon \in (0, 1)$, let ρ_ϵ denote the function $x \to (1/\epsilon^n)\,\rho(x/\epsilon)$. As $\epsilon \to 0$, ρ_ϵ converges in the distribution sense to the Dirac distribution and it is a classical result that[1]

$$\rho_\epsilon * v \to v \text{ in } L^2(\mathbb{R}^n), \forall\, v \in L^2(\mathbb{R}^n). \tag{1.14}$$

Now $\rho_\epsilon * u$ belongs to $\mathcal{D}(\mathbb{R}^n)$ since this function has a compact support ($\subset$(support ρ_ϵ) + support u) and components which are $\mathscr{C}^\infty$. According to (1.14) $\rho_\epsilon * u$ converges to u in $L^2(\mathbb{R}^n)$ as $\epsilon \to 0$, and

$$\operatorname{div}(\rho_\epsilon * u) = \rho_\epsilon * \operatorname{div} u \text{ converges to } \operatorname{div} u \text{ in } L^2(\mathbb{R}^n),$$

as $\epsilon \to 0$. Hence u is the limit in $E(\mathbb{R}^n)$ of functions of $\mathcal{D}(\mathbb{R}^n)$.

---

(1) $*$ is the convolution operator

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y$$

If $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then $f * g$ makes sense and belongs to $L^p(\mathbb{R}^n)$.

(iii) For the general case, $\Omega \neq \mathbb{R}^n$, we use the remark after (1.14): Ω is locally star-shaped. The sets Ω, $(\mathcal{O}_j)_{j\in J}$, form an open covering of $\bar{\Omega}$. Let us consider a partition of unity subordinated to this covering.

$$1 = \phi + \sum_{j\in J} \phi_j, \text{ where } \phi \in \mathscr{D}(\Omega),\ \phi_j \in \mathscr{D}(\mathcal{O}_j). \tag{1.15}$$

We may write

$$u = \phi u + \sum_{j\in J} \phi_j u,$$

the sum Σ_j is actually finite since the support of u is compact (in $\bar{\Omega}$).

Since the function ϕu has compact support in Ω it can be shown as in (ii) that ϕu is the limit in $E(\Omega)$ of functions belonging to $\mathscr{D}(\Omega)$ (the function ϕu extended by 0 outside Ω belongs to $E(\mathbb{R}^n)$ and for ϵ sufficiently small, $\rho_\epsilon * (\phi u)$ has compact support in Ω).

Let us consider now one of the functions $u_j = \phi_j u$ not identically equal to zero. The set $\mathcal{O}_j' = \mathcal{O}_j \cap \Omega$ is star-shaped with respect to one of its points; after a translation in $\mathbb{R}^n$ we can suppose this point is 0. Let σ_λ, $\lambda \neq 0$, be the linear (homothetic) transformation $x \to \lambda x$. It is clear, since $\mathcal{O}_j'$ is Lipschitz, and star-shaped with respect to 0, that:

$$\mathcal{O}_j' \subset \bar{\mathcal{O}}_j' \subset \sigma_\lambda \mathcal{O}_j' \qquad \text{for } \lambda > 1,$$

$$\sigma_\lambda \mathcal{O}_j' \subset \overline{\sigma_\lambda \mathcal{O}_j'} \subset \mathcal{O}_j' \quad \text{for } 0 < \lambda < 1.$$

Let $\sigma_\lambda \circ v$ denote the function $x \mapsto v(\sigma_\lambda(x))$; because of Lemma 1.1 below, the restriction to $\mathcal{O}_j'$ of the function $\sigma_\lambda \circ u_j$, $\lambda > 1$, converges to u_j in $E(\mathcal{O}_j')$ (or $E(\Omega)$) as $\lambda \mapsto 1$. But if $\psi_j \in \mathscr{D}(\sigma_\lambda(\mathcal{O}_j'))$ and $\psi_j = 1$ on $\mathcal{O}_j'$ the function $\psi_j(\sigma_\lambda \circ u)$ clearly belongs to $E(\mathbb{R}^n)$. Hence we must only approximate in place of the function u_j, a function $v_j \in E(\Omega)$ which is the restriction to Ω of a function $w_j \in E(\mathbb{R}^n)$ with compact support (take $w_j = \psi_j(\sigma_\lambda \circ u)$). The result follows then from point (ii). □

It remains only to prove Lemma 1.1 giving some results that we used in the above proof and other results which will be needed later.

Lemma 1.1. *Let $\mathcal{O}$ be an open set which is star-shaped with respect to* 0

(i) *If $p \in \mathcal{D}'(\mathcal{O})$ is a distribution in $\mathcal{O}$, then a distribution $\sigma_\lambda \circ p$ can be defined in $\mathcal{D}'(\sigma_\lambda \mathcal{O})$ by*

$$\langle \sigma_\lambda \circ p, \phi \rangle = \frac{1}{\lambda^n} \langle p, \sigma_{1/\lambda} \circ \phi \rangle, \ \forall \phi \in \mathcal{D}(\sigma_\lambda \mathcal{O}) \quad (\lambda > 0). \tag{1.16}$$

The derivatives of $\sigma_\lambda \circ p$ are related to the derivatives of p by the formula

$$D_i(\sigma_\lambda \circ p) = \lambda \sigma_\lambda \circ (D_i p), \quad 1 \leqslant i \leqslant n. \tag{1.17}$$

If $\lambda > 1$, $\lambda \to 1$, the restriction to $\mathcal{O}$ of $\sigma_\lambda \circ p$ converges in the distribution sense to p.

(ii) *If $p \in L^\alpha(\mathcal{O})$, $1 \leqslant \alpha < +\infty$, then $\sigma_\lambda \circ p \in L^\alpha(\sigma_\lambda \mathcal{O})$. For $\lambda > 1$, $\lambda \to 1$, the restriction to $\mathcal{O}$ of $\sigma_\lambda \circ p$ converges to p in $L^\alpha(\mathcal{O})$.*

Proof. (i) It is clear that the mapping $\phi \mapsto (1/\lambda^n) \langle p, \sigma_{1/\lambda} \circ \phi \rangle$ is linear and continuous on $\mathcal{D}(\sigma_\lambda \mathcal{O})$ and hence defines a distribution which is denoted by $\sigma_\lambda \circ p$.

The formula (1.17) is easy,

$$\begin{aligned} \langle D_i(\sigma_\lambda \circ p), \phi \rangle &= -\langle (\sigma_\lambda \circ p), D_i \phi \rangle \\ &= -\frac{1}{\lambda^n} \langle p, \sigma_{1/\lambda} \circ (D_i \phi) \rangle \\ &= -\frac{1}{\lambda^{n-1}} \langle p, D_i(\sigma_{1/\lambda} \circ \phi) \rangle \\ &= +\frac{1}{\lambda^{n-1}} \langle D_i p, \sigma_{1/\lambda} \circ \phi \rangle \\ &= \lambda \ \langle \sigma_\lambda \circ D_i p, \phi \rangle \end{aligned}$$

When $\lambda > 1$, $\lambda \to 1$, the functions $\sigma_{1/\lambda} \circ \phi$ have compact support in $\mathcal{O}$ for $\lambda - 1$ small enough, and converge to ϕ in $\mathcal{D}(\mathcal{O})$ as $\lambda \to 1$.

(ii) It is clear that

$$\int_{\sigma_\lambda \mathcal{O}} |(\sigma_\lambda p)(y)|^\alpha \, dy = \lambda^n \int_{\mathcal{O}} |p(x)|^\alpha \, dx,$$

$$\|\sigma_\lambda \circ p\|_{L^\alpha(\sigma_\lambda \mathcal{O})} = \|p\|_{L^\alpha(\mathcal{O})}.$$

It is then sufficient to prove that $\sigma_\lambda \circ p$ restricted to $\mathcal{O}$ converges to p, for the p's belonging to some dense subspace of $L^\alpha(\mathcal{O})$. But $\mathcal{D}(\mathcal{O})$ is dense in $L^\alpha(\mathcal{O})$, and the result is obvious if $p \in \mathcal{D}(\mathcal{O})$.

1.3. *A trace theorem*

We suppose here that Ω is an open bounded set of class $\mathscr{C}^2$. It is known that there exists a linear continuous operator $\gamma_0 \in \mathcal{L}(H^1(\Omega), L^2(\Gamma))$ (the trace operator), such that $\gamma_0 u$ = the restriction of u to Γ for every function $u \in H^1(\Omega)$ which is twice continuously differentiable in $\overline{\Omega}$. The space $H_0^1(\Omega)$ is equal to the kernel of γ_0. The image space $\gamma_0(H^1(\Omega))$ is a dense subspace of $L^2(\Gamma)$ denoted $H^{1/2}(\Gamma)$; the space $H^{1/2}(\Gamma)$ can be equipped with the norm carried from $H^1(\Omega)$ by γ_0. There exists moreover a linear continuous operator $\ell_\Omega \in \mathcal{L}(H^{1/2}(\Gamma), H^1(\Omega))$ (which is called a lifting operator), such that $\gamma_0 \circ \ell_\Omega$ = the identity operator in $H^{1/2}(\Gamma)$. All these results are given in Lions [1], Lions & Magenes [1].

We want to prove a similar result for the vector functions in $E(\Omega)$.

Let $H^{-1/2}(\Gamma)$ be the dual space of $H^{1/2}(\Gamma)$; since $H^{1/2}(\Gamma) \subset L^2(\Gamma)$ with a stronger topology, $L^2(\Gamma)$ is contained in $H^{-1/2}(\Gamma)$ with a stronger topology. We have the following trace theorem (which means that we can define $\boldsymbol{u}\cdot\nu|_\Gamma$ when $\boldsymbol{u} \in E$):

Theorem 1.2. *Let Ω be an open bounded set of class $\mathscr{C}^2$. Then there exists a linear continuous operator $\gamma_\nu \in \mathcal{L}(E(\Omega), H^{-1/2}(\Gamma))$ such that*

$$\gamma_\nu \boldsymbol{u} = \text{the restriction of } \boldsymbol{u}\cdot\nu \text{ to } \Gamma, \text{ for every } \boldsymbol{u} \in \mathcal{D}(\overline{\Omega}). \tag{1.18}$$

The following generalized Stokes formula is true for all $\boldsymbol{u} \in E(\Omega)$ and $w \in H^1(\Omega)$

$$(\boldsymbol{u}, \operatorname{grad} w) + (\operatorname{div} \boldsymbol{u}, w) = \langle \gamma_\nu \boldsymbol{u}, \gamma_0 w \rangle \tag{1.19}$$

Proof. Let $\phi \in H^{1/2}(\Gamma)$ and let $w \in H^1(\Omega)$ with $\gamma_0 w = \phi$. For $\boldsymbol{u} \in E(\Omega)$, let us set

$$X_{\boldsymbol{u}}(\phi) = \int_\Omega [\operatorname{div} \boldsymbol{u}(x)\, w(x) + \boldsymbol{u}(x) \operatorname{grad} w(x)]\, dx$$

$$= (\operatorname{div} \boldsymbol{u}, w) + (\boldsymbol{u}, \operatorname{grad} w).$$

Lemma 1.2. *$X_{\boldsymbol{u}}(\phi)$ is independent of the choice of w, as long as $w \in H^1(\Omega)$ and $\gamma_0 w = \phi$.*

Proof. Let w_1 and w_2 belong to $H^1(\Omega)$, with

$$\gamma_0 w_1 = \gamma_0 w_2 = \phi$$

and let $w = w_1 - w_2$.

We must prove that

$$(\operatorname{div} \boldsymbol{u}, w_1) + (\boldsymbol{u}, \operatorname{grad} w_1) = (\operatorname{div} \boldsymbol{u}, w_2) + (\boldsymbol{u}, \operatorname{grad} w_2),$$

that is to say

$$(\operatorname{div} \boldsymbol{u}, w) + (\boldsymbol{u}, \operatorname{grad} w) = 0. \tag{1.20}$$

But since $w \in H^1(\Omega)$ and $\gamma_0 w = 0$, w belongs to $H_0^1(\Omega)$ and is the limit in $H^1(\Omega)$ of smooth functions with compact support: $w = \lim w_m$, $w_m \in \mathscr{D}(\Omega)$. It is obvious that

$$(\operatorname{div} \boldsymbol{u}, w_m) + (\boldsymbol{u}, \operatorname{grad} w_m) = 0, \quad \forall\, w_m \in \mathscr{D}(\Omega)$$

and (1.20) follows as $m \to \infty$. $\square$

Let us take now $w = \ell_\Omega \phi$ (see above). Then by the Schwarz inequality

$$|X_{\boldsymbol{u}}(\phi)| \leqslant \|\boldsymbol{u}\|_{E(\Omega)} \|w\|_{H^1(\Omega)},$$

and since $\ell_\Omega \in \mathcal{L}(H^{1/2}(\Gamma), H^1(\Omega))$

$$|X_{\boldsymbol{u}}(\phi)| \leqslant c_0 \|\boldsymbol{u}\|_{E(\Omega)} \|\phi\|_{H^{1/2}(\Gamma)}, \tag{1.21}$$

where c_0 = the norm of the linear operator ℓ_Ω.

Therefore the mapping $\phi \mapsto X_{\boldsymbol{u}}(\phi)$ is a linear continuous mapping from $H^{1/2}(\Gamma)$ into $\mathcal{R}$. Thus there exists $g = g(\boldsymbol{u}) \in H^{-1/2}(\Gamma)$ such that

$$X_{\boldsymbol{u}}(\phi) = \langle g, \phi \rangle. \tag{1.22}$$

It is clear that the mapping $\boldsymbol{u} \mapsto g(\boldsymbol{u})$ is linear and, by (1.21),

$$\|g\|_{H^{-1/2}(\Gamma)} \leqslant c_0 \|\boldsymbol{u}\|_{E(\Omega)}; \tag{1.23}$$

this proves that the mapping $\boldsymbol{u} \mapsto g(\boldsymbol{u}) = \gamma_\nu \boldsymbol{u}$ is continuous from $E(\Omega)$ into $H^{-1/2}(\Gamma)$.

The last point is to prove (1.18) since (1.19) follows directly from the definition of $\gamma_\nu \boldsymbol{u}$.

Lemma 1.3. *If $\boldsymbol{u} \in \mathscr{D}(\bar{\Omega})$, then*

$$\gamma_\nu \boldsymbol{u} = \textit{the restriction of } \boldsymbol{u} \cdot \nu \textit{ on } \Gamma.$$

Proof. For such a smooth u and for any $w \in \mathscr{D}(\bar{\Omega})$ (or if u and w are twice continuously differentiable in $\bar{\Omega}$),

$$X_u(\gamma_0 w) = \int_\Omega \operatorname{div}(u\, w)\, dx$$

$$= \int_\Gamma w(u \cdot \nu) d\Gamma = \int_\Gamma (u \cdot \nu)(\gamma_0 w) d\Gamma \quad \text{(by the Stokes formula)}$$

$$= \langle u \cdot \nu, \gamma_0 w \rangle .$$

Since for these functions w, the traces $\gamma_0 w$ form a dense subset of $H^{1/2}(\Gamma)$, the formula

$$X_u(\phi) = \langle u \cdot \nu, \phi \rangle$$

is also true by continuity for every $\phi \in H^{1/2}(\Gamma)$. By comparison with (1.22), we get $\gamma_\nu u = u \cdot \nu|_\Gamma$.

Remark 1.1. Theorem 1.1 is not explicitly used in the proof of Theorem 1.2, but the density theorem combined with Lemma 1.3 shows that the operator γ_ν is unique since its value on a dense subset is known.

Remark 1.2. The operator γ_ν actually maps $E(\Omega)$ *onto* $H^{-1/2}(\Gamma)$.

Let ϕ be given in $H^{-1/2}(\Gamma)$, such that $\langle \phi, 1 \rangle = 0$. Then the Neumann problem

$$\begin{aligned} &\Delta p = 0 \text{ in } \Omega \\ &\frac{\partial p}{\partial \nu} = \phi \text{ on } \Gamma \end{aligned} \tag{1.24}$$

has a weak solution $p = p(\phi) \in H^1(\Omega)$ which is unique up to an additive constant (See Lions & Magenes [1]). For one of these solutions p let

$$u = \operatorname{grad} p$$

It is clear that $u \in E(\Omega)$ and $\gamma_\nu u = \phi$. In addition it is clear that there exists a vector function u_0 with components in $\mathscr{C}^1(\bar{\Omega})$ such that $\gamma_\nu u_0 = 1$. Then for any ψ in $H^{1/2}(\Gamma)$, writing

$$\psi = \phi + \frac{\langle \psi, 1\rangle}{\text{mes}\,\Gamma}, \quad \phi = \psi - \frac{\langle \psi, 1\rangle}{\text{mes}\,\Gamma}, \tag{1.25}$$

one can define a $u = u(\phi)$ such that $\gamma_\nu u = \psi$ by setting

$$u = \operatorname{grad} p(\phi) + \frac{\langle \psi, 1\rangle}{\text{mes}\,\Gamma} u_0. \tag{1.26}$$

Moreover the mapping $\psi \mapsto u(\psi)$ is a linear continuous mapping from $H^{-1/2}(\Gamma)$ into $E(\Omega)$ (i.e., a *lifting operator as* ℓ_Ω). $\square$

Let $E_0(\Omega)$ be the closure of $\mathscr{D}(\Omega)$ in $E(\Omega)$. We have

Theorem 1.3. *The kernel of* γ_ν *is equal to* $E_0(\Omega)$.

Proof. If $u \in E_0(\Omega)$, then by the definition of this space, there exists a sequence of functions $u_m \in \mathscr{D}(\Omega)$ which converges to u in $E(\Omega)$ as $m \to \infty$. Theorem 1.2 implies that $\gamma_\nu u_m = 0$ and hence $\gamma_\nu \cdot u = \lim_{m\to\infty} \gamma_\nu \cdot u_m = 0$.

Conversely let us prove that if $u \in E(\Omega)$ and $\gamma_\nu \cdot u = 0$, then u is the limit in $E(\Omega)$ of vector functions in $\mathscr{D}(\Omega)^n$.

Let Φ be any function in $\mathscr{D}(\mathbb{R}^n)$, and ϕ the restriction of Φ to Ω. Since $\gamma_\nu \cdot u = 0$, we have $\langle \gamma_\nu \cdot u, \gamma_0 \phi\rangle = 0$ which means

$$\int_\Omega [\operatorname{div} u \cdot \phi + u \cdot \operatorname{grad} \phi]\, dx = 0.$$

Hence

$$\int_{\mathbb{R}^n} [\widetilde{\operatorname{div} u} \cdot \Phi + \tilde{u} \cdot \operatorname{grad} \Phi]\, dx = 0, \quad \forall \Phi \in \mathscr{D}(\mathbb{R}^n)$$

and so

$$\operatorname{div} \tilde{u} = \widetilde{\operatorname{div} u}, \tag{1.27}$$

where $\tilde{v}$ denotes the function equal to v in Ω and to 0 in $\complement\Omega$. This implies that $\tilde{u} \in E(\mathbb{R}^n)$.

Following exactly the same steps as in proving Theorem 1.1 (in particular points (i) and (ii)) we may reduce the general case to the case where the function u has its support included in one of the sets $\mathcal{O}_j \cap \bar{\Omega}$. For such a function u we remark that $\tilde{u} \in E(\mathbb{R}^n)$ and that $\sigma_\lambda \cdot \tilde{u}$ has a compact support in $\mathcal{O}'_j$, for $0 < \lambda < 1$ ($\mathcal{O}'_j$ is supposed star-shaped with respect to 0). According to Lemma 1.1, $\sigma_\lambda \circ \tilde{u}$ restricted to $\mathcal{O}'_j$ (or Ω) converges to u in $E(\mathcal{O}'_j)$ (or $E(\Omega)$) as $\lambda \to 1$. The problem is then reduced to approximating a function u with compact support in Ω in terms of elements of $\mathcal{D}(\mathbb{R}^n)$; this is obvious by regularization (as in point (ii) in the proof of Theorem 1.1).

Remark 1.3. If the set Ω is unbounded or if its boundary is not smooth, some partial results remain true: for example, if $u \in E(\Omega)$, we can define $\gamma_\nu u$ on each bounded part Γ_0 of Γ of class $\mathscr{C}^2$, and $\gamma_\nu u \in H^1(\Gamma_0)$. If Ω is smooth but unbounded or if its boundary is the union of a finite number of bounded $(n-1)$-dimensional manifolds of class $\mathscr{C}^2$, then $\gamma_\nu u$ is defined, in this way, on all Γ. Nevertheless the generalized Stokes formula (1.19) does not hold here.

The results will be more precise if we know more about the trace of functions in $H^1(\Omega)$. Let us assume that the following results hold:

– there exists $\gamma_0 \in \mathcal{L}(H^1(\Omega), L^2(\Gamma))$ such that $\gamma_0 u = u_{|\Gamma}$ for every $u \in \mathcal{D}(\Omega)$. We denote $\gamma_0(H^1(\Gamma))$ by $\mathcal{H}^{1/2}(\Gamma)$.

– there exists a lifting operator $\ell_\Omega \in \mathcal{L}(\mathcal{H}^{1/2}(\Gamma), H^1(\Omega))$, such that $\gamma_0 \circ \ell_\Omega =$ the identity; $\mathcal{H}^{1/2}(\Gamma)$ is equipped with the norm carried by γ_0.

– Ω is a Lipschitz set.

Then all the preceding results can be extended to this case. Theorems 1.1 and 1.3 are true. The proof of Theorem 1.2 leads to a definition of $\gamma_\nu \cdot u$ as an element of $\mathcal{H}^{-1/2}(\Gamma) =$ the dual space of $\mathcal{H}^{1/2}(\Gamma)$. The generalized Stokes formula (1.19) is valid.

1.4. *Characterization of the spaces H and V*

We recall the notation at the end of Section 1.1:

$$\mathscr{V} = \{u \in \mathcal{D}(\Omega), \operatorname{div} u = 0\},$$

$$H = \text{the closure of } \mathscr{V} \text{ in } L^2(\Omega),$$

$$V = \text{the closure of } \mathscr{V} \text{ in } H_0^1(\Omega).$$

Characterization of the gradient of a distribution.

Let Ω be an open set in $\mathbb{R}^n$ and let p be a distribution on Ω, $p \in \mathscr{D}'(\Omega)$. It is easy to check that for any $v \in \mathscr{V}$ we have

$$\langle \operatorname{grad} p,\ v\rangle = \sum_{i=1}^{n} \langle D_i p,\ v_i\rangle = -\sum_{i=1}^{n} \langle p, D_i v_i\rangle = -\langle p, \operatorname{div} v\rangle = 0. \quad (1.28)$$

The converse of this property is true. This important property can be proved by using a profound result of De Rham(1):

Proposition 1.1. *Let Ω be an open set of $\mathbb{R}^n$ and $f = \{f_1, \ldots, f_n\}$, $f_i \in \mathscr{D}'(\Omega)$, $i = 1, \ldots, n$. A necessary and sufficient condition that*

$$f = \operatorname{grad} p, \quad (1.29)$$

for some p in $\mathscr{D}'(\Omega)$, is that

$$\langle f, v\rangle = 0 \quad \forall\, v \in \mathscr{V}. \quad (1.30)$$

This result is essential for the interpretation, later, of the variational formulation of Navier-Stokes and related equations.

Let us now state a number of connected results.

Proposition 1.2. *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^n$.*

(i) *If a distribution p has all its first-order derivatives $D_i p$, $1 \leqslant i \leqslant n$, in $L^2(\Omega)$, then $p \in L^2(\Omega)$ and*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant c(\Omega)\|\operatorname{grad} p\|_{L^2(\Omega)}. \quad (1.31)$$

---

(1) We sketch an outline of this proof which lies outside the scope of this book. We consider the currents

$$f = \sum_{i=1}^{n} f_i \mathrm{d}x_i,\ f_i \in \mathfrak{D}'(\Omega),$$

and the $\mathscr{C}^\infty$ forms of degree $(n-1)$ with compact support:

$$\tilde{v} = \sum_{i=1}^{n} (-1)^i v_i\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \ldots \wedge \mathrm{d}x_n,\ v_i \in \mathfrak{D}(\Omega).$$

Then $\mathrm{d}\tilde{v} = 0$ if and only if $v = \{v_1, \ldots, v_n\} \in V$ (i.e. $\operatorname{div} v = 0$). Theorem 17′ on p.114 of De Rham [1] states that for $f \in \mathfrak{D}'(\Omega)$, $f = \mathrm{d}g$, $g \in \mathfrak{D}'(\Omega)$ (i.e. $f_i = D_i g$, $1 \leqslant i \leqslant n$), if and only if $\langle \tilde{f}, \tilde{v}\rangle = 0$ for every $\tilde{v}$ of the previous type. This is equivalent to the statement in Proposition 1.1. (J.L. Lions, private communication).

Cf. in Remark 1.9 an alternate proof of Proposition 1.1 based on elementary arguments and Proposition 1.2. Cf. also the comments.

(ii) *If a distribution p has all its first derivatives $D_i p$, $1 \leqslant i \leqslant n$, in $H^{-1}(\Omega)$, then $p \in L^2(\Omega)$ and*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant c(\Omega)\|\operatorname{grad} p\|_{H^{-1}(\Omega)}. \tag{1.32}$$

In both cases, if Ω is any open set in $\mathbb{R}^n$, $p \in L^2_{\mathrm{loc}}(\Omega)$.

Proof. Point (i) and (1.31) are proved in Deny & Lions [1] for a bounded star-shaped open set Ω. In our case, because of this result, p is L^2 on every sphere contained in Ω with its closure, and on all the sets $\mathcal{O}_j'$ defined following (1.4). Since a finite number of these balls and sets $\mathcal{O}_j'$ cover Ω the result follows.

Point (ii) is proved in Magenes & Stampacchia [1] if Ω is of class $\mathscr{C}^1$ and in J. Nečas [2] if Ω is only lipschitnian.

For a set without any regularity property, we apply the foregoing results on each ball contained in Ω with its closure, and we obtain merely that $p \in L^2_{\mathrm{loc}}(\Omega)$.

Remark 1.4. (i) Combining the results of Propositions 1.1 and 1.2, we see that if $f \in \boldsymbol{H}^{-1}(\Omega)$ (or $\boldsymbol{L}^2_{\mathrm{loc}}(\Omega)$) and $(f, v) = 0$, $\forall\, \boldsymbol{v} \in \mathscr{V}$, then $f = \operatorname{grad} p$ with $p \in L^2_{\mathrm{loc}}(\Omega)$. If moreover, Ω is a Lipschitz open bounded set, then $p \in L^2(\Omega)$ (or $H^1(\Omega)$).

(ii) Point (ii) in Proposition 1.2 implies that the gradient operator is an isomorphism from $L^2(\Omega)/\mathbb{R}$ into $H^{-1}(\Omega)$; hence the range of this linear operator is closed. We recall (Ω bounded) that $L^2(\Omega)/\mathbb{R}$ is isomorphic to the subspace of $L^2(\Omega)$ orthogonal to the constants

$$L^2(\Omega)/\mathbb{R} = \left\{ p \in L^2(\Omega),\ \int_\Omega p(x)\mathrm{d}x = 0 \right\}.$$

See also in (6.12), a different version of (1.32).

Characterization of the space H

We can now give the following characterization of H and $H^\perp$ (the orthogonal complement of H in $\boldsymbol{L}^2(\Omega)$).

Theorem 1.4. *Let Ω be a Lipschitz open bounded set in $\mathbb{R}^n$. Then:*

$$H^\perp = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{u} = \operatorname{grad} p,\ p \in H^1(\Omega)\}, \tag{1.33}$$

$$H = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \operatorname{div} \boldsymbol{u} = 0,\ \gamma_\nu \boldsymbol{u} = 0\}. \tag{1.34}$$

Proof of (1.33). Let $\boldsymbol{u}$ belong to the space in the right-hand side of (1.33). Then for all $\boldsymbol{v} \in \mathscr{V}$,

$$(\boldsymbol{u}, \boldsymbol{v}) = (\operatorname{grad} p, \boldsymbol{v}) = -(p, \operatorname{div} \boldsymbol{v}) = 0\,. \tag{1.35}$$

Hence, $\boldsymbol{u} \in H^{\perp}$.

Conversely, $H^{\perp}$ is contained in the space in the right-hand side of (1.33). Indeed if $\boldsymbol{u} \in H^{\perp}$, then,

$$(\boldsymbol{u}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \mathscr{V}$$

and according to Proposition 1.1, $\boldsymbol{u} = \operatorname{grad} p$, $p \in \mathscr{D}'(\Omega)$. Then, by Proposition 1.2, $p \in H^1(\Omega)$.

Proof of (1.34). Let $H^{\bullet}$ be the space in the right-hand term of (1.34) and let us prove that $H \subset H^{\bullet}$. If $\boldsymbol{u} \in H$, then $\boldsymbol{u} = \lim_{m\to\infty} \boldsymbol{u}_m$, $\boldsymbol{u}_m \in \mathscr{V}$, this convergence in $\boldsymbol{L}^2(\Omega)$ implies convergence in the distribution sense; since differentiation is a continuous operator in distribution space and $\operatorname{div} \boldsymbol{u}_m = 0$, we see that $\operatorname{div} \boldsymbol{u} = 0$. Since $\operatorname{div} \boldsymbol{u}_m = \operatorname{div} \boldsymbol{u} = 0$, $\boldsymbol{u}_m$ and $\boldsymbol{u}$ belong to $E(\Omega)$ and

$$\|\boldsymbol{u}-\boldsymbol{u}_m\|_{E(\Omega)} = \|\boldsymbol{u}-\boldsymbol{u}_m\|_{\boldsymbol{L}^2(\Omega)} \tag{1.36}$$

so that $\boldsymbol{u}_m$ converges to $\boldsymbol{u}$ in $E(\Omega)$ and $\gamma_\nu \boldsymbol{u} = \lim_{m\to\infty} \gamma_\nu \boldsymbol{u}_m = 0$ ($\gamma_\nu \boldsymbol{u}_m = \boldsymbol{u}_m \cdot \boldsymbol{\nu}_{|\Gamma} = 0 \ \forall \boldsymbol{u}_m \in \mathscr{V}$). Hence $\boldsymbol{u} \in H^{\bullet}$.

Let us suppose that H is not the whole space $H^{\bullet}$ and let $H^{\bullet\bullet}$ be the orthogonal complement of H in $H^{\bullet}$. By (1.33) every $\boldsymbol{u} \in H^{\bullet\bullet}$ is the gradient of some $p \in H^1(\Omega)$; moreover p satisfies

$$\Delta p = \operatorname{div} \boldsymbol{u} = 0, \qquad \boldsymbol{u} \cdot \boldsymbol{\nu}_{|\Gamma} = \left.\frac{\partial p}{\partial \nu}\right|_{\Gamma} = 0, \tag{1.37}$$

and this implies that p is a constant and $\boldsymbol{u} = 0$; therefore $H^{\bullet\bullet} = \{0\}$ and $H = H^{\bullet}$.

Remark 1.5. If Ω is any open set in $\mathbb{R}^n$, the proof of (1.33) with very slight modifications shows that

$$H^{\perp} = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{u} = \operatorname{grad} p,\ p \in L^2_{\mathrm{loc}}(\Omega)\}. \tag{1.38}$$

If Ω is unbounded but satisfies condition (1.4) then

$$H^{\perp} = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{u} = \operatorname{grad} p,\ p \in L^2_{\mathrm{loc}}(\bar{\Omega})\}. \tag{1.39}$$

Theorem 1.5. *Let Ω be an open bounded set of class $\mathscr{C}^2$. Then*

$$L^2(\Omega) = H \oplus H_1 \oplus H_2, \tag{1.40}$$

where H, H_1, H_2 *are mutually orthogonal spaces*

$$H_1 = \{u \in L^2(\Omega),\ u = \operatorname{grad} p,\ p \in H^1(\Omega),\ \Delta p = 0\}, \tag{1.41}$$

$$H_2 = \{u \in L^2(\Omega),\ u = \operatorname{grad} p,\ p \in H_0^1(\Omega)\}. \tag{1.42}$$

Proof. It is clear that H_1 and H_2 are included in $H^\perp$, and that the intersection of any two of the spaces H, H_1, H_2, is effectively reduced to $\{0\}$.

The spaces H_1 and H_2 are orthogonal; if $u = \operatorname{grad} p \in H_1$, $v = \operatorname{grad} q \in H_2$, then $u \in E(\Omega)$ and by using the generalized Stokes formula (1.19),

$$(u, v) = (u, \operatorname{grad} q) = \langle \gamma_\nu u, \gamma_0 q \rangle - (\operatorname{div} u, q)$$

and this vanishes since $\gamma_0 q = 0$ and $\operatorname{div} u = \Delta p = 0$.

It remains for us to prove that any element u of $L^2(\Omega)$ can be written as the sum of elements u_0, u_1, u_2 of H, H_1, H_2. For such a u let p be the unique solution of the Dirichlet problem

$$\Delta p = \operatorname{div} u \in H^{-1}(\Omega), \quad p \in H_0^1(\Omega).$$

We take

$$u_2 = \operatorname{grad} p. \tag{1.43}$$

Let then q be the solution of the Neumann problem

$$\Delta q = 0, \quad \left.\frac{\partial q}{\partial \nu}\right|_\Gamma = \gamma_\nu (u - \operatorname{grad} p). \tag{1.44}$$

We notice that $\operatorname{div}(u - \operatorname{grad} p) = 0$, so that $u - \operatorname{grad} p \in E(\Omega)$, and $\gamma_\nu(u - \operatorname{grad} p)$ is defined as an element of $H^{-1/2}(\Gamma)$, and by the Stokes formula (1.19)

$$\langle \gamma_\nu(u - \operatorname{grad} p), 1 \rangle = \int_\Omega \operatorname{div}(u - \operatorname{grad} p)\, dx = 0.$$

According to a result of Lions & Magenes [1], the Neumann problem (1.44) possesses a solution which is unique to within an additive constant We take

$$u_1 = \operatorname{grad} q, \tag{1.45}$$

$$u_0 = u - u_1 - u_2. \tag{1.46}$$

We have now to show that $u_0 \in H$. But $\operatorname{div} u_0 = \operatorname{div}(u - u_1 - u_2) = \operatorname{div} u - \Delta p = 0$, and

$$\gamma_\nu u_0 = \gamma_\nu(u - u_1 - u_2) = \gamma_\nu(u - \operatorname{grad} p) - \partial q/\partial\nu = 0.$$

Remark 1.6. (i) Let us write P_H for the orthogonal projector in $L^2(\Omega)$ onto H; obviously P_H is continuous into $L^2(\Omega)$. In fact P_H also maps $H^1(\Omega)$ into itself and is continuous for the norm of $H^1(\Omega)$. In the proof of Theorem 1.5, let us assume that $u \in H^1(\Omega)$; then $\operatorname{div} u \in H_0^1(\Omega) \cap H^2(\Omega)$; $u - \operatorname{grad} p$ belongs to $H^1(\Omega)$ and $\gamma_\nu(u - \operatorname{grad} p)$ belongs to $H^{1/2}(\Gamma)$. Finally we infer from (1.44) that $q \in H^2(\Omega)$, and

$$P_H u = u - \operatorname{grad}(p+q) \in H^1(\Omega).$$

It is clear also (see for instance Lions & Magenes [1]) that the mappings $u \mapsto p$, $u - \operatorname{grad} p \mapsto q$, are continuous in the appropriate spaces and we conclude that P_H is continuous from $H_0^1(\Omega)$ into $H^1(\Omega)$:

$$\|P_H u\|_{H^1(\Omega)} \leqslant c(\Omega)\, \|u\|_{H^1(\Omega)},\ \forall\, u \in H^1(\Omega). \tag{1.47}$$

If Ω is of class $\mathscr{C}^{r+1}$, r integer $\geqslant 1$, a similar argument shows that if $u \in H^r(\Omega)$ then $P_H u \in H^r(\Omega)$ and P_H is linear and continuous for the norm of $H^r(\Omega)$:

$$\|P_H u\|_{H^r(\Omega)} \leqslant c(r, \Omega)\, \|u\|_{H^r(\Omega)},\ \forall\, u \in H^r(\Omega). \tag{1.48}$$

(ii) An orthogonal decomposition of H (appearing when Ω is multi-connected) is given in Appendix I.

Characterization of the space V.

Theorem 1.6. *Let Ω be an open bounded Lipschitz set. Then*

$$V = \{u \in H_0^1(\Omega),\ \operatorname{div} u = 0\}. \tag{1.49}$$

Proof. Let $V^\bullet$ be the space in the right-hand side of (1.49). It is clear that $V \subset V^\bullet$, for if $u \in V$, $u = \lim u_m$, $u_m \in \mathscr{V}$; this convergence in $H_0^1(\Omega)$ implies that $\operatorname{div} u_m$ converges to $\operatorname{div} u$ as $m \to \infty$, and since $\operatorname{div} u_m = 0$, $\operatorname{div} u = 0$.

To prove that $V = V^\bullet$, we will show that any continuous linear form L on $V^\bullet$ which vanishes on V is identically equal to 0. We first observe that L admits a (non-unique) representation of the type

$$L(v) = \sum_{i=1}^{n} \langle \ell_i, v_i \rangle,\quad \ell_i \in H^{-1}(\Omega). \tag{1.50}$$

Indeed $V^{\bullet}$ is a closed subspace of $\boldsymbol{H}_0^1(\Omega) = [H_0^1(\Omega)]^n$; any linear continuous form on $V^{\bullet}$ can be extended as a linear continuous form on $\boldsymbol{H}_0^1(\Omega)$ and such form is of the same type as the form in the right-hand side of (1.50).

Now, the vector distribution $\ell = (\ell_1, \ldots, \ell_n)$ belongs to $\boldsymbol{H}^{-1}(\Omega)$, and $\langle \ell, v\rangle = 0$, $\forall\, v \in \mathscr{V}$. Propositions 1.1 and 1.2 are applicable and show that $\ell = \operatorname{grad} p, p \in L^2(\Omega)$; thus

$$\langle \ell_i, v_i\rangle = \langle D_i p, v_i\rangle = -(p, D_i v_i),\ \ \forall\, v_i \in H_0^1(\Omega).$$

Therefore L vanishes on $V^{\bullet}$, since

$$L(\boldsymbol{v}) = \sum_{i=1}^{n} \langle \ell_i, v_i\rangle = -(p, \operatorname{div} \boldsymbol{v}) = 0,\ \ \forall \boldsymbol{v} \in \boldsymbol{V}^{\bullet}.$$

Remark 1.7 (i) Let us assume that Ω is a bounded open set which is globally star-shaped with respect to one of its points, say 0, and such that $\sigma_\lambda \Omega \subset \Omega$, for every $0 < \lambda < 1$, where σ_λ denotes as before the linear transformation $\boldsymbol{x} \to \lambda \boldsymbol{x}$.

In such a case we can give a more direct and much simpler proof of (1.49).

Let $\boldsymbol{u} \in V^{\bullet}$. Then the function $\sigma_\lambda \circ \boldsymbol{u}$ belongs to $\boldsymbol{H}_0^1(\sigma_\lambda \Omega)$ and $\operatorname{div} \sigma_\lambda \boldsymbol{u} = 0$. The function $\boldsymbol{u}_\lambda$, equal to $\sigma_\lambda \circ \boldsymbol{u}$ in $\sigma_\lambda \Omega$ and to 0 in $\Omega - \sigma_\lambda \Omega\ (0 < \lambda < 1)$, is in $\boldsymbol{H}_0^1(\Omega)$ and $\operatorname{div} \boldsymbol{u}_\lambda$ equals $\lambda \sigma_\lambda (\operatorname{div} \boldsymbol{u})$ in $\sigma_\lambda \Omega$ and 0 in $\Omega - \sigma_\lambda \Omega$; hence $\operatorname{div} \boldsymbol{u}_\lambda = 0$, $\boldsymbol{u}_\lambda \in V^{\bullet}$ and has a compact support in Ω. In this case it is easy to check by regularization that $\boldsymbol{u}_\lambda \in V$, and since $\boldsymbol{u}_\lambda$ converges to $\boldsymbol{u}$ in $\boldsymbol{H}_0^1(\Omega)$ as $\lambda \to 1$, $\boldsymbol{u} \in V$ and $V = V^{\bullet}$.

(ii) If the assumptions of Theorem 1.6 are not satisfied (Ω bounded and lipschitzian) then (1.49) may not be true. As in the proof of Theorem 1.6, let us denote by $V^{\bullet}$ the space in the right-hand side of (1.49). We have $V \subset V^{\bullet}$ and it is not known whether $V = V^{\bullet}$ when Ω is bounded but not lipschitzian. If Ω is not bounded, it follows from an example of J.G. Heywood in [4] that V may be different from $V^{\bullet}$, and actually $\dim V^{\bullet}/V = 1$ in his example. In Ladyzhenskaya–Solonnikov [2] the authors give examples of other unbounded open sets Ω such that $\dim V^{\bullet}/V = k$, k an arbitrary integer.

Remark 1.8. The results which will be extensively used in the book are Propositions 1.1 and 1.2, Theorem 1.6 and less frequently, Theorem 1.5.

Remark 1.9. A result weaker than Proposition 1.1 but sufficient for what follows can be proved by a different method avoiding the utilization of De Rham's theory (cf. p. 14).

Assume that Ω is a Lipschitz bounded open set in $\mathbb{R}^n$ and that $f \in H^{-1}(\Omega)$, satisfies $\langle f, v\rangle = 0$ $\forall\, v \in \mathcal{V}$ (or V). *Then* $f = \operatorname{grad} p$, $p \in L^2(\Omega)$. (1.51)

For an arbitrary open set $\Omega \subset R^n$, the same result holds, with only $p \in L^2_{\text{loc}}(\Omega)$. (1.52)

Let us sketch the proof of this result due to L. Tartar [1] and based on Proposition 1.2 and Remark 1.4 (ii).

It is clear that there exists an increasing sequence of open sets Ω_m ($\Omega_m \subset \Omega_{m+1}$), which are Lipschitz and whose union is Ω. Let A (or A_m) be the gradient operator $\in \mathcal{L}(L^2(\Omega), H^{-1}(\Omega))$ (or $\in \mathcal{L}(L^2(\Omega_m)$, $H^{-1}(\Omega_m)))$, and let $A^* \in \mathcal{L}(H^1_0(\Omega), L^2(\Omega))$ (or $A^*_m \in \mathcal{L}(H^1_0(\Omega_m)$, $L^2(\Omega_m)))$ be its adjoint.

By Remark 1.4 (ii), the range of A (written $R(A)$) is a closed subspace of $H^{-1}(\Omega))$. Now it is known from linear operator theory that the orthogonal of Ker A^* is the closure of $R(A)$ and this is therefore equal to $R(A)$; Ker A^* is the Kernel of A^*

$$\operatorname{Ker} A^* = \{u \in H^1_0(\Omega), \operatorname{div} u = 0\}^{(1)}$$

Similar remarks hold for A_m.

Now let f satisfy the condition in (1.51) and let $u \in \operatorname{Ker} A^*_m$. If $\tilde{u}$ is the function u extended by 0 outside Ω_m, then $\tilde{u} \in H^1_0(\Omega)$ and $\operatorname{div} \tilde{u} = \operatorname{div} u = 0$. Since $\tilde{u}$ has a compact support in Ω, it is clear by regularization that $\tilde{u}$ is the limit in $H^1_0(\Omega)$ of elements of $\mathcal{V}$, hence $\tilde{u} \in V$, and $\langle f, \tilde{u}\rangle = 0$. Therefore the restriction of f to Ω_m is orthogonal to Ker A^*_m, and thus belongs to $R(A_m)$: $f = \operatorname{grad} p_m$ on Ω_m, $p_m \in L^2(\Omega_m)$. Since the Ω_m are increasing sets, $p_{m+1} - p_m = \text{const}$ on Ω_m, and we can choose p_{m+1} so that this constant is zero. Hence $f = \operatorname{grad} p$, $p \in L^2_{\text{loc}}(\Omega)$.

This is sufficient for (1.52). For obtaining (1.51) ($p \in L^2(\Omega)$), we use the fact that Ω is locally star shaped (see Section 1.1). Let $(\mathcal{O}_j)_{1 \leqslant j \leqslant J}$ be a finite covering of Γ, such that $\mathcal{O}'_j = \mathcal{O}_j \cap \Omega$ is star-shaped, $\forall j$. If $u \in H^1_0(\mathcal{O}'_j)$ and $\operatorname{div} u = 0$, then for $0 < \lambda < 1$, $\sigma_\lambda u$ belongs to Ω. As before $\langle f, \tilde{u}\rangle = 0$, and then $f = \operatorname{grad} q_j$ in $\mathcal{O}'_j$, $q_j \in L^2(\mathcal{O}'_j)$; $q_j = p$ on $\mathcal{O}'_j$, and $p \in L^2(\mathcal{O}'_j)$ $\forall j$, so that $p \in L^2(\Omega)$.

§2. Existence and uniqueness for the Stokes equations

The Stokes equations are the linearized stationary form of the full Navier–Stokes equations. We give here the variational formulation of

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(1) It is proved in Theorem 1.6 that this space is V, but this result is not yet known when we prove Proposition 1.1.

Stokes problem, an existence and uniqueness result using the projection theorem, and a few other remarks concerning the case of an unbounded domain and the regularity of solutions.

2.1. *Variational formulation of the problem*

Let Ω be an open bounded set in $\mathbb{R}^n$ with boundary Γ, and let $f \in L^2(\Omega)$ be a given vector function in Ω. We seek a vector function $u = (u_1, \ldots, u_n)$ representing the velocity of the fluid, and a scalar function p representing the pressure, which are defined in Ω and satisfy the following equations and boundary conditions (ν is the coefficient of kinematic viscosity, a constant):

$$-\nu \Delta \boldsymbol{u} + \operatorname{grad} p = \boldsymbol{f} \quad \text{in } \Omega \quad (\nu > 0) \tag{2.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma. \tag{2.3}$$

If $\boldsymbol{f}$, $\boldsymbol{u}$, and p are smooth functions satisfying (2.1) – (2.3) then, taking the scalar product of (2.1) with a function $\boldsymbol{v} \in \mathscr{V}$ we obtain,

$$(-\nu \Delta \boldsymbol{u} + \operatorname{grad} p, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})$$

and, integrating by parts, the term $(-\Delta \boldsymbol{u}, \boldsymbol{v})$ gives[1]

$$\sum_{i=1}^{n} (\operatorname{grad} u_i, \operatorname{grad} v_i) = ((\boldsymbol{u}, \boldsymbol{v})) \tag{2.4}$$

and the term $(\operatorname{grad} p, \boldsymbol{v})$ gives

$$-(p, \operatorname{div} \boldsymbol{v}) = 0,$$

and there results

$$\nu ((\boldsymbol{u}, \boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathscr{V}. \tag{2.5}$$

Since each side of (2.5) depends linearly and continuously on $\boldsymbol{v}$ for the $H_0^1(\Omega)$ topology, the equality (2.5) is still valid by continuity for each $\boldsymbol{v} \in V$

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[1] See the notation at the end of Section 1.1.

the closure of $\mathscr{V}$ in $\boldsymbol{H}_0^1(\Omega)$. If the set Ω is of class $\mathscr{C}^2$ then due to (2.3) the (smooth) function $\boldsymbol{u}$ belongs to $\boldsymbol{H}_0^1(\Omega)$, and because of (2.2) and Theorem 1.6, $\boldsymbol{u} \in V$. We arrive then at the following conclusion:

$$\textit{u belongs to } V \textit{ and satisfies } \nu((\boldsymbol{u}, \boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v}),\ \forall\ \boldsymbol{v} \in V. \tag{2.6}$$

Conversely, let us suppose that $\boldsymbol{u}$ satisfies (2.6) and let us then show that $\boldsymbol{u}$ satisfies (2.1) – (2.3) in some sense. Since $\boldsymbol{u}$ belongs only to $\boldsymbol{H}_0^1(\Omega)$, we have less regularity than before and can only expect $\boldsymbol{u}$ to satisfy (2.1) – (2.3) in a sense weaker than the classical sense. Actually, $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ implies that the traces $\gamma_0 \boldsymbol{u}_i$ of its components are zero in $H^{1/2}(\Gamma)$; $\boldsymbol{u} \in V$ implies (using Theorem 1.6) that $\operatorname{div} \boldsymbol{u} = 0$ in the distribution sense; and using (2.6) we have

$$\langle -\nu\Delta\boldsymbol{u} - \boldsymbol{f}, \boldsymbol{v}\rangle = 0,\ \forall\ \boldsymbol{v} \in \mathscr{V}.$$

Then by virtue of Propositions 1.1 and 1.2, there exists some distribution $p \in L^2(\Omega)$ such that

$$-\nu\Delta\boldsymbol{u} - \boldsymbol{f} = -\operatorname{grad} p$$

in the distribution sense in Ω.

We have thus proved

Lemma 2.1. *Let Ω be an open bounded set of class $\mathscr{C}^2$.*
The following conditions are equivalent

(i) *$\boldsymbol{u} \in V$ satisfies* (2.6)
(ii) *$\boldsymbol{u}$ belongs to $\boldsymbol{H}_0^1(\Omega)$ and satisfies* (2.1) – (2.3) *in the following weak sense:*

$$\begin{aligned}&\textit{there exists } p \in L^2(\Omega) \textit{ such that } -\nu\Delta\boldsymbol{u} + \operatorname{grad} p = \boldsymbol{f}\\ &\textit{in the distribution sense in } \Omega\,;\end{aligned} \tag{2.7}$$

$$\operatorname{div} \boldsymbol{u} = 0 \textit{ in the distribution sense in } \Omega; \tag{2.8}$$

$$\gamma_0 \boldsymbol{u} = 0. \tag{2.9}$$

Definition 2.1. The problem "find $\boldsymbol{u} \in V$ satisfying (2.6)" is called the *variational formulation* of problem (2.1) – (2.3).

Remark 2.1. Before studying existence and uniqueness problems for (2.6), let us make a few observations.

(i) The variational formulation of problem (2.1) – (2.3) was introduced by J. Leray [1] [2] [3]. It reduces the classical problem (2.1) – (2.3) to the problem of finding only $\boldsymbol{u}$; the existence of p is then a consequence of Proposition 1.1.

(ii) When the set Ω is not smooth or unbounded, we have two spaces which we called V and $V^{\bullet}$ in the proof of Theorem 1.6 and which may be different ($V \subset V^{\bullet}$; cf. Remark 1.7 (ii)):

$$V = \text{the closure of } \mathscr{V} \text{ in } \boldsymbol{H}_0^1(\Omega), \quad V^{\bullet} = \{u \in \boldsymbol{H}_0^1(\Omega), \operatorname{div} u = 0\}.$$

We may then pose two (possibly different) variational formulations: either (2.6) exactly, or the same problem with V replaced by $V^{\bullet}$. In the general case the relation between V and $V^{\bullet}$ and a fortiori between these two variational problems is not known. The relations between these problems is studied in J.G. Heywood [2], O.A. Ladyzhenskaya–V.A. Solonnikov [4] for the cases considered by these authors (Remark 1.7 (ii)).

For technical reasons, particularly important in the non-linear case, we will allways work with the space V and consider only the variational problem (2.6).

Let us remark as a complement to Lemma 2.1 that for any set Ω, if $\boldsymbol{u}$ satisfies (2.6) (or (2.6) with V replaced by $V^{\bullet}$), then it satisfies (2.7) with the restriction that $p \in L^2_{loc}(\Omega)$ only; it satisfies (2.8) without any modification; and it satisfies (2.9) in the sense that $u \in H_0^1(\Omega)$; a more precise meaning depends on the trace theorems available for Ω.

2.2. *The projection theorem*

Let Ω be any open set of $\mathbb{R}^n$ such that

$$\Omega \text{ is bounded in some direction.} \tag{2.10}$$

According to (1.10), $\boldsymbol{H}_0^1(\Omega)$ is a Hilbert space for the scalar product (2.4); $\mathscr{V}$ is defined in (1.12) and V is the closure of $\mathscr{V}$ in $\boldsymbol{H}_0^1(\Omega)$.

Theorem 2.1. *For any open set $\Omega \subset \mathbb{R}^n$ which is bounded in some direction, and for every $f \in L^2(\Omega)$, the problem* (2.6) *has a unique solution $\boldsymbol{u}$. (The result is also valid if f is given in $H^{-1}(\Omega)$.)*

Moreover, there exists a function $p \in L^2_{loc}(\Omega)$ *such that* (2.7) – (2.8) *are satisfied.*

If Ω *is an open bounded set of class* $\mathscr{C}^2$, *then* $p \in L^2(\Omega)$ *and* (2.7) – (2.9) *are satisfied by* $\boldsymbol{u}$ *and* p.

This theorem is a simple consequence of the preceding lemma and the following classical projection theorem.

Theorem 2.2. *Let* W *be a separable real Hilbert space (norm* $\|\cdot\|_W$*) and let* $a(\boldsymbol{u}, \boldsymbol{v})$ *be a bilinear continuous form on* $W \times W$, *which is coercive,* i.e., *there exists* $\alpha > 0$ *such that*

$$a(\boldsymbol{u}, \boldsymbol{u}) \geqslant \alpha \|\boldsymbol{u}\|^2_W, \; \forall\, \boldsymbol{u} \in W. \tag{2.11}$$

Then for each ℓ *in* W', *the dual space of* W, *there exists one and only one* $\boldsymbol{u} \in W$ *such that*

$$a(\boldsymbol{u}, \boldsymbol{v}) = \langle \ell, \boldsymbol{v} \rangle, \; \forall\, \boldsymbol{v} \in W. \tag{2.12}$$

Of course to apply this theorem to (2.6), we take W = the space V equipped with the norm associated with (2.4), $a(\boldsymbol{u}, \boldsymbol{v}) = \nu((\boldsymbol{u}, \boldsymbol{v}))$, and for $\boldsymbol{v} \to \langle \ell, \boldsymbol{v} \rangle$ the form $\boldsymbol{v} \to (\boldsymbol{f}, \boldsymbol{v})$ which is obviously linear and continuous on V. The space V is separable as a closed subspace of the separable space $\boldsymbol{H}^1_0(\Omega)$.

Proof of theorem 2.2. *Uniqueness.* Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be two solutions of (2.12) and let $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$. We have

$$a(\boldsymbol{u}_1, \boldsymbol{v}) = a(\boldsymbol{u}_2, \boldsymbol{v}) = \langle \ell, \boldsymbol{v} \rangle, \quad \forall \boldsymbol{v} \in W,$$
$$a(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in W.$$

Taking $\boldsymbol{v} = \boldsymbol{u}$ in this equality we see from (2.11) that

$$\alpha \|\boldsymbol{u}\|^2_W \leqslant a(\boldsymbol{u}, \boldsymbol{u}) = 0,$$

and hence $\boldsymbol{u} = 0$.

Existence. Since W is separable, there exists a sequence of elements $\boldsymbol{w}_1, \ldots \boldsymbol{w}_m, \ldots,$ of W which is free and total in W. Let W_m be the space spanned by $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m$. For each fixed integer m we define an approximate solution of (2.12) in W_m; that is, a vector $\boldsymbol{u}_m \in W_m$

$$\boldsymbol{u}_m = \sum_{i=1}^{m} \xi_{i,m} \boldsymbol{w}_i, \quad \xi_{i,m} \in \mathbb{R} \tag{2.13}$$

satisfying

$$a(\boldsymbol{u}_m, \boldsymbol{v}) = \langle \ell, \boldsymbol{v} \rangle, \quad \forall\, \boldsymbol{v} \in W_m. \tag{2.14}$$

Let us show that there exists one and only one $\boldsymbol{u}_m$ such that (2.14) holds. Equation (2.14) is equivalent to the set of m equations

$$a(\boldsymbol{u}_m, \boldsymbol{w}_j) = \langle \ell, \boldsymbol{w}_j \rangle, \quad j = 1, \dots, m, \tag{2.15}$$

and (2.15) is a linear system of m equations for the m components $\xi_{i,m}$ of $\boldsymbol{u}_m$:

$$\sum_{i=1}^{m} \xi_{i,m}\, a(\boldsymbol{w}_i, \boldsymbol{w}_j) = \langle \ell, \boldsymbol{w}_j \rangle, \quad j = 1, \dots, m. \tag{2.16}$$

The existence and uniqueness of $\boldsymbol{u}_m$ will be proved once we show that the linear system (2.16) is regular. To show this it is sufficient to prove that the homogeneous linear system associated with (2.16), i.e.,

$$\sum_{i=1}^{m} \xi_i\, a(\boldsymbol{w}_i, \boldsymbol{w}_j) = 0, \quad j = 1, \dots, m, \tag{2.17}$$

has only one solution $\xi_1 = \dots = \xi_m = 0$. But if $\xi_1, \dots, \xi_m$, satisfy (2.17), then by multiplying each equation (2.17) by the corresponding ξ_j and adding these equations, we obtain

$$\sum_{i,j=1}^{m} \xi_i \xi_j\, a(\boldsymbol{w}_i, \boldsymbol{w}_j) = 0$$

or, because of the bilinearity of a,

$$a\left(\sum_{i=1}^{m} \xi_i \boldsymbol{w}_i, \sum_{j=1}^{m} \xi_j \boldsymbol{w}_j \right) = 0;$$

using (2.11) we find

$$\sum_{i=1}^{m} \xi_i \boldsymbol{w}_i = 0.$$

and finally $\xi_1 = \dots = \xi_m = 0$ since $\boldsymbol{w}_1, \dots, \boldsymbol{w}_m$ are linearly independent.

Passage to the limit: When we put $\boldsymbol{v} = \boldsymbol{u}_m$ in (2.14), we obtain

$$a(\boldsymbol{u}_m, \boldsymbol{u}_m) = \langle \ell, \boldsymbol{u}_m \rangle, \tag{2.18}$$

from which, by (2.11), it follows that

$$\alpha \|\boldsymbol{u}_m\|_W^2 \leqslant a(\boldsymbol{u}_m, \boldsymbol{u}_m) = \langle \ell, \boldsymbol{u}_m \rangle \leqslant \|\ell\|_{W'} \|\boldsymbol{u}_m\|,$$

$$\|\boldsymbol{u}_m\|_W \leqslant \frac{1}{\alpha} \|\ell\|_{W'}, \tag{2.19}$$

which proves that the sequence $\boldsymbol{u}_m$ is bounded independently of m in W. Since the closed balls of Hilbert space are weakly compact, there exists an element $\boldsymbol{u}$ of W and a sequence $\boldsymbol{u}_{m'}$, $m' \to \infty$, extracted from $\boldsymbol{u}_m$, such that

$$\boldsymbol{u}_{m'} \to \boldsymbol{u} \text{ in the weak topology of } W, \text{ as } m' \to \infty. \tag{2.20}$$

Let $\boldsymbol{v}$ be a fixed element of W_j for some j. As soon as $m' \geqslant j$, $\boldsymbol{v} \in W_{m'}$, and according to (2.14) we have

$$a(\boldsymbol{u}_{m'}, \boldsymbol{v}) = \langle \ell, \boldsymbol{v} \rangle.$$

By using the following lemma, we can take the limit in this equation as $m' \to \infty$, and obtain:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \langle \ell, \boldsymbol{v} \rangle. \tag{2.21}$$

Equality (2.21) holds for each $\boldsymbol{v} \in \bigcup_{j=1}^{\infty} W_j$, and as this set is dense in W, equality (2.21) still holds by continuity for $\boldsymbol{v}$ in W. This proves that $\boldsymbol{u}$ is a solution of (2.12).

Lemma 2.2. *Let $a(\boldsymbol{u}, \boldsymbol{v})$ be a bilinear continuous form on a Hilbert space W.*

Let ϕ_m (or ψ_m) be a sequence of elements of W which converges to ϕ (or ψ) in the weak (or strong) topology of W. Then

$$\lim_{m \to \infty} a(\psi_m, \phi_m) = a(\psi, \phi). \tag{2.22}$$

$$\lim_{m \to \infty} a(\phi_m, \psi_m) = a(\phi, \psi). \tag{2.23}$$

Proof. We write

$$a(\psi_m, \phi_m) - a(\psi, \phi) = a(\psi_m - \psi, \phi_m) + a(\psi, \phi_m - \phi).$$

As the form a is continuous and the sequence ϕ_m is bounded,

$$|a(\psi_m - \psi, \phi_m)| \leqslant c\|\psi_m - \psi\|_W \|\phi_m\|_W \leqslant c'\|\psi_m - \psi\|_W,$$

and this term converges to 0 as $m \mapsto \infty$.

We notice next that the linear operator $\boldsymbol{v} \to a(\psi, \boldsymbol{v})$ is continuous on W and hence there exists some element of W', depending on ψ and denoted by $A(\psi)$,

$$a(\psi, \boldsymbol{v}) = \langle A(\psi), \boldsymbol{v}\rangle, \quad \forall\, \boldsymbol{v} \in W. \tag{2.24}$$

We can now write

$$a(\psi, \phi_m - \phi) = \langle A(\psi), \phi_m - \phi\rangle$$

and this converges to 0 as $m \to \infty$, as a consequence of the weak convergence of ϕ_m.

This proves (2.22). For (2.23) we have only to apply (2.22) to the bilinear form

$$a^*(\boldsymbol{u}, \boldsymbol{v}) = a(\boldsymbol{v}, \boldsymbol{u}).$$

Remark 2.2. (i) Theorem 2.1 is also true if $\boldsymbol{f}$ is given $H^{-1}(\Omega)$.

(ii) It can be proved that the sequence $\{\boldsymbol{u}_m\}$ constructed in the proof of Theorem 2.2, as a whole, converges to the solution $\boldsymbol{u}$ of (2.12) in the strong topology of W. We do not prove this result here; it will appear as a consequence of Theorem 3.1.

(iii) Using the form $A(\psi)$ introduced in the proof of Lemma 2.2 (cf. (2.24)), one can write equation (2.12) in the form

$$\langle A(\boldsymbol{u}), \boldsymbol{v}\rangle = \langle \ell, \boldsymbol{v}\rangle$$

which is equivalent to

$$A(\boldsymbol{u}) = \ell \text{ in } W'.$$

An alternative classical proof of the projection theorem is to show that the operator $u \mapsto A(u)$ is an isomorphism from W onto W'; cf. R. Temam [8].

A variational property.

Proposition 2.1. *The solution* u *of* (2.6) *is also the unique element of* V *such that*

$$E(u) \leqslant E(v), \quad \forall v \in V, \tag{2.26}$$

where

$$E(v) = \nu\|v\|^2 - 2(f, v). \tag{2.27}$$

Proof. Let u be the solution of (2.6). Then as

$$\|u-v\|^2 \geqslant 0, \quad \forall v \in V,$$

we have

$$\nu\|u\|^2 + \nu\|v\|^2 - 2\nu((u, v)) \geqslant 0. \tag{2.28}$$

Because of (2.6) we have

$$-\nu\|u\|^2 = \nu\|u\|^2 - 2(f, u) = E(u),$$

$$-2\nu((u, v)) = -2(f, v)$$

and thus (2.28) gives exactly (2.26).

Conversely, if $u \in V$ satisfies (2.26), then for any $v \in V$ and $\lambda \in \mathbb{R}$, one has

$$E(u) \leqslant E(u+\lambda v).$$

This may be reduced to

$$\nu\lambda^2\|v\|^2 + 2\lambda\nu((u,v)) - 2\lambda(f, v) \geqslant 0, \quad \forall \lambda \in \mathbb{R}. \tag{2.29}$$

This inequality can hold for each $\lambda \in \mathbb{R}$ only if

$$\nu((u, v)) = (f, v),$$

and thus u is indeed a solution of (2.6).

Remark 2.3. If the spaces V and V^* of Remark 2.1 are different, Theorem 2.2 also gives the existence and uniqueness of a $\tilde{u} \in V^{\bullet}$ such that

$$\nu((\tilde{u}, v)) = (f, v), \quad \forall v \in V^{\bullet}. \tag{2.30}$$

Proposition 2.1 is also valid with u and V replaced by $\tilde{u}$ and $V^{\bullet}$.

2.3. *The unbounded case*

We consider here the case where Ω is unbounded and does not satisfy (2.10). If Ω is of class $\mathscr{C}^2$, problem (2.1) – (2.3) is not equivalent to (2.6) as in Lemma 2.1. The reason is that if u is a classical solution of (2.1) – (2.3), it is not clear, without further information about the behaviour of u at infinity, that $u \in H^1(\Omega)$; hence perhaps $u \notin V$ and equation (2.5) cannot be extended by continuity to the closure V of $\mathscr{V}$. Further, there is a difficulty in trying to solve problem (2.6) directly using Theorem 2.2; hypothesis (2.11) is not satisfied, V is a Hilbert space for the norm

$$[\![u]\!] = \sqrt{|u|^2 + \|u\|^2},$$

which is not equivalent to the norm $\|u\|$ since we lose the Poincaré inequality.

In order to pose and solve a variational problem in the general case, let us introduce the space

$$Y = \text{the completion of } \mathscr{V} \text{ under the norm } \|\cdot\|. \tag{2.31}$$

It is clear, since $\|u\| \leqslant [\![u]\!]$, that Y is a larger space than V

$$V \subset Y. \tag{2.32}$$

Lemma 2.3. $Y \subset \{u \in L^{\alpha}(\Omega): D_i u \in L^2(\Omega),\ 1 \leqslant i \leqslant n\}$ (2.33)

with $\alpha = 2n/(n-2)$ *if* $n \geqslant 3$, *and*

$$Y \subset \{u \in L^{\alpha}_{\text{loc}}(\Omega),\ D_i u \in L^2(\Omega),\ i = 1, 2\}, \quad \forall \alpha \geqslant 1, \tag{2.34}$$

for $n = 2$. *The injections are continuous.*

Proof. Let us prove (2.33). This is a consequence of the Sobolev inequality (see Sobolev [1], Lions [1], and also Chapter II):

$$\|\phi\|_{L^\alpha(\Omega)} \leqslant c(\alpha, n)\, \|\operatorname{grad} \phi\|_{L^2(\Omega)}, \quad \forall\, \phi \in \mathscr{D}(\Omega), \tag{2.35}$$

where $\alpha = 2n/(n-2)$.

If $\boldsymbol{u} \in Y$ there exists a sequence of elements $\boldsymbol{u}_m \in \mathscr{V}$ converging to $\boldsymbol{u}$ in Y; by (2.35)

$$\|\boldsymbol{u}_m - \boldsymbol{u}_p\|_{L^\alpha(\Omega)} \leqslant c'(\alpha, n)\, \|\boldsymbol{u}_m - \boldsymbol{u}_p\|,$$

$$\|\boldsymbol{u}_m - \boldsymbol{u}_p\|_Z \leqslant c''\|\boldsymbol{u}_m - \boldsymbol{u}_p\|, \tag{2.36}$$

where Z stands for the space on the right-hand side of inclusion (2.33) and $\|\cdot\|_Z$ is the natural norm of Z:

$$\|\boldsymbol{u}\|_Z = \|\boldsymbol{u}\|_{L^\alpha(\Omega)} + \|\boldsymbol{u}\|.$$

As m and p tend to infinity, the right-hand side of (2.36) converges to 0. Thus $\boldsymbol{u}_m$ is a Cauchy sequence in Z; its limit $\boldsymbol{u}$ belongs to Z. It is clear also that

$$\|\boldsymbol{u}\|_Z \leqslant c''\, \|\boldsymbol{u}\|$$

with the same c'' as in (2.36) ($c'' = c''(n)$).

The proof is similar for (2.34), by considering Cauchy sequences in $L^\alpha(\Omega')$ where Ω' is a compact subset of Ω.

Theorem 2.3. *Let Ω be any open set in $\mathbb{R}^n$, and let f be given in Y', the dual of the space Y in* (2.31).

Then, there exists a unique $\boldsymbol{u} \in Y$ such that

$$\nu((\boldsymbol{u}, \boldsymbol{v})) = \langle f, \boldsymbol{v}\rangle, \quad \forall\, \boldsymbol{v} \in Y. \tag{2.37}$$

There exists $p \in L^2_{\mathrm{loc}}(\Omega)$, such that (2.7) *is satisfied and* (2.8) *is true.*

Proof. We apply Theorem 2.2 with the space Y, $a(\boldsymbol{u}, \boldsymbol{v}) = \nu((\boldsymbol{u}, \boldsymbol{v}))$, and ℓ replaced by f; we get a unique $\boldsymbol{u}$ satisfying (2.37).

Then, Remark 1.4 (i) shows the existence of some $p \in L^2_{\mathrm{loc}}(\Omega)$ such that (2.7) is satisfied; (2.8) is of course easily verified. Finally, (2.9) is satisfied in some sense depending on the trace theorem available for Y (or for the space Z on the right-hand side of inclusions (2.33) and (2.34)).

If Ω is locally Lipschitz, Proposition 1.2 shows that $p \in L^2_{loc}(\bar{\Omega})$.

Remark 2.4. By Lemma 2.3, for $f \in L^{\alpha'}(\Omega)$ $(1/\alpha + 1/\alpha' = 1)$,

$$v \mapsto \int_\Omega f \cdot v \, dx \tag{2.38}$$

is a linear continuous form on Y. Thus one can take in Theorem 2.3, any $f \in L^{\alpha'}(\Omega)$ $(\alpha = 2n/(n-2),\ n \geqslant 3)$:

2.4. *The non-homogeneous Stokes Problem*

We consider a non-homogeneous Stokes problem.

Theorem 2.4. *Let Ω be an open bounded set of class $\mathscr{C}^2$ in $\mathbb{R}^n$. Let there be given $f \in H^{-1}(\Omega)$, $g \in L^2(\Omega)$, $\phi \in H^{1/2}(\Gamma)$, such that*

$$\int_\Omega g \, dx = \int_\Gamma \phi \cdot \nu \, d\Gamma. \tag{2.39}$$

Then there exists

$$u \in H^1(\Omega),\ p \in L^2(\Omega),$$

which are solutions of the non homogeneous Stokes problem

$$-\nu \Delta u + \operatorname{grad} p = f \text{ in } \Omega \tag{2.40}$$

$$\operatorname{div} u = g \text{ in } \Omega \tag{2.41}$$

$$\gamma_0 u = \phi \text{ i.e. } u = \phi \text{ on } \Gamma, \tag{2.42}$$

u is unique and p is unique up to the addition of a constant.

Proof. Since $H^{1/2}(\Gamma) = \gamma_0 H^1(\Omega)$, there exists $u_0 \in H^1(\Omega)$, such that $\gamma_0 u_0 = \phi$. Then, from (2.39) and Stokes' formula,

$$\int_\Omega (g - \operatorname{div} u_0) \, dx = 0.$$

Using Lemma 2.4 below we see that there exists a $u_1 \in H^1_0(\Omega)$ such that $\operatorname{div} u_1 = g - \operatorname{div} u_0$.

Setting $v = u - u_0 - u_1$, (2.40) -- (2.42) reduces to a homogeneous Stokes problem for v:

$$-\nu\Delta v + \operatorname{grad} p = f - \nu\Delta(u_0 + u_1) \in H^{-1}(\Omega),$$
$$\operatorname{div} v = 0 \ in\ \Omega,$$
$$v = 0 \ on\ \partial\Omega.$$

The existence and uniqueness of v and p (and therefore u and p) then follows.

Lemma 2.4. *Let Ω be a Lipschitz open bounded set in $\mathbb{R}^n$.*
Then the divergence operator maps $H_0^1(\Omega)$ onto the space $L^2(\Omega)/\mathbb{R}$, i.e

$$\left\{ g \in L^2(\Omega),\ \int_\Omega g(x)\,dx = 0 \right\}. \tag{2.43}$$

Proof. From Proposition 1.2, and Remark 1.4 (ii), $A = \operatorname{grad} \in \mathcal{L}(L^2(\Omega), H^{-1}(\Omega))$ is an isomorphism from (2.43) onto its range, $R(A)$. By transposition, its adjoint $A^* = -\operatorname{div} \in \mathcal{L}(H_0^1(\Omega), L^2(\Omega))$ is an isomorphism from the orthogonal of $R(A)$ onto the space (2.43); in particular A^* maps $H_0^1(\Omega)$ onto $L^2(\Omega)/\mathbb{R}$.

Remark 2.5. Theorem 2.4 is easily extended to the case where Ω is only a Lipschitz open bounded set in $\mathbb{R}^n$, provided ϕ is given as the trace of a function ϕ_0 in $H^1(\Omega)$ and (2.39) and (2.42) are understood as follows

$$\int_\Omega g\,dx = \int_\Omega \operatorname{div} \phi_0\,dx, \tag{2.39'}$$

$$u - \phi_0 \in H_0^1(\Omega) \tag{2.42'}$$

We just take $u_0 = \phi_0$.

2.5. *Regularity results*

A classical result is that the solution $u \in H_0^1(\Omega)$ of the Dirichlet problem $-\Delta u + u = f$ belongs to $H^{m+2}(\Omega)$ whenever $f \in H^m(\Omega)$ (and Ω is sufficiently smooth). One naturally wonders whether similar regularity results exist for the Stokes problem.

This result and the similar one in L^p-spaces is given by the following Proposition.

Proposition 2.2. *Let Ω be an open bounded set of class $\mathscr{C}^r$, $r = \max(m+2, 2)$, m integer > 0. Let us suppose that*

$$u \in W^{2,\alpha}(\Omega), \qquad p \in W^{1,\alpha}(\Omega), \qquad 1 < \alpha < +\infty, \tag{2.44}$$

are solutions of the generalized Stokes problem (2.40) – (2.42):

$$-\nu\Delta u + \operatorname{grad} p = f \quad \text{in } \Omega, \tag{2.40}$$

$$\operatorname{div} u = g \text{ in } \Omega, \tag{2.41}$$

$$\gamma_0 u = \phi; \text{ i.e., } u = \phi \text{ on } \Gamma. \tag{2.42}$$

If $f \in W^{m,\alpha}(\Omega)$, $g \in W^{m+1,\alpha}(\Omega)$ and $\phi \in W^{m+2-1/\alpha,\alpha}(\Gamma)$, (1) *then*

$$u \in W^{m+2,\alpha}(\Omega),\ p \in W^{m+1,\alpha}(\Omega) \tag{2.45}$$

and there exists a constant $c_0(\alpha, \nu, m, \Omega)$ such that

$$\begin{aligned} &\|u\|_{W^{m+2,\alpha}(\Omega)} + \|p\|_{W^{m+1,\alpha}(\Omega)/\mathbb{R}} \\ &\qquad \leq c_0 \{\|f\|_{W^{m,\alpha}(\Omega)} + \|g\|_{W^{m+1,\alpha}(\Omega)} \\ &\qquad + \|\phi\|_{W^{m+2-1/\alpha,\alpha}(\Gamma)} + d_\alpha \|u\|_{L^\alpha(\Omega)}\} \end{aligned} \tag{2.46}$$

$d_\alpha = 0$ for $\alpha \geq 2$, $d_\alpha = 1$ for $1 < \alpha < 2$.

Proof. This proposition results from the paper of Agmon-Douglis-Nirenberg [2] (hereafter referred to as A.D.N.), giving *a priori* estimates of solutions of general elliptic systems.

Let $u_{n+1} = 1/\nu\, p$, $u = (u_1, \ldots, u_{n+1})$, $f = (f_1/\nu, \ldots, f_n/\nu, g)$. Then equations (2.42) and (2.43) become

$$\sum_{j=1}^{n+1} \ell_{ij}(\mathrm{D})u_j = f_j,\ 1 \leq i \leq n+1, \tag{2.47}$$

---

(1) $W^{m+2-1/\alpha,\alpha}(\Gamma) = \gamma_0 W^{m+2,\alpha}(\Omega)$ and is equipped with the image norm

$$\|\psi\|_{W^{m+2-1/\alpha,\,\alpha}(\Gamma)} = \operatorname*{Inf}_{\gamma_0 u=\psi} \|u\|_{W^{m+2,\,\alpha}(\Omega)}.$$

where $\ell_{ij}(\xi)$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, is the matrix

$$\begin{aligned}
&\ell_{ij}(\xi) = |\xi|^2 \delta_{ij}, \ 1 \leqslant i, j \leqslant n, \\
&\ell_{n+1,j}(\xi) = -\ell_{j,n+1}(\xi) = \xi_j, \ 1 \leqslant j \leqslant n \\
&\ell_{n+1,n+1}(\xi) = 0, \\
&|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2.
\end{aligned} \tag{2.48}$$

We take (see p. 38, of A.D.N.), $s_i = 0$, $t_i = 2$, $1 \leqslant i \leqslant n$, $s_{n+1} = -1$, $t_{n+1} = 1$. As requested, degree $\ell_{ij}(\xi) \leqslant s_i + t_j$, $1 \leqslant i, j \leqslant n+1$, and we have $\ell'_{ij}(\xi) = \ell_{ij}(\xi)$. We easily compute $L(\xi) = \det \ell'_{ij}(\xi) = |\xi|^{2n}$, so that $L(\xi) \neq 0$ for real $\xi \neq 0$, and this ensures the ellipticity of the system (Condition (1.5), p. 39). It is clear that (1.7) on p. 39 holds with $m = n$. The Supplementary Condition on L is satisfied: $L(\xi + \tau\xi') = 0$ has exactly n roots with positive imagiriary part and these roots are all equal to

$$\tau^+(\xi, \xi') = -\xi \cdot \xi' + i\sqrt{|\xi|^2 |\xi'|^2 - |\xi \cdot \xi'|^2}.$$

Concerning the boundary conditions (see p. 42), there are n boundary conditions and

$$B_{hj} = \delta_{hj} \text{ (the Kronecker symbol) for } 1 \leqslant h \leqslant n, \ 1 \leqslant j \leqslant n+1.$$

We take $r_h = -2$ for $h = 1, \ldots, n$. Then, as requested, degree $B_{hj} \leqslant r_h + t_j$ and we have $B'_{hj} = B_{hj}$.

It remains to check the Complementing Boundary Condition. It is easy to verify that

$$M^+(\xi) = (\tau - \tau^+(\xi))^n$$

where $\tau^+(\xi) = \tau^+(\xi, \nu)$. The matrix with elements $\Sigma_{j=1}^N B'_{hj}(\xi) L^{jk}(\xi)$ is simply the matrix with elements $\ell_{hk}(\xi)$, $1 \leqslant h, k \leqslant n$, $-\ell_{h,n+1}(\xi)$, $1 \leqslant h \leqslant n$. A combination $\Sigma_{h=1}^n C_h \Sigma_{j=1}^N B'_{hj} L^{jk}$ is then equal to

$$\left(C_1(\xi + \tau\nu)^2, \ldots, C_n(\xi + \tau\nu)^2, \sum_{i=1}^n C_i(\xi_i + \tau\nu_i)\right)$$

and this is zero modulo M^+ only if $C_1 = \cdots = C_n = 0$, and the Complementing Condition holds.

We then apply Theorem 10.5, page 78 of A.D.N. in order to get (2.45) and (2.46) with $d_\alpha = 1$ for all α. According to the remark after Theorem

10.5, one can take $d_\alpha = 0$ for $\alpha \geqslant 2$ since the solutions $\boldsymbol{u}$ and p of (2.41) (2.44) are necessarily unique (p is unique up to an additive constant): if $(\boldsymbol{u}_*, p_*)$, $(\boldsymbol{u}_{**}, p_{**})$ are two solutions, then $\boldsymbol{u} = \boldsymbol{u}_* - \boldsymbol{u}_{**}$, $p = p_* - p_{**}$ are solutions of (2.7) – (2.9) with $f = 0$ and hence $\boldsymbol{u} = 0$ and p = constant.

□

Remark 2.6. For $\alpha = 2$ and $m \in \mathbb{R}$, $m \geqslant -1$, one has results similar to those in Proposition 2.2 by using the interpolation techniques of Lions-Magenes [1]. □

Proposition 2.2 does not assert the existence of $\boldsymbol{u}$, p satisfying (2.42) – (2.46) (for given f, g, ϕ) but gives only a result on the regularity of an eventual solution. The existence is ensured by Theorems 2.1 and 2.4 if $\alpha = 2$. The following proposition gives a general existence and regularity result for $n = 2$ or 3.

Proposition 2.3. *Let Ω be an open set of $\mathbb{R}^n$, $n = 2$ or 3, of class $\mathscr{C}^r$, $r = \max(m + 2, 2)$, m integer $\geqslant -1$, and let $f \in W^{m,\alpha}(\Omega)$, $g \in W^{m+1,\alpha}(\Omega)$, $\phi \in W^{m+2-1/\alpha,\alpha}(\Gamma)$ be given satisfying the compatibility condition*

$$\int_\Omega g\,dx = \int_\Gamma \phi \cdot \nu\, d\Gamma. \tag{2.49}$$

Then there exist unique functions $\boldsymbol{u}$ and p (p is unique up to a constant) which are solutions of (2.40) – (2.42) *and satisfy* (2.45) *and* (2.46) *with $d_\alpha = 0$ for any α, $1 < \alpha < \infty$.*

Proof. This full existence and regularity result does not follow from the theory of Agmon–Douglis–Nirenberg [1] and this is the reason for limiting the dimension of space to $n = 2$ or 3. Proposition 2.3 is completely proved in Cattabriga [1] when $n = 3$ (and even for $m = -1$). For $n = 2$ one can reduce the problem to a classical biharmonic problem. There exists $\nu \in W^{m+2,\alpha}(\Omega)$, such that

$$\operatorname{div} \boldsymbol{\nu} = g, \tag{2.50}$$

$$\gamma_0 \boldsymbol{\nu} = \phi. \tag{2.51}$$

Such $\boldsymbol{\nu}$ may be defined by

$$\boldsymbol{\nu} = \operatorname{grad} \theta + \left\{ \frac{\partial \sigma}{\partial x_2}, -\frac{\partial \sigma}{\partial x_1} \right\} \tag{2.52}$$

where $\theta \in W^{m+3,\alpha}(\Omega)$ is a solution of the Neumann problem

$$\Delta\theta = g \text{ in } \Omega, \tag{2.53}$$

$$\frac{\partial\theta}{\partial\nu} = \phi\cdot\nu \text{ on } \Gamma \tag{2.54}$$

and $\sigma \in W^{m+3,\alpha}(\Omega)$ will be chosen later.

The Neumann problem (2.53) – (2.54) has a solution θ because of (2.49), and $\theta \in W^{m+3,\alpha}(\Omega)$ by the usual regularity results for the Neumann problem.

The conditions on σ are only boundary conditions on Γ and these are

$$\frac{\partial\sigma}{\partial\tau} = \text{the tangential derivative of } \sigma = 0,$$

$$\frac{\partial\sigma}{\partial\nu} = \text{the normal derivative of } \sigma = \phi\cdot\tau - \frac{\partial\theta}{\partial\tau}.$$

Since $\phi\cdot\tau - \partial\theta/\partial\tau \in W^{m+2-1/\alpha,\alpha}(\Gamma)$, there exists a $\sigma \in W^{m+3,\alpha}(\Omega)$ with $\gamma_0\sigma = 0$, $\gamma_1\sigma = \phi\cdot\tau - \partial\theta/\partial\tau$. With these definitions of σ and θ, the vector v in (2.52) belongs to $W^{m+2,\alpha}(\Omega)$ and satisfies (2.50) – (2.51). Moreover, the mapping $\{g, \phi\} \mapsto v$ is linear and continuous.

Setting $w = u - v$, the problem (2.40) – (2.42) reduces to the problem of finding $w \in W^{m+2,\alpha}(\Omega)$, $p \in W^{m+1,\alpha}(\Omega)$ such that

$$-\nu\Delta w + \operatorname{grad} p = f', f' = f + \nu\Delta v, \tag{2.55}$$

$$\operatorname{div} w = 0, \tag{2.56}$$

$$\gamma_0 w = 0. \tag{2.57}$$

If Ω is simply connected then, because of (2.56), there exists a function ρ such that

$$w = (D_2\rho, -D_1\rho). \tag{2.58}$$

Condition (2.57) amounts to saying that $\rho = \partial\rho/\partial\nu = 0$ on Γ and (2.55) gives after differentiation

$$-\nu\Delta(D_2 w_1 - D_1 w_2) = D_2 f_1' - D_1 f_2'.$$

Thus we obtain

$$\nu \Delta^2 \rho = D_1 f_2' - D_2 f_1' \in W^{m-1,\alpha}(\Omega) \tag{2.59}$$

$$\rho = 0, \frac{\partial \rho}{\partial \nu} = 0 \text{ on } \Gamma. \tag{2.60}$$

The biharmonic problem (2.59) (2.60) has a unique solution $\rho \in W^{m+3,\alpha}(\Omega)$, and the function w defined by (2.58) is a solution of (2.55) – (2.57) and belongs to $W^{m+2,\alpha}(\Omega)$.

If Ω is not simply connected, the condition (2.56) allows us to obtain (2.58) locally, i.e. ρ might not be a single valued function. A further

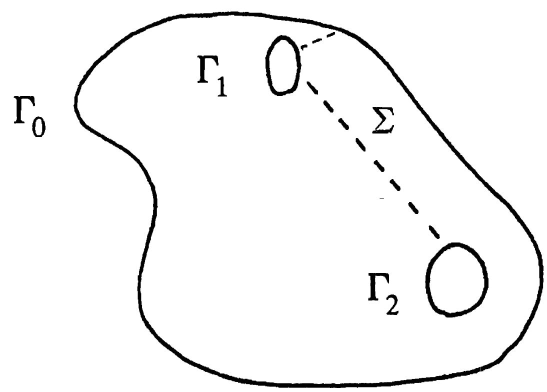


Fig. 2

argument using (2.57) and proved in the next Lemma shows that ρ is necessarily a single-valued function, such that

$$\rho = 0 \text{ on } \Gamma_0, \rho = \text{constant on } \Gamma_i, i = 1, 2, \ldots, \tag{2.61}$$

$$\frac{\partial \rho}{\partial \nu} = 0 \text{ on } \Gamma, \tag{2.62}$$

where $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \ldots$, is the decomposition of $\partial\Omega$ into its connected components, Γ_0 is one of these components, (the outside boundary of Ω if Ω is bounded for example), and $\Gamma_1, \Gamma_2, \ldots$, are the other components.

Then the problem (2.55)–(2.57) is reduced for ρ to the equation (2.59) associated with the boundary condition (2.61), (2.62). As previously, the biharmonic problem (2.59), (2.61), (2.62) has a unique solution $\rho \in W^{m+3,\alpha}(\Omega)$ and the function w defined by (2.58) is a solution of (2.55) – (2.57) and belongs to $W^{m+2,\alpha}(\Omega)$.

Lemma 2.5. *Assume that a vector function w in $W_0^{1,1}(\Omega)$, Ω open set of $\mathbb{R}^2$, is divergence free. Then there exists a unique single-valued function $\rho \in W^{2,1}(\Omega)$ which satisfies* (2.61), (2.62) *and*

$$w = (D_2\, \rho, -D_1\, \rho). \tag{2.63}$$

Proof. Let us first show this result for smooth functions w. The result can be extended for the more general vector functions w by a density argument.

Let $\Gamma_0, \Gamma_1, \Gamma_2, \ldots,$ denote the connected components of $\Gamma = \partial\Omega$ and let us make some smooth cuts Σ in Ω such that $\Omega = \Sigma \cup \dot{\Omega}$, where $\dot{\Omega}$ is a simply connected open set. Then for the given smooth function w, there exists a function ρ such that (2.63) holds in $\dot{\Omega}$. It is clear that ρ is smooth in the closure of Ω and since $w = 0$ on $\partial\Omega$, grad ρ vanishes on $\partial\Omega$ so that ρ is constant on each Γ_i, and $\partial\rho/\partial\nu$ vanishes on $\partial\Omega$. We make ρ unique by choosing the value of ρ on Γ_0: $\rho = 0$ on Γ_0.

Now let $\rho_\pm$ denote the values of ρ on both sides of Σ, Σ_+ and Σ_-. For an arbitrary w they could be different; but if w vanishes on $\partial\Omega$ and ρ is constant on each Γ_i, they must be the same: if $M_\pm$, $P_\pm$ are two points of $\Sigma_\pm$, $M_\pm \in \partial\Omega$, we have

$$\rho(P_+) - \rho(M_+) = \int_{\Sigma(M_+ \to P_+)} w\, dx = \int_{\Sigma(M_- \to P_-)} w\, dx$$

$$= \rho(P_-) - \rho(M_-),$$

and since $\rho(M_+) = \rho(M_-)$, we see that $\rho(P_+) = \rho(P_-)$ on Σ.

2.6. *Eigenfunctions of the Stokes Problem*

Let Ω be any open bounded set in $\mathbb{R}^n$. The mapping $\Lambda : f \to 1/\nu\, u$ defined by Theorem 2.1 is clearly linear and continuous from $L^2(\Omega)$ into V and hence into $H_0^1(\Omega)$. Since Ω is bounded the natural injection of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact[1] and therefore Λ considered as a linear operator in $L^2(\Omega)$ is compact. This operator is also self-adjoint as

$$(\Lambda f_1, f_2) = \nu\,((u_1, u_2)) = (f_1, \Lambda f_2)$$

when $\Lambda f_i = u_i$, $i = 1, 2$.

[1] The Compactness Theorems in Sobolev Spaces are recalled in Chapter II, § 1.

Hence, this operator Λ possesses an orthonormal sequence of eigenfunctions w_j: $\Lambda w_j = \lambda_j w_j$, $j \geq 1$, $\lambda_j > 0$, $\lambda_j \to +\infty$, $j \to +\infty$, :

$$w_j \in V,\ ((w_j, v)) = \lambda_j (w_j, v), \quad \forall v \in V. \tag{2.64}$$

As usual

$$(w_j, w_k) = \delta_{jk},$$
$$((w_j, w_k)) = \lambda_j \delta_{jk}, \quad \forall j, k. \tag{2.65}$$

Using again Theorem 2.1 we can interpret (2.64) as follows: for each j, there exists $p_j \in L^2_{\mathrm{loc}}(\Omega)$ such that

$$-\nu \Delta w_j + \operatorname{grad} p_j = \lambda_j w_j \text{ in } \Omega,$$
$$\operatorname{div} w_j = 0 \text{ in } \Omega,$$
$$\gamma_0 w_j = 0 \text{ on } \Gamma.$$

If Ω is a Lipschitz set then $p_j \in L^2(\Omega)$, due to Proposition 1.2. If Ω is of class $\mathscr{C}^m$ (m an integer ≥ 2), a reiterated application of Proposition 2.2 shows that

$$w_j \in \boldsymbol{H}^m(\Omega), \quad p_j \in \boldsymbol{H}^{m-1}(\Omega), \qquad \forall j \geq 1. \tag{2.67}$$

If Ω is of class $\mathscr{C}^\infty$ then, by (2.65),

$$w_j \in \mathscr{C}^\infty(\overline{\Omega}), \quad p_j \in \mathscr{C}^\infty(\overline{\Omega}), \qquad \forall j \geq 1. \tag{2.68}$$

§3. Discretization of the Stokes equations (I)

Section 3.1 deals with the general concept of approximation of a normed space and Section 3.2 contains a general convergence theorem for the approximation of a general variational problem. In the last section and throughout all of Section 4 we describe some particular approximations of the basic space V of the Navier-Stokes equations. We give the corresponding numerical scheme for the Stokes equations and then apply the general convergence theorem to this case.

In Section 3.3 we consider the finite difference method. Finite element methods will be treated in the next section (Section 4; 4.1 to 4.5).

The approximations of the space V introduced here will be used throughout subsequent chapters, and they will be referred to as (APX1), (APX2), ...

3.1. *Approximation of a normed space*

When computational methods are involved, a normed space W must be approximated by a family $(W_h)_{h \in \mathcal{H}}$ of normed spaces W_h. The set $\mathcal{H}$ of indices depends on the type of approximation considered: we will consider below the main situations for $\mathcal{H}$, i.e., $\mathcal{H} = \mathbb{N}$ (= positive integers) for the Galerkin method, $\mathcal{H} = \Pi_{j=1}^{n}(0, h_j^0]$ for finite differences, and $\mathcal{H}$ = a set of triangulations of the domain Ω for finite element methods. The precise form of $\mathcal{H}$ need not be known; we need only to know that there exists a filter on $\mathcal{H}$, and we are concerned with passing to the limit through this filter. For the sake of simplicity we will always speak about passage to the limit as "$h \to 0$", which is, strictly speaking, the correct terminology for finite differences; definitions and results can be easily adapted to the other cases.

Definition 3.1. An *internal approximation* of a normed vector space W is a set consisting of a family of triples $\{W_h, p_h, r_h\}$, $h \in \mathcal{H}$ where
(i) W_h is a normed vector space;
(ii) p_h is a linear continuous operator from W_h into W;
(iii) r_h is a (perhaps nonlinear) operator from W into W_h.

The natural way to compare an element $\boldsymbol{u} \in W$ and an element $\boldsymbol{u}_h \in W_h$ is either to compare $p_h \boldsymbol{u}_h$ and $\boldsymbol{u}$ in W or to compare $\boldsymbol{u}_h$ and $r_h \boldsymbol{u}$ in W_h. The first point of view is certainly more interesting as we make comparisons in a fixed space. Nevertheless comparisons in W_h can also be useful. (See Fig. 3.)

Another way to compare an element $\boldsymbol{u} \in W$ is to compare a certain image $\bar{\omega}\boldsymbol{u}$ of $\boldsymbol{u}$ in some other space F, with a certain image $p_h \boldsymbol{u}_h$ of $\boldsymbol{u}_h$

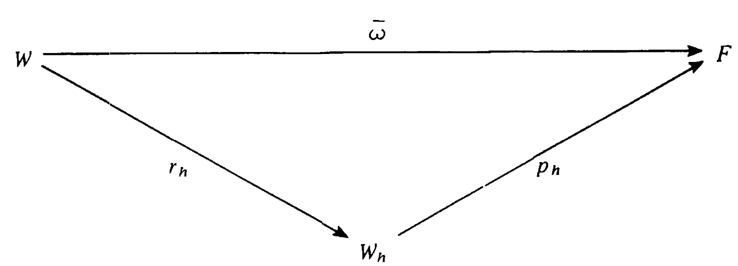


Fig. 3

in F. This leads to the concept of external approximation of a space W, which contains the concept of internal approximation as a particular case.

Definition 3.2. An external approximation of a normed space W is a set consisting of

(i) a normed space F and an isomorphism $\bar{\omega}$ of W into F.

(ii) a family of triples $\{W_h, p_h, r_h\}_{h \in \mathcal{H}}$, in which, for each h,

- W_h is a normed space
- p_h a linear continuous mapping of W_h into F
- r_h a (perhaps nonlinear) mapping of W into W_h.

When $F = W$ and $\bar{\omega}$ = identity, we get of course an internal approximation of W. It is easy to specialize what follows to internal approximations.

In most cases, W_h are finite dimensional spaces; rather often the operators p_h are injective. In some cases the operators r_h are linear, or linear only on some subspace of W, but there is no need to impose this condition in the general case; also, no continuity property of the r_h is required.

The operators p_h and r_h are called *prolongation* and *restriction operators,* respectively. When the spaces W and F are Hilbert spaces, and when the spaces W_h are likewise Hilbert spaces, the approximation is said to be a Hilbert approximation.

Definition 3.3. For given h, $\boldsymbol{u} \in W$, $\boldsymbol{u}_h \in W_h$, we say that

(i) $\|\bar{\omega}\boldsymbol{u} - p_h \boldsymbol{u}_h\|_F$ is the error between $\boldsymbol{u}$ and $\boldsymbol{u}_h$,

(ii) $\|\boldsymbol{u}_h - r_h \boldsymbol{u}\|_{W_h}$ is the discrete error between $\boldsymbol{u}$ and $\boldsymbol{u}_h$,

(iii) $\|\bar{\omega}\boldsymbol{u} - p_h r_h \boldsymbol{u}\|_F$ is the truncation error of $\boldsymbol{u}$.

We now defi e stable and convergent approximations.

Definition 3.4. The prolongation operators p_h are said to be stable if their norms

$$\|p_h\| = \sup_{\substack{u_h \in W_h \\ \|u_h\|_{W_h} = 1}} \|p_h u_h\|_F$$

can be majorized independently of h.

The approximation of the space W is said to be stable if the prolongation operators are stable.

Let us now consider what happens when "$h \to 0$".

Definition 3.5. We will say that a family $\boldsymbol{u}_h$ converges strongly (or weakly) to $\boldsymbol{u}$ if $p_h \boldsymbol{u}_h$ converges to $\bar{\omega}\boldsymbol{u}$ when $h \to 0$ in the strong (or weak) topology of F.

We will say that the family $\boldsymbol{u}_h$ converges discretely to $\boldsymbol{u}$ if

$$\lim_{h \to 0} \|\boldsymbol{u}_h - r_h \boldsymbol{u}\|_{W_h} = 0.$$

Definition 3.6. We will say that an external approximation of a normed space W is convergent if the two following conditions hold:

(C1) for all $\boldsymbol{u} \in W$

$$\lim_{h \to 0} p_h r_h \boldsymbol{u} = \bar{\omega}\boldsymbol{u}$$

in the strong topology of F.

(C2) for each sequence $\boldsymbol{u}_{h'}$ of elements of $W_{h'}$, $(h' \to 0)$, such that $p_{h'}\boldsymbol{u}_{h'}$ converges to some element ϕ in the weak topology of F, we have, $\phi \in \bar{\omega}\, W$; i.e., $\phi = \bar{\omega}\boldsymbol{u}$ for some $\boldsymbol{u} \in W$. □

Remark 3.1. Condition (C2) disappears when $\bar{\omega}$ is surjective and especially in the case of internal approximations. □

The following proposition shows that condition (C1) can in some sense be weakened for internal and external approximations.

Proposition 3.1. *Let there be given a stable external approximation of a space W which is convergent in the following restrictive sense: the operators r_h are defined only on a dense subset $\mathscr{W}$ of W and condition* (C1) *in Definition 3.6 holds only for the $\boldsymbol{u}$ belonging to $\mathscr{W}$ (condition* (C2) *remains unchanged).*

Then it is possible to extend the definition of the restriction operators r_h to the whole space W so that condition (C1) *is valid for each $\boldsymbol{u} \in W$ and hence the approximation of W is stable and convergent without any restriction.*

Proof. Let $\boldsymbol{u} \in W$, $\boldsymbol{u} \notin \mathscr{W}$; we must define in some way $r_h \boldsymbol{u} \in W_h, \forall\, h$, so that $p_h r_h \boldsymbol{u} \to \bar{\omega}\boldsymbol{u}$ as $h \to 0$. This element $\boldsymbol{u}$ can be approximated by elements in $\mathscr{W}$, and these elements in turn can be approximated by elements in the space $p_h W_h$; we have only to suitably combine these two approximations.

For each integer $n \geqslant 1$, there exists $\boldsymbol{u}_n \in \mathscr{W}$ such that $\|\boldsymbol{u}_n - \boldsymbol{u}\|_W \leqslant 1/n$ and hence

$$\|\bar{\omega}\boldsymbol{u}_n - \bar{\omega}\boldsymbol{u}\|_F \leqslant \frac{c_0}{n}, \tag{3.1}$$

where c_0 is the norm of the isomorphism $\bar{\omega}$.

For each fixed integer n, $p_h r_h u_n$ converges to $\bar{\omega}\boldsymbol{u}_n$ in F as $h \to 0$. Thus there exists some $\eta_n > 0$, such that $|h| \leqslant \eta_n$ implies

$$\|p_h r_h \boldsymbol{u}_n - \bar{\omega}\boldsymbol{u}_n\|_F \leqslant \frac{1}{n}.$$

We can suppose that η_n is less than both η_{n-1} and $1/n$ so that the η_n form a strictly decreasing sequence converging to 0:

$$0 < \eta_{n+1} < \ldots < \eta_1\,;\;\; \eta_n \to 0.$$

Let us define $r_h \boldsymbol{u}$ by

$$r_h \boldsymbol{u} = r_h \boldsymbol{u}_n \quad \text{for} \quad \eta_{n+1} < |h| \leqslant \eta_n.$$

It is clear that for $\eta_{n+1} < |h| \leqslant \eta_n$,

$$\|\bar{\omega}\boldsymbol{u} - p_h r_h \boldsymbol{u}\|_F \leqslant \|\bar{\omega}\boldsymbol{u} - \bar{\omega}\boldsymbol{u}_n\|_F + \|\bar{\omega}\boldsymbol{u}_n - p_h r_h \boldsymbol{u}_n\|_F$$
$$+ \|p_h r_h \boldsymbol{u}_n - p_h r_h \boldsymbol{u}\|_F \leqslant \frac{(1+c_0)}{n}$$

and consequently

$$\|\bar{\omega}\boldsymbol{u} - p_h r_h \boldsymbol{u}\|_F \leqslant \frac{1+c_0}{n}.$$

for $|h| \leqslant \eta_n$. This implies the convergence of $p_h r_h \boldsymbol{u}$ to $\bar{\omega}\boldsymbol{u}$ as $h \to 0$ and completes the proof.

Remark 3.2. If the mappings r_h are defined on the whole space W and condition (C1) holds for all $u \in \mathscr{W}$, Proposition 3.1 shows us that we can modify the value of $r_h u$ on the complement of $\mathscr{W}$ so that condition (C1) is satisfied for all $u \in W$.

Galerkin approximation of a normed space.

As a very easy example we can define a Galerkin approximation of a separable normed space W.

Let W_h, $h \in N = \mathcal{H}$, be an increasing sequence of finite dimensional subspaces of W whose union is dense in W. For each h, let p_h be the canonical injection of W_h into W, and for any $u \in W_{h_0}$ which does not belong to W_{h_0-1}, let $r_h u = 0$ if $h \leqslant h_0$, and $r_h u = u$ if $h > h_0$. It is clear that $p_h r_h u \to u$ as $h \to \infty$, for any $u \in \cup_{h\in N} W_h$. The operator r_h is defined only on $\mathscr{W} = \cup_{h \in N} W_h$ which is dense in W. Since the prolongation operators have norm one they are stable, and according to Proposition 3.1 the definition of the operators r_h can be extended in some way (which does not matter) to the whole space W so that we get a stable convergent internal approximation of W; this is a Galerkin approximation of W.

3.2. *A general convergence theorem*

Let us now discuss the approximation of the general variational problem (2.12). W is a Hilbert space, $a(u, v)$ is a coercive bilinear continuous form on $W \times W$,

$$a(u, u) \geqslant \alpha \|u\|_W^2, \quad \forall u \in W, (\alpha > 0), \tag{3.2}$$

and ℓ is a linear continuous form on W.

Let u denote the unique solution in W of

$$a(u, v) = \langle \ell, v \rangle, \quad \forall v \in W. \tag{3.3}$$

With respect to the approximation of this element u, let there be given an external stable and convergent Hilbert approximation of the space W, say $\{W_h, p_h, r_h\}_{h \in \mathcal{H}}$. Likewise, for each $h \in \mathcal{H}$, let there be given

(i) a continuous bilinear form $a_h(u_h, v_h)$ on $W_h \times W_h$ which is coercive and which, more precisely, satisfies

$\exists \alpha_0 > 0$, independent of h, such that

$$a_h(u_h, u_h) \geqslant \alpha_0 \;\|u_h\|_h^2, \; \forall \; u_h \in W_h \tag{3.4}$$

where $\|\cdot\|_h$ stands for the norm in W_h,

(ii) a continuous linear form on W_h, $\ell_h \in W_h'$, such that

$$\|\ell_h\|_{*h} \leqslant \beta, \tag{3.5}$$

in which $\|\cdot\|_{*h}$ stands for the norm in W_h', and in which β is independent of h.

We now associate with equation (3.3) the following family of approximate equations:

For fixed $h \in \mathcal{H}$, find $u_h \in W_h$ such that

$$a_h(u_h, v_h) = \langle \ell_h, v_h \rangle, \forall \; v_h \in W_h. \tag{3.6}$$

By the preceding hypotheses, Theorem 2.2 (in which W, W', a and ℓ are replaced by W_h, W'_h, a_h, and ℓ_h respectively) now asserts that equation (3.6) has a unique solution; we will say that u_h is the approximate solution of equation (3.3).

A general theorem on the convergence of the approximate solutions u_h to the exact solution will be given after defining precisely the manner in which the forms a_h and ℓ_h are consistent with the forms a and ℓ. We make the following consistency hypotheses:

If the family v_h converges weakly to v as $h \to 0$, and if the family w_h converges strongly to w as $h \to 0$, then

$$\lim_{h \to 0} a_h(v_h, w_h) = a(v, w)$$
$$\lim_{h \to 0} a_h(w_h, v_h) = a(w, v). \tag{3.7}$$

If the family v_h converges weakly to v as $h \to 0$, then

$$\lim_{h \to 0} \langle \ell_h, v_h \rangle = \langle \ell, v \rangle. \tag{3.8}$$

The general convergence theorem is then

Theorem 3.1. *Under the hypotheses* (3.2), (3.4), (3.5), (3.7) *and* (3.8), *the solution* u_h *of* (3.6) *converges strongly to the solution* u *of* (3.3), *as* $h \to 0$.

Proof. Putting $v_h = u_h$ in (3.6) and using (3.4) and (3.5), we find

$$a_h(u_h, u_h) = \langle \ell_h, u_h \rangle, \tag{3.9}$$

$$\alpha_0 \|u_h\|_h^2 \leqslant \|\ell_h\|_{*h} \|u_h\|_h \leqslant \beta \|u_h\|_h\,;$$

hence

$$\|u_h\|_h \leqslant \beta/\alpha_0. \tag{3.10}$$

As the operators p_h are stable, there exists a constant c_0 which majorizes the norm of these operators

$$\|p_h\| = \|p_h\|_{\mathcal{L}(W_h,\, F)} \leqslant c_0\,; \tag{3.11}$$

and hence

$$\|p_h u_h\|_F \leqslant \frac{c_0 \beta}{\alpha_0}. \tag{3.12}$$

Under these conditions, there exists some $\phi \in F$ and a sequence h' converging to 0, such that

$$\lim_{h' \to 0} p_{h'} u_{h'} = \phi$$

in the weak topology of F; according to hypothesis (C2) in Definition 3.6, $\phi \in \bar{\omega} W$, whence $\phi = \bar{\omega} u_*$ for some $u_* \in W$:

$$\lim_{h' \to 0} p_{h'} u_{h'} = \bar{\omega} u_*, \text{ (weak topology of } F). \tag{3.13}$$

Let us show that $u_* = u$. For a fixed $v \in W$, we write (3.6) with $v_h = r_h v$ and then take the limit with the sequence h' which gives, by using (3.7), (3.8), and (3.13):

$$a_h(u_h, r_h v) = \langle \ell_h, r_h v \rangle$$

$$\lim_{h' \to 0} a_{h'}(u_{h'}, r_{h'} v) = a(u_*, v)$$

$$\lim_{h' \to 0} \langle \ell_{h'}, r_{h'} v \rangle = \langle \ell, v \rangle.$$

Finally

$$a(u_*, v) = \langle \ell, v \rangle,$$

and because $v \in W$ is arbitrary, u_* is a solution of (3.3) and thus $u_* = u$.

One may show in exactly the same way, that from every subsequence of $p_h u_h$, one can extract a subsequence which converges in the weak topology of F to $\bar{\omega} u$. This proves that $p_h u_h$ as a whole converges to $\bar{\omega} u$ in the weak topology, as $h \to 0$.

Proof of the Strong Convergence. Let us consider the expression

$$X_h = a_h(u_h - r_h u, u_h - r_h u),$$

or

$$X_h = a_h(u_h, u_h) - a_h(u_h, r_h u) - a_h(r_h u, u_h) + a_h(r_h u, r_h u).$$

By (3.7), (3.8), and (3.9),

$$\lim_{h \to 0} a_h(u_h, u_h) = \langle \ell, u \rangle$$

$$\lim_{h \to 0} a_h(u_h, r_h u) = \lim_{h \to 0} a_h(r_h u, u_h) = \lim_{h \to 0} a_h(r_h u, r_h u) = a(u, u).$$

Finally

$$\lim_{h \to 0} X_h = -a(u, u) + \langle \ell, u \rangle = 0, \tag{3.14}$$

according to (3.3) (when $v = u$).

With (3.4) and (3.11) we now get

$$0 \leqslant \alpha_0 \|u_h - r_h u\|_h^2 \leqslant X_h, \text{ whence}$$

$$0 \leqslant \|p_h u_h - p_h r_h u\|_F^2 \leqslant \frac{c_0^2}{\alpha_0} X_h \to 0.$$

Using now condition (C1) of Definition 3.6 and

$$\|p_h u_h - \bar{\omega} u\|_F \leqslant \|p_h u_h - p_h r_h u\|_F + \|p_h r_h u - \bar{\omega} u\|_F,$$

we see that this converges to 0 as $h \to 0$.

The theorem is proved.

Remark 3.3. As announced in Remark 2.2, point (ii), Theorem 3.1 is applicable to the Galerkin approximation of (3.3) used in the proof of Theorem 2.2. One takes $W_h = W_m$, $\forall\, h = m \in \mathcal{H}$, $\mathcal{H} = \mathcal{N}$, and as in the example at the end of Section 3.2 one obtains a Galerkin approximation of W. With

$$a_h(v, w) = a(v, w), \; \langle \ell_h, v\rangle = \langle \ell, v\rangle, \; \forall\, v, \mathbf{w} \in W,$$

Theorem 3.1 is applicable and shows that u_m converges to u in the strong topology of W as $m \to \infty$.

Remark 3.4. If $\{W_{ih}\}_{1 \leqslant i \leqslant N(h)}$ constitutes a basis of W_h, then the approximate problem (3.6) is equivalent to a regular linear system for the components of u_h in this basis; i.e., if

$$u_h = \sum_{i=1}^{N(h)} \xi_{ih} w_{ih}$$

$$\sum_{i=1}^{N(h)} \xi_{ih} a_h(w_{ih}, w_{jh}) = \langle \ell_h, w_{jh}\rangle, \; 1 \leqslant j \leqslant N(h). \tag{3.15}$$

The solution of (3.15) is obtained by the usual methods for algebraic linear systems.

When a basis of W_h cannot be easily constructed (and this happens sometimes for the Stokes problem), some special method must be found to actually solve (3.6).

3.3. *Approximation by finite differences*

We study the approximation by finite differences of the space $H_0^1(\Omega)$, then the same for the space V, and finally the approximation of Stokes problem by the corresponding scheme. The approximation of V considered here will be denoted by (APX1).

3.3.1. *Notation*

When working with finite differences, h denotes the vector-mesh, $h = (h_1, \ldots, h_n)$ where h_i is the mesh in the x_i direction and thus

$$0 < h_i \leqslant h_i^0,$$

for some strictly positive numbers h_i^0; hence

$$\mathcal{H} = \prod_{i=1}^{n} (0, h_i^0). \tag{3.16}$$

We are interested in passing to the limit $h \to 0$.

For all $h \in \mathcal{H}$ we define:

(i) $\boldsymbol{h}_i$ is the vector $h_i e_i$, where the j^{th} coordinate of e_i is δ_{ij} = the Kronecker delta.

(ii) R_h is the set of points of $\mathbb{R}^n$ of the form $j_1 \boldsymbol{h}_1 + \ldots + j_n \boldsymbol{h}_n$, in which the j_i are integers of arbitrary sign ($j_i \in \mathbb{Z}$).

(iii) $\sigma_h(M)$, $M = (\mu_1, \ldots, \mu_n)$, is the set

$$\prod_{i=1}^{n} \left(\mu_i - \frac{h_i}{2}, \mu_i + \frac{h_i}{2} \right)$$

and is called a *block.*

(iv) $\sigma_h(M, r)$ is the set

$$\bigcup_{\substack{1 \leqslant i \leqslant n \\ -r \leqslant \alpha \leqslant +r}} \sigma_h\left(M + \frac{\alpha}{2} \boldsymbol{h}_i\right);$$

of course $\sigma_h(M, 0) = \sigma_h(M)$.

(v) w_{hM} is the characteristic function of the block $\sigma_h(M)$.

(vi) δ_{ih} (or δ_i if no confusion can arise) is the finite difference operator

$$(\delta_i \phi)(x) = \frac{\phi(x + \frac{1}{2}\boldsymbol{h}_i) - \phi(x - \frac{1}{2}\boldsymbol{h}_i)}{h_i} \tag{3.17}$$

If $j = (j_1, \ldots, j_n) \in N^n$ is a multi-index, then δ_h^j (or simply δ^j) will denote the operator

$$\delta^j = \delta_1^{j_1} \ldots \delta_n^{j_n}. \tag{3.18}$$

(vii) With each open set Ω of $\mathbb{R}^n$ and each non-negative integer r we associate the following point sets

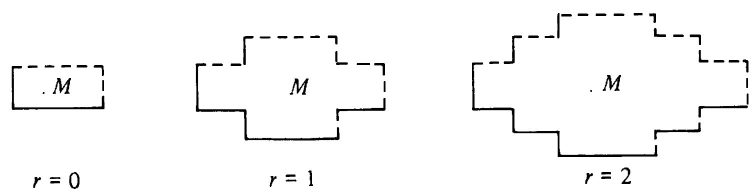
$$\mathring{\Omega}_h^r = \{M \in R_h, \sigma_h(M, r) \subset \Omega\} \tag{3.19}$$

$$\Omega_h^r = \{M \in \mathcal{R}_h, \sigma_h(M, r) \cap \Omega \neq \phi\} \tag{3.20}$$

(viii) Sometimes we will use other finite difference operators such as ∇_{ih} and $\overline{\nabla}_{ih}$ (also denoted ∇_i and $\overline{\nabla}_i$);

$$\nabla_{ih}\, \phi(x) = \frac{\phi(x+h_i) - \phi(x)}{h_i} \tag{3.21}$$

$$\overline{\nabla}_{ih}\, \phi(x) = \frac{\phi(x) - \phi(x-h_i)}{h_i} \tag{3.22}$$



Examples of Sets $\sigma_h(M, r)$ in the Plane.

3.3.2. *External approximation of $H_0^1(\Omega)$*

Let Ω be a Lipschitz open bounded set in $\mathbb{R}^n$. Let $W = H_0^1(\Omega)$, $F = L^2(\Omega)^{n+1}$ equipped with the natural Hilbert scalar product, and let $\bar{\omega}$ be the mapping

$$u \to \bar{\omega} u = (u, D_1 u, \ldots, D_n u) \tag{3.23}$$

from W into F. It is clear that

$$\|\bar{\omega} u\|_F = [\![u]\!]_{H_0^1(\Omega)}$$

so that $\bar{\omega}$ is an isomorphism from W into F.

Space W_h: With the preceding notation, W_h will be the space of step functions

$$u_h(x) = \sum_{M \in \mathring{\Omega}_h^1} u_h(M) w_{hM}(x),\; u_h(M) \in \mathbb{R}^n. \tag{3.24}$$

The functions w_{hM} for $M \in \mathring{\Omega}_h^1$ are linearly independent and span the whole space W_h; they form a basis of W_h. The dimension of W_h is n

times the number $N(h)$ of points $M \in \Omega_h^1$; W_h is finite dimensional. This space is provided with the scalar product

$$[\![u_h, v_h]\!]_h = \int_\Omega u_h(x) v_h(x)\,dx + \sum_{i=1}^{n} \int_\Omega \delta_i u_h(x)\, \delta_i v_h(x)\,dx \tag{3.25}$$

which makes it a Hilbert space.

The functions u_h and $\delta_i u_h$, $1 \leqslant i \leqslant n$, have compact supports in Ω, by the definition of W_h and the set Ω_h^1. Hence they will be considered as vector functions defined on Ω or on $\mathbb{R}^n$.

Operators p_h. The prolongation operators p_h are the discrete analogue of $\bar{\omega}$:

$$p_h u_h = (u_h, \delta_1 u_h, \ldots, \delta_n u_h), \ \forall\, u_h \in W_h. \tag{3.26}$$

The norm of p_h is exactly one,

$$\| p_h u_h \|_F = [\![u_h]\!]_h$$

and they are stable.

Operators r_h: As a consequence of Proposition 3.1 we need only define the operator r_h on $\mathscr{W} = \mathscr{D}(\Omega)$ which is a dense subspace of $H_0^1(\Omega)$; we put

$$(r_h u)(M) = u(M), \quad \forall\, M \in \mathring{\Omega}_h^1, \quad \forall\, u \in \mathscr{D}(\Omega) \tag{3.27}$$

which completely defines $r_h u \in W_h$.

Proposition 3.2. *The preceding external approximation of* $H_0^1(\Omega)$ *is stable and convergent.*

Proof. The approximation is stable since the prolongation operators are stable.

We must now check conditions (C1) and (C2) of Definition 3.6.

Lemma 3.1. *Condition* (C1) *is satisfied:* $\forall\, u \in \mathscr{D}(\Omega)$,

$$r_h u \to u \text{ in } L^2(\Omega), \tag{3.28}$$

$$\delta_i r_h u \to D_i u \text{ in } L^2(\Omega), \tag{3.29}$$

as $h \to 0$.

Proof. Let $u \in \mathscr{D}(\Omega)$ and let h be sufficiently small for the support of u to be included in the set

$$\Omega(h) = \bigcup_{M \in \mathring{\Omega}_h^1} \sigma_h(M) \tag{3.30}$$

For any $M \in \mathring{\Omega}_h^1$, for any $x \in \sigma_h(M)$, the Taylor formula gives (with $u_h = r_h u$)

$$|u_h(x) - u(x)| = |u(M) - u(x)| \leqslant c_1(u)|M - x| \leqslant \frac{c_1(u)}{2}|h|$$

where

$$c_1(u) = \operatorname*{Sup}_{x \in \Omega} |\text{grad}\, u(x)| \tag{3.31}$$

$$|h| = \left(\sum_{i=1}^{n} h_i^2\right)^{1/2}. \tag{3.32}$$

Then

$$\operatorname*{Sup}_{x \in \Omega(h)} |u_h(x) - u(x)| \leqslant \frac{c_1(u)}{2}|h|. \tag{3.33}$$

On the set $\Omega - \Omega(h)$,

$$|u(x)| \leqslant c_1(u)\mathrm{d}(x, \Gamma),$$

$$|u_h(x) - u(x)| \leqslant c_1(u)\mathrm{d}(\Omega(h), \Gamma). \tag{3.34}$$

Hence

$$\operatorname*{Sup}_{x \in \Omega} |u_h(x) - u(x)| \leqslant c_1(u)\left\{\frac{|h|}{2} + \mathrm{d}(\Omega(h), \Gamma)\right\} \tag{3.35}$$

which converges to 0 as $h \to 0$; u_h converges to u in $L^\infty(\Omega)$ and then in $L^2(\Omega)$ since Ω is bounded.

To prove (3.29), we use again Taylor's formula: $\forall\ M \in \mathring{\Omega}_h^1$, $\forall\ x \in \sigma_h(M)$,

$$
\begin{aligned}
&|\delta_i u_h(x) - D_i u(x)| \\
&= \left| \frac{1}{h_i} \left[u_h\left(M + \frac{1}{2} h_i\right) - u_h\left(M - \frac{1}{2} h_i\right)\right] - \mathrm{D}_i u(x) \right| \\
&\leqslant c_2(u)|h|,
\end{aligned} \tag{3.36}
$$

where $c_2(u)$ depends only on the maximum norm of the second derivatives of u.

On the set $\Omega - \Omega(h)$,

$$
|\mathrm{D}_i u(x)| \leqslant c'_2(u)\mathrm{d}(\Omega(h), \Gamma)
$$

and then on the whole set Ω

$$
|\delta_i u_h(x) - \mathrm{D}_i u(x)| \leqslant c_2(u)|h| + c'_2(u)\,\mathrm{d}(\Omega(h), \Gamma). \tag{3.37}
$$

This converges to 0 as $h \to 0$, and shows the convergence of $\delta_i u_h$ to $\mathrm{D}_i u$ in the uniform and L^2 norms.

Lemma 3.2. *Condition* (C2) *is satisfied.*

Proof. Let there be given a sequence $u_{h'} \in W_{h'}$, $h' \to 0$, such that $p_{h'} u_{h'}$ converges to ϕ in the weak topology of F, as $h' \to 0$. This means

$$
\begin{aligned}
&\lim_{h' \to 0} u_{h'} = \phi_0 \\
&\lim_{h' \to 0} \delta_{ih'} u_{h'} = \phi_i, \qquad 1 \leqslant i \leqslant n
\end{aligned} \tag{3.38}
$$

in the weak topology of $L^2(\Omega)$; $\phi = (\phi_0, \dots, \phi_n)$. As the functions $u_{h'}$, $\delta_i u_{h'}$ have compact support in Ω, we also have

$$
\begin{aligned}
&\lim_{h' \to 0} u_{h'} = \tilde{\phi}_0 \\
&\lim_{h' \to 0} \delta_i u_{h'} = \tilde{\phi}_i, \qquad 1 \leqslant i \leqslant n,
\end{aligned} \tag{3.39}
$$

in the weak topology of $L^2(\mathcal{R}^n)$; here $\tilde{g}$ means the function equal to g in Ω and equal to 0 in the complement of Ω.

A discrete integration by parts formula gives

$$\int_{\mathbb{R}^n} \delta_{ih}\, u_{h'}(x)\sigma(x)\mathrm{d}x = -\int_{\mathbb{R}^n} u_{h'}(x)\delta_{ih'}\sigma(x)\mathrm{d}x, \tag{3.40}$$

for each $\sigma \in \mathscr{D}(\mathbb{R}^n)$.

As $h' \to 0$, the left-hand side of (3.40) converges to

$$\int_{\mathbb{R}^n} \tilde{\phi}_i(x)\sigma(x)\,\mathrm{d}x;$$

the right-hand side converges to

$$-\int_{\mathbb{R}^n} \tilde{\phi}_0(x)\,D_i\sigma(x)\,\mathrm{d}x,$$

since $\delta_{ih}\sigma$ converges to $D_i\sigma$ in $L^2(\mathbb{R}^n)$ as shown in Lemma 3.1. So

$$\int_{\mathbb{R}^n} \tilde{\phi}_i(x)\sigma(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} \tilde{\phi}_0(x)D_i\sigma(x)\,\mathrm{d}x,\ \forall\ \sigma \in \mathscr{D}(\mathbb{R}^n)$$

which amounts to saying that

$$\tilde{\phi}_i = D_i\tilde{\phi}_0,\quad 1 \leqslant i \leqslant n, \tag{3.41}$$

in the distribution sense.

It is clear now that $\tilde{\phi}_0 \in H^1(\mathbb{R}^n)$, and since $\tilde{\phi}_0$ vanishes in the complement of Ω, ϕ_0 belongs to $H_0^1(\Omega)$. Thus $\phi \in \bar{\omega}\, W$;

$$\phi = \bar{\omega}\phi_0,\ \phi_0 \in H_0^1(\Omega). \tag{3.42}$$

3.3.3. *Discrete Poincaré inequality*

The following discrete Poincaré inequality (see (1.9)) will allow us to equip the space W_h in (3.24) with another scalar product $((\cdot,\cdot))_h$, the discrete analogue of the scalar product $((\cdot,\cdot))$ (see (1.11)).

Proposition 3.3. *Let Ω be a set bounded in the x_i direction, and let u_h be a scalar step function of type* (3.24) (*with $u_h(M) \in \mathbb{R}$*). *Then*

$$|u_h| \leqslant 2\,\ell|\delta_{ih} u_h| \tag{3.43}$$

where ℓ is the width of Ω in this direction.

Proof. For the sake of simplicity we take $i = 1$. Since u_h has a compact support, for any $M \in R_h$,

$$u_h(M)^2 = \sum_{j=0}^{\infty} \{[u_h(M - jh_1)]^2 - [u_h(M - (j+1)h_1]^2\}$$

$$= h_1 \sum_{j=0}^{\infty} [\delta_{1h} u_h(M - (j+\tfrac{1}{2})h_1] [u_h(M - jh_1) +$$

$$u_h(M - (j+1)h_1)]$$

$$u_h(M)^2 \leqslant I = h_1 \sum_{j=-\infty}^{+\infty} |\delta_{1h} u_h(M - (j+\frac{1}{2})h_1)| [\,|u_h(M - jh_1)|$$

$$+ |u_h(M - (j+1)h_1)|\,]. \tag{3.44}$$

The sums are actually finite. Now for any $i \in Z$, $u_h\,(M+ih_1)^2$ is majorized by exactly the same expression I. There are less than ℓ/h_1 values of i such that $u_h(M+ih_1) \neq 0$ since the x_1-width of Ω is less than ℓ. Hence

$$\sum_{i=-\infty}^{+\infty} u_h(M+ih_1)^2 \leqslant \frac{\ell}{h_1} I. \tag{3.45}$$

Let $\mathscr{T}_h(M)$ denote the tube $\cup_{i=-\infty}^{+\infty} \sigma_h(M+ih_1)$. We can interpret (3.45) as follows:

$$\int_{\mathscr{T}h(M)} u_h(x)^2 \mathrm{d}x = (h_1 \ldots h_n) \sum_{i=-\infty}^{+\infty} u_h(M+ih_1)^2$$

$$\leqslant \ell(h_1 \ldots h_n)\ \frac{I}{h_1}$$

$$= \ell \int_{\mathscr{T}h\,(M)} |\delta_{1h} u_h(x)| \left\{ |u_h(x+\frac{1}{2}h_1)| + \right.$$

$$\left. |u_h(x-\frac{1}{2}h_1)| \right\} \mathrm{d}x.$$

We take summations of the last inequality for all tubes $\mathscr{F}_h(M)$ and obtain

$$\int_{\mathbb{R}^n} u_h(x)^2 \,\mathrm{d}x \leqslant \ell \int_{\mathbb{R}^n} |\delta_{1h} u_h(x)| \left\{ |u_h(x+\tfrac{1}{2}h_1)| + |u_h(x-\tfrac{1}{2}h_1)| \right\} \mathrm{d}x.$$

Applying Schwarz's inequality, we obtain

$$|u_h|^2 = \int_{\mathbb{R}^n} u_h(x)^2 \,\mathrm{d}x$$

$$\leqslant \ell |\delta_{1h} u_h| \cdot \left\{ 2 \int_{\mathbb{R}^n} [|u_h(x+\tfrac{1}{2}h_1)|^2 + |u_h(x-\tfrac{1}{2}h_1)|^2]\,\mathrm{d}x \right\}^{\frac{1}{2}}$$

$$\leqslant 2\ell |\delta_{1h} u_h| \cdot |u_h|$$

and (3.43) follows.

Proposition 3.4. *Let Ω be a bounded Lipschitz set. If we equip the space W_h with the scalar product*

$$((u_h, v_h))_h = \sum_{i=1}^{n} \int_{\Omega} \delta_{ih} u_h \, \delta_{ih} v_h \,\mathrm{d}x, \tag{3.46}$$

we have again a stable convergent approximation of $\boldsymbol{H}_0^1(\Omega)$.

Proof. The prolongation operators are stable as a consequence of Proposition 3.3.

Remark 3.5. Using the difference operators ∇_{ih}, or $\overline{\nabla}_{ih}$, or even any "reasonable" approximation of the differentiation operator $\partial/\partial x_i$, one can define many other similar approximations of the space $H_0^1(\Omega)$. The modifications arise then in (3.25), where δ_{ih} is replaced by $\nabla_{ih}, \dots$, in the set of points $\mathring{\Omega}_h^1$ which must be suitably defined and in some points of the proof of Lemma 3.1 and 3.2.

The same Poincaré inequality is valid for the operators ∇_{ih} and $\overline{\nabla}_{ih}$ but not for more general operators.

Remark 3.6. When Ω is unbounded, one can define an external approximation of $\boldsymbol{H}_0^1(\Omega)$ with a space W_h consisting of either:

– step functions $\Sigma_{M\in\mathring{\Omega}_h^1} \lambda_M w_{hM}$, which have compact support (we restrict the sum to a finite number of points $M\in\mathring{\Omega}_h^1$),

– or step functions $\Sigma\lambda_M w_{hM}$ for the M in the intersection of $\mathring{\Omega}_h^1$ with some "large" ball: $|\boldsymbol{x}|\leqslant\rho(h)$, where $\rho(h)\to+\infty$ as $h\to 0$.

In the second case W_h is finite dimensional but not in the first case.

Without any modification for the other elements of the approximation, it is clear that we obtain a stable convergent approximation of $\boldsymbol{H}_0^1(\Omega)$ for an unbounded locally Lipschitzian set Ω.

The discrete Poincaré inequality is available if Ω is bounded in one of the directions $x_1,\ldots,x_n$.

3.3.4. *Approximation of the space V (APX1).*

Let Ω be a Lipschitzian bounded set in $\mathbb{R}^n$ and let $\mathscr{V}$ be the usual space (1.12) and V its closure in $\boldsymbol{H}_0^1(\Omega)$.

We define now an approximation of V using finite differences (which will be denoted by (APX1)).

Let $F=\mathrm{L}^2(\Omega)^{n+1}$ equipped with the natural Hilbertian scalar product and let us define the mapping $\bar{\omega}\in\mathscr{L}(V,F)$:

$$u\to\bar{\omega}u=(u, D_1u,\ldots,D_nu).$$

It is clear that

$$\|\bar{\omega}u\|_F=[\![u]\!]$$

so that $\bar{\omega}$ is an isomorphism from V into F.

Space V_h. We take in the following V_h as the space W_h: V_h is the space of step functions

$$u_h(x)=\sum_{M\in\mathring{\Omega}_h^1} u_h(M)w_{hM}(x),\ \ u_h(M)\in\mathbb{R}^n, \tag{3.47}$$

which are discretely divergence free in the following sense:

$$\sum_{i=1}^n \nabla_{ih}u_{ih}(M)=0,\ \ \forall\, M\in\mathring{\Omega}_h^1 \tag{3.48}$$

which amounts to saying

$$\sum_{i=1}^{n} \nabla_{ih} u_{ih}(x) = 0, \ \forall\, x \in \Omega(h). \tag{3.49}$$

No basis of V_h is available; it is clear that V_h is a finite dimensional space with dimension less than or equal to $nN(h)-N(h) = (n-1)N(h)$ since all the functions in (3.47) form a space of dimension $nN(h)$ and there are at most $N(h)$ independent linear constraints in (3.48); it is not clear whether the constraints (3.48) are always linearly independent so that V_h is not necessarily of dimension $(n-1)N(h)$.

The space V_h is equipped with one of the scalar products

$$((u_h, v_h))_h = \sum_{i=1}^{n} \int_\Omega \delta_i u_h(x) \cdot \delta_i u_h(x) \mathrm{d}x \tag{3.50}$$

$$[\![u_h, v_h]\!]_h = \int_\Omega u_h(x) v_h(x) \mathrm{d}x + ((u_h, v_h))_h. \tag{3.51}$$

Because of Proposition 3.3 (discrete Poincaré inequality), V_h equipped with either one of these scalar products is a Hilbert space.
Operators p_h. These are the discrete analogue of $\bar{\omega}$:

$$p_h u_h = \{u_h, D_1 u_h, \ldots, D_n u_h\}. \tag{3.52}$$

These operators are stable since by (3.43)

$$\|p_h u_h\|_F = [\![u_h]\!]_h \leqslant c\,\|u_h\|_h, \ \forall\, u_h \in V_h. \tag{3.53}$$

Operators r_h. We define $r_h u$ only for $u \in \mathscr{V}$, a dense subspace of V; $\forall\, M = (m_1 h_1, \ldots, m_n h_n) \in \mathring{\Omega}_h^1$, $(r_h u)(M)$ is defined by

$$\begin{aligned} u_{ih}(M) &= i^{\text{th}} \text{ component of } u_h \\ &= \text{the average value of } u_i \text{ on the face} \\ &\quad x_i = (m_i - \tfrac{1}{2})h_i \text{ of } \sigma_h(M). \end{aligned} \tag{3.54}$$

This complicated definition of $r_h u$ is necessary if we want u_h to belong to V_h; actually

Lemma 3.3. *For each* $u \in \mathscr{V}$, $r_h u \in V_h$.

Proof. We have

$$\nabla_{ih} u_{ih}(M) = \frac{1}{h_1 \ldots h_n} \left\{ \int_{\Sigma_i} u_i(x) \mathrm{d}x - \int_{\Sigma_i'} u_i(x) \mathrm{d}x \right\}$$

where Σ_i and Σ'_i respectively are the faces $x_i = (m_i + 1/2)h_i$ and $x_i = (m_i - 1/2)h_i$ of $\sigma_h(M)$.

This gives also

$$\nabla_{ih} u_{ih}(M) = \frac{1}{h_1 \ldots h_n} \int_{\Sigma_i \cup \Sigma_i'} u . \nu \mathrm{d}\Gamma,$$

where ν stands for the unit vector normal to the boundary of $\sigma_h(M)$ and pointing in the outward direction. Then, for each $M \in \mathring{\Omega}_h^1$,

$$\begin{aligned}\sum_{i=1}^{n} \nabla_{ih} u_{ih}(M) &= \frac{1}{h_1 \ldots h_n} \int_{\partial \sigma_h(M)} u . \nu \mathrm{d}\Gamma \\ &= \text{(by Stokes formula)} \\ &= \frac{1}{h_1 \ldots h_n} \int_{\sigma_h(M)} \operatorname{div} u \, \mathrm{d}x = 0, \quad \text{since } \operatorname{div} u = 0.\end{aligned}$$

Conditions (3.48) are met.

Proposition 3.5. *The preceding external approximation of V is stable and convergent.*

Proof. Stability has been shown already.

The proof of condition (C1) is very similar to the proof of Lemma 3.1 and we do not repeat all the details; for example, for $x \in \sigma_h(M)$, $M \in \mathring{\Omega}_h^1$,

$$|u_{ih}(x) - u_i(x)| = |u_{ih}(\xi) - u_i(x)|,$$

where ξ is some point of the face $x_i = (m_i - 1/2)h_i$ of $\sigma_h(M)$, and hence

$$|u_{ih}(x) - u_i(x)| \leqslant c_1(u_i)|x - \xi| \leqslant c_1(u_i)|h|,$$

and (3.35) is replaced by

$$\sup_{x\in\Omega} |u_h(x)-u(x)| \leqslant c'_1(u)\{|h| + d(\Omega(h), \Gamma)\}. \tag{3.55}$$

The proof for condition (C2) is similar to the proof of Lemma 3.2; more precisely, the same proof as for Lemma 3.2 shows that if

$$p_{h'}u_{h'} \to \phi \text{ in the weak topology of } F,$$

as $h' \to 0$, then $\phi = \bar{\omega}\boldsymbol{u} = (\boldsymbol{u}, D_1\boldsymbol{u}, \ldots, D_n\boldsymbol{u})$, where $\boldsymbol{u} \in H_0^1(\Omega)$. Because of Theorem 1.6, proving that $\boldsymbol{u} \in V$ now amounts to proving that $\operatorname{div} \boldsymbol{u} = 0$. This follows from (3.49) as we will now show. Let σ be any test function in $\mathscr{D}(\Omega)$, and let us suppose that h is small enough for the support of σ to be included in $\Omega(h)$; then (3.49) shows that

$$\int_\Omega \left(\sum_{i=1}^n (\nabla_{ih}u_{ih})(x)\right)\sigma(x)\mathrm{d}x = 0$$

or

$$\int_{\mathcal{R}^n} \left(\sum_{i=1}^n (\nabla_{ih}u_{ih})(x)\right)\sigma(x)\mathrm{d}x = 0. \tag{3.56}$$

It is easy to check the discrete integration by parts formula

$$\int_{\mathcal{R}^n} (\nabla_{ih}\theta)(x)\cdot\sigma(x)\mathrm{d}x = -\int_{\mathcal{R}^n} \theta(x)(\overline{\nabla}_{ih}\sigma)(x)\mathrm{d}x; \tag{3.57}$$

then (3.56) becomes

$$\int_{\mathcal{R}^n} \sum_{i=1}^n [u_{ih}(x)\cdot(\overline{\nabla}_{ih}\sigma)(x)]\,\mathrm{d}x = 0. \tag{3.58}$$

With a proof similar to that of Lemma 3.1 (based on Taylor's formula) we see that

$$\overline{\nabla}_{ih}\sigma \to -D_i\sigma, \text{ as } h\to 0, \tag{3.59}$$

in the (uniform and) L^2 norm. Since $\boldsymbol{u}_{ih}$ converges to $\boldsymbol{u}_i$ for the weak topology of $L^2(\mathcal{R}^n)$, letting $h \to 0$ in (3.58) gives the result

$$\int_{\mathcal{R}^n} \left[\sum_{i=1}^n \boldsymbol{u}_i(x)\cdot D_i\sigma(x)\right]\mathrm{d}x = 0. \tag{3.60}$$

The equality (3.60), true for any $\sigma \in \mathscr{D}(\Omega)$, implies that $\operatorname{div} \boldsymbol{u} = 0$ and then $\phi = \bar{\omega}\boldsymbol{u}$, with $\boldsymbol{u} \in V$.

Remark 3.7. The Remark 3.5 can be extended to the present case with, however, one restriction: condition (3.48) – (3.49) in the definition of the spaces V_h cannot be replaced by similar relationships involving other finite difference operators; for example, it seems impossible to replace (3.49) by

$$\sum_{i=1}^{n} \delta_{ih} u_{ih}(x) = 0, \tag{3.61}$$

since (3.61) requires many more algebraic relations than (3.49) and probably too many relations (in which case $V_h = \{0\}$).

Remark 3.8. When Ω is unbounded one can define, using the methods mentioned in Remark 3.5, a stable and convergent external approximation of the space W introduced in (2.31).

3.3.5. *Approximation of Stokes problem*

Using the above approximation of V and the results of Section 3.2, we can propose a finite difference scheme for the approximation of Stokes' problem. Let us take, for (3.6),

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h \tag{3.62}$$

$$\langle \ell_h, \boldsymbol{v}_h \rangle = (f, \boldsymbol{v}_h), \tag{3.63}$$

where V_h and $((\cdot,\cdot))_h$ are the space and scalar product just defined, and ν and f are given as in Section 2.1.

The approximate problem corresponding to (2.6) is then:

$$\begin{aligned} &\textit{To find } \boldsymbol{u}_h \in V_h \textit{ such that} \\ &\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h = (f, \boldsymbol{v}_h), \ \forall \ \boldsymbol{v}_h \in V_h. \end{aligned} \tag{3.64}$$

Proposition 3.6. *For all $h \in \mathcal{H}$ the solution u_h of* (3.64) *exists and is unique; moreover as $h \to 0$ the solution $\boldsymbol{u}_h$ of* (3.64) *converges to the* *solution* $\boldsymbol{u}$ *of* (2.6) *in the following sense:*

$$\boldsymbol{u}_h \to \boldsymbol{u} \text{ in } \boldsymbol{L}^2(\Omega), \tag{3.65}$$

$$\delta_{ih}\boldsymbol{u}_h \to D_i\boldsymbol{u} \text{ in } \boldsymbol{L}^2(\Omega). \tag{3.66}$$

Proof. We have only to check that Theorem 3.1 is applicable. Condition (3.4) is obvious ($\alpha_0 = 1$); for (3.5) we notice that

$$\begin{aligned} |\langle \ell_h, \boldsymbol{v}_h\rangle| = |(\boldsymbol{f}, \boldsymbol{v}_h)| &\leqslant |\boldsymbol{f}|\cdot|\boldsymbol{v}_h| \\ &\leqslant \text{(by the discrete Poincaré inequality)} \\ &\leqslant c(\Omega)|\boldsymbol{f}|\,\|\boldsymbol{v}_h\|_h, \quad \forall\, \boldsymbol{v}_h \in V_h. \end{aligned}$$

Hence

$$\|\ell_h\|_{*h} \leqslant c(\Omega)|\boldsymbol{f}|, \tag{3.67}$$

and (3.5) is satisfied.

For (3.7) – (3.8) we notice that

$$p_h\boldsymbol{v}_h \to \bar{\omega}\boldsymbol{v} \text{ weakly (respectively } p_h\boldsymbol{w}_h \to \bar{\omega}\boldsymbol{w} \text{ strongly)}$$

means

$$\boldsymbol{v}_h \to \boldsymbol{v}, \text{and } \delta_{ih}\boldsymbol{v}_h \to D_i\boldsymbol{v} \text{ in } \boldsymbol{L}^2(\Omega) \text{ weakly,}$$

(respectively

$$\boldsymbol{w}_h \to \boldsymbol{w}, \text{ and } \delta_{ih}\boldsymbol{w}_h \to D_i\boldsymbol{w} \text{ in } \boldsymbol{L}^2(\Omega) \text{ strongly,}$$

and it is clear that this implies,

$$(\delta_{ih}\boldsymbol{v}_h, \delta_{ih}\boldsymbol{w}_h) \to (D_i\boldsymbol{v}, D_i\boldsymbol{w})$$

$$((\boldsymbol{v}_h, \boldsymbol{w}_h))_h \to ((\boldsymbol{v}, \boldsymbol{w})),$$

$$(\boldsymbol{f}, \boldsymbol{v}_h) \to (\boldsymbol{f}, \boldsymbol{v}).$$

Approximation of the Pressure.

We want to present the "approximate" pressure which is implicitly contained in (3.64) as well as the exact pressure p is implicitly contained in (2.6).

The space V_h in (3.47) is a subspace of the space W_h in (3.24); namely, the space of $v_h \in W_h$ satisfying the linear constraints (3.48).

The form $v_h \to \nu((u_h, v_h))_h - (f, v_h)$ appears as a linear form defined on W_h which vanishes on V_h. Hence introducing the Lagrange multipliers corresponding to the linear constraints (3.48) we find, with the aid of a classical theorem of linear algebra, that there exist numbers $\lambda_M \in \mathcal{R}$, $M \in \overset{\circ}{\Omega}{}_h^1$, such that the equation

$$\nu((u_h,v_h))_h - (f, v_h) = \sum_{M \in \overset{\circ}{\Omega}{}_h^1} \lambda_M \sum_{i=1}^{n} (\nabla_{ih} v_{ih}(M)), \tag{3.68}$$

holds for each $v_h \in W_h$.

Let us now introduce the operator $D_h \in \mathcal{L}(W_h, L^2(\Omega))$:

$$D_h v_h(x) = \sum_{i=1}^{n} \nabla_{ih} v_{ih}(x), \ \forall\ v_h \in W_h, \tag{3.69}$$

its adjoint $D_h^* \in \mathcal{L}(L^2(\Omega), W_h)$ is defined by

$$(D_h^*\theta, v_h) = (\theta, D_h v_h), \ \forall\ v_h \in W_h, \ \forall\ \theta \in L^2(\Omega). \tag{3.70}$$

Let π_h be the step function which vanishes outside $\Omega(h) = \cup_{M \in \overset{\circ}{\Omega}{}_h^1} \sigma_h(M)$ and which satisfies

$$\pi_h(x) = \pi_h(M) = \frac{\lambda_M}{h_1 \dots h_n}, \ \forall\, x \in \sigma_h(M), \ M \in \overset{\circ}{\Omega}{}_h^1. \tag{3.71}$$

Then (3.68) can be written as

$$\nu((u_h, v_h))_h - (f, v_h) = (\pi_h, D_h v_h),$$

or equivalently,

$$\nu((u_h, v_h))_h - (D_h^*\pi_h, v_h) = (f, v_h), \forall\ v_h \in W_h. \tag{3.72}$$

Taking successively $v_h = w_{hM} e_j$ for $M \in \overset{\circ}{\Omega}{}_h^1$, $j = 1, \dots, n$, we can interpret (3.72) as

$$-\nu \sum_{i=1}^{n} \delta_{ih}^2 u_h(M) + (\overline{\nabla}_h \pi_h)(M) = f_h(M), \quad M \in \mathring{\Omega}_h^1 \tag{3.73}$$

where $\overline{\nabla}_h(\pi_h(M))$ is the vector $(\overline{\nabla}_{1h}\pi_h(M), \ldots, \overline{\nabla}_{nh}\pi_h(M))$ and

$$f_h(M) = \frac{1}{h_1 \ldots h_n} \int_{\sigma_h(M)} f(x)\mathrm{d}x. \tag{3.74}$$

The equations (3.73), and

$$\sum_{i=1}^{n} (\nabla_{ih} u_{ih})(M) = 0, \quad M \in \mathring{\Omega}_h^1, \tag{3.75}$$

are the discrete form of the equations (2.7) – (2.8); $-D_h^* \pi_h$ is the "approximation" of grad p, $-D_h^*$ is a discrete gradient operator.

Remark 3.9. As indicated in Remark 3.4, the solution of (3.64) is not easy since we do not know any simple basis of V_h. One possibility for solving (3.64) would be to solve the system (3.73), (3.75), which is a linear system with unknowns

$$u_{1h}(M), \ldots, u_{nh}(M), \;\; \pi_h(M), \;\; M \in \mathring{\Omega}_h^1.$$

This system has a unique solution up to an additive constant for the $\pi_h(M)$; this non-uniqueness makes the resolution of this linear system difficult; moreover, the matrix of the system is ill-conditioned.

More efficient ways for actually computing the approximate solution will be given in Section 5.

The Error.

Let us suppose that the exact solution satisfies $\boldsymbol{u} \in \mathscr{C}^3(\overline{\Omega})$ and $p \in \mathscr{C}^2(\overline{\Omega})$. Then by using Taylor's formula, we have

$$-\nu \sum_{i=1}^{n} (\delta_{ih}^2 r_h \boldsymbol{u})(M) - (\overline{\nabla}_h p)(M) = \boldsymbol{f}(M) + \epsilon_h(M) \tag{3.76}$$

where $r_h \boldsymbol{u}$ is the function of V_h defined by (3.54) and where $\epsilon_h(M)$ is a "small" vector:

$$|\epsilon_h(M)| \leqslant c(\boldsymbol{u}, p)|h|, \tag{3.77}$$

$c(\boldsymbol{u}, p)$ depending only on the maximum norms of third derivatives of $\boldsymbol{u}$ and second derivatives of p. Let us denote by π'_h the function $\sum_{M \in \mathring{\Omega}_h^1} p(M) w_{hM}$. Then, the equality (3.76) is equivalent to

$$\nu((r_h \boldsymbol{u}, \boldsymbol{v}_h))_h + (\pi'_h, D_h \boldsymbol{v}_h) = (\boldsymbol{f} + \epsilon_h, \boldsymbol{v}_h) \tag{3.78}$$

for each $\boldsymbol{v}_h \in W_h$ (space (3.24)) and implies

$$\nu((r_h \boldsymbol{u}, \boldsymbol{v}_h))_h = (\boldsymbol{f} + \epsilon_h, \boldsymbol{v}_h), \tag{3.79}$$

for each $\boldsymbol{v}_h \in V_h$.

Subtracting this equality from (3.64) we obtain

$$\nu((\boldsymbol{u}_h - r_h \boldsymbol{u}, \boldsymbol{v}_h))_h = (\epsilon_h, \boldsymbol{v}_h), \tag{3.80}$$

and then taking $\boldsymbol{v}_h = \boldsymbol{u}_h - r_h \boldsymbol{u}$, we see that

$$\begin{aligned} \nu \|\boldsymbol{u}_h - r_h \boldsymbol{u}\|_h^2 &= (\epsilon_h, \boldsymbol{u}_h - r_h \boldsymbol{u}) \\ &\leqslant c(\Omega, \boldsymbol{u}, p)|h| \, \|\boldsymbol{u}_h - r_h \boldsymbol{u}\|_h . \end{aligned}$$

Hence we find the following estimates for the discrete error:

$$\|\boldsymbol{u}_h - r_h \boldsymbol{u}\|_h \leqslant \frac{1}{\nu} c(\Omega, \boldsymbol{u}, p)|h| \tag{3.81}$$

$$|\boldsymbol{u}_h - r_h \boldsymbol{u}| \leqslant \frac{1}{\nu} c'(\Omega, \boldsymbol{u}, p)|h|. \tag{3.82}$$

§4. Discretization of Stokes equations (II)

We study here the discretization of Stokes' equations by means of finite element methods. The results are less general here than in the previous section and vary according to the dimension. We successively consider conforming finite elements which are: piecewise polynomials of degree two in the two-dimensional case (Section 4.2), piecewise polynomials of degree three in the three-dimensional case (Section 4.3), and piecewise polynomials of degree four in the two-dimensional case (Section 4.4). Finally we consider an external approximation by non-conforming finite elements (any dimension) in Section 4.5.

4.1. *Preliminary results*

We will have to work with piecewise polynomial functions defined on n-simplices. For that purpose, we recall here some definitions and introduce some notations adapted to the situation.

Barycentric coordinates

Let there be given in $\mathbb{R}^n$, $(n+1)$ points $A_1, \ldots, A_{n+1}$,(1) with coordinates $a_{1,i}, \ldots,\ 1 \leqslant i \leqslant n+1$, which do not lie in the same hyperplane; this amounts saying that the n vectors $A_1A_2, \ldots, A_1A_{n+1}$ are independent, or that the matrix

$$\mathscr{A} = \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n+1} \\ \vdots & & & \vdots \\ a_{n,1} & a_{n,2} & & a_{n,n+1} \\ 1 & 1 & & 1 \end{vmatrix} \tag{4.1}$$

is non-singular. Given any point $P \in \mathbb{R}^n$, with coordinates $x_1, \ldots, x_n$, there exist $(n+1)$ real numbers

$$\lambda_i = \lambda_i(P), \quad 1 \leqslant i \leqslant n+1$$

such that

$$OP = \sum_{i=1}^{n+1} \lambda_i OA_i, \tag{4.2}$$

$$\sum_{i=1}^{n+1} \lambda_i = 1, \tag{4.3}$$

---

(1) In this section dealing with finite elements, the capital letters $A, B, M, P, \ldots$, (sometimes with subscripts) will denote points of the affine space $\mathbb{R}^n$. Couples of such letters, like $AB, \ldots$, denote the vector of $\mathbb{R}^n$ with origin A and terminal point B.

where O is the origin of $\mathbb{R}^n$.

To see tnis it suffices to remark that (4.2) and (4.3) are equivalent to the linear system

$$\mathscr{A} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix} . \tag{4.4}$$

which has a unique solution since the matrix $\mathscr{A}$ is non-singular, by hypothesis. The quantities λ_i are called the *barycentric coordinates of P, with respect to the* $(n+1)$ *points* $A_1, \ldots, A_{n+1}$. As a consequence of (4.4), the numbers λ_i appear as linear, generally non-homogeneous functions of the coordinates $x_1, \ldots, x_n$ of P:

$$\lambda_i = \sum_{j=1}^{n} b_{i,j}x_j + b_{i,n+1}, \; 1 \leqslant i \leqslant n+1, \tag{4.5}$$

where the matrix $\mathscr{B} = (b_{i,j})$ is the inverse of the matrix $\mathscr{A}$. It is easy to see that the point O in (4.2) can be replaced by another point of $\mathbb{R}^n$ without changing the value of the barycentric coordinates; hence

$$\sum_{i=1}^{n+1} \lambda_i PA_i = 0. \tag{4.6}$$

Clearly the barycentric coordinates are also independent of the choice of a basis in $\mathbb{R}^n$.

The convex hull of the $(n+1)$ points A_i is exactly the set of points of $\mathbb{R}^n$ with barycentric coordinates satisfying the conditions:

$$0 \leqslant \lambda_i \leqslant 1, \;\; 1 \leqslant i \leqslant n+1. \tag{4.7}$$

This convex hull $\mathscr{S}$ is the *n-simplex generated by the points* A_i, *which are called the vertices of the n-simplex.* The *barycenter G* of $\mathscr{S}$ is the point of $\mathscr{S}$ whose barycentric coordinates are all equal and hence equal $1/n{+}1$. An *m-dimensional face* of $\mathscr{S}$ is any m-simplex $(1 \leqslant m \leqslant n-1)$ generated by $m+1$ of the vertices of $\mathscr{S}$ (of course these vertices do not lie in an $(m\text{-}1)$ - dimensional subspace of $\mathbb{R}^n$). A *1-dimensional face is an edge.*

In the two-dimensional case ($n = 2$) the 2-simplices are triangles; the vertices and edges of the simplex are simply the vertices and edges of the triangle. In the three-dimensional case, the 3-simplices are tetrahedrons, the two-faces are the four triangles which form its boundary.

An interpolation result

Proposition 4.1. *Let $A_1, \ldots, A_{n+1}$ be $(n+1)$ points of $\mathbb{R}^n$ which are not included in a hyperplane. Given $(n+1)$ real numbers $\alpha_1, \ldots, \alpha_{n+1}$, there exists one and only one linear function u such that $u(A_i) = \alpha_i$, $1 \leqslant i \leqslant n+1$, and*

$$u(P) = \sum_{i=1}^{n+1} \alpha_i \lambda_i(P), \ \forall\, P \in \mathbb{R}^n, \tag{4.8}$$

where the $\lambda_i(P)$ are the barycentric coordinates of P with respect to $A_1, \ldots, A_{n+1}$.

Proof. Let

$$u(x) = \sum_{j=1}^{n} \beta_j x_j + \beta_{n+1},$$

be this function. The unknowns are $\beta_1, \ldots, \beta_{n+1}$ which satisfy the following equations asserting that $u(A_i) = \alpha_i$:

$$\sum_{j=1}^{n} \beta_j a_{j,i} + \beta_{n+1} = \alpha_i, \quad 1 \leqslant i \leqslant n+1.$$

The matrix of the system is the transposed matrix ${}^t\mathscr{A}$ of $\mathscr{A}$ and thus the function u exists and is unique.

It remains to see that (4.8) is the required function; actually

$$u(A_j) = \alpha_j, \ \ 1 \leqslant j \leqslant n+1,$$

since $\lambda_i(A_j) = \delta_{ij}$, the Kronecker delta, for each i and j.

Remark 4.1. Higher order interpolation formulas using the barycentric coordinates will be given later (see Sections 4.2, 4.3, 4.4).

Differential properties

We give some differential properties of the λ_i considered as functions of the cartesian coordinates $x_1, \ldots, x_n$, of P; here we denote by D the gradient operator $D = (D_1, \ldots, D_n)$.

Lemma 4.1.

$$\sum_{i=1}^{n+1} D\lambda_i = 0 \tag{4.9}$$

$$D\lambda_i(P) \cdot PA_j = \delta_{ij} - \lambda_i(P), \quad 1 \leqslant i, \; j \leqslant n+1. \tag{4.10}$$

Proof. The identity (4.3) immediately implies (4.9). According to (4.5)

$$\frac{\partial \lambda_i}{\partial x_k} = b_{i,k}, \qquad 1 \leqslant i \leqslant n+1, \quad 1 \leqslant k \leqslant n,$$

and then

$$\begin{aligned} D\lambda_i(P) \cdot PA_j &= D\lambda_i(P) \cdot OA_j - D\lambda_i(P) \cdot OP \\ &= \sum_{k=1}^{n} b_{i,k} a_{k,j} - \sum_{k=1}^{n} b_{i,k} x_k \\ &= \sum_{k=1}^{n} b_{i,k} a_{k,j} + b_{i,n+1} - \lambda_i \\ &= \delta_{ij} - \lambda_i; \end{aligned}$$

for the last equality we note that $\mathscr{B} = \mathscr{A}^{-1}$.

Lemma 4,2. *Let $\mathscr{S}$ be an n-simplex with vertices $A_1, \ldots, A_{n+1}$ and let ρ' be the least upper bound of the diameters of all balls included in*

$\mathscr{S}$. *Then*

$$|D\lambda_i| \leqslant \frac{1}{\rho'}, \quad 1 \leqslant i \leqslant n+1, \tag{4.11}$$

where $|D\lambda_i|$ *is the Euclidian norm of the constant vector* $D\lambda_i$.

Proof. We have

$$|D\lambda_i| = D\lambda_i \cdot x, \tag{4.12}$$

where x is the unit vector parallel to $D\lambda_i$; but we may write

$$x = \frac{1}{\rho'} PQ,$$

where P and Q belong to $\mathscr{S}$; denoting by $\mu_1, \ldots, \mu_{n+1}$, the barycentric coordinates of Q with respect to $A_1, \ldots, A_{n+1}$, we have, because of (4.2) – (4.3),

$$PQ = \sum_{j=1}^{n+1} \mu_j PA_j,$$

$$\sum_{j=1}^{n+1} \mu_j = 1.$$

Then

$$\begin{aligned} D\lambda_i \cdot x &= \frac{1}{\rho'} \left(D\lambda_i\right) \cdot \left(\sum_{j=1}^{n+1} \mu_j PA_j\right) \\ &= \frac{1}{\rho'} \sum_{j=1}^{n+1} \mu_j D\lambda_i \cdot PA_j \\ &= \text{(according to (4.10))} \\ &= \frac{1}{\rho'} \sum_{j=1}^{n+1} \mu_j(\delta_{ij} - \lambda_i) = \frac{1}{\rho'} (\mu_i - \lambda_i). \end{aligned}$$

Since P and Q belong to $\mathscr{S}$, $0 \leqslant \lambda_i \leqslant 1$, $0 \leqslant \mu_i \leqslant 1$ for each i,

$1 \leqslant i \leqslant n+1$, and then $-1 \leqslant \mu_i - \lambda_i \leqslant 1$, so that

$$|D\lambda_i \cdot \boldsymbol{x}| \leqslant \frac{1}{\rho'}$$

and (4.11) follows.

Norms of some linear transformations

Let $\mathscr{S}$ and $\bar{\mathscr{S}}$ be two n-simplices with vertices $A_1, \dots, A_{n+1}$, and $\bar{A}_1, \dots, \bar{A}_{n+1}$. We denote by ρ (resp. ρ') the diameter of the smallest ball containing $\mathscr{S}$ (resp. the diameter of the largest ball contained in $\mathscr{S}$); $\bar{\rho}, \bar{\rho}'$, has a similar meaning.

We can suppose that, up to a translation, $A_1 = \bar{A}_1 = 0$, the origin in $\mathbb{R}^n$, and we denote then by Λ the linear mapping in $\mathbb{R}^n$ such that

$$A_i = \Lambda\, \bar{A}_i, \quad 2 \leqslant i \leqslant n+1. \tag{4.13}$$

The norms of Λ and Λ^{-1} can be majorized as follows in terms of $\rho, \rho', \bar{\rho}, \bar{\rho}'$:

Lemma 4.3.

$$\|\Lambda\| \leqslant \frac{\bar{\rho}}{\rho'}, \quad \|\Lambda^{-1}\| \leqslant \frac{\rho}{\bar{\rho}'}$$

Proof. As in the proof of Lemma 4.2, let $\boldsymbol{x}$ be any vector in $\mathbb{R}^n$ with norm 1. Then

$$\boldsymbol{x} = \frac{1}{\rho'} PQ,$$

where P and Q belong to $\mathscr{S}$. It is clear that

$$\Lambda\boldsymbol{x} = \frac{1}{\rho'} \bar{P}\bar{Q},$$

where $\bar{P} = \Lambda P$, $\bar{Q} = \Lambda Q$. But $\bar{P}$ and $\bar{Q}$ belong to $\bar{\mathscr{S}}$ too, since

$$OP = \sum_{i=1}^{n+1} \lambda_i OA_i, \quad 0 \leqslant \lambda_i \leqslant 1,$$

implies

$$\Lambda OP = \sum_{i=1}^{n+1} \lambda_i \, O\bar{A}_i,$$

so that the barycentric coordinates of $\bar{P}$ with respect to $\bar{A}_1, \ldots, \bar{A}_{n+1}$ are the same as the barycentric coordinates of P with respect to $A_1, \ldots, A_{n+1}$. Hence $|\bar{P}\bar{Q}| \leqslant \bar{\rho}$, and

$$|\Lambda \boldsymbol{x}| \leqslant \frac{\bar{\rho}}{\rho'} .$$

The first inequality (4.14) is proved. The second inequality is obvious when interchanging the role of $\mathscr{S}$ and $\bar{\mathscr{S}}$. $\square$

When handling divergence free vector functions, the following lemma will be useful:

Lemma 4.4. *Let $\boldsymbol{x} \to \boldsymbol{u}(\boldsymbol{x})$ be a divergence free vector function defined on $\mathscr{S}$ (or on $\mathbb{R}^n_x$) and let $\bar{\boldsymbol{x}} \to \bar{\boldsymbol{u}}(\bar{\boldsymbol{x}})$ be defined on $\bar{\mathscr{S}}$ by*

$$\bar{\boldsymbol{u}}(\bar{\boldsymbol{x}}) = \Lambda \boldsymbol{u}(\Lambda^{-1}\bar{\boldsymbol{x}}), \forall \ \bar{\mathrm{x}} \in \bar{\mathscr{S}} \text{ (or } \mathbb{R}^n_{\bar{x}}). \tag{4.15}$$

Then $\bar{\boldsymbol{u}}$ is a divergence free vector function too.

Proof. Let (α_{ij}) and $(\beta_{k\ell})$ denote the elements of Λ and Λ^{-1}. Then

$$\begin{aligned}
\frac{\partial \bar{\boldsymbol{u}}_i}{\partial \bar{\boldsymbol{x}}_j}(\bar{\boldsymbol{x}}) &= \frac{\partial}{\partial \bar{\boldsymbol{x}}_j} \sum_{\ell} \alpha_{i\ell} u_\ell (\Lambda^{-1}\bar{\boldsymbol{x}}) \\
&= \sum_{\ell,k} \alpha_{i\ell} \frac{\partial u_\ell}{\partial x_k} \cdot \frac{\partial x_k}{\partial \bar{x}_j} \\
&= \sum_{\ell,k} \alpha_{i\ell}\beta_{kj} \frac{\partial u_\ell}{\partial x_k} (\Lambda^{-1}\bar{\boldsymbol{x}})
\end{aligned}$$

and

$$\begin{aligned}
(\operatorname{div} \bar{u})(\bar{x}) &= \sum_i \frac{\partial \bar{u}_i}{\partial \bar{x}_i}(\bar{x}) = \sum_{i,k,\ell} \alpha_{i\ell}\beta_{ki} \frac{\partial \boldsymbol{u}_\ell}{\partial x_k} (\Lambda^{-1}\bar{\boldsymbol{x}}) \\
&= \sum_k \frac{\partial u_k}{\partial x_k} (\Lambda^{-1}\bar{\boldsymbol{x}}) = 0.
\end{aligned}$$

Regular triangulations of an open set Ω

Let Ω be an open bounded set in $\mathbb{R}^n$.

Let $\mathcal{T}_h$ be a family of n–simplices; such a family will be called *an admissible triangulation of* Ω if the following conditions are satisfied:

$$\Omega(h) = \bigcup_{\mathcal{S} \in \mathcal{T}_h} \mathcal{S} \subset \Omega \tag{4.16}$$

If $\mathcal{S}$ and $\mathcal{S}' \in \mathcal{T}_h$, then $\mathring{\mathcal{S}} \cap \mathring{\mathcal{S}}' = \phi$, (where $\mathring{\mathcal{S}}$ is the interior of $\mathcal{S}$[1]) and, either $\mathcal{S} \cap \mathcal{S}'$ is empty or $\mathcal{S} \cap \mathcal{S}'$ is exactly a whole m–face for both $\mathcal{S}$ and $\mathcal{S}'$ (any m, $0 \leqslant m \leqslant n-1$). (4.17)

We will denote by $\{\mathcal{T}_h\}_{h \in \mathcal{H}}$ the family of all admissible triangulations of Ω; with each admissible triangulation $\mathcal{T}_h$ we associate the following three numbers:

$$\rho(h) = \operatorname*{Sup}_{\mathcal{S} \in \mathcal{T}_h} \rho_{\mathcal{S}} \tag{4.18}$$

$$\rho'(h) = \operatorname*{Inf}_{\mathcal{S} \in \mathcal{T}_h} \rho'_{\mathcal{S}} \tag{4.19}$$

$$\sigma(h) = \operatorname*{Sup}_{\mathcal{S} \in \mathcal{T}_h} \frac{\rho_{\mathcal{S}}}{\rho'_{\mathcal{S}}} \tag{4.20}$$

where, as before, $\rho = \rho_{\mathcal{S}}$ is the diameter of the smallest ball containing $\mathcal{S}$, and $\rho' = \rho'_{\mathcal{S}}$ is the diameter of the greatest ball contained in $\mathcal{S}$.

For finite element methods, we are concerned with passage to the limit, $\rho(h) \to 0$. It will appear later that some restrictions on $\sigma(h)$ are necessary to obtain convergent approximations.

A sub-family of the family of admissible triangulations $\{\mathcal{T}_h\} \in \mathcal{H}$ will be called a *regular triangulation* of Ω if $\sigma(h)$ remains bounded as $\rho(h) \to 0$,

$$\sigma(h) \leqslant \alpha < +\infty, \quad \rho(h) \to 0 \tag{4.21}$$

and Ω(h) converges to Ω in the following sense:

---

[1] i.e., the points of $\mathcal{S}$ with barycentric coordinates, with respect to the vertices of $\mathcal{S}$, satisfying $0 < \lambda_i < 1$, $1 \leqslant i \leqslant n+1$.

For each compact set $K \subset \Omega$, there exists
$\delta = \delta(K) > 0$ such that $\rho(h) \leqslant \delta(K) \Rightarrow \Omega(h) \supset K$. (4.22)

$\mathcal{H}_\alpha$ will denote the set of admissible triangulations of Ω satisfying (4.21) and (4.22).

Remark 4.2. In the two dimensional case the 2–simplex is a triangle and it is known that

$$\frac{1}{2\tan\dfrac{\theta}{2}} \leqslant \frac{\rho_{\mathscr{S}}}{\rho'_{\mathscr{S}}} \leqslant \frac{2}{\sin\theta}$$

where θ is the smallest angle of $\mathscr{S}$.

The condition (4.21) thus amounts to saying that the smallest angle of all the triangles $\mathscr{S} \in \mathscr{T}_h$ remains bounded from below:

$$\theta \geqslant \theta_0 > 0. \tag{4.23}$$

Our purpose now will be to associate to a regular family of triangulations $\{\mathscr{T}_h\}_{h\in\mathcal{H}_\alpha}$ of Ω, various types of approximations of the function spaces with which we are concerned.

4.2. *Finite elements of degree* 2 ($n = 2$)

Let Ω be a Lipschitzian open bounded set in $\mathbb{R}^n$. We describe an internal approximation of $H_0^1(\Omega)$ (any n) and then an external approximation of V ($n = 2$ only). The approximate functions are piecewise polynomials of degree 2.

4.2.1. *Approximation of* $H_0^1(\Omega)$

Let $\mathscr{T}_h$ be any admissible triangulation of Ω.

Space W_h.

This is the space of continuous vector functions, which vanish outside $\Omega(h)$

$$\Omega(h) = \bigcup_{\mathscr{S}\in\mathscr{T}_h} \mathscr{S} \tag{4.24}$$

and whose components are polynomials of degree two[1] on each simplex $\mathscr{S} \in \mathscr{T}_h$.

This space W_h is a finite dimensional subspace of $H_0^1(\Omega)$. We equip it with the scalar product induced by $H_0^1(\Omega)$:

$$((u_h, v_h))_h = ((u_h, v_h)), \forall\ u_h, v_h \in W_h. \tag{4.25}$$

A basis of W_h.

If $\mathscr{S}$ is an n-simplex we denote, as before, the vertices of $\mathscr{S}$ by $A_1, \ldots, A_{n+1}$; we denote also by A_{ij} the mid-point of A_iA_j.

Firstly we have

Lemma 4.5. *A polynomial of degree less than or equal to two is uniquely defined by its values at the points A_i, A_{ij}, $1 \leqslant i, j \leqslant n+1$ (the vertices and the mid-points of the edges of an n-simplex $\mathscr{S}$).*

Moreover, this polynomial is given in terms of the barycentric coordinates with respect to $A_1, \ldots, A_{n+1}$, by the formula:

$$\phi(x) = \sum_{i=1}^{n+1} (2(\lambda_i(x))^2 - \lambda_i(x))\phi(A_i)$$

$$+ 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n+1} \lambda_i(x)\lambda_j(x)\phi(A_{ij}). \tag{4.26}$$

Proof. Let us show first that (4.26) satisfies the requirements. The function on the right-hand side of (4.26) is a polynomial of degree two since the $\lambda_i(x)$ are linear non-homogeneous functions of $x_1, \ldots, x_n$ (see (4.5)). Besides, if $\psi(x)$ denotes this function,

$$\psi(A_k) = \phi(A_k) \text{ since } \lambda_i(A_k) = \delta_{ik}$$

$$\psi(A_{k\ell}) = \phi(A_{k\ell}) \text{ since } \lambda_i(A_{k\ell}) = \frac{\delta_{ik} + \delta_{i\ell}}{2}.$$

Thus ψ is as required.

---

[1] Roughly speaking, a polynomial of degree two means a polynomial of degree less than or equal to two.

Now, a polynomial of degree two has the form

$$\phi(x) = \alpha_0 + \sum_{i=1}^{n} (\alpha_i x_i + \beta_i x_i^2) + \sum_{\substack{i,j=1 \\ i<j}}^{n} \alpha_{ij} x_i x_j, \tag{4.27}$$

and ϕ is defined by $(n+1)(n+2)/2$ unknown coefficients $\alpha_0, \alpha_i, \beta_i, \alpha_{ij}$. There are $(n+1)$ points A_i, $n(n+1)/2$ points A_{ij}, and hence the conditions on ϕ:

$$\phi(A_i) = \text{given}, \quad \phi(A_{ij}) = \text{given}, \tag{4.28}$$

are $(n+1)(n+2)/2$ linear equations for the unknown coefficients. According to (4.26) this system has a solution for any set of data in (4.28); thus the linear system is a regular system[1], and the solution found in (4.26) is unique. □

Now let us denote by $\mathscr{U}_h$ the set of vertices and mid-edges of the n-simplices $\mathscr{S} \in \mathscr{T}_h$. We denote also by $\mathring{\mathscr{U}}_h$ those points of $\mathscr{U}_h$ which belong to the interior of $\Omega(h)$. According to the preceding lemma there is at most one function u_h in W_h which takes given values at the points $A \in \mathring{\mathscr{U}}_h$. Actually we have more.

Lemma 4.6. *There exists one and only one function u_h in W_h which takes given values at the points $M \in \mathring{\mathscr{U}}_h$.*

Proof. We saw that such a function is necessarily unique. Now, by Lemma 4.5, there exists a function u_h whose components are piecewise polynomials of degree two, which takes given values at the points $M \in \mathring{\mathscr{U}}_h$ and which vanishes at the points $M \in \mathscr{U}_h - \mathring{\mathscr{U}}_h$ and outside $\Omega(h)$. We just have to check that this function is continuous. On each $(n-1)$-face $\mathscr{S}'$ of a simplex $\mathscr{S} \in \mathscr{T}_h$, each component u_{ih} of u_h is a polynomial of degree two which has two (perhaps different) values u_{ih}^+ and u_{ih}^-. But u_{ih}^+ and u_{ih}^- are polynomials of degree less than or equal to two in $(n-1)$ variables, which are equal at the vertices and the mid-points of the edges of $\mathscr{S}'$; Lemma 4.5 applied to an $(n-1)$-dimensional simplex shows that $u_{ih}^+ = u_{ih}^-$ on $\mathscr{S}'$. Therefore u_h is continuous, and u_h belongs to W_h. □

---

[1] We use the well-known property that, in finite dimensional spaces, the linear operators which are onto, are one-to-one and onto.

Repeating the argument of the preceding proof, we see that there exists a unique scalar continuous function, which is a polynomial of degree two on each simplex $\mathscr{S} \in \mathscr{T}_h$, and which takes on given values at the points $M \in \mathring{\mathscr{U}}_h$, and which vanishes outside $\Omega(h)$. Let us denote by w_{hM} the function of this type defined by

$$w_{hM}(M) = 1, \qquad w_{hM}(P) = 0, \qquad \forall\, P \in \mathring{\mathscr{U}}_h,\ P \neq M,\ (M \in \mathring{\mathscr{U}}_h). \tag{4.29}$$

Finally, we have

Lemma 4.7. *The functions $w_{hM}e_i$, $M \in \mathring{\mathscr{U}}_h$, $i = 1, \ldots, n$, form a basis of W_h, and the dimension of W_h is $nN(h)$ where $N(h)$ is the number of points in $\mathring{\mathscr{U}}_h$.*

Proof. These functions are linearly independent and, clearly, each function $\boldsymbol{u}_h \in W_h$ can be written

$$\boldsymbol{u}_h(x) = \sum_{M \in \mathring{\mathscr{U}}_h} \sum_{i=1}^{n} u_{ih}(M) e_i w_{hM}(x)$$

or

$$\boldsymbol{u}_h = \sum_{M \in \mathring{\mathscr{U}}_h} \boldsymbol{u}_h(M) w_{hM}. \tag{4.30}$$

Operator p_h.

The prolongation operator p_h is the identity,

$$p_h \boldsymbol{u}_h = \boldsymbol{u}_h,\ \forall\, u_h \in W_h. \tag{4.31}$$

The p_h's have norm one and thus are stable.

Operator r_h.

We define $r_h\boldsymbol{u}$ for $\boldsymbol{u} \in \mathscr{D}(\Omega)$; we set:

$$(r_h\boldsymbol{u})(M) = \boldsymbol{u}(M),\ \forall\, M \in \mathring{\mathscr{U}}_h. \tag{4.32}$$

Proposition 4.2. *The preceding internal approximation of $\boldsymbol{H}_0^1(\Omega)$ is stable and convergent, provided h belongs to a regular triangulation $\mathcal{H}_\alpha$ of Ω.*

Proof. We only have to prove that for each $\boldsymbol{u} \in \mathscr{D}(\Omega)$,

$$p_h r_h \boldsymbol{u} \to \boldsymbol{u} \text{ in } H_0^1(\Omega),$$

as $\rho(h) \to 0$, $h \in \mathcal{H}_\alpha$.

If h is sufficiently small, $\Omega(h)$ contains the support of $\boldsymbol{u}$, and then as shown by the next lemma

$$\|p_h r_h \boldsymbol{u} - \boldsymbol{u}\| \leqslant c(\boldsymbol{u}) \rho^2(h) \cdot \sigma(h) \leqslant c(\boldsymbol{u}) \alpha \rho^2(h), \tag{4.33}$$

and the result follows.

Lemma 4.8. *Let $\mathscr{S}$ be an n-simplex, ϕ a scalar function in $\mathscr{C}^3(\mathscr{S})$, and let $\tilde{\phi}$ be the interpolating polynomial of degree two such that,*

$$\tilde{\phi}(A_i) = \phi(A_i),\ \tilde{\phi}(A_{ij}) = \phi(A_{ij})$$

for $1 \leqslant i, j \leqslant n+1$.

Then, we have

$$\operatorname*{Sup}_{x \in \mathscr{S}} |\phi(x) - \tilde{\phi}(x)| \leqslant c(\phi)\, \rho_{\mathscr{S}}^3 \tag{4.34}$$

$$\operatorname*{Sup}_{x \in \mathscr{S}} \left| \frac{\partial \phi}{\partial x_i}(x) - \frac{\partial \tilde{\phi}}{\partial x_i}(x) \right| \leqslant c(\phi)\, \frac{\rho_{\mathscr{S}}^3}{\rho'_{\mathscr{S}}} \tag{4.35}$$

where $c(\phi)$ depends on the maximum norm of the third derivatives of ϕ.

This lemma is a particular case of general theorems concerning polynomial interpolation on a simplex in connection with finite elements.

Polynomial Interpolation on a Simplex.

Let $\mathscr{S}$ be an n-simplex and let $\mathscr{E}$ be a finite set of points of $\mathscr{S}$ having the following property: for any family of given numbers $\gamma_M \in \mathbb{R}$, $M \in \mathscr{E}$, there exists a unique polynomial p of degree less than or equal to k such that

$$p(M) = \gamma_M,\ \forall\, M \in \mathscr{E}. \tag{4.36}$$

Such a set $\mathscr{E}$ is called *k-unisolvent* by Ciarlet-Raviart [1]; for example, according to Proposition 4.1 and Lemma 4.5, the vertices $A_1, \ldots, A_{n+1}$

of $\mathscr{S}$ are 1-unisolvent, the points A_i, A_{ij}, $1 \leqslant i, j \leqslant n+1$, are 2-unisolvent.

Let us denote by p_i the polynomial of degree k such that

$$p_i(M_i) = 1,\ p_i(M_j) = 0,\ M_j \neq M_i,\ M_j \in \mathscr{E}. \tag{4.37}$$

Then the polynomial p in (4.36) can be written as

$$p = \sum_{M_i \in \mathscr{E}} \gamma_{M_i} p_i \tag{4.38}$$

Now let us suppose that a function ϕ is given, $\phi \in \mathscr{C}^{k+1}(\mathscr{S})$, and let $\tilde{\varphi}$ be the interpolating polynomial of degree k defined by

$$\tilde{\phi}(M) = \phi(M),\ \forall\, M \in \mathscr{E} \tag{4.39}$$

i.e.,

$$\tilde{\phi} = \sum_{M_i \in \mathscr{E}} \phi(M_i) p_i. \tag{4.40}$$

Using Taylor's formula it is proved that for any multi-index $j = (j_1, \ldots, j_n)$ with $[j] = j_1 + \cdots + j_n \leqslant k$, one has

$$D^j \tilde{\phi}(P) = D^j \phi(P) \tag{4.41}$$

$$+ \frac{1}{(k+1)!} \sum_{M_i \in \mathscr{E}} \sum_{[\ell] = k+1} \{D^\ell \phi(P_i) \cdot M_i P^\ell\}\, D^j p_i(P),$$

where P_i is some point of the open interval (M_i, P),

$$D^\ell = D_1^{\ell_1} \ldots D_n^{\ell_n};\ M_i P^\ell = \epsilon_{1i}^{\ell_1} \ldots \epsilon_{ni}^{\ell_n}$$

for $M_i P = (\epsilon_{1i}, \ldots, \epsilon_{ni})$, $\ell = (\ell_1, \ldots, \ell_n)$.

The error between ϕ and $\tilde{\phi}$ is majorized on $\mathscr{S}$ by

$$\operatorname*{Sup}_{x \in \mathscr{S}} |D^j \phi(P) - D^j \tilde{\phi}(P)| \leqslant c\, \eta_{k+1}(\phi)\, \frac{\rho_{\mathscr{S}}^{k+1}}{\rho'^{m}_{\mathscr{S}}}, \tag{4.42}$$

for $[j] = j_1 + \cdots j_n = m \leqslant k$, where $\rho_{\mathscr{S}}$ and $\rho'_{\mathscr{S}}$ are defined in Section 4.1[(1)].

This is a consequence of (4.41) and the following estimation of p_i

$$\operatorname*{Sup}_{x \in \mathscr{S}} |D^j p_i(x)| \leqslant \frac{c}{\rho'^m_{\mathscr{S}}} \text{ for } [j] = m \leqslant \mathrm{k}.$$

For the proofs of (4.41) and (4.43), the reader is referred to Ciarlet-Raviart [1], Raviart [4] and Strang-Fix [1]; for the particular case of Lemma 4.8, see also Ciarlet-Wagshall [1]. □

4.2.2. *Approximation of* V *(APX 2)*

Here Ω is an open bounded set in $\mathcal{R}^2$; we shall define an external approximation of the space V.

Space F, *Operator* $\bar{\omega}$.

The space F is $H_0^1(\Omega)$ and $\bar{\omega}$ is the identity

$$\bar{\omega} \boldsymbol{u} = \boldsymbol{u}, \forall \ \boldsymbol{u} \in V; \tag{4.44}$$

$\bar{\omega}$ is an isomorphism from V into F.

Let $\mathscr{T}_h$ be any admissible triangulation of Ω.

Space V_h.

V_h is a subspace of the space W_h previously defined. It is the space of continuous vector functions which vanish outside

$$\Omega(h) = \bigcup_{\mathscr{S} \in \mathscr{T}_h} \mathscr{S} \tag{4.45}$$

and whose components are polynomials of degree two on each simplex $\mathscr{S} \in \mathscr{T}_h$ and such that

$$\int_{\mathscr{S}} \operatorname{div} \boldsymbol{u}_h \, \mathrm{d}x = 0, \ \forall \ \mathscr{S} \in \mathscr{T}_h. \tag{4.46}$$

The condition (4.46) is a discrete form of the condition div $\boldsymbol{u} = 0$.

(1)

$$\eta_k(\phi) = \operatorname*{Sup}_{x} \operatorname*{Sup}_{[j]=k} \left\{ | D^j \phi(x) | \right\}.$$

The supremum in x is taken on $\mathscr{S}$; elsewhere when using this notation, the supremum is understood on the whole support of ϕ.

The functions $u_h \in V_h$ belong to $H_0^1(\Omega)$, but not to V, $V_h \not\subset V$. We do not have a simple basis of V_h; according to Lemma 4.7, any function $u_h \in V_h$ can be written as

$$u_h = \sum_{M \in \mathring{\mathscr{V}}_h} u_h(M) w_{hM}$$

but the functions w_{hM} do not belong to V_h. Lemma 4.7 and (4.46) show also that

$$\dim V_h \leqslant 2N(h) - N'(h),$$

where $N(h)$ is the number of points in $\mathring{\mathscr{V}}_h$ and $N'(h)$ is the number of triangles $\mathscr{S} \in \mathscr{T}_h$.

We provide the space V_h with the scalar product of $H_0^1(\Omega)$ (as W_h):

$$((u_h, v_h))_h = ((u_h, v_h)). \tag{4.47}$$

Operator p_h.

The operator p_h is the identity (recall that $V_h \subset H_0^1(\Omega)$). The prolongation operators have norm one and are thus stable.

Operator r_h.

The restriction operators are more difficult to define because of condition (4.46) which must be satisfied by $r_h u$.

Let u be an element of $\mathscr{V}$; we set

$$r_h u = u_h = u_h^1 + u_h^2, \tag{4.48}$$

where u_h^1 and u_h^2 belong separately to W_h; u_h^1 is defined as in (4.32) by

$$u_h^1(M) = u(M), \ \forall \ M \in \mathring{\mathscr{V}}_h. \tag{4.49}$$

There is no reason for u_h^1 to belong to V_h, and actually u_h^2 will be a "small corrector" so that $u_h^1 + u_h^2 \in V_h$. We define u_h^2 by its values at the points $M \in \mathring{\mathscr{V}}_h$; if $M = A_i$ is the vertex of a triangle then $u_h^2(A_i) = 0$; if $M = A_{ij}$ is the mid-point of an edge, then, letting ν_{ij} denote one of the two unit vectors orthogonal to A_iA_j, we set:

$$\begin{aligned}
&u_h^2(A_{ij}) \cdot A_i A_j = 0 \\
&u_h^2(A_{ij}) \cdot \nu_{ij} \\
&\quad = - \{ u(A_{ij}) + \frac{1}{4} u(A_i) + \frac{1}{4} u(A_j) \} \cdot \nu_{ij} \\
&\quad + \frac{3}{2} \int_0^1 u(tA_i + (1-t)A_j) \cdot \nu_{ij} \, dt
\end{aligned} \tag{4.50}$$

Lemma 4.9. *u_h defined by* (4.48) – (4.50) *belongs to* V_h.

Proof. The main idea in (4.50) was to choose $\boldsymbol{u}_h^2$ so that

$$\int_{A_i}^{A_j} \boldsymbol{u}_h \cdot \nu_{ij} \mathrm{d}\ell = \int_{A_i}^{A_j} \boldsymbol{u} \cdot \nu_{ij} \mathrm{d}\ell. \tag{4.51}$$

If we show that (4.51) is satisfied, we will then have for any triangle $\mathscr{S}$

$$\int_{\mathscr{S}} \operatorname{div} \boldsymbol{u}_h \, \mathrm{d}x = \int_{\partial \mathscr{S}} \boldsymbol{u}_h \cdot \nu \mathrm{d}\ell = \int_{\partial \mathscr{S}} \boldsymbol{u} \cdot \nu \mathrm{d}\ell = \int_{\mathscr{S}} \operatorname{div} \boldsymbol{u} \, \mathrm{d}x = 0,$$

since $\boldsymbol{u} \in \mathscr{V}$ (ν = unit vector normal to $\partial\mathscr{S}$ pointing outward with respect to $\mathscr{S}$).

Let us prove (4.51). The function $\boldsymbol{u}_h^2$ is equal to

$$\boldsymbol{u}_h^2 = \sum_{\substack{M \in \mathscr{U}_h \\ M = A_{k\ell}}}^{\circ} \boldsymbol{u}_h^2(M) w_{hM}. \tag{4.52}$$

On the segment $\overline{A_i A_j}$, the function $w_{hA_{ij}}$ is the only function w_{hM} in the preceding sum which is not identically equal to 0. By the definition of $w_{hA_{ij}}$ one easily checks that

$$w_{hA_{ij}}(tA_i + (1-t)A_j) = 4t(1-t), \quad 0 < t < 1. \tag{4.53}$$

Likewise

$$\boldsymbol{u}_h^1 = \sum_{M \in \mathscr{U}_h}^{\circ} \boldsymbol{u}_h^1(M) w_{hM}$$

where the only functions w_{hM} which do not vanish on $A_i A_j$ are w_{hA_i}, w_{hA_j}, $w_{hA_{ij}}$. It is easily shown that

$$w_{hA_i}(tA_i + (1-t)A_j) = (t-1)(2t-1)$$

$$w_{hA_j}(tA_i + (1-t)A_j) = t(2t-1). \tag{4.54}$$

Then

$$\frac{1}{|A_iA_j|}\int_{A_i}^{A_j} u_h(x)\cdot \nu_{ij}\mathrm{d}\ell = \int_0^1 u_h(tA_i+(1-t)A_j)\cdot \nu_{ij}\mathrm{d}t$$

$$= \frac{2}{3}\,u_h^2\,(A_{ij})\cdot\nu_{ij} + \frac{2}{3}\,u_h^1\,(A_{ij})\cdot\nu_{ij} + \frac{1}{6}\,\{u_h^1(A_i)+u_h^1\,(A_j)\}\cdot\nu_{ij}$$

$$= \text{(by 4.50))}$$

$$= \int_0^1 u(tA_i+(1-t)A_j)\cdot\nu_{ij}\mathrm{d}t = \frac{1}{|A_iA_j|}\int_{A_i}^{A_j} u(x)\cdot\nu_{ij}\,\mathrm{d}\ell.$$

The lemma is proved.

Proposition 4.3. *The preceding external approximation of V is stable and convergent, provided h belongs to a regular triangulation $\mathcal{H}_\alpha$ of Ω.*

Proof. Let us check first the condition $(C2)$ of Definition 3.6.

We have to show that if a sequence $p_{h'}u_{h'}$, $u_{h'}\in V_{h'}$, converges weakly to ϕ in F, then $\phi = u \in V$. According to Theorem 1.6, we need only to show that

$$\operatorname{div} u = 0. \tag{4.55}$$

Let θ be any function of $\mathscr{D}(\Omega)$; by (4.46), we have

$$\int_\Omega (\operatorname{div} u_h)\theta_h\,\mathrm{d}x = 0, \tag{4.56}$$

where θ_h is the step function defined above, which is equal on each $\mathscr{S}\in\mathscr{T}_h$ to the average value of θ on $\mathscr{S}$, and which vanishes outside $\Omega(h)$. It is easy to see that when support $\theta \subset \Omega(h)$,

$$\sup_{x\in\Omega}|\theta_h(x) - \theta(x)| \leqslant c(\theta)\rho(h),$$

so that θ_h converges to θ in the L^∞ and L^2 norms; thus we can pass to the limit in (4.56) and obtain

$$\int_\Omega \operatorname{div} \boldsymbol{u} \cdot \theta \, \mathrm{d}x = 0, \quad \forall \theta \in \mathscr{D}(\Omega).$$

This proves (4.55).

The condition $(C1)$ of Definition 3.6 is

$$\lim_{h \to 0} p_h r_h \boldsymbol{u} = \overline{\omega} \boldsymbol{u}, \forall\, \boldsymbol{u} \in \mathscr{V}. \tag{4.57}$$

This is equivalent to

$$\lim_{h \to 0} \|\boldsymbol{u} - r_h \boldsymbol{u}\| = 0, \forall\, \boldsymbol{u} \in \mathscr{V}. \tag{4.58}$$

Let us suppose that $\rho(h)$ is sufficiently small for $\Omega(h)$ to contain the support of $\boldsymbol{u}$. Because of Lemma 4.8 and (4.42), on each triangle $\mathscr{S} \in \mathscr{T}_h$:

$$\left.\begin{aligned} &\operatorname*{Sup}_{x \in \mathscr{S}} |\boldsymbol{u}(x) - \boldsymbol{u}_h^1(x)| \leqslant c\eta_3(\boldsymbol{u}) \rho_{\mathscr{S}}^3 \\ &\operatorname*{Sup}_{x \in \mathscr{S}} |D_i \boldsymbol{u}(x) - D_i \boldsymbol{u}_h^1(x)| \leqslant c\eta_3(\boldsymbol{u}) \frac{\rho_{\mathscr{S}}^3}{\rho'_{\mathscr{S}}}. \end{aligned}\right\} \tag{4.59}$$

By the proof of Lemma 4.9,

$$\begin{aligned} \frac{2}{3} \boldsymbol{u}_h^2(A_{ij}) \cdot \boldsymbol{v}_{ij} &= \frac{1}{|A_i A_j|} \int_{A_i}^{A_j} \boldsymbol{u}_h^2(x) \cdot \boldsymbol{v}_{ij} \, \mathrm{d}l \\ &= \frac{1}{|A_i A_j|} \int_{A_i}^{A_j} [\boldsymbol{u}(x) - \boldsymbol{u}_h^1(x)] \cdot \boldsymbol{v}_{ij} \, \mathrm{d}l \\ &= \int_0^1 (\boldsymbol{u} - \boldsymbol{u}_h^1)(tA_i + (1-t)A_j) \cdot \boldsymbol{v}_{ij} \, \mathrm{d}t. \end{aligned} \tag{4.60}$$

Hence, with (4.60) estimated by (4.59),

$$|\boldsymbol{u}_h^2(A_{ij})| = |\boldsymbol{u}_h^2(A_{ij}) \cdot \boldsymbol{v}_{ij}| \leqslant c\eta_3(\boldsymbol{u}) \rho_{\mathscr{S}}^3. \tag{4.61}$$

Now by (4.43), we obtain

$$\begin{aligned}&\operatorname*{Sup}_{x\in\mathscr{S}} |w_{hM}(x)| \leqslant c\\&\operatorname*{Sup}_{x\in\mathscr{S}} |D_i w_{hM}(x)| \leqslant \frac{c}{\rho'_{\mathscr{S}}}, \qquad i=1,\ldots,n\end{aligned} \tag{4.62}$$

Next, combining (4.61)–(4.62) with (4.52), we get

$$\begin{aligned}&\operatorname*{Sup}_{x\in\mathscr{S}} |u_h^2(x)| \leqslant c\eta_3(u)\rho_{\mathscr{S}}^3\\&\operatorname*{Sup}_{x\in\mathscr{S}} |D_i u_h^2(x)| \leqslant c\eta_3(u)\frac{\rho^3}{\rho'_{\mathscr{S}}}.\end{aligned} \tag{4.63}$$

Finally, combining (4.59) and these last inequalities, it follows that

$$\left.\begin{aligned}&\operatorname*{Sup}_{x\in\Omega} |u(x)-u_h(x)| \leqslant c\eta_3(u)\rho(h)^3\\&\operatorname*{Sup}_{x\in\Omega} |D_i u(x)-D_i u_h(x)| \leqslant c\eta_3(u)\rho(h)^2\sigma(h) \leqslant c\eta_3(u)\alpha\rho(h)^2\end{aligned}\right\} \tag{4.64}$$

The proof is completed.

Remark 4.3. If Ω is a polygon, it is possible to choose the triangulation $\mathscr{T}_h$ such that $\Omega(h)=\Omega$, and this is usually done in practical computations. In this case we can extend the preceding computation to any $u \in H_0^1(\Omega) \cap \mathscr{C}^3(\overline{\Omega})$ and we find

$$\|u - r_h u\| \leqslant c\eta_3\,(u)\sigma(h)\rho(h)^2. \tag{4.65}$$

4.2.3. *Approximation of Stokes problem*

Using the preceding approximation of V and Section 3.2 we can propose a finite element scheme for the approximation of a two-dimensional Stokes problem.

Let us take in (3.6)

$$\begin{aligned}&a_h(u_h, v_h) = \nu((u_h, v_h)),\\&\langle l_h, v_h\rangle = (f, v_h),\end{aligned} \tag{4.66}$$

where ν and f are given as in Section 2.1 (see Theorem 2.1).

The approximate problem (3.6) is then

$$\textit{To find } u_h \in V_h \textit{ such that } \nu((u_h, v_h)) = (f, v_h), \forall v_h \in V_h. \tag{4.67}$$

The solution u_h of (4.67) exists and is unique; moreover, we have

Proposition 4.4. *If* $\rho(h) \to 0$ *with* $\sigma(h) \leqslant \alpha$ *(i.e.,* $h \in \mathcal{H}_\alpha$*), the solution* u_h *of* (4.67) *converges to the solution* u *of* (2.6) *in the* $H_0^1(\Omega)$ *norm.*

Proof. It is easy to see that Theorem 3.1 is applicable, and the conclusion gives exactly the convergence result announced. $\square$

Approximation of the pressure

We introduce the approximation of the pressure, as in Section 3.3.

The form

$$v_h \to \nu((u_h, v_h)) - (f, v_h)$$

is a linear form on W_h, which vanishes on V_h. Since V_h is characterized by the set of linear constraints (4.46), we know that there exists a family of numbers $\lambda_{\mathscr{S}}$, $\mathscr{S} \in \mathscr{T}_h$, which are the Lagrange multipliers associated with the constraints (4.46), such that

$$\nu((u_h, v_h)) - (f, v_h)$$

$$= \sum_{\mathscr{S} \in \mathscr{T}_h} \lambda_{\mathscr{S}} \left(\int_{\mathscr{S}} \operatorname{div} v_h \, dx \right), \forall v_h \in W_h. \tag{4.68}$$

Let $\chi_{h\mathscr{S}}$ denote the characteristic function of $\mathscr{S}$ and let π_h denote the step function

$$\begin{aligned} \pi_h &= \sum_{\mathscr{S} \in \mathscr{T}_h} \pi_h(\mathscr{S}) \chi_{h\mathscr{S}} \\ \pi_h(\mathscr{S}) &= \frac{\lambda_{\mathscr{S}}}{(\operatorname{meas} \mathscr{S})} \end{aligned} \tag{4.69}$$

We then have

$$\nu((\boldsymbol{u}_h, \boldsymbol{v}_h)) - (\pi_h, \operatorname{div} \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in W_h, \tag{4.70}$$

which is the discrete analogue of equation

$$\nu((\boldsymbol{u}, \boldsymbol{v})) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \ \forall \boldsymbol{v} \in H_0^1(\Omega). \tag{4.71}$$

Remark 4.4. Since no basis of V_h is available, the solution of (4.67) is not easy. The computation can be effected by the algorithms studied in Section 5.

The error between $\boldsymbol{u}$ and $\boldsymbol{u}_h$

Let us suppose that Ω has a polygonal boundary ($\Omega \subset \mathbb{R}^2$) and that $\boldsymbol{u} \in \mathscr{C}^3(\overline{\Omega})$ and $p \in \mathscr{C}^1(\overline{\Omega})$. Then according to Remark 4.3,

$$\|\boldsymbol{u} - r_h \boldsymbol{u}\| \leqslant c(\boldsymbol{u}, \alpha)\, \rho(h)^2. \tag{4.72}$$

We can take $\boldsymbol{v} = \boldsymbol{v}_h = \boldsymbol{u}_h - r_h \boldsymbol{u}$ in (4.69) and (4.70); subtracting then (4.71) from (4.70) there remains

$$\nu((\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{u}_h - r_h \boldsymbol{u})) = (\pi_h - p, \operatorname{div}(\boldsymbol{u}_h - r_h \boldsymbol{u})). \tag{4.73}$$

Let π'_h denote the step function

$$\pi'_h = \sum_{\mathscr{S} \in \mathscr{T}_h} \frac{1}{(\operatorname{meas} \mathscr{S})} \left(\int_{\mathscr{S}} p(x) \mathrm{d}x \right) \chi_{h\mathscr{S}}. \tag{4.74}$$

Then the right-hand side of (4.73) is equal to

$$(\pi'_h - p, \operatorname{div}(\boldsymbol{u}_h - r_h \boldsymbol{u}))$$

and is majorized by

$$|\pi'_h - p| \cdot |\operatorname{div}(\boldsymbol{u}_h - r_h \boldsymbol{u})| \leqslant |\pi'_h - p| \ \|\boldsymbol{u}_h - r_h \boldsymbol{u}\|.$$

Hence

$$\nu((u_h - u, u_h - r_h u)) \leqslant |\pi'_h - p| \, \|u_h - r_h u\|,$$

$$\nu\|u_h - r_h u\|^2 \leqslant \{|\pi'_h - p| + \nu\|u - r_h u\|\} \|u_h - r_h u\|$$

$$\|u_h - r_h u\| \leqslant \frac{1}{\nu} |\pi'_h - p| + \|u - r_h u\|. \tag{4.75}$$

It is easy to see that $|\pi'_h - p|$ is majorized by $c\eta_1(p)\rho(h)$ and then we have at least

$$\|u_h - r_h u\| \leqslant c(u, p)\rho(h).$$

$$\|u_h - u\| \leqslant c(u, p)\rho(h) \tag{4.76}$$

and therefore, using (4.64),

$$\|u_h - u\| \leqslant c\eta_1(u, p)\rho(h). \tag{4.77}$$

When the boundary of Ω is not a polygon, an additional error of order $\rho(h)$ appears as usual, in the right-hand side of (4.76)–(4.77).

Remark 4.5. The estimation (4.77) is not satisfying because it indicates that the error between u and u_h is of order $\rho(h)$ (in the $H_0^1(\Omega)$ norm), while the distance between u and V_h is of order $\rho(h)^2$ since this distance is majorized by $\|u - r_h u\|$ (see (4.64)).

We do not know if this is due to the fact that the estimation (4.76) is not optimal, or if indeed, the error $\|u - u_h\|$ is of order $\rho(h)$. The purpose of the next subsection is to give an improvement to the algorithm which allows us to obtain an approximate solution for which the $H_0^1(\Omega)$ norm error is surely of order $\rho(h)^2$.

4.2.4. *Utilisation of the bulb functions*

We will now give another approximation of the spaces $H_0^1(\Omega)$ and V, which will lead to an algorithm of approximation of Stokes problem slightly different from (4.67) and for which the error will be of the optimal order (cf. Remark 4.5). The corresponding approximation of $\underset{\sim}{V}$ will be denoted by (APX2′), the approximate spaces will be denoted $\widetilde{W}_h$, $\widetilde{V}_h$, ..., leaving the notation W_h, V_h, ..., for the spaces introduced in the study of Approximation (APX2).

Approximation of $H_0^1(\Omega)$.

Let $\mathscr{T}_h$ be any admissible triangulation of Ω, Ω open bounded set of $\mathbb{R}^2$.

The space $\widetilde{W}_h$ is the space of continuous vector functions which vanish outside $\Omega(h)$, and whose components are equal on each triangle $\mathscr{S} \in \mathscr{T}_h$, to the sum of a polynomial of degree 2 and of a so-called bulb function: the bulb function on $\mathscr{S}$ is the function

$$\beta(x) = \lambda_1(x)\,\lambda_2(x)\,\lambda_3(x),$$

where the λ_i are the barycentric coordinates with respect to the vertices of $\mathscr{S}$. We observe that $\beta(x) = 0$ on $\partial\mathscr{S}$, and that $\beta(x) > 0$ on $\mathscr{S}$. The graph of β looks like a bulb attached to the boundary of $\mathscr{S}$.

For each $\mathscr{S} \in \mathscr{T}_h$, we denote by $\beta_{h\mathscr{S}}$ the continuous real valued function on Ω, equal to the bulb function on $\mathscr{S}$ and to 0 outside $\mathscr{S}$. Let $\mathscr{B}_h$ be the space of functions of type

$$x \to \sum_{\mathscr{S} \in \mathscr{T}_h} \sigma_h(\mathscr{S})\,\beta_{h\mathscr{S}}(x), \qquad \sigma_h(\mathscr{S}) \in \mathbb{R}^2. \tag{4.78}$$

Then the space $\widetilde{W}_h$ can be written as

$$\widetilde{W}_h = W_h + \mathscr{B}_h, \tag{4.79}$$

where W_h is the space introduced in section 4.2.1. Since $W_h \cap \mathscr{B}_h = \{0\}$, any element $\widetilde{\boldsymbol{v}}_h$ of $\widetilde{W}_h$ admits a unique decomposition of the form

$$\widetilde{\boldsymbol{v}}_h = \boldsymbol{v}_h + \boldsymbol{t}_h, \qquad \boldsymbol{v}_h \in W_h, \qquad \boldsymbol{t}_h \in \mathscr{B}_h. \tag{4.80}$$

We provide the space $\widetilde{W}_h$ (included in $H_0^1(\Omega)$) with the scalar product induced by $H_0^1(\Omega)$

$$((\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{v}}_h)) = ((\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{v}}_h)), \quad \forall\, \widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{v}}_h \in \widetilde{W}_h. \tag{4.81}$$

Let us set p_h = the identity operator, and let us simply define $r_h\boldsymbol{u}$, $\boldsymbol{u} \in \mathscr{D}(\Omega)$ by setting

$$r_h\boldsymbol{u} \in W_h \;(\subset \widetilde{W}_h),$$

$$r_h\boldsymbol{u}(M) = \boldsymbol{u}(M), \;\; \forall M \in \mathring{\mathscr{V}}_h.$$

Then the proof of Proposition 4.2, immediately shows the following

Proposition 4.5. *The preceding internal approximation of $H_0^1(\Omega)$ is stable and convergent, provided h belongs to a regular triangulation $\mathcal{H}_\alpha$ of Ω.*

Approximation of V (APX2').

As before the space F is $H_0^1(\Omega)$ and $\bar{\omega}$ is the identity, $\bar{\omega}\boldsymbol{u} = \boldsymbol{u}$, $\forall\, \boldsymbol{u} \in V$; $\bar{\omega}$ is an isomorphism from V into F.

Let $\mathscr{T}_h$ be an admissible triangulation of Ω.

Space $\widetilde{V}_h$.

$\widetilde{V}_h$ is a subspace of the space $\widetilde{W}_h$, more precisely it is the space of functions $\widetilde{\boldsymbol{u}}_h \in \widetilde{W}_h$ such that (c.f. (4.46)):

$$\int_{\mathscr{S}} q \operatorname{div} \widetilde{\boldsymbol{u}}_h \, \mathrm{d}x = 0, \forall \mathscr{S} \in \mathscr{T}_h, \forall \text{ linear function } q. \tag{4.82}$$

While the space W_h is clearly imbedded in $\widetilde{W}_h$, the space V_h is not included in $\widetilde{V}_h$; the functions in $\widetilde{V}_h$ are of a more general type (because of the bulb functions) but they satisfy the algebraic relations (4.82) which are more restrictive than (4.46). We will point out, later on, some very special relations between the spaces V_h and $\widetilde{V}_h$, and we will show in particular that dim V_h = dim $\widetilde{V}_h$.

We equip the space $\widetilde{V}_h$, included in $H_0^1(\Omega)$, with the scalar product (4.81).

Operator p_h.

The operator p_h is the identity. The prolongation operators have norm one and are thus stable.

Operator r_h.

The restriction operators denoted $\widetilde{r}_h$ are of course rather difficult to construct.

Let $\boldsymbol{u}$ be an element of $\mathscr{V}$; we set

$$\widetilde{r}_h \boldsymbol{u} = \widetilde{\boldsymbol{u}}_h = \boldsymbol{u}_h + \boldsymbol{t}_h, \tag{4.83}$$

where $\boldsymbol{u}_h + \boldsymbol{t}_h$ is the decomposition (4.80) of $\widetilde{r}_h \boldsymbol{u}$; now $\boldsymbol{u}_h$ is defined as

$$u_h = r_h u, \tag{4.84}$$

r_h being the restriction operator of Approximation (APX2) (see section 4.2.2).

It remains to choose $t_h \in \mathscr{B}_h$, so that $u_h + t_h \in \tilde{V}_h$. Next lemma shows that the conditions (4.82) define a unique t_h, and that

$$t_h = \sum_{\mathscr{S} \in \mathscr{T}_h} \sigma_h(\mathscr{S})\, \beta_{h\mathscr{S}}, \tag{4.85}$$

$$\sigma_{h_i}(\mathscr{S}) = \frac{60}{\text{area}\,\mathscr{S}} \left\{ \int_{\partial \mathscr{S}} x_i (u_h \cdot \nu)\, \mathrm{d}\Gamma - \int_{\mathscr{S}} u_{h_i}\, \mathrm{d}x \right\}, \; i = 1, 2 \tag{4.86}$$

Lemma 4.10. *$\tilde{u}_h$ defined by the relations* (4.83) *to* (4.86) *belongs to $\tilde{V}_h$.*

Proof. We must show that the relations (4.82) hold for $q(x) = 1, x_1, x_2$, and for each $\mathscr{S} \in \mathscr{T}_h$. For $q = 1$, we observe that

$$\int_{\mathscr{S}} \operatorname{div} \tilde{u}_h\, \mathrm{d}x = \int_{\mathscr{S}} \operatorname{div} u_h\, \mathrm{d}x + \int_{\mathscr{S}} \operatorname{div} t_h\, \mathrm{d}x;$$

$\int_{\mathscr{S}} \operatorname{div} u_h\, \mathrm{d}x$ vanishes since $u_h = r_h u \in V_h$, and

$$\int_{\mathscr{S}} \operatorname{div} t_h\, \mathrm{d}x = \int_{\partial \mathscr{S}} t_h \cdot \nu\, \mathrm{d}\Gamma$$

vanishes for any bulb function $t_h \in \mathscr{B}_h$, since $t_h = 0$ on $\partial \mathscr{S}$.

Let us now examine the condition (4.82) for $q = x_i$, $i = 1, 2$. It can be written as

$$0 = \int_{\mathscr{S}} x_i \operatorname{div} \tilde{u}_h\, \mathrm{d}x = \int_{\mathscr{S}} x_i \operatorname{div} u_h\, \mathrm{d}x + \int_{\mathscr{S}} x_i \operatorname{div} t_h\, \mathrm{d}x, \tag{4.87}$$

or, since $\beta_{h\mathscr{S}}$ vanishes outside $\mathscr{S}$,

$$\int_{\mathscr{S}} x_i \operatorname{div}(\sigma_h(\mathscr{S})\beta_{h\mathscr{S}})\,\mathrm{d}x = -\int_{\mathscr{S}} x_i \operatorname{div} \boldsymbol{u}_h \,\mathrm{d}x,$$

$$\int_{\mathscr{S}} x_i \left(\sigma_{h_1}(\mathscr{S}) \frac{\partial\beta_{h\mathscr{S}}}{\partial x_1} + \sigma_{h_2}(\mathscr{S}) \frac{\partial\beta_{h\mathscr{S}}}{\partial x_2}\right)\mathrm{d}x$$

$$= -\int_{\mathscr{S}} \operatorname{div}(x_i\boldsymbol{u}_h)\,\mathrm{d}x + \int_{\mathscr{S}} u_{h_i}\,\mathrm{d}x$$

$$= -\int_{\partial\mathscr{S}} x_i\, \boldsymbol{u}_h\cdot\boldsymbol{\nu}\,\mathrm{d}\Gamma + \int_{\mathscr{S}} u_{h_i}\,\mathrm{d}x.$$

In order to transform the left-hand side of the last equality we observe that

$$\int_{\mathscr{S}} x_i \frac{\partial\beta_{h\mathscr{S}}}{\partial x_j}\,\mathrm{d}x = \int_{\mathscr{S}} \frac{\partial}{\partial x_j}(x_i\beta_{h\mathscr{S}})\,\mathrm{d}x$$

$$-\,\delta_{ij}\int_{\mathscr{S}} \beta_{h\mathscr{S}}\,\mathrm{d}x \quad (\delta_{ij} = \text{the Kronecker delta})$$

By Green's formula and since $\beta_{h\mathscr{S}} = 0$ on $\partial\mathscr{S}$, there remains

$$\int_{\mathscr{S}} x_i \frac{\partial\beta_{h\mathscr{S}}}{\partial x_j}\,\mathrm{d}x = -\,\delta_{ij}\int_{\mathscr{S}} \beta_{h\mathscr{S}}\,\mathrm{d}x,$$

and according to the next lemma, this quantity is equal to

$$-\,\frac{\operatorname{area}\mathscr{S}}{60}\,\delta_{ij}.$$

Finally, the relations are equivalent to the relations

$$\frac{\operatorname{area}\mathscr{S}}{60}\,\sigma_{h_j}(\mathscr{S}) = \int_{\partial\mathscr{S}} x_i(\boldsymbol{u}_h\cdot\nu)\mathrm{d}x - \int_{\mathscr{S}} u_{h_i}\,\mathrm{d}x, \quad j = 1, 2$$

which are exactly the relations (4.86). $\square$

In the last proof we used the following result.

Lemma 4.11.

$$\int_{\mathscr{S}} \beta_{h\mathscr{S}}(x)\, \mathrm{d}x = \frac{\text{area}\,\mathscr{S}}{60}. \tag{4.88}$$

Proof. For $x \in \mathscr{S}$, $\beta_{h\mathscr{S}}(x) = \lambda_1(x)\,\lambda_2(x)\,\lambda_3(x)$, where the λ_i are the barycentric coordinates with respect to the vertices of $\mathscr{S}$.

We consider the linear transformation Λ in $\mathbb{R}^2$,

$$x = (x_1, x_2) \to \bar{x} = (\bar{x}_1, \bar{x}_2) = (\lambda_1(x), \lambda_2(x)),$$

which maps $\mathscr{S}$ into the triangle

$$\bar{\mathscr{S}} = \{\bar{x}: 0 \leqslant \bar{x}_i,\ i = 1, 2, \bar{x}_1 + \bar{x}_2 \leqslant 1\}.$$

We have $\mathrm{d}x = J\,\mathrm{d}\bar{x}$, J denoting the Jacobian of Λ,

$$J = \frac{\int_{\mathscr{S}} \mathrm{d}x}{\int_{\bar{\mathscr{S}}} \mathrm{d}\bar{x}} = \frac{\text{area}\,\mathscr{S}}{\text{area}\,\bar{\mathscr{S}}} = 2\ \text{area}\,\mathscr{S}.$$

Then,

$$\int_{\mathscr{S}} \beta_{h\mathscr{S}}(x)\, \mathrm{d}x = \int_{\bar{\mathscr{S}}} \bar{x}_1\,\bar{x}_2\,(1 - \bar{x}_1 - \bar{x}_2) J\, \mathrm{d}\bar{x}.$$

The computation of the integral

$$\int_{\bar{\mathscr{S}}} \bar{x}_1\,\bar{x}_2\,(1 - \bar{x}_1 - \bar{x}_2)\, \mathrm{d}\bar{x}$$

is elementary; the value of this integral is $1/120$ and (4.88) follows.

Proposition 4.6. *The preceding external approximation of V is stable and convergent provided h belongs to a regular triangulation $\mathcal{H}_\alpha$ of Ω.*

Proof. The condition (C2) of Definition 3.6 is proved by exactly the same arguments as in Proposition 4.3.

The condition (C1) of Definition 3.6 is proved if we establish that

$$\lim_{h \to 0} \|\boldsymbol{u} - \tilde{r}_h \boldsymbol{u}\| = 0, \quad \forall\, \boldsymbol{u} \in \mathscr{V}.$$

Due to (4.58) and the definition of $\tilde{r}_h$, it suffices to show that

$$\lim_{h \to 0} \|t_h\| = 0, \tag{4.89}$$

t_h being defined by (4.85) (4.86). Since div $\boldsymbol{u} = 0$,

$$\int_{\partial\mathscr{S}} x_i(\boldsymbol{u}, \boldsymbol{\nu})\, \mathrm{d}\Gamma = \int_{\mathscr{S}} \operatorname{div}(x_i \boldsymbol{u})\, \mathrm{d}x = \int_{\mathscr{S}} u_i \, \mathrm{d}x,$$

and this gives an alternative (equivalent) expression of the σ_{h_i}:

$$\sigma_{h_i}(\mathscr{S}) = \frac{60}{\text{area}\,\mathscr{S}} \left\{ \int_{\partial\mathscr{S}} x_i\,(\boldsymbol{u}_h - \boldsymbol{u}) \cdot \boldsymbol{\nu})\mathrm{d}\Gamma - \int_{\mathscr{S}} (u_{h_i} - u_i)\, \mathrm{d}x \right\}, \quad i = 1, 2. \tag{4.90}$$

The majorations contained in the proof of Proposition 4.3, imply that

$$\operatorname*{Sup}_{x \in \mathscr{S}} |\boldsymbol{u}_h(x) - \boldsymbol{u}(x)| \leqslant c' \eta_3(\boldsymbol{u})\, \rho_{\mathscr{S}}^3, \tag{4.91}$$

and combining this with (4.90), we get

$$|\sigma_{h_i}(\mathscr{S})| \leqslant \frac{c'' \eta_3(\boldsymbol{u})}{\text{area}\,\mathscr{S}} \left(3 + \frac{\pi^2}{4}\right) \rho_{\mathscr{S}}^5$$

$$\left(\text{since } \frac{\pi}{4} \rho_{\mathscr{S}}'^2 \leqslant \text{area}\,\mathscr{S}\right) \leqslant c''' \eta_3(\boldsymbol{u}) \frac{\rho_{\mathscr{S}}^5}{\rho_{\mathscr{S}}'^2}$$

$$|\sigma_h(\mathscr{S})| \leqslant c\,\alpha^2\,\eta_3(u)\,\rho_{\mathscr{S}}^3\,. \tag{4.92}$$

One can now estimate $\|t_h\|$. On $\mathscr{S}$,

$$Dt_h = \sigma_h(\mathscr{S})\,[D\lambda_1\cdot\lambda_2\cdot\lambda_3 + \lambda_1\cdot D\lambda_2\cdot\lambda_3 + \lambda_1\cdot\lambda_2\cdot D\lambda_3]$$

and since $|\lambda_i(x)| \leqslant 1$ and $|D\lambda_i| \leqslant 1/\rho'_{\mathscr{S}}$ (see (4.11)),

$$|Dt_h| \leqslant c\,\alpha^2\,\eta_3(u)\,\frac{\rho_{\mathscr{S}}^3}{\rho'_{\mathscr{S}}} \leqslant c\,\alpha^3\,\eta_3(u)\,\rho(h)^2$$

$$\int_\Omega |Dt_h|^2\,\mathrm{d}x \leqslant c^2\,\alpha^6\,\eta_3^2(u)\,\rho(h)^4\,(\text{area }\Omega). \tag{4.93}$$

Finally,

$$\|t_h\| = |Dt_h|_{L^2(\Omega)} \leqslant c\,\alpha^3\,\eta_3\,(u)\,\rho\,(h)^2\,, \tag{4.94}$$

and (4.89) is proved.

Remark 4.6. From (4.83), and the majorations (4.59) (4.63) (4.93) we infer that

$$\sup_{x\in\Omega} |u(x) - \widetilde{r}_h u(x)| \leqslant c\,\eta_3(u)(1+\alpha^3)\rho(h)^3$$

$$\sup_{x\in\Omega} |D_i u(x) - D_i\,\widetilde{r}_h u(x)| \leqslant c\,\eta_3(u)\,\alpha(1+\alpha^3)\rho(h)^2\,, \tag{4.95}$$

$i = 1, \ldots, n$, $\forall\, u \in \mathscr{V}$, h sufficiently small.

If $\Omega = \Omega(h)$, the relations (4.95) are valid for each $u \in V \cap \mathscr{C}^3(\bar{\Omega})$ and we deduce in particular

$$\|u - \widetilde{r}_h u\| \leqslant c\eta_3(u)\,\alpha(1+\alpha^3)\,\rho(h)^2. \tag{4.96}$$

Approximation of Stokes problem.

Using the preceding approximation of V and Section 3.2 we can propose another finite element scheme for the approximation of a two-dimensional Stokes problem.

We take in (3.6)

$$a_h(\widetilde{u}_h, \widetilde{v}_h) = \nu((\widetilde{u}_h, \widetilde{v}_h)) \tag{4.97}$$

$$\langle \ell_h, \tilde{v}_h \rangle = (f, \tilde{v}_h). \tag{4.98}$$

The approximate problem (3.6) is then

To find $u_h \in \tilde{V}_h$ *such that* $\nu((\tilde{u}_h, \tilde{v}_h)) = (f, \tilde{v}_h), \ \forall\, \tilde{v}_h \in \tilde{V}_h$. (4.99)

The solution $\tilde{u}_h$ of (4.99) exists and is unique; moreover by application of Theorem 3.1 we obtain the convergence result.

Proposition 4.7. *If* $\rho(h) \to 0$, *with* $\sigma(h) \leqslant \alpha$ *(i.e.* $h \in \mathcal{H}_\alpha$*), the solution* $\tilde{u}_h$ *of* (4.99) *converges to the solution* u *of* (2.6) in the $H_0^1(\Omega)$ *norm.*

The advantage of this scheme on scheme (4.67) will appear after we have described the approximation of the pressure and given the estimation of the error.

Approximation of the Pressure.

We introduce the approximation of the pressure as in (4.70). The approximate pressure π_h will now be a function piecewise linear on each triangle $\mathscr{S}$.

The form

$$\tilde{v}_h \to \nu((\tilde{u}_h, \tilde{v}_h)) - (f, \tilde{v}_h),$$

is a linear form on $\tilde{W}_h$, which vanishes on $\tilde{V}_h$. Since $\tilde{V}_h$ is characterised by the set of linear constraints (4.82) with $q = 1, x_1, x_2$, there exists a family of Lagrange multipliers, $\lambda^i_{\mathscr{S}}$, $i = 0, 1, 2, \mathscr{S} \in \mathscr{T}_h$, such that

$$\begin{aligned}\nu((\tilde{u}_h, \tilde{v}_h)) - (f, \tilde{v}_h) = &\sum_{\mathscr{S}\in\mathscr{T}_h} \lambda^0_{\mathscr{S}} \left(\int_{\mathscr{S}} \operatorname{div} \tilde{v}_h \, dx \right) \\ &+ \sum_{i=1}^{2} \sum_{\mathscr{S}\in\mathscr{T}_h} \lambda^i_h \left(\int_{\mathscr{S}} x_i \operatorname{div} \tilde{v}_h \, dx \right).\end{aligned}$$

Let $\tilde{\pi}_h$ denote the scalar function on Ω, which vanishes outside $\Omega(h)$ and is a.e. equal to $\lambda^0_{\mathscr{S}} + x_1 \lambda^1_{\mathscr{S}} + x_2 \lambda^2_{\mathscr{S}}$ on each $\mathscr{S} \in \mathscr{T}_h$. Then the right hand side of the last relation is equal to

$$\int_\Omega \tilde{\pi}_h \operatorname{div} \tilde{v}_h \, dx = (\tilde{\pi}_h, \operatorname{div} \tilde{v}_h)$$

and we have

$$\nu((\tilde{u}_h, \tilde{v}_h)) - (\tilde{\pi}_h, \operatorname{div} \tilde{v}_h) = (f, \tilde{v}_h), \forall\, \tilde{v}_h \in \tilde{W}_h, \tag{4.100}$$

and this relation is again the discrete analogue of (4.71).

The error between $\boldsymbol{u}$ *and* $\tilde{u}_h$.

Assume that $\Omega \subset \mathbb{R}^2$ has a polygonal boundary, $\Omega = \Omega(h)$, and that $u \in \mathscr{C}^3(\bar{\Omega})$ $p \in \mathscr{C}^2(\bar{\Omega})$. Then, according to Remark 4.6

$$\|u - \tilde{r}_h u\| \leqslant c(u, \alpha)\, \rho(h)^2. \tag{4.101}$$

Let us set $v = \tilde{v}_h = \tilde{u}_h - \tilde{r}_h u$ in (4.100) and (4.71). Subtracting (4.71) from (4.100) now gives

$$\nu((\tilde{u}_h - u, \tilde{u}_h - \tilde{r}_h u)) = (\tilde{\pi}_h - p, \operatorname{div}(\tilde{u}_h - \tilde{r}_h u)). \tag{4.102}$$

Let π'_h denote the piecewise linear function, equal on each triangle $\mathscr{S}$ to

$$p(G) + (x - G)\cdot \nabla p(G),$$

where G represents the barycenter of $\mathscr{S}$. By Taylor formula

$$\sup_{x \in \mathscr{S}} |\pi'_h(x) - p(x)| \leqslant c(p)\, \rho^2$$

$$\sup_{x \in \Omega} |\pi'_h(x) - p(x)| \leqslant c(p)\, \rho(h)^2. \tag{4.103}$$

However, the right-hand side of (4.102) is also equal to

$$(\pi'_h - p, \operatorname{div}(\tilde{u}_h - \tilde{r}_h u))$$

and because of (4.103) we can majorize the absolute value of this expression by

$$|\pi'_h - p|\, |\operatorname{div}(\tilde{u}_h - \tilde{r}_h u)| \leqslant |\pi'_h - p| \cdot \|\tilde{u}_h - \tilde{r}_h u\| \leqslant$$

$$\leqslant c(p)\, \rho(h)^2 \|\tilde{u}_h - \tilde{r}_h u\|.$$

Hence

$$\nu\|\tilde{u}_h - \tilde{r}_h u\|^2 \leqslant \{|\pi'_h - p| + \nu\|u - \tilde{r}_h u\|\}\, \|\tilde{u}_h - \tilde{r}_h u\|$$

$$\|\tilde{u}_h - \tilde{r}_h u\| \leqslant \frac{1}{\nu} |\pi'_h - p| + \|u - \tilde{r}_h u\|$$

$$\|\tilde{u}_h - \tilde{r}_h u\| \leqslant \frac{1}{\nu} c(p)\, \rho(h)^2 + c(u, p)\, \rho(h)^2.$$

Finally,

$$\|\tilde{u}_h - u\| \leqslant c(u, p)\, \rho(h)^2, \tag{4.104}$$

and we get an error of the optimal order i.e. *of the order of the distance between* u *and* $\tilde{V}_h$. □

Algebraic relation between V_h *and* $\tilde{V}_h$.

We would like to exhibit now a simple algebraic relation between V_h and $\tilde{V}_h$: we will show that there exists a simple isomorphism Λ_h from $\tilde{V}_h$ onto V_h. This implies that V_h and $\tilde{V}_h$ have exactly the same dimension and that the scheme (4.99) can be interpreted as a scheme in V_h. In other words, we finally exhibit a scheme in V_h giving in some sense the optimal order error[(1)]. It is also interesting to observe that (4.99) is equivalent to a scheme in V_h: a scheme in V_h is simpler to solve than a scheme in $\tilde{V}_h$, since we have one instead of three linear constraints on each $\mathscr{S} \in \mathscr{T}_h$ ((4.46) and (4.82)).

Any $\tilde{u}_h$ belonging to $\tilde{W}_h$ can be uniquely written as $u_h + t_h$, $u_h \in W_h$, $t_h \in \mathscr{B}_h$. The mapping $\tilde{u}_h \to u_h$ is a linear projection from $\tilde{W}_h$ onto W_h. We denote as Λ_h the restriction of this projection to $\tilde{V}_h$.

Lemma 4.12. *Λ_h is an isomorphism from $\tilde{V}_h$ onto V_h.*

Proof. We see first that $\Lambda_h \tilde{u}_h = u_h = \tilde{u}_h - t_h \in V_h$ if $\tilde{u}_h \in \tilde{V}_h$. Indeed, on each $\mathscr{S} \in \mathscr{T}_h$:

$$\int_{\mathscr{S}} \operatorname{div} u_h \, dx = \int_{\mathscr{S}} \operatorname{div} \tilde{u}_h \, dx - \int_{\mathscr{S}} \operatorname{div} t_h \, dx,$$

and this vanishes since $\tilde{u}_h \in \tilde{V}_h$ and we already observed that

(1) We get $\|u - \tilde{u}_h\| \leqslant c(u, p)\, \rho(h)^2$, but apparently not $\|u - \Lambda_h \tilde{u}_h\| \leqslant c(u, p)\, \rho(h)^2$. We do not know if $\|\Lambda_h \tilde{u}_h - \tilde{u}_h\| \leqslant c\, \rho(h)^2$.

$$\int_{\mathscr{S}} \operatorname{div} t_h \, dx = 0, \forall t_h \in \mathscr{B}_h.$$

As a projection operator, Λ_h is one to one. There remains to show that Λ_h is onto: given $u_h \in V_h$, we have to find $t_h = t_h(u_h) \in \mathscr{B}_h$ such that $\widetilde{u}_h \in \widetilde{V}_h$ and $\widetilde{u}_h = u_h + t_h(u_h)$, i.e. $\Lambda_h \widetilde{u}_h = u_h$. This is proved as in Lemma 4.10 and the relations giving t_h are exactly the same as (4.85), (4.86). □

As a consequence of Lemma 4.12, we write the scheme (4.99) in the following form:

To find $u_h \in V_h$ *such that*

$$\nu((u_h + t_h(u_h), v_h + t_h(v_h))) = (f, v_h + t_h(v_h)), \forall v_h \in V_h \quad (4.105)$$

where $t_h(u_h) \in \mathscr{B}_h$ is given in terms of u_h by Lemma 4.11 and formulas (4.85), (4.86). The solution u_h of (4.105) differs from the solution u_h of (4.67) and the improvement lies in the fact that $\|u - \Lambda_h^1 u_h\|$ is of optimal order (in the usual case, $\Omega(h) = \Omega$ and $u \in \mathscr{C}^3(\overline{\Omega})$, $p \in \mathscr{C}^2(\overline{\Omega})$).

4.3. *Finite Elements of Degree* 3(n = 3)

Let Ω be a Lipschitz open bounded set in $\mathbb{R}^3$. We first describe an internal approximation of $H_0^1(\Omega)$ and then an external approximation of V. The approximate functions are piecewise polynomials of degree 3.

4.3.1. *Approximation of* $H_0^1(\Omega)$

Let $\mathscr{T}_h$ be an admissible triangulation of Ω and let

$$\Omega(h) = \bigcup_{\mathscr{S} \in \mathscr{T}_h} \mathscr{S}. \quad (4.106)$$

If $\mathscr{S}$ is a 2-simplex (i.e., a tetrahedron) we denote by $A_1, \ldots, A_4$, the vertices of $\mathscr{S}$ and by $B_1, \ldots, B_4$, the barycenter of the 2-faces $\mathscr{S}'_1, \ldots, \mathscr{S}'_4$. We denote by $\mathscr{E}_h^1$ the set of vertices of the simplices $\mathscr{S} \in \mathscr{T}_h$ and by $\mathscr{E}_h^2$ the set of barycenters of the 2-faces of the simplices $\mathscr{S}$ belonging to $\mathscr{T}_h$; $\mathscr{E}_h = \mathscr{E}_h^1 \cup \mathscr{E}_h^2$.

We first prove the following result.

Lemma 4.13. *A polynomial of degree three in* $\mathbb{R}^3$ *is uniquely defined* *by its values at the points A_i, B_i, $1 \leqslant i \leqslant 4$, and the values of its first derivatives at the points A_i. Moreover, the polynomial is given in terms of the barycentric coordinates with respect to $A_1, \ldots, A_4$, by the formula*

$$\phi(x) = \sum_{i=1}^{4} [1-2(\lambda_i(x))^3 + 3(\lambda_i(x))^2]\,\phi(A_i)$$

$$+\frac{1}{6}\sum_{i=1}^{4} \frac{\lambda_1(x)\ldots\lambda_4(x)}{\lambda_i(x)}\left[27\phi(B_i) - 7\sum_{\substack{\alpha=1\\ \alpha\neq i}}^{4}\phi(A_\alpha)\right]$$

$$+\sum_{\substack{i,j=1\\ i\neq j}}^{4} (\lambda_i(x))^2\lambda_j(x)[D\phi(A_i)\cdot A_iA_j]$$

$$-\sum_{i=1}^{4} \frac{\lambda_1(x)\ldots\lambda_4(x)}{\lambda_i(x)}\,[D\phi(A_i)\cdot A_iA_j] \tag{4.107}$$

Proof. The proof is exactly the same as the proof of Lemma 4.5. The coefficients of ϕ are the solutions of a linear system with as many equations as unknowns; we just have to check that the polynomial on the right-hand side of (4.107) fulfills all the required conditions for any set of given datas $\phi(A_i)$, $\phi(B_i)$, $D\phi(A_i)$. □

It follows from this lemma that a scalar function ϕ_h which is defined on $\Omega(h)$ and is a polynomial of degree three on each simplex $\mathscr{S} \in \mathscr{T}_h$ is completely known if the values of ϕ_h are given at the points $A_i \in \mathscr{E}_h^1$ and $B_i \in \mathscr{E}_h^2$ and also the values of $D\phi_h$ are given at the points $A_i \in \mathscr{E}_h^1$. Such a function ϕ_h is differentiable on each $\mathscr{S}$, $\mathscr{S} \in \mathscr{T}_h$, but there is no reason for such a function to be differentiable or even continuous in all of $\Omega(h)$. Actually, this function ϕ_h is at least continuous: on a two face $\mathscr{S}'$ of a tetrahedron $\mathscr{S} \in \mathscr{T}_h$, ϕ_h has two – perhaps different – values ϕ_h^+ and ϕ_h^-; but ϕ_h^+ and ϕ_h^- are polynomials of degree three which take the same values at the vertices and at the barycenter of $\mathscr{S}'$; the first derivatives of ϕ_h^+ and ϕ_h^- at the vertices of $\mathscr{S}'$ are also equal (these are the derivatives of ϕ_h which are tangential with respect to $\mathscr{S}'$). Now, it can be proved exactly as in Lemma 4.5 and 4.6 that $\phi_h^+ = \phi_h^-$.

We denote then by w_{hM}, $M \in \mathscr{E}_h$, the scalar function which is a piecewise polynomial of degree three on $\Omega(h)$ with

$$w_{hM}(M) = 1,\ w_{hM}(P) = 0,\ \forall\ P \in \mathcal{E}_h,\ P \neq M$$

$$Dw_{hM}(P) = 0,\ \forall P \in \mathcal{E}_h^1. \tag{4.108}$$

For $M \in \mathcal{E}_h$, $i = 1, 2, 3$, $w_{hM}^{(i)}$ is the scalar function which is a piecewise polynomial of degree three on $\Omega(h)$ such that

$$\begin{aligned} &w_{hM}^{(i)}(P) = 0,\ \forall\ P \in \mathcal{E}_h,\\ &Dw_{hM}^{(i)}(P) = 0,\ \forall\ P \in \mathcal{E}_h^1,\ P \neq M,\\ &Dw_{hM}^{(i)}(M) = e_i,\ i = 1, 2, 3. \end{aligned} \tag{4.109}$$

All of the functions w_{hM}, $w_{hM}^{(i)}$ are continuous on $\Omega(h)$.

Space W_h

The space W_h is the space of continuous vector functions $\boldsymbol{u}_h$ from Ω into $\mathbb{R}^3$, of type

$$\boldsymbol{u}_h = \sum_{M \in \mathcal{E}_h} \boldsymbol{u}_h(M) w_{hM} + \sum_{M \in \mathcal{E}_h^1} \sum_{i=1}^{3} D_i \boldsymbol{u}_h(M) w_{hM}^{(i)}, \tag{4.110}$$

which vanish outside $\Omega(h)$.

It is clear that $\boldsymbol{u}_h(M) = 0$ for any $M \in \mathcal{E}_h \cup \partial\Omega(h)$; but since the tangential derivatives of $\boldsymbol{u}_h$ vanish on the faces of the tetrahedrons $\mathcal{S}$ which are included in $\Omega(h)$, the derivatives $D_i \boldsymbol{u}_h(M)$, $M \in \mathcal{E}_h^1 \cup \partial\Omega(h)$ are not independent.

The space W_h is a finite dimensional subspace of $\boldsymbol{H}_0^1(\Omega)$; we provide it with the scalar product induced by $\boldsymbol{H}_0^1(\Omega)$

$$((\boldsymbol{u}_h, \boldsymbol{v}_h))_h = ((\boldsymbol{u}_h, \boldsymbol{v}_h)), \forall\, \boldsymbol{u}_h, \boldsymbol{v}_h \in W_h. \tag{4.111}$$

Operator p_h.

The prolongation operator p_h is the identity; the p_h are stable.

Operator r_h.

For $\boldsymbol{u} \in \mathscr{D}(\Omega)$, we define $\boldsymbol{u}_h = r_h \boldsymbol{u}$ by $\boldsymbol{u}_h = 0$ if the support of $\boldsymbol{u}$ is not included in $\Omega(h)$, and if the support is included in $\Omega(h)$,

$$\begin{aligned} &\boldsymbol{u}_h(M) = \boldsymbol{u}(M), \forall\, M \in \mathcal{E}_h,\\ &D\boldsymbol{u}_h(M) = D\boldsymbol{u}(M), \forall\, M \in \mathcal{E}_h^1. \end{aligned} \tag{4.112}$$

Proposition 4.8. *The preceding internal approximation of $H_0^1(\Omega)$ is stable and convergent, provided h belongs to a regular triangulation $\mathcal{H}_\alpha$ of Ω.*

Proof. We have only to prove that for each $u \in \mathcal{D}(\Omega)$,

$$u_h = r_h u \to u \text{ in } H_0^1(\Omega)$$

as $\rho(h) \to 0$, $h \in \mathcal{H}_\alpha$.

This is proved in a similar fashion to Proposition 4.2. The analogue of (4.42) for Hermite type interpolation polynomials (see Ciarlet & Raviart [1]) shows that

$$\begin{aligned} &\sup_{x \in \mathscr{S}} |u(x) - u_h(x)| \leqslant c\,\eta_4(u)\,\rho_{\mathscr{S}}^4 \\ &\sup_{x \in \mathscr{S}} |Du(x) - Du_h(x)| \leqslant c\,\eta_4(u)\,\frac{\rho^4}{\rho'_{\mathscr{S}}}\ . \end{aligned} \tag{4.113}$$

Hence

$$\begin{aligned} &\sup_{x \in \Omega} |u(x) - u_h(x)| \leqslant c(u)\,\rho(h)^4 \\ &\sup_{x \in \Omega} |Du(x) - Du_h(x)| \leqslant c(u)\,\rho(h)^3\,\sigma(h) \end{aligned} \tag{4.114}$$

and in particular

$$\|u - u_h\| \leqslant \alpha c(u)\,\rho(h)^3 \tag{4.115}$$

provided $\operatorname{supp} u \subset \Omega(h)$.

Approximation of V (APX 3).

We recall that Ω is a bounded set in $\mathcal{R}^3$.

Space F, Operator $\bar{\omega}$.

The space F is $H_0^1(\Omega)$ and $\bar{\omega}$ is the identity. Let $\mathcal{T}_h$ be an admissible triangulation of Ω.

Space V_h.

V_h is a subspace of the previous space W_h; it is the space of $u_h \in W_h$ such that

$$\int_{\mathscr{S}} \operatorname{div} u_h \,dx = 0, \forall\, \mathscr{S} \in \mathcal{T}_h. \tag{4.116}$$

We provide the space V_h with the scalar product (4.111) induced by $H_0^1(\Omega)$ and W_h.

Operator p_h.
This operator is the identity (recall that $V_h \subset H_0^1(\Omega)$).

Operator r_h.
The construction of r_h is based on the same principle as in Section 4.2.
Let u be an element of $\mathscr{V}$; we set

$$r_h u = u_h^1 + u_h^2, \tag{4.117}$$

where u_h^1 and u_h^2 separately belong to W_h; u_h^1 is defined exactly as in (4.112)

$$\left.\begin{aligned} &u_h^1(M) = u(M), \ \forall M \in \mathcal{E}_h, \\ &Du_h^1(M) = Du(M), \ \forall M \in \mathcal{E}_h^1. \end{aligned}\right\} \tag{4.118}$$

The corrector u_h^2 is defined by

$$u_h^2(M) = 0, \ Du_h^2(M) = 0, \ \forall M \in \mathcal{E}_h^1 \tag{4.119}$$

and at the points $M \in \mathcal{E}_h^2$, the component of $u_h^2(M)$ which is tangent to the face $\mathscr{S}'$ whose M is the barycenter, is zero; the normal component $u_h^2(M)\cdot\nu$ is characterized by the condition that

$$\int_{\mathscr{S}'} u_h(x)\cdot\nu \,\mathrm{d}\Gamma = \int_{\mathscr{S}'} u(x)\cdot\nu\,\mathrm{d}\Gamma. \tag{4.120}$$

One can prove [1] that there exists some constant d such that

$$\int_{\mathscr{S}'} u_h^2(x)\cdot\nu\,\mathrm{d}\Gamma = d(\text{area } \mathscr{S}')u_h^2(M)\cdot\nu$$

---

[1] The principle of the proof is similar to that of Lemma 4.9.

and (4.120) means that

$$u_h^2(M)\cdot\nu = \frac{1}{d(\text{area}\,\mathscr{S}')}\int_{\mathscr{S}'}(u - u_h^1)(x)\cdot\nu\,\mathrm{d}\Gamma. \tag{4.121}$$

It is clear then that u_h belongs to V_h since, for each $\mathscr{S}\in\mathscr{T}_h$,

$$\int_{\mathscr{S}}\operatorname{div} u_h\,\mathrm{d}x = \int_{\partial\mathscr{S}} u_h\cdot\nu\,\mathrm{d}\Gamma = \int_{\partial\mathscr{S}} u\cdot\nu\,\mathrm{d}\Gamma = \int\operatorname{div} u\,\mathrm{d}x = 0.$$

Proposition 4.9. *The preceding external approximation of V is stable and convergent, provided h belongs to a set of regular triangulations $\mathcal{H}_\alpha$ of Ω.*

The proof of this proposition follows the same lines as the proof of Proposition 4.3.

The approximation of Stokes problem can then be studied exactly as in Section 4.2.

4.4. *An internal approximation of V*

We assume here that Ω is an open bounded subset of $\mathbb{R}^2$ with a Lipschitz boundary. We suppose for simplicity that Ω is simply connected; for the multi-connected case see Remark 4.7.

In the two-dimensional case, the condition $\operatorname{div} u = 0$ is

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \tag{4.122}$$

and implies that there exists a function ψ (the stream function), such that

$$u_1 = \frac{\partial\psi}{\partial x_2},\ u_2 = -\frac{\partial\psi}{\partial x_1}. \tag{4.123}$$

The function ψ is defined locally for any set Ω, and globally for a simply connected set Ω.

In the present case we can associate with each function $\boldsymbol{u}$ in V the corresponding stream function ψ. The condition $\boldsymbol{u} = 0$ on $\partial\Omega$ amounts to saying that the tangential and normal derivatives of ψ on $\partial\Omega$ vanish. Then ψ is constant on $\partial\Omega$ and since ψ is only defined up to an additive constant, we can suppose that $\psi = 0$ on Γ and hence $\psi \in H_0^2(\Omega)$.

Therefore the mapping

$$\psi \to \boldsymbol{u} = \left\{ \frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1} \right\} \tag{4.124}$$

is an isomorphism from $H_0^2(\Omega)$ onto V.

Our purpose now is to construct an approximation of $H_0^2(\Omega)$ by piecewise polynomial functions of degree 5 and then to obtain with the isomorphism (4.124) an *internal* approximation of V.

Internal Approximation of $H_0^2(\Omega)$.

Let $\mathscr{T}_h$ be an admissible triangulation of Ω, and let

$$\Omega(h) = \bigcup_{\mathscr{S} \in \mathscr{T}_h} \mathscr{S} \tag{4.125}$$

A 2-simplex is a triangle. Let $\mathscr{S}$ be some triangle with vertices A_1, A_2, A_3; we denote by B_1, B_2, B_3, or by A_{23}, A_{13}, A_{12}, the mid-points of the edges A_2A_3, A_1A_3 and A_1A_2; $\boldsymbol{\nu}_{ij}$ denotes one of the unit vectors normal to the edge A_iA_j, $1 \leqslant i, j \leqslant 3$.

We first notice the following result:

Lemma 4.14. *A polynomial ϕ of degree 5 in $\mathbb{R}^2$ is uniquely defined by the following values of ϕ and its derivatives:*

$$D^\alpha \phi(A_i), \quad 1 \leqslant i \leqslant 3, \quad [\alpha] \leqslant 2, \tag{4.126}$$

$$\frac{\partial \phi}{\partial \nu_{ij}}(A_{ij}), \quad 1 \leqslant i, j \leqslant 3, \quad i \neq j, \tag{4.127}$$

where the A_i are the vertices of a triangle $\mathscr{S}$ and the A_{ij} are the mid-points of the edges.

Principle of the Proof.

We see that there are as many unknowns (21 coefficients for ϕ) as linear equations for these unknowns (the 21 conditions corresponding to (4.126)–(4.127)).

As in Lemmas 4.5 and 4.13 it is then sufficient to show that a solution does in fact exist for any set of data in (4.126)–(4.127) and this can be proved by an explicit construction of ϕ leading to a formula similar to (4.26) or (4.107). We omit the very technical proof of this point which can be found in A. Ženišek [1] or M. Zlámal [1].(1)

It follows from this lemma that a scalar function ψ_h which is defined on $\Omega(h)$ and is a polynomial of degree five on each triangle $\mathscr{S}\in\mathscr{T}_h$, is completely known if the values of ψ_h are given at the points $A_i \in \mathscr{E}_h^1$, $B_i \in \mathscr{E}_h^2$, and if also the values of the first and second derivatives are given at the points $A_i \in \mathscr{E}_h^1$, where

$$\mathscr{E}_h^1 = \text{set of vertices of the triangles } \mathscr{S}\in\mathscr{T}_h, \text{ belonging to the interior of } \Omega(h)$$

$$\mathscr{E}_h^2 = \text{set of mid-points of the edges of the triangles } \mathscr{S}\in\mathscr{T}_h, \text{ belonging to the interior of } \Omega(h)$$

$$\mathscr{E}_h = \mathscr{E}_h^1 \cup \mathscr{E}_h^2. \tag{4.128}$$

Such a function ϕ_h is infinitely differentiable on each $\bar{\mathscr{S}}, \mathscr{S}\in\mathscr{T}_h$, but there is no reason for such a function to be as smooth in all of $\Omega(h)$. Actually, the function ϕ_h is continuously differentiable in $\Omega(h)$. Let ϕ_h^+ and ϕ_h^- denote the values of ϕ_h on two sides of the edge A_1A_2 of a triangle $\mathscr{S}\in\mathscr{T}_h$; ϕ_h^+, ϕ_h^- are polynomials of degree less than or equal to five on A_1A_2 and they are equal together with their first and second derivatives at the points A_1 and A_2 (six independent conditions) and hence $\phi_h^+ = \phi_h^-$. The tangential derivatives $\partial\phi_h^+/\partial\tau$ and $\partial\phi_h^-/\partial\tau$, $\tau = A_1A_2/|A_1A_2|$ are also necessarily equal. Let us show then that the normal derivatives $\partial\phi_h^+/\partial\nu_{12}$ and $\partial\phi_h^-/\partial\nu_{12}$ are equal on A_1A_2. These derivatives are polynomials of degree less than or equal to four on

(1) The principle of the construction is the following: let $\lambda_1, \lambda_2, \lambda_3$ denote the barycentric coordinates with respect to A_1, A_2, A_3. The affine mapping $x \to y = (\lambda_1(x), \lambda_2(x))$, maps the triangle $\mathscr{S}$, on the triangle $\hat{\mathscr{S}}$:

$$y_1 = \lambda_1 \geqslant 0,\ y_2 = \lambda_2 \geqslant 0,\ 0 \leqslant y_1 + y_2 = \lambda_1 + \lambda_2 \leqslant 1.$$

The construction of $\phi(x(y))$ on $\hat{\mathscr{S}}$ is elementary; then using the inverse mapping $y \to x$, we obtain the function $\phi(x)$.

A_1A_2; they are equal at A_1 and A_2 together with their first derivatives, and they are equal at A_{12}. Therefore they are equal on A_1A_2. This shows that ϕ_h is continuously differentiable on $\Omega(h)$.

To each point $M \in \mathcal{E}_h^2$ we associate the function ψ_{hM}^0, which is a piecewise polynomial of degree five on $\Omega(h)$, such that,

$$\frac{\partial}{\partial \nu}\,\psi_{hM}^0(M) = 1 \text{ and all the other nodal values of } \psi_{hM}^0 \text{ are zero, i.e.,}$$

$$\begin{aligned} &\frac{\partial \psi_{hM}^0}{\partial \nu}(P) = 0,\ \ \forall P \in \mathcal{E}_h^2,\ \ P \neq M, \\ &D^\alpha \psi_{hM}^0(P) = 0,\ \ \forall P \in \mathcal{E}_h^1,\ \ [\alpha] \leqslant 2. \end{aligned} \tag{4.129}$$

To each point $M \in \mathcal{E}_h^1$, we associate the six functions $\psi_{hM}^1, \ldots, \psi_{hM}^6$ defined as follows: they are piecewise polynomials of degree five on $\Omega(h)$ and

$$\psi_{hM}^1(M) = 1, \text{ all the other nodal values of } \psi_{hM}^1 \text{ are zero} \tag{4.130}$$

$$\text{for } i = 1 \text{ or } 2,\ D_j\psi_{hM}^{i+1}(M) = \delta_{ij}, \text{ and all the other nodal values of } \psi_{hM}^{i+1} \text{ are zero} \tag{4.131}$$

$$\begin{aligned} &D_1^2\psi_{hM}^4(M) = 1,\ \ D_1D_2\psi_{hM}^5(M) = 1,\ \ D_2^2\psi_{hM}^6(M) = 1, \\ &\text{and all the other nodal values of } \psi_{hM}^4 \\ &\text{and of } \psi_{hM}^5 \text{ and } \psi_{hM}^6 \text{ respectively are zero.} \end{aligned} \tag{4.132}$$

All these functions are continuously differentiable on $\Omega(h)$.

Space X_h.

The space X_h is the space of continuously differentiable scalar functions on Ω (or $\mathcal{R}^2$) of type:

$$\psi_h = \sum_{M \in \mathcal{E}_h^2} \xi_M^0\,\psi_{hM}^0 + \sum_{i=1}^{6}\ \sum_{M \in \mathcal{E}_h^1} \xi_M^i\,\psi_{hM}^i,\ \ \xi_M^i \in \mathcal{R} \tag{4.133}$$

These functions vanish outside $\Omega(h)$, and since they are continuously differentiable in Ω,

$$\begin{aligned} &D^\alpha \psi_h(M) = 0, \ \forall M \in \mathcal{E}_h^1 \cap \partial\Omega(h), \ |\alpha| \leqslant 1, \\ &\frac{\partial \psi_h}{\partial \nu}(M) = 0, \ \forall M \in \mathcal{E}_h^2 \cap \partial\Omega(h). \end{aligned} \tag{4.134}$$

The space X_h is a finite dimensional subspace of $H_0^2(\Omega)$; we provide it with the scalar product induced by $H_0^2(\Omega)$:

$$((\psi_h, \phi_h))_h = ((\psi_h, \phi_h))_{H_0^2(\Omega)}, \ \ \forall \psi_h, \phi_h \in X_h. \tag{4.135}$$

Operator p_h. p_h = the identity as $X_h \subset H_0^2(\Omega)$.
Operator r_h. For $\psi \in \mathscr{D}(\Omega)$ (a dense subspace of $H_0^2(\Omega)$), we define $r_h \psi = \psi_h$ by its nodal values

$$\left.\begin{aligned} &D^\alpha \psi_h(M) = D^\alpha \psi(M), \ \ \forall M \in \mathcal{E}_h^1, \ [\alpha] \leqslant 2 \\ &\frac{\partial \psi_h}{\partial \nu_{ij}}(A_{ij}) = \frac{\partial \psi}{\partial \nu_{ij}}(A_{ij}), \ \forall A_{ij} \in \mathcal{E}_h^2. \end{aligned}\right\} \tag{4.136}$$

Proposition 4.10. (X_h, p_h, r_h) *defines a stable and convergent internal approximation of* $H_0^2(\Omega)$, *provided h belongs to a regular triangulation* $\mathcal{H}_\alpha$ *of* Ω.

Proof. We only have to prove that, for each $\psi \in \mathscr{D}(\Omega)$,

$$r_h \psi \to \psi \text{ in } H_0^2(\Omega), \ \text{ as } \rho(h) \to 0. \tag{4.137}$$

This follows from an analog of (4.42) and (4.113), for Hermite type polynomial interpolation (see Ciarlet & Raviart [1], G. Strang and G. Fix [1], A. Ženišek [1], cf. M. Zlámal [1]):

$$\operatorname*{Sup}_{x \in \mathscr{T}} |\psi_h(x) - \psi(x)| \leqslant c\eta_6(\psi)\rho_{\mathscr{T}}^5 \tag{4.138}$$

$$\operatorname*{Sup}_{x \in \mathscr{T}} |D_i\psi_h(x) - D_i\psi(x)| \leqslant c\eta_6(\psi)\frac{\rho_{\mathscr{T}}^5}{\rho'_{\mathscr{T}}}, \ i = 1, 2 \tag{4.139}$$

$$\operatorname*{Sup}_{x \in \mathscr{T}} |D^\alpha\psi_h(x) - D^\alpha\psi(x)| \leqslant c\eta_6(\psi)\frac{\rho_{\mathscr{T}}^5}{\rho'^2_{\mathscr{T}}}, \ [\alpha] = 2. \tag{4.140}$$

Therefore

$$\|\psi_h - \psi\|_{H_0^2(\Omega(h))} \leqslant c(\psi)\alpha^2\rho(h)^3 \tag{4.141}$$

and it is clear by (4.22) that

$$\|\psi\|_{H_0^2(\Omega - \Omega(h))} \to 0 \text{ as } \rho(h) \to 0. \tag{4.142}$$

Internal approximation of V (APX4).

We recall that Ω is a bounded simply connected set of $\mathbb{R}^2$. We define an *internal* approximation of V, using the preceding approximation of $H_0^2(\Omega)$ and the isomorphism (4.124).

Let there be given an admissible triangulation $\mathcal{T}_h$ of Ω. We associate with $\mathcal{T}_h$, the space V_h, and the operators p_h, r_h, as follows.

Space V_h.

It is the space of continuous vector functions u_h defined on Ω (or $\mathbb{R}^2$), of type

$$u_h = \left\{ \frac{\partial \psi_h}{\partial x_2}, - \frac{\partial \psi_h}{\partial x_1} \right\}, \tag{4.143}$$

ψ_h belonging to the previous space X_h.

It is clear that u_h vanishes outside $\Omega(h)$ and is continuous since ψ_h is continuously differentiable, and that $\operatorname{div} u_h = 0$. Therefore $u_h \in V$, and V_h is *a finite dimensional subspace of V.* We provide it with the scalar product induced by V

$$((u_h, v_h))_h = ((u_h, v_h)). \tag{4.144}$$

In particular,

$$\|u_h\|_h = \left(\sum_{[\alpha] = 2} |D^\alpha \psi_h|^2_{L^2(\Omega)} \right)^{1/2} \leqslant \|\psi_h\|_{H_0^2(\Omega)}. \tag{4.145}$$

Operator p_h: the identity.

Operator r_h.

Let u belong to $\mathcal{V}$, and let ψ denote the corresponding stream function (see (4.124)); clearly, $\psi \in \mathcal{D}(\Omega)$ and we can define $\psi_h \in X_h$ by (4.136). Then we set

$$u_h = r_h u = \left\{ \frac{\partial \psi_h}{\partial x_2}, - \frac{\partial \psi_h}{\partial x_1} \right\} \in V_h. \tag{4.146}$$

Proposition 4.11. *The preceding internal approximation of V is stable and convergent if $\rho(h) \to 0$, with h belonging to a regular triangulation $\mathcal{H}_\alpha$ of Ω.*

Proof. We have only to show that

$$u_h = r_h u \to u \text{ in } V, \ \forall u \in \mathscr{V}. \tag{4.147}$$

According to (4.124), (4.145), (4.145), we have

$$\|u_h - u\| \leqslant \|\psi_h - \psi\|_{H_0^2(\Omega)}.$$

The convergence (4.147) follows then from (4.141) and (4.142).

Approximation of Stokes Problem.
We take for (3.6),

$$a_h(u_h, v_h) = \nu((u_h, v_h))_h = \nu((u_h, v_h)) \tag{4.148}$$

$$\langle \ell_h, v_h \rangle = \langle f, v_h \rangle, \tag{4.149}$$

where ν and f are given as in Section 2.1 (see Theorem 2.1).
The approximate problem associated with (2.6) is

To find $u_h \in V_h$ such that

$$\nu((u_h, v_h)) = \langle f, v_h \rangle, \quad \forall v_h \in V_h. \tag{4.150}$$

The solution u_h of (4.150) exists and is unique and, according to Theorem 3.1 and Proposition 4.11,

$$u_h \to u \text{ in } V \text{ strongly if } \rho(h) \to 0, \ h \in \mathcal{H}_\alpha. \tag{4.151}$$

The Error between u and u_h.
Let us suppose that Ω has a polygonal boundary, so that we can choose triangulations $\mathcal{T}_h$ such that $\Omega(h) = \Omega$. Let us suppose that the solution u of Stokes problem is so smooth that $u \in \mathscr{C}^5(\bar{\Omega})$; then, by (4.141) and (4.145),[1]

$$\|u - r_h u\| \leqslant c(u, \alpha)\, \rho(h)^3. \tag{4.152}$$

---

(1) For the sake of simplicity (4.141) was proved for $\psi \in \mathscr{D}(\Omega)$; the proof is valid for any $\psi \in \mathscr{C}^6(\bar{\Omega}) \cap H_0^2(\Omega)$.

The equations

$$\nu((u, v)) = \langle f, v\rangle, \ \forall\ v \in V,$$

$$\nu((u_h, v_h)) = \langle f, v_h\rangle, \ \forall\ v_h \in V,$$

give

$$\nu((u - u_h, r_h u - u_h)) = 0.$$

Therefore,

$$\|u - u_h\|^2 = ((u - u_h, u - r_h u)) \leqslant \|u - u_h\|\,\|u - r_h u\|$$

$$\|u - u_h\| \leqslant \|u - r_h u\|,$$

and by (4.152), we obtain

$$\|u - u_h\| \leqslant c(u, \alpha)\, \rho(h)^3. \tag{4.153}$$

Remark 4.7. If Ω is a multi-connected open subset of $\mathbb{R}^2$, and if u belongs to V, then according to Lemma 2.5 there exists a function ψ satisfying (4.123). Since u vanishes on $\partial\Omega$, the function ψ is necessarily single valued, $\partial\psi/\partial\nu$ vanishes on $\partial\Omega$, ψ vanishes on Γ_0 the external part of $\partial\Omega$ and is constant on the internal components of $\partial\Omega$, $\Gamma_1, \Gamma_2, \ldots$.

The mapping (4.124),

$$\psi \to u = \left\{ \frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1} \right\}$$

gives here an isomorphism between the space

$$X = \{\psi \in H^2(\Omega),\ \psi = 0 \text{ on } \Gamma_0,$$

$$\psi = \text{const. on the } \Gamma_i' s,\ \frac{\partial\psi}{\partial\nu} = 0 \text{ on } \partial\Omega\} \tag{4.154}$$

and V.

An easy adaption of Proposition 4.10, leads to an approximation of X, and, as in Proposition 4.11 we deduce from this an approximation of the space V in the multi-connected case.

Remark 4.8. (i) An internal approximation of V with functions piecewise polynomials of degree 6 is constructed in F. Thomasset [1].

(ii) Internal approximations of V are not available if $n = 3$.

4.5. *Non-conforming finite elements*

Because of the condition div $\boldsymbol{u} = 0$, it is not possible to approximate V by the most simple finite elements, the piecewise linear continuous functions. This was shown by M. Fortin [2]. Our purpose here is to describe an approximation of V by linear, non-conforming finite elements, which in this case, means, piecewise linear but discontinuous functions. This leads to the approximation of V denoted by (APX 5). Then we associate with this approximation of V a new approximation scheme for Stokes problem.

4.5.1. *Approximation of $H_0^1(\Omega)$.*

We suppose that Ω is a bounded Lipschitz open set in $\mathbb{R}^n$ and in this section we will approximate $H_0^1(\Omega)$ by non-conforming piecewise linear finite elements.

Let $\mathcal{T}_h$ denote an admissible triangulation of Ω. If $\mathcal{S} \in \mathcal{T}_h$, we denote by $A_1, \ldots, A_{n+1}$ its vertices, by $\mathcal{S}_i$ the $(n-1)$-face which does not contain A_i, and by B_i the barycentre of the face $\mathcal{S}_i$. If G denotes the barycenter of $\mathcal{S}$, then since the barycentric coordinates of B_i with respect to the A_j, $j \neq i$, are equal to $1/n$ we have

$$GB_i = \sum_{j \neq i} \frac{GA_j}{n} = \sum_{j=1}^{n+1} \frac{GA_j}{n} - \frac{GA_i}{n} \tag{4.155}$$

or

$$GB_i = -\frac{1}{n}\, GA_i, \tag{4.156}$$

since $\Sigma_{j=1}^{n+1}\, GA_j = 0$ (the barycentric coordinates of G with respect to $A_1, \ldots, A_{n+1}$, are equal to $1/(n+1)$). We deduce from this, that

$$nB_iB_j = n(GB_j - GB_i) = GA_i - GA_j = -A_iA_j, \tag{4.157}$$

and therefore the vectors B_1B_j, $j = 2, \ldots, n+1$, are linearly independent like the vectors A_1A_j, $j = 2, \ldots, n+1$. Because of this, the barycentric coordinates of a point P, with respect to $B_1, \ldots, B_{n+1}$, can be defined, and we denote by $\mu_1, \ldots, \mu_{n+1}$, these coordinates. We remark also that for each given set of $(n+1)$ numbers $\beta_1, \ldots, \beta_{n+1}$, there exists one and only one linear function taking on at the points $B_1, \ldots, B_{n+1}$, the values $\beta_1, \ldots, \beta_{n+1}$, and this function $\boldsymbol{u}$ is

$$\boldsymbol{u}(P) = \sum_{i=1}^{n+1} \beta_i \mu_i(P); \tag{4.158}$$

(see Proposition 4.1).

Space W_h.

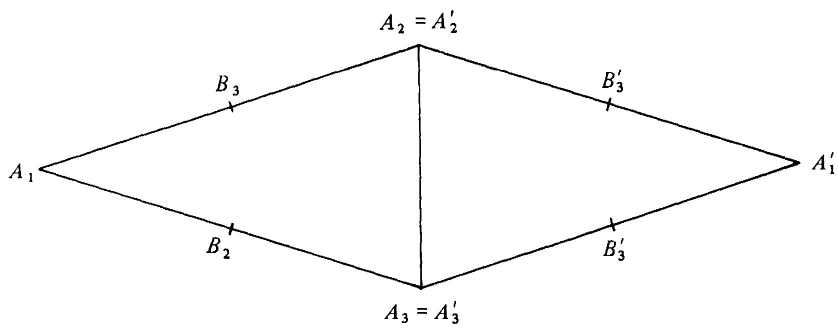
W_h is the space of vector-functions $\boldsymbol{u}_h$ which are linear on each $\mathscr{S} \in \mathscr{T}_h$, vanish outside $\Omega(h)$[1] and are such that the value of $\boldsymbol{u}_h$ at the barycenter B_i of some $(n-1)$-dimensional face $\mathscr{S}_i$ of a simplex $\mathscr{S} \in \mathscr{T}_h$ is zero, if this face belongs to the boundary of $\Omega(h)$; if this face intersects the interior of $\Omega(h)$ then the values of $\boldsymbol{u}_h$ at B_i are the same when B_i is considered as a point of the two different adjacent simplices.

Let $\mathscr{U}_h$ denote the set of points B_i which are barycenters of an $(n-1)$ dimensional face of a simplex $\mathscr{S} \in \mathscr{T}_h$ and which belong to the interior of $\Omega(h)$. A function $\boldsymbol{u}_h \in W_h$ is completely characterized by its values at the points $B_i \in \mathscr{U}_h$.

We denote by w_{hB}, B being a point of $\mathscr{U}_h$, the *scalar* function which is linear on each simplex $\mathscr{S} \in \mathscr{T}_h$, satisfies all the boundary and matching conditions that the functions of W_h satisfy and, moreover,

$$w_{hB}(B) = 1, \quad w_{hB}(M) = 0 \quad \forall M \in \mathscr{U}_h, \ M \neq B. \tag{4.159}$$

Such a function w_{hB} has a support equal to the two simplices which are adjacent to B.



Two adjacent triangles ($n = 2$).

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[1] As before, $\Omega(h) = \bigcup_{\mathscr{S} \in \mathscr{T}_h} \mathscr{S}$.

Lemma 4.15. *The functions $w_{hB}e_i$ of W_h, where $B \in \mathscr{U}_h$ and $1 \leqslant i \leqslant n$, form a basis of W_h. Hence the dimension of W_h is $nN(h)$, $N(h)$ being the number of points in $\mathscr{U}_h$.*

Proof. It is clear that these functions are linearly independent and that they span the whole space W_h: by Proposition 4.1, any $u_h \in W_h$ can be written as

$$u_h = \sum_{B \in \mathscr{U}_h} u_h(B) w_{hB}. \tag{4.160}$$

The space W_h is not included in $H_0^1(\Omega)$; actually the derivative $D_i u_h$ of some function $u_h \in W_h$ is the sum of Dirac distributions located on the faces of the simplices and of a step function $D_{ih} u_h$ defined almost everywhere by

$$D_{ih} u_h(x) = D_i u_h(x), \ \forall\, x \in \mathscr{S}, \ \forall\, \mathscr{S} \in \mathscr{T}_h. \tag{4.161}$$

Since u_h is linear on $\mathscr{S}$, $D_{ih} u_h$ is constant on each simplex.

We equip W_h with the following scalar product:

$$[\![u_h, v_h]\!]_h = (u_h, v_h) + \sum_{i=1}^{n} (D_{ih} u_h, D_{ih} v_h) \tag{4.162}$$

which is the discrete analogue of the scalar product of $H_0^1(\Omega)$:

$$[\![u, v]\!] = (u, v) + \sum_{i=1}^{n} (D_i u, D_i v). \tag{4.163}$$

Space F, operators $\bar{\omega}$, p_h.

We take, as in Section 3.3, $F = L^2(\Omega)^{n+1}$, and for $\bar{\omega}$ the isomorphism

$$u \in H_0^1(\Omega) \mapsto \bar{\omega} u = (u, D_1 u, \ldots, D_n u) \in F. \tag{4.164}$$

Similarly, the operator p_h is defined by

$$u_h \in W_h \mapsto p_h u_h = (u_h, \widetilde{D}_{1h} u_h, \ldots, \widetilde{D}_{nh} u_h) \in F. \tag{4.165}$$

The operators p_h each have norm equal to 1 and are stable.

Operator r_h.

We define $r_h \boldsymbol{u} = \boldsymbol{u}_h$, for $\boldsymbol{u} \in \mathscr{D}(\Omega)$, by

$$\boldsymbol{u}_h(B) = \boldsymbol{u}(B), \quad \forall B \in \mathscr{V}_h. \tag{4.166}$$

Proposition 4.12. *If h belongs to a regular triangulation* $\mathcal{H}_\alpha$ *of* Ω, *the preceding approximation of* $H_0^1(\Omega)$ *is stable and convergent.*

Proof. We have to check the conditions (C1) and (C2) of Definition 3.6.

For condition (C2), we have to prove that, for each $\boldsymbol{u} \in \mathscr{D}(\Omega)$,

$$\boldsymbol{u}_h \to \boldsymbol{u} \text{ in } \boldsymbol{L}^2(\Omega) \text{ as } \rho(h) \to 0, \tag{4.167}$$

$$D_{ih}\boldsymbol{u}_h \to D_i\boldsymbol{u} \text{ in } \boldsymbol{L}^2(\Omega) \text{ as } \rho(h) \to 0. \tag{4.168}$$

On each simplex $\mathscr{S}$ we can apply the result (4.42) to each component of $\boldsymbol{u}$; this gives:

$$\sup_{x \in \mathscr{S}} |\boldsymbol{u}(x) - \boldsymbol{u}_h(x)| \leqslant c\eta_2(\boldsymbol{u})\rho_{\mathscr{S}}^2$$

$$\sup_{x \in \mathscr{S}} |D_i\boldsymbol{u}(x) - D_i\boldsymbol{u}_h(x)| \leqslant c\eta_2(\boldsymbol{u})\frac{\rho_{\mathscr{S}}^2}{\rho'_{\mathscr{S}}}. \tag{4.169}$$

Therefore

$$\|p_h\boldsymbol{u}_h - \bar{\omega}\boldsymbol{u}\|_F \leqslant c(\boldsymbol{u})\alpha\rho(h) + [\![\boldsymbol{u}]\!]_{H_0^1(\Omega - \Omega(h))} \tag{4.170}$$

and this goes to 0 as $\rho(h) \to 0$.

To prove the condition (C1) let us suppose that $p_{h'}\boldsymbol{u}_{h'}$ converges weakly in F to $\phi = (\phi_0, \ldots, \phi_n)$; this means that

$$\boldsymbol{u}_{h'} \to \phi_0 \text{ in } \boldsymbol{L}^2(\Omega) \text{ weakly}, \tag{4.171}$$

$$D_{ih}\boldsymbol{u}_{h'} \to \phi_i \text{ in } \boldsymbol{L}^2(\Omega) \text{ weakly}, \ 1 \leqslant i \leqslant n. \tag{4.172}$$

Since the functions have compact supports included in Ω, (4.171) and (4.172) amount to saying that:

$$\tilde{\boldsymbol{u}}_{h'} \to \tilde{\phi}_0 \text{ in } \boldsymbol{L}^2(\mathscr{R}^n) \text{ weakly}, \tag{4.173}$$

$$D_{ih}\tilde{\boldsymbol{u}}_{h'} \to \tilde{\phi}_i \text{ in } \boldsymbol{L}^2(\mathscr{R}^n) \text{ weakly}, \ 1 \leqslant i \leqslant n, \tag{4.174}$$

($\tilde{g}$ is the function equal to g in Ω and to 0 in $\complement\Omega$)).

If we show that

$$\tilde{\phi}_i = D_i \tilde{\phi}_0, \quad 1 \leqslant i \leqslant n, \tag{4.175}$$

it will follow that $\tilde{\phi}_0 \in H^1(\mathbb{R}^n)$ and hence $\phi_0 \in H_0^1(\Omega)$ with $\phi_i = D_i \phi_0$, which amounts to saying that $\phi = \overline{\omega u}$, $u = \phi_0$.

Let θ be any test function in $\mathscr{D}(\mathbb{R}^n)$; then:

$$\int_{\mathbb{R}^n} \tilde{u}_{h'} (D_i \theta) \, \mathrm{d}x \to \int_{\mathbb{R}^n} \phi_0 (D_i \theta) \, \mathrm{d}x,$$

$$\int_{\mathbb{R}^n} (D_{ih} \tilde{u}_{h'}) \, \theta \, \mathrm{d}x \to \int_{\mathbb{R}^n} \phi_i \theta \, \mathrm{d}x.$$

The equality (4.175) is proved if we show that

$$\int_{\mathbb{R}^n} \phi_i \theta \, \mathrm{d}x = - \int_{\mathbb{R}^n} \phi_0 (D_i \theta) \mathrm{d}x, \quad \forall \, \theta \in \mathscr{D}(\mathbb{R}^n),$$

or that

$$\mathscr{I}^i_{h'} = \int_{\mathbb{R}^n} \tilde{u}_{h'} (D_i \theta) \, \mathrm{d}x + \int_{\mathbb{R}^n} (D_{ih} \tilde{u}_{h'}) \theta \, \mathrm{d}x \to 0, \text{ as } \rho(h') \to 0,$$

for each $\theta \in \mathscr{D}(\mathbb{R}^n)$. But according to a technical estimate proved in section 4.5.4 (see (4.218)), we have

$$|\mathscr{I}^i_h| \leqslant c(n, \Omega) \, \alpha \rho(h) \, \|\theta\|_{H^1(\Omega)} \|u_h\|_h,$$

and it is clear that $\mathscr{I}^i_h \to 0$ as $\rho(h) \to 0$.

Discrete Poincaré Inequality.

The following discrete Poincaré inequality will allow us to endow the space W_h described above, with another scalar product $((.,.))_h$, the discrete analogue of the scalar product $((.,.))$ of $H_0^1(\Omega)$ (see (1.11)) and Proposition 3.3).

Proposition 4.13. *Let us suppose that Ω is a bounded set in $\mathbb{R}^n$. Then there exists a constant $c(\Omega, \alpha)$ depending only on Ω and the constant α*

in (4.21) *such that the inequality*

$$|u_h|^2_{L^2(\Omega)} \leqslant c(\Omega, \alpha) \sum_{i=1}^{n} |D_{ih} u_h|^2_{L^2(\Omega)}, \tag{4.177}$$

holds for any scalar function of type (4.160):

$$u_h = \sum_{B \in \mathscr{V}_h} u_h(B) w_{hB}. \tag{4.178}$$

A similar inequality holds for the vector functions of type (4.160):

$$|u_h|_{L^2(\Omega)} \leqslant c'(\Omega, \alpha) \|u_h\|_h, \quad \forall\, u_h \in W_h \tag{4.179}$$

where

$$\|u_h\|_h = \left\{ \sum_{i=1}^{n} |D_{ih} u_h|^2_{L^2(\Omega)} \right\}^{1/2}. \tag{4.180}$$

Proof. The inequality (4.179) follows immediately from (4.177). In order to prove (4.177), we will show that

$$\left| \int_\Omega u_h \theta \, \mathrm{d}x \right| \leqslant c(\Omega, \alpha) |\theta|_{L^2(\Omega)} \|u_h\|_h, \tag{4.181}$$

for each u_h of type (4.178) and for each θ in $\mathscr{D}(\Omega)$; (4.181) implies (4.177) since $\mathscr{D}(\Omega)$ is dense in $L^2(\Omega)$.

Let us denote by χ the solution of the Dirichlet problem

$$\Delta\chi = \theta \text{ in } \Omega, \quad \chi \in H^1_0(\Omega).$$

The function χ is $\mathscr{C}^\infty$ on Ω and

$$\|\chi\|_{H^2(\Omega)} \leqslant c_0(\Omega) |\theta|_{L^2(\Omega)} {}^{(1)}. \tag{4.182}$$

(1) Strictly speaking this inequality is true only if Ω is smooth enough; in the general case (4.182) is valid if we define χ by $\Delta\chi = \theta$, $\chi \in H^1_0(\Omega')$ where Ω' is smooth and $\Omega' \supset \bar{\Omega}$. This makes no change in the following.

We then have

$$\int_\Omega u_h \theta \, \mathrm{d}x = \sum_{\mathscr{S} \in \mathscr{T}_h} \int u_h \cdot \Delta\chi \, \mathrm{d}x,$$

and the Green formula implies

$$\int_{\mathscr{S}} u_h \, \Delta\chi \, \mathrm{d}x = \int_{\partial\mathscr{S}} u_h \frac{\partial\chi}{\partial\nu} \, \mathrm{d}\Gamma - \int_{\mathscr{S}} \operatorname{grad} u_h \cdot \operatorname{grad} \chi \, \mathrm{d}x.$$

Hence

$$\left| \int_\Omega u_h \theta \, \mathrm{d}x \right| \leqslant |((u_h, \chi))_h| + |\mathscr{J}_h|$$

$$\left| \int_\Omega u_h \theta \, \mathrm{d}x \right| \leqslant \|u_h\|_h \|\chi\|_{H^1(\Omega)} + |\mathscr{J}_h| \leqslant c(\Omega) |\theta|_{L^2(\Omega)} \|u_h\|_h + |\mathscr{J}_h| \tag{4.183}$$

where

$$|\mathscr{J}_h| = \sum_{\mathscr{S} \in \mathscr{T}_h} \int_{\partial\mathscr{S}} u_h \frac{\partial\chi}{\partial\nu} \, \mathrm{d}\Gamma.$$

It is clear that

$$\mathscr{J}_h = \sum_{i=1}^{n} \mathscr{J}_h^i, \tag{4.184}$$

with

$$\mathscr{J}_h^i = \sum_{\mathscr{S} \in \mathscr{T}_h} \int_{\partial\mathscr{S}} u_h \, \chi_i \, \nu_i \, \mathrm{d}\Gamma$$

where

$$\chi_i = \frac{\partial\chi}{\partial x_i} \tag{4.185}$$

and $\nu_1, \ldots, \nu_n$, are the components of the unit vector ν normal to $\partial\mathscr{S}$. Because of the Green formula,

$$\int_{\partial\mathscr{S}} u_h \, \chi_i \, \nu_i \, \mathrm{d}\Gamma = \int_{\mathscr{S}} \frac{\partial}{\partial x_i} (u_h \chi_i) \, \mathrm{d}x,$$

so that

$$\mathcal{J}_h^i = \int_\Omega \frac{\partial}{\partial x_i}(u_h \chi_i)\, \mathrm{d}x.$$

We have already considered this expression in the proof of Proposition 4.12. Using the majoration (4.176) we get

$$|\mathcal{J}_h^i| \leqslant c(n, \Omega)\, \alpha\, \rho(h) \|\chi_i\|_{H^1(\Omega)} \|u_h\|_h,$$

$$|\mathcal{J}_h^i| \leqslant c(n, \Omega)\, \alpha\, \rho(h) \|\chi\|_{H^2(\Omega)} \|u_h\|_h.$$

Due to (4.182) and (4.184), we get

$$|\mathcal{J}_h| \leqslant c(n, \Omega)\, \alpha\, \rho(h)\, |\theta|_{L^2(\Omega)} \|u_h\|_h. \tag{4.186}$$

The combination of (4.183) and (4.186) gives precisely (4.181) with

$$c(\Omega, \alpha) = c(\Omega) + c(n, \Omega)\, \alpha\, \rho(h).$$ (1)

Proposition 4.14. *Let Ω be a bounded Lipschitz set. Let us suppose that we equip the space W_h with the scalar product*

$$((u_h, v_h))_h = \sum_{i=1}^{n} (D_{ih} u_h, D_{ih} v_h), \tag{4.187}$$

and leave the other unchanged in the statement of Proposition 4.12. Then this approximation of $H_0^1(\Omega)$ is again stable and convergent.

Proof. The only difference between this and Proposition 4.12 comes from the stability of the operators p_h and this difficulty is completely overcome by Proposition 4.13 and (4.179), which give

$$\|u_h\|_h \leqslant [\![u_h]\!]_h \leqslant c(\Omega)\, \|u_h\|_h, \quad \forall\, u_h \in W_h. \tag{4.188}$$

4.5.2 *Approximation of V* (APX5).

Let Ω be a Lipschitz bounded set in $\mathbb{R}^n$ and let $\mathcal{V}$ be the usual space (1.12) and V its closure in $H_0^1(\Omega)$.

We now define an approximation of V similar to the preceding approximation of $H_0^1(\Omega)$.

---

(1) $\rho(h)$ is obviously bounded; for example $\rho(h) \leqslant$ diameter of Ω.

As previously we take $F = L^2(\Omega)^{n+1}$, and $\bar{\omega} \in \mathscr{L}(V, F)$ as the linear operator

$$u \in V \to \bar{\omega} u = \{u, D_1 u, \ldots, D_n u\}. \tag{4.189}$$

Space V_h.

V_h is a subspace of the preceding space W_h:

$$V_h = \{u_h \in W_h \mid \sum_{i=1}^{n} D_{ih} u_{ih} = 0\}. \tag{4.190}$$

The condition in (4.190) concerning the divergence of u_h is equivalent to

$$\operatorname{div} u_h = 0 \text{ in } \mathscr{S}, \ \forall\, \mathscr{S} \in \mathscr{T}_h. \tag{4.191}$$

We equip the space V_h with the scalar product $((u_h, v_h))_h$ induced by W_h.

Operator p_h.

As before,

$$p_h u_h = \{u_h, D_{1h} u_h, \ldots, D_{nh} u_h\}.$$

The operators p_h are stable because of the inequality (4.179) (or (4.188)).

Operator r_h.

We have to define $r_h u = u_h \in V_h$, for $u \in \mathscr{V}$. Since u_h must satisfy the condition (4.191), the operator r_h used for the approximation of $H_0^1(\Omega)$ does not satisfy all the requirements. We choose the following operator r_h instead: $u_h = r_h u$ is characterized by the values of $u_h(B)$, $B \in \mathscr{U}_h$; if $B \in \mathscr{U}_h$, B is the barycenter of some $(n - 1)$-face $\mathscr{S}'$ of some n-simplex $\mathscr{S} \in \mathscr{T}_h$, and we set

$$u_h(B) = \frac{1}{\operatorname{meas}_{n-1}(\mathscr{S}')} \int_{\mathscr{S}'} u \, d\Gamma. \tag{4.192}$$

Let us show that $u_h \in V_h$; since $\operatorname{div} u_h$ is constant on each simplex $\mathscr{S}$, the condition (4.191) is equivalent to

$$\int_{\mathscr{S}} \operatorname{div} u_h \, dx = 0, \quad \forall \mathscr{S} \in \mathscr{T}_h.$$

Applying the Green formula, we get

$$\int_{\mathscr{S}} \operatorname{div} \boldsymbol{u}_h \, \mathrm{d}x = \sum_{\mathscr{S}' \in \partial^+ \mathscr{S}} \int_{\mathscr{S}'} \boldsymbol{u}_h \cdot \nu_{\mathscr{S}'} \, \mathrm{d}\Gamma$$

where $\partial^+ \mathscr{S}$ is the set of $(n-1)$-dimensional faces of $\mathscr{S}$; by (4.192) this is equal to

$$\sum_{\mathscr{S}' \in \partial^+ \mathscr{S}} \int_{\mathscr{S}'} \boldsymbol{u} \cdot \nu_{\mathscr{S}'} \, \mathrm{d}\Gamma = \int_{\partial \mathscr{S}} \boldsymbol{u} \cdot \nu \mathrm{d}\Gamma = \int_{\mathscr{S}} \operatorname{div} \boldsymbol{u} \, \mathrm{d}x,$$

and this last integral is zero since $\operatorname{div} \boldsymbol{u} = 0$.

Proposition 4.15. *The previous external approximation of V is stable and convergent, provided h belongs to a regular triangulation $\mathcal{H}_\alpha$ of Ω.*

Proof. We have noted already that the p_h are stable. Let us check the condition (C2) of Definition 3.6. For that, let us suppose that

$$p_{h'} \boldsymbol{u}_{h'} \to \phi \text{ in } F, \text{ weakly.} \tag{4.193}$$

Exactly as in Proposition 4.12. we see that

$$\phi = \bar{\omega} \boldsymbol{u}, \ \boldsymbol{u} \in H_0^1(\Omega) \tag{4.194}$$

Here, moreover, we must prove that $\boldsymbol{u} \in V$, i.e., $\operatorname{div} \boldsymbol{u} = 0$. But (4.193) means in particular

$$\sum_{i=1}^{n} D_{ih'} u_{ih'} \to \operatorname{div} \boldsymbol{u} \text{ in } L^2(\Omega), \text{ weakly,}$$

and since $\sum_{i=1}^{n} D_{ih'} u_{ih'}$ is identically zero, $\operatorname{div} \boldsymbol{u}$ is zero.

Let us check the condition (C1); if $\boldsymbol{u} \in \mathscr{V}$ we denote by $\boldsymbol{u}_h$ the function $r_h \boldsymbol{u}$ and by $\boldsymbol{v}_h$ the function of W_h defined by

$$\boldsymbol{v}_h(B) = \boldsymbol{u}(B), \ \ \forall \, B \in \mathscr{U}_h .$$

It was proved in Proposition 4.12 that

$$\|p_h v_h - \bar{\omega} u\|_F \leqslant c(u)\alpha\rho(h) + [\![u]\!]_{H^1(\Omega - \Omega(h))}. \tag{4.195}$$

It suffices now to show that

$$\|p_h u_h - p_h v_h\|_F = [\![u_h - v_h]\!]_h \to 0, \text{ as } \rho(h)\to 0. \tag{4.196}$$

Because of the inequality (4.188), it suffices to prove that

$$\|u_h - v_h\|_h \to 0, \text{ as } \rho(h)\to 0.$$

Each B of $\mathscr{U}_h$ is the barycenter of some face $\mathscr{S}'$ of some simplex $\mathscr{S}$; we can write

$$u(x) = u(B) + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(B)\cdot(x_i - \beta_i) + \sigma(x), \tag{4.197}$$

where $(\beta_1, \ldots, \beta_n)$ are the coordinates of B and

$$|\sigma(x)| \leqslant c(u)\,\rho_{\mathscr{S}}^2, \quad \forall x \in \mathscr{S}. \tag{4.198}$$

Integrating (4.197) on $\mathscr{S}'$, we find

$$u_h(B) = v_h(B) + \left(\int_{\mathscr{S}'} \sigma(x)\,\mathrm{d}x\right)\left(\int_{\mathscr{S}'} \mathrm{d}x\right)^{-1}$$

since $\int_{\mathscr{S}'} (x_i - \beta_i)\,\mathrm{d}x = 0$. Because of (4.198),

$$u_h(B) - v_h(B) = \epsilon_h(B), \tag{4.199}$$

with

$$|\epsilon_h(B)| \leqslant c(u)\,\rho_{\mathscr{S}}^2. \tag{4.200}$$

Inside the simplex $\mathscr{S}$ with faces $\mathscr{S}_1, \ldots, \mathscr{S}_{n+1}$,

$$u_h(x) - v_h(x) = \sum_{i=1}^{n+1} \epsilon_h(B_i)\,\mu_i(x)$$

where $\mu_1, \ldots, \mu_{n+1}$, are the barycentric coordinates of x with respect to $B_1, \ldots, B_{n+1}$. Therefore, in $\mathscr{S}$,

$$|\operatorname{grad}(u_h - v_h)| \leqslant c(u)\rho_{\mathscr{S}}^2 \sum_{i=1}^{n+1} |\operatorname{grad}\mu_i|$$

and by Lemma 4.2 and (4.21),

$$|\operatorname{grad}(u_h - v_h)| \leqslant c(u) \frac{\rho_{\mathscr{S}}^2}{\rho'_{\mathscr{S}}}.$$

Therefore in all Ω,

$$|D_{ih}(u_h - v_h)(x)| \leqslant c(u)\,\alpha\rho(h). \tag{4.201}$$

and this implies

$$\|u_h - v_h\|_h \leqslant c(u, \alpha, \Omega)\,\rho(h) \tag{4.202}$$

so that

$$\|p_h u_h - \bar{\omega} u\|_F \leqslant c(u)\,\alpha\rho(h) + [\![u]\!]_{H^1(\Omega - \Omega(h))}. \tag{4.203}$$

4.5.3. *Approximation of the Stokes Problem.*

Using the preceding approximation of V and the general results of Section 3.2, we can propose another scheme for the Stokes problem.

Let f belong to $L^2(\Omega)$, and $\nu > 0$. We set, with the preceding notations,

$$a_h(u_h, v_h) = \nu((u_h, v_h))_h, \quad \forall\, u_h, v_h \in V_h \tag{4.204}$$

$$\langle \ell_h, v_h \rangle = (f, v_h), \quad \forall\, v_h \in V_h. \tag{4.205}$$

The approximate problem is

$$\begin{aligned} &\text{To find } u_h \in V_h, \text{ such that} \\ &\nu((u_h, v_h))_h = (f, v_h), \quad \forall\, v_h \in V_h. \end{aligned} \tag{4.206}$$

The solution u_h of (4.206) exists and is unique. If $\rho(h) \to 0$, with h belonging to a regular triangulation $\mathscr{H}_\alpha$, then the following convergences hold

$$\begin{aligned} &u_h \to u \text{ in } L^2(\Omega) \text{ strongly,} \\ &D_{ih} u_h \to D_i u \text{ in } L^2(\Omega) \text{ strongly, } 1 \leqslant i \leqslant n. \end{aligned} \tag{4.207}$$

This follows, of course, from Theorem 3.1.

We can, as in Section 3.3 and as for other approximations, introduce the discrete pressure. It is a step function π_h of the type

$$\pi_h = \sum_{\mathscr{S} \in \mathscr{T}_h} \pi_h(\mathscr{S})\,\chi_{h\mathscr{S}}, \tag{4.208}$$

where $\pi_h(\mathscr{S})$ is the value of π_h on $\mathscr{S}$, $\pi_h(\mathscr{S}) \in \mathbb{R}$, and $\chi_{h\mathscr{S}}$ is the characteristic function of $\mathscr{S}$. This function π_h is such that

$$\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h - (\pi_h, \mathrm{div}_h\, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in W_h, \tag{4.209}$$

where

$$\mathrm{div}_h\, \boldsymbol{v}_h = \sum_{i=1}^{n} D_{ih} v_{ih}. \tag{4.210}$$

The error between $\boldsymbol{u}$ and $\boldsymbol{u}_h$, the solutions of (2.6) and (4.209) respectively, can be estimated as in Section 3.3. Let us suppose that Ω has a polygonal boundary, that $\Omega(h) = \Omega$, and that $\boldsymbol{u} \in \mathscr{C}^3(\bar{\Omega})$, $p \in \mathscr{C}^1(\bar{\Omega})$. We can define an approximation $r_h \boldsymbol{u}$ by a formula similar to (4.192), and it is not difficult to see that the estimation (4.203) still holds:

$$\|p_h r_h \boldsymbol{u} - \bar{\omega} \boldsymbol{u}\|_F \leqslant c(\boldsymbol{u}, \alpha)\, \rho(h). \tag{4.211}$$

We will prove later the following lemma.

Lemma 4.16. *Let $\boldsymbol{u}$, p denote the exact solution of* (2.6)–(2.8) *and let us suppose that $\boldsymbol{u} \in \mathscr{C}^3(\bar{\Omega})$. Then,*

$$a_h(\boldsymbol{u}, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) + \mathfrak{L}_h(\boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in V_h, \tag{4.212}$$

where

$$|\mathfrak{L}_h(\boldsymbol{v}_h)| \leqslant c(\boldsymbol{u}, p)\, \rho(h)\, \|\boldsymbol{v}_h\|_h. \tag{4.213}$$

If we accept this lemma temporarily, we see that

$$a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) = -\mathfrak{L}_h(\boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in V_h,$$

$$a_h(\boldsymbol{u}_h - r_h \boldsymbol{u}, \boldsymbol{v}_h) = a_h(\boldsymbol{u} - r_h \boldsymbol{u}, \boldsymbol{v}_h) - \mathfrak{L}_h(\boldsymbol{v}_h).$$

Taking $\boldsymbol{v}_h = \boldsymbol{u}_h - r_h \boldsymbol{u}$ and using (4.205) we obtain

$$\nu \|\boldsymbol{u}_h - r_h \boldsymbol{u}\|_h^2 \leqslant \nu \|\boldsymbol{u} - r_h \boldsymbol{u}\|_h \|\boldsymbol{u}_h - r_h \boldsymbol{u}\|_h + |\mathfrak{L}_h(\boldsymbol{u}_h - r_h \boldsymbol{u})|.$$

The estimates (4.211) and (4.213) then give

$$\nu \|\boldsymbol{u}_h - r_h \boldsymbol{u}\|_h \leqslant c(\boldsymbol{u}, p, \alpha, \Omega)\, \rho(h). \tag{4.214}$$

More precisely, the constant c in (4.214) depends only on the norms of $\boldsymbol{u}$ in $\mathscr{C}^3(\bar{\Omega})$ and of p in $\mathscr{C}^1(\bar{\Omega})$.

Proof of Lemma 4.16.

We take the scalar product in $L^2(\Omega)$, of the equation

$$-\nu\Delta \boldsymbol{u} + \operatorname{grad} p = \boldsymbol{f}, \tag{4.215}$$

with $\boldsymbol{v}_h$; since $\Omega = \Omega(h)$, we find

$$\sum_{\mathcal{S}\in\mathcal{T}_h} \left\{ -\nu(\Delta \boldsymbol{u}, \boldsymbol{v}_h)_{\mathcal{S}} + (\operatorname{grad} p, \boldsymbol{v}_h)_{\mathcal{S}} - (\boldsymbol{f}, \boldsymbol{v}_h)_{\mathcal{S}} \right\} = 0.$$

Green's formula applied in each simplex $\mathcal{S}$ gives

$$\sum_{\mathcal{S}\in\mathcal{T}_h} \left\{ -\nu(\Delta \boldsymbol{u}, \boldsymbol{v}_h)_{\mathcal{S}} + (\operatorname{grad} p, \boldsymbol{v}_h)_{\mathcal{S}} - (\boldsymbol{f}, \boldsymbol{v}_h)_{\mathcal{S}} \right\}$$

$$= a_h(\boldsymbol{u}, \boldsymbol{v}_h) - (\boldsymbol{f}, \boldsymbol{v}_h) - \ell_h(\boldsymbol{v}_h) = 0,$$

where

$$\ell_h(\boldsymbol{v}_h) = \sum_{\mathcal{S}\in\mathcal{T}_h} \int_{\partial\mathcal{S}} \left(\nu \frac{\partial \boldsymbol{u}}{\partial \vec{\nu}} \boldsymbol{v}_h - p\boldsymbol{v}_h\vec{\nu}\right) \mathrm{d}\Gamma \text{ (1)}$$

The estimate (4.213) of ℓ_h is then proved exactly as in Lemmas 4.17, 4.18 and 4.19 (see section 4.5.4).

Remark 4.9. A simple basis for V_h is available in the two-dimensional case. See the work of Crouzeix [1].

Non-conforming finite elements which are piecewise polynomials of degree $k > 1$ have also been studied for the approximation of, either $H_0^1(\Omega)$, or the space V.

M. Fortin [2] has pointed out that the space V cannot be approximated by conforming finite elements of degree one (i.e., piecewise linear functions). For this reason the approximation studied in this section is certainly very useful for Stokes and Navier–Stokes problems.

Several numerical computations of viscous incompressible flows, using these elements have been performed by F. Thomasset [2].

---

(1) The unit vector normal to $\partial\mathcal{S}$ is denoted $\vec{\nu}$ in this formula, to avoid any confusion with the constant $\nu > 0$.

4.5.4. *Auxiliary Estimates.*

We prove here some auxiliary technical estimates used before and which will be needed also in Chapter II. These estimates concern the expression

$$\mathscr{I}_h^i = \int_\Omega \frac{\partial}{\partial x_i} (\boldsymbol{u}_h \phi) \, \mathrm{d}x, \tag{4.217}$$

for $\boldsymbol{u}_h$ in W_h, $i = 1, \ldots, n$, and for different types of functions ϕ.

The following proofs are straightforward. They can be simplified by using general results on finite elements.

Proposition 4.16.

$$|\mathscr{I}_h^i| \leqslant c(n, \Omega) \, \alpha \, \rho(h) \, \|\phi\|_{H^1(\Omega)} \|\boldsymbol{u}_h\|_h, \tag{4.218}$$

for $\phi \in H^1(\Omega)$ *and* $\boldsymbol{u}_h \in W_h$.

The proof is given in the following lemmas.

Lemma 4.17.

$$\mathscr{I}_h^i = \sum_{\mathscr{S} \in \mathscr{T}_h} \sum_{\mathscr{S}' \in \partial^+ \mathscr{S}} \int_{\mathscr{S}'} \boldsymbol{u}_h \phi \nu_{i,\mathscr{S}'} \, \mathrm{d}\Gamma, \tag{4.219}$$

where $\partial^+ \mathscr{S}$ *is the set of the* $(n-1)$*-dimensional faces of* $\mathscr{S}$ (1), *and* $\nu_{i,\mathscr{S}'}$ *is the* i^{th} *component of the unit vector* $\boldsymbol{\nu}_{\mathscr{S}'}$ *which is normal to* $\mathscr{S}'$ *and is pointing outward with respect to* $\mathscr{S}$.

Proof. Since the functions vanish outside $\Omega(h)$, we have

$$\mathscr{I}_h = \int_{\Omega(h)} [\boldsymbol{u}_h (D_i \phi) + (D_{ih} \boldsymbol{u}_h) \, \phi] \, \mathrm{d}x =$$

(1) There are $(n-1)$ such faces.

$$\sum_{\mathscr{S}\in\mathscr{T}_h}\int_{\mathscr{S}}[u_h(D_i\phi)+(D_iu_h)\phi]\,\mathrm{d}x=\sum_{\mathscr{S}\in\mathscr{T}_h}\int_{\mathscr{S}}D_i(u_h\phi)\,\mathrm{d}x.$$

The Green-Stokes formula gives

$$\int_{\mathscr{S}}D_i(u_h\phi)\,\mathrm{d}x=\sum_{\mathscr{S}'\in\partial^+\mathscr{S}}\int_{\mathscr{S}'}u_h\phi\nu_{i,\mathscr{S}'}\,\mathrm{d}\Gamma.$$

•

Lemma 4.18. *With the notations of Lemma 4.17,*

$$\mathscr{J}_h^i=\sum_{\mathscr{S}\in\mathscr{T}_h}\sum_{\mathscr{S}'\in\partial^+\mathscr{S}}\int_{\mathscr{S}'}(u_h(x)-u_h(B))(\phi(x)-\phi(\mathscr{S}'))\nu_{i,\mathscr{S}'}\,\mathrm{d}\Gamma,\tag{4.220}$$

where $B=B_{\mathscr{S}'}$ is the barycenter of $\mathscr{S}'$, and $\phi(\mathscr{S}')$ is the average value of ϕ on $\mathscr{S}'$.

Proof. We first show that

$$\mathscr{J}_h^i=\sum_{\mathscr{S}\in\mathscr{T}_h}\sum_{\mathscr{S}'\in\partial^+\mathscr{S}}\int_{\mathscr{S}'}(u_h(x)-u_h(B))\,\phi(x)\nu_{i,\mathscr{S}'}\,\mathrm{d}\Gamma\tag{4.221}$$

To prove this equality we just have to show that

$$\sum_{\mathscr{S}\in\mathscr{T}_h}\sum_{\mathscr{S}'\in\partial^+\mathscr{S}}\int_{\mathscr{S}'}u_h(B)\,\phi\,(x)\nu_{i,\mathscr{S}'}\,\mathrm{d}\Gamma=0.\tag{4.222}$$

/

But for a face $\mathscr{S}'$ belonging to the boundary of $\Omega(h)$, $u_h(B)=0$ and the contribution of this face to the sum is zero. If $\mathscr{S}'$ belongs to the boundary of two adjacent simplices, the face contributes to the sum in two opposite terms: the $u_h(B)$ and $\phi(x)$ are the same and the $\nu_{i,\mathscr{S}'}$ are equal but with opposite signs when $\mathscr{S}'$ is considered as a part of the boundary of the two simplices. Hence the sum (4.222) is zero.

The equality (4.220) is then easily deduced from (4.221) if we prove that

$$\sum_{\mathscr{S}\in\mathscr{T}_h}\ \sum_{\mathscr{S}'\in\partial^+\mathscr{S}} \int_{\mathscr{S}'} (u_h(x)-u_h(B))\phi(\mathscr{S}')\nu_{i,\mathscr{S}'}\,\mathrm{d}\Gamma = 0. \quad (4.223)$$

But to prove (4.223) we simply note that

$$\int_{\mathscr{S}'} [u_h(x)-u_h(B)]\ \phi(\mathscr{S}')\,\nu_{i,\mathscr{S}'}\,\mathrm{d}\Gamma = 0,$$

since $\phi(\mathscr{S}')$ and $\nu_{i,\mathscr{S}'}$ are constant on $\mathscr{S}'$ and since

$$\int_{\mathscr{S}'} u_h(x)\,\mathrm{d}\Gamma = u_h(B)\int_{\mathscr{S}'}\mathrm{d}\Gamma, \quad (4.224)$$

because u_h is linear on $\mathscr{S}'$ and B is the barycenter of $\mathscr{S}'$.

Lemma 4.19.

$$|\mathscr{J}_h^i| \leqslant c(n)\sqrt{\alpha\rho(h)}\,\|u_h\|_h\cdot$$

$$\left(\sum_{\mathscr{S}\in\mathscr{T}_h}\ \sum_{\mathscr{S}'\in\partial^+\mathscr{S}} \int_{\mathscr{S}'} (\phi(x)-\phi(\mathscr{S}'))^2\,\mathrm{d}\Gamma\right)^{1/2} \quad (4.225)$$

Proof. Since $u_h(x)-u_h(B)$ is a linear function on $\mathscr{S}'$ which vanishes at $x=B$, we can write it on $\mathscr{S}$ as

$$u_h(x)-u_h(B) = \sum_{i=1}^n \frac{\partial u_h}{\partial x_i}(x_i-\beta_i),$$

where $\beta_1,\ldots,\beta_n$, are the coordinates of B. Hence

$$|u_h(x)-u_h(B)| \leqslant \rho_{\mathscr{S}}\,|\mathrm{grad}\,u_h|,\quad \forall x\in\mathscr{S},$$

and

$$\left|\int_{\mathscr{S}'} (u_h(x)-u_h(B))\,(\phi(x)-\phi(\mathscr{S}'))\nu_{i,\mathscr{S}'}\,\mathrm{d}\Gamma\right|$$

$$\leqslant \rho_{\mathscr{S}} \left(\int_{\mathscr{S}'} |\text{grad } u_h|^2 \, d\Gamma \right)^{1/2} \left(\int_{\mathscr{S}'} (\phi(x) - \phi(\mathscr{S}'))^2 \, d\Gamma \right)^{1/2}.$$

But, since grad u_h is constant on $\mathscr{S}$,

$$\int_{\mathscr{S}'} (\text{grad } u_h)^2 \, d\Gamma = \text{meas}_{n-1}(\mathscr{S}') \cdot |\text{grad } u_h|^2$$

Let us denote by ξ the distance between $\mathscr{S}'$ and the opposite vertex of $\mathscr{S}$. It is well known that

$$\text{meas}_n(\mathscr{S}) = \frac{1}{n} \, \xi \, \text{meas}_{(n-1)}(\mathscr{S}').$$

Hence

$$\text{meas}_{n-1}(\mathscr{S}') = \frac{n}{\xi} \text{meas}_n(\mathscr{S}) \leqslant \frac{n}{\rho'_{\mathscr{S}}} \text{meas}_n(\mathscr{S}) \tag{4.226}$$

since

$$\rho'_{\mathscr{S}} \leqslant \xi. \tag{4.227}$$

The reason (4.227) holds is that the largest ball included in $\mathscr{S}$ has a diameter equal to $\rho'_{\mathscr{S}}$ and this ball is included in the set bounded by the hyperplane containing $\mathscr{S}'$ and the parallel hyperplane containing the opposite vertex.

Therefore

$$\int_{\mathscr{S}'} (\text{grad } u_h)^2 \, d\Gamma \leqslant \frac{n}{\rho'_{\mathscr{S}}} \text{meas}_n(\mathscr{S}) \cdot |\text{grad } u_h|^2 \leqslant \frac{n}{\rho'_{\mathscr{S}}} \int_{\mathscr{S}} |\text{grad } u_h|^2 \, dx$$

and

$$|\mathscr{J}_h^i| \leqslant c(n) \sum_{\mathscr{S} \in \mathscr{T}_h} \sum_{\mathscr{S}' \in \partial^+\mathscr{S}} \frac{\rho_{\mathscr{S}}}{\sqrt{\rho'_{\mathscr{S}}}} \cdot$$

$$\left(\int_{\mathscr{S}} |\text{grad } u_h|^2 \, dx \right)^{1/2} \left(\int_{\mathscr{S}'} |\phi(x) - \phi(\mathscr{S}')|^2 \, d\Gamma \right)^{1/2}. \tag{4.228}$$

We then obtain (4.225) by using (4.20), (4.21) and applying the Schwarz inequality to (4.228).

Lemma 4.20

$$\int_{\mathscr{S}'} (\phi(x) - \phi(\mathscr{S}'))^2 \,\mathrm{d}\Gamma \leqslant c(n)\alpha\, \rho(h) \int_{\mathscr{S}} (\operatorname{grad}(\phi)^2 \mathrm{d}x, \qquad (4.229)$$

$$\sum_{\mathscr{S}\in\mathscr{T}_h} \sum_{\mathscr{S}'\in\partial^+\mathscr{S}} \int_{\mathscr{S}'} (\phi(x) - \phi(\mathscr{S}'))^2 \,\mathrm{d}\Gamma$$
$$\leqslant c(n)\,\alpha\,\rho(h) \int_{\Omega} (\operatorname{grad}\phi)^2 \,\mathrm{d}x \qquad (4.230)$$

Proof. The inequality (4.230) follows directly from (4.229).

The inequality (4.229) is an obvious consequence of the trace theorems in $H^1(\mathscr{S})$ if we replace the constant $c(n)\alpha^2$ in the right-hand side of (4.229) by some constant $c(\mathscr{S})$ depending on the particular simplex $\mathscr{S}$; the interest of (4.229) is that this inequality is uniformly valid with respect to the simplices $\mathscr{S}$ in $\mathscr{T}_h$.

To prove (4.229) we make some transformation in the coordinates which maps $\mathscr{S}$ on a fixed simplex $\overline{\mathscr{S}}$ and then we apply the trace theorem inequality in $\overline{\mathscr{S}}$ and come back to $\mathscr{S}$.

For simplicity we suppose that $A_1 = 0$, that $\mathscr{S}'$ is contained in the hyperplane $x_n = 0$, and that the vertices of $\mathscr{S}'$ are $A_1, \ldots, A_n$; the referential simplex is the simplex $\overline{\mathscr{S}}$ with vertices $\overline{A}_1, \ldots, \overline{A}_{n+1}$, $\overline{A}_1 = 0$, and $\overline{A_1 A_{i+1}} = e_i$, $i = 1, \ldots, n$. The face corresponding to $\mathscr{S}'$ is the face $\overline{\mathscr{S}}'$ with vertices $\overline{A}_1, \ldots, \overline{A}_n$. Let Λ denote the linear operator in $\mathbb{R}^n$ which is defined by

$$A_i = \Lambda \overline{A}_i, \;\; i = 2, \ldots, n+1$$

and let Λ' be the linear mapping in $\mathbb{R}^{n-1}$, which is defined by

$$A_i = \Lambda' \overline{A}_i, \;\; i = 2, \ldots, n.$$

A change of coordinates for the integral in the left hand side of (4.229) gives

$$\int_{\mathscr{S}'} \sigma(x)^2 \,\mathrm{d}\Gamma_{\mathscr{S}'} = \frac{1}{|\det \Lambda'|} \int_{\overline{\mathscr{S}}} \overline{\sigma}^2(\overline{x}) \,\mathrm{d}\Gamma_{\overline{\mathscr{S}}}$$

where

$$\sigma(x) = \phi(x) - \phi(\mathscr{S}') \tag{4.231}$$

and

$$\bar{\sigma}(\bar{x}) = \sigma(x), \quad \bar{x} = \Lambda^{-1}x. \tag{4.232}$$

For the simplex $\bar{\mathscr{S}}$, the trace theorem inequality and the Poincaré inequality give (recall that $\sigma(B) = 0$)

$$\int_{\bar{\mathscr{S}}'} \bar{\sigma}^2(\bar{x})\, \mathrm{d}\Gamma \leqslant c(\bar{\mathscr{S}}) \sum_{j=1}^{n} \cdot \int_{\bar{\mathscr{S}}} \frac{\partial\bar{\sigma}}{\partial\bar{x}_j}(\bar{x})^2\, \mathrm{d}\bar{x}\,.$$

We come back to $\mathscr{S}$ and the coordinates x_i. We write

$$\frac{\partial\bar{\sigma}}{\partial\bar{x}_j}(\bar{x}) = \sum_{k=1}^{n} \Lambda_{kj}^{-1} \left(\frac{\partial\sigma}{\partial x_k}\right)(\Lambda\bar{x}),$$

$$\sum_{j=1}^{n} \left| \frac{\partial\bar{\sigma}}{\partial\bar{x}_j}(\bar{x}) \right|^2 \leqslant \|\Lambda^{-1}\|^2 \sum_{k=1}^{n} \left| \frac{\partial\sigma}{\partial x_k}(\Lambda\bar{x}) \right|^2,$$

and hence

$$\int_{\bar{\mathscr{S}}'} \bar{\sigma}^2(\bar{x})\, \mathrm{d}\Gamma \leqslant c(\bar{\mathscr{S}})\, \|\Lambda^{-1}\|^2 \int_{\bar{\mathscr{S}}} \left(\sum_{k=1}^{n} \left| \frac{\partial\sigma}{\partial x_k}(\Lambda\bar{x}) \right|^2 \right) \mathrm{d}\bar{x}$$

$$\leqslant c(\bar{\mathscr{S}})\, |\det\Lambda|\, \|\Lambda^{-1}\|^2 \int_{\mathscr{S}} (\operatorname{grad}\sigma)^2\, \mathrm{d}x$$

We arrive to

/ $$\int_{\mathscr{S}'} (\phi(x) - \phi(\mathscr{S}')^2\, \mathrm{d}\Gamma \leqslant c\, \frac{|\det\Lambda|}{|\det\Lambda'|}\, \|\Lambda^{-1}\|^2 \int_{\mathscr{S}} (\operatorname{grad}\phi)^2\, \mathrm{d}x.$$

In order to prove (4.229), it remains to show that

$$\frac{|\det\Lambda|}{|\det\Lambda'|}\, \|\Lambda^{-1}\|^2 \leqslant c\alpha\rho(h) \tag{4.233}$$

Since

$$\frac{\det \Lambda}{\det \Lambda'} = \frac{\det(\Lambda')^{-1}}{\det \Lambda^{-1}} = \Lambda_{nn} \text{ (1)}$$

and

$$|\Lambda_{nn}| \leqslant \|\Lambda\|,$$

the left-hand side of (4.233) is majorized by

$$\|\Lambda\| \ \|\Lambda^{-1}\|^2$$

Because of Lemma 4.3, this term is majorized by

$$\frac{\rho_{\overline{\mathscr{S}}}}{\rho'_{\mathscr{S}}} \left(\frac{\rho_{\mathscr{S}}}{\rho'_{\overline{\mathscr{S}}}} \right)^2 \leqslant c(\overline{\mathscr{S}})\ \alpha\ \rho_{\mathscr{S}} \leqslant c(\overline{\mathscr{S}})\ \alpha\ \rho(h),$$

and (4.233) follows.

The proof of Lemma 4.20 is complete.

The estimation (4.218) now follows obviously from (4.225) and (4.230). □

Next inequality proved by similar methods will be useful in Chapter II.

Proposition 4.17.

$$|\mathscr{J}_h^i| \leqslant c(n, p, \Omega)\alpha^2 \left(\sum_{j=1}^n |D_j\phi|_{L^{q'}(\Omega)} \right) \cdot \left(\sum_{j=1}^n |D_{jh} u_h|_{L^p(\Omega)} \right) \tag{4.234}$$

for $\phi \in W^{1,q'}(\Omega)$ *and* $u_h \in W_h$, $1/q' = 1 - 1/q$, $1/q = 1/p - 1/n$ *for* $1 < p < n$.

Starting with the expression (4.220) of $\mathscr{J}_h^i$, we will proceed essentially as in Lemmas 4.19 and 4.20, with the main difference that Schwarz inequalities for integrals are replaced by Hölder inequalities with suitable exponents.

Lemma 4.21

$$|\mathscr{J}_h^i| \leqslant \sum_{\mathscr{S} \in \mathscr{T}_h} \sum_{\mathscr{S}' \in \partial^+ \mathscr{S}} \rho_{\mathscr{S}} (\mathrm{meas}_{n-1}(\mathscr{S}'))^{1/\gamma'}.$$

---

(1) Λ_{nn} is the (n, n) element of Λ; note that $\Lambda_{in} = 0$, $1 \leqslant i \leqslant n-1$, due to our choice of the coordinate axes.

$$|\operatorname{grad}_{\mathscr{S}} u_h| \cdot \left(\int_{\mathscr{S}'} |\phi(x) - \phi(\mathscr{S}')|^{\gamma} \mathrm{d}\Gamma \right)^{1/\gamma} \tag{4.235}$$

where $\operatorname{grad}_{\mathscr{S}} u_h$ *denotes the value of* $\operatorname{grad} u_h$ *on* $\mathscr{S}$ $(1 \leqslant i \leqslant n)$.

Proof. As for Lemma 4.19, we write on the face $\mathscr{S}'$

$$u_h(x) - u_h(B) = \sum_{i=1}^{n} \frac{\partial u_h}{\partial x_i} \cdot (x_i - \beta_i),$$

and thus

$$|u_h(x) - u_h(B)| \leqslant \rho_{\mathscr{S}} \; |\operatorname{grad} u_h|, \forall \; x \in \mathscr{S}. \tag{4.236}$$

Let γ be some real number, $1 < \gamma < \infty$, which will be specified later, and let γ' be the conjugate exponent $1/\gamma + 1/\gamma' = 1$. We have

$$\left| \int_{\mathscr{S}'} (u_h(x) - u_h(B))(\phi(x) - \phi(\mathscr{S}'))\, \nu_{i,\mathscr{S}'} \mathrm{d}\Gamma \right|$$

$$\leqslant \rho_{\mathscr{S}} \, |\operatorname{grad} u_h| \cdot \int_{\mathscr{S}'} |\phi(x) - \phi(\mathscr{S}')| \mathrm{d}\Gamma \leqslant \text{(with Holder inequality)}$$

$$\leqslant \rho_{\mathscr{S}} \; (\operatorname{meas}_{n-1}(\mathscr{S}'))^{1/\gamma'} |\operatorname{grad} u_h| \left(\int_{\mathscr{S}'} |\phi(x) - \phi(\mathscr{S}')|^{\gamma} \mathrm{d}\Gamma \right)^{1/\gamma},$$

and (4.235) follows by summation.

Lemma 4.22. *Let*

$$\gamma' = \frac{q(n-1)}{n} \text{ and } \gamma = \frac{\gamma'}{\gamma' - 1}.$$

Then for $1 \leqslant i \leqslant n$

$$\left(\int_{\mathscr{S}'} |\phi(x) - \phi(\mathscr{S}')|^{\gamma} \mathrm{d}\Gamma \right)^{1/\gamma}$$

$$\leqslant c(n, p) \cdot \frac{(\operatorname{meas}_{n-1}(\mathscr{S}'))^{1/\gamma}}{(\operatorname{meas}_n(\mathscr{S}))^{1/q'}} \, \rho_{\mathscr{S}} \left(\int_{\mathscr{S}} |\operatorname{grad} \phi_i|^{q'} \mathrm{d}x \right)^{1/q'}. \tag{4.237}$$

Proof. We proceed exactly as in Lemma 4.20 and use the same notations Setting $\sigma(x) = \phi(x) - \phi(\mathscr{S}')$, we have

$$\left(\int_{\mathscr{S}'} |\sigma(x)|^\gamma \, d\Gamma_{\mathscr{S}'}\right)^{1/\gamma} = |\det \Lambda'|^{-1/\gamma} \left(\int_{\overline{\mathscr{S}'}} |\bar{\sigma}(\bar{x})|^\gamma \, d\Gamma_{\overline{\mathscr{S}'}}\right)^{1/\gamma}$$

We apply Poincaré inequality and Sobolev imbedding theorem on the reference simplex $\mathscr{S}$:

$$\begin{aligned}
|\bar{\sigma}|_{L^\gamma(\overline{\mathscr{S}'})} &\leqslant (\text{by the Sobolev inequality on } \overline{\mathscr{S}}' \subset \mathbb{R}^{n-1}) \\
&\leqslant c_0(n, p)\, |\bar{\sigma}|_{W^{1/q, q'}(\overline{\mathscr{S}'})} \\
&\leqslant (\text{by a trace theorem, see Lions [1]}) \\
&\leqslant c_1(n, p)\, |\bar{\sigma}|_{W^{1,q'}(\bar{\mathscr{S}})} \\
&\leqslant (\text{by the Poincaré inequality}) \\
&\leqslant c(\bar{\mathscr{S}})\, c_2(n, p) \left(\sum_{j=1}^{n} \int_{\bar{\mathscr{S}}} \left|\frac{\partial \bar{\sigma}}{\partial \bar{x}_j}(\bar{x})\right|^{q'} d\bar{x}\right)^{1/q'} \text{(1)}
\end{aligned}$$

Using a majoration given in Lemma 4.20, we see that

$$\begin{aligned}
\left(\sum_{j=1}^{n} \left|\frac{\partial \bar{\sigma}}{\partial \bar{x}_j}(\bar{x})\right|^{q'}\right)^{1/q'} &\leqslant c(q') \left(\sum_{j=1}^{n} \left|\frac{\partial \bar{\sigma}}{\partial \bar{x}_j}(\bar{x})\right|^2\right)^{1/2} \\
&\leqslant c(q') \|\Lambda^{-1}\| \left(\sum_{j=1}^{n} \left|\frac{\partial \sigma}{\partial x_j}(\Lambda \bar{x})\right|^2\right)^{1/2} \\
&\leqslant c'(q') \|\Lambda^{-1}\| \left(\sum_{j=1}^{n} \left|\frac{\partial \sigma}{\partial x_j}(\Lambda \bar{x})\right|^{q'}\right)^{1/q'}
\end{aligned}$$

We get

$$|\bar{\sigma}|_{L^\gamma(\overline{\mathscr{S}'})} \leqslant c(\bar{\mathscr{S}})\, c_3(n, p)\, \|\Lambda^{-1}\| \left(\sum_{j=1}^{n} \int_{\bar{\mathscr{S}}} \left|\frac{\partial \sigma}{\partial x_j}(\Lambda \bar{x})\right|^{q'} d\bar{x}\right)^{1/q'}$$

(1) The $c_i(n, p)$'s also depend on $\overline{\mathscr{S}'}$ but $\overline{\mathscr{S}}'$ is fixed.

and coming back to the simplex $\mathscr{S}$,

$$|\sigma|_{L^\gamma(\overline{\mathscr{S}'})} \leqslant c(\overline{\mathscr{S}})\, c_3(n, p)\, |\det \Lambda|^{1/q'} \|\Lambda^{-1}\| \, .$$

$$\left(\sum_{j=1}^{n} \int_{\mathscr{S}} \left| \frac{\partial \sigma}{\partial x_j}(x) \right|^{q'} \mathrm{d}x \right)^{1/q} .$$

Finally

$$\left(\int_{\mathscr{S}'} |\sigma(x)|^\gamma \, \mathrm{d}\Gamma_{\mathscr{S}'} \right)^{1/\gamma} \leqslant c(\overline{\mathscr{S}})\, c_3(n, p) \frac{|\det \Lambda|^{1/q'}}{|\det \Lambda'|^{1/\gamma}} \|\Lambda^{-1}\| \, \cdot$$

$$\left(\sum_{j=1}^{n} \int_{\mathscr{S}} \left| \frac{\partial \phi}{\partial x_j} \right|^{q'} \mathrm{d}x \right)^{1/q} .$$

We achieve the proof by observing that

$$\det \Lambda = \frac{\mathrm{meas}_n(\overline{\mathscr{S}})}{\mathrm{meas}_n(\mathscr{S})}, \quad \det \Lambda' = \frac{\mathrm{meas}_{n-1}(\overline{\mathscr{S}'})}{\mathrm{meas}_{n-1}(\mathscr{S}')},$$

and by Lemma 4.3

$$\|\Lambda^{-1}\| \leqslant \frac{\rho_{\mathscr{S}}}{\rho'_{\mathscr{S}}} .$$

Lemma 4.23.

$$|\mathscr{J}_h^i| \leqslant c(n, p)\, d^2 \left(\sum_{j=1}^{n} |D_{jh} u_h|_{L^p(\Omega)} \right) \left(\sum_{j=1}^{n} |D_j \phi|_{L^{q'}(\Omega)} \right). \quad (4.238)$$

Proof. In order to prove (4.238), we combine (4.235) and (4.237). We obtain ($1/\gamma + 1/\gamma' = 1$):

$$|\mathscr{J}_h^i| \leqslant c(n, p, \Omega) \sum_{\mathscr{S} \in \mathscr{T}_h} \rho_{\mathscr{S}}^2 (\mathrm{meas}_n(\mathscr{S}))^{-1/q'} (\mathrm{meas}_{n-1} \mathscr{S}')) \cdot$$

$$\cdot |\mathrm{grad}_{\mathscr{S}} u_h| \cdot \left(\int_{\mathscr{S}} |\mathrm{grad}\, \phi_i|^{q'} \, \mathrm{d}x \right)^{1/q'}$$

and, with (4.226),

$$|\mathcal{J}_h^i| \leqslant c(n,p,\Omega) \sum_{\mathscr{S} \in \mathcal{T}_h} \alpha \rho_{\mathscr{S}} (\mathrm{meas}_n(\mathscr{S}))^{1/q} |\mathrm{grad}\, \boldsymbol{u}_h| \cdot \cdot \left(\int_{\mathscr{S}} |\mathrm{grad}\, \phi|^{q'} \mathrm{d}x \right)^{1/q'} \tag{4.239}$$

since

$$\mathrm{meas}_{n-1}(\mathscr{S}') \leqslant \frac{n}{\rho'_{\mathscr{S}}} \mathrm{meas}_n(\mathscr{S}) \leqslant \frac{n\alpha}{\rho_{\mathscr{S}}} \mathrm{meas}_n(\mathscr{S}).$$

The n-dimensional measure of $\mathscr{S}$ is larger than the n-dimensional measure of a ball of diameter $\rho'_{\mathscr{S}}$; therefore

$$(\rho'_{\mathscr{S}})^n \leqslant c(n)\, \mathrm{meas}_n(\mathscr{S})$$

and

$$\rho_{\mathscr{S}} \leqslant \alpha \rho'_{\mathscr{S}} \leqslant c(n)\, \alpha\, (\mathrm{meas}_n(\mathscr{S}))^{1/n}.$$

With this majoration, we infer from (4.239) that

$$|\mathcal{J}_h^i| \leqslant c(n,p,\Omega)\, \alpha^2 \sum_{\mathscr{S} \in \mathcal{T}_h} (\mathrm{meas}_n(\mathscr{S}))^{1/p} \cdot |\mathrm{grad}_{\mathscr{S}}\, \boldsymbol{u}_h| \cdot \left(\int_{\mathscr{S}} |\mathrm{grad}\, \phi|^{q'} \mathrm{d}x \right)^{1/q'}.$$

Applying Hölder inequality with exponent q and q' to this sum, we get

$$|\mathcal{J}_h^i| \leqslant c(n,p,\Omega)\alpha^2 \left(\sum_{\mathscr{S} \in \mathcal{T}_h} \mathrm{meas}_n(\mathscr{S})^{q/p} |\mathrm{grad}_{\mathscr{S}}\, \boldsymbol{u}_h|^q \right)^{1/q} \cdot \left(\int_{\Omega(h)} |\mathrm{grad}\, \phi|^{q'} \mathrm{d}x \right)^{1/q'}. \tag{4.240}$$

The problem of obtaining (4.238) is now reduced to proving the following inequality:

$$\left(\sum_{\mathscr{S} \in \mathscr{T}_h} \operatorname{meas}_n (\mathscr{S})^{q/p} |\operatorname{grad}_{\mathscr{S}} u_h|^q \right)^{1/q} \leqslant$$

$$\left(\sum_{\mathscr{S} \in \mathscr{T}_h} \operatorname{meas}_n (\mathscr{S}) |\operatorname{grad} u_h|^p \right)^{1/p} \tag{4.241}$$

since the right-hand side of this inequality is equal to

$$\left(\sum_{\mathscr{S} \in \mathscr{T}_h} \int_{\mathscr{S}} |\operatorname{grad} u_h|^p \, \mathrm{d}x \right)^{1/p}$$

and this is easily bounded by

$$c(p) \left(\sum_{j=1}^{n} |D_{jh} u_h|_{L^p(\Omega)} \right).$$

Up to a modification of notation, the proof of (4.241) is the purpose of the next lemma.

Lemma 4.24. *Let a_j, z_j, be N pairs of non negative numbers, and let p, n, q be as before. Then*

$$\left(\sum_{j=1}^{N} a_j^{q/p} z_j^q \right)^{1/q} \leqslant \left(\sum_{j=1}^{N} a_j z_j^p \right)^{1/p}. \tag{4.242}$$

Proof. For $N = 2$, the proof is elementary. We set

$$a_1 z_1^p + a_2 z_2^p = \rho, \quad z_1 = \left(\frac{\rho - a_2 z_2^p}{a_1} \right)^{1/p},$$

and we observe that the function

$$z_2 \to a_1^{q/p} z_1^q + a_2^{q/p} z_2^q = a_1^{q/p} \left(\frac{\rho - a_2 z_2^p}{a_1} \right)^{q/p} + a_2^{q/p} z_2^q$$

attains its maximum, on the interval $[0, (\rho/a_2)^{1/p}]$, at the end points, and this maximum is equal to $\rho^{q/p}$.

We then proceed by induction on N; assuming that (4.242) is true for $N - 1$ terms, we write

$$\left(\sum_{j=1}^{N} a_j^{q/p} z_j^q\right)^{1/q} = \left(a_1^{q/p} z_1^q + \sum_{j=2}^{N} a_j^{q/p} z_j^q\right)^{1/q} \leqslant \tag{4.243}$$

$$\leqslant (a_1^{q/p} z_1^q + \tilde{a}_2^{q/p} \tilde{z}_2^q)^{1/q}$$

where

$$\tilde{a}_2 = \tilde{z}_2 = \left(\sum_{j=2}^{N} a_j z_j^p\right)^{1/(p+1)}.$$

Using the inequality previously established for $N = 2$, we majorize the right-hand side of (4.243) by

$$(a_1 z_1^p + \tilde{a}_2 \tilde{z}_2^p)^{1/p} = \left(\sum_{j=1}^{N} a_j z_j^p\right)^{1/p},$$

and (4.242) is proved. □

§5. Numerical Algorithms

We saw that discretization of the Stokes equations does not solve completely the problem of numerical approximation of these equations; for the actual computation of the solution, we must have a basis of the space V_h such that the analogue of (3.6) leads to an algebraic linear system for the components of $\boldsymbol{u}_h$, with a sufficiently sparse matrix. This occurs only with the schemes corresponding to (APX4) and (APX5); for the discrete problem associated with (APX1)–(APX3) we do not even have an explicit basis of V_h.

In Sections 5.1 to 5.3 we will study two algorithms which are very useful for the practical solution of the discretized equations. In Sections 5.1 and 5.2 we consider the continuous case and in Section 5.3 we show briefly how they can be adapted to the discrete problems.

The results proved in Section 5.4 are related to this problem but they also show how incompressible fluids can be considered as the limit of "slightly" compressible fluids.

5.1. *Uzawa Algorithm*

In Proposition 2.1 we interpreted the Stokes problem as a variational problem, an optimization problem with linear constraints. The algorithms described in this and the following sections are classical algorithms of optimization. We will present these algorithms and study their convergence without any direct reference to optimization theory, although the idea of the algorithm and the proof of the convergence result from optimization theory.

Let us consider the functions $\boldsymbol{u}$ and p defined by Theorem 2.1; we will obtain $\boldsymbol{u}$, p as limits of sequences $\boldsymbol{u}^m$, p^m which are much easier to compute than $\boldsymbol{u}$ and p.

We start the algorithm with an arbitrary element p^0,

$$p^0 \in L^2(\Omega). \tag{5.1}$$

When p^m is known, we define $\boldsymbol{u}^{m+1}$ and p^{m+1} $(m \geqslant 0)$, by the conditions

$$\boldsymbol{u}^{m+1} \in \boldsymbol{H}_0^1(\Omega) \text{ and}$$

$$\nu((\boldsymbol{u}^{m+1}, \boldsymbol{v})) - (p^m, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \ \forall \ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{5.2}$$

$$p^{m+1} \in L^2(\Omega) \text{ and}$$

$$(p^{m+1} - p^m, q) + \rho(\operatorname{div} \boldsymbol{u}^{m+1}, q) = 0, \ \forall \ q \in L^2(\Omega). \tag{5.3}$$

We suppose that $\rho > 0$ is a fixed number; other conditions on ρ will be given later.

The existence and uniqueness of the solution $\boldsymbol{u}^{m+1}$ of (5.2) is very easy, because of the projection theorem (Theorem 2.2). Actually $\boldsymbol{u}^{m+1}$ is simply a solution of the Dirichlet problem

$$\boldsymbol{u}^{m+1} \in \boldsymbol{H}_0^1(\Omega)$$

$$-\nu\Delta\boldsymbol{u}^{m+1} = \operatorname{grad} p^m + \boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega). \tag{5.4}$$

When $\boldsymbol{u}^{m+1}$ is known, p^{m+1} is explicitly given by (5.3) which is equivalent to

$$p^{m+1} = p^m - \rho \operatorname{div} \boldsymbol{u}^{m+1} \in L^2(\Omega). \tag{5.5}$$

Convergence of the Algorithm

Theorem 5.1. *If the number ρ satisfies*

$$0 < \rho < 2\nu \tag{5.6}$$

then as $m \to \infty$, $\boldsymbol{u}^{m+1}$ converges to $\boldsymbol{u}$ in $\boldsymbol{H}_0^1(\Omega)$ and p^{m+1} converges to *p weakly in* $L^2(\Omega)/\mathcal{R}$.

Proof. The equation (2.7) which is satisfied by $\boldsymbol{u}$ and p is equivalent to

$$\nu((\boldsymbol{u}, \boldsymbol{v})) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \ \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{5.7}$$

Let us take $\boldsymbol{v} = \boldsymbol{u}^{m+1} - \boldsymbol{u}$ in equations (5.2) and (5.7) and then let us subtract the resulting equations; this gives:

$$\nu \|\boldsymbol{u}^{m+1} - \boldsymbol{u}\|^2 = (p^m - p, \operatorname{div} \boldsymbol{u}^{m+1})$$

or

$$\nu \|\boldsymbol{v}^{m+1}\|^2 = (q^m, \operatorname{div} \boldsymbol{v}^{m+1}), \tag{5.8}$$

where we have set

$$\boldsymbol{v}^{m+1} = \boldsymbol{u}^{m+1} - \boldsymbol{u}, \tag{5.9}$$

$$q^m = p^m - p. \tag{5.10}$$

Taking $q = p^{m+1} - p$ in (5.3), we get:

$$(q^{m+1} - q^m, q^{m+1}) + \rho(\operatorname{div} \boldsymbol{v}^{m+1}, q^{m+1}) = 0,$$

or equivalently

$$|q^{m+1}|^2 - |q^m|^2 + |q^{m+1} - q^m|^2 = -2\rho(\operatorname{div} \boldsymbol{v}^{m+1}, q^{m+1}). \tag{5.11}$$

We next multiply equation (5.8) by 2ρ, and then add equation (5.11), obtaining

$$\begin{aligned} |q^{m+1}|^2 - |q^m|^2 + |q^{m+1} - q^m|^2 + 2\rho\nu \|\boldsymbol{v}^{m+1}\|^2 \\ = -2\rho(\operatorname{div} \boldsymbol{v}^{m+1}, q^{m+1} - q^m). \end{aligned} \tag{5.12}$$

We majorize the right-hand side of (5.12) by

$$2\rho\, |\operatorname{div} \boldsymbol{v}^{m+1}|\ |q^{m+1} - q^m|$$

which is less than or equal to

$$2\rho\, \|\boldsymbol{v}^{m+1}\|\ |q^{m+1} - q^m|$$

since

$$|\operatorname{div} \boldsymbol{v}| \leqslant \|\boldsymbol{v}\|, \ \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \text{ [1]} \tag{5.13}$$

[1] This inequality easily established if $\boldsymbol{v} \in (\mathscr{D})(\Omega)$ is valid by a continuity argument for each $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$. By the same method one can check that if $n = 2$ or 3

$$\|\boldsymbol{v}\|^2 = |\operatorname{div} \boldsymbol{v}|^2_{L^2(\Omega)} + |\operatorname{rot} \boldsymbol{v}|^2_{\boldsymbol{L}^2(\Omega)}, \quad \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega).$$

We can then majorize the last expression by

$$\delta|q^{m+1}-q^m|^2+\frac{\rho^2}{\delta}\|v^{m+1}\|^2,$$

where $0<\delta<1$ is arbitrary at the present time. Hence

$$\begin{aligned}&|q^{m+1}|^2-|q^m|^2+(1-\delta)|q^{m+1}-q^m|^2\\&\qquad+\rho(2\nu-\frac{\rho}{\delta})\|v^{m+1}\|^2\leqslant 0.\end{aligned}\tag{5.14}$$

If we add the inequalities (5.14) for m = 0, ..., N, we find

$$\begin{aligned}&|q^{N+1}|^2+(1-\delta)\sum_{m=0}^{N}|q^{m+1}-q^m|^2\\&\qquad+(2\nu-\frac{\rho}{\delta})\rho\sum_{m=0}^{N}\|v^{m+1}\|^2\leqslant|q^0|^2.\end{aligned}\tag{5.15}$$

Because of condition (5.6), there exists some δ such that

$$0<\frac{\rho}{2\nu}<\delta<1,$$

and hence

$$(2\nu-\frac{\rho}{\delta})>0.$$

With such a δ fixed, the inequality (5.15) shows that

$$\begin{aligned}&|q^{m+1}-q^m|^2=|p^{m+1}-p^m|^2\to 0 \text{ as } m\to\infty,\\&\|v^{m+1}\|^2=\|u^{m+1}-u\|^2\to 0 \text{ as } m\to\infty.\end{aligned}\tag{5.16}$$

The convergence of u^{m+1} to u is thereby proved. Now by (5.14) we see also that the sequence p^m is bounded in $L^2(\Omega)$. We can then extract from p^m a subsequence $p^{m'}$ converging weakly in $L^2(\Omega)$ to some element p_*. The equation (5.2) gives in the limit

$$\nu((u,v))-(p_*,\operatorname{div}v)=(f,v),\ \ \forall\, v\in H_0^1(\Omega),$$

and by comparison with (5.7), we get

$$(p-p_*,\operatorname{div}v)=0,\ \ \forall\, v\in H_0^1(\Omega),$$

whence

$$\text{grad}\,(p - p_*) = 0, \quad p_* = p + \text{const.}$$

From any subsequence of p^m, we can extract a subsequence converging weakly in $L^2(\Omega)$ to $p + c$; hence the sequence p^m converges as a whole to p for the weak topology of $L^2(\Omega)/\mathcal{R}$.

Remark 5.1. Let us define p by imposing the condition

$$\int_\Omega p(x)\mathrm{d}x = 0.$$

Let us suppose that p^0 in $L^2(\Omega)$ is chosen so that

$$\int_\Omega p^0(x)\mathrm{d}x = 0.$$

Then clearly we have

$$\int_\Omega p^m(x)\mathrm{d}x = 0, \qquad m \geqslant 1,$$

and the whole sequence p^m converges to p, weakly, in the space $L^2(\Omega)$. □

5.2. *Arrow-Hurwicz Algorithm.*

In this case too, the functions $\boldsymbol{u}$ and p are the limits of two sequences $\boldsymbol{u}^m$, p^m which are recursively defined.

We start the algorithm with arbitrary elements $\boldsymbol{u}^0$, p^0,

$$\boldsymbol{u}^0 \in \boldsymbol{H}_0^1(\Omega), \quad p^0 \in L^2(\Omega). \tag{5.17}$$

When p^m and $\boldsymbol{u}^m$ are known, we define p^{m+1} and $\boldsymbol{u}^{m+1}$ as the solutions of

$$\left.\begin{aligned} &\boldsymbol{u}^{m+1} \in \boldsymbol{H}_0^1(\Omega) \text{ and} \\ &((\boldsymbol{u}^{m+1} - \boldsymbol{u}^m, \boldsymbol{v})) + \rho\nu((\boldsymbol{u}^m, \boldsymbol{v})) - \rho(p^m, \operatorname{div}\boldsymbol{v}) \\ &\qquad\qquad = \rho(\boldsymbol{f}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \end{aligned}\right\} \tag{5.18}$$

$$\left.\begin{aligned} &p^{m+1} \in L^2(\Omega) \text{ and} \\ &\alpha(p^{m+1} - p^m, q) + \rho(\operatorname{div}\boldsymbol{u}^{m+1}, q) = 0, \quad \forall\, q \in L^2(\Omega). \end{aligned}\right\} \tag{5.19}$$

We suppose that ρ and α are two strictly positive numbers; conditions on ρ and α will appear later.

The existence and uniqueness of $\boldsymbol{u}^{m+1} \in \boldsymbol{H}_0^1(\Omega)$ satisfying (5.18) is easily established with the projection theorem; $\boldsymbol{u}^{m+1}$ is the solution of the Dirichlet problem

$$\begin{aligned} &-\Delta \boldsymbol{u}^{m+1} = -\Delta \boldsymbol{u}^m + \rho\nu\Delta \boldsymbol{u}^m - \rho \operatorname{grad} p^m + \rho \boldsymbol{f} \\ &\boldsymbol{u}^{m+1} \in \boldsymbol{H}_0^1(\Omega). \end{aligned} \tag{5.20}$$

Then p^{m+1} is explicitly given by (5.19) which is equivalent to

$$p^{m+1} = p^m - \frac{\rho}{\alpha} \operatorname{div} \boldsymbol{u}^{m+1} \in \dot{L}^2(\Omega). \tag{5.21}$$

Convergence of the Algorithm.

Theorem 5.2. *If the numbers α and ρ satisfy*

$$0 < \rho < \frac{2\alpha\nu}{\alpha\nu^2 + 1}, \tag{5.22}$$

then, as $m \to \infty$, u^m converges to u in $H_0^1(\Omega)$ and p^m converges to p weakly in $L^2(\Omega)/\mathcal{R}$

Proof. Let

$$\boldsymbol{v}^m = \boldsymbol{u}^m - \boldsymbol{u}, \tag{5.23}$$

$$q^m = p^m - p. \tag{5.24}$$

Equations (5.18) and (5.7) give

$$((\boldsymbol{v}^{m+1} - \boldsymbol{v}^m, \boldsymbol{v})) + \rho\nu((\boldsymbol{v}^m, \boldsymbol{v})) = \rho(q^m, \operatorname{div} \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),$$

and taking $\boldsymbol{v} = \boldsymbol{v}^{m+1}$ we obtain

$$\begin{aligned} \|\boldsymbol{v}^{m+1}\|^2 - \|\boldsymbol{v}^m\|^2 + \|\boldsymbol{v}^{m+1} - \boldsymbol{v}^m\|^2 + 2\rho\nu\|\boldsymbol{v}^{m+1}\|^2 \\ = 2\rho\nu((\boldsymbol{v}^{m+1}, \boldsymbol{v}^{m+1} - \boldsymbol{v}^m)) + 2\rho(q^m, \operatorname{div} \boldsymbol{v}^{m+1}) \\ \leqslant \delta\|\boldsymbol{v}^{m+1} - \boldsymbol{v}^m\|^2 + \frac{\rho^2\nu^2}{\delta}\|\boldsymbol{v}^{m+1}\|^2 \\ + 2\rho(q^m, \operatorname{div} \boldsymbol{v}^{m+1}), \end{aligned} \tag{5.25}$$

where $\delta > 0$ is arbitrary at the present time.

Equation (5.19), with $q = q^{m+1}$ can be written as

$$\begin{aligned}\alpha|q^{m+1}|^2 - \alpha|q^m|^2 + \alpha|q^{m+1} - q^m|^2 &= -2\rho(q^{m+1}, \operatorname{div} \boldsymbol{u}^{m+1})\\ &= -2\rho(q^m, \operatorname{div} \boldsymbol{v}^{m+1}) - 2\rho(q^{m+1} - q^m, \operatorname{div} \boldsymbol{v}^{m+1})\\ &\leqslant -2\rho(q^m, \operatorname{div} \boldsymbol{v}^{m+1}) + 2\rho|q^{m+1} - q^m|\,|\operatorname{div} \boldsymbol{v}^{m+1}|\\ &\leqslant (\text{by } (5.13))\\ &\leqslant -2\rho(q^m, \operatorname{div} \boldsymbol{v}^{m+1}) + 2\rho|q^{m+1} - q^m|\,\|\boldsymbol{v}^{m+1}\|.\end{aligned}$$

Finally, with the same δ as before,

$$\begin{aligned}\alpha|q^{m+1}|^2 - \alpha|q^m|^2 + \alpha|q^{m+1} - q^m|^2 &\leqslant -2\rho(q^m, \operatorname{div} \boldsymbol{v}^{m+1})\\ &+ \alpha\delta|q^{m+1} - q^m|^2 + \frac{\rho^2}{\alpha\delta}\|\boldsymbol{v}^{m+1}\|^2.\end{aligned} \tag{5.26}$$

Adding inequalities (5.25) and (5.26), we get

$$\begin{aligned}&\alpha|q^{m+1}|^2 + \|\boldsymbol{v}^{m+1}\|^2 - \alpha|q^m|^2 - \|\boldsymbol{v}^m\|^2 + \alpha(1-\delta)|q^{m+1} - q^m|^2\\ &+ (1-\delta)\|\boldsymbol{v}^{m+1} - \boldsymbol{v}^m\|^2 + \rho\left(2\nu - \frac{\rho\nu^2}{\delta} - \frac{\rho}{\alpha\delta}\right)\|\boldsymbol{v}^{m+1}\|^2 \leqslant 0.\end{aligned} \tag{5.27}$$

If condition (5.22) holds, then

$$2\nu > \rho\nu^2 + \frac{\rho}{\alpha},$$

and for some $0 < \delta < 1$ sufficiently close to 1, we have again

$$2\nu > \frac{1}{\delta}\left(\rho\nu^2 + \frac{\rho}{\alpha}\right)$$

so that

$$\rho\left(2\nu - \frac{\rho\nu^2}{\delta} - \frac{\rho}{\alpha\delta}\right) > 0.$$

By adding inequalities (5.27) for $m = 0, \ldots, N$, we obtain an inequality of the same type as (5.15), and the proof is completed as for Theorem 5.1.

Remark 5.2. It is easy to extend the Remark 5.1 to this algorithm.

5.3. *Discrete Form of these Algorithms.*

We describe the discrete form of these algorithms in the case of finite differences (approximation APX1).

In order to actually compute the step functions $\boldsymbol{u}_h$ and π_h which are solutions of (3.64), (3.71) and (3.73), we define two sequences of step functions $\boldsymbol{u}_h^m$, π_h^m, of the type

$$\boldsymbol{u}_h^m = \sum_{M \in \mathring{\Omega}_h^1} \xi_M w_{hM}, \ \ \xi_M \in \mathcal{R}^n \ \text{(i.e., } \boldsymbol{u}_h^m \in W_h) \tag{5.29}$$

$$\pi_h^m = \sum_{M \in \mathring{\Omega}_h^1} \eta_M w_{hM}, \ \ \eta_M \in \mathcal{R}, \tag{5.30}$$

which are recursively defined by the analogue of one of the preceding algorithms.

Uzawa Algorithm.

We start with an aribtrary π_h^0 of type (5.30). When π_h^m is known, we define $\boldsymbol{u}_h^{m+1}$ and π_h^{m+1} by

$$\boldsymbol{u}_h^{m+1} \in W_h \text{ and}$$

$$\nu((\boldsymbol{u}_h^{m+1}, \boldsymbol{v}_h))_h - (\pi_h^m, D_h \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \ \ \forall \boldsymbol{v}_h \in W_h \tag{5.31}$$

$$\pi_h^{m+1}(M) = \pi_h^m(M) - \rho(D_h \boldsymbol{u}_h^{m+1})(M) \tag{5.32}$$

where D_h is the discrete divergence operator defined by (3.69).

If ρ satisfies the same condition (5.6), a repetition of the proof of Theorem 5.1 shows that, as $m \to \infty$

$$\boldsymbol{u}_h^m \to \boldsymbol{u}_h \text{ in } W_h, \tag{5.33}$$

$$\pi_h^m \to \pi_h \text{ up to a constant;} \tag{5.34}$$

the convergence holds for any norm on the finite dimensional spaces considered.

Arrow-Hurwicz Algorithm.

We start with arbitrary $\boldsymbol{u}_h^0$, π_h^0 of type (5.29) and (5.30) respectively.

When u_h^m, π_h^m are known, we define u_h^{m+1}, π_h^{m+1} by

$$u_h^{m+1} \in W_h \text{ and}$$

$$((u_h^{m+1} - u_h^m, v_h))_h + \rho\nu((u_h^m, v_h))_h \tag{5.35}$$

$$- \rho(\pi_h^m, D_h v_h) = (f, v_h), \ \forall \, v_h \in W_h.$$

$$\pi_h^{m+1}(M) = \pi_h^m(M) - \frac{\rho}{\alpha} D_h u_h^{m+1}(M), \ \forall \, M \in \mathring{\Omega}_h^1. \tag{5.36}$$

If ρ satisfies the condition (5.22), an extension of the proof of Theorem 5.2 gives the convergence (5.33) – (5.34).

Discrete Arrow-Hurwicz Algorithm.

The problems (5.31) and (5.35) are discrete Dirichlet problems and their solution is easy and quite standard. Nevertheless, it is interesting to notice that in the finite dimensional case, we can use another form of Arrow-Hurwicz algorithm, for which we do not have any boundary value problem to solve during the iteration process.

When u_h^m, π_h^m are known, we define u_h^{m+1} by

$$u_h^{m+1} \in W_h \text{ and,}$$

$$(u_h^{m+1} - u_h^m, v_h) + \rho\nu((u_h^m, v_h))_h - \rho(\pi_h^m, D_h v_h) \tag{5.37}$$

$$= \rho(f, v_h), \ \forall \, v_h \in W_h,$$

and then π_h^{m+1} is again defined by (5.36). The variational equation (5.37) is equivalent to the following equations.

$$u_h^{m+1}(M) = u_h^m(M) + \rho\nu \sum_{i=1}^{n} (\delta_{ih}^2 u_h^m)(M)$$

$$- \rho(\overline{\nabla}_h \pi_h^m)(M) + \rho f_h(M), \quad \forall \, M \in \mathring{\Omega}_h^1, \tag{5.38}$$

where $\overline{\nabla}_h$ and f_h were defined in (3.74).

The proof of Theorem 5.2 can be extended to this situation as follows. Since W_h is a finite dimensional space, all the norms defined on W_h are equivalent, and hence there exists some constant $S(h)$ depending on h, such that

$$\|u_h\|_h \leqslant S(h)|u_h|, \ \forall \, u_h \in W_h. \tag{5.39}$$

We will compute $S(h)$, and use this remark extensively in Chapter III ($S(h) = 2(\Sigma_{i=1}^{n} 1/h_i^2)^{1/2}$).

Now, if ρ satisfies :

$$0 < \rho < \frac{2\alpha\nu}{\alpha\nu^2 S^2(h) + 1}, \tag{5.40}$$

then the convergences (5.33) – (5.34) are also true for the algorithm (5.36) – (5.37).

The proof is the same as for Theorem 5.2. The inequality (5.25) is just replaced by ($v_h^m = u_h^m - u_h$, $\kappa_h^m = \pi_h^m - \pi_h$) :

$$\begin{aligned}
|v_h^{m+1}|^2 - |v_h^m|^2 + |v_h^{m+1} - v_h^m| + 2\rho\nu \|v_h^{m+1}\|_h^2 \\
&= 2\rho\nu((v_h^{m+1}, v_h^{m+1} - v_h^m))_h + 2\rho(\kappa_h^m, D_h v_h^{m+1}) \\
&\leqslant 2\rho\nu\|v_h^{m+1}\|_h \|v_h^{m+1} - v_h^m\|_h + 2\rho(\kappa_h^m, D_h v_h^{m+1}) \\
&\leqslant 2\rho\nu S(h)\|v_h^{m+1}\|_h |v_h^{m+1} - v_h^m| + 2\rho(\kappa_h^m, D_h v_h^{m+1}) \\
&\leqslant \delta|v_h^{m+1} - v_h^m|^2 + \frac{\rho^2\nu^2 S^2(h)}{\delta}\|v_h^{m+1}\|_h^2 \\
&\qquad + 2\rho(\kappa_h^m, D_h v_h^{m+1}).
\end{aligned} \tag{5.41}$$

The inequality (5.27) is accordingly changed to

$$\begin{aligned}
&\alpha|\kappa_h^{m+1}|^2 + |v_h^{m+1}|^2 - \alpha|\kappa_h^m|^2 - |v_h^m|^2 \\
&\quad + \alpha(1-\delta)|\kappa_h^{m+1} - \kappa_h^m|^2 + (1-\delta)|v_h^{m+1} - v_h^m|^2 \\
&\quad + \rho\Big(2\nu - \frac{\rho\nu^2 S^2(h)}{\delta} - \frac{\rho}{\alpha\delta}\Big)\|v_h^{m+1}\|_h^2 \leqslant 0,
\end{aligned} \tag{5.42}$$

and because of (5.40), inequality (5.42) leads to the same conclusion as (5.27).

§6. Slightly Compressible Fluids

The stationary linearized equations of slightly compressible fluids are

$$-\nu\Delta \boldsymbol{u}_\epsilon - \frac{1}{\epsilon}\,\mathrm{grad\ div}\ \boldsymbol{u}_\epsilon = f \quad \text{in } \Omega, \tag{6.1}$$

$$\boldsymbol{u}_\epsilon = 0 \quad \text{on } \partial\Omega, \tag{6.2}$$

where $\epsilon > 0$ is "small". Equations (6.1) – (6.2) are also the stationary Lamé equations of elasticity.

We will show that equations (6.1) – (6.2) have a unique solution $\boldsymbol{u}_\epsilon$ for $\epsilon > 0$ fixed, and that $\boldsymbol{u}_\epsilon$ converges to the solution $\boldsymbol{u}$ of the Stokes equations as $\epsilon \to 0$.

At first, equations (6.1) – (6.2) were used as "approximate" equations for Stokes equations – one way to overcome the difficulty "div $\boldsymbol{u} = 0$", was to solve equations (6.1) – (6.2) with ϵ sufficiently small in place of solving the Stokes equations. Nowadays, since many efficient algorithms are known for solving the Stokes equations, and since the discretization of (6.1) – (6.2) leads to a very ill-conditioned matrix for very small ϵ, one can try to do the converse: compute $\boldsymbol{u}_\epsilon$ for small ϵ by using Stokes equations.

In Section 6.1 we show the relation between $\boldsymbol{u}_\epsilon$ and $\boldsymbol{u}$ and in Section 6.2 we give an asymptotic development of $\boldsymbol{u}_\epsilon$ as $\epsilon \to 0$. Then in Section 6.3 we show how one can proceed to compute $\boldsymbol{u}_\epsilon$, for small ϵ, using this asymptotic expansion.

6.1. *Convergence of $\boldsymbol{u}_\epsilon$ to $\boldsymbol{u}$.*

Theorem 6.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.*

For $\epsilon > 0$ fixed, there exists a unique $\boldsymbol{u}_\epsilon \in \boldsymbol{H}_0^1(\Omega)$ which satisfies (6.1).

When $\epsilon \to 0$,

$$\boldsymbol{u}_\epsilon \to \boldsymbol{u} \text{ in the norm of } \boldsymbol{H}_0^1(\Omega), \tag{6.3}$$

$$-\frac{\mathrm{div}\ \boldsymbol{u}_\epsilon}{\epsilon} \to p \text{ in the norm of } L^2(\Omega), \tag{6.4}$$

where $\boldsymbol{u}$ and p are defined by (2.6)–(2.9) *and moreover*

$$\int_\Omega p(x)\,\mathrm{d}x = 0. \tag{6.5}$$

Proof. It is easy to show that the problem (6.1)–(6.2) is equivalent to the following variational problem:

To find $u_\epsilon \in H_0^1(\Omega)$ *such that*

$$\nu((u_\epsilon, v)) + \frac{1}{\epsilon}(\operatorname{div} u_\epsilon, \operatorname{div} v) = (f, v), \ \forall\, v \in H_0^1(\Omega). \tag{6.6}$$

Actually, if u_ϵ satisfies (6.1)–(6.2) then $u_\epsilon \in H_0^1(\Omega)$ and satisfies (6.6) for each $v \in \mathscr{D}(\Omega)$. By a continuity argument, it satisfies (6.6) too, for each $v \in H_0^1(\Omega)$. Conversely, if $u_\epsilon \in H_0^1(\Omega)$ is a solution of (6.6) then u_ϵ satisfies (6.1) in the distribution sense and (6.2) in the sense of trace theorems.

The existence and uniqueness of u_ϵ satisfying (6.6) results from the projection theorem: we apply Theorem 2.2 with

$$W = H_0^1(\Omega),\ a(u, v) = \nu((u, v)) + \frac{1}{\epsilon}(\operatorname{div} u, \operatorname{div} v),$$

$$\langle \ell, v \rangle = (f, v).$$

The coercivity of a and the continuity of a and ℓ are obvious.

To prove (6.3) let us subtract (2.7) from (6.1); this gives

$$-\nu \Delta (u_\epsilon - u) - \frac{1}{\epsilon} \operatorname{grad} \operatorname{div} u_\epsilon = + \operatorname{grad} p \tag{6.7}$$

and thus

$$\nu((u_\epsilon - u, v)) + \frac{1}{\epsilon}(\operatorname{div} u_\epsilon, \operatorname{div} v) = -(p, \operatorname{div} v), \forall\, v \in H_0^1(\Omega). \tag{6.8}$$

Equation (6.8) follows easily from (6.7) for $v \in \mathscr{D}(\Omega)$; by a continuity argument, (6.8) is satisfied for each $v \in H_0^1(\Omega)$.

Let us put $v = u_\epsilon - u$ in (6.8); we obtain

$$\begin{aligned} \nu\|u_\epsilon - u\|^2 &+ \frac{1}{\epsilon}|\operatorname{div} u_\epsilon|^2 \\ &= -(p, \operatorname{div} u_\epsilon) \leqslant |p|\,|\operatorname{div} u_\epsilon| \\ &\leqslant \frac{1}{2\epsilon}|\operatorname{div} u_\epsilon|^2 + \frac{\epsilon}{2}|p|^2 \end{aligned}$$

so that

$$\nu\|u_\epsilon - u\|^2 + \frac{1}{2\epsilon}|\operatorname{div} u_\epsilon|^2 \leqslant \frac{\epsilon}{2}|p|^2. \tag{6.9}$$

This proves (6.3). Consequently, (6.7) shows that

$$\frac{\partial}{\partial x_i}\left(\frac{\operatorname{div} \boldsymbol{u}_\epsilon}{\epsilon}\right) \to -\frac{\partial p}{\partial x_i}, \tag{6.10}$$

in the norm of $H^{-1}(\Omega)$, $i = 1, \ldots, n$ ($\Delta(\boldsymbol{u}_\epsilon - \boldsymbol{u})$ converges to 0 in $\boldsymbol{H}^{-1}(\Omega)$, because of (6.3)).

According to the following lemma,

$$|p + \frac{\operatorname{div} \boldsymbol{u}_\epsilon}{\epsilon}| \leqslant \sum_{i=1}^{n} \| \frac{\partial}{\partial x_i} (p + \frac{\operatorname{div} \boldsymbol{u}_\epsilon}{\epsilon}) \|_{H^{-1}(\Omega)}, \tag{6.11}$$

since, because $\boldsymbol{u}_\epsilon$ vanishes on $\partial\Omega$ and (6.5) holds,

$$\int_\Omega (p + \frac{\operatorname{div} \boldsymbol{u}_\epsilon}{\epsilon})\, \mathrm{d}x = 0.$$

The convergence (6.4) is proved.

Lemma 6.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists a constant $c = c(\Omega)$ depending only on Ω, such that*

$$|\sigma|_{L^2(\Omega)} \leqslant c(\Omega) \left\{ \left| \int_\Omega \sigma\, \mathrm{d}x \right| + \sum_{i=1}^{n} \left\| \frac{\partial \sigma}{\partial x_i} \right\|_{H^{-1}(\Omega)} \right\}, \tag{6.12}$$

for every σ in $L^2(\Omega)$.

Proof. Let us denote by $[\sigma]$ the expression between the brackets on the right-hand side of (6.12); $[\sigma]$ is a norm on $L^2(\Omega)$; it is obviously a semi-norm and, if $[\sigma] = 0$, then σ is a constant since $\partial\sigma/\partial x_i = 0$, $i = 1, \ldots, n$, and this constant is zero since $\int_\Omega \sigma\, \mathrm{d}x = 0$.

It is clear that there exists a constant $c' = c'(\Omega)$ such that

$$[\sigma] \leqslant c'(\Omega)|\sigma|_{L^2(\Omega)}, \quad \forall \sigma \in L^2(\Omega). \tag{6.13}$$

If we show that $L^2(\Omega)$ is complete for the norm $[\sigma]$ then, by the closed graph theorem, $[\sigma]$ and $|\sigma|$ will be two equivalent norms on $L^2(\Omega)$ and (6.12) will be proved.

In order to show that $L^2(\Omega)$ is complete for the norm $[\sigma]$, let us consider a sequence σ_m, which is a Cauchy sequence for this norm. Then the integrals $\int_\Omega \sigma_m\, \mathrm{d}x$ form a Cauchy sequence in R and the derivatives $\partial\sigma_m/\partial x_i$ are Cauchy sequences in $H^{-1}(\Omega)$:

$$\int_\Omega \sigma_m \, dx \to \lambda \text{ as } m\to\infty, \tag{6.14}$$

$$\frac{\partial \sigma_m}{\partial x_i} \to \chi_i \text{ as } m\to\infty, \text{ in } H^{-1}(\Omega), \ 1 \leqslant i \leqslant n. \tag{6.15}$$

It is clear that $\langle \operatorname{grad} \sigma_m, v\rangle = 0, \quad \forall\, v \in \mathscr{V}$ and, because of (6.15),

$$\sum_{i=1}^{m} \langle \chi_i, v_i \rangle = 0, \quad \forall\, v \in \mathscr{V}.$$

By Proposition 1.1, there exists some distribution σ such that

$$\chi_i = \frac{\partial \sigma}{\partial x_i}, \quad 1 \leqslant i \leqslant n.$$

Proposition 1.2 shows that $\sigma \in L^2(\Omega)$. We can choose σ so that

$$\int_\Omega \sigma \, dx = \lambda$$

and it is easy to see that the sequence σ_m converges to this element σ of $L^2(\Omega)$ in the norm $[\sigma]$.

Remark 6.1. If Ω is not connected, (6.12) is true if we replace $|\int_\Omega \sigma \, dx|$ by

$$\sum_j \left| \int_{\Omega_j} \sigma \, dx \right|$$

where the Ω_j are the connected components of Ω. For extending Theorem 6.1 to this case we just have to define p by imposing:

$$\int_{\Omega_j} p \, dx = 0, \quad \forall\, \Omega_j. \tag{6.16}$$

6.2. *Asymptotic Expansion of* $\boldsymbol{u}_\epsilon$.

From now on we denote by $\boldsymbol{u}^0$ and p^0 the solution of Stokes problem (2.6)–(2.9) which satisfies (6.5) (in place of $\boldsymbol{u}$ and p)

We will show that $\boldsymbol{u}_\epsilon$ has an asymptotic development

$$\boldsymbol{u}_\epsilon = \boldsymbol{u}^0 + \epsilon \boldsymbol{u} + \epsilon^2 \boldsymbol{u}^2 + \cdots + \epsilon^N \boldsymbol{u}^N + \cdots . \tag{6.17}$$

where all the $\boldsymbol{u}^i$ belong to the space $\boldsymbol{H}_0^1(\Omega)$.

The functions $\boldsymbol{u}^i$ and some auxiliary functions p^i are recursively defined as follows:

$$\boldsymbol{u}^0, p^0, \text{ are already known;} \tag{6.18}$$

when $\boldsymbol{u}^{m-1}$, p^{m-1} are known ($m \geqslant 1$), we define $\boldsymbol{u}^m$ and p^m as the solutions of the nonhomogeneous Stokes problem

$$\boldsymbol{u}^m \in \boldsymbol{H}_0^1(\Omega),\ p^m \in L^2(\Omega), \tag{6.19}$$

$$-\nu \Delta \boldsymbol{u}^m + \operatorname{grad} p^m = 0 \tag{6.20}$$

$$\operatorname{div} \boldsymbol{u}^m = -p^{m-1} \tag{6.21}$$

$$\int_\Omega p^m(x)\, \mathrm{d}x = 0. \tag{6.22}$$

The existence and uniqueness of $\boldsymbol{u}^m$ and p^m follow from Theorem 2.4. The condition (6.22) is useful in two ways: it ensures the complete uniqueness of p^m which is otherwise only unique up to the addition of a constant; it also ensures the compatibility condition necessary for the level $m + 1$:

$$\int_\Omega \operatorname{div} \boldsymbol{u}^{m+1}\, \mathrm{d}x = \int_\Gamma \boldsymbol{u}^{m+1} \cdot \nu\, \mathrm{d}\Gamma = 0 = \int_\Omega p^m\, \mathrm{d}x. \tag{6.23}$$

We denote by $\boldsymbol{u}_\epsilon^N$, p_ϵ^N, $N \geqslant 1$, the quantities

$$\boldsymbol{u}_\epsilon^N = \sum_{m=0}^{N} \epsilon^m \boldsymbol{u}^m, \tag{6.24}$$

$$p_\epsilon^N = \sum_{m=0}^{N} \epsilon^m p^m. \tag{6.25}$$

Theorem 6.2. *Let Ω be a bounded domain of class $\mathscr{C}^2$ in $\mathbb{R}^n$.*

Then for each $m \geqslant 1$, there exist functions $\boldsymbol{u}^m$, p^m, uniquely defined by (6.19)–(6.22).

For each $N \geqslant 0$, *as* $\epsilon \to 0$,

$$\frac{u_\epsilon - u_\epsilon^N}{\epsilon^N} \to 0 \text{ in the } H_0^1(\Omega) \text{ norm,} \tag{6.26}$$

$$\frac{1}{\epsilon^N}\left(-\frac{\operatorname{div} u_\epsilon}{\epsilon} - p_\epsilon^N\right) \to 0 \text{ in the } L^2(\Omega) \text{ norm.} \tag{6.27}$$

Proof. The existence and uniqueness results have been established as previously remarked.

We multiply (6.20) by ϵ^m and add these equations for $m = 1, \ldots, N$. We then add the resulting equation to the equation satisfied by u^0 and p^0 (formerly denoted u and p)

$$-\nu\Delta u^0 + \operatorname{grad} p^0 = f.$$

After expanding we obtain

$$-\nu\Delta u_\epsilon^N - \frac{1}{\epsilon} \operatorname{grad} \operatorname{div} u_\epsilon^N = f - \epsilon^N \operatorname{grad} p^N.$$

By comparison with (6.3) we find

$$u_\epsilon - u_\epsilon^N \in H_0^1(\Omega), \; p^N \in L^2(\Omega), \tag{6.28}$$

$$-\nu\Delta(u_\epsilon - u_\epsilon^N) - \frac{1}{\epsilon} \operatorname{grad} \operatorname{div} (u_\epsilon - u_\epsilon^N) = + \epsilon^N \operatorname{grad} p^N. \tag{6.29}$$

As for (6.8), we show that (6.29) is equivalent to

$$\begin{aligned} &\nu((u_\epsilon - u_\epsilon^N, v)) + \frac{1}{\epsilon} (\operatorname{div} (u_\epsilon - u_\epsilon^N), \operatorname{div} v) = \\ &\qquad -\epsilon^N (p^N, \operatorname{div} v), \; \forall\, v \in H_0^1(\Omega). \end{aligned} \tag{6.30}$$

Putting $v = u_\epsilon - u_\epsilon^N$ in (6.30) we get

$$\begin{aligned} &\nu\|u_\epsilon - u_\epsilon^N\|^2 + \frac{1}{\epsilon} |\operatorname{div}(u_\epsilon - u_\epsilon^N)|^2 \\ &\qquad = -\epsilon^N (p^N, \operatorname{div}(u_\epsilon - u_\epsilon^N)) \\ &\qquad \leqslant \epsilon^N |p^N| \, |\operatorname{div}(u_\epsilon - u_\epsilon^N)| \\ &\qquad \leqslant \frac{1}{2\epsilon} |\operatorname{div}(u_\epsilon - u_\epsilon^N)|^2 + \frac{\epsilon^{2N+1}}{2} |p^N|^2 \end{aligned}$$

so that

$$\nu \|u_\epsilon - u_\epsilon^N\|^2 + \frac{1}{2\epsilon} |\operatorname{div}(u_\epsilon - u_\epsilon^N)|^2 \leqslant \frac{\epsilon^{2N+1}}{2} |p^N|^2 . \tag{6.31}$$

The inequality (6.31) clearly implies (6.26). This, in turn, implies that $1/\epsilon^N \Delta(u_\epsilon - u_\epsilon^N) \to 0$ in $H^{-1}(\Omega)$ and hence (6.29) shows that

$$\frac{1}{\epsilon^{N+1}} \frac{\partial}{\partial x_i} \operatorname{div}(u_\epsilon - u_\epsilon^N) \to \frac{\partial p^N}{\partial x_i} \text{ in } H^{-1}(\Omega), \ 1 \leqslant i \leqslant n. \tag{6.32}$$

But

$$-\frac{1}{\epsilon} \operatorname{grad} \operatorname{div} u_\epsilon^N = -\frac{1}{\epsilon} \sum_{m=1}^{N} \epsilon^m \operatorname{grad} \operatorname{div} u^m$$

$$= \frac{1}{\epsilon} \sum_{m=1}^{N} \epsilon^m \operatorname{grad} p^{m-1} = \operatorname{grad}(p_\epsilon^N - \epsilon^N p^N),$$

which along with (6.32) implies that

$$\frac{1}{\epsilon^N} \frac{\partial}{\partial x_i} \left(-\frac{\operatorname{div} u_\epsilon}{\epsilon} - p_\epsilon^N \right) \to 0 \text{ in } H^{-1}(\Omega)$$

as $\epsilon \to 0$. Finally (6.27) results from (6.22), (6.25) and Lemma 6.1.

Remark 6.2. Remark 6.1 can be easily adapted to Theorem 6.2: this theorem holds for non-connected sets Ω, provided we replace condition (6.22) by the conditions:

$$\int_{\Omega_j} p^m \, dx = 0 \tag{6.33}$$

on each connected component Ω_j of Ω.

Remark 6.3. A discrete version of the result given by Theorem 6.2 is studied by R. S. Falk [1].

6.3. *Numerical Algorithms*

Let us show briefly how one can extend the algorithms described in Section 5 to the solution of the nonhomogeneous Stokes problems (6.19) – (6.22) which is, at this point, the only difficulty for practical computation of the asymptotic expansion (6.17) of u_ϵ.

We only describe the adaptation of the Uzawa algorithm.

Changing our notation, we write problem (6.19) – (6.22) as the problem: to find $\boldsymbol{v}$, p such that

$$\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),\ p \in L^2(\Omega) \tag{6.34}$$

$$-\nu\Delta\boldsymbol{v} + \operatorname{grad} p = 0, \tag{6.35}$$

$$\operatorname{div} \boldsymbol{v} = \phi, \tag{6.36}$$

$$\int_\Omega p(x)\,\mathrm{d}x = 0, \tag{6.37}$$

where ϕ is given with

$$\int_\Omega \phi(x)\,\mathrm{d}x = 0. \tag{6.38}$$

The existence and uniqueness of $\boldsymbol{v}$ and p are known.

We start the algorithm with any

$$p^0 \in L^2(\Omega), \text{ such that } \int_\Omega p^0(x)\,\mathrm{d}x = 0. \tag{6.39}$$

When p^m is known, we define $\boldsymbol{v}^{m+1}$ $(m \geqslant 0)$ by

$$\boldsymbol{v}^{m+1} \in \boldsymbol{H}_0^1(\Omega) \text{ and}$$

$$\nu((\boldsymbol{v}^{m+1}, \boldsymbol{w})) - (p^m, \operatorname{div} \boldsymbol{w}) = 0,\ \forall\, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \tag{6.40}$$

$$p^{m+1} \in L^2(\Omega) \text{ and}$$

$$(p^{m+1} - p^m, \theta) + \rho(\operatorname{div} \boldsymbol{v}^{m+1} - \phi, \theta) = 0, \qquad \theta \in L^2(\Omega). \tag{6.41}$$

The equation (6.40) is a Dirichlet problem for $\boldsymbol{v}^{m+1}$:

$$\boldsymbol{v}^{m+1} \in \boldsymbol{H}_0^1(\Omega)$$

$$-\nu\Delta\boldsymbol{v}^{m+1} = -\operatorname{grad} p^m \in H^{-1}(\Omega) \tag{6.42}$$

and (6.41) gives p^{m+1} directly as

$$p^{m+1} = p^m - \rho(\operatorname{div} \boldsymbol{v}^{m+1} - \phi) \in L^2(\Omega). \tag{6.43}$$

We notice that

$$\int_\Omega p^m \, \mathrm{d}x = 0, \ \forall \ m \geqslant 0. \tag{6.44}$$

Exactly as for Theorem 5.1 (see also Remark 5.1), one can prove the following result.

Theorem 6.3. *If the number ρ satisfies*

$$0 < \rho < 2\nu \tag{6.45}$$

then, as $m \to \infty$, v^{m+1} converges to v in the norm of $H_0^1(\Omega)$ and p^{m+1} converges to p weakly in $L^2(\Omega)$.

CHAPTER II

STEADY-STATE NAVIER–STOKES EQUATIONS

Introduction

In this chapter we will be concerned with the steady-state Navier-Stokes equations from the same point of view as in the previous chapter, i.e., existence, uniqueness and numerical approximation of the solution. However, there are three important differences with the linear case; these are:

– The introduction of the *compactness methods.* For passing to the limit in the nonlinear term we need strong convergence results; these are obtained by compactness arguments.

– Some technical difficulties related to the nonlinear term and connected with the Sobolev inequalities. Their consequence is a treatment of the equation which varies slightly according to the dimension of the space.

– *The non-uniqueness of solutions,* in general. Uniqueness occurs only when "the data are small enough, or the viscosity is large enough".

In Section 1 we describe some existence, uniqueness, and regularity results in various situations (Ω bounded or not, homogeneous or non-homogeneous equations, ...). In Section 2 we prove a discrete Sobolev inequality and a discrete compactness theorem for step function spaces considered in the approximation (APX1) of V (approximation of V by finite differences) and for nonconforming element function appearing in the approximation (APX5) of V. The similar results for the approximations (APX2), ... , (APX4), are already available as consequences of the theorem in the continuous case. Section 3 deals with the approximation of the stationary problem: discretization and resolution of the discretized problems. The purpose of Section 4 is to show an example of non-uniqueness of solutions for the steady-state Navier-Stokes equations. The proof is based on a topological degree argument; the presentation is essentially self-contained.

§1. Existence and Uniqueness Theorems.

In this section we study some existence and uniqueness results for the steady-state (nonlinear) Navier-Stokes equations. The existence results are obtained by constructing approximate solutions to the equation by the Galerkin Method, and then passing to the limit, as in the linear case. As we have already said, for passing to the limit we need, in the nonlinear case, some strong convergence properties of the sequence, and these are obtained by compactness methods.

In Section 1.1 we recall the Sobolev inequalities and a compactness theorem for the Sobolev spaces; this theorem is of course the basic tool for the compactness method. In Section 1.2 we give a variational formulation of the homogeneous Navier-Stokes equations (i.e., the Navier-Stokes equations with homogeneous boundary conditions); we study some properties of a nonlinear (trilinear) form which occurs in the variational formulation. We then give a general existence theorem and a rather restricted uniqueness result. In Section 1.3 we consider the case where the set Ω is unbounded and we give regularity results for solutions. Section 1.4 deals with inhomogeneous Navier-Stokes equations.

1.1. *Sobolev Inequalities and Compactness Theorems.*

Imbedding Theorems.

We recall the Sobolev imbedding theorems which will be used frequently from now on. Let m be an integer and p any finite number greater than or equal to one, $p \geqslant 1$; then, if $1/p - m/n = 1/q > 0$ the space $W^{m,p}(\mathcal{R}^n)$ is included in $L^q(\mathcal{R}^n)$ and the injection is continuous. If $u \in W^{m,p}(\mathcal{R}^n)$ and $1/p - m/n = 0$ then u belongs to $L^q(\mathcal{O})$ for any bounded set $\mathcal{O}$ and any q, $1 \leqslant q < \infty$. If $1/p - m/n < 0$ then a function in $W^{m,p}(\mathcal{R}^n)$ is almost everywhere equal to a continuous function; such a function has also some Hölder or Lipschitz continuity properties but these properties will not be used here; if a function belongs to $W^{m,p}(R^n)$ with $1/p - m/n < 0$ then the derivatives of order α belong to $W^{m-\alpha,p}(\mathcal{R}^n)$ and some embedding results of preceding type hold for these derivatives if $1/p - (m - \alpha)/n > 0$.

For $u \in W^{m,p}(\mathcal{R}^n)$, $m \geqslant 1$, $1 \leqslant p < \infty$

$$
\left.\begin{aligned}
&\text{if } \frac{1}{p} - \frac{m}{n} = \frac{1}{q} > 0,\ |u|_{L^q(\mathcal{R}^n)} \leqslant c(m, p, n)\|u\|_{W^{m,p}(\mathcal{R}^n)},\\
&\text{if } \frac{1}{p} - \frac{m}{n} = 0,\ |u|_{L^q(\mathcal{O})} \leqslant c(m, p, n, q, \mathcal{O})\ \|u\|_{W^{m,p}(\mathcal{R}^n)}\\
&\qquad \forall \text{ bounded set } \mathcal{O} \subset \mathcal{R}^n,\ \forall\, q,\ 1 \leqslant q < \infty,
\end{aligned}\right\} \tag{1.1}
$$

$$\text{if } \frac{1}{p} - \frac{m}{n} < 0, \ |u|_{\mathscr{C}^0(\mathcal{O})} \leqslant c(m, n, p, \mathcal{O}) \|u\|_{W^{m,p}(\mathbb{R}^n)},$$

$$\forall \text{ bounded set } \mathcal{O}, \ \mathcal{O} \subset \mathbb{R}^n.$$

If Ω is any open set of $\mathbb{R}^n$, results similar to (1.1) can usually be obtained if Ω is sufficiently smooth so that:

There exists a continuous linear prolongation operator

$$\Pi \in \mathcal{L}(W^{m,p}(\Omega), W^{m,p}(\mathbb{R}^n)). \tag{1.2}$$

Property (1.2) is satisfied by a locally Lipschitz set Ω. When (1.2) is satisfied, the properties (1.1) applied to Πu, $u \in W^{m,p}(\Omega)$ give in particular, assuming that $u \in W^{m,p}(\Omega)$, $m \geqslant 1$, $1 < p < \infty$, and (1.2) holds:

$$\text{if } \frac{1}{p} - \frac{m}{n} = \frac{1}{q} > 0, \ |u|_{L^q(\Omega)} \leqslant c(m, p, n, \Omega) \|u\|_{W^{m,p}(\Omega)},$$

$$\text{if } \frac{1}{p} - \frac{m}{n} = 0, \ |u|_{L^q(\mathcal{O})} \leqslant c(m, p, n, q, \mathcal{O}, \Omega) \|u\|_{W^{m,p}(\Omega)},$$

$$\text{any } q, \ 1 \leqslant q < \infty, \text{ any bounded set } \mathcal{O} \subset \bar{\Omega},$$

$$\text{if } \frac{1}{p} - \frac{m}{n} < 0, \ |u|_{\mathscr{C}^0(\mathcal{O})} \leqslant c(m, p, n, q, \Omega, \mathcal{O}) \|u\|_{W^{m,p}(\Omega)}, \tag{1.3}$$

$$\text{any bounded set } \mathcal{O}, \ \mathcal{O} \subset \bar{\Omega}.$$

When $u \in \overset{\circ}{W}{}^{m,p}(\Omega)$, the function $\tilde{u}$ which is equal to u in Ω and to 0 in $\complement\Omega$, belongs to $W^{m,p}(\mathbb{R}^n)$, and hence the properties (1.3) are valid without any hypothesis on Ω.

The case of particular interest for us is the case $p = 2$, $m = 1$, i.e., the case $H_0^1(\Omega)$. Without any regularity property required for Ω we have for $u \in H_0^1(\Omega)$

$$n = 2, \ |u|_{L^q(\Omega)} \leqslant c(q, \mathcal{O}, \Omega) \|u\|_{H_0^1(\Omega)}$$

$$\forall \text{ bounded set } \mathcal{O} \subset \Omega, \ \forall \ q, \ 1 \leqslant q < \infty$$

$$n = 3, \ |u|_{L^6(\Omega)} \leqslant c(\Omega) \|u\|_{H_0^1(\Omega)}$$

$$n = 4, \ |u|_{L^4(\Omega)} \leqslant c(\Omega) \|u\|_{H_0^1(\Omega)}$$

$$n \geqslant 3, \ |u| \ L^{2n/(n-2)}(\Omega) \leqslant c(\Omega) \|u\|_{H_0^1(\Omega)}. \tag{1.4}$$

Compactness Theorems.

Theorem 1.1. *Let Ω be any bounded open set of $\mathbb{R}^n$ satisfying* (1.2). *Then the embedding*

$$W^{1,p}(\Omega) \subset L^{q_1}(\Omega) \tag{1.5}$$

is compact for any q_1, $1 \leqslant q_1 < \infty$, if $p \geqslant n$, and for any q_1, $1 \leqslant q_1 < q$ (q given by $1/p - 1/n = 1/q$) if $1 \leqslant p < n$.

With the same values of p and q_1, the embedding

$$\mathring{W}^{1,p}(\Omega) \subset L^{q_1}(\Omega) \tag{1.6}$$

is compact for any bounded open set Ω.

As a particular case of Theorem 1.1 we notice that for any unbounded set Ω, if $u \in W^{1,p}(\Omega)$, then the restriction of u to $\mathcal{O}$, $\mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$, $\mathcal{O}$ bounded, belongs to $L^{q_1}(\mathcal{O})$ and this restriction mapping is compact

$$W^{1,p}(\Omega) \to L^{q_1}(\Omega) \tag{1.7}$$

(same values of p and q_1).

For all the preceding properties of Sobolev spaces, the reader is referred to the references mentioned in the first section of Chapter I (see also at the end the comments on Chapter I).

1.2. *The Homogeneous Navier-Stokes Equations.*

Let Ω be a Lipschitz, bounded open set in $\mathbb{R}^n$ with boundary Γ, let $f \in \boldsymbol{L}^2(\Omega)$ be a given vector function. We are looking for a vector function $\boldsymbol{u} = (u_1, \ldots, u_n)$ and a scalar function p, representing the velocity and the pressure of the fluid, which are defined in Ω and satisfy the following equations and boundary conditions:

$$-\nu\Delta\boldsymbol{u} + \sum_{i=1}^{n} u_i D_i \boldsymbol{u} + \operatorname{grad} p = f \quad \text{in } \Omega \tag{1.8}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega \tag{1.9}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma. \tag{1.10}$$

Exactly as in Section 2.1 of Chapter I, if $\boldsymbol{f}$, $\boldsymbol{u}$, p are smooth functions satisfying (1.8) - (1.10) then $\boldsymbol{u} \in V$ and, for each $\boldsymbol{v} \in \mathscr{V}$,

$$\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \tag{1.11}$$

where

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i,j=1}^{n} \int_{\Omega} u_i (D_i v_j) w_j \, dx. \tag{1.12}$$

A continuity argument shows moreover that equation (1.11) is satisfied by any $\boldsymbol{v} \in V$. Conversely, if $\boldsymbol{u}$ is a smooth function in V such that (1.11) holds for each $\boldsymbol{v} \in \mathscr{V}$, then because of Proposition 1.3, Chapter I, there exists a distribution p such that (1.8) is satisfied, and (1.9) - (1.10) are satisfied since $\boldsymbol{u} \in V$.

For $\boldsymbol{u}$ and $\boldsymbol{v}$ in V, the expression $b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})$ does not necessarily make sense and then the variational formulation of (1.8) - (1.10) is not exactly "to find $\boldsymbol{u} \in V$ such that (1.11) holds for each $\boldsymbol{v} \in V$." The variational formulation will be slightly different, and this will be stated after studying some properties of the form $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$.

Let us introduce first the following spaces:

$$\widetilde{V} = \text{the closure of } \mathscr{V} \text{ in } \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^n(\Omega); \tag{1.13}$$

of course $\boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^n(\Omega)$ and $\widetilde{V}$ are equipped with the norm

$$\|\boldsymbol{u}\|_{\boldsymbol{H}_0^1(\Omega)} + |\boldsymbol{u}|_{\boldsymbol{L}^n(\Omega)}. \tag{1.14}$$

In general $\widetilde{V}$ is a subspace of V, different from V but, because of (1.4), $\widetilde{V} = \mathrm{V}$ for $n = 2, 3$ or 4 (and Ω bounded);

$$V_s = \text{the closure of } \mathscr{V} \text{ in } \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^s(\Omega), \ (s \geqslant 1); \tag{1.15}$$

it is understood again that $\boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^s(\Omega)$ and V_s are equipped with the Hilbertian norm

$$\{\|\boldsymbol{u}\|^2_{\boldsymbol{H}_0^1(\Omega)} + \|\boldsymbol{u}\|^2_{\boldsymbol{H}^s(\Omega)}\}^{1/2}; \tag{1.16}$$

V_s is included in V.

The Trilinear Form b.

The form b is trilinear and continuous on various spaces among the spaces V, $\widetilde{V}$, V_s. The most convenient result concerning b is the following result which is independent of any property of Ω.

Lemma 1.1. *The form b is defined and trilinear continuous on* $\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times (\boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^n(\Omega))$, *$\Omega$ bounded or unbounded, any* *dimension of space* $\mathbb{R}^n$.

Proof. If $\boldsymbol{u}, \boldsymbol{v} \in V$ and $\boldsymbol{w} \in \widetilde{V}$, then because of (1.4) ($n \geqslant 3$):

$$u_i \in L^{2n/(n-2)}(\Omega),\ D_i v_j \in L^2(\Omega),\ w_j \in L^n(\Omega),\ 1 \leqslant i, j \leqslant n.$$

By the Hölder inequality, $u_i(D_i v_j) w_j$ belongs to $L^1(\Omega)$ and

$$\left| \int_\Omega u_i D_i v_j w_j \,\mathrm{d}x \right| \leqslant |u_i|_{L^{2n/(n-2)}(\Omega)} \, |D_i v_j|_{L^2(\Omega)} |w_j|_{L^n(\Omega)}. \tag{1.17}$$

Then $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is well defined and

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant c(n) \|\boldsymbol{u}\|_{H_0^1(\Omega)} \|\boldsymbol{v}\|_{H_0^1(\Omega)} \|\boldsymbol{w}\|_{H_0^1(\Omega) \cap L^n(\Omega)}. \tag{1.18}$$

The form b is obviously trilinear and (1.18) ensures the continuity of b.

When $n = 2$, we have the same result, ($\boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^2(\Omega) = \boldsymbol{H}_0^1(\Omega)$), but (1.17) must be replaced by

$$\left| \int_\Omega u_i D_i v_j w_j \,\mathrm{d}x \right| \leqslant |u_i|_{L^4(\Omega)} |D_i v_j|_{L^2(\Omega)} |w_j|_{L^4(\Omega)}. \tag{1.19}$$

In particular one has

Lemma 1.2. *For any open set Ω, b is a trilinear continuous form on $V \times V \times \widetilde{V}$. If Ω is bounded and $n \leqslant 4$, b is trilinear continuous on $V \times V \times V$.*

We will prove, when needed, other properties of b similar to those given before; the proof will be always the same as in Lemma 1.1 (use of Hölder's inequality and the embedding theorem (1.4)).

We denote by $B(\boldsymbol{u}, \boldsymbol{v})$, $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$, the linear continuous form on $\widetilde{V}$ defined by

$$\langle B(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w} \rangle = b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}),\ \ \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),\ \forall\, \boldsymbol{w} \in \widetilde{V}. \tag{1.20}$$

For $\boldsymbol{u} = \boldsymbol{v}$, we write

$$B(\boldsymbol{u}) = B(\boldsymbol{u}, \boldsymbol{u}),\ \ \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega). \tag{1.21}$$

Another fundamental property of b is the following:

Lemma 1.3. *For any open set Ω,*

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0, \ \forall \ \boldsymbol{u} \in V, \ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^n(\Omega) \tag{1.22}$$

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}), \ \forall \ \boldsymbol{u} \in V, \ \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^n(\Omega). \tag{1.23}$$

Proof. Property (1.23) is a consequence of (1.22) when we replace $\boldsymbol{v}$ by $\boldsymbol{v} + \boldsymbol{w}$, and we use the multilinear properties of b.

In order to prove (1.22), it suffices to show this equality for $\boldsymbol{u} \in \mathscr{V}$ and $\boldsymbol{v} \in \mathscr{D}(\Omega)$. But for such $\boldsymbol{u}$ and $\boldsymbol{v}$

$$\int_\Omega u_i D_i v_j v_j \, \mathrm{d}x = \int_\Omega u_i D_i \frac{(v_j)^2}{2} \, \mathrm{d}x$$

$$= -\frac{1}{2} \int_\Omega D_i u_i (v_j)^2 \, \mathrm{d}x,$$

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = -\frac{1}{2} \sum_{j=1}^{n} \int_\Omega \operatorname{div} \boldsymbol{u} \, (v_j)^2 \, \mathrm{d}x = 0. \tag{1.24}$$

Variational Formulation.

For Ω bounded, and n arbitrary, we associate with (1.8) - (1.10) the problem

$$\begin{aligned} &\text{To find } \boldsymbol{u} \in V \text{ such that} \\ &\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \ \forall \ \boldsymbol{v} \in \widetilde{V}. \end{aligned} \tag{1.25}$$

($\boldsymbol{f}$ given in $\boldsymbol{L}^2(\Omega)$). It is clear from (1.11) and (1.13) that if $\boldsymbol{u}$ and p are smooth functions satisfying (1.8) - (1.10), then $\boldsymbol{u}$ satisfies (1.25). Conversely if $\boldsymbol{u} \in V$ satisfies (1.25), then

$$\langle -\nu \Delta \boldsymbol{u} + \sum_i u_i D_i \boldsymbol{u} - \boldsymbol{f}, \boldsymbol{v} \rangle = 0, \ \forall \ \boldsymbol{v} \in \mathscr{V} \tag{1.26}$$

$\Delta \boldsymbol{u} \in \boldsymbol{H}^{-1}(\Omega)$, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, and $u_i D_i \boldsymbol{u} \in \boldsymbol{L}^{n'}(\Omega)$ $(1/n' = 1 - 1/n)$, since $u_i \in L^{2n/n-2}(\Omega)$ by (1.4) and $D_i \boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$. Now, according to Proposition 1.1 and 1.2, Chapter I, there exists a distribution $p \in L^1_{\mathrm{loc}}(\Omega)$(1),

(1) Proposition I.1.1 shows the existence of p as a distribution, $p \in \mathscr{D}'(\Omega)$. Proposition I.1.2 and a further regularization argument show that p is a function belonging (at least) to $L^1_{\mathrm{loc}}(\Omega)$.

such that (1.8) is satisfied in the distribution sense, then (1.9) and (1.10) are satisfied respectively in the distribution and the trace theorem senses.

Theorem 1.2. *Let Ω be a bounded set in $\mathcal{R}^n$ and let f be given in $H^{-1}(\Omega)$*

Then Problem (1.25) *has at least one solution $\boldsymbol{u} \in V$ and there exists a distribution $p \in L^1_{\mathrm{loc}}(\Omega)$ such that* (1.8) - (1.9) *are satisfied.*

Proof. We have only to prove the existence of $\boldsymbol{u}$; the existence of p and the interpretation of (1.8) - (1.9) have already been shown.

The existence of $\boldsymbol{u}$ is proved by the Galerkin method: we construct an approximate solution of (1.25) and then pass to the limit.

The space $\widetilde{V}$ is separable as a subspace of $H^1_0(\Omega)$. Because of (1.13) there exists a sequence $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m, \ldots$, of linearly independent elements of $\mathscr{V}$ which is total in $\widetilde{V}$. This sequence is also free and total in V.

For each fixed integer $m \geqslant 1$, we would like to define an approximate solution $\boldsymbol{u}_m$ of (1.25) by

$$\boldsymbol{u}_m = \sum_{i=1}^{m} \xi_{i,m} \boldsymbol{w}_i, \quad \xi_{i,m} \in \mathcal{R} \tag{1.27}$$

$$\nu((\boldsymbol{u}_m, \boldsymbol{w}_k)) + b(\boldsymbol{u}_m, \boldsymbol{u}_m, \boldsymbol{w}_k) = \langle f, \boldsymbol{w}_k \rangle, \quad k = 1, \ldots, m. \tag{1.28}$$

The equations (1.27) - (1.28) are a system of nonlinear equations for $\xi_{1,m}, \ldots, \xi_{m,m}$, and the existence of a solution of this system is not obvious, but follows from the next lemma.

Lemma 1.4. *Let X be a finite dimensional Hilbert space with scalar product $[\cdot,\cdot]$ and norm $[\cdot]$ and let P be a continuous mapping from X into itself such that*

$$[P(\xi), \xi] > 0 \text{ for } [\xi] = k > 0. \tag{1.29}$$

Then there exists $\xi \in X$, $[\xi] \leqslant k$, such that

$$P(\xi) = 0. \tag{1.30}$$

The proof of Lemma 1.4 follows the proof of Theorem 1.2. We apply this lemma for proving the existence of $\boldsymbol{u}_m$, as follows:

X = the space spanned by $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m$; the scalar product on X is the scalar product $((\cdot,\cdot))$ induced by V, and $P = P_m$ is defined by

$$[P_m(\boldsymbol{u}), \boldsymbol{v}] = ((P_m(\boldsymbol{u}), \boldsymbol{v})) = \nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) -$$

$$(f, \boldsymbol{v}), \ \forall\, \boldsymbol{u}, \boldsymbol{v} \in X.$$

The continuity of the mapping P_m is obvious; let us show (1.29).

$$\begin{aligned}[P_m(u), u] &= \nu\|u\|^2 + b(u, u, u) - \langle f, u\rangle \\ &= (\text{by } (1.22)) \\ &= \nu\|u\|^2 - \langle f, u\rangle \\ &\geqslant \nu\|u\|^2 - \|f\|_{V'}\|u\|,\end{aligned}$$

$$[P_m(u), u] \geqslant \|u\|\,(\nu\|u\| - \|f\|_{V'}). \tag{1.31}$$

It follows that $[P_m u, u] > 0$ for $\|u\| = k$, and k sufficiently large: more precisely, $k > 1/\nu\|f\|_{V'}$. The hypotheses of Lemma 1.4 are satisfied and there exists a solution u_m of (1.27) - (1.28).

Passage to the Limit.

We multiply (1.28) by $\xi_{k,m}$ and add corresponding equalities for $k = 1, \ldots, m$; this gives

$$\nu\|u_m\|^2 + b(u_m, u_m, u_m) = \langle f, u_m\rangle$$

or, because of (1.24),

$$\nu\|u_m\|^2 = \langle f, u_m\rangle \leqslant \|f\|_{V'}\|u_m\|.$$

We obtain then the *a priori* estimate:

$$\|u_m\| \leqslant \frac{1}{\nu}\|f\|_{V'}. \tag{1.32}$$

Since the sequence u_m remains bounded in V, there exists some u in V and a subsequence $m' \to \infty$ such that

$$u_{m'} \to u \text{ for the weak topology of } V. \tag{1.33}$$

The compactness theorem 1.2 shows in particular that the injection of V into $L^2(\Omega)$ is compact, so we have also

$$u_{m'} \to u \text{ in the norm of } L^2(\Omega). \tag{1.34}$$

Let us admit for a short time the following lemma.

Lemma 1.5. *If u_μ converges to u in V weakly and in $L^2(\Omega)$ strongly, then*

$$b(u_\mu, u_\mu, v) \to b(u, u, v), \quad \forall\, v \in \mathscr{V}. \tag{1.35}$$

Then we can pass to the limit in (1.28) with the subsequence $m' \to \infty$. From (1.33), (1.34), (1.35) we find that

$$\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \tag{1.36}$$

for any $\boldsymbol{v} = \boldsymbol{w}_1, \ldots, \boldsymbol{w}_m \ldots$. Equation (1.36) is also true for any $\boldsymbol{v}$ which is a linear combination of $\boldsymbol{w}_1, \ldots, \mathrm{w}_m, \ldots$. Since these combinations are dense in $\widetilde{V}$, a continuity argument finally shows that (1.36) holds for each $\boldsymbol{v} \in \widetilde{V}$ and that $\boldsymbol{u}$ is a solution of (1.25).

Proof of Lemma 1.4. This is an easy consequence of the Brouwer fixed point theorem.

Suppose that P has no zero in the ball D of X centered at O and with radius k. Then the following application

$$\xi \to S(\xi) = -k \frac{P(\xi)}{[P(\xi)]}$$

maps D into itself and is continuous. The Brouwer theorem implies then that S has a fixed point in D: there exists $\xi_0 \in D$, such that

$$-k \frac{P(\xi_0)}{[P(\xi_0)]} = \xi_0 .$$

If we take the norm of both sides of this equation we see that $[\xi_0] = k$, and if we take the scalar product of each side with ξ_0, we find

$$[\xi_0]^2 = k^2 = -k \frac{[P(\xi_0), \xi_0]}{[P(\xi_0)]}$$

This equality contradicts (1.29) and thus $P(\xi)$ must vanish at some point of D.

Proof of Lemma 1.5. It is easy to show, as for (1.22) - (1.23), that

$$\begin{aligned} b(\boldsymbol{u}_\mu, \boldsymbol{u}_\mu, \boldsymbol{v}) &= -b(\boldsymbol{u}_\mu, \boldsymbol{v}, \boldsymbol{u}_\mu) \\ &= -\sum_{i,j=1}^{n} \int_\Omega \boldsymbol{u}_{\mu i} \boldsymbol{u}_{\mu j} D_i \boldsymbol{v}_j \, \mathrm{d}x. \end{aligned}$$

But $\boldsymbol{u}_{\mu i}$ converges to $\boldsymbol{u}_i$ in $L^2(\Omega)$ strongly; since $D_i \boldsymbol{v}_j \in L^\infty(\Omega)$, it is easy to check that

$$\int_\Omega \boldsymbol{u}_{\mu i} \boldsymbol{u}_{\mu j} D_i \boldsymbol{v}_j \, \mathrm{d}x \to \int_\Omega \boldsymbol{u}_i \boldsymbol{u}_j D_i \boldsymbol{v}_j \, \mathrm{d}x.$$

Hence $b(\boldsymbol{u}_\mu, \boldsymbol{v}, \boldsymbol{u}_\mu)$ converges to $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}) = -b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})$.

Uniqueness.

For uniqueness we only have the following result:

Theorem 1.3. *If $n \leqslant 4$ and if ν is sufficiently large or f "sufficiently small" so that*

$$\nu^2 > c(n)\,\|f\|_{V'} \tag{1.37}$$

then there exists a unique solution $\boldsymbol{u}$ of (1.25).

The constant $c(n)$ in (1.37) is the constant $c(n)$ in (1.18); its estimation is connected with the estimation of the constants in (1.4) and this is given for instance in Lions [1].

Proof of Theorem 1.3. We can take $\boldsymbol{v} = \boldsymbol{u}$ in (1.25) since $\widetilde{V} = V$ for $n \leqslant 4$; we obtain with (1.22)

$$\nu\,\|\boldsymbol{u}\|^2 = \langle f, \boldsymbol{u}\rangle \leqslant \|f\|_{V'}\|\boldsymbol{u}\| \tag{1.38}$$

so that any solution $\boldsymbol{u}$ of (1.25) satisfies

$$\|\boldsymbol{u}\| \leqslant \frac{1}{\nu}\,\|f\|_{V'}. \tag{1.39}$$

Now let $\boldsymbol{u}_*$ and $\boldsymbol{u}_{**}$ be two different solutions of (1.25) and let $\boldsymbol{u} = \boldsymbol{u}_* - \boldsymbol{u}_{**}$. We subtract the equations (1.25) corresponding to $\boldsymbol{u}_*$ and $\boldsymbol{u}_{**}$ and we obtain

$$\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}_*, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}_*, \boldsymbol{v}) = 0, \ \forall\, \boldsymbol{v} \in V. \tag{1.40}$$

We take $\boldsymbol{v} = \boldsymbol{u}$ in (1.40) and use again (1.22); hence:

$$\nu\|\boldsymbol{u}\|^2 = -\,b(\boldsymbol{u}, \boldsymbol{u}_*, \boldsymbol{u}).$$

With (1.18) and (1.39) this gives (for $\boldsymbol{u} = \boldsymbol{u}_*$)

$$\begin{aligned}\nu\,\|\boldsymbol{u}\|^2 &\leqslant c(n)\,\|\boldsymbol{u}\|^2\,\|\boldsymbol{u}_*\| \\ &\leqslant \frac{c(n)}{\nu}\,\|f\|_{V'}\|\boldsymbol{u}\|^2,\end{aligned}$$

$$\left(\nu - \frac{c(n)}{\nu}\,\|f\|_{V'}\right)\|\boldsymbol{u}\|^2 \leqslant 0.$$

Because of (1.37) this inequality implies $\|\boldsymbol{u}\| = 0$, which means $\boldsymbol{u}_* = \boldsymbol{u}_{**}$.

Remark 1.1. The solution of (1.25) is probably not unique if (1.37) is not satisfied or at least for ν small enough (f fixed). A non-uniqueness result for ν small will be proved in Section 4 for a problem very similar to (1.25).

Remark 1.2. For $n > 4$, Theorem 1.2 shows the existence of solutions $\boldsymbol{u}$ of (1.25) satisfying (1.39): the majorations (1.32) and (1.33) give indeed:

$$\|\boldsymbol{u}\| \leqslant \lim_{m'\to\infty} \|\boldsymbol{u}_{m'}\| \leqslant \frac{1}{\nu}\|f\|_{V'}. \tag{1.41}$$

Nevertheless the proof of Theorem 1.3 cannot be extended to this case; (1.40) holds for each $\boldsymbol{v} \in \widetilde{V}$ and it is not possible to take $\boldsymbol{v} = \boldsymbol{u}$.

1.3. *The Homogeneous Navier-Stokes Equations (continued).*

The Unbounded Case.

We can study the case Ω unbounded by introducing the same spaces as in Section 2.3, Chapter I. Let us recall that

$$Y = \text{the completed space of } \mathscr{V} \text{ for the norm } \|\cdot\|. \tag{1.42}$$

Let us consider also the space $\widetilde{Y}$

$$\begin{aligned} \widetilde{Y} = {} & \text{the closure of } \mathscr{V} \text{ in the space} \\ & \quad Y \cap L^n(\Omega) \text{ equipped with the norm:} \end{aligned} \tag{1.43}$$

$$\|\boldsymbol{u}\| + \|\boldsymbol{u}\|_{L^n(\Omega)}{}^{(1)}. \tag{1.44}$$

We recall that because of Lemma 2.3, Chapter I, we have the continuous injection

$$Y \subset \{\boldsymbol{u} \in L^\alpha(\Omega),\ D_i\boldsymbol{u} \in L^2(\Omega),\ 1 \leqslant i \leqslant n\} \tag{1.45}$$

for $n \geqslant 3$, where

$$\alpha = \frac{2n}{n-2}. \tag{1.46}$$

---

(1) For a smooth open set Ω, Y is probably equal to $Y \cap L^n(\Omega)$, but this result is not proved.

Since Ω is unbounded the spaces $L^\gamma(\Omega)$ are not decreasing when γ increases as in the bounded case and $\widetilde{Y} \neq Y$ even for $n \leqslant 4$. Lemma 1.2 cannot be extended to the unbounded case; however, we have:

Lemma 1.6. *For $n \geqslant 3$, the form b is defined and trilinear continuous on $Y \times Y \times \widetilde{Y}$ and*

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0, \ \ \forall \boldsymbol{u} \in Y, \ \boldsymbol{v} \in \widetilde{Y}, \tag{1.47}$$

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}), \ \forall \boldsymbol{u} \in Y, \boldsymbol{v}, \boldsymbol{w} \in \widetilde{Y}. \tag{1.48}$$

Proof. The inequality (1.17) ($n \geqslant 3$) is valid; for $\boldsymbol{u}, \boldsymbol{v} \in Y$, $\boldsymbol{w} \in \widetilde{Y}$ we then have

$$\left| \int_\Omega u_i D_i v_j w_j \,\mathrm{d}x \right| \leqslant c \, \|\boldsymbol{u}\|_Y \, \|\boldsymbol{v}\|_Y \, \|\boldsymbol{w}\|_{\widetilde{Y}}$$

so that

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant c(n) \, \|\boldsymbol{u}\|_Y \, \|\boldsymbol{v}\|_Y \, \|\boldsymbol{w}\|_{\widetilde{Y}}. \tag{1.49}$$

The relations (1.47) and (1.48) are proved exactly as (1.22) and (1.23): we prove them for $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathscr{V}$ and then pass to the limit.

The variational formulation of Problem (1.8) - (1.10) for Ω unbounded and $n \geqslant 3$ is set as follows:

To find $\boldsymbol{u} \in Y$ such that

$$\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \ \ \forall \ \boldsymbol{v} \in \widetilde{Y}. \tag{1.50}$$

Theorem 1.4. *Let Ω be an open set in $\mathscr{R}^n$, $n \geqslant 3$, and let $\boldsymbol{f}$ be given in Y', the dual space of Y.*

Then there exists at least one $\boldsymbol{u}$ in Y which satisfies (1.50).

Proof. The proof is very similar to the proof of Theorem 1.2 (the bounded case). There exists a sequence $w_1, \ldots, w_m, \ldots,$ of elements of $\mathscr{V}$ which is free and total in $\widetilde{Y}$ and hence in Y; this sequence is not necessarily the same sequence as before.

We define an approximate solution $\boldsymbol{u}_m$ by

$$u_m = \sum_{i=1}^{m} \xi_{i,m} w_i, \quad \xi_{i,m} \in \mathbb{R}, \tag{1.51}$$

$$\nu((u_m, w_k)) + b(u_m, u_m, w_k) = \langle f, w_k \rangle, \quad k = 1, \ldots, m. \tag{1.52}$$

The existence of u_m satisfying (1.51) - (1.52) is proved exactly as before, using Lemma 1.4. We have then an *a priori* estimate analogous to (1.32):

$$\|u_m\| \leqslant \frac{1}{\nu} \|f\|_{Y'} \quad (\|\cdot\| = \text{the norm in } Y). \tag{1.53}$$

There exists therefore a subsequence $m' \to \infty$ and an element $u \in Y$ such that

$$u_m \to u \text{ weakly in } Y. \tag{1.54}$$

The proof finishes as in the bounded case, except for the passage to the limit in the nonlinear term $b(u_{m'}, u_{m'}, v)$; it is not true that $u_{m'}$ converges to u in $L^2(\Omega)$ strongly since u does not even belong to $L^2(\Omega)$ in general ($Y \subset L^2(\Omega)$). Nevertheless, we have

Lemma 1.7. *If u_μ converges to u in Y weakly, then,*

$$b(u_\mu, u_\mu, v) \to b(u, u, v), \ \forall \ v \in \mathscr{V}. \tag{1.55}$$

Proof. We can show that

$$u_\mu \to u \text{ strongly in } L^2_{\mathrm{loc}}(\Omega), \tag{1.56}$$

which means that

$$u_\mu \to u \text{ in } L^2(\mathscr{O}), \tag{1.57}$$

for each bounded set $\mathscr{O} \subset \Omega$.

Actually, let $\psi \in \mathscr{D}(\mathbb{R}^n)$, $\psi = 1$ on $\mathscr{O}$ and let Ω' be a bounded subset of Ω containing the support of ψ. Then the functions ψu_μ belong to $H_0^1(\Omega')$ and since u_μ converges to u weakly in Y,

$$\psi u_\mu \to \psi u, \text{ weakly in } H_0^1(\Omega').$$

Hence $\psi u_\mu \to \psi u$ strongly in $L^2(\Omega)$; in particular

$$\int_{\mathscr{O}} |u_\mu - u|^2 \,\mathrm{d}x \leqslant \int_{\Omega'} \psi^2 |u_\mu - u|^2 \,\mathrm{d}x \to 0,$$

and (1.57) follows.

Since u_μ converges to u for the L^2 norm, on the support of v, the convergence (1.55) is now proved as in the bounded case:

$$b(u_\mu, u_\mu, v) = -b(u_\mu, v, u_\mu) \to -b(u, v, u) = b(u, u, v) \quad \square$$

Remark 1.3. For $n = 2$, an element u of Y does not belong in general to any space $L^\beta(\Omega)$. For this reason the proof of Lemma 1.6 fails and b is not defined on $Y \times Y \times Y$.

We can replace (1.50) by the problem: to find $u \in Y$ such that

$$\nu((u, v)) + b(u, u, v) = \langle f, v\rangle, \ \forall\ v \in \mathscr{V}.$$

The same proof as for Theorem 1.4 shows that such a u always exists provided f is given in Y'. $\square$

Remark 1.4. Since $Y \neq \tilde{Y}$ (for any n), we cannot set $v = u$ in (1.50). Therefore the proof of Theorem 1.2 cannot be extended to the unbounded case even for $n \leqslant 4$. $\square$

Regularity of the Solution.

If the dimension n of the space is less than or equal to three, we can obtain some information about any solution u of (1.25) or (1.50) by reiterating the following simple procedure: the information we have on u gives us some regularity property of the nonlinear term

$$\sum_{i=1}^{n} u_i D_i u.$$

We then write (1.8) - (1.10) as

$$-\nu\Delta u + \operatorname{grad} p = f - \sum_{i=1}^{n} u_i D_i u, \text{ in } \Omega, \tag{1.58}$$

$$\operatorname{div} u = 0, \text{ in } \Omega, \tag{1.59}$$

$$u = 0 \text{ on } \Gamma; \tag{1.60}$$

using the available regularity properties of f and Proposition 2.2 of Chapter I we obtain new informations on the regularity of u. If the properties of u thus obtained are better than before, we can reiterate the procedure.

Let us show, for example, the following result:

Proposition 1.1. *Let Ω be an open set of class $\mathscr{C}^\infty$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let f be given in $\mathscr{C}^\infty(\bar{\Omega})$.*

Then any solution $\{\boldsymbol{u}, \boldsymbol{p}\}$ of (1.8)–(1.10) belongs to $\mathscr{C}^\infty(\bar{\Omega}) \times \mathscr{C}^\infty(\bar{\Omega})$.

Proof. Let us begin with the case where Ω is bounded.

The nonlinear term $\Sigma_{i=1}^n\, \boldsymbol{u}_i D_i \boldsymbol{u}$ is also equal to $\Sigma_{i=1}^n D_i(\boldsymbol{u}_i \boldsymbol{u})$, because of (1.59). If $n = 2$, $\boldsymbol{u}_i$ belongs to $L^\alpha(\Omega)$ for any α, $1 \leqslant \alpha < +\infty$ (by (1.4)), and then $\boldsymbol{u}_i \boldsymbol{u}_j$ belongs to $L^\alpha(\Omega)$ for any such α, and $D_i(\boldsymbol{u}_i \boldsymbol{u}_j)$ belongs to $W^{-1,\alpha}(\Omega)$ for any such α. Proposition 2.2, Chapter I, shows us that $\boldsymbol{u}$ belongs then to $W^{1,\alpha}(\Omega)$, and p belongs to $L^\alpha(\Omega)$, for any α. For $\alpha > 2$, $W^{1,\alpha}(\Omega) \subset L^\infty(\Omega)$ because of (1.3); hence $\boldsymbol{u}_i D_i \boldsymbol{u} \in \boldsymbol{L}^\alpha(\Omega)$ for any α. Then, Proposition I.2.2 shows us that $\boldsymbol{u} \in W^{2,\alpha}(\Omega)$, $p \in W^{1,\alpha}(\Omega)$ for any α. It is easy to check that $\boldsymbol{u}_i D_i \boldsymbol{u} \in W^{1,\alpha}(\Omega)$, so that $\boldsymbol{u} \in W^{3,\alpha}(\Omega)$. Repeating this procedure we find in particular that

$$\boldsymbol{u} \in H^m(\Omega), \quad p \in H^m(\Omega), \quad \text{for any } m \geqslant 1. \tag{1.61}$$

The same properties hold for any derivative of $\boldsymbol{u}$ or p; (1.3) implies therefore that any derivative of $\boldsymbol{u}$ or p belongs to $\mathscr{C}(\bar{\Omega})$, and this is the property announced.

For $n = 3$, we notice that $\boldsymbol{u}_i \in L^6(\Omega)'$ (by (1.4)) and then

$$\boldsymbol{u}_i D_i \boldsymbol{u}_j \in L^{3/2}(\Omega).$$

Proposition I.2.2 implies that $\boldsymbol{u} \in W^{2,3/2}(\Omega)$; but (1.3) shows us that $\boldsymbol{u} \in \boldsymbol{L}^\alpha(\Omega)$ for any α, $1 \leqslant \alpha < +\infty$ ($p = 3/2$, $m = 2$, $n = 3$). Therefore

$$D_i(\boldsymbol{u}_i \boldsymbol{u}_j) \in W^{-1,\alpha}(\Omega),$$

for any α, and at this point we only need to repeat the proof given for $n = 2$.

If Ω is unbounded, we obtain the same regularity on any compact subset of $\bar{\Omega}$ by applying the preceding technique to $\psi \boldsymbol{u}$ where ψ is a cut-off function, $\psi \in \mathscr{D}(\mathbb{R}^n)$, $\psi = 1$ on the compact subset of Ω.

Remark 1.4.

(i) It is clear that we can assume less regularity for $\boldsymbol{f}$ and obtain less regularity for $\boldsymbol{u}$ and p.

(ii) The same technique fails when $n \geqslant 4$. For instance, for Ω bounded and $n = 4$, if we write the nonlinear term as $D_i(\boldsymbol{u}_i \boldsymbol{u})$, we just have $\boldsymbol{u}_i \boldsymbol{u}_j \in L^2(\Omega)$, $D_i(\boldsymbol{u}_i \boldsymbol{u}) \in \boldsymbol{H}^{-1}(\Omega)$, so that $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$; if we write the non-linear terms as $\boldsymbol{u}_i D_i \boldsymbol{u}$, we have $\boldsymbol{u}_i D_i \boldsymbol{u} \in \boldsymbol{L}^{4/3}(\Omega)$, so that

$u \in W^{2,4/3}(\Omega)$; but this gives nothing more than $u_i \in L^4(\Omega)$, $D_i u_j \in L^2(\Omega)$, which was known before.

1.4. *The Non-homogeneous Navier-Stokes Equations.*

Let Ω be an open bounded set in $\mathbb{R}^n$. We consider here the following non-homogeneous Navier-Stokes problem: let there be given two vector functions f and ϕ defined respectively on Ω and Γ and satisfying some conditions which will be specified later; to find $\boldsymbol{u}$ and p such that

$$-\nu\Delta \boldsymbol{u} + \sum_{i=1}^{n} u_i D_i \boldsymbol{u} + \operatorname{grad} p = f, \text{ in } \Omega, \tag{1.62}$$

$$\operatorname{div} \boldsymbol{u} = 0, \text{ in } \Omega, \tag{1.63}$$

$$\boldsymbol{u} = \phi, \text{ on } \Gamma. \tag{1.64}$$

We will suppose that Ω is of class $\mathscr{C}^2$, that f is given in $H^{-1}(\Omega)$ and that ϕ is given in the following slightly restrictive form [1]:

$$\phi = \operatorname{curl} \zeta \tag{1.65}$$

where

$$\zeta \in H^2(\Omega), D_i\zeta \in L^n(\Omega), \zeta \in L^\infty(\Omega), \tag{1.66}$$

and curl denotes the usual operator for $n = 2, 3$; for $n \geq 4$, curl denotes a linear differential operator with constant coefficients, such that div (curl ζ) $\equiv 0$ [2].

Theorem 1.5. *Under the above hypotheses, there exists at least one* $u \in H^1(\Omega)$, *and a distribution* p *on* Ω, *such that* (1.62) – (1.64) *hold* [3]

Proof. Let ψ be any vector function belonging to $H^1(\Omega) \cap L^n(\Omega)$ such that

$$\begin{gathered} \psi \in H^1(\Omega) \cap L^n(\Omega), \operatorname{div} \psi = 0 \\ \psi = \phi \text{ on } \Gamma. \end{gathered} \tag{1.67}$$

Let us set

$$\hat{\boldsymbol{u}} = \boldsymbol{u} - \psi.$$

---

(1) Cf. condition (2.1) and Remark 2.1 in Appendix I.
(2) curl $\zeta = (R_1\zeta), R_n\zeta), R_i\zeta = \Sigma_{j,k}\alpha_{ijk}D_j\zeta_k$; it suffices that $\Sigma_{i=1}^n \alpha_{ijk} = 0$, $\forall j, k, 1 \leq j, k \leq n$.
(3) Cf. an improved form of Theorem 1.5 in Appendix I.

Then $\boldsymbol{u}$ belongs to $\boldsymbol{H}^1(\Omega)$ and satisfies (1.63); (1.64) amounts to saying that

$$\hat{\boldsymbol{u}} \in V. \tag{1.68}$$

Equation (1.62) is equivalent to

$$-\nu\Delta\hat{\boldsymbol{u}} + \sum_{i=1}^{n} \hat{u}_i D_i \hat{\boldsymbol{u}} + \sum_{i=1}^{n} \hat{u}_i D_i \psi + \sum_{i=1}^{n} \psi_i D_i \hat{\boldsymbol{u}} + \operatorname{grad} p = \hat{f} \tag{1.69}$$

where

$$\hat{f} = f + \nu\Delta\psi - \sum_{i=1}^{n} \psi_i D_i \psi.$$

We remark that

$$\hat{f} \in H^{-1}(\Omega),$$

which we show as follows. It is clear that $\boldsymbol{f} + \nu\Delta\psi \in \boldsymbol{H}^{-1}(\Omega)$; we notice moreover that $\psi_i D_i \in \boldsymbol{L}^{\alpha'}(\Omega)$, $1/\alpha' + 1/\alpha = 1$, $\alpha = 2n/(n-2)$ if $n \geqslant 3$, any $\alpha > 2$ if $n = 2$; since $H_0^1(\Omega) \subset L^\alpha(\Omega)$, $\psi_i D_i \psi$ belongs to $\boldsymbol{H}^{-1}(\Omega)$ too.

As in Section 1.2 we can show that Problem (1.68) - (1.69) is solved if we find a $\hat{\boldsymbol{u}}$ in V such that

$$\begin{aligned} \nu((\hat{\boldsymbol{u}}, \boldsymbol{v})) + b(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}, \boldsymbol{v}) + b(\hat{\boldsymbol{u}}, \psi, \boldsymbol{v}) + b(\psi, \hat{\boldsymbol{u}}, \boldsymbol{v}) \\ = \langle \hat{f}, \boldsymbol{v}\rangle, \ \forall\ \boldsymbol{v} \in \widetilde{V}. \end{aligned} \tag{1.70}$$

The existence of a $\hat{\boldsymbol{u}} \in V$ satisfying (1.70) can be proved exactly as in Theorem 1.2, provided there exists some $\beta > 0$, such that

$$\nu\|\boldsymbol{v}\|^2 + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v}) + b(\boldsymbol{v}, \psi, \boldsymbol{v}) + b(\psi, \boldsymbol{v}, \boldsymbol{v}) \geqslant \beta\|\boldsymbol{v}\|^2, \ \forall\ \boldsymbol{v} \in \widetilde{V},$$

or because of (1.22)

$$\nu\|\boldsymbol{v}\|^2 + b(\boldsymbol{v}, \psi, \boldsymbol{v}) \geqslant \beta\|\boldsymbol{v}\|^2, \ \forall\ \boldsymbol{v} \in \widetilde{V}. \tag{1.71}$$

Now (1.71) will certainly be satisfied if we can find ψ which satisfies (1.67) and

$$|b(\boldsymbol{v}, \psi, \boldsymbol{v})| \leqslant \frac{\nu}{2}\|\boldsymbol{v}\|^2, \quad \forall \boldsymbol{v} \in \widetilde{V}. \tag{1.72}$$

In order to show this, we will prove the following lemma:

Lemma 1.8. *For any $\gamma > 0$, there exists some $\psi = \psi(\gamma)$ satisfying* (1.67)

and

$$|b(v, \psi, v| \leqslant \gamma \|v\|^2, \ \forall \ v \in \widetilde{V}. \tag{1.73}$$

Before this we prove two other lemmas.

Lemma 1.9. *Let $\rho(x) = d(x, \Gamma)$ = the distance from x to Γ. For any $\epsilon > 0$, there exists a function $\theta_\epsilon \in \mathscr{C}^2(\overline{\Omega})$ such that*

$$\theta_\epsilon = 1 \textit{ in some neighbourhood of } \Gamma \textit{ (which depends on } \epsilon). \tag{1.74}$$

$$\theta_\epsilon = 0 \textit{ if } \rho(x) \geqslant 2\delta(\epsilon), \ \delta(\epsilon) = \exp\left(-\frac{1}{\epsilon}\right) \tag{1.75}$$

$$|D_k \theta_\epsilon(x)| \leqslant \frac{\epsilon}{\rho(x)} \textit{ if } \rho(x) \leqslant 2\delta(\epsilon), \ k = 1, \ldots, n. \tag{1.76}$$

Proof. Let us consider with E. Hopf [2], the function $\lambda \to \xi_\epsilon(\lambda)$ defined for $\lambda \geqslant 0$ by

$$\xi_\epsilon(\lambda) = \begin{cases} 1 \text{ if } \lambda < \delta(\epsilon)^2 \\ \epsilon \log\left(\dfrac{\delta(\epsilon)}{\lambda}\right) \text{ if } \delta(\epsilon)^2 < \lambda < \delta(\epsilon) \\ 0 \text{ if } \lambda > \delta(\epsilon) \end{cases} \tag{1.77}$$

and let us denote by χ_ϵ the function

$$\chi_\epsilon(x) = \xi_\epsilon(\rho(x)). \tag{1.78}$$

Since the function ρ belongs to $\mathscr{C}^2(\overline{\Omega})$, the function χ_ϵ satisfies (1.74) – (1.76) and θ_ϵ is obtained by regularization of χ_ϵ.

Lemma 1.10. *There exists a positive constant c_1 depending only on Ω such that*

$$\left|\frac{1}{\rho} v\right|_{L^2(\Omega)} \leqslant c_1 \|v\|_{H_0^1(\Omega)}, \ \forall \ v \in H_0^1(\Omega). \tag{1.79}$$

Proof. By using a partition of unity subordinated to a covering of Γ, and local coordinates near the boundary, we reduce the problem to the same problem with Ω = a half-space = $\{x = (x_n, x'), \ x_n > 0, x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\}$. In this case $\rho(x) = \chi_n$, and it is sufficient to check that

$$\int_\Omega \frac{v(x)}{x_n^2}^2 \mathrm{d}x \leqslant c_1 \int_\Omega |D_n v(x)|^2 \,\mathrm{d}x, \quad \forall v \in \mathscr{D}(\Omega). \tag{1.80}$$

This inequality is obvious if one proves the following one-dimensional inequality:

$$\int_0^{+\infty} |\frac{v(s)}{s}|^2 \mathrm{d}s \leqslant 2 \int_0^{+\infty} |v'(s)|^2 \mathrm{d}s, \quad \forall v \in \mathscr{D}(0, +\infty). \tag{1.81}$$

This is a classical Hardy inequality. In order to prove it, we write $s = e^\sigma$, $t = e^\tau$ and

$$\frac{v(s)}{s} = \frac{1}{s} \int_0^s w(t)\,\mathrm{d}t, \quad v' = w,$$

$$\int_0^{+\infty} \frac{|v(s)|^2}{|s|^2} \mathrm{d}x = \int_{-\infty}^{+\infty} e^{-\sigma} \left(\int_0^{e^\sigma} w(t)\,\mathrm{d}t \right)^2 \mathrm{d}\sigma$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \mathscr{Y}(\sigma - \tau) e^{-(\sigma-\tau)/2} w(e^\tau) e^{\tau/2} \,\mathrm{d}\tau \right)^2 \mathrm{d}\sigma,$$

where $\mathscr{Y}$ represents the Heaviside function, $\mathscr{Y}(\sigma) = 1$, for $\sigma > 0$ and $\mathscr{Y}(\sigma) = 0$ for $\sigma < 0$. By the usual convolution inequality we majorize the last quantity by

$$\left(\int_{-\infty}^{+\infty} \mathscr{Y}(\sigma)\, e^{-\sigma/2} \,\mathrm{d}\sigma \right)^2 \cdot \int_{-\infty}^{+\infty} |w(e^\tau)|^2 e^\tau \,\mathrm{d}\tau = 4 \int_0^{+\infty} |w(t)|^2 \,\mathrm{d}t,$$

and (1.81) follows.

Proof of Lemma 1.8. Let us now show that

$$\psi = \operatorname{curl}(\theta_\epsilon \zeta)$$

satisfies (1.67) and (1.73); (1.67) is obvious because of (1.65) and (1.74),

$$\psi_j(x) = 0 \text{ if } \rho(x) > 2\delta(\epsilon)$$

and

$$|\psi_j(x)| \leqslant c_2 \left(\frac{\epsilon}{\rho(x)} |\zeta(x)| + |D\zeta(x)| \right) \text{ if } \rho(x) \leqslant 2\delta(\epsilon) \tag{1.82}$$

where

$$|D\zeta(x)| = \left\{ \sum_{i,j=1}^{n} |D_i \zeta_j(x)|^2 \right\}^{1/2} .$$

As we supposed that $\zeta_i \in L^\infty(\Omega)$, we deduce from (1.82) that

$$|\psi_j(x)| \leqslant c_3 \left(\frac{\epsilon}{\rho(x)} + |D\rho(x)| \right), \forall\ j,\ \rho(x) \leqslant 2\delta(\epsilon).$$

We have therefore

$$|v_i \psi_j|_{L^2(\Omega)} \leqslant c_4 \left\{ \epsilon \left| \frac{v_i}{\rho} \right|_{L^2} + \left(\int_{\rho \leqslant 2\delta(\epsilon)} v_i^2 |D\zeta|^2 \, \mathrm{d}x \right)^{1/2} \right\} \tag{1.83}$$

But, using Hölder's inequality, we see that

$$\left(\int_{\rho \leqslant 2\delta(\epsilon)} v_i^2 |D\zeta|^2 \, \mathrm{d}x \right)^{1/2} \leqslant \mu(\epsilon) |v_i|_{L^\alpha(\Omega)}$$

where $1/\alpha = 1/2 - 1/n$ and

$$\mu(\epsilon) = \left\{ \int_{\rho(x) \leqslant 2\delta(\epsilon)} |D\zeta(x)|^n \, \mathrm{d}x \right\}^{1/n};$$

$\mu(\epsilon) \to 0$ as $\epsilon \to 0$ since $D_i \zeta_j \in L^n(\Omega)$, $1 \leqslant i, j \leqslant n$.

With this last majorization, (1.4), and Lemma 1.10, (1.83) gives

$$\begin{aligned} |v_i \psi_j|_{L^2(\Omega)} &\leqslant c_5 (\epsilon \|v\| + \mu(\epsilon) |v|_{L^\alpha(\Omega)} \leqslant \\ &\qquad c_6 (\epsilon + \mu(\epsilon)) \|v\|, \ 1 \leqslant i, j \leqslant n. \end{aligned} \tag{1.84}$$

Now it is easy to check (1.73); for each $v \in \mathscr{V}$,

$$b(v, \psi, v) = -b(v, v, \psi)$$

$$|b(v, v, \psi)| \leqslant \|v\| \left\{ \sum_{i,j=1}^{n} |v_i \psi_j| \right\}$$

$$\leqslant (\text{by } (1.84))$$

$$\leqslant c_7(\epsilon + \mu(\epsilon)) \|v\|^2. \tag{1.85}$$

If ϵ is sufficiently small for

$$c_7(\epsilon + \mu(\epsilon)) \leqslant \gamma,$$

we obtain (1.73) for each $v \in \mathscr{V}$ and by continuity, for each $v \in \widetilde{V}$.

Remark 1.5.
(i) For $n \leqslant 3$, the conditions (1.66) reduce to

$$\zeta \in H^2(\Omega).$$

because of the Sobolev imbedding theorems (see (1.3)).
(ii) It is easy to write the boundary condition in the form (1.65) for the classical problems of hydrodynamics such as the cavitation problem, the Taylor problem, etc. *Cf. also Appendix I.*

Remark 1.6. It is easy to extend Proposition 1.1 to non-homogeneous problems. With the hypotheses of this proposition and moreover assuming that $\phi \in \mathscr{C}^\infty(\Gamma)$ the solution $\{u, p\}$ of (1.62) - (1.64) belongs to $\mathscr{C}^\infty(\bar{\Omega}) \times \mathscr{C}^\infty(\bar{\Omega})$.
To prove this we proceed as in Proposition 1.1, directly on the equations (1.62) - (1.64) (i.e., without introducing $\hat{u}$). $\square$

A uniqueness result similar to Theorem 1.3 holds: for $n \leqslant 4$, ν "large", and f "small", there is uniqueness:

Theorem 1.6. *We suppose that $n \leqslant 4$, that the norm of ϕ in $L^n(\Omega)$*[1] *is sufficiently small so that*

$$|b(v, \phi, v)| \leqslant \frac{\nu}{2} \|v\|^2, \ \forall v \in V, \tag{1.86}$$

and ν is sufficiently large so that

$$\nu^2 > 4c(n)\|\hat{f}\|_{V'}, \tag{1.87}$$

where $c(n)$ is the constant in (1.18) *and*

$$\hat{f} = f + \nu\Delta\phi - \sum_{i=1}^{n} \phi_i D_i \phi. \tag{1.88}$$

---

(1) For $n = 2$ replace $L^n(\Omega)$ by $L^\alpha(\Omega)$ for some $\alpha > 2$.

Then, there exists a unique solution $\boldsymbol{u}$, p *of* (1.62) - (1.64) [1].

Proof. It was proved in Lemmas 1.1, 1.2, 1.3 that

$$b(\boldsymbol{v}, \phi, \boldsymbol{v}) = -b(\boldsymbol{v}, \boldsymbol{v}, \phi),$$

and

$$|b(\boldsymbol{v}, \boldsymbol{v}, \phi)| \leqslant c\, \|\boldsymbol{v}\|^2 |\phi|_{L^n(\Omega)}.$$

Therefore condition (1.86) is satisfied if $|\phi|_{L^n(\Omega)}$ is small enough: this means that (1.72) is satisfied with $\psi = \phi$ and *we do not need in this case the previous construction of* ψ. Nevertheless, the proof of existence goes along the same lines, with $\psi = \phi$.

If $\boldsymbol{u}_1$ is a solution of (1.62) - (1.64), then $\hat{\boldsymbol{u}}_1 = \boldsymbol{u}_1 - \phi$ is a solution of (1.70) with $\psi = \phi$. Taking $\boldsymbol{v} = \hat{\boldsymbol{u}}_1$ in (1.70) we get

$$\nu\|\hat{\boldsymbol{u}}_1\|^2 = -b(\hat{\boldsymbol{u}}_1, \phi, \hat{\boldsymbol{u}}_1) + \langle \hat{\boldsymbol{f}}, \hat{\boldsymbol{u}}_1 \rangle \leqslant \frac{\nu}{2}\|\hat{\boldsymbol{u}}_1\|^2 + \|\hat{\boldsymbol{f}}\|_{V'}\|\hat{\boldsymbol{u}}_1\|$$

by using (1.86), and therefore

$$\|\hat{\boldsymbol{u}}_1\| \leqslant \frac{2}{\nu}\|\hat{\boldsymbol{f}}\|_{V'}. \tag{1.89}$$

Let us suppose that $\boldsymbol{u}_0$, $\boldsymbol{u}_1$ are two solutions of (1.62) - (1.64); let $\hat{\boldsymbol{u}}_0 = \boldsymbol{u}_0 - \phi$, $\hat{\boldsymbol{u}}_1 = \boldsymbol{u}_1 - \phi$, $\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}_0 - \hat{\boldsymbol{u}}_1$; $\hat{\boldsymbol{u}}_0$ and $\hat{\boldsymbol{u}}_1$ satisfy (1.70) with $\psi = \phi$:

$$\nu((\hat{\boldsymbol{u}}_0, \boldsymbol{v})) + b(\hat{\boldsymbol{u}}_0, \hat{\boldsymbol{u}}_0, \boldsymbol{v}) + b(\hat{\boldsymbol{u}}_0, \phi, \boldsymbol{v}) + b(\phi, \hat{\boldsymbol{u}}_0, \boldsymbol{v}) = \langle \hat{\boldsymbol{f}}, \boldsymbol{v}\rangle, \quad \boldsymbol{v} \in V.$$

$$\nu((\hat{\boldsymbol{u}}_1, \boldsymbol{v})) + b(\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_1, \boldsymbol{v}) + b(\hat{\boldsymbol{u}}_1, \phi, \boldsymbol{v}) + b(\phi, \hat{\boldsymbol{u}}_1, \boldsymbol{v}) = \langle \hat{\boldsymbol{f}}, \boldsymbol{v}\rangle, \quad \boldsymbol{v} \in V.$$

We take $\boldsymbol{v} = \hat{\boldsymbol{u}}$ in these equations, and subtract the second one from the first one; after expanding and using (1.22) we find:

$$\nu\|\hat{\boldsymbol{u}}\|^2 = -b(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}) - b(\hat{\boldsymbol{u}}, \phi, \hat{\boldsymbol{u}}). \tag{1.90}$$

Because of (1.86),

$$-b(\hat{\boldsymbol{u}}, \phi, \hat{\boldsymbol{u}}) \leqslant \frac{\nu}{2}\|\hat{\boldsymbol{u}}\|^2.$$

By (1.18),

$$-b(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}) \leqslant c(n)\|\hat{\boldsymbol{u}}_1\|\,\|\hat{\boldsymbol{u}}\|^2,$$

[1] As always, p is unique up to a constant.

and because of (1.89), this is majorized by

$$\frac{2}{\nu}\, c(n) \|\hat{f}\|_{V'} \|\hat{u}\|^2.$$

We finally arrive at the inequality

$$\left(\frac{\nu}{2} - \frac{2}{\nu}\, c(n) \|\hat{f}\|_{V'}\right) \|\hat{u}\|^2 \leqslant 0,$$

and because of (1.87), this implies $\hat{u} = 0$.

If $\hat{u}_0 = \hat{u}_1$, it is clear that grad p_0 = grad p_1, and so the difference between p_0 and p_1 is constant.

Remark 1.7. Non-uniqueness results for Problem (1.62) - (1.64) have been proved, in the two-dimensional case, for certain configurations; *c.f.* Rabinowitz [2], Velte [1] [2] and Section 4 of this Chapter.

§2. Discrete inequalities and compactness theorems.

Before going through the numerical approximation of the stationary Navier-Stokes equations, we must introduce new tools: the discrete analogue, for step functions and non-conforming finite elements of the Sobolev inequalities and of the compactness theorem, Theorem 1.1. These and some further Sobolev-type inequalities are the goals set for this section This section is rather technical and the details of the proofs will not be needed in the sequel.

2.1 *Discrete Sobolev Inequalities for Step Functions.*

The notations are those used for finite differences; see Section 3.3, Chapter I. We recall in particular that R_h is the set of points with coordinates $m_1 h_1, \ldots, m_h h_n$, $m_i \in \mathscr{Z}$, $h = (h_1, \ldots, h_n)$, $h_i > 0$; w_{hM} is the characteristic function of the block

$$\sigma_h(M) = \prod_{i=1}^{n} \left(\mu_i - \frac{h_i}{2}, \mu_i + \frac{h_i}{2}\right), \quad M = (\mu_1, \ldots, \mu_n), \tag{2.1}$$

and δ_{ih} is the difference operator

$$\delta_{ih}\, \phi(x) = \frac{1}{h_i}\, [\phi(x + \frac{1}{2}\vec{h}_i) - \phi(x - \frac{1}{2}\vec{h}_i)] \tag{2.2}$$

where $\vec{h}_i$ is the vector with i^{th} component h_i, and all other components 0.

Theorem 2.1. *Let p denote some number such that* $1 \leqslant p < n$, *and let* q *be defined by* $1/q = 1/p - 1/n$.

There exists a constant $c = c(n, p)$ *depending only on n and p such that*

$$|u_h|_{L^q(\mathbb{R}^n)} \leqslant c(n, p) \sum_{i=1}^{n} |\delta_{ih} u_h|_{L^p(\mathbb{R}^n)}, \tag{2.3}$$

for each step function u_h

$$u_h = \sum_{M \in R_h} u_h(M) w_{hM}, \tag{2.4}$$

with compact support.

Proof.

(i) Let us consider the scalar function

$$s \to g(s) = |s|^{(n-1)p/n-p}.$$

Since $(n-1)p/(n-p) \geqslant 1$, this function is differentiable with derivative

$$g'(s) = \frac{(n-1)p}{n-p} |s|^{(n(p-2)+p)/(n-p)} s.$$

The Taylor formula can be written

$$g(s_1) - g(s_2) = (s_1 - s_2) g'(\lambda s_1 + (1-\lambda)s_2), \ \lambda \in (0, 1)$$

and gives

$$|g(s_1) - g(s_2)| \leqslant |s_1 - s_2| \frac{(n-1)p}{n-p} |\lambda s_1 + (1-\lambda)s_2|^{n(p-1)/n-p}$$

$$\leqslant \frac{(n-1)p}{n-p} |s_1 - s_2| \left\{ |s_1| + |s_2| \right\}^{n(p-1)/n-p},$$

$$|g(s_1) - g(s_2)| \leqslant c_1(n, p) |s_1 - s_2| \cdot$$

$$\cdot \{ |s_1|^{n(p-1)/n-p} + |s_2|^{n(p-1)/n-p} \}. \tag{2.5}$$

(ii) Let M belong to R_h; we apply (2.5) with

$$s_1 = u_h(M - r\vec{h}_i), \ s_2 = u_h(M - (r+1)\vec{h}_i);$$

$$|u_h(M - r\vec{h}_i)|^{(n-1)p/n-p} - |u_h(M - (r+1)\vec{h}_i)|^{(n-1)p/n-p}$$
$$\leqslant c_1(n,p)\,|u_h(M - r\vec{h}_i) - u_h(M-(r+1)\vec{h}_i)|$$
$$\cdot \{|u_h(M - rh_i)|^{n(p-1)/n-p} + |u_h(M-(r+1)\vec{h}_i)|^{n(p-1)/n-p}\}.$$

Summing these inequalities for $r \geqslant 0$, we find (the sum is actually finite):

$$|u_h(M)|^{(n-1)p/n-p} \leqslant c_1(n,p)h_i \sum_{r=0}^{+\infty} |\delta_{ih} u_h(M - (r+\frac{1}{2})\vec{h}_i)|$$
$$\cdot \{|u_h(M - r\vec{h}_i)|^{n(p-1)/n-p}$$
$$+ |u_h(M-(r+1)\vec{h}_i)|^{n(p-1)/n-p}\}. \tag{2.6}$$

We strengthen inequality (2.6) by replacing the sum on the right-hand side by the sum for $r \in \mathscr{Z}$; we can then interpret the sum as an integral and majorize it by

$$c_1(n,p)\int_{-\infty}^{+\infty} |\delta_{ih} u_h(\hat{\mu}_i, \xi_i)| \cdot$$
$$\left\{ \sum_{\alpha=-1}^{1} |u_h(\hat{\mu}_i, \xi_i + \frac{\alpha}{2} h_i)|^{n(p-1)/n-p} \right\} \mathrm{d}\xi_i,$$

where $(\mu_1, \ldots, \mu_n)$ are the coordinates of M and $\hat{\mu}_i = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n)$. In a similar way we denote by $\hat{x}_i$ the vector $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and then write $x = (\hat{x}_i, x_i)$.

For any $x \in \sigma_h(M)$, inequality (2.6) gives now

$$|u_h(x)|^{(n-1)p/n-p} = |u_h(M)|^{(n-1)p/n-p} \tag{2.7}$$
$$\leqslant c_1(n,p)\int_{-\infty}^{+\infty} |\delta_{ih} u_h(\hat{x}_i, \xi_i)|$$
$$\cdot \left\{ \sum_{\alpha=-1}^{1} |u_h(\hat{x}_i, \xi_i + \frac{\alpha h_i}{2})|^{n(p-1)/n-p} \right\} \mathrm{d}\xi_i.$$

Let us now set

$$w_i(x) = w_i(\hat{x}_i) = \sup_{x_i \in \mathbb{R}} |u_h(x)|^{p/n-p} \tag{2.8}$$

Then, $|w_i(\hat{x}_i)|^{n-1}$ is majorized by the right-hand side of (2.7), hence

$$\int_{\mathbb{R}^{n-1}} w_i(\hat{x}_i)^{n-1} \mathrm{d}\hat{x}_i \leqslant c_1(n,p) \int_{\mathbb{R}^n} |\delta_{ih} u_h(\hat{x}_i, \xi_i)|$$

$$\cdot \left\{ \sum_{\alpha=-1}^{1} |u_h(\hat{x}_i, \xi_i + \frac{\alpha h_i}{2})|^{n(p-1)/n-p} \right\} \mathrm{d}\hat{x}_i \mathrm{d}\xi_i.$$

$\leqslant$ (by Hölder's inequality)

$$\leqslant c_1(n,p) |\delta_{ih} u_h|_{L^p(\mathbb{R}^n)} \cdot \left(\sum_{\alpha=-1}^{+1} \int_{\mathbb{R}^n} |u_h(\hat{x}_i, \xi_i + \alpha h_i)|^q \, \mathrm{d}\hat{x}_i \mathrm{d}\xi_i \right)^{(p-1)/p};$$

therefore

$$\int_{\mathbb{R}^{n-1}} w_i(\hat{x}_i)^{n-1} \, \mathrm{d}\hat{x}_i \leqslant c_2(n,p) |\delta_{ih} u_h|_{L^p(\mathbb{R}^n)} |u_h|^{(p-1)q/p}_{L^q(\mathbb{R}^n)} \tag{2.9}$$

Now we have

$$\int_{\mathbb{R}^n} |u_h|^q \, \mathrm{d}x \leqslant \int_{\mathbb{R}^n} \prod_{i=1}^{n} \operatorname{Sup}_{x_i} |u_h(\hat{x}_i, x_i)|^{p/n-p} \mathrm{d}x$$

$$\leqslant \int_{\mathbb{R}^n} \prod_{i=1}^{n} w_i(\hat{x}_i) \, \mathrm{d}x.$$

According to the inequality given in the next lemma, this is majorized by

$$\prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^{n-1}} |w_i(\hat{x}_i)|^{n-1} \, \mathrm{d}\hat{x}_i \right\}^{1/n-1}$$

and, because of (2.9),

$$|u_h|^q_{L^q(\mathbb{R}^n)} \leqslant c_3(n,p) \cdot \left\{ \prod_{i=1}^{n} |\delta_{ih} u_h|^{1/(n-1)}_{L^p(\mathbb{R}^n)} \right\} |u_h|^{nq(p-1)/(n-1)p}_{L^q(\mathbb{R}^n)}$$

$$|u_h|_{L^q(\mathbb{R}^n)}^{n/(n-1)} \leqslant c_3(n, p) \left\{ \prod_{i=1}^{n} |\delta_{ih} u_h|_{L^p(\mathbb{R}^n)}^{1/(n-1)} \right\} .$$

$$|u_h|_{L^q(\mathbb{R}^n)} \leqslant c_4(n, p) \left\{ \prod_{i=1}^{n} |\delta_{ih} u_h|_{L^p(\mathbb{R}^n)} \right\}^{1/n}$$

$$|u_h|_{L^q(\mathbb{R}^n)} \leqslant c_5(n, p) \sum_{i=1}^{n} |\delta_{ih} u_h|_{L^p(\mathbb{R}^n)}.$$

Lemma 2.1. *Let $w_1, \ldots, w_n$, be n measurable bounded functions on $\mathbb{R}^n$, with compact supports, with w_i independent of x_i. Then*

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^{n} w_i(\hat{x}_i) \right) \mathrm{d}x \leqslant \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^{n-1}} |w_i(\hat{x}_i)|^{n-1} \mathrm{d}\hat{x}_i \right\}^{1/n-1} . \tag{2.10}$$

This is a particular case of an inequality of E. Gagliardo [1]; see also Lions [1], page 31.

Remark 2.1. For $p = n$, if the support of u_h is included in a bounded set Ω, then

$$|u_h|_{L^q(\mathbb{R}^n)} \leqslant c(n, q, \Omega) \sum_{i=1}^{n} |\delta_{ih} u_h|_{L^q(\mathbb{R}^n)}, \tag{2.11}$$

for each u_h of type (2.4), and for any q, $1 \leqslant q < +\infty$. Actually any such q greater than p can be written as $p_1 n/(n-p_1)$ with $1 \leqslant p_1 < n$. The inequality (2.3) is then applicable, with $c = c(n, p_1) = c'(n, q)$:

$$|u_h|_{L^q(\mathbb{R}^n)} \leqslant c'(n, q) \sum_{i=1}^{n} |\delta_{ih} u_h|_{L^{p_1}(\mathbb{R}^n)}.$$

The Hölder inequality shows us that

$$|\delta_{ih} u_h|_{L^{p_1}(\mathbb{R}^n)} \leqslant (\text{meas}\, \Omega')^{(1/p_1) - (1/p)} |\delta_{ih} u_h|_{L^p(\mathbb{R}^n)}$$

where Ω' contains the support of $\delta_{ih} u_h$. If we suppose that $|h|$ is bounded by 1 (or by some constant d), (meas Ω') is bounded by (meas Ω) $\times$ Const.; then combining the last two inequalities, we obtain (2.11). □

In the two and three dimensional cases, we prove another related inequality which will be useful.

Proposition 2.1. *Let us suppose that the dimension of the space is two or three.*

For any step function u_h *of type* (2.4) *with compact support, we have:*

$$\left.\begin{aligned}&|u_h|_{L^4(\mathbb{R}^2)} \leqslant 2^{1/4}\cdot 3^{1/2}\cdot |u_h|_{L^2(\mathbb{R}^2)}\cdot\\ &\left\{\sum_{i=1}^{2} |\delta_{ih}u_h|^2_{L^2(\mathbb{R}^2)}\right\}^{1/4}, \ \text{if } n = 2,\end{aligned}\right\} \tag{2.12}$$

$$\left.\begin{aligned}&|u_h|_{L^4(\mathbb{R}^3)} \leqslant 2^{1/2}\cdot 3^{3/4}\cdot |u_h|^{1/4}_{L^2(\mathbb{R}^3)}\cdot\\ &\left\{\sum_{i=1}^{3} |\delta_{ih}u_h|^2_{L^2(\mathbb{R}^3)}\right\}^{3/8} \ \text{if } n = 3 \text{ [1]}\end{aligned}\right\} \tag{2.13}$$

Proof. We use the inequality (2.7) with $n = 2$ and $p = 4/3$; a more precise analysis of the proof of (2.7) shows that $c_1(n, p) = 2$ in the present case; actually $g(s) = s^2$ and for (2.5) we have clearly

$$|g(s_1) - g(s_2)| \leqslant 2|s_1 - s_2| \ \{|s_1| + |s_2|\}.$$

We then have, for any $x \in \sigma_h(M)$, and $M \in \mathcal{R}_h$,

$$\begin{aligned}|u_h(x)|^2 \leqslant 2 \int_{-\infty}^{+\infty} &|\delta_{ih} u_h(\hat{x}_i, \xi_i)|\cdot\\ &\left\{\sum_{\alpha=-1}^{+1} |u_h(\hat{x}_i, \xi_i + \frac{\alpha h_i}{2})|\right\} \mathrm{d}\xi_i.\end{aligned} \tag{2.14}$$

Since the right-hand side of (2.14) is independent of x_i, we obtain

$$\begin{aligned}\operatorname*{Sup}_{x_i} |u_h(x)|^2 \leqslant 2 \int_{-\infty}^{+\infty} &|\delta_{ih} u_h(\hat{x}_i, \xi_i)|\cdot\\ &\left\{\sum_{\alpha=-1}^{+1} |u_h(\hat{x}_i, \xi_i + \frac{\alpha h_i}{2})|\right\} \mathrm{d}\xi_i.\end{aligned} \tag{2.15}$$

---

(1) Similar inequalities are given for the continuous case in Chapter III, Lemma 3.3.

Now we may write, for the two dimensional case,

$$\int_{\mathcal{R}^2} |u_h(x)|^4 \, dx \leqslant \int_{\mathcal{R}^2} [\operatorname*{Sup}_{x_1} |u_h(x)|^2] \, [\operatorname*{Sup}_{x_2} |u_h(x)|^2] \, dx$$

$$\leqslant \left\{ \int_{-\infty}^{+\infty} [\operatorname*{Sup}_{x_1} |u_h(x)|^2] \, dx_2 \right\} \left\{ \int_{-\infty}^{+\infty} [\operatorname*{Sup}_{x_2} |u_h(x)|^2] \, dx_1 \right\}$$

$\leqslant$ (because of (2.15))

$$\leqslant 4 \left\{ \int_{\mathcal{R}^2} |\delta_{1h} u_h(\xi_1, x_2)| \left[\sum_{\alpha=-1}^{+1} |u_h(\xi_1 + \frac{\alpha h_i}{2}, x_2)| \right] d\xi_1 \, dx_2 \right\}$$

$$\left\{ \int_{\mathcal{R}^2} |\delta_{2h} u_h(x_1, \xi_2)| \left[\sum_{\alpha=-1}^{+1} |u_h(x_1, \xi_2 + \frac{\alpha h_2}{2})| \right] dx_1 \, d\xi_2 \right\}. \tag{2.16}$$

By the Schwarz inequality and since

$$\int_{\mathcal{R}^2} |u_h(\xi_1 + \frac{\alpha h_1}{2}, x_2)|^2 \, d\xi_1 \, dx_2 =$$

$$\int_{\mathcal{R}^2} |u_h(x)|^2 \, dx_1 \, dx_2 = |u_h|^2_{L^2(\mathcal{R}^2)}, \tag{2.17}$$

the last expression is majorized by

$$36 \{ |\delta_{1h} u_h|_{L^2(\mathcal{R}^2)} |u_h|_{L^2(\mathcal{R}^2)} \} \{ |\delta_{2h} u_h|_{L^2(\mathcal{R}^2)} |u_h|_{L^2(\mathcal{R}^2)} \}$$

$$\leqslant 18 |u_h|^2_{L^2(\mathcal{R}^2)} \left\{ \sum_{i=1}^{2} |\delta_{ih} u_h|_{L^2(\mathcal{R}^2)} \right\}.$$

Hence (2.12) is proved.

In the three dimensional case, using (2.12) and (2.15), we write

$$\int_{\mathcal{R}^3} |u_h(x)|^4 \, dx \leqslant 18 \int \left\{ \left[\int |u_h|^2 \, dx_1 \, dx_2 \right] \cdot \right.$$

$$\cdot\left[\sum_{i=1}^{2}\int|\delta_{ih}u_h|^2\,\mathrm{d}x_1\,\mathrm{d}x_2\right]\Bigg\}\,\mathrm{d}x_3$$

$$\leqslant 18\left\{\operatorname*{Sup}_{x_3}\int|u_h|^2\,\mathrm{d}x_1\,\mathrm{d}x_2\right\}\left\{\sum_{i=1}^{2}|\delta_{ih}u_h|^2_{L^2(\mathbb{R}^3)}\right\}. \tag{2.18}$$

$\leqslant$ (because of (2.15))

$$\leqslant 36\left\{\sum_{i=1}^{2}|\delta_{ih}u_h|^2_{L^2(\mathbb{R}^3)}\right\}\left\{\int_{\mathbb{R}^3}|\delta_{3h}\,u_h(\hat{x}_3,x_3)|\cdot\right.$$

$$\left.\left[\sum_{\alpha=-1}^{+1}|u_h(\hat{x}_3,x_3+\frac{\alpha h_3}{2})|\right]\mathrm{d}x\right\}$$

$\leqslant$ (by Schwarz's inequality)

$$\leqslant 3^3\,2^2\,|u_h|_{L^2(\mathbb{R}^3)}|\delta_{3h}u_h|_{L^2(\mathbb{R}^3)}\left\{\sum_{i=1}^{2}|\delta_{ih}u_h|^2_{L^2(\mathbb{R}^3)}\right\}$$

$$\leqslant 3^3\,2^2\,|u_h|_{L^2(\mathbb{R}^3)}\left\{\sum_{i=1}^{3}|\delta_{ih}u_h|^2_{L^2(\mathbb{R}^3)}\right\}^{3/2},$$

and (2.13) is proved. □

Remark 2.2. The inequalities (2.3), (2.11), and (2.12) can be extended by continuity to classes of step functions with unbounded support.

2.2. *A Discrete Compactness Theorem for Step Functions.*

We give here a discrete analogue of Theorem 1.1, more precisely of the fact that the injection of $\overset{\circ}{W}{}^{1,p}(\Omega)$ into $L^{q_1}(\Omega)$ is compact if

$1 \leqslant p < n$ *and* q_1 *is any number, such that*

$$1 \leqslant q_1 < q,\ \textit{where}\ \frac{1}{q}=\frac{1}{p}-\frac{1}{n}, \tag{2.19}$$

$$p = n\ \textit{and}\ q_1\ \textit{is any number},\ 1 \leqslant q_1 < +\infty. \tag{2.20}$$

Theorem 2.2. *Let $\mathscr{E}_h$ be a family, may be empty, of step functions of type* (2.4) *and let*

$$\mathscr{E} = \bigcup_{|h| \leqslant c_0} \mathscr{E}_h. \tag{2.21}$$

Let us suppose that

the functions u_h of $\mathscr{E}$ have their supports included in some fixed bounded subset of $\mathbb{R}^n$, say Ω, (2.22)

$$\operatorname*{Sup}_{u_h \in \mathscr{E}} \{|u_h|_{L^p(\mathbb{R}^n)} + \sum_{i=1}^{n} |\delta_{ih} u_h|_{L^p(\mathbb{R}^n)}\} < +\infty. \tag{2.23}$$

Then if p and q_1 satisfy conditions (2.19) - (2.20), *the family $\mathscr{E}$ is relatively compact in $L^{q_1}(\mathbb{R}^n)$ (or $L^{q_1}(\Omega)$).*

Proof. According to a theorem of M. Riesz [1], we must prove the following two properties:

(i) For each $\epsilon > 0$, there exists a compact set $K \subset \Omega$, such that

$$\int_{\Omega - K} |u_h|^{q_1}\, dx \leqslant \epsilon, \ \forall\, u_h \in \mathscr{E}. \tag{2.24}$$

(ii) For each $\epsilon > 0$, there exists $\eta > 0$ such that,

$$|\tau_\ell u_h - u_h|_{L^{q_1}(\mathbb{R}^n)} \leqslant \epsilon, \tag{2.25}$$

for any $u_h \in \mathscr{E}$ and any $\ell = (\ell_1, \ldots, \ell_n)$, with $|\ell| \leqslant \eta$; τ_ℓ denotes the translation operator

$$(\tau_\ell \phi)(x) = \phi(x + \ell). \tag{2.26}$$

Proof of (i). Because of the Sobolev inequalities (2.3) and (2.11), the family $\mathscr{E}$ is bounded in $L^q(\Omega)$ where q is given by (2.19) if $p < n$, and q is some fixed number, $q > q_1$, otherwise ($p = n$).

By the Hölder inequality, we then get

$$\int_{\Omega - K} |u_h|^{q_1}\, dx \leqslant \left(\int_{\Omega - K} dx\right)^{1 - (q_1/q)} \left(\int_{\Omega - K} |u_h|^q\, dx\right)^{q_1/q}$$

$$\int_{\Omega-K} |u_h|^{q_1}\, dx \leqslant c(\text{meas}(\Omega-K))^{1-(q_1/q)}, \ \forall\, u_h \in \mathscr{E}. \tag{2.27}$$

The right-hand side of (2.27) (and hence the left-hand side) can be made less that ϵ, by choosing the compact K sufficiently large; (i) is proved.

Proof of (ii). First, we show that (2.25) may be replaced by a similar condition on $|\tau_\ell u_h - u_h|_{L^p}$ (condition (2.30) below).

Case (a): $q_1 \leqslant p$. For any $f \in L^q(\Omega)$ we have $f \in L^{q_1}(\Omega)$ as well as $f \in L^p(\Omega)$ since Ω is bounded and $q_1 \leqslant p < q$. Also, $0 \leqslant 1/q_1 - 1/p < 1$. By the Hölder inequality,

$$|f|_{L^{q_1}(\Omega)} \leqslant (\text{meas}\,\Omega)^{1/q_1 - 1/p} \cdot |f|_{L^p(\Omega)} = \text{Const.} \cdot |f|_{L^p(\Omega)}.$$

Case (b): $q_1 > p$. For any function $f \in L^q(\Omega)$ we can write, using the Hölder inequality,

$$\int |f|^{q_1}\, dx = \int |f|^{\theta q_1} |f|^{(1-\theta)q_1}\, dx$$
$$\leqslant \left(\int |f|^{q_1\theta\rho}\, dx\right)^{1/\rho} \left(\int |f|^{q_1(1-\theta)\rho'}\, dx\right)^{1/\rho'}$$

where $\theta \in (0, 1)$, $\rho > 1$, and as usual $1/\rho' + 1/\rho = 1$. We can choose θ and ρ so that

$$q_1\theta\rho = p, \ \ q_1(1-\theta)\rho' = q;$$

this defines ρ and θ uniquely, and these numbers belong to the specified intervals, $(\theta q_1(q-q_1)(q-p) = p(q-q_1)$, and $\rho(q-q_1) = q-p)$.

Then

$$\int |f|^{q_1}\, dx \leqslant \left(\int |f|^p\, dx\right)^{1/\rho} \left(\int |f|^q\, dx\right)^{1/\rho'}. \tag{2.28}$$

In particular, for any ℓ and u_h,

$$|\tau_\ell u_h - u_h|_{L^{q_1}(\mathbb{R}^n)} \leqslant |\tau_\ell u_h - u_h|^{1-\theta}_{L^q(\mathbb{R}^n)} \cdot |\tau_\ell u_h - u_h|^{\theta}_{L^p(\mathbb{R}^n)}$$

$$\leqslant \{|\tau_\ell u_h|_{L^q(\mathbb{R}^n)} + |u_h|_{L^q(\mathbb{R}^n)}\}^{1-\theta} |\tau_\ell u_h - u_h|^\theta_{L^p(\mathbb{R}^n)}$$

$$\leqslant 2^{1-\theta} |u_h|^{1-\theta}_{L^q(\mathbb{R}^n)} |\tau_\ell u_h - u_h|^\theta_{L^p(\mathbb{R}^n)}.$$

Since the family $\mathscr{E}$ is bounded in $L^q(\mathbb{R}^n)$,

$$|\tau_\ell u_h - u_h|_{L^{q_1}(\mathbb{R}^n)} \leqslant c|\tau_\ell u_h - u_h|^\theta_{L^p(\mathbb{R}^n)}. \tag{2.29}$$

Inequality (2.29) shows us that it suffices to prove condition (ii) with q_1 replaced by p:

$$\left.\begin{aligned} &\forall\ \epsilon > 0,\ \exists\ \eta, \text{ such that} \\ &|\tau_\ell u_h - u_h|_{L^p(\mathbb{R}^n)} \leqslant \epsilon \\ &\text{for } |\ell| \leqslant \eta \text{ and } u_h \in \mathscr{E}. \end{aligned}\right\} \tag{2.30}$$

The proof of (2.30) follows easily from (2.23) and the next two lemmas.

Lemma 2.2.

$$|\tau_\ell u_h - u_h|_{L^p(\mathbb{R}^n)} \leqslant \sum_{i=1}^n |\tau_{\vec{\ell}_i} u_h - u_h|_{L^p(\mathbb{R}^n)}. \tag{2.31}$$

where $\vec{\ell}_i$ denotes the vector $\ell_i \vec{h}_i$

Proof. Denoting by I the identity operator, one can check easily the identity

$$\tau_\ell - I = \sum_{i=1}^n \tau_{\vec{\ell}_i} \ldots \tau_{\vec{\ell}_{i-1}} (\tau_{\vec{\ell}_i} - I). \tag{2.32}$$

This identity allows us to majorize the norm $|\tau_\ell u_h - u_h|_{L^p(\mathbb{R}^n)}$ by

$$\sum_{i=1}^n |\tau_{\vec{\ell}_i} \ldots \tau_{\vec{\ell}_{i-1}} (\tau_{\vec{\ell}_i} u_h - u_h)|_{L^p(\mathbb{R}^n)}.$$

We obtain (2.31) recalling that

$$|\tau_\alpha f|_{L^p(\mathbb{R}^n)} = |f|_{L^p(\mathbb{R}^n)}. \tag{2.33}$$

for any $\alpha \in \mathbb{R}^n$ and any function $f \in L^p(\mathbb{R}^n)$.

Lemma 2.3.

$$|\tau_{\vec{\ell}_i} u_h - u_h|_{L^p(\mathcal{R}^n)} \leqslant c(|\ell_i| + |\ell_i|^{1/p})\,|\delta_{ih} u_h|_{L^p(\mathcal{R}^n)},\quad 1 \leqslant i \leqslant n. \tag{2.34}$$

Proof. Since

$$|\tau_{\vec{\ell}_i} u_h - u_h|_{L^p(\mathcal{R}^n)} = |\tau_{-\vec{\ell}_i} u_h - u_h|_{L^p(\mathcal{R}^n)},$$

we can suppose that $\ell_i \geqslant 0$ and we then set

$$\left.\begin{aligned} &\ell_i = (\alpha_i + \beta_i)\,h_i,\ \text{ where } \alpha_i \text{ is an integer} \geqslant 0,\\ &\quad\text{and } 0 \leqslant \beta_i < 1,\ \ 1 \leqslant i \leqslant n. \end{aligned}\right\} \tag{2.35}$$

We write

$$\tau_{\vec{\ell}_1} u_h - u_h = \sum_{j=0}^{\alpha_1 - 1} \tau_{j\vec{h}_1}(\tau_{\vec{h}_1} u_h - u_h) + \tau_{\alpha_1 h_1}(\tau_{\beta_1 \vec{h}_1} u_h - u_h). \tag{2.36}$$

From (2.36) and (2.33), we get the majoration

$$|\tau_{\vec{\ell}_1} u_h - u_h|_{L^p(\mathcal{R}^n)} \leqslant \sum_{j=0}^{\alpha_1 - 1} |\tau_{\vec{h}_1} u_h - u_h|_{L^p(\mathcal{R}^n)} + |\tau_{\beta_1 \vec{h}_1} u_h - u_h|_{L^p(\mathcal{R}^n)}. \tag{2.37}$$

But

$$\tau_{\vec{h}_1} - I = h_1 \tau_{\vec{h}_1/2}\, \delta_{1h}$$

and the sum on the right-hand side of (2.37) is equal to

$$\alpha_1 h_1\, |\delta_{1h} u_h|_{L^p(\mathcal{R}^n)},$$

since $\alpha_1 h_1 \leqslant \ell_1$, we obtain

$$|\tau_{\vec{\ell}_1} u_h - u_h|_{L^p(\mathcal{R}^n)} \leqslant \ell_1 |\delta_{1h} u_h|_{L^p(\mathcal{R}^n)} + |\tau_{\beta_1 \vec{h}_1} u_h - u_h|_{L^p(\mathcal{R}^n)}. \tag{2.38}$$

Let us now majorize the norm of $\tau_{\beta_1 h_1} u_h - u_h$.

For $x \in \sigma_h(M)$, $x = (x_1, \ldots, x_n)$, and $M \in \mathcal{R}_h$, $M = (m_1 h_1, \ldots, m_n h_n)$, we have

$$\tau_{\beta_1 \vec{h}_1} u_h(x) - u_h(x) = \begin{cases} 0 \text{ if } (m_1 - \frac{1}{2})h_1 < x_1 < (m_1 - \beta_1 + \frac{1}{2})h_1 \\[2ex] h_1\, \delta_{1h} u_h(M + \frac{\vec{h}_1}{2}) \text{ if } (m_1 - \beta_1 + \frac{1}{2})h_1 < \\ \qquad\qquad x_1 < (m + \frac{1}{2})h_1. \end{cases}$$

Hence

$$\int_{\sigma_h(M)} |\tau_{\beta_1 \vec{h}_1} u_h - u_h|^p \, dx = \beta_1 h_1^p \int_{\sigma_h(M)} |\delta_{1h} u_h (x + \frac{\vec{h}_1}{2})|^p \, dx,$$

and summing these equations for the different points M of $\mathcal{R}_h$, we obtain

$$|\tau_{\beta_1 \vec{h}_1} u_h - u_h|_{L^p(\mathcal{R}^n)} = \beta_1^{1/p} h_1 |\tau_{\vec{h}_1/2} \delta_{1h} u_h|_{L^p(\mathcal{R}^n)}.$$

Since h is bounded and $\beta_1 h_1 \leqslant \ell_1$, this is less than

$$c \ell_1^{1/p} |\delta_{1h} u_h|_{L^p(\mathcal{R}^n)}$$

and (2.34) follows for $i = 1$; the proof is the same for $i = 2, \ldots, n$. □

Remark 2.3. In the most common applications of Theorem 2.2, the family $\mathscr{E}$ is a sequence of elements u_{h_m}, where h_m is converging to zero. Hence $\mathscr{E}_{h_m} = \{u_{h_m}\}$ and $\mathscr{E}_h$ is empty if h is not an h_m.

Remark 2.4. Let us suppose that Ω is bounded and that

$$\mathscr{E} = \bigcup_{|h| < c_0} \mathscr{E}_h, \quad \mathscr{E}_h = \{u_h \in W_h, \ \|u_h\|_h \leqslant 1\} \tag{2.39}$$

where W_h is the approximation of $H_0^1(\Omega)$ by finite differences, (APX1). We infer from Theorem 2.2, that $\mathscr{E}$ is a relatively compact set in $L^2(\Omega)$. The following set $\mathscr{E}'$, which is a subset of $\mathscr{E}$, is also relatively compact in $L^2(\Omega)$:

$$\mathscr{E}' = \bigcup_{|h| \leqslant c_0} \mathscr{E}'_h, \quad \mathscr{E}'_h = \{u_h \in V_h, \ \|u_h\|_h \leqslant 1\}; \tag{2.40}$$

V_h corresponds to the approximation (APX1) of V. □

2.3. *Discrete Sobolev Inequalities for Non-conforming Finite Elements.*

For the remainder of this section, the notations are those used in Chapter I, Section 4.5, for non-conforming finite elements. We recall in particular that $\mathscr{T}_h$ is a regular triangulation of an open bounded set Ω; $\mathscr{U}_h$ is the set of points B which are barycenters of an $(n-1)$ dimensional face of a simplex $\mathscr{S} \in \mathscr{T}_h$ and which belong to the interior of $\Omega(h)$ [1]; w_{hB}, $B \in \mathscr{U}_h$, is the scalar function which is linear on each

[1] $\Omega(h) = \bigcup_{\mathscr{S} \in \mathscr{T}_h} \mathscr{S}$.

simplex $\mathscr{S} \in \mathscr{T}_h$, vanishes outside $\Omega(h)$, and takes the following nodal values: $w_{hB}(B) = 1$, $w_{hB}(M) = 0$ for all M different from B which are barycenters of an $(n-1)$ face of an $\mathscr{S} \in \mathscr{T}_h$.

For any function u_h of type

$$u_h = \sum_{M \in \mathscr{U}_h} u_h(M)\, w_{hM}, \tag{2.41}$$

$\mathscr{S}$

$D_{ih} u_h$ is the step function vanishing outside $\Omega(h)$ and such that

$$D_{ih} u_h(x) = D_i u_h(x), \ \forall\, x \in \ , \ \forall\, \mathscr{S} \in \mathscr{T}_h.$$

Theorem 2.3. *Let p denote some number such that $1 \leqslant p < n$, and let q be defined by $1/q = 1/p - 1/n$. There exists a constant $c = c(n, p, \Omega)$ depending only on n, p, Ω, such that*

$$|u_h|_{L^q(\Omega)} \leqslant c(n, p, \Omega, \alpha) \sum_{i=1}^{n} |D_{ih} u_h|_{L^p(\Omega)}, \tag{2.42}$$

for each function of type (2.41).

Proof. The proof relies strongly on Proposition 4.17, Chapter I.

In order to prove (2.42) we must show that for each θ in $\mathscr{D}(\Omega)$,

$$\left| \int_\Omega u_h\, \theta \, dx \right| \leqslant c(n, p, \Omega) |\theta|_{L^{q'}(\Omega)} \left(\sum_{j=1}^{n} |D_{jh} u_h|_{L^p(\Omega)} \right) \tag{2.43}$$

where q' is defined by $1/q + 1/q' = 1$.

We denote by χ the solution of the Dirichlet problem

$$\Delta\chi = \theta \text{ in } \Omega, \ \chi = 0 \text{ on } \partial\Omega.$$

The function χ is $\mathscr{C}^\infty$ on Ω and according to the regularity results for elliptic equations,

$$|\chi|_{W^{2,q'}(\Omega)} \leqslant c(n, p, \Omega)\, |\theta|_{L^q(\Omega)} \tag{2.44}$$

(observe that $1 < q, q' < \infty$). Due to the Sobolev inequality the derivatives $\chi_i = D_i\chi$ of χ $(1 \leqslant i \leqslant n)$ belong to $W^{1,q'}(\Omega)$, and thus to $L^{p'}(\Omega)$, where $1/p' + 1/p = 1$:

$$|\chi_i|_{L^{q'}(\Omega)}, \leqslant c(n, p, \Omega) |\theta|_{L^{q'}(\Omega)}, \ \chi_i = D_i\chi, \ 1 \leqslant i \leqslant n. \tag{2.45}$$

Using this function ψ we write

$$\int_\Omega u_h\,\theta\,\mathrm{d}x = \sum_{\mathscr{S}\in\mathscr{T}_h}\int_{\mathscr{S}} u_h\Delta\chi\,\mathrm{d}x = \sum_{i=0}^{n}\mathscr{J}_h^i,$$

where

$$\mathscr{J}_h^0 = -\sum_{\mathscr{S}\in\mathscr{T}_h}\int_{\mathscr{S}} D_i u_h D_i\chi\,\mathrm{d}x = -\int_\Omega D_{ih}u_h D_i\chi\,\mathrm{d}x,$$

and

$$\mathscr{J}_h^i = \sum_{\mathscr{S}\in\mathscr{T}_h}\int_{\mathscr{S}} D_i(u_h D_i\chi)\,\mathrm{d}x.$$

We have

$$\left|\int_\Omega u_h\theta\,\mathrm{d}x\right| \leqslant \sum_{i=0}^{n}|\mathscr{J}_h^i| \tag{2.46}$$

and in order to prove (2.43) we will establish some majorations of type (2.43) for the expressions $|\mathscr{J}_h^i|$, $0\leqslant i\leqslant n$. For $\mathscr{J}_h^0$ this majoration is easily obtained since the Hölder inequality and (2.45) allow us to set

$$\begin{aligned}|\mathscr{J}_h^0| &\leqslant \sum_{i=1}^{n}|D_{ih}u_h|_{L^p(\Omega)}|D_i\chi|_{L^{p'}(\Omega)}\\ &\leqslant c(n,p,\Omega)\,|\theta|_{L^{q'}(\Omega)}\left(\sum_{i=0}^{n}|D_{ih}u_h|_{L^p(\Omega)}\right).\end{aligned} \tag{2.47}$$

For $_h^i$, $1\leqslant i\leqslant n$, we use I (4.234) with $\phi=\chi_i=D_i\chi$, and we obtain

$$|\mathscr{J}_h^i| \leqslant c(n,p,\Omega,\alpha)\left(\sum_{j=1}^{n}|D_j\chi_i|_{L^{q'}(\Omega)}\right)\left(\sum_{j=1}^{n}|D_{jh}u_h|_{L^p(\Omega)}\right)$$

$$|\mathscr{J}_h^i| \leqslant c(n,p,\Omega,\alpha)\,|\chi|_{W^{2,q'}(\Omega)}\left(\sum_{j=1}^{n}|D_{jh}u_h|_{L^p(\Omega)}\right)$$

$$\leqslant \text{(by 2.44))}$$

$$\leqslant c(n, p, \Omega, \alpha)\, |\theta|_{L^{q'}(\Omega)} \left(\sum_{j=1}^{n} |D_{jh}\, u_h|_{L^p(\Omega)} \right). \qquad \square$$

Remark 2.5. In contrast to the usual Sobolev inequalities (see Section 1, and Theorem 2.1, in the discrete case), the inequality (2.42) contains a coefficient $c(n, p, \Omega)$ which depends on Ω. Indeed, the usual Sobolev inequalities contain coefficients which are independent of the support of the concerned functions. We have lost some information, but this is not very important for what follows. $\square$

Remark 2.6. The case $n = p$ can be treated as in Remark 2.1. Let p be any number such that $1 \leqslant p < n$, and let q be defined by $1/q = 1/p - 1/n$ For any function u_h of type (2.41), the relation (2.42) gives:

$$|u_h|_{L^q(\Omega)} \leqslant c(n, p, \Omega, \alpha) \sum_{i=1}^{n} |D_{ih} u_h|_{L^p(\Omega)}.$$

According to Hölder's inequality,

$$|D_{ih} u_h|_{L^p(\Omega)} \leqslant (\text{meas}\ \Omega)^{1-p/n}\, |D_{ih} u_h|_{L^n(\Omega)}.$$

Hence, with another constant $c(n, p, \Omega, \alpha)$,

$$|u_h|_{L^q(\Omega)} \leqslant c(n, p, \Omega, \alpha) \sum_{i=1}^{n} |D_{ih} u_h|_{L^n(\Omega)}. \tag{2.48}$$

Changing now the name of the constant $c(n, p, \Omega)$ and observing that q may be any number $\geqslant 1$, we write (2.48) in the following form

$$|u_h|_{L^q(\Omega)} \leqslant c(n, q, \Omega, \alpha) \cdot \sum_{i=1}^{n} |D_{ih} u_h|_{L^n(\Omega)} \tag{2.49}$$

for each u_h of type (2.41) with support included in Ω, and for each q, $1 \leqslant q \leqslant \infty$. $\square$

Remark 2.7. An inequality similar to (2.13) ($n = 3$) is given in Chapter III. We do not know if an inequality similar to (2.12) ($n = 2$) is valid for non-conforming finite elements, in the two-dimensional case.

A weaker inequality is given in Chapter III. □

2.4. *A discrete compactness theorem for non-conforming finite elements*

We will now give a discrete analogue of Theorem 1.3, more precisely of the fact that the injection of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact for any bounded set Ω. However, because of a difficulty of a technical nature, the result is proved only in the two-and three-dimensional cases.

We denote by $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ a regular family of triangulations of Ω. For each h, we consider the space Y_h of scalar functions u_h of type (2.41), i.e.,

$$u_h = \sum_{M\in\mathcal{V}_h} u_h(M)\, w_{hM}\,,\ u_h(M)\in\mathcal{R}. \tag{2.50}$$

These functions are linear on each simplex $\mathcal{S}\in\mathcal{T}_h$, vanish outside $\Omega(h) = \bigcup_{\mathcal{S}\in\mathcal{T}_h}\mathcal{S}$, and are continuous at the barycenter of an $(n-1)$ face of a simplex $\mathcal{S}\in\mathcal{T}_h$. The space Y_h is endowed with the scalar product

$$((u_h, v_h))_h = \sum_{i=1}^{n} (D_{ih}u_h, D_{ih}v_h) = \sum_{i=1}^{n}\ \sum_{\mathcal{S}\in\mathcal{T}_h} \int_{\mathcal{S}} \frac{\partial u_h}{\partial x_i}\frac{\partial v_h}{\partial x_i}\,dx\,. \tag{2.51}$$

Now we state the Theorem:

Theorem 2.4. *Assume that $n = 2$ or 3 and that $\mathcal{T}_h$, $h\in\mathcal{H}_a$, is a regular family of triangulations of Ω.*

Let $\mathcal{E}_h$ be a family which may be empty of functions of type (2.50), *and let*

$$\mathcal{E} = \bigcup_{\rho(h)\leqslant c_0} \mathcal{E}_h\,. \tag{2.52}$$

Let us suppose that

$$\operatorname*{Sup}_{u_h\in\mathcal{E}} \|u_h\|_h < +\infty. \tag{2.53}$$

Then the family $\mathcal{E}$ is relatively compact in $L^2(\Omega)$.

Proof. The main idea of the proof consists of observing that the space of (piecewise linear) conforming finite element functions is a subspace of the space of (piecewise linear) non-conforming finite element functions. We then obtain the result, using the "nondiscrete" compactness theorem (Theorem 1.1) and the previous *a priori* estimate.

The space Y_h defined by (2.50) contains the space X_h of continuous piecewise linear functions which vanish on the boundary of $\Omega(h)$ (the confirming elements). Since X_h and Y_h are finite dimensional, both are Hilbert spaces for the scalar product $((.,.))_h$ (see (2.51)) [(1)]; one can therefore consider the orthogonal projector π_h from Y_h onto X_h, and by definition of such a projector,

$$((\pi_h u_h, v_h))_h = ((u_h, v_h))_h, \ \forall\ u_h \in Y_h, \ \forall\ v_h \in X_h \tag{2.54}$$

$$\|\pi_h u_h\|_h \leqslant \|u_h\|_h, \ \forall\ u_h \in Y_h. \tag{2.55}$$

Let us now consider the elements u_h of the family $\mathscr{E}$. Due to (2.55) the family

$$\{\pi_h u_h \mid u_h \in \mathscr{E}\}$$

is a bounded family in $H_0^1(\Omega)$ since $X_h \subset H_0^1(\Omega)$ and the $\|\cdot\|_h$ –norm and the $H_0^1(\Omega)$ norm ($\|\cdot\|$) coincide for conforming elements:

$$\operatorname*{Sup}_{u_h \in \mathscr{E}} \|\pi_h u_h\|_{H_0^1(\Omega)} = \operatorname*{Sup}_{u_h \in \mathscr{E}} \|\pi_h u_h\|_h \leqslant \operatorname*{Sup}_{u_h \in \mathscr{E}} \|u_h\|_h < +\infty.$$

By virtue of Theorem 1.1, the family $\pi_h u_h$ is relatively compact in $L^2(\Omega)$ and therefore there exists a subsequence $h' \to 0$, and an u in $L^2(\Omega)$ such that

$$\pi_{h'} u_{h'} \to u \text{ in } L^2(\Omega), \text{ as } h' \to 0.$$

The proof will be complete when we show that

$$\pi_{h'} u_{h'} - u_{h'} \to 0, \text{ as } h' \to 0, \tag{2.56}$$

and (2.56) follows immediately from the next Lemma.

Lemma 2.9. *For each v_h in Y_h,*

$$|\pi_h v_h - v_h|_{L^2(\Omega)} \leqslant c(\alpha, \Omega)\, \rho(h)\, \|v_h\|_h. \tag{2.57}$$

Proof. As in the proof of Theorem 2.3, let θ be an element of $\mathscr{D}(\Omega)$ and let χ denote the solution of the Dirichlet problem

$$\Delta\chi = \theta \text{ in } \Omega, \ \chi = 0 \text{ on } \partial\Omega.$$

---

(1) These properties have already been mentioned and used several times.

In order to estimate the norm $|v_h - \pi_h v_h|_{L^2(\Omega)}$ we consider the expression

$$(v_h - \pi_h v_h, \theta) = (v_h - \pi_h v_h, \Delta\chi).$$

As in the proof of Theorem 2.3,

$$(v_h - \pi_h v_h, \Delta\chi) = ((\pi_h v_h - v_h, \chi))_h + \sum_{i=1}^{n} \mathscr{J}_h^i \tag{2.58}$$

$$\mathscr{J}_h^i = \sum_{\mathscr{S} \in \mathscr{T}_h} \int_{\mathscr{S}} D_i((v_h - \pi_h v_h)\, D_i\chi)\, \mathrm{d}x.$$

Due to Proposition 4.16, Chapter I,

$$|\mathscr{J}_h^i| \leqslant c(\Omega, \alpha)\, \rho(h)\, \|v_h - \pi_h v_h\|_h\, |\theta|_{L^2(\Omega)}{}^{(1)}, \tag{2.59}$$

and with (2.53) and (2.55)

$$|\mathscr{J}_h^i| \leqslant c\, \rho(h). \tag{2.60}$$

In order to estimate $((v_h - \pi_h, \chi))_h$, we consider the functions χ_h characterized by $\chi_h \in X_h$, $\chi_h(M) = \chi(M), \forall\, M \in \mathscr{U}_h$ (the functions interpolating χ. Note that χ is continuous in $\bar{\Omega}$, $H^2(\Omega) \subset \mathscr{C}^0(\bar{\Omega})$ for $n \leqslant 3$, see (1.3)). Because of the general results on interpolation recalled in Chapter I[2] we have

$$\begin{aligned} \|\chi - \chi_h\| &\leqslant c(\alpha)\, \rho(h)\, |\chi|_{H^2(\Omega)} \leqslant \text{(by definition of } \chi) \\ &\leqslant c\, \rho(h)\, |\theta|_{L^2(\Omega)}. \end{aligned} \tag{2.61}$$

Using (2.54) we observe that

$$((v_h - \pi_h v_h, \chi))_h = ((v_h - \pi_h v_h, \chi - \chi_h))_h$$

---

(1) It is essential to assume that the family of triangulation is regular, $\sigma_h \leqslant \alpha < +\infty$, $\forall\, h$.

(2) See I. (4.42), (4.43). These results were stated for L^∞– norms. We use here similar results valid for L^2– norms, see Ciarlet & Raviart [1], Theorem 5, p. 196.

and thus

$$|((v_h - \pi_h v_h, \chi))_h \leqslant \|v_h - \pi_h v_h\|_h \|\chi - \chi_h\|_h \leqslant c\, p(h).$$

The relation (2.57) is established.

Remark 2.8. In the most common applications of Theorem 2.4, the family $\mathscr{E}$ is a sequence of elements u_{h_m}, $\rho(h_m) \to 0$, $\sigma(h_m) \leqslant \alpha$. Hence $\mathscr{E}_{h_m} = \{u_{h_m}\}$ and $\mathscr{E}_h = \phi$ if h is not an h_m. If $\|u_{h_m}\|_{h_m} \leqslant c$, the sequence u_{h_m} is relatively compact in $L^2(\Omega)$.

§ 3. Approximation of the Stationary Navier–Stokes Equations

We discuss here the approximation of the stationary Navier–Stokes equations by numerical schemes of the same type as those used for the linear Stokes problem. We give in Section 3.1 a general convergence theorem; we then apply it in Section 3.2 to numerical schemes based on the approximations (APX 1), . . ., (APX 5) of the space V. In Section 3.3 we extend to the nonlinear case the numerical algorithms discussed in Section 5, Chapter I.

All of this section appears to be an extension to the nonlinear case of the results obtained in the linear case in Sections 3, 4 and 5. Nevertheless the results are not as strong here as in the linear case due particularly to the nonuniqueness of solutions of the exact problem.

3.1. *A General Convergence Theorem*

Let Ω be a bounded Lipschitz open set in $\mathscr{R}^n$ and let f be given in $L^2(\Omega)$. By Theorem 1.2 there exists at least one u in V such that

$$\nu((u, v)) + b(u, u, v) = (f, v), \ \forall\, v \in \widetilde{V}. \tag{3.1}$$

Because of Theorem 1.3, this u is unique if $n \leqslant 4$ and if ν is sufficiently large.

Our purpose here is to discuss the approximation of Problem 3.1.

Let there be given first an external, stable and convergent Hilbert approximation of the space V, say $(\bar{\omega}, F)$, $(V_h, p_h, r_h)_{h \in \mathcal{H}}$, where the V_h are finite dimensional; at this point this approximation could be any of the approximations (APX1), . . ., (APX5), described in Chapter I.

Let us suppose that we are given some consistent approximation of the bilinear form $\nu((u, v))$, and of the linear form (f, v), satisfying the same hypotheses as in Section 3, Chapter I:

(i) for each $h \in \mathcal{H}$, $a_h(u_h, v_h)$ is a bilinear continuous form on $V_h \times V_h$, uniformly coercive in the sense

$\exists\, \alpha_0 > 0$ independent of h, such that

$$a_h(u_h, u_h) \geqslant \alpha_0 \, \|u_h\|_h^2, \quad u_h \in V_h, \tag{3.2}$$

where $\|\cdot\|_h$ stands for the norm in V_h.

(ii) for each $h \in \mathcal{H}$, ℓ_h is a linear continuous form on V_h, such that

$$\|\ell_h\|_{V'_h} \leqslant \beta. \tag{3.3}$$

The required consistency hypotheses are:

If the family v_h converges weakly to v as $h \to 0$ and if the family w_h converges strongly to w as $h \to 0$,[1] then

$$\lim_{h \to 0} a_h(v_h, w_h) = \nu((v, w)),$$

$$\lim_{h \to 0} a_h(w_h, v_h) = \nu((w, v)). \tag{3.4}$$

If the family v_h converges weakly to v as $h \to 0$, then

$$\lim_{h \to 0} \langle \ell_h, v_h \rangle = (f, v) \tag{3.5}$$

For the approximation of the form b, we suppose that we are given a trilinear continuous form $b_h(u_h, v_h, w_h)$, on V_h, such that:

$$b_h(u_h, v_h, v_h) = 0, \ \forall u_h, v_h \in V_h. \tag{3.6}$$

---

[1]We recall that this means

$p_h v_h \to \bar{\omega} v$ in F weakly,

$p_h w_h \to \bar{\omega} w$ in F strongly.

if the family v_h converges weakly to v, as $h \to 0$, and if w belongs to $\mathscr{V}$, then

$$\lim_{h\to 0} b_h(v_h, v_h, r_h w) = b(v, v, w). \tag{3.7}$$

Sometimes it will be useful to be more precise about the continuity of b_h and we shall require

$$|b_h(u_h, v_h, w_h)| \leqslant c(n, \Omega)\ \|u_h\|_h\ \|v_h\|_h\ \|w_h\|_h,$$
$$\forall u_h, v_h, w_h \in V_h, \tag{3.8}$$

where the constant $c = c(n, \Omega)$ depends on n and Ω *but not on h*: an inequality such as (3.8) with c depending on h is obvious, since such an inequality is equivalent to the continuity of the trilinear form b_h.

We can now define an approximate problem for (3.1):

$$\left.\begin{aligned} &\textit{To find } u_h \in V_h, \textit{ such that}\\ &a_h(u_h, v_h) + b_h(u_h, u_h, v_h) = \langle \ell_h, v_h\rangle. \end{aligned}\right\} \tag{3.9}$$

We then have

Proposition 3.1. *For each h, there exists at least one u_h in V_h, which is a solution of* (3.9).

If (3.8) *holds and if*

$$\alpha_0^2 > c(n, \Omega)\beta, \tag{3.10}$$

then, u_h is unique.

Proof. The existence of u_h follows from Lemma 1.4. We apply this lemma with $X = V_h$ which is a finite dimensional Hilbert space for the scalar product $((\cdot, \cdot))_h$. We define the operator P from V_h into V_h by,

$$\begin{aligned} ((P(u_h), v_h))_h &= a_h(u_h, v_h) + b_h(u_h, u_h, v_h)\\ &\quad - \langle \ell_h, v_h\rangle,\ \forall u_h, v_h \in V_h. \end{aligned} \tag{3.11}$$

The operator P is continuous and there remains only to check (1.29); but, with (3.6),

$$\begin{aligned} ((P(u_h), u_h))_h &= a_h(u_h, u_h) - \langle \ell_h, u_h\rangle\\ &\geqslant (\text{by } (3.2)\text{–}(3.3))\\ &\geqslant \alpha_0\ \|u_h\|_h^2 - \|\ell_h\|_{V'_h}\ \|u_h\|_h\\ &\geqslant (\alpha_0\ \|u_h\|_h - \beta)\ \|u_h\|_h. \end{aligned}$$

Therefore

$$((P(u_h), u_h))_h > 0,$$

provided

$$\|u_h\|_h = k, \text{ and } k > \frac{\beta}{\alpha_0}.$$

Lemma 1.4 gives the existence of at least one u_h such that

$$P(u_h) = 0$$

or

$$((P(u_h), v_h))_h = 0, \ \forall v_h \in V_h, \tag{3.12}$$

which is exactly equation (3.9).

Let us suppose that (3.8) and (3.10) hold and let us show that u_h is unique. If u_h^* and u_h^{**} are two solutions of (3.9) and if $u_h = u_h^* - u_h^{**}$, then

$$a_h(u_h, v_h) + b_h(u_h^*, u_h^*, v_h) - b_h(u_h^{**}, u_h^{**}, v_h) = 0, \ \forall v_h \in V_h;$$

taking $v_h = u_h$ and using (3.6) we find

$$a_h(u_h, u_h) = b_h(u_h, u_h^*, u_h).$$

Because of (3.2) and (3.8), we have

$$\alpha_0 \|u_h\|_h^2 \leqslant c \|u_h^*\|_h \ \|u_h\|_h^2 . \tag{3.13}$$

If we set $v_h = u_h^*$ in the equation (3.9) satisfied by u_h^*, we find

$$a_h(u_h^*, u_h^*) = \langle \ell_h, u_h^* \rangle ,$$

and with (3.2) and (3.3),

$$\alpha_0 \|u_h^*\|_h^2 \leqslant \beta \|u_h^*\|_h ,$$

$$\|u_h^*\|_h \leqslant \frac{\beta}{\alpha_0} . \tag{3.14}$$

Using this majoration and (3.13) we obtain,

$$\left(\alpha_0 - \frac{c\beta}{\alpha_0}\right) \|u_h\|_h^2 \leqslant 0; \tag{3.15}$$

if (3.10) holds, this shows that $u_h = 0$.

Theorem 3.1. *We assume that conditions* (3.2) *to* (3.7) *are satisfied;* u_h

is some solution of (3.9).

If $n \leqslant 4$, *the family* $\{p_h \boldsymbol{u}_h\}$ *contains subsequences which are strongly convergent in F. Any such subsequence converges to* $\overline{\omega}\boldsymbol{u}$, *where* $\boldsymbol{u}$ *is some solution of* (3.1). *If the solution of* (3.1) *is unique, the whole family* $\{p_h \boldsymbol{u}_h\}$ *converges to* $\overline{\omega}\boldsymbol{u}$.

If $n \geqslant 5$, *we have the same conclusions, with only weak convergences in F.*

Proof. We suppose that n is arbitrary.

Putting $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (3.9), and using (3.2), (3.3) and (3.6) we find

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) = \langle \ell_h, \boldsymbol{u}_h \rangle,$$

$$\alpha_0 \|\boldsymbol{u}_h\|_h \leqslant \beta. \tag{3.16}$$

Since the p_h are stable, the family $p_h \boldsymbol{u}_h$ is bounded in F; therefore there exists some subsequence $h' \to 0$, and some $\phi \in F$ such that

$$p_{h'}\boldsymbol{u}_{h'} \to \phi \text{ in } F \text{ weakly.}$$

The condition (C2) for the approximation of a space shows that, necessarily $\phi \in \overline{\omega} V$, or $\phi = \overline{\omega}\boldsymbol{u}$, $\boldsymbol{u} \in V$:

$$p_{h'}\boldsymbol{u}_{h'} \to \overline{\omega}\boldsymbol{u} \text{ in } F \text{ weakly, } h' \to 0. \tag{3.17}$$

Let $\boldsymbol{v}$ be an element of V and let us write (3.9) with $\boldsymbol{v}_h = r_h \boldsymbol{v}$:

$$a_h(\boldsymbol{u}_h, r_h\boldsymbol{v}) + b_h(\boldsymbol{u}_h, \boldsymbol{u}_h, r_h\boldsymbol{v}) = \langle \ell_h, r_h\boldsymbol{v} \rangle. \tag{3.18}$$

As $h' \to 0$, according to (3.4), (3.5), (3.7),

$$a_{h'}(\boldsymbol{u}_{h'}, r_{h'}\boldsymbol{v}) \to \nu((\boldsymbol{u}, \boldsymbol{v})),$$

$$b_{h'}(\boldsymbol{u}_{h'}, \boldsymbol{u}_{h'}, r_{h'}\boldsymbol{v}) \to b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}),$$

$$\langle \ell_{h'}, r_{h'}\boldsymbol{v} \rangle \to (\boldsymbol{f}, \boldsymbol{v}).$$

Hence $\boldsymbol{u}$ belongs to V and satisfies

$$\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \ \forall \boldsymbol{v} \in \mathscr{V}. \tag{3.19}$$

If $n \leqslant 4$, equation (3.19) holds, by continuity, for each $\boldsymbol{v} \in V$; if $n \geqslant 5$, we find, by continuity, that (3.19) is satisfied for each $\boldsymbol{v} \in \widetilde{V}$. In both cases, $\boldsymbol{u}$ is a solution of the stationary Navier–Stokes equations.

It can be proved by exactly the same method, that any convergent subsequence of $p_h \boldsymbol{u}_h$ converges to $\overline{\omega}\boldsymbol{u}$, where $\boldsymbol{u}$ is some solution of (3.1). If this solution is unique, the whole family $p_h \boldsymbol{u}_h$ converges to $\overline{\omega}\boldsymbol{u}$ in F weakly.

Let us show the strong convergence when $n \leqslant 4$.

As in the linear case (see Theorem I.3.1) we shall consider the expression

$$X_h = a_h(u_h - r_h u, u_h - r_h u).$$

Expanding this expression and using (3.9) with $v_h = u_h$ we find

$$X_h = \langle \ell_h, u_h \rangle - a_h(u_h, r_h u) - a_h(r_h u, u_h) + a(r_h u, r_h u).$$

Because of (2.4), (3.5), and (3.17),

$$X_{h'} \to \langle f, u \rangle - a(u, u), \quad \text{as } h' \to 0.$$

We take $v = u$ in (3.1) and use (1.22); this gives

$$\langle f, u \rangle = a(u, u),^{(1)}$$

hence

$$X_{h'} \to 0 \text{ as } h' \to 0. \tag{3.20}$$

We now finish the proof as in the linear case; the inequality (3.2) shows that

$$\|u_{h'} - r_{h'} u\|_{h'} \to 0.$$

Since the p_h are stable, this implies

$$\|p_{h'}u_{h'} - p_{h'}r_{h'}u\|_F \leqslant \|p_{h'}\|_{\mathcal{L}(V_{h'}, F)} \|u_{h'} - r_{h'}u\|_{h'} \to 0.$$

Now we write

$$\|p_{h'}u_{h'} - \bar{\omega}u\|_F \leqslant \|p_{h'}u_{h'} - p_{h'}r_{h'}u\|_F + \|p_{h'}r_{h'}u - \bar{\omega}u\|_F,$$

and the two terms on the right-hand side of this inequality converge to 0 as $h' \to 0$.(2)

---

(1) This equation does not hold for $n \geqslant 5$, and this is the reason the proof cannot be extended to these cases.

(2) We recall that for each $u \in V$, and each $h \in \mathcal{H}$, there exists $r_h u \in V_h$, such that

$$p_h r_h u \to \bar{\omega} u, \quad \text{in } F \text{ strongly}, \quad \text{as } h \to 0$$

(see Proposition 3.1).

3.2. *Applications*

In this section we apply the general convergence theorem to approximation schemes corresponding to the approximations (APX1), . . ., (APX5) of V.

Approximation (APX1). We choose a_h and ℓ_h as in the linear case (see I. (3.62) and (3.63)),

$$a_h(u_h, v_h) = \nu((u_h, v_h))_h \tag{3.21}$$

$$\langle \ell_h, v_h \rangle = (f, v_h). \tag{3.22}$$

Before defining b_h, we introduce the trilinear form $\hat{b}(u, v, w)$,

$$\hat{b}(u, v, w) = b'(u, v, w) + b''(u, v, w) \tag{3.23}$$

$$b'(u, v, w) = \frac{1}{2} \sum_{i,j=1}^{n} \int_\Omega u_i(D_i v_j) w_j \mathrm{d}x \tag{3.24}$$

$$b''(u, v, w) = -\frac{1}{2} \sum_{i,j=1}^{n} \int_\Omega u_i v_j (D_i w_j) \mathrm{d}x. \tag{3.25}$$

It is not difficult to see that

$$\hat{b}(u, u, v) = b(u, u, v), \ \forall u \in V, \ \forall v \in \mathcal{V}; \tag{3.26}$$

but $\hat{b}$ and b are otherwise different.

We now define b_h as,

$$b_h(u_h, v_h, w_h) = b'_h(u_h, v_h, w_h) + b''_h(u_h, v_h, w_h) \tag{3.27}$$

$$b'_h(u_h, v_h, w_h) = \frac{1}{2} \sum_{i,j=1}^{n} \int_\Omega u_{ih}(\delta_{ih} v_{jh}) w_{jh} \mathrm{d}x \tag{3.28}$$

$$b''_h(u_h, v_h, w_h) = -\frac{1}{2} \sum_{i,j=1}^{n} \int_\Omega u_{ih} v_{jh}(\delta_{ih} w_{jh}) \mathrm{d}x. \tag{3.29}$$

It is clear that b'_h, b''_h, and hence b_h are trilinear forms on V_h; since

V_h has a finite dimension, these forms are continuous.

We have to check (3.6) and (3.7); (3.6) is obvious with our choice of the form b_h, and (3.7) is the purpose of the next lemma.

Lemma 3.1. *If $p_h u_h$ converges to $\overline{\omega} u$, then*

$$b_h(u_h, u_h, r_h v) \to b(u, u, v),\ \forall v \in \mathscr{V}. \tag{3.30}$$

Proof. Saying that $p_h u_h$ converges weakly to $\overline{\omega} u$ means that

$$u_h \to u \text{ weakly in } L^2(\Omega) \tag{3.31}$$

and

$$\delta_{ih}\, u_h \to D_i u \text{ weakly in } L^2(\Omega),\ 1 \leqslant i \leqslant n. \tag{3.32}$$

The Compactness Theorem 2.2 is applicable and shows that

$$u_h \to u \text{ strongly in } L^2(\Omega). \tag{3.33}$$

We know that if $v \in \mathscr{V}$, $p_h r_h v$ converges to $\overline{\omega} v$ in F strongly; but the proofs of Lemma I.3.1 and of Proposition I.3.5 show that actually

$$r_h v \to v \text{ in the norm of } L^\infty(\Omega), \tag{3.34}$$

$$\delta_{ih}\, r_h\, v \to D_i v \text{ in the norm of } L^\infty(\Omega). \tag{3.35}$$

If we prove that

$$b_h'(u_h, u_h, r_h v) \to b'(u, u, v) \tag{3.36}$$

$$b_h''(u_h, u_h, r_h v) \to b''(u, u, v), \tag{3.37}$$

then, according to (3.27), the proof of (3.7) will be complete.

For (3.36) we write

$$|b_h'(u_h, u_h, r_h v) - b'(u, u, v)|$$

$$\leqslant c_0 \sum_{i,\,j=1}^{n} \left| \int_\Omega (u_{ih} v_{jh} - u_i v_j)\, \delta_{ih} u_{jh}\, \mathrm{d}x \right| +$$

$$c_0 \sum_{i,j=1}^{n} \left| \int_\Omega u_i v_j\, (\delta_{ih} u_{jh} - D_i u_j) \mathrm{d}x \right|.$$

All the preceding integrals converge to 0, and (3.36) is proved; the proof of (3.37) is similar. □

The convergence result given by Theorem 3.1 is as follows ($n \leqslant 4$):

$$u_{h'} \to u \text{ in } L^2(\Omega) \text{ strongly,} \tag{3.38}$$

$$\delta_{ih'} u_{h'} \to D_i u \text{ in } L^2(\Omega) \text{ strongly, } 1 \leqslant i \leqslant n. \tag{3.39}$$

Exactly as in the linear case it can be shown that there exists some step function

$$\pi_h = \sum_{M \in \mathring{\Omega}_h^1} \pi_h(M) w_{hM} \tag{3.40}$$

such that

$$\nu((u_h, v_h))_h + (\overline{\nabla}_h \pi_h, v_h) = (f, v_h), \ \forall v_h \in W_h ; \tag{3.41}$$

therefore a solution u_h of (3.9) is a step function

$$u_h = \sum_{M \in \mathring{\Omega}_h^1} u_h(M) w_{hM} \tag{3.42}$$

such that

$$\sum_{i=1}^{n} (\nabla_{ih} u_{ih})(M) = 0, \ \forall M \in \mathring{\Omega}_h^1 \tag{3.43}$$

and

$$-\nu \sum_{i=1}^{n} \delta_{ih}^2 u_h(M) + \frac{1}{2} \sum_{i=1}^{n} u_{ih}(M) \delta_{ih} u_h(M) \tag{3.44}$$

$$-\frac{1}{2} \sum_{i=1}^{n} \delta_{ih}(u_{ih} u_h)(M) + \overline{\nabla}_h \pi_h(M) = f_h(M), \ \forall M \in \mathring{\Omega}_h^1$$

where

$$f_h(M) = \frac{1}{h_1 \ldots h_n} \int_{\sigma_h(M)} f(x) \mathrm{d}x. \tag{3.45}$$

When condition (3.8) and some condition similar to (3.10) are satisfied, u_h and u are unique and the error between u and u_h can be estimated as in the linear case, if, moreover, $u \in \mathscr{C}^3(\bar{\Omega})$ and $p \in \mathscr{C}^2(\bar{\Omega})$.

Using the Taylor formula we can write,

$$-\nu \sum_{i=1}^{n} (\delta_{ih} r_h u)(M) + \frac{1}{2} \sum_{i=1}^{n} (r_h u)_i(M) \delta_{ih}(r_h u)(M)$$

$$-\frac{1}{2} \sum_{i=1}^{n} \delta_{ih}((r_h u)_i r_h u)(M) + (\bar{\nabla}_h p)(M)$$
$$= f(M) + \epsilon_h(M), \ \forall M \in \mathring{\Omega}_h^1, \tag{3.46}$$

with

$$|\epsilon_h(M)| \leqslant c(u, p)\,|h|, \tag{3.47}$$

where $c(u, p)$ depends only on the maximum norms of the second and third derivatives of u, and of the second derivatives of p. Equations (3.46) show that

$$\nu((r_h u, v_h))_h + b_h(r_h u, r_h u, v_h) - (\bar{\nabla}_h \pi'_h, v_h)$$
$$= (f + \epsilon_h, v_h), \tag{3.48}$$

for each $v_h \in W_h$ where π'_h is the step function

$$\pi'_h = \sum_{M \in \mathring{\Omega}_h^1} p(M) w_{hM}. \tag{3.49}$$

Subtracting (3.48) from (3.9) gives

$$\nu((u_h - r_h u, v_h))_h = -b_h(u_h, u_h, v_h) + b_h(r_h u, r_h u, v_h)$$
$$+ (\epsilon_h, v_h), \ \forall v_h \in W_h.$$

We take $v_h = u_h - r_h u$ and use (3.6) to get:

$$\nu \|u_h - r_h u\|_h^2 = -b_h(u_h - r_h u, u_h, u_h - r_h u)$$
$$+ (\epsilon_h, u_h - r_h u),$$

and with (3.8) and (3.47),

$$\nu \|u_h - r_h u\|_h^2 \leqslant c(n, \Omega) \|u_h\|_h \, \|u_h - r_h u\|_h^2$$
$$+ c(u, p) \, \|u_h - r_h u\|_h \, |h|,$$
$$\nu \|u_h - r_h u\|_h \leqslant c(n, \Omega) \|u_h\|_h \, \|u_h - r_h u\|_h + c(u, p)\,|h|$$

$$\leqslant \text{(by (3.16))}$$

$$\leqslant \frac{\beta}{\alpha_0} c(n, \Omega) \|u_h - r_h u\|_h + c(u, p) |h|.$$

Finally

$$\left(\nu - \frac{\beta}{\alpha_0} c(n, \Omega) \right) \|u_h - r_h u\|_h \leqslant c(u, p) |h|. \tag{3.50}$$

If

$$\nu\alpha_0 > \beta c(n, \Omega), \tag{3.51}$$

(the constant $c(n, \Omega)$ in (3.8)), this gives the following majoration of the error

$$\|u_h - r_h u\|_h \leqslant \frac{c(u, p)}{\left[\nu - \dfrac{\beta}{\alpha_0} c(n, \Omega) \right]} |h|. \quad \square \tag{3.52}$$

Approximation (APX2). In the two-dimensional case we can associate to the approximation (APX2) of V described in Section I.4.2, a new discretization scheme for the Navier–Stokes equations.

We recall that for this approximation V_h is a subspace of $H_0^1(\Omega)$, and we take, as in the linear case,

$$a_h(u_h, v_h) = \nu((u_h, v_h)), \tag{3.53}$$

$$\langle \ell_h, v_h \rangle = (f, v_h), \tag{3.54}$$

where $((\cdot,\cdot))$ is the scalar product in V_h and in $H_0^1(\Omega)$.

We define the form b_h by

$$b_h(u_h, v_h, w_h) = \hat{b}(u_h, v_h, w_h),\ \forall\, u_h, v_h, w_h \in V_h, \tag{3.55}$$

where $\hat{b}$ is defined by (3.23) – (3.25); since V_h is a space of bounded vector functions, the forms $\hat{b}$ are defined on V_h; they are trilinear, and hence continuous.

Condition (3.6) is obviously satisfied by our choice of b_h; condition (3.7) is the purpose of the next lemma.

Lemma 3.2. *If $p_h u_h = u_h$ converges to $\bar{\omega} u$, then*

$$b_h(u_h, u_h, r_h v) \to b(u, u, v),\ \forall\, v \in \mathscr{V}. \tag{3.56}$$

Proof. We recall that $F = H_0^1(\Omega)$ and $\bar{\omega}$ and p_h are the identity. Saying that $p_h \boldsymbol{u}_h$ converges weakly in F to $\bar{\omega}\boldsymbol{u}$, amounts to saying that

$$\boldsymbol{u}_h \to \boldsymbol{u} \text{ weakly in } \boldsymbol{H}_0^1(\Omega). \tag{3.57}$$

The Compactness Theorem 1.1 then shows that

$$\boldsymbol{u}_h \to \boldsymbol{u} \text{ strongly in } \boldsymbol{L}^2(\Omega). \tag{3.58}$$

The proof of Lemma 3.2 will be the same as that of Lemma 3.1, if we observe that

$$r_h \boldsymbol{v} \to \boldsymbol{v} \text{ in the norm of } \boldsymbol{L}^\infty(\Omega) \tag{3.59}$$

$$D_i r_h \boldsymbol{v} \to D_i \boldsymbol{v} \text{ in the norm of } \boldsymbol{L}^\infty(\Omega),\ 1 \leqslant i \leqslant n, \tag{3.60}$$

which was actually proved in Proposition I.4.3 (see (4.63)). □

The weak (or strong) convergence result given by Theorem 3.1 is the following one:

If $\rho(h) \to 0$, with $\sigma(h) \leqslant \alpha$ (i.e. $h \in \boldsymbol{H}_\alpha$), then

$$\boldsymbol{u}_{h'} \to \boldsymbol{u} \textit{ in } \boldsymbol{H}_0^1(\Omega) \textit{ weakly (or strongly)}. \tag{3.61}$$

Exactly as in the linear case (see Section I.4.2), we can show that there exists a step function π_h,

$$\pi_h = \sum_{\mathscr{S} \in \mathscr{T}_h} \pi_h(\mathscr{S}) \chi_{h\mathscr{S}}, \tag{3.62}$$

(χ_h = the characteristic function of the simplex $\mathscr{S}$), such that

$$\begin{aligned} \nu((\boldsymbol{u}_h, \boldsymbol{v}_h)) + b_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - (\pi_h, \operatorname{div} \boldsymbol{v}_h) \\ = (\boldsymbol{f}, \boldsymbol{v}_h),\ \forall \boldsymbol{v}_h \in W_h. \end{aligned} \tag{3.63}$$

This equation is the discrete analogue of

$$\begin{aligned} &\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \\ &\forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^n(\Omega). \end{aligned} \tag{3.64}$$

Since the dimension of the spaces is $n = 2$, the form $\hat{b}$ is trilinear continuous on $\boldsymbol{H}_0^1(\Omega)$ and there exists some constant $\hat{c}$ such that

$$|\hat{b}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant \hat{c} \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|,\ \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega). \tag{3.65}$$

This can be shown by the same method as (1.18); hence (3.8) holds with

$$c(n, \Omega) = \hat{c}\,.$$

We will get an estimation of the error assuming that $u \in \mathscr{C}^3(\bar{\Omega})$, $p \in \mathscr{C}^2(\bar{\Omega})$, and a hypothesis similar to (3.51). We take $v_h = r_h u - u_h$ in (3.63) and $v = r_h u - u_h$ in (3.64); subtracting these equations we find

$$\nu((u - u_h, r_h u - u_h)) + \hat{b}(u, u, r_h u - u_h) - \hat{b}(u_h, u_h, r_h u - u_h)$$
$$- (p - \pi_h, \operatorname{div}(r_h u - u_h)) = 0. \tag{3.66}$$

Since $\operatorname{div}(r_h u - u_h)$ is a step function which is constant on each simplex $\mathscr{S} \in \mathscr{T}_h$, we have

$$(p - \pi_h, \operatorname{div}(r_h u - u_h)) = (p - \pi'_h, \operatorname{div}(r_h u - u_h))$$

where π'_h is defined by

$$\pi'_h = \sum_{\mathscr{S} \in \mathscr{T}_h} \frac{1}{(\operatorname{meas} \mathscr{S})} \left(\int_{\mathscr{S}} p(x)\mathrm{d}x \right) \chi_{h_{\mathscr{S}}}. \tag{3.67}$$

Hence

$$|(p - \pi_h, \operatorname{div}(r_h u - u_h)| \leqslant |\pi'_h - p|\, |\operatorname{div}(r_h u - u_h)|$$
$$\leqslant |\pi'_h - p|\, \|u_h - r_h u\|. \tag{3.68}$$

We then estimate the difference

$$\hat{b}(u_h, u_h, r_h u - u) - \hat{b}(u, u, r_h u - u)$$
$$= \hat{b}(u_h - r_h u, u_h, r_h u - u_h) + \hat{b}(r_h u, u_h, r_h u - u_h)$$
$$- \hat{b}(u, u, r_h u - u) = \hat{b}(u_h - r_h u, u_h, r_h u - u_h)$$
$$+ \hat{b}(r_h u, r_h u - u, r_h u - u_h) + \hat{b}(r_h u - u, u, r_h u - u_h)$$

The absolute value of this sum can be majorized because of (3.65) by

$$\hat{c}\|u_h\|\, \|r_h u - u_h\|^2 + \hat{c}\{\|r_h u\| + \|u\|\}\, \|r_h u - u\| \cdot$$
$$\|r_h u - u_h\|\,. \tag{3.69}$$

We recall that

$$\nu \|u_h\|^2 = \langle f, u_h \rangle$$

and therefore

$$\|u_h\| \leqslant \frac{1}{\nu}\, |f|\,.$$

Hence the sum (3.69) is less than or equal to

$$\frac{\hat{c}}{\nu}\,|f|\;\|u_h - r_h u\|^2 + \hat{c}\{\|r_h u\| + \|u\|\}\;\|r_h u - u\|\cdot$$

$$\|r_h u - u_h\|\,. \tag{3.70}$$

With the majorations (3.68) and (3.70), we get from (3.66)

$$\left(\nu - \frac{\hat{c}}{\nu}|f|\right)\|u_h - r_h u\|^2 \leqslant \nu\|u_h - r_h u\|\;\|u - r_h u\|$$

$$+\,|\pi_h' - p|\;\|u_h - r_h u\| + \hat{c}\;\{\|r_h u\| + \|u\|\}\,\|u - r_h u\|\,,$$

and finally

$$\left(\nu - \frac{\hat{c}}{\nu}|f|\right)\|u_h - r_h u\| \leqslant \nu\|u - r_h u\|$$

$$+\,|\pi_h' - p| + \hat{c}\;\{\|r_h u\| + \|u\|\}\;\|u - r_h u\|\,. \tag{3.71}$$

Under the assumption

$$\nu^2 > \hat{c}\,|f|\,, \tag{3.72}$$

the inequality (3.71) gives a majoration of the error between u_h and $r_h u$ and hence between u and u_h.

Approximation (APX3). As in the linear case, the method here is very similar to the method used for the approximation (APX2).

Approximation (APX4). We recall that Ω must be a simply connected open set in $\mathbb{R}^2$.

Since (APX4) is an internal approximation of V, the simplest scheme (3.9) associated with this approximation is

To find $u_h \in V_h$ such that

$$\nu((u_h, v_h)) + b(u_h, u_h, v_h) = \langle f, v_h\rangle,\ \forall v_h \in V_h\,. \tag{3.73}$$

Theorem 3.1 is applicable and shows that

$$u_h \to u \text{ in } V, \text{ as } \rho(h) \to 0, \tag{3.74}$$

provided $\sigma(h) \leqslant \alpha$ (*i.e.* $h \in \mathcal{H}_\alpha$).

The error between u and u_h can be estimated as follows (if we have uniqueness): we take $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_h$ in the variational equation satisfied by $\boldsymbol{u}$,

$$\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \ \forall \boldsymbol{v} \in V.$$

We then take $\boldsymbol{v} = r_h \boldsymbol{u} - \boldsymbol{u}_h$ in (3.73); we subtract these equations and find

$$\nu \|\boldsymbol{u}_h - \boldsymbol{u}\|^2 = \nu((\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - r_h \boldsymbol{u})) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, r_h \boldsymbol{u} - \boldsymbol{u}_h) - b(\boldsymbol{u}, \boldsymbol{u}, r_h \boldsymbol{u} - \boldsymbol{u}_h) . \tag{3.75}$$

The difference

$$b(\boldsymbol{u}_h, \boldsymbol{u}_h, r_h \boldsymbol{u} - \boldsymbol{u}_h) - b(\boldsymbol{u}, \boldsymbol{u}, r_h \boldsymbol{u} - \boldsymbol{u}_h)$$

is equal to

$$\begin{aligned} b(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{u}_h, r_h \boldsymbol{u} - \boldsymbol{u}_h) &+ b(\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}, r_h \boldsymbol{u} - \boldsymbol{u}_h) \\ &= b(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{u}_h, r_h \boldsymbol{u} - \boldsymbol{u}) + b(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) \\ &+ b(\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}, r_h \boldsymbol{u} - \boldsymbol{u}) = b(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{u}_h, r_h \boldsymbol{u} - \boldsymbol{u}) \\ &+ b(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h) + b(\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}, r_h \boldsymbol{u} - \boldsymbol{u}) . \end{aligned}$$

Because of (1.18) this is majorized by

$$c\{\|\boldsymbol{u}\| + \|\boldsymbol{u}_h\|\} \ \|\boldsymbol{u}_h - \boldsymbol{u}\| \ \|r_h \boldsymbol{u} - \boldsymbol{u}\| + c\|\boldsymbol{u}\| \ \|\boldsymbol{u} - \boldsymbol{u}_h\|^2.$$

We recall that

$$\nu \|\boldsymbol{u}\|^2 = \langle \boldsymbol{f}, \boldsymbol{u} \rangle ,$$

$$\|\boldsymbol{u}\| \leqslant \frac{1}{\nu} \ \|\boldsymbol{f}\|_{V'} ,$$

and similarly

$$\nu \|\boldsymbol{u}_h\|^2 = \langle \boldsymbol{f}, \boldsymbol{u}_h \rangle$$

$$\|\boldsymbol{u}_h\| \leqslant \frac{1}{\nu} \ \|\boldsymbol{f}\|_{V'} .$$

Therefore the last expression is majorized by

$$\frac{2c}{\nu} \ |\boldsymbol{f}| \ \|\boldsymbol{u}_h - \boldsymbol{u}\| \ \|r_h \boldsymbol{u} - \boldsymbol{u}\| + \frac{c}{\nu} \ |\boldsymbol{f}| \ \|\boldsymbol{u} - \boldsymbol{u}_h\|^2.$$

We deduce then from (3.75)

$$\left(\nu - \frac{c}{\nu}|f|\right)\|u_h - u\|^2 \leqslant \left(\nu + \frac{2c}{\nu}|f|\right)\|u_h - u\|\;\|r_h u - u\|,$$

$$\left(\nu - \frac{c}{\nu}|f|\right)\|u_h - u\| \leqslant \left(\nu + \frac{2c}{\nu}|f|\right)\|r_h u - u\|. \tag{3.76}$$

If

$$\nu^2 > c\;\|f\|_{V'}, \tag{3.77}$$

inequality (3.76) shows that the error $\|u_h - u\|$ has the same order as $\|r_h u - u\|$. Since c is the constant $c(n)$, $n = 2$, in (1.18), the inequality (3.77) is exactly inequality (1.37) which ensures the uniqueness of the solution u of the exact problem.

Approximation (APX5). If we consider the approximation (APX5) of V, we can set

$$a_h(u_h, v_h) = \nu((u_h, v_h))_h \tag{3.78}$$

$$\langle \ell_h, v_h \rangle = (f, v_h) \tag{3.79}$$

$$b_h(u_h, v_h, w_h) = b'_h(u_h, v_h, w_h) + b''_h(u_h, v_h, w_h), \tag{3.80}$$

$$b'_h(u_h, v_h, w_h) = \frac{1}{2} \sum_{i,j=1}^{n} \int_\Omega u_{ih}(D_{ih} v_{jh}) w_{jh}\, dx \tag{3.81}$$

$$b''_h(u_h, v_h, w_h) = -\frac{1}{2} \sum_{i,j=1}^{n} \int_\Omega u_{ih}\, v_{jh}\,(D_{ih}\, w_{jh})\, dx. \tag{3.82}$$

It is clear that b'_h, b''_h and b_h are trilinear forms on V_h and since V_h has a finite dimension, these forms are continuous.

We have to check (3.6) and (3.7); (3.6) is obvious with our choice of the form b_h, and (3.7) is the purpose of next lemma.

Lemma 3.3. *Assume that $n \leqslant 3$. If $p_h u_h$ converges weakly to $\overline{\omega} u$, then*

$$b_h(u_h, u_h, r_h v) \to b(u, u, v),\;\; \forall v \in \mathscr{V}. \tag{3.83}$$

Proof. By definition, we are assuming that

$$u_h \to u \text{ in } L^2(\Omega) \text{ weakly,} \tag{3.84}$$

and

$$D_{ih}\, u_h \to D_i u \text{ in } L^2(\Omega) \text{ weakly, } 1 \leqslant i \leqslant n. \tag{3.85}$$

The Compactness Theorem 2.4 shows that

$$u_h \to u \text{ in } L^2(\Omega) \text{ strongly.} \tag{3.86}$$

We know that if $v \in \mathscr{V}$, $p_h\, r_h v$ converges to $\overline{\omega} v$ in F strongly; but the proofs of Propositions I.4.12 and I.4.15 show that furthermore

$$r_h\, v \to v \text{ in the norm of } L^\infty(\Omega), \tag{3.87}$$

$$D_{ih}\, r_h\, v \to D_i v \text{ in the norm of } L^\infty(\Omega). \tag{3.88}$$

The proof of (3.7) will be complete if we prove that

$$b'_h(u_h, u_h, r_h v) \to b'(u, u, v), \tag{3.89}$$

$$b''_h(u_h, u_h, r_h v) \to b''(u, u, v)\,. \tag{3.90}$$

For (3.89) we write

$$\left| b'_h(u_h, u_h, r_h v) - b'(u, u, v) \right| \leqslant$$

$$c_0 \sum_{i,j=1}^{n} \left| \int_\Omega (u_{ih}\, v_{jh} - u_i\, v_j) D_{ih}\, u_{jh}\, \mathrm{d}x \right| +$$

$$c_0 \sum_{i,j=1}^{n} \left| \int_\Omega u_i\, v_j (D_{ih}\, u_{jh} - D_i\, u_j)\, \mathrm{d}x \right|.$$

All the preceding integrals converge to 0 and (3.89) follows. The proof of (3.90) is similar. $\square$

The convergence result given by Theorem 3.1 states that

$$u_{h'} \to u \text{ in } L^2(\Omega) \text{ strongly,} \tag{3.91}$$

$$D_{ih'}\, u_{h'} \to D_i u \text{ in } L^2(\Omega) \text{ strongly, } 1 \leqslant i \leqslant n \tag{3.92}$$

(we recall that $n = 2$ or 3 only).

Exactly as in the linear case it can be shown that there exists some step function π_h constant on each $\mathscr{S}$, $\mathscr{S} \in \mathscr{T}_h$, and vanishing outside $\Omega(h)$ such that

$$\nu((u_h, v_h))_h + b_h(u_h, u_h, v_h) - (\pi_h, \operatorname{div}_h v_h) = (f, v_h), \ \forall v_h \in W_h\,. \tag{3.93}$$

When condition (3.8) and some condition similar to (3.10) are satisfied, u_h and u are unique and the error between u and u_h can be estimated as in the linear case, if moreover $u \in \mathscr{C}^3(\bar{\Omega})$ and $p \in \mathscr{C}^2(\bar{\Omega})$.

Lemma 3.4. *Let u and p denote the exact solution of* (1.8)–(1.10) *and let us suppose that $u \in \mathscr{C}^3(\bar{\Omega})$, $p \in \mathscr{C}^2(\bar{\Omega})$, and that $\Omega = \Omega(h)$. Then*

$$\nu((u, v_h))_h + b_h(u, u, v_h) - (p, \operatorname{div} v_h) = (f, v_h) + \ell_h(v_h), \tag{3.94}$$

where

$$|\ell_h(v_h)| \leqslant c(u, p)\, \rho(h)\, \|v_h\|_h\,. \tag{3.95}$$

Proof. We take the scalar product in $L^2(\Omega)$ of $v_h \in W_h$ with the equation (1.8) written in the form

$$-\nu\Delta u + \frac{1}{2}\sum_{i=1}^{n} (u_i D_i u - D_i(u_i u)) + \operatorname{grad} p = f\,.$$

Since $\Omega = \Omega(h)$, we find

$$\sum_{\mathscr{S}} \left\{ -\nu(\Delta u, v_h)_{\mathscr{S}} + \frac{1}{2}(u_i D_i u - D_i(u_i u), v_h)_{\mathscr{S}} + (\operatorname{grad}_p, v_h)_{\mathscr{S}} - (f, v_h)_{\mathscr{S}} \right\} = 0.$$

The Green formula applied several times in each simplex $\mathscr{S}$ shows that the left–hand side of this relation is equal to

$$\{\nu((u, v_h))_h + \hat{b}(u, u, v_h) - (p, \operatorname{div}_h v_h) - (f, v_h) - \ell_h(v_h)\} = 0$$

where

$$\ell_h(v_h) = \sum_{\mathscr{S} \in \mathscr{T}_h} \left\{ \int_{\partial \mathscr{S}} \left(\nu \frac{\partial u}{\partial \vec{\nu}} \cdot v_h - \frac{1}{2}(u \cdot \vec{\nu})(u \cdot v_h) + \frac{\partial p}{\partial \vec{\nu}} v_h \right) d\Gamma \right\}.$$

The estimation (3.95) of $\ell_h(v_h)$ follows easily from Proposition I.4.16. $\square$

We now proceed as for (3.71). We take $v_h = u_h - r_h u$ in (3.93) and (3.94) and subtract these relations. We get

$$\begin{aligned} \nu((u - u_h, u_h - r_h u))_h &+ b_h(u, u, u_h - r_h u) \\ &- b_h(u_h, u_h, u_h - r_h u) - (p - \pi_h, \operatorname{div}_h(r_h u - u_h)) \\ &= \ell_h(v_h)\,. \end{aligned} \tag{3.96}$$

Since $\operatorname{div}_h(r_h u - u_h)$ vanishes, the corresponding term disappears. We then estimate the difference

$$\begin{aligned} b_h(u_h, u_h, u_h - r_h u) - b_h(u, u, u_h - r_h u) \\ &= b_h(u_h - r_h u, u_h, u_h - r_h u) + b_h(r_h u, u_h, u_h - r_h u) \\ &- b_h(u, u, u_h - r_h u) = \\ &= b_h(u_h - r_h u, u_h, u_h - r_h u) \\ &+ b_h(r_h u, r_h u - u, u_h - r_h u) + b_h(r_h u - u, u, u_h - r_h u] \end{aligned}$$

The absolute value of this sum can be majorized, because of (1.18) and (3.65), by

$$\begin{aligned} c\|u_h\|\ \|u_h - r_h u\|^2 + c\{\|r_h u\|_h + \|u\|\}\ \|r_h u - u\|_h \cdot \\ \|u_h - r_h u\|_h\,. \end{aligned} \tag{3.97}$$

We recall that

$$\nu\|u_h\|_h^2 = \langle f, u_h\rangle$$

and therefore

$$\|u_h\|_h \leqslant \frac{1}{\nu}\,|f|\,.$$

Hence the sum (3.97) is less or equal to

$$\begin{aligned} \frac{c}{\nu}\,|f|\ \|u_h - r_h u\|_h^2 + c\{\|r_h u\|_h + \|u\|\}\ \|r_h u - u\|_h \cdot \\ \|r_h u - u_h\|_h \end{aligned} \tag{3.98}$$

With this majoration and (3.95) we get from (3.96)

$$\begin{aligned} \left(\nu - \frac{c}{\nu}\,|f|\right)\|u_h - r_h u\|_h^2 &\leqslant \nu\|u_h - r_h u\|_h\ \|u - r_h u\|_h \\ &+ c\{\|r_h u\|_h + \|u\|\}\ \|u_h - r_h u\|_h\ \|u - r_h u\|_h \\ &+ c(u, p)\,\rho(h)\ \|u_h - r_h u\|_h\,. \end{aligned} \tag{3.99}$$

Finally

$$\left(\nu - \frac{c}{\nu} |f| \right) \|u_h - r_h u\|_h \leqslant \nu \|u - r_h u\|_h$$

$$+ c\{\|r_h u\|_h + \|u\|\} \|u - r_h u\|_h + c(u, p)\, \rho(h)\,. \tag{3.100}$$

With the assumption

$$\nu^2 > c|f|\,, \tag{3.101}$$

the inequality (3.100) gives a majoration of the error between u_h and $r_h u$ and hence between u and u_h.

3.3. *Numerical Algorithms.*

The following analysis is restricted to the dimensions $n \leqslant 4$. We wish to extend to the nonlinear case the numerical algorithms described, for the linear case, in Section 5, Chapter I.

We observe that the stationary Navier–Stokes equations are not the Euler equations of an optimization problem like the Stokes equations. The following algorithms then are some extension of the Uzawa and Arrow–Hurwicz algorithms classically related to optimization problems.

In the remainder of this section we will always use the trilinear form $\hat{b}(u, v, w)$ defined by (3.23)–(3.25). This is a trilinear continuous form on $H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ and there exists some constant $\hat{c} = \hat{c}(n)$ such that

$$|\hat{b}(u, v, w)| \leqslant \hat{c}(n) \|u\| \|v\| \|w\|, \forall u, v, w \in H_0^1(\Omega),$$

$$(n \leqslant 4)\,. \tag{3.102}$$

We have already noticed that

$$\hat{b}(u, v, w) = b(u, v, w) \text{ if } u \in V,\ v, w \in H_0^1(\Omega) \tag{3.103}$$

$$\hat{b}(u, v, v) = 0,\ u, v \in H_0^1(\Omega). \tag{3.104}$$

Uzawa Algorithm. In order to approximate the solutions of (1.8)–(1.11) we shall construct, as in Section I.5, two sequences of elements

$$u^m \in H_0^1(\Omega),\ p^m \in L^2(\Omega). \tag{3.105}$$

This construction is relatively easy because the condition div $u = 0$ will disappear in the approximate equations.

We start the algorithm with an arbitrary element p^0:

$$p^0 \in L^2(\Omega)\,. \tag{3.106}$$

When p^m is known ($m \geqslant 0$), we define u^{m+1} as some solution of

$$\begin{aligned} &u^{m+1} \in H_0^1(\Omega), \text{ and} \\ &\nu((u^{m+1}, v)) + \hat{b}(u^{m+1}, u^{m+1}, v) = p^m, \operatorname{div} v) \\ &\qquad\qquad + (f, v),\ \forall\, v \in H_0^1(\Omega). \end{aligned} \tag{3.107}$$

We then define p^{m+1} by

$$\begin{aligned} &p^{m+1} \in L^2(\Omega), \text{ and} \\ &(p^{m+1} - p^m, q) + \rho(\operatorname{div} u^{m+1}, q) = 0,\ \forall\, q \in L^2(\Omega). \end{aligned} \tag{3.108}$$

Later we will give the conditions that the number $\rho > 0$ must satisfy.

The existence of u^{m+1} satisfying (3.107) is not obvious, but can be proved using the Galerkin method, exactly as in Theorem 1.2. Therefore we will skip the proof. It is not difficult to see that u^{m+1} is the solution of the following nonlinear Dirichlet problem:

$$\left.\begin{aligned} &u^{m+1} \in H_0^1(\Omega) \\ &-\nu\Delta u^{m+1} + \sum_{i=1}^{n} u_i^{m+1} D_i u^{m+1} + \frac{1}{2}(\operatorname{div} u^{m+1}) u^{m+1} \\ &= -\operatorname{grad} p^m + f \in H^{-1}(\Omega). \end{aligned}\right\} \tag{3.109}$$

The solution of (3.107)–(3.109) is not, in general, unique. When u^{m+1} is known, p^{m+1} is explicitly given by (3.108) which is equivalent to

$$p^{m+1} = p^m - \rho \operatorname{div} u^{m+1} \in L^2(\Omega)\,. \tag{3.110}$$

To investigate convergence we will assume that

$$\nu - \frac{c(n)}{\nu} \|f\|_{V'} = \bar{\nu} > 0; \tag{3.111}$$

with (3.102), (3.103) and Theorem 1.3, the condition (3.111) implies the uniqueness of the solution of (1.8)–(1.11); p is unique up to an additive constant; we fix this constant by requiring that

$$\int_\Omega p(x)\mathrm{d}x = 0\,. \tag{3.112}$$

Proposition 3.2. *We assume that $n \leqslant 4$ and that condition* (3.00) *holds. We suppose also that the number ρ satisfies*

$$0 < \rho < 2\bar{\nu} \tag{3.113}$$

Then, as $m \to \infty$,

$$u^m \to u \text{ in the norm of } H_0^1(\Omega), \tag{3.114}$$

$$p^m \to p \text{ in } L^2(\Omega), \text{ weakly}, \tag{3.115}$$

where $\{u, p\}$ is the unique solution of (1.8)–(1.11) *which satisfies* (3.112).

Proof. We set

$$\begin{aligned} v^{m+1} &= u^{m+1} - u \\ q^{m+1} &= p^{m+1} - p \end{aligned} \tag{3.116}$$

and we proceed as in the proof of Theorem I.5.1.

We subtract equation (3.107) from the equation

$$\nu((u, v)) + \hat{b}(u, u, v) - (p, \operatorname{div} v) = \langle f, v \rangle, \ \forall\, v \in H_0^1(\Omega) \tag{3.117}$$

and we take $v = v^{m+1}$ to obtain

$$\begin{aligned} \nu \|v^{m+1}\|^2 &= (q^m, \operatorname{div} v^{m+1}) + \hat{b}(u, u, v^{m+1}) \\ &\quad - \hat{b}(u^{m+1}, u^{m+1}, v^{m+1}) = -(q^{m+1} - q^m, \operatorname{div} v^{m+1}) \\ &\quad - (q^{m+1}, \operatorname{div} v^{m+1}) + \hat{b}(v^{m+1}, u, v^{m+1}) \\ &\quad \leqslant (\text{by } (3.102) \text{ and } (1.39)) \\ &\quad \leqslant -(q^{m+1} - q^m, \operatorname{div} v^{m+1}) - (q^{m+1}, \operatorname{div} v^{m+1}) \\ &\quad + \frac{\hat{c}}{\nu} \|f\|_{V'} \|v^{m+1}\|^2 . \end{aligned} \tag{3.118}$$

We take $q = q^{m+1}$ in (3.108) and we find,

$$\begin{aligned} &|q^{m+1}|^2 - |q^m|^2 + |q^{m+1} - q^m|^2 \\ &\quad = -2\rho(\operatorname{div} v^{m+1}, q^{m+1}) . \end{aligned} \tag{3.119}$$

We multiply the last inequality in (3.118) by 2ρ and add it to (3.119); this gives

$$\begin{aligned} &|q^{m+1}|^2 - |q^m|^2 + |q^{m+1} - q^m|^2 \\ &\quad + \left(\nu - \frac{\hat{c}}{\nu} \|f\|_{V'} \right) \|v^{m+1}\|^2 \leqslant \end{aligned}$$

$$\leqslant -2\rho(q^{m+1} - q^m, \operatorname{div} v^{m+1}). \tag{3.120}$$

This inequality is similar to equation (5.12) in the proof of Theorem I.5.1 with v replaced by $\bar{v}$ (see (3.111)). The proof can be completed exactly as in Theorem I.5.1.

Remark 3.1. In the general case, when uniqueness is not assumed, we can prove weak convergence results for the average values

$$\frac{1}{N}\sum_{m=1}^{N} u_m, \quad \frac{1}{N}\sum_{m=1}^{N} p^m .$$

These sequences are bounded in $H_0^1(\Omega)$ and $L^2(\Omega)$, and every weakly convergent subsequence converges to a couple $\{u, p\}$ which is a solution of (1.8)–(1.11).

Arrow–Hurwicz Algorithm. We construct a sequence of couples $\{u^m, p^m\}$ defined as follows.

We start the algorithm with arbitrary elements

$$u^0 \in H_0^1(\Omega),\ p^0 \in L^2(\Omega). \tag{3.121}$$

When p^m, u^m are known, we define p^{m+1}, u^{m+1}, as solutions of

$$\left.\begin{aligned} &u^{m+1} \in H_0^1(\Omega) \text{ and} \\ &((u^{m+1} - u^m, v)) + \rho\nu((u^m, v)) + b(u^m, u^{m+1}, v) \\ &\qquad - \rho(p^m, \operatorname{div} v) = \rho(f, v),\ \forall\, v \in H_0^1(\Omega) \end{aligned}\right\} \tag{3.122}$$

$$\left.\begin{aligned} &p^{m+1} \in L^2(\Omega) \text{ and} \\ &\alpha(p^{m+1} - p^m, q) + \rho(\operatorname{div} u^{m+1}, q) = 0,\ \forall\, q \in L^2(\Omega). \end{aligned}\right\} \tag{3.123}$$

We suppose that ρ and α are two strictly positive numbers; conditions on ρ and α will be given later.

The existence and uniqueness of $u^{m+1} \in H_0^1(\Omega)$ satisfying (3.122) is easy with the projection theorem; (3.122) is a linear variational equation equivalent to the Dirichlet problem

$$u^{m+1} \in H_0^1(\Omega)$$

$$-\Delta u^{m+1} + \rho \sum_{i=1}^{n} u_i^m D_i u^{m+1} + \frac{\rho}{2}(\operatorname{div} u^m) u^{m+1}$$

$$= -\Delta u^m + \rho\nu\Delta u^m + \rho \operatorname{grad} p^m + f. \tag{3.124}$$

Hence p^{m+1} is explicitly given by (3.123) which is equivalent to

$$p^{m+1} = p^m - \frac{\rho}{\alpha} \operatorname{div} u^{m+1} \in L^2(\Omega)\,. \tag{3.125}$$

Convergence can be proved under stronger conditions than those used in Proposition 3.2.

Proposition 3.3. *We assume that* $n \leqslant 4$, *that*

$$\nu - \frac{2\hat{c}}{\nu} \|f\|_{V'} - \frac{4\hat{c}^2}{\nu^2} \|f\|_{V'}^2 = \nu^* > 0 \tag{3.126}$$

and that

$$0 < \rho < \frac{\alpha\nu^*}{2(1+\nu^2\alpha)}\,. \tag{3.127}$$

Then, as $m \to \infty$,

$$u^m \to u \text{ in the norm of } H_0^1(\Omega), \tag{3.128}$$

$$p^m \to p \text{ in } L^2(\Omega) \text{ weakly}, \tag{3.129}$$

where $\{u, p\}$ *is the unique solution of* (1.8)–(1.11) *which satisfies* (3.112).

Proof. We use again the notation (3.116). We take $v = 2v^{m+1}$ in (3.117) and (3.122) and subtract these equations; this gives

$$\begin{aligned}
&\|v^{m+1}\|^2 - \|v^m\|^2 + \|v^{m+1} - v^m\|^2 + 2\rho\nu\|v^{m+1}\|^2 \\
&\quad = 2\rho\nu((v^{m+1}, v^{m+1} - v^m)) + 2\rho(q^m, \operatorname{div} v^{m+1}) \\
&\quad + 2b(u^m, u^{m+1}, v^{m+1}) - 2b(u, u, v^{m+1}) \\
&\quad \leqslant \frac{1}{4} \|v^{m+1} - v^m\|^2 + 4\rho^2\nu^2 \|v^{m+1}\|^2 \\
&\quad + 2\rho(q^m, \operatorname{div} v^{m+1}) + 2b(v^m - v^{m+1}, u, v^{m+1}) \\
&\quad + 2b(v^{m+1}, u, v^{m+1}) \\
&\quad \leqslant \text{(because of (3.102) and (1.39))} \\
&\quad \leqslant \frac{1}{4} \|v^{m+1} - v^m\|^2 + 4\rho^2\nu^2 \|v^{m+1}\|^2 +
\end{aligned}$$

$$+ 2\rho(q^m, \operatorname{div} v^{m+1}) + \frac{2\hat{c}}{\nu} \|f\|_{V'} \|v^{m+1} - v^m\| \|v^{m+1}\|$$

$$+ \frac{2\hat{c}}{\nu} \|f\|_{V'} \|v^{m+1}\|^2 \leqslant \frac{1}{2} \|v^{m+1} - v^m\|^2$$

$$+ (4\rho^2\nu^2 + \frac{4\hat{c}^2}{\nu^2} \|f\|_{V'}^2 + \frac{2\hat{c}}{\nu} \|f\|_{V'}) \|v^{m+1}\|^2$$

$$+ 2\rho(q^m, \operatorname{div} v^{m+1}) . \tag{3.130}$$

We take $q = 2q^{m+1}$ in (3.123):

$$\alpha|q^{m+1}|^2 - \alpha|q^m|^2 = 2\rho(q^{m+1}, \operatorname{div} v^{m+1})$$

$$= -2\rho(q^m, \operatorname{div} v^{m+1}) - 2\rho(q^{m+1} - q^m, \operatorname{div} v^{m+1})$$

$$\leqslant -2\rho(q^m, \operatorname{div} v^{m+1}) + 2\rho|q^{m+1} - q^m| \|v^{m+1}\|$$

$$\leqslant \frac{\alpha}{2} |q^{m+1} - q^m|^2 + \frac{2\rho^2 n}{\alpha} \|v^{m+1}\|^2$$

$$- 2\rho(q^m, \operatorname{div} v^{m+1}) \tag{3.131}$$

(see (5.13), (5.25), (5.26), Chapter I). We add the last inequality (3.130) to the last inequality (3.131) and obtain

$$\alpha|q^{m+1}|^2 - \alpha|q^m|^2 + \frac{1}{2} |q^{m+1} - q^m|^2 + \|v^{m+1}\|^2$$

$$- \|v^m\|^2 + \frac{1}{2} \|v^{m+1} - v^m\|^2 + 2\rho \Big(\nu - \frac{2\hat{c}}{\nu} \|f\|_{V'}$$

$$- \frac{4\hat{c}^2}{\nu^2} \|f\|_{V'}^2 - 2\rho\nu^2 - \frac{2\rho}{\alpha} \Big) \|v^{m+1}\|^2 \leqslant 0 . \tag{3.132}$$

The conditions (3.126)–(3.127) ensure that the coefficient of $\|v^{m+1}\|^2$ in (3.132) is strictly positive; this inequality is then similar to the inequality (5.27) in the proof of Theorem I.5.2; we finish the proof as in that theorem. □

§4. Bifurcation Theory and Non-Uniqueness Results

The uniqueness of solution of the Stationary Navier–Stokes equations has only been proved under the assumptions that ν is sufficiently large, or that the given forces and boundary values of the velocity are sufficiently small. It is expected that otherwise the solution is not unique, and this has been proved by V. I. Iudovich [1], [2], P. Rabinowitz [2] and W. Velte [1], [2]. The first two papers deal with the Benard problem (Navier–Stokes and heat conduction equations) and the third one shows the non-uniqueness of solutions of the Taylor problem.

In this section we will establish the non-uniqueness of solutions of the Taylor problem – following W. Velte [2]. Subsection 4.1 contains the description of the problem and preliminary results. Subsection 4.2 recalls the main results of the topological degree theory that we need and then Subsection 4.3 gives the proof of the non-uniqueness theorem.

4.1. *The Taylor Problem. Preliminary Results.*

4.1.1. *The Equations*

The Taylor problem is the study of the flow of a viscous incompressible liquid in a doman of $\mathcal{R}^3$ bounded by two infinite cylinders of radius r_1 and r_2 ($r_2 > r_1 > 0$), having the same vertical axis. The inner cylinder is rotating with an angular velocity α, while the other is at rest.

Since we are looking for axi-symmetrical solutions, we will use cylindrical coordinates in $\mathcal{R}^3$, say r, θ, z, where the Oz axis is the axis of the cylinders. The fluid thus fills the domain Ω:

$$r_1 < r < r_2, \quad -\infty < z < +\infty. \tag{4.1}$$

We denote by $\widetilde{u}$, $\widetilde{v}$, $\widetilde{w}$, the components of the velocity vector in the cylindrical coordinates, and $\widetilde{p}$ denotes the pressure. Then the non-dimensional form of the equations of motion for an axi-symmetrical flow are:

$$\begin{aligned}
&-\frac{1}{\lambda}\left(A\widetilde{u} - \frac{\widetilde{u}}{r^2}\right) + \widetilde{u}\,\frac{\partial \widetilde{u}}{\partial r} + \widetilde{w}\,\frac{\partial \widetilde{u}}{\partial z} - \frac{1}{r}\,\widetilde{v}^2 + \frac{\partial \widetilde{p}}{\partial r} = 0\\
&-\frac{1}{\lambda}\left(A\widetilde{v} - \frac{\widetilde{v}}{r^2}\right) + \widetilde{u}\,\frac{\partial \widetilde{v}}{\partial r} + \widetilde{w}\,\frac{\partial \widetilde{v}}{\partial z} + \frac{1}{r}\,\widetilde{u}\,\widetilde{v} = 0\\
&-\frac{1}{\lambda}\,A\widetilde{w} + \widetilde{u}\,\frac{\partial \widetilde{w}}{\partial z} + \widetilde{w}\,\frac{\partial \widetilde{w}}{\partial z} + \frac{\partial \widetilde{p}}{\partial z} = 0
\end{aligned} \tag{4.2}$$

$$\frac{\partial}{\partial r}\,(r\widetilde{u}) + \frac{\partial}{\partial z}\,(r\widetilde{w}) = 0 \tag{4.3}$$

where

$$A = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{4.4}$$

and $\lambda = Re$ is the Reynolds numbers

$$\lambda = Re = \alpha r_1^2 \nu^{-1} \tag{4.5}$$

The boundary conditions on the lateral surfaces of the cylinders are

$$\begin{aligned} &\tilde{u} = \tilde{w} = 0, \ \tilde{v} = 1, \ \text{for } r = r_1 \\ &\tilde{u} = \tilde{v} = \tilde{w} = 0 \text{ for } r = r_2. \end{aligned} \tag{4.6}$$

A very simple solution of (4.2), (4.3), (4.6) is known, which we denote by u_0, v_0, w_0, p_0:

$$\begin{aligned} &u_0 = w_0 = 0, v_0(r) = \frac{1}{r_2^2 - r_1^2}\left(\frac{r_2^2}{r} - r\right) \\ &p_0(r) = v_0^2 \log r + \text{const.} \end{aligned} \tag{4.7}$$

We can then look for the solutions to (4.2), (4.3), (4.6) of the form

$$\tilde{u} = u_0 + u, \tilde{v} = v_0 + v, \tilde{w} = w_0 + w, \tilde{p} = p_0 + p,$$

and we obtain for u, v, w, p, the following equations:

$$\begin{aligned} &-\frac{1}{\lambda}\left(Au - \frac{u}{r^2}\right) + \frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{1}{r}v^2 - \frac{2}{r}v v_0 + \frac{\partial p}{\partial r} = 0 \\ &-\frac{1}{\lambda}\left(Av - \frac{v}{r^2}\right) + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{1}{r}uv + \frac{1}{r}(v_0 + rv_0')u = 0 \\ &-\frac{1}{\lambda}Aw + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = 0 \end{aligned} \tag{4.8}$$

where $v_0' = \dfrac{\mathrm{d}v_0}{\mathrm{d}r}$,

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{4.9}$$

and the following boundary conditions deduced from (4.6):

$$u = v = w = 0 \text{ for } r = r_1 \text{ and } r = r_2. \tag{4.10}$$

We observe that the boundary conditions (4.10) are not sufficient to obtain a well set problem (as Condition (4.6) is not sufficient in the case of Equations (4.2), (4.3)): some boundary conditions at $z = \pm\infty$ should be added. We can either look for solutions u, v, w, which vanish at $z = \pm\infty$, or for solutions which are independent of z, or for solutions which are periodic in z. The first possibility corresponds exactly to the type of problem studied in Section 1, the second possibility also, as this amounts to looking for bidimensional solutions. The third type of problem (z–periodic solutions), is not exactly one of the problems studied in Section 1 but is very close to them, and this is actually the type of solution experimentally observed. *We will seek the solutions of* (4.8), (4.9), (4.10) *which are periodic in z, with period L. We remember that*

$$u = v = w = p = 0,$$

is a trivial solution, and we want to show, in some cases, the existence of a non-trivial solution.

4.1.2. *The stream function*

Because of (4.9), there exists a function $f = f(r, z)$, such that

$$\frac{\partial}{\partial z}(rf) = ru, \quad \frac{\partial}{\partial r}(rf) = -rw. \tag{4.11}$$

The boundary condition (4.10) ensures that f is a single valued function.

It is interesting to write Equations (4.8) using only the dependent variables f, v. To do this we differentiate the first equation in (4.8) with respect to z, the third one with respect to r, and then subtract the resultant equations. The pressure disappears; observing that

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} = \mathcal{L}f, \quad \frac{\partial}{\partial r}(Aw) = \mathcal{L}\left(\frac{\partial w}{\partial r}\right)$$

where

$$\mathcal{L} = A - \frac{1}{r^2} = \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - \frac{1}{r^2},$$

we obtain after expanding

$$\mathcal{L}^2 f + \lambda\left(a\frac{\partial v}{\partial z} + M(f, v)\right) = 0, \tag{4.12}$$

with

$$M(f, \nu) = -\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial z}\,\mathcal{L}f\right) + \frac{\partial}{\partial z}\left[\frac{1}{r}\left(\frac{\partial}{\partial r}(rf)\right)\mathcal{L}f\right] + \frac{1}{r}\frac{\partial}{\partial z}(\nu^2),$$

$$a(r) = \frac{2\nu_0}{r} = \frac{2}{r_2^2 - r_1^2}\left(\frac{r_2^2}{r^2} - 1\right).$$

The second equation in (4.8) can be written as follows:

$$\mathcal{L}\nu + \lambda\left(b\,\frac{\partial f}{\partial z} + N(f, \nu)\right) = 0 \tag{4.13}$$

with

$$N(f, \nu) = -\frac{1}{r}\frac{\partial}{\partial r}(r\nu)\cdot\frac{\partial f}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rf)\cdot\frac{\partial \nu}{\partial z}$$

$$b = -\left(\frac{\nu_0}{r} + \nu_0'\right) = \frac{2}{r_2^2 - r_1^2}.$$

The boundary conditions (4.10) become:

$$f = \frac{\partial f}{\partial r} = \nu = 0 \text{ for } r = r_1 \text{ and } r = r_2. \tag{4.14}$$

The problem is thus reduced to finding the non-trivial solutions f, ν, of (4.12), (4.13), (4.14) which are periodic in z with period L.

4.1.3. *The Associated Functional Equation*

Since the functions f and ν are z-periodic with period L, it is sufficient to consider the restriction of these functions to the open set

$$\mathcal{O}_L = \left\{(r, z)\,\middle|\, r_1 < r < r_2,\ -\frac{L}{2} < z < \frac{L}{2}\right\}.$$

We will denote by $H^k(\mathcal{O}; L)$ the space of functions characterized as follows: if they are prolongated to the whole set $\mathcal{O}$,

$$\mathcal{O} = \{(r, z) | r_1 < r < r_2,\ z \in \mathbb{R}\}$$

as periodic functions of period L, the prolongated function belongs to H^k of any bounded subset of $\mathcal{O}$ (1). Actually $H^k(\mathcal{O}; L)$ is exactly the subspace of functions ν of $H^k(\mathcal{O}_L)$ such that

$$\frac{\partial^j \nu}{\partial z^j}\left(r, \frac{L}{2}\right) = \frac{\partial^j \nu}{\partial z^j}\left(r, \frac{L}{2}\right),\ j = 0, \ldots, k-1.$$

(1) Such a function will never belong to $H^k(\mathcal{O})$ except if it is the zero function.

Let us now introduce the spaces $W = W_1 \times W_2$ and $V = V_1 \times V_2$:

$$W_1 = \left\{ f \in H^2(\mathcal{O}; L) | f = \frac{\partial f}{\partial r} = 0 \text{ at } r = r_1 \text{ and } r = r_2 \right\}$$

$$W_2 = \{ v \in H^1(\mathcal{O}; L) | v = 0 \text{ at } r = r_1, \ r = r_2 \}.$$

$$V_1 = H^3(\mathcal{O}; L) \cap W_1, \quad V_2 = W_2 .$$

It is clear that W_1, W_2, V_1, V_2, equipped with the norms induced respectively by $H^2(\mathcal{O}_L)$, $H_1(\mathcal{O}_L)$, $H^3(\mathcal{O}_L)$, $H^2(\mathcal{O}_L)$ are Hilbert spaces. We equip V with the product norm induced by $H^3(\mathcal{O}_L) \times H^2(\mathcal{O}_L)$, but we shall equip W with another scalar product:

$$((\phi_1, \phi_2))_W = ((f_1, f_2))_{W_1} + ((v_1, v_2))_{W_2}$$

$$((f_1, f_2))_{W_1} = \int_{\mathcal{C}_L} \mathcal{L}f_1 \cdot \mathcal{L}f_2 \, r \, dr \, dz$$

$$((v_1, v_2))_{W_2} = \int_{\mathcal{C}_L} \left(\frac{\partial v_1}{\partial z} \frac{\partial v_2}{\partial z} + \frac{\partial v_1}{\partial r} \frac{\partial v_2}{\partial r} \right) r \, dr \, dz \tag{4.15}$$

where $\phi_i = \{f_i, v_i\}$ are elements of W. The corresponding norm on W_2 is obviously equivalent to the norm induced by $H^1(\mathcal{O}_L)$; the corresponding norm on W_1 is equivalent to the norm of $H^2(\mathcal{O}_L)$ according to Lemma 4.1 below. Thus W is a Hilbert space for the scalar product (4.15).

Lemma 4.1. *The norm* $\|f\|_{W_1}$ *is equivalent to the norm induced by* $H^2(\mathcal{O}_L)$ *on* W_1.

Proof. It is easily seen that if f and g belong to W_1,

$$\int_{\mathcal{C}_L} \mathcal{L}f \cdot g \, r \, dr \, dz = - \int_{\mathcal{C}_L} \left(\frac{\partial f}{\partial r} \frac{\partial g}{\partial r} + \frac{\partial f}{\partial z} \frac{\partial g}{\partial z} \right) r \, dr \, dz \, . \tag{4.16}$$

Thus if $f \in W_1$ and $\mathcal{L}f = 0$, then

$$\int_{\mathcal{C}_L} \mathcal{L}f \cdot f \, r \, dr \, dz = 0$$

and $f = 0$ by (4.16). This shows that $\|f\|_{W_1}$ is a norm on W_1. This norm is now equivalent to the norm $|\mathcal{L}f|_{L^2(\mathcal{O}_L)}$ and this last norm is equivalent to the norm of $H^2(\mathcal{O}_L)$ by application of the regularity theorems to the second order elliptic operator $-\mathcal{L}$ (see for example Agmon–Douglis–Niremberg [1]).

The operator T. The definition of the operator T will be given after the next two lemmas.

Lemma 4.2. *For g given in $L^2(\mathcal{O}_L)$, there exists a unique u in W_2 (resp. v in W_1) such that*

$$\mathcal{L}u = g \tag{4.17}$$

(resp.

$$\mathcal{L}^2 v = g \tag{4.18}$$

Moreover $u \in H^2(\mathcal{O}_L)$ and $v \in H^4(\mathcal{O}_L)$.

Proof. It is easy to see with (4.15) that these problems are respectively equivalent to the following ones:

– To find u in W_2 such that

$$-\int_{\mathcal{O}_L}\left(\frac{\partial u}{\partial r}\frac{\partial u_1}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial u_1}{\partial z}\right) r\,\mathrm{d}r\,\mathrm{d}z$$

$$= \int_{\mathcal{O}_L} g\,u_1\,r\,\mathrm{d}r\,\mathrm{d}z,\ \forall\ u_1 \in W_2,$$

– To find v in W_1 such that

$$\int_{\mathcal{O}_L}\mathcal{L}v\cdot\mathcal{L}v_1\cdot r\,\mathrm{d}r\,\mathrm{d}z = \int_{\mathcal{O}_L} g\,v_1\,r\,\mathrm{d}r\,\mathrm{d}z,\ \forall\ v_1 \in W_1.$$

The existence and uniqueness of solutions of these variational problems follow from Lemma 4.1 and the Projection Theorem (Theorem I.2.2). Then, it follows from the regularity theorems for elliptic operators ($\mathcal{L}$ and $\mathcal{L}^2$ here), that $u \in H^2(\mathcal{O}_L)$ and $v \in H^4(\mathcal{O}_L)$.

Lemma 4.3. *For $\{f, v\}$ given in V, the functions*

$$a\frac{\partial v}{\partial z}+M(f, v), \quad b\frac{\partial f}{\partial z}+N(f, v),$$

belong to $L^2(\mathcal{O}_L)$.

Proof. We have only to prove that $M(f, v)$ and $N(f, v)$ belong to $L^2(\mathcal{O}_L)$. This can be shown using the Sobolev imbedding Theorem which gives in particular (see Subsection 1.1):

$$H^2(\mathcal{O}_L) \subset \mathscr{C}^0(\bar{\mathcal{O}}_L), \quad H^1(\mathcal{O}_L) \subset L^4(\mathcal{O}_L).$$

Now the first terms in $M(f, v)$ are

$$-\frac{\partial^2 f}{\partial r\,\partial z}\cdot \mathcal{L}f-\frac{\partial f}{\partial z}\frac{\partial}{\partial r}(\mathcal{L}f);$$

if $\{f, v\}$ belongs to V, then f belongs to $H^3(\mathcal{O}_L)$, $\partial^2 f/\partial r\,\partial z$ and $\mathcal{L}f$ belong to $H^1(\mathcal{O}_L)$ and thus to $L^4(\mathcal{O}_L)$ and their product is in $L^2(\mathcal{O}_L)$; $\partial f/\partial z$ is in $H^2(\mathcal{O}_L)$ and thus in $\mathscr{C}^0(\bar{\mathcal{O}}_L)$, $\partial/\partial r\,(\mathcal{L}f)$ is in $L^2(\mathcal{O}_L)$, and the product of these terms is in $L^2(\mathcal{O}_L)$.

The proof is similar for the other terms in M and N. $\square$

We can now introduce the operator T. According to Lemmas 4.2 and 4.3, for $\{f, v\}$ given in V there exists a unique pair $\{f', v'\}$ belonging to V and also to $H^4(\mathcal{O}_L) \times H^2(\mathcal{O}_L)$, such that

$$\left.\begin{aligned} &\mathcal{L}^2 f' = -\left(a\frac{\partial v}{\partial z}+M(f, v)\right)\\ &\mathcal{L}v' = -\left(b\frac{\partial f}{\partial z}+N(f, v)\right)\\ &\{f', v'\} \in W \text{ (or } V). \end{aligned}\right\} \tag{4.19}$$

Definition 4.1. We denote by T the mapping from V into itself defined by

$$\{f, v\} \to \{f', v'\}.$$

The relation of this operator with our problem is the following:

Proposition 4.1. *The two following problems are equivalent:*

– *To find* $\{f, v\}$ *in* V *which satisfies* (4.12), (4.13), (4.14).
– *To find* $\phi = \{f, v\}$ *in* V *such that*

$$\phi = \lambda T \phi . \tag{4.20}$$

The proof is obvious. □

Henceforth we will study the problem (4.12)–(4.14) in its functional form (4.20). The next subsection is devoted to the study of the operator T. □

4.1.4. *Properties of the operator T.*

Lemma 4.4. *T is a compact operator in V.*

Proof. Let $\phi_n = \{f_n, v_n\}$ be a bounded sequence in V and let $\phi'_n = \{f'_n, v'_n\} = T\phi_n$. The proof of Lemma 4.3 shows more precisely that

$$a \frac{\partial v_n}{\partial z} + M(f_n, v_n), \ b \frac{\partial f_n}{\partial z} + N(f_n, v_n),$$

are bounded sequences in $L^2(\mathcal{O}_L)$. Thus $\mathcal{L}^2 f'_n$ and $\mathcal{L} v'_n$ are bounded sequences in $L^2(\mathcal{O}_L)$, f'_n and v'_n are bounded sequences in $H^4(\mathcal{O}_L)$ and $H^2(\mathcal{O}_L)$, and they are thus relatively compact sequences respectively in $H^3(\mathcal{O}_L)$ (or V_1) and $H^1(\mathcal{O}_L)$ (or V_2) (see Theorem 1.1). □

Let us consider the linear operator B defined as follows:

$$\phi = \{f, v\} \to \phi'' = \{f'', v''\} = B\phi ,$$

$$\mathcal{L}^2 f'' = -a \frac{\partial v}{\partial z} \tag{4.21}$$

$$\mathcal{L} v'' = -\frac{\partial f}{\partial z}$$

$$\{f'', v''\} \in W .$$

For ϕ given in W, there exists a unique pair $\{f'', v''\}$ in W which satisfies (4.21) (se Lemma 4.2). Thus B is a linear continuous mapping from W into $W \cap \{H^4(\mathcal{O}_L) \times H^2(\mathcal{O}_L)\}$. Using again Theorem 1.1, we obtain

Lemma 4.5. *B is a linear compact operator in W and in V.*

The relation between the operators T and B is the following one.

Lemma 4.6. *The operator B is the Fréchet differential of T at point O.*

Proof. Let us write

$$R\phi = \phi^* = T\phi - A\phi = \phi' - \phi'', \ \phi \in V .$$

We have

$$\mathcal{L}^2 f^* = -M(f, v)$$

$$\mathcal{L} v^* = -N(f, v) \tag{4.22}$$

Since $TO = O$, we must prove that

$$\frac{\|\phi^*\|_V}{\|\phi\|_V} \to 0, \text{ as } \|\phi\|_V \to 0 . \tag{4.23}$$

With (4.22) and the methods of Lemma 4.3 one easily sees that

$$|M(f, v)|_{L^2(\mathcal{O}_L)} \leqslant c_0 \|\phi\|_V^2 ,$$

$$|N(f, v)|_{L^2(\mathcal{O}_L)} \leqslant c_1 \|\phi\|_V^2 ,$$

(c_i = constants). Then, by Lemma 4.2,

$$\|f^*\|_{H^2(\mathcal{O}_L)} \leqslant c_2 |M(f, v)|_{L^2(\mathcal{O}_L)} \leqslant c_3 \|\phi\|_V^2 ,$$

$$\|v^*\|_{H^2(\mathcal{O}_L)} \leqslant c_4 |N(f, v)|_{L^2(\mathcal{O}_L)} \leqslant c_5 \|\phi\|_V^2 .$$

In particular

$$\|\phi^*\|_V \leqslant c_6 \|\phi\|_V^2 ,$$

and (4.23) follows. □

4.1.5. *A uniqueness result.*

Before starting the proof of the non-uniqueness of solutions, we establish a simple uniqueness result (for λ "small"), which is exactly the adaptation of Theorem 1.6. to the Taylor problem.

Proposition 4.2. *If λ is sufficiently small*

$$0 \leqslant \lambda < c(r_1, r_2) \ ^{(1)} \tag{4.24}$$

the problem (4.12)–(4.14) (*or* (4.20)) *possesses no solution in V other than the trivial one.*

---

(1) $c(r_1, r_2)$ is a constant depending on r_1 and r_2, made partly explicit in the proof of Proposition 4.2.

Proof. Let $\phi = \{f, \nu\}$ be a solution of (4.12)–(4.14). We multiply (4.12) by f, (4.13) by ν, and integrate these equations in $\mathcal{O}_L$ with respect to the measure $r\,\mathrm{d}r\,\mathrm{d}z$. We have

$$\int_{\mathcal{O}_L} [M(f, \nu)\cdot f - N(f, \nu)\cdot \nu]\; r\,\mathrm{d}r\,\mathrm{d}z = 0\,.$$

Using then (4.16) we get

$$\int_{\mathcal{O}_L} \left[|\mathcal{L}f|^2 + \left|\frac{\partial \nu}{\partial r}\right|^2 + \left|\frac{\partial \nu}{\partial z}\right|^2 \right] r\cdot\mathrm{d}r\,\mathrm{d}z =$$

$$\lambda \int_{\mathcal{O}_L} \left(- a\,\frac{\partial \nu}{\partial z} f + b\,\frac{\partial f}{\partial z}\, \nu \right) r\,\mathrm{d}r\,\mathrm{d}z\,.$$

Integrating by parts, we see that the right–hand side of this equation is equal to

$$\lambda \int_{\mathcal{O}_L} (a + b)\, \nu\, \frac{\partial f}{\partial z}\, r\,\mathrm{d}r\,\mathrm{d}z\,.$$

This is bounded by

$$\lambda r_2 \ \mathrm{Sup}|a + b|\ |\nu|_{L^2(\mathcal{O}_L)} \left|\frac{\partial f}{\partial z}\right|_{L^2(\mathcal{O}_L)} \leqslant$$

$$\lambda c(r_1, r_2) \int_{\mathcal{O}_L} \left[|\mathcal{L}f|^2 + \left|\frac{\partial \nu}{\partial r}\right|^2 + \left|\frac{\partial \nu}{\partial z}\right|^2 \right] r\,\mathrm{d}r\,\mathrm{d}z\,,$$

since

$$\operatorname*{Sup}_{r_1 \leqslant r \leqslant r_2} |a + b| = \frac{2}{r_2^2 - r_1^2} \cdot \frac{r_2^2}{r_1^2}\,,$$

and

$$|\nu|^2_{L^2(\mathcal{O}_L)} \leqslant \text{const.} \int_{\mathcal{O}_L} \left(\left|\frac{\partial \nu}{\partial r}\right|^2 + \left|\frac{\partial \nu}{\partial z}\right|^2 \right) \mathrm{d}r\,\mathrm{d}z$$

(Poincaré Inequality),

$$\left|\frac{\partial f}{\partial z}\right|^2_{L^2(\mathcal{O}_L)} \leqslant \text{const } |\mathcal{L}f|^2_{L^2(\mathcal{O}_L)}$$

(see Lemma 4.1).

Finally,

$$[1 - \lambda\, c(r_1, r_2)] \cdot \int_{\mathcal{O}_L} \left[|\mathcal{L}f|^2 + \left|\frac{\partial v}{\partial r}\right|^2 + \left|\frac{\partial v}{\partial z}\right|^2 \right] r \, dr \, dz \leqslant 0$$

and $f = v = 0$ when (4.24) is satisfied. □

Remark 4.1. With a slight modification of the preceding proof, one can show the uniqueness of the solution for

$$\lambda < \overline{\lambda},$$

where $\overline{\lambda}$ is the smallest eigenvalue of the operator $(B + B^*)/2$ which is compact and self-adjoint in W.

Remark 4.2. Our goal is to show, in some cases, the non-uniqueness of solution for sufficiently large λ.

4.2. *A spectral property of B.*

4.2.1. *Fourier Series Expansions*

If f belongs to $L^2(\mathcal{O}_L)$ then f possesses a Fourier series expansion

$$f = \sum_{n=0}^{\infty} (f_n(r) \cos(n\,\sigma\, z) + \overline{f}_n(r) \sin(n\,\sigma\, z)), \tag{4.25}$$

$(\sigma = 2\pi/L)$, where f_n and $\overline{f}_n$ are L^2 on the interval $[r_1, r_2]$. Moreover the sum

$$\left\{ \sum_{n=0}^{\infty} \left(|f_n|^2_{L^2(r_1, r_2)} + |\overline{f}_n|^2_{L^2(r_1, r_2)} \right) \right\}^{1/2}$$

is finite and defines on $L^2(\mathcal{O}_L)$, a norm equivalent to the usual norm.

The functions in $H^k(\mathcal{O}; L)$ $(k \geqslant 1)$ possess alike a Fourier series expansion of type (4.25), where $f_n, \overline{f}_n$ are in $H^k(r_1, r_2)$. The sum

$$\left\{ \sum_{n=0}^{\infty} \sum_{j=0}^{k} n^{2j} \left(|f_n|^2_{H^{k-j}(r_1, r_2)} + |\bar{f}_n|^2_{H^{k-j}(r_1, r_2)} \right) \right\}^{1/2} \tag{4.26}$$

is finite and defines on $H^k(\mathcal{O}; L)$ a norm equivalent to the norm induced by $H^k(\mathcal{O}_L)$.

We shall consider the subspace $\bar{V}$ of V containing all the pairs $\{f, v\}$ such that

$$f(r, -z) = -f(r, z)$$

$$v(r, -z) = v(r, z)\ .$$

These functions f and v admit Fourier series expansion of type

$$f = \sum_{n=0}^{\infty} f_n \sin(n\,\sigma\,z) \tag{4.27}$$

$$v = \sum_{n=0}^{\infty} v_n \cos(n\,\sigma\,z)\ .$$

It is easy to particularise the preceding remarks to such functions (V, and thus $\bar{V}$, is a closed subspace of $H^3(\mathcal{O}; L) \times H^1(\mathcal{O}; L)$).

Now, if $\phi = \{f, v\} \in \bar{V}$, then $\phi'' = B\phi = \{f'', v''\}$ also belongs to $\bar{V}$ and the Fourier series of f'' and v'' are given by solving the following one–dimensional boundary value problems

$$(\mathcal{M} - (n\,\sigma)^2)^2\, f_n''(r) = a\,n\,\sigma\, v_n(r) \tag{4.28}$$

$$-(\mathcal{M} - (n\,\sigma)^2)\, v_n''(r) = b\,n\,\sigma\, f_n(r)$$

$$\mathcal{M} = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2}\ ,$$

with the boundary conditions

$$f_n''(r) = \frac{\mathrm{d}f_n''}{\mathrm{d}r}(r) = v_n(r) = 0 \text{ at } r = r_1 \text{ and } r = r_2\ . \tag{4.29}$$

We will denote by B_n the linear mapping

$$\{f_n, v_n\} \to \{f_n'', v_n''\},$$

which is continuous from $L^2(r_1, r_2) \times L^2(r_1, r_2)$ into $H^4(r_1, r_2) \times H^2(r_1, r_2)$.

4.2.2. *Properties of the* B_n.

We note the following result which can be found for example in S. Karlin [1] or K. Kirchgässner [1].

Lemma 4.7. *Let*

$$M = \alpha(r)\frac{d^2}{dr^2} + \beta(r)\frac{d}{dr} + \gamma(r)$$

where α, β, γ, *are four times continuously differentiable on* $[-1,+1]$, *and* $\alpha > 0$ *and* $\gamma < 0$. *Then*

(i) *The Green's function* $G(r, s)$ *of* M *under the boundary conditions* $G(\pm 1, s) = 0$, *is negative for* $r, s \in (-1, +1)$

(ii) *The Green's function* $H(r, s)$ of M^2 *under the boundary conditions* $H(\pm 1, s) = (\partial H)/\partial z\ (\pm 1, s) = 0$, *is positive for* $r, s \in (-1, +1)$.

We infer from this the

Lemma 4.8. *The Green's functions* $H(k; r, s)$ *and* $G(k; r, s)$ of $(\mathscr{M} - k^2)^2$ *and* $-(\mathscr{M} - k^2)$ *on* (r_1, r_2), *under the boundary conditions* (4.29) *are positive on* $(r_1, r_2) \times (r_1, r_2)$.

With these kernels, we can convert (4.28), (4.29), into integral equations

$$f''_n(r) = n\,\sigma \int_{r_1}^{r_2} G(n\,\sigma; r, s)\, a(s)\, v_n(s)\, ds, \tag{4.30}$$

$$v''_n(r) = n\,\sigma \int_{r_1}^{r_2} H(n\,\sigma; r, s)\, b\, f_n(s)\, ds\,. \tag{4.31}$$

An eigenvector of B_n is a pair of functions $\{f_n, v_n\}$, such that

$$B_n\{f_n, v_n\} = \lambda\{f_n, v_n\}$$

for some $\lambda \in \mathbb{R}$. Therefore the relations (4.28) (4.29) hold with $f''_n = \lambda f_n$, $v''_n = \lambda v_n$. These are equivalent to the following ones deduced from (4.30), (4.31):

$$f_n(r) = \lambda n\,\sigma \int_{r_1}^{r_2} G(n\,\sigma; r, s)\, a(s)\, v_n(s)\, ds\,, \tag{4.32}$$

$$v_n(r) = \lambda n\, \sigma \int_{r_1}^{r_2} H(n\,\sigma; r, s) b\, f_n(s)\, \mathrm{d}s\,. \tag{4.33}$$

Eliminating v_n, we also get,

$$f_n(r) = \mu \int_{r_1}^{r_2} K(n\,\sigma; r, s)\, f_n(s)\, \mathrm{d}s\,, \tag{4.34}$$

where $\mu = \lambda^2$ and

$$K(k; r, s) = k^2 \int_{r_1}^{r_2} G(k; r, t) H(k; t, s)\, a(t) b\, \mathrm{d}t\,. \tag{4.35}$$

We shall now give some properties of the eigenvalues of the operators B_n. They are based on the following result whose proof can be found for example in Witting [1].[1]

Lemma 4.9. *Let $K(r, s)$ denote a real continuous function defined on the square $[r_1, r_2] \times [r_1, r_2]$, which is strictly positive on the interior of this square.*

The eigenvalue problem

$$f(r) = \lambda \int_{r_1}^{r_2} K(r, s)\, f(s)\, \mathrm{d}s, \quad r_1 < r < r_2\,, \tag{4.36}$$

possesses a solution $\lambda_1 > 0$ which corresponds to an eigenfunction $f \in \mathscr{C}^0([r_1, r_2]), f(r) > 0$ for $r_1 < r < r_2$.

Any other eigensolution of (4.36) *will correspond to an eigenvalue λ, such that $|\lambda| > \lambda_1$.*

---

[1] This is a particular case of a general result of Krein–Rutman [1] concerning linear compact operators leaving invariant a cone of a Banach space. These results are infinite dimensional extensions of the Perron–Frobenius theorem for positive matrices, well–known in linear algebra (see for instance R. S. Varga [1]).

Lemma 4.10. *The operator B_n possesses an eigenvalue $\lambda_n^1 > 0$, which corresponds to an eigenvector $\{f_n^1, v_n^1\}$ with $f_n^1(r) > 0$, $v_n^1(r) > 0$ for $r_1 < r < r_2$.*

Any other eigenvalue λ of B_n satisfies $|\lambda| > \lambda_n^1$.

Proof. Due to (4.34), $\{f_n, v_n\}$ is an eigenvector of B_n with eigenvalue λ_n, if and only if, f_n is an eigensolution of (4.34) with $\mu = \lambda_n^2$. Lemma 4.9 is applicable to the equation (4.34) and the present Lemma is proved, except for the positiveness of v_n^1. The positiveness of v_n^1 is a consequence of Lemma 4.8, the positiveness of f_n^1 and the relation (4.33) written with $v_n = v_n^1, f_n = f_n^1$. □

Another spectral property of B_n will be useful.

Lemma 4.11. *Let λ_n be an eigenvalue of B_n. Then*

$$|\lambda_n| > \lambda_n^1 \geqslant \frac{2}{\max|a+b|}\, n^2\, \sigma^2 \,. \tag{4.37}$$

Proof. Lemma 4.10 gives $|\lambda_n| > \lambda_n^1$. Let us prove the second inequality in (4.37). We have

$$\begin{aligned} &(\mathcal{M} - (n\sigma)^2)^2\, f_n^1(r) = \lambda_n^1\, a\, n\sigma\, v_n^0(r) \\ &-(\mathcal{M} - (n\sigma))\, v_n^1(r) = \lambda_n^1\, b\, n\sigma\, f_n^1(r) \,. \end{aligned} \tag{4.38}$$

We multiply the first relation (4.38) by $r\, f_n^1(r)$, the second by $r\, v_n^1(r)$, then we integrate and integrate by parts. We obtain:

$$J(f_n^1, v_n^1) = \lambda_n^1\, n\sigma \int_{r_1}^{r_2} (a+b)\, f_n^1\, v_n^1 r\, \mathrm{d}r \,, \tag{4.39}$$

where

$$J(f, v) = \int_{r_1}^{r_2} \left[(\mathcal{M}f)^2 + 2(n\sigma)^2 \left(\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + \frac{1}{r^2} f^2 \right) + \right.$$
$$\left. + (n\sigma)^4 f^2 + \left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^2 + \frac{1}{r^2}\, v^2 + (n\sigma)^2\, v^2 \right] r\, \mathrm{d}r \,.$$

The right-hand side of (4.39) is bounded by

$$\lambda_n^1\ n\ \sigma\ \max|a+b| \int_{r_1}^{r_2} |f_n^1\ v_n^1| r\,dr$$

$$\leqslant \lambda_n^1\ n\sigma \max \frac{|a+b|}{2} \left\{ n\sigma \int_{r_1}^{r_2} |f_n^1|^2 r\,dr + \frac{1}{n\sigma} \int_{r_1}^{r_2} |v_n^1|^2 r\,dr \right\}$$

$$\leqslant \frac{\lambda_n^1}{(n\sigma)^2} \max \frac{|a+b|}{2} J(f_n^1, v_n^1)\,.$$

We can divide by $J(f_n^1, v_n^1)$ which is non-zero, and (4.37) follows. □

4.2.3. *Spectral properties of B.*

We first observe that if $\{f, v\}$ is an eigenvector of B with eigenvalue λ, then the $\{f_n, v_n\}$ corresponding to the expansion (4.27) of $\{f, v\}$ are respectively eigenvectors of the operators B_n, with eigenvalue λ. We will conversely deduce from Section 4.2.2 a spectral property of B.

We consider all the λ_n^1, $n \geqslant 1$, given by Lemma 4.10. Due to (4.37), $\lambda_n^1 \to \infty$ as $n \to \infty$, and therefore

$$\operatorname*{Inf}_{n \geqslant 1} \lambda_n^1$$

is finite and strictly positive. We denote by m the largest integer n such that $\lambda_n^1 = \inf\limits_{p \geqslant 1} \lambda_p^1$. It may happen that $\lambda_m^1 = \lambda_n^1$ for some other values of n, $n < m$. We would like to avoid this situation and actually we have:

Lemma 4.12. *One can choose the period L, so that* $\lambda_1^1 > \lambda_n^1$, $\forall\, n > 1$.

In this case $\lambda_1^1 = \operatorname*{Inf}\limits_{n \geqslant 1} \lambda_n^1$ *is denoted* λ_1; λ_1 *is a simple eigenvalue of B in* $\overline{V}$ *and any other eigenvalue* λ *of B satisfies* $|\lambda| > \lambda_1$. *The eigenvector* $\phi^1 = \{f^1, v^1\}$ *corresponding to* λ^1 *admits a Fourier expansion of type* (4.27) *with* $f_n^1 = v_n^1 = 0$, $\forall\, n > 1$.

Proof. Let L be arbitrarily chosen and let $\sigma = 2\pi/L$ and m be defined as above (m = the largest n such that $\lambda_m^1 = \operatorname*{Inf}\limits_{n \geqslant 1} \lambda_n^1$). We set $\sigma^* = m\sigma$, $L^* = L/m$. For the corresponding operator B, λ_1 is an eigenvalue of B_1, and $\lambda_1 < \lambda_n^1$, $\forall\, n > 1$. The other properties stated in the Lemma are now obvious. $\square$

Until the end of this Section we assume that L is chosen so that Lemma 4.12 holds.

Our last result concerns the degree of λ_1 as an eigenvalue of B. The degree of λ_1 is the dimension of $Ker(I - \lambda_1 B)^p$, which is independent of p, for large p.

Lemma 4.13. *Under the conditions of Lemma 4.12, λ_1 is an eigenvalue of B of degree* 1.

Proof. We will show that if $(I - \lambda_1 B)^p \phi = 0$ for $p \geqslant 2$, then $(I - \lambda_1 B) = 0$ so that $Ker(I - \lambda_1 B)^p$ is equal for each p to $Ker(I - \lambda_1 B)$, and its dimension is one, because of Lemma 4.12.

We proceed by induction on p and actually we just have to show that $(I - \lambda_1 B)^2 \phi = 0$ implies $(I - \lambda_1 B)\phi = 0$.

Let us consider some function ϕ^0 such that

$$(I - \lambda_1 B)^2 \phi^0 = 0 .$$

We argue by contradiction and assume that $(I - \lambda_1 B)\phi^0$ is not equal to 0. This vector is then equal, within a multiplicative constant, to the previous eigenfunction ϕ^1:

$$\phi^1 = (I - \lambda_1 B)\phi^0, \quad \phi^1 = \lambda_1 B \phi^1 . \tag{4.40}$$

We have,

$$\phi^0 = \lambda_1 B(\phi^0 + \phi^1),$$

which amounts to saying that

$$\mathcal{L}^2 f^0 = \lambda_1 \, a \frac{\partial}{\partial z} (v^0 + v^1),$$

$$\mathcal{L} v^0 = \lambda_1 b \frac{\partial}{\partial z} (f^0 + f^1). \tag{4.41}$$

We recall that

$$f^1(r, z) = f_1^1(r) \sin(\sigma z), \quad v^1(r, z) = v_1^1(r) \cos(\sigma z).$$

Let us consider the Fourier series of f^0 and v^0;

$$f^0 = \sum_{n=1}^{\infty} f_n^0 \sin(n\sigma z), \quad v^0 = \sum_{n=1}^{\infty} v_n^0 \cos(n\sigma z).$$

The relations (4.41) imply

$$\begin{aligned} (\mathcal{M} - \sigma^2)^2 f_1^0 &= \lambda_1 \, a \, \sigma \, (v_1^0 + v_1^1), \\ -(\mathcal{M} - \sigma^2) v_1^0 &= \lambda_1 \, b \, \sigma \, (f_1^0 + f_1^1), \end{aligned} \tag{4.42}$$

and for $n \geqslant 2$:

$$\begin{aligned} (\mathcal{M} - \sigma^2)^2 f_n^0 &= \lambda_1 \, a \, \sigma \, n \, v_n^0 \\ -(\mathcal{M} - \sigma^2) \, v_n^0 &= \lambda_1 \, b \, \sigma \, n \, f_n^0. \end{aligned}$$

Since λ_1 is not an eigenvalue of B_n for $n \geqslant 2$, we see that

$$f_n^0 = v_n^0 = 0 \text{ for } n \geqslant 2.$$

We now convert (4.42) into integral equations, as in (4.30), (4.31). Since $\{f_1^1, v_1^1\}$ is an eigenvector of B_1, we obtain

$$f_1^0(r) = \lambda_1 \int_{r_1}^{r_2} G(\sigma; r, s) \, a(s) \sigma \, v_1^0(s) \, \mathrm{d}s + f_1^1(r),$$

$$v_1^0(r) = \lambda_1 \int_{r_1}^{r_2} H(\sigma; r, s) \, b \, \sigma \, f_1^0(s) \, \mathrm{d}s + v_1^1(r).$$

By elimination of v_1^0, and using the kernel K introduced in (4.35), we get

$$f_1^0(r) - \lambda_1^2 \int_{r_1}^{r_2} K(\sigma; r, s) \, f_1^0(s) \, \mathrm{d}s = 2 f_1^1(r). \tag{4.43}$$

The equation (4.43) satisfies the Fredholm alternative. Thus f_1^1 is orthogonal to the eigenfunction g_1 of the adjoint equation:

$$g_1(r) - \lambda_1^2 \int_{r_1}^{r_2} K(\sigma; s, r) \, g_1(s) \, \mathrm{d}s = 0.$$

By Lemma 4.9, f_1^1 and g_1 are positive on (r_1, r_2) and this contradicts the orthogonality condition

$$\int_{r_1}^{r_2} f_1^1(s)\, g_1(s)\, \mathrm{d}s = 0.$$

Thus $(I - \lambda_1 B)\phi^0 = 0$, and the proof is complete. $\square$

4.3. *Elements of Topological Degree Theory*

We recall a few definitions and properties of topological degree theory. For the proofs and further results, the reader is referred to the basic work of J. Leray and J. Schauder [1], or M. A. Krasnoselskii [1], L. Nirenberg [1], P. Rabinowitz [4].

4.3.1. *The topological degree.*

Let T be a compact operator in a normed space V, and let $S = I - T$ (I = the identity in V). We denote by ω, ω_i, bounded domains of V; $\bar{\omega}$ and $\partial\omega$ denote the closure and the boundary of ω.

If ω is a bounded domain of V, if $v \in V$ and

$$v \notin S(\partial\omega),$$

one can define an integer $d(S, \omega, v)$ which is called the topological degree of S, in ω, at the point v.

The main properties of the degree are the following ones:

(i) If $\omega = \omega_1 \cup \omega_2$, and $\omega_1 \cap \omega_2 = \phi$, if $v \notin S(\partial\omega_1)$, and $v \notin S(\partial\omega_2)$, then $v \notin S(\partial\omega)$ and

$$d(S, \omega, v) = d(S, \omega_1, v) + d(S, \omega_2, v).$$

(ii) If $d(S, \omega, v) \neq 0$, then $v \in S(\omega)$, which amounts to saying that the equation

$$(I - T)(u) = v$$

has at least one solution in ω.

(iii) $d(S, \omega, v)$ remains constant if S, ω, v, varies continuously, in such a way that v never belongs to $S(\partial\omega)$.(1)

---

(1) A continuous variation of S is defined as follows:

$S = S(\lambda) = I - T(\lambda)$, $\lambda \in \mathbb{R}$ (or any topological space), and $\lambda \to T(\lambda)\ \phi$ is a mapping uniformly continuous with respect to $\phi (\phi \in V)$.

4.3.2. *The index.*

Let u_0 be a point of V, $v = Su_0$, and let us assume that the equation $Su = v$ admits only the solution u_0, in some neighbourhood of u_0.

In this case, one can define for ϵ small enough, the degree $d(S, \omega_\epsilon(u_0), v)$, where $\omega_\epsilon(u_0)$ is the open ball of radius ϵ, centred at u_0. According to the property (iii) of the degree, this number is independent of ϵ, as $\epsilon \to 0$.

We define then, the index of S at u_0 as this degree, for ϵ sufficiently small:

$$i(S, u_0) = d(S, \omega_\epsilon(u_0), v), \quad \epsilon < \epsilon_0.$$

Some fundamental properties of this index are listed below:

(i) If the equation $Su = v$ posseses a finite number of solutions u_k in a bounded domain ω, and has no solutions on $\partial\omega$, then

$$d(S, \omega, v) = \sum_k i(S, u_k).$$

(ii) The index of the identity ($T = 0$) at any point u_0 is one: $i(I, u_0) = 1$.

(iii) Let us assume that T admits at the point u_0, a Fréchet differential A. Then A is compact like T. If $I - A$ is one to one (i.e. 1 is not an eigenvalue of A), then u_0 is an isolated solution of the equation $Su = Su_0$, and one can define the index $i(S, u_0)$.

One has

$$i(S, u_0) = i(I - A, 0) = i(I - A).$$

(iv) If A is a linear compact operator in V and if $I - A$ is one to one, the index of $I - A$ is ± 1.(1)

Similarly the index of $I - \lambda A$ is defined on any interval $\lambda' \leqslant \lambda \leqslant \lambda''$ containing no eigenvalue of A; the index is constant on such intervals and is equal to ± 1. In particular $i(I - \lambda A) = 1$ on the interval $(0, \lambda_1)$, where λ_1 is the smallest positive eigenvalue of A.

When λ crosses a spectral value λ_i of A, the index $i(I - \lambda A)$ is multiplied by $(-1)^m$ where m is the degree of λ_i, i.e. the dimension of $Ker(I - \lambda_i A)^k$ which is independent of k, when k is sufficiently large.

---

(1) One can define the index of $I - A$ if and only if $I - A$ is one to one; when it is defined, the index is the same at every point u_0.

4.4 *The non-uniqueness theorem.*

Our purpose is to prove the following result.

Theorem 4.1. *For λ sufficiently large, and for suitable values of L, the problem* (4.2), (4.3), (4.6) *possesses z–periodic solutions of period L which are different from the trivial solution* (4.7).

Proof. We will prove that the equation (4.20) has a non-trivial solution in V, when λ is sufficiently large. According to Lemma 4.12, we can choose L so that λ_1 is a simple eigenvalue of B in V: these are the values of L mentioned in Theorem 4.1.

It is known from Proposition 4.2 that (4.20) possesses only the trivial solution for λ sufficiently small ($\lambda \leqslant c(r_1, r_2)$, see (4.24)).

The theorem is already proved if the equation (4.20) has a non trivial solution for some $\lambda \in [0, \lambda_1]$. Therefore we will assume from now on that

$$\phi = 0 \textit{ is the only solution of } \phi = \lambda\, T\,\phi \textit{ for any } \phi \in [0, \lambda_1] \quad (4.44)$$

With this assumption, the next lemmas, using the degree theory, will show the existence of non zero ϕ of $\overline{V}$, satisfying $\phi = \lambda\, T\,\phi$, with $\lambda > \lambda_1$.

Lemma 4.14. *Let ω be some open ball of V centered at* 0.

There exists some $\delta > 0$ such that $\phi = \lambda\, T\,\phi$ has no solution on the boundary $\partial\omega$ of ω, for each λ in the interval $[\lambda_1, \lambda_1 + \delta]$.

Proof. We argue by contradiction. If this statement is false, there exists a sequence of λ_n decreasing to λ_1, and a sequence of u_k belonging to $\partial\omega$. such that

$$u_n = \lambda_n\, T\, u_n .$$

Since the sequence u_n is bounded, the sequence $T\,u_n$ is relatively compact (by Lemma 4.4), and there exists a subsequence $T\,u_{n_i}$ converging to some limit v in V. Then $u_{n_i} = \lambda_{n_i}\, T\, u_{n_i}$ converges to $\lambda_1 v$. Since T is continuous, we must have

$$\lambda_1 v = \lambda_1\; T(\lambda_1 v).$$

Thus $\lambda_1 v$ is a solution of $\phi = \lambda_1\; T\phi$, and because of (4.44), $\lambda_1 v = 0$, $v = 0$. This contradicts the fact that $\|\lambda_1 v\|_V$ is equal to the radius of the ball ω ($\|u_n\|$ = radius of ω, $\forall\, n$).

Lemma 4.15. *Under the assumption* (4.44), *if* ω *and* δ *are as in Lemma* 4.14, *the equation* $\phi = \lambda\, T\, \phi$ *has no solution on* $\partial\omega$, *for any* $\lambda \in [0, \lambda_1 + \delta)$.

Obvious Corollary of Lemma 4.14.

This lemma allows us to define the degree $d(I - \lambda T, \omega, 0)$ for $\lambda \in [0, \lambda_1 + \delta)$.

Lemma 4.16. *With* δ *and* ω *as before,*

$$d(I - \lambda T, \omega, 0) = 1, \text{ for } \lambda \in [0, \lambda_1 + \delta).$$

Proof. It follows from the property (iii) of the degree that $d(I - \lambda T, \omega, 0) = d(I, \omega, 0) = i(I)$ and this index is equal to one (the index of the identity).

Lemma 4.17. *Under the assumption* (4.44), *there exists for any* $\lambda \in (\lambda_1, \lambda_1 + \delta)$ *at least one non-trivial solution of* $\phi = \lambda\, T\, \phi$.

Proof. According to Lemma 4.13, and the properties (iv) of the index, $i(I - \lambda B)$ is equal to 1 on $[0, \lambda_1)$ and is equal to -1 on $(\lambda_1, \lambda_1 + \delta)$. According to the property (iii) of the index, $i(I - \lambda T, 0)$ is +1 for $\lambda \in (0, \lambda_1)$ and -1 for $\lambda \in (\lambda_1, \lambda_1 + \delta)$. If $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and if zero is the only solution of $\phi = \lambda\, T\, \phi$ in ω, we should have

$$d(I - \lambda T, \omega, 0) = i(I - \lambda T, 0)$$

according to the property (i) of the index. But we proved that

$$d(I - \lambda T, \omega, 0) = +1, \; i(I - \lambda T, 0) = -1, \; \lambda \in (\lambda_1, \lambda_1 + \delta).$$

Thus the equation $\phi = \lambda\, T\, \phi$ has a non-trivial solution for any $\lambda \in (\lambda_1, \lambda_1 + \delta)$.

The proof of Theorem 4.1 is complete.

Remark 4.3. The condition "λ sufficiently large", amounts to saying that the angular velocity α is large or that the viscosity ν is small (for fixed r_1, r_2).

Remark 4.4. Under condition (4.44), there exists for each $\lambda \in (\lambda_1, \lambda_1 + \delta)$ a non-trivial solution ϕ_λ of (4.20). One can prove that $\phi_\lambda \to 0$ in V, as λ decreases to λ_1. This is the bifurcation

In case of the Benard problem the situation is very similar, but it can be proved that there only exists the trivial solution for $\lambda \in [0, \lambda_1]$. Thus the assumption (4.44) is unnecessary, and one does prove the occurrence of a bifurcation (see V. I. Iudovich [2], Rabinowitz [1], Velte [1]).

A study of the Taylor problem by analytical methods is developed in Rabinowitz [5]. $\square$

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CHAPTER III

THE EVOLUTION NAVIER–STOKES EQUATION

Introduction

This final chapter deals with the full Navier–Stokes Equations; i.e., the evolution nonlinear case. First we describe a few basic results concerning the existence and uniqueness of solutions, and then we study the approximation of these equations by several methods.

In Section 1 we briefly examine the linear evolution equations (evolution Stokes equations). This section contains some technical lemmas appropriate for the study of evolution equations. Section 2 gives compactness theorems which will enable us to obtain strong convergence results in the evolution case, and to pass to the limit in the nonlinear terms. Section 3 contains the variational formulation of the problem (weak or turbulent solutions, according to J. Leray [1], [2], [3]; E. Höpf [2]) and the main results of existence and uniqueness of solution (the dimension of the space is $n = 2$ or 3); the existence is based on the construction of an approximate solution by the Galerkin method. In Section 4 further existence and uniqueness results are presented; here existence is obtained by semi–discretization in time, and is valid for any dimension of the space.

In the final section we study the approximation of the evolution Navier–Stokes equations, in the two- and three-dimensional cases. Several schemes are considered corresponding to a classical discretization in the time variable (implicit, Crank–Nicholson, explicit) associated with any of the discretizations in the space variables introduced in Chapter I (finite differences, finite elements). We conclude with a study of the nonlinear stability of these schemes, establishing sufficient conditions for stability and proving the convergence of all these schemes when they are stable.

§ 1. The Linear Case

In this section we develop some results of existence, uniqueness, and regularity of the solutions of the linearized Navier–Stokes equations. After introducing some notation useful in the linear as well as in the nonlinear case (Section 1.1), we give the classical and variational formulations of the problem and the statement of the main existence and uniqueness result (Section 1.2); the proofs of the existence and of the uniqueness are then given in Sections 1.3 and 1.4.

1.1. *Notations*

Let Ω be an open Lipschitz set in $\mathbb{R}^n$; for simplicity we suppose Ω bounded, and we refer to the remarks in Section 1.5 for the unbounded case. We recall the definition of the spaces $\mathscr{V}$, V, H, used in the previous chapters and which will be the basic spaces in this chapter too:

$$\mathscr{V} = \{\boldsymbol{u} \in \mathscr{D}(\Omega), \operatorname{div} \boldsymbol{u} = 0\} \tag{1.1}$$

$$V = \text{the closure of } \mathscr{V} \text{ in } \boldsymbol{H}_0^1(\Omega), \tag{1.2}$$

$$H = \text{the closure of } \mathscr{V} \text{ in } \boldsymbol{L}^2(\Omega). \tag{1.3}$$

The space H is equipped with the scalar product $(\cdot\,,\,\cdot)$ induced by $\boldsymbol{L}^2(\Omega)$; the space V is a Hilbert space with the scalar product

$$((\boldsymbol{u}, \boldsymbol{v})) = \sum_{i=1}^{n} (D_i\boldsymbol{u}, D_i\boldsymbol{v}), \tag{1.4}$$

since Ω is bounded.

The space V is contained in H, is dense in H, and the injection is continuous. Let H' and V' denote the dual spaces of H and V, and let i denote the injection mapping from V into H. The adjoint operator i' is linear continuous from H' into V', and is one to one since $i(V) = V$ is dense in H and $i'(H')$ is dense in V' since i is one to one; therefore H' can be identified with a dense subspace of V'. Moreover, by the Riesz representation theorem, we can identify H and H', and we arrive at the inclusions

$$V \subset H \equiv H' \subset V', \tag{1.5}$$

where each space is dense in the following one and the injections are continuous.

As a consequence of the previous identifications, the scalar product in H of $\boldsymbol{f} \in H$ and $\boldsymbol{u} \in V$ is the same as the scalar product of $\boldsymbol{f}$ and $\boldsymbol{u}$ in the duality between V' and V:

$$\langle \boldsymbol{f}, \boldsymbol{u}\rangle = (\boldsymbol{f}, \boldsymbol{u}), \forall\, \boldsymbol{f} \in H, \ \forall\, \boldsymbol{u} \in V. \tag{1.6}$$

For each $\boldsymbol{u}$ in V, the form

$$v \in V \to ((u, v)) \in \mathbb{R} \tag{1.7}$$

is linear and continuous on V; therefore, there exists an element of V' which we denote by Au such that

$$\langle Au, v \rangle = ((u, v)), \forall v \in V. \tag{1.8}$$

It is easy to see that the mapping $u \to Au$ is linear and continuous, and, by Theorem I.2.2 and Remark I.2.2, is an isomorphism from V onto V'.

If Ω is unbounded, the space V is equipped with the scalar product

$$[\![u, v]\!] = ((u, v)) + (u, v); \tag{1.9}$$

the inclusions (1.5) hold. The operator A is linear continuous from V into V' but it not in general an isomorphism; for every $\epsilon > 0$, $A + \epsilon I$ is an isomorphism from V onto V'.

Let a, b be two extended real numbers, $-\infty \leqslant a < b \leqslant \infty$, and let X be a Banach space. For given α, $1 \leqslant \alpha < +\infty$, $L^\alpha(a, b; X)$ denotes the space of L^α-integrable functions from $[a, b]$ into X, which is a Banach space with the norm

$$\left\{ \int_a^b \|f(t)\|_X^\alpha \, dt \right\}^{1/\alpha}. \tag{1.10}$$

The space $L^\infty(a, b; X)$ is the space of essentially bounded functions from $[a, b]$ into X, and is equipped with the Banach norm

$$\operatorname*{Ess\,Sup}_{[a,b]} \|f(t)\|_X. \tag{1.11}$$

The space $\mathscr{C}([a, b]; X)$ is the space of continuous functions from $[a, b]$ into X and if $-\infty < a < b < \infty$ is equipped with the Banach norm

$$\operatorname*{Sup}_{t \in [a,b]} \|f(t)\|_X. \tag{1.12}$$

Most often the interval $[a, b]$ will be the interval $[0, T]$, $T > 0$ fixed; when no confusion can arise, we will use the following more condensed notations,

$$L^\alpha(X) = L^\alpha(0, T; X), \quad 1 \leqslant \alpha \leqslant +\infty \tag{1.13}$$

$$\mathscr{C}(X) = \mathscr{C}([0, T]; X). \tag{1.14}$$

The remainder of this Section 1.1 is devoted to the proof of the following technical lemma concerning the derivatives of functions with values in a Banach space:

Lemma 1.1. *Let X be a given Banach space with dual X' and let $\boldsymbol{u}$ and $\boldsymbol{g}$ be two functions belonging to $L^1(a, b; X)$. Then, the following three conditions are equivalent*

(i) *$\boldsymbol{u}$ is a.e. equal to a primitive function of g,*

$$u(t) = \xi + \int_0^t g(s)\,\mathrm{d}s,\ \xi \in X,\ a.e.\ t \in [a, b] \tag{1.15}$$

(ii) *For each test function $\phi \in \mathscr{D}((a, b))$,*

$$\int_a^b \boldsymbol{u}(t)\phi'(t)\mathrm{d}t = -\int_a^b \boldsymbol{g}(t)\phi(t)\mathrm{d}t \quad \left(\phi' = \frac{\mathrm{d}\phi}{\mathrm{d}t}\right); \tag{1.16}$$

(iii) *For each $\eta \in X'$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{u}, \eta\rangle = \langle \boldsymbol{g}, \eta\rangle, \tag{1.17}$$

in the scalar distribution sense, on (a, b).

If (i) – (iii) *are satisfied $\boldsymbol{u}$, in particular, is a.e. equal to a continuous function from $[a, b]$ into X.*

Proof. We suppose for simplicity that the interval $[a, b]$ is the interval $[0, T]$. A legitamate integration by parts shows that (i) implies (ii) and (iii); it remains to check that the property (iii) implies the property (ii) and that (ii) implies (i).

If (iii) is satisfied and $\phi \in \mathscr{D}((0, T))$, then by definition,

$$\int_0^T \langle \boldsymbol{u}(t), \eta\rangle\, \phi'(t)\mathrm{d}t = -\int_0^T \langle \boldsymbol{g}(t), \eta\rangle\, \phi(t)\mathrm{d}t \tag{1.18}$$

or

$$\langle \int_0^T \boldsymbol{u}(t)\,\phi'(t)\,\mathrm{d}t + \int_0^T \boldsymbol{g}(t)\,\phi(t)\,\mathrm{d}t, \eta\rangle = 0,\ \forall\, \eta \in X',$$

so that (1.16) is satisfied.

Let us now prove that (ii) implies (i). We can reduce the general case to the case $\boldsymbol{g} = 0$. To see this, we set $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_0$ with

$$u_0(t) = \int_0^t g(s)\mathrm{d}s; \tag{1.19}$$

it is clear that u_0 is an absolutely continuous function and that $u_0' = g$; hence (1.16) holds with u replaced by $u_0 + v$ and

$$\int_0^T v(t)\phi'(t)\mathrm{d}t = 0,\ \forall\ \phi \in \mathscr{D}((0, T)). \tag{1.20}$$

The proof of (i) will be achieved if we show that (1.20) implies that v is a constant element of X.

Let ϕ_0 be some function in $\mathscr{D}((0, T))$, such that

$$\int_0^T \phi_0(t)\mathrm{d}t = 1.$$

Any function ϕ in $\mathscr{D}((0, T))$ can be written as

$$\phi = \lambda\phi_0 + \psi',\ \ \lambda = \int_0^T \phi(t)\mathrm{d}t,\ \ \psi \in \mathscr{D}((0, T)); \tag{1.21}$$

indeed since

$$\int_0^T (\phi(t) - \lambda\phi_0(t))\mathrm{d}t = 0,$$

the primitive function of $\phi - \lambda\phi_0$ vanishing at 0, belongs to $\mathscr{D}((0, T))$, and ψ is precisely this primitive function. According to (1.20) and (1.21),

$$\int_0^T (v(t) - \xi)\phi(t)\mathrm{d}t = 0,\ \forall\ \phi \in \mathscr{D}((0, T)) \tag{1.21a}$$

where

$$\xi = \int_0^T v(s)\phi_0(s)\mathrm{d}s.$$

To achieve the proof, it remains to show that (1.21a) implies that

$$v(t) = \xi \text{ a.e.},$$

i.e., that a function $w \in L^1(X)$ such that

$$\int_0^T w(t)\phi(t)\mathrm{d}t = 0, \ \forall\, \phi \in \mathscr{D}((0,T)), \tag{1.22}$$

is zero almost everywhere. This well-known result is proved by regularization: if $\tilde{w}$ is the function equal to w on $[0, T]$ and to 0 outside this interval, and if ρ_ϵ is an even regularizing function, then for ϵ small enough, $\rho_\epsilon * \phi$ belongs to $\mathscr{D}((0, T))$, $\forall\, \phi \in \mathscr{D}((0, T))$, and

$$\int_0^T w(t)\,(\rho_\epsilon * \phi)\,(t)\mathrm{d}t = \int_{-\infty}^{+\infty} \tilde{w}(t)\,(\rho_\epsilon * \phi)\,(t)\mathrm{d}t$$

$$= \int_{-\infty}^{+\infty} (\rho_\epsilon * \tilde{w})\,(t)\phi(t)\mathrm{d}t = 0.$$

Hence, for any $\eta > 0$ fixed, $\rho_\epsilon * \tilde{w}$ is equal to 0 on the interval $[\eta, T\text{-}\eta]$, for ϵ small enough; as $\epsilon \to 0$, $\rho_\epsilon * \tilde{w}$ converges to $\tilde{w}$ in $L^1(-\infty, +\infty; X)$. Thus w is zero on $[\eta, T - \eta]$; since $\eta > 0$ is arbitrarily small, w is zero on the whole interval $[0, T]$. $\square$

1.2. *The Existence and Uniqueness Theorem*

Let Ω be a Lipschitz open bounded set in $\mathbb{R}^n$ and let $T > 0$ be fixed. We denote by Q the cylinder $\Omega \times (0, T)$. The linearized Navier–Stokes equations are the evolution equations corresponding to the Stokes problem:

To find a vector function

$$u : \Omega \times [0, T] \to \mathbb{R}^n$$

and a scalar function

$$p : \Omega \times [0, T] \to \mathbb{R},$$

respectively equal to the velocity of the fluid and to its pressure, such that

$$\frac{\partial u}{\partial t} - \nu \Delta u + \operatorname{grad} p = f \text{ in } Q = \Omega \times (0, T), \tag{1.23}$$

$$\operatorname{div} u = 0 \text{ in } Q, \tag{1.24}$$

$$u = 0 \text{ on } \partial\Omega \times [0, T], \tag{1.25}$$

$$u(x, 0) = u_0(x), \text{ in } \Omega, \tag{1.26}$$

where the vector functions f and u_0 are given, f defined on $\Omega \times [0, T]$, u_0 defined on Ω; the equations (1.25) and (1.26) give respectively the boundary and initial conditions.

Let us suppose that u and p are classical solutions of (1.23)–(1.26), say $u \in \mathscr{C}^2(\bar{Q})$, $p \in \mathscr{C}^1(\bar{Q})$. If v denotes any element of $\mathscr{V}$, it is easily seen that

$$\left(\frac{\partial u}{\partial t}, v\right) + \nu((u, v)) = (f, v). \tag{1.27}$$

By continuity, the equality (1.27) holds also for each $v \in V$; we observe also that

$$\left(\frac{\partial u}{\partial t}, v\right) = \frac{\mathrm{d}}{\mathrm{d}t}(u, v).$$

This leads to the following weak formulation of the problem (1.23)–(1.26):

For f and u_0 given,

$$f \in L^2(0, T; V') \tag{1.28}$$

$$u_0 \in H, \tag{1.29}$$

to find u, satisfying

$$u \in L^2(0, T; V) \tag{1.30}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(u, v) + \nu((u, v)) = \langle f, v\rangle, \ \forall v \in V. \tag{1.31}$$

$$u(0) = u_0. \tag{1.32}$$

If u belongs to $L^2(0, T; V)$ the condition (1.32) does not make sense in general; its meaning will be explained after the following two remarks

(i) The spaces in (1.28), (1.29), and (1.30) are the spaces for which existence and uniqueness will be proved; it is clear at least that a smooth solution u of (1.23)–(1.26) satisfies (1.30).

(ii) We cannot check now that a solution of (1.30)–(1.32) is a solution, in some weak sense, of (1.23)–(1.26); hence we postpone the investigation of this point until Section 1.5.

By (1.6) and (1.8), we can write (1.31) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u, v\rangle = \langle f - \nu A u, v\rangle, \ \forall v \in V. \tag{1.33}$$

Since A is linear and continuous from V into V' and $u \in L^2(V)$, the function Au belongs to $L^2(V')$; hence $f - \nu Au \in L^2(V')$ and (1.33) and Lemma 1.1 show that

$$u' \in L^2(0, T; V') \tag{1.34}$$

and that u is a.e. equal to an (absolutely) continuous function from $[0, T]$ into V'. Any function satisfying (1.30) and (1.31) is, after modification on a set of measure zero, a continuous function from $[0, T]$ into V', and therefore the condition (1.32) makes sense.

Let us suppose again that f is given in $L^2(V')$ as in (1.28). If u satisfies (1.30) and (1.31) then, as observed before, u satisfies (1.34) and (1.33). According to Lemma 1.1 the equality (1.33) is itself equivalent to

$$u' + \nu Au = f. \tag{1.35}$$

Conversely if u satisfies (1.30), (1.34), and (1.35), then u clearly satisfies (1.31), $\forall\, v \in V$.

An alternative formulation of the weak problem is the following: *Given f and u_0 satisfying* (1.28)–(1.29), *to find u satisfying*

$$u \in L^2(0, T; V), \ u' \in L^2(0, T; V'), \tag{1.36}$$

$$u' + \nu Au = f, \ \text{on } (0, T), \tag{1.37}$$

$$u(0) = u_0. \tag{1.38}$$

Any solution of (1.36)–(1.38) is a solution of (1.30)–(1.32) and conversely.

Concerning the existence and uniqueness of solution of these problems, we will prove the following result.

Theorem 1.1. *For given f and u_0 which satisfy* (1.28) *and* (1.29), *there exists a unique function u which satisfies* (1.36)–(1.38). *Moreover*

$$u \in \mathscr{C}([0, T]; H). \tag{1.39}$$

The proof of the existence is given in Section 1.3, that of the uniqueness and of (1.39) are in Section 1.4.

1.3. *Proof of the Existence in Theorem* 1.1

We use the Faedo–Galerkin method. Since V is separable there exists a sequence of linearly independent elements, $w_1, \ldots, w_m, \ldots$, which is total in V. For each m we define an approximate solution u_m of (1.37) or (1.31) as follows:

$$u_m = \sum_{i=1}^{m} g_{im}(t) w_i, \tag{1.40}$$

and

$$(u_m', w_j) + \nu((u_m, w_j)) = \langle f, w_j \rangle, \; j = 1, \ldots, m, \tag{1.41}$$

$$u_m(0) = u_{0m}, \tag{1.42}$$

where u_{0m} is, for example, the orthogonal projection in H of u_0 on the space spanned by $w_1, \ldots, w_m$.[1]

The functions g_{im}, $1 \leqslant i \leqslant m$, are scalar functions defined on $[0, T]$, and (1.41) is a linear differential system for these functions; indeed we have

$$\sum_{i=1}^{m} (w_i, w_j) g_{im}'(t) + \nu \sum_{i=1}^{m} ((w_i, w_j)) g_{im}(t)$$
$$= \langle f(t), w_j \rangle, j = 1, \ldots, m; \tag{1.43}$$

since the elements $w_1, \ldots, w_m$, are linearly independent, it is well-known that the matrix with elements (w_i, w_j) $(1 \leqslant i, j \leqslant m)$ is non-singular; hence by inverting this matrix we reduce (1.43) to a linear system with constant coefficients

$$g_{im}'(t) + \sum_{j=1}^{m} \alpha_{ij} g_{jm}(t) = \sum_{j=1}^{m} \beta_{ij} \langle f(t), w_j \rangle$$
$$1 \leqslant i \leqslant m, \tag{1.44}$$

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[1] u_{0m} can be any element of the space spanned by $w_1, \ldots, w_m$ such that $u_{0m} \to u_0$ in the norm of H, as $m \to \infty$.

where $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$.

The condition (1.42) is equivalent to m equations

$$g_{im}(0) = \text{the } i^{\text{th}} \text{ component of } u_{0m}. \tag{1.45}$$

The linear differential system (1.44) together with the initial conditions (1.45) defines uniquely the g_{im} on the whole interval $[0, T]$.

Since the scalar functions $t \to \langle f(t), w_j \rangle$ are square integrable, so are the functions g_{im} and therefore, for each m

$$u_m \in L^2(0, T; V), \ u'_m \in L^2(0, T; V). \tag{1.46}$$

We will obtain *a priori* estimates independent of m for the functions u_m and then pass to the limit.

A Priori Estimates.

We multiply equation (1.41) by $g_{jm}(t)$ and add these equations for $j = 1, \ldots, m$. We get

$$(u'_m(t), u_m(t)) + \nu \|u_m(t)\|^2 = \langle f(t), u_m(t) \rangle.$$

But, because of (1.46),

$$2(u'_m(t), u_m(t)) = \frac{\mathrm{d}}{\mathrm{d}t} |u_m(t)|^2,$$

and this gives

$$\frac{\mathrm{d}}{\mathrm{d}t} |u_m(t)|^2 + 2\nu \|u_m(t)\|^2 = 2\langle f(t), u_m(t) \rangle \tag{1.47}$$

The right–hand side of (1.47) is majorized by

$$2\|f(t)\|_{V'} \ \|u_m(t)\| \leqslant \nu \|u_m(t)\|^2 + \frac{1}{\nu} \ \|f(t)\|^2_{V'}.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} |u_m(t)|^2 + \nu \|u_m(t)\|^2 \leqslant \frac{1}{\nu} \ \|f(t)\|^2_{V'}. \tag{1.48}$$

Integrating (1.48) from 0 to s, $0 < s < T$, we obtain in particular

$$|u_m(s)|^2 \leqslant |u_{0m}|^2 + \frac{1}{\nu} \int_0^s \|f(t)\|^2_{V'} \, \mathrm{d}t \leqslant |u_0|^2 +$$

$$+\frac{1}{\nu}\int_0^T \|f(t)\|_{V'}^2\,\mathrm{d}t. \tag{1.49}$$

Hence:

$$\operatorname*{Sup}_{s\in[0,T]} |u_m(s)|^2 \leqslant |u_0|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|_{V'}^2\,\mathrm{d}t. \tag{1.50}$$

The right–hand side of (1.50) is finite and independent of m; therefore

The sequence u_m *remains in a bounded set of*

$L^\infty(0, T; H)$. (1.51)

We then integrate (1.48) from 0 to T and get

$$|u_m(T)|^2 + \nu\int_0^T \|u_m(t)\|^2\,\mathrm{d}t \leqslant$$

$$|u_{0m}|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|_{V'}^2\,\mathrm{d}t \leqslant$$

$$|u_0|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|_{V'}^2\,\mathrm{d}t. \tag{1.52}$$

This shows that

The sequence u_m *remains in a bounded set of*

$L^2(0, T; V)$. (1.53)

Passage to the Limit.

The *a priori* estimate (1.51) shows the existence of an element u in $L^\infty(0, T; H)$ and a subsequence $m' \to \infty$, such that

$u_{m'}$ converges to u, for the weak–star topology of

$L^\infty(0, T; H)$; (1.54)

(1.54) means that for each $v \in L^1(0, T; H)$,

$$\int_0^T (u_{m'}(t) - u(t), v(t))\mathrm{d}t \to 0,\ \ m' \to \infty. \tag{1.55}$$

By (1.53) the subsequence $u_{m'}$ belongs to a bounded set of $L^2(0, T; V)$; therefore another passage to a subsequence shows the existence of some u_* in $L^2(0, T; V)$ and some subsequence (still denoted $u_{m'}$) such that

$$u_{m'} \text{ converges to } u_*, \text{ for the weak topology of } L^2(0, T; V). \tag{1.56}$$

The convergence (1.56) means

$$\int_0^T \langle u_{m'}(t) - u_*(t), v(t)\rangle \mathrm{d}t \to 0, \forall\ v \in L^2(0, T; V').$$

In particular, by (1.6),

$$\int_0^T (u_{m'}(t), v(t))\mathrm{d}t \to \int_0^T (u_*(t), v(t))\mathrm{d}t, \tag{1.57}$$

for each v in $L^2(0, T; H)$. Comparing with (1.55) we see that

$$\int_0^T (u(t) - u_*(t), v(t))\mathrm{d}t = 0, \tag{1.58}$$

for each v in $L^2(0, T; H)$; hence

$$u = u_* \in L^2(0, T; V) \cap L^\infty(0, T; H). \tag{1.59}$$

In order to pass to the limit in equations (1.41) and (1.42), let us consider scalar functions ψ continuously differentiable on $[0, T]$ and such that

$$\psi(T) = 0 \tag{1.60}$$

For such a function ψ we multiply (1.41) by $\psi(t)$, integrate with respect to t and integrate by parts:

$$\int_0^T (u_m'(t), w_j)\psi(t)\mathrm{d}t = -\int_0^T (u_m(t)\psi'(t), w_j)\mathrm{d}t - (u_m(0), w_j)\psi(0).$$

Hence we find,

$$-\int_0^T (u_m(t), \psi'(t)w_j)\mathrm{d}t + \nu\int_0^T ((u_m(t), \psi(t)w_j))\mathrm{d}t$$

$$= (u_{0m}, w_j)\psi(0) + \int_0^T \langle f(t), w_j\rangle \mathrm{d}t. \tag{1.61}$$

The passage to the limit for $m = m' \to \infty$ in the integrals on the left-hand side is easy using (1.54), (1.57), and (1.59); we observe also that

$$u_{0m} \to u_0, \text{ in } H, \text{ strongly}. \tag{1.62}$$

Hence we find in the limit

$$-\int_0^T (u(t), \psi'(t)w_j)\mathrm{d}t + \nu\int_0^T ((u(t), \psi(t)w_j))\mathrm{d}t$$

$$= (u_0, w_j)\psi(0) + \int_0^T \langle f(t), w_j\rangle\psi(t)\mathrm{d}t. \tag{1.63}$$

The equality (1.63) which holds for each j, allows us to write by a linearity argument:

$$-\int_0^T (u(t), v)\psi'(t)\mathrm{d}t + \nu\int_0^T ((u(t), v))\psi(t)\mathrm{d}t$$

$$= (u_0, v)\psi(0) + \int_0^T \langle f(t), v\rangle\psi(t)\mathrm{d}t, \tag{1.64}$$

for each v which is a finite linear combination of the w_j's. Since each term of (1.64) depends linearly and continuously on v, for the norm of V, the equality (1.64) is still valid, by continuity, for each v in V.

Now, writing in particular (1.64) with $\psi = \phi \in \mathscr{D}((0, T))$, we find the following equality which is valid in the distribution sense on $(0, T)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u, v) + \nu((u, v)) = \langle f, v\rangle, \forall\ v \in V; \tag{1.65}$$

which is exactly (1.31). As proved before the statement of Theorem 1.1, this equality and (1.59) imply that u' belongs to $L^2(0, T; V')$ and

$$u' + \nu A u = f. \tag{1.66}$$

Finally, it remains to check that $u(0) = u_0$ (the continuity of u is proved in Section 1.4). For this, we multiply (1.65) by $\psi(t)$, (the same ψ as before), integrate with respect to t, and integrate by parts:

$$\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}(u(t), v)\psi(t)\mathrm{d}t = -\int_0^T (u(t), v)\psi'(t)\mathrm{d}t + (u(0), v)\psi(0).$$

We get

$$-\int_0^T (u(t), v)\psi'(t)\mathrm{d}t + \nu\int_0^T ((u(t), v))\psi(t)\mathrm{d}t$$

$$= (u(0), v)\psi(0) + \int_0^T \langle f(t), v\rangle\psi(t)\mathrm{d}t. \tag{1.67}$$

By comparison with (1.64), we see that

$$(u_0 - u(0), v)\psi(0) = 0,$$

for each $v \in V$, and for each function ψ of the type considered. We can choose ψ such that $\psi(0) \neq 0$, and therefore

$$(u(0) - u_0, v) = 0, \forall\, v \in V.$$

This equality implies that

$$u(0) = u_0$$

and achieves the proof of the existence. □

1.4. *Proof of the Continuity and Uniqueness*

This proof is based on the following lemma which is a particular case of a general theorem of interpolation of Lions–Magenes [1]:

Lemma 1.2. *Let V, H, V' be three Hilbert spaces, each space included* *in the following one as in* (1.5), V' *being the dual of* V. *If a function* $\boldsymbol{u}$ *belongs to* $L^2(0, T; V)$ *and its derivative* $\boldsymbol{u}'$ *belongs to* $L^2(0, T; V')$, *then* $\boldsymbol{u}$ *is almost everywhere equal to a function continuous from* $[0, T]$ *into* H *and we have the following equality, which holds in the scalar distribution sense on* $(0, T)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}|^2 = 2\langle \boldsymbol{u}', \boldsymbol{u}\rangle. \tag{1.68}$$

The equality (1.68) is meaningful since the functions

$$t \to |\boldsymbol{u}(t)|^2, \quad t \to \langle \boldsymbol{u}'(t), \boldsymbol{u}(t)\rangle$$

are both integrable on $[0, T]$.

A proof of this lemma more elementary than that of Lions–Magenes [1] is given below.

If we assume this lemma, (1.39) becomes obvious and it only remains to check the uniqueness. Let us assume that $\boldsymbol{u}$ and $\boldsymbol{v}$ are two solutions of (1.36)–(1.38) and let $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$. Then $\boldsymbol{w}$ belongs to the same spaces as $\boldsymbol{u}$ and $\boldsymbol{v}$, and

$$\boldsymbol{w}' + \nu A\boldsymbol{w} = 0, \quad \boldsymbol{w}(0) = 0. \tag{1.69}$$

Taking the scalar product of the first equality (1.69) with $\boldsymbol{w}(t)$, we find

$$\langle \boldsymbol{w}'(t), \boldsymbol{w}(t)\rangle + \nu\|\boldsymbol{w}(t)\|^2 = 0 \text{ a.e.}$$

Using then (1.68) with $\boldsymbol{u}$ replaced by $\boldsymbol{w}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{w}(t)|^2 + 2\nu\|\boldsymbol{w}(t)\|^2 = 0$$

$$|\boldsymbol{w}(t)|^2 \leqslant |\boldsymbol{w}(0)|^2 = 0, \quad t \in [0, T].$$

and hence $\boldsymbol{u}(t) = \boldsymbol{v}(t)$ for each t. □

Proof of Lemma 1.2. The elementary proof of Lemma 1.2 which was announced before, is now given in the two following lemmas.

Lemma 1.3. *Under the assumptions of Lemma* 1.2, *the equality* (1.68) *is satisfied.*

Proof. By regularizing the function $\tilde{\boldsymbol{u}}$, from $\mathbb{R}$ into V, which is equal to $\boldsymbol{u}$ on $[0, T]$ and to 0 outside this interval, we easily obtain a sequence of functions $\boldsymbol{u}_m$ such that

$$\forall\ m,\ \boldsymbol{u}_m \text{ is infinitely differentiable from } [0, T] \text{ onto } V, \tag{1.70}$$

as $m \to \infty$,

$$u_m \to u \text{ in } L^2_{\text{loc}}(]0, T[; V),$$

$$u'_m \to u' \text{ in } L^2_{\text{loc}}(]0, T[; V'). \tag{1.71}$$

Because of (1.6) and (1.70), the equality (1.68) for u_m is obvious:

$$\frac{\mathrm{d}}{\mathrm{d}t} |u_m(t)|^2 = 2(u'_m(t), u_m(t)) = 2\langle u'_m(t), u_m(t)\rangle, \forall\, m. \tag{1.72}$$

As $m \to \infty$, it follows from (1.71) that

$$|u_m|^2 \to |u|^2 \text{ in } L^1_{\text{loc}}(]0, T[)$$

$$\langle u'_m, u_m\rangle \to \langle u', u\rangle \text{ in } L^1_{\text{loc}}(]0, T[)$$

These convergences also hold in the distribution sense; therefore we are allowed to pass to the limit in (1.72) in the distribution sense; in the limit we find precisely (1.68).

Since the function

$$t \to \langle u'(t), u(t)\rangle$$

is integrable on $[0, T]$, the equality (1.68) shows us that the function u of Lemma 1.3 satisfies

$$u \in L^\infty(0, T; H). \tag{1.73}$$

In the particular case of the function u satisfying (1.36)–(1.38), this was proved directly in Section 1.3.

According to Lemma 1.1., u is continuous from $[0, T]$ into V'. Therefore, with this and (1.73), the following Lemma 1.4 shows us that u is weakly continuous from $[0, T]$ into H, i.e.,

$$\forall\, v \in H, \text{ the function } t \to (u(t), v) \text{ is continuous.} \tag{1.74}$$

Admitting temporarily this point we can achieve the proof of Lemma 1.2. We must prove that for each $t_0 \in [0, T]$,

$$|u(t) - u(t_0)|^2 \to 0, \text{ as } t \to t_0. \tag{1.75}$$

Expanding this term, we find

$$|u(t)|^2 + |u(t_0)|^2 - 2(u(t), u(t_0)).$$

When $t \to t_0$, $|u(t)|^2 \to |u(t_0)|^2$ since by (1.68),

$$|u(t)|^2 = |u(t_0)|^2 + 2\int_{t_0}^{t} \langle u'(s), u(s)\rangle \mathrm{d}s;$$

and because of (1.74)

$$(u(t), u(t_0)) \to |u(t_0)|^2,$$

so (1.75) is proved.

The proof of Lemma 1.2 is achieved as soon as we prove the next lemma. This lemma is stated in a slightly more general form.

Lemma 1.4. *Let X and Y be two Banach spaces, such that*

$$X \subset Y \tag{1.76}$$

with a continuous injection.

If a function ϕ belongs to $L^\infty(0, T; X)$ and is weakly continuous with values in Y, then ϕ is weakly continuous with values in X.

Proof. If we replace Y by the closure of X in Y, we may suppose that X is dense in Y. Hence the dense continuous imbedding of X into Y gives by duality a dense continuous imbedding of Y' (dual of Y), into X' (dual of X):

$$Y' \subset X'. \tag{1.77}$$

By assumption, for each $\eta \in Y'$,

$$\langle \phi(t), \eta\rangle \to \langle \phi(t_0), \eta\rangle, \text{ as } t \to t_0, \forall\, t_0, \tag{1.78}$$

and we must prove that (1.78) is also true for each $\eta \in X'$.

We first prove that $\phi(t) \in X$ for each t and that

$$\|\phi(t)\|_X \leqslant \|\phi\|_{L^\infty(0, T; X)}, \forall\, t \in [0, T]. \tag{1.79}$$

Indeed, by regularizing the function $\tilde{\phi}$ equal to ϕ on $[0, T]$ and to 0 outside this interval, we find a sequence of smooth functions ϕ_m from $[0, T]$ into X such that

$$\|\phi_m(t)\|_X \leqslant \|\phi\|_{L^\infty(X)}, \forall\, m, \forall\, t \in [0, T]$$

and

$$\langle \phi_m(t), \eta\rangle \to \langle \phi(t), \eta\rangle, \ m \to \infty, \forall\, \eta \in Y'.$$

Since

$$|\langle \phi_m(t), \eta\rangle| \leqslant \|\phi\|_{L^\infty(X)} \|\eta\|_{X'}, \forall\, m, \forall\, t,$$

we obtain in the limit

$$|\langle \phi(t), \eta\rangle| \leqslant \|\phi\|_{L^\infty(X)} \|\eta\|_{X'}, \forall\, t \in [0, T], \forall\, \eta \in Y'.$$

This inequality shows that $\phi(t) \in X$ and that (1.79) holds.

Finally let us prove (1.78) for η in X'. Since $Y'$is dense in X', there exists, for each $\epsilon > 0$, some $\eta_\epsilon \in Y'$ such that

$$\|\eta - \eta_\epsilon\|_{X'} \leqslant \epsilon.$$

We then write

$$\langle \phi(t) - \phi(t_0), \eta\rangle = \langle \phi(t) - \phi(t_0), \eta - \eta_\epsilon\rangle + \langle \phi(t) - \langle \phi(t_0), \eta_\epsilon\rangle$$

$$|\langle \phi(t) - \phi(t_0), \eta\rangle| \leqslant 2\epsilon \|\phi\|_{L^\infty(X)} + |\langle \phi(t) - \phi(t_0), \eta_\epsilon\rangle.$$

As $t \to t_0$, since $\eta_\epsilon \in Y'$, the continuity assumption implies that

$$\langle \phi(t) - \phi(t_0), \eta_\epsilon\rangle \to 0$$

and hence

$$\overline{\lim}\, |\langle \phi(t) - \phi(t_0), \eta\rangle| \leqslant 2\epsilon \|\phi\|_{L^\infty(X)}.$$

Since $\epsilon > 0$ is arbitrarily small, the preceding upper limit is zero, and (1.78) is proved. □

1.5. *Miscellaneous Remarks*

We give in this section some remarks and complements to Theorem 1.1.

An Extension of Theorem 1.1.

Theorem 1.1 is a particular case of an abstract theorem, involving abstract spaces V and H, and an abstract operator A; see Lions–Magenes [1].

If instead of (1.28) we assume that

$$f = f_1 + f_2,\ \ f_1 \in L^2(0, T; V'),\ \ f_2 \in L^1(0, T; H), \tag{1.80}$$

then all the conclusions of Theorem 1.1 are true with only one modification:

$$u' \in L^2(0, T; V') + L^1(0, T; H). \tag{1.81}$$

In the proof of the existence, we write after (1.47):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} |u_m(t)|^2 + 2\nu \|u_m(t)\|^2 &\leqslant 2\|f_1(t)\|_{V'}\ \|u_m(t)\| \\ &+ 2|f_2(t)|\ |u_m(t)| \leqslant \nu \|u_m(t)\|^2 \\ &+ \frac{1}{\nu} \|f_1(t)\|_{V'}^2 + |f_2(t)|\ \{1 + |u_m(t)|^2\}. \end{aligned} \tag{1.82}$$

Hence, in particular

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\{1+|u_m(t)|^2\} \leqslant \frac{1}{\nu}\,\|f_1(t)\|_{V'}^2 + |f_2(t)|\,\{1+|u_m(t)|^2\}. \tag{1.83}$$

Multiplying this by

$$\exp\left\{-\int_0^t |f_2(\sigma)|\,\mathrm{d}\sigma\right\},$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\exp\left(-\int_0^t |f_2(\sigma)|\mathrm{d}\sigma\right)\cdot(1+|u_m(t)|^2)\right\}$$

$$\leqslant \frac{1}{\nu}\,\|f_1(t)\|_{V'}^2\,\exp\left(-\int_0^t |f_2(\sigma)|\mathrm{d}\sigma\right).$$

Integrating this inequality from 0 to s, $s > 0$, we obtain a majoration similar to (1.50) which implies (1.51). Then integrating (1.82) from 0 to T we obtain (1.53).

The proof of the existence is then conducted exactly as in Section 1.3.

Concerning the derivative u', we have

$$u' = -\nu A u + f_1 + f_2 \in L^2(0, T; V') + L^1(0, T; H). \tag{1.84}$$

It is easy to see that Lemma 1.2 is also valid if

$$\begin{aligned} &u \in L^2(0, T; V) \cap L^\infty(0, T; H),\\ &u' \in L^2(0, T; V') + L^1(0, T; H). \end{aligned} \tag{1.85}$$

Noting this, we can prove the uniqueness and the continuity of u, $u \in \mathscr{C}([0, T]; H)$, exactly as in Section 1.4. □

The Case Ω Unbounded.

For the evolution problem, when Ω is unbounded, *the introduction of the space Y considered in the stationary unbounded case* (Chapter I, Section 2.3) is no longer necessary. All the previous results hold if Ω is unbounded and V is equipped with the norm (1.9). Let us assume, in the most general case, that f satisfies (1.80). We have exactly the same results as in Theorem 1.1, if f satisfies (1.28), and the same results as in

(1.81) if f satisfies (1.80). The only difference is that we must replace (1.82) by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,&|\boldsymbol{u}_m(t)|^2 + 2\nu\|\boldsymbol{u}_m(t)\|^2\\ &\leqslant 2\|f_1(t)\|_{V'}\|\boldsymbol{u}_m(t)\| + 2|f_2(t)|\,|\boldsymbol{u}_m(t)|\\ &\leqslant \nu\|\boldsymbol{u}_m(t)\|^2 + \nu|\boldsymbol{u}_m(t)|^2 + \frac{1}{\nu}\|f_1(t)\|_{V'}^2\\ &\quad + |f_2(t)|(1+|\boldsymbol{u}_m(t)|^2)\end{aligned} \tag{1.86}$$

Hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\{1+|\boldsymbol{u}_m(t)|^2\} &\leqslant (|f_2(t)|+\nu)\,\{1+|\boldsymbol{u}_m(t)|^2\}\\ &\quad + \frac{1}{\nu}\,\|f_1(t)\|_{V'}^2.\end{aligned} \tag{1.87}$$

This inequality is then treated exactly as (1.83), to obtain (1.51). After that, integrating (1.86) from 0 to T, we obtain

$$\int_0^T \|\boldsymbol{u}_m(t)\|^2\,\mathrm{d}t \leqslant \text{Const.}$$

This estimation, together with (1.51), gives (1.53).

The proofs of the existence, the uniqueness, and the continuity are then exactly the same as before. □

Interpretation of the Variational Problem

We wish to make precise in what sense the function $\boldsymbol{u}$ defined by Theorem 1.1 is a solution of the initial problem (1.23)–(1.26).

Proposition 1.1. *Under the assumptions of Theorem* 1.1, *there exists a distribution p on $Q = \Omega \times (0, T)$, such that the function $\boldsymbol{u}$ defined by Theorem* 1.1 *and p satisfy* (1.23) *in the distribution sense in Q;* (1.24) *is satisfied in the distribution sense too and* (1.26) *is satisfied in the sense*

$$\boldsymbol{u}(t) \to \boldsymbol{u}_0 \text{ in } L^2(\Omega), \text{ as } t \to 0. \tag{1.88}$$

Proof. The equality (1.24) is an easy consequence of $\boldsymbol{u} \in L^2(0, T; V)$; (1.26) and (1.88) follow also immediately from Theorem 1.1; (1.25) is satisfied in a sense which depends on the trace theorems available for Ω since $\boldsymbol{u}$ is in $L^2(0, T; \boldsymbol{H}_0^1(\Omega))$.

To introduce the pressure, let us set

$$\boldsymbol{U}(t) = \int_0^t \boldsymbol{u}(s)\mathrm{d}s,\quad \boldsymbol{F}(t) = \int_0^t f(s)\mathrm{d}s. \tag{1.89}$$

It is clear that, at least,

$$U \in \mathscr{C}([0, T]; V), \ \ F \in \mathscr{C}([0, T]; V').$$

Integrating (1.31), we see that

$$(u(t) - u_0, v) + \nu((U(t), v)) = \langle F(t), v\rangle, \forall\ t \in [0, T], \quad \forall\ v \in V, \tag{1.90}$$

or

$$\langle u(t) - u_0 - \nu\Delta U(t) - F(t), v\rangle = 0, \quad \forall t \in [0, T], \forall\ v \in V.$$

By an application of Proposition I.1.1 and I.1.2, we find, for each $t \in [0, T]$, the existence of some function $P(t)$,

$$P(t) \in L^2(\Omega),$$

such that

$$u(t) - u_0 - \nu\Delta U(t) + \operatorname{grad} P(t) = F(t). \tag{1.91}$$

We infer from Remark I.1.4 (ii) that the gradient operator is an isomorphism from $L^2(\Omega)/\mathbb{R}$ into $H^{-1}(\Omega)$. Observing that

$$\operatorname{grad} P = F + \nu\Delta U - u + u_0, \tag{1.92}$$

we conclude that grad P belongs to $\mathscr{C}([0, T]; H^{-1}(\Omega))$ as does the right-hand side of (1.92); therefore

$$P \in \mathscr{C}([0, T]; L^2(\Omega)). \tag{1.93}$$

This enables us to differentiate (1.91) in the t variable, in the distribution sense in $Q = \Omega \times (0, T)$; setting

$$p = \frac{\partial P}{\partial t}, \tag{1.94}$$

we obtain precisely (1.23).

We do not have in general any information on p better than (1.93)–(1.94). In the next proposition, we will get more regularity on p after assuming more regularity on the data f and u_0. $\square$

Some Results of Regularity

Assuming that the data Ω, f, u_0, are sufficiently smooth, we can obtain as much regularity as desired for u and p. We will only prove a simple result of this type:

Proposition 1.2. *Let us assume that Ω is of class $\mathscr{C}^2$, that*

$$f \in L^2(0, T; H)$$

and

$$u_0 \in V. \tag{1.96}$$

Then

$$u \in L^2(0, T; H^2(\Omega)), \tag{1.97}$$

$$u' \in L^2(0, T; H), \text{ i.e., } u' \in L^2(Q), \tag{1.98}$$

$$p \in L^2(0, T; H^1(\Omega)). \tag{1.99}$$

Proof. The first point is to obtain (1.98); this is proved by getting another *a priori* estimate for the approximate solution u_m constructed by the Galerkin method.

Using the notation of Section 1.3, we multiply (1.41) by $g'_{jm}(t)$, and add these equalities for $j = 1, \ldots, m$; this gives

$$|u'_m(t)|^2 + \nu((u_m(t), u'_m(t))) = (f(t), u'_m(t))$$

or

$$2|u'_m(t)|^2 + \nu \frac{\mathrm{d}}{\mathrm{d}t} \|u_m(t)\|^2 = 2(f(t), u'_m(t)). \tag{1.100}$$

We then integrate (1.100) from 0 to T, and use the Schwarz inequality; we obtain

$$\begin{aligned} 2\int_0^T |u'_m(t)|^2 \mathrm{d}t + \nu\|u_m(T)\|^2 &= \nu\|u_{0m}\|^2 \\ &+ 2\int_0^T (f(t), u'_m(t))\mathrm{d}t \leqslant \nu\|u_{0m}\|^2 \\ &+ \int_0^T |f(t)|^2 \mathrm{d}t + \int_0^T |u'_m(t)|^2 \mathrm{d}t, \end{aligned}$$

$$\int_0^T |u'_m(t)|^2 \mathrm{d}t \leqslant \nu\|u_{0m}\|^2 + \int_0^T |f(t)|^2 \mathrm{d}t. \tag{1.101}$$

The basis w_j used for the Galerkin method may be chosen so that $w_j \in V$ for each j and we can take

u_{0m} = the projection in V of u_0 on the space spanned by $w_1, \ldots, w_m$.

Therefore

$$u_{0m} \to u_0 \text{ in } V \text{ strongly as } m \to \infty, \tag{1.102}$$

and

$$\|u_{0m}\| \leqslant \|u_0\|. \tag{1.103}$$

With these choices of the w_j's and of u_{0m}, (1.101) shows that

$$u_m' \textit{ belongs to a bounded set of } L^2(0, T; H), \tag{1.104}$$

and (1.98) is proved.

We then come back to the equalities (1.23), (1.24), (1.25) and we apply the regularity theorem of the stationary case (Theorem I.2.4): For almost every t in $[0, T]$,

$$\left.\begin{aligned} &-\nu\Delta u(t) + \operatorname{grad} p(t) = f - u' \in L^2(0, T; L^2(\Omega)) \\ &\operatorname{div} u(t) = 0 \text{ in } \Omega \\ &u(t) = 0 \text{ on } \partial\Omega \end{aligned}\right\} \tag{1.105}$$

so that $u(t)$ belongs to $H^2(\Omega)$ and $p(t)$ belongs to $H^1(\Omega)$. Moreover, since the mapping

$$f(t) - u'(t) \to \{u(t), p(t)\}$$

is linear continuous from $L^2(\Omega)$ into $H^2(\Omega) \times H^1(\Omega)$, due to I.(2.40), and since

$$f - u' \in L^2(0, T; L^2(\Omega)),$$

it is clear that (1.97) and (1.99) are satisfied.

§2. Compactness Theorems

The compactness theorems presented in Chapter II are not sufficient for the nonlinear evolution problems. Our goal now is to prove some compactness theorems which are appropriate for the nonlinear problems of the remainder of this chapter.

After a preliminary result given in Section 2.1, we prove in Section 2.2 a compactness theorem in the frame of Banach spaces. In Section 2.3 we prove two other compactness theorems in the frame of Hilbert spaces; one of them involves fractional derivatives in time of the functions.

Some related discrete forms of these theorems will be studied later on. □

2.1. *A Preliminary Result*

The proofs of the compactness theorems of the next two sections will be based on the following lemma.

Lemma 2.1. *Let X_0, X, and X_1 be three Banach spaces such that*

$$X_0 \subset X \subset X_1, \tag{2.1}$$

the injection of X into X_1 being continuous, and:

$$\text{the injection of } X_0 \text{ into } X \text{ is compact.} \tag{2.2}$$

Then for each $\eta > 0$, there exists some constant c_η depending on η (and on the spaces X_0, X, X_1) such that:

$$\|v\|_X \leqslant \eta \|v\|_{X_0} + c_\eta \|v\|_{X_1}, \quad \forall v \in X_0. \tag{2.3}$$

Proof. The proof is by contradiction. Saying that (2.3) is not true amounts to saying that there exists some $\eta > 0$ such that for each c in $\mathbb{R}$,

$$\|v\|_X \geqslant \eta \|v\|_{X_0} + c \|v\|_{X_1},$$

for at least one v. Taking $c = m$, we obtain a sequence of elements v_m satisfying

$$\|v_m\|_X \geqslant \eta \|v_m\|_{X_0} + m \|v_m\|_{X_1}, \ \forall m.$$

We consider then the normalized sequence

$$w_m = \frac{v_m}{\|v_m\|_{X_0}},$$

which satisfies

$$\|w_m\|_X \geqslant \eta + m \|w_m\|_{X_1}, \ \forall m. \tag{2.4}$$

Since $\|w_m\|_{X_0} = 1$, the sequence w_m is bounded in X and (2.4) shows that

$$\|w_m\|_{X_1} \to 0, \ \text{ as } m \to \infty. \tag{2.5}$$

In addition, by (2.2), the sequence $\boldsymbol{w}_m$ is relatively compact in X; hence we can extract from $\boldsymbol{w}_m$ a subsequence $\boldsymbol{w}_\mu$ strongly convergent in X. From (2.5) the limit of $\boldsymbol{w}_\mu$ must be 0, but this contradicts (2.4) as:

$$\|\boldsymbol{w}_\mu\|_X \geqslant \eta > 0, \forall\ \mu.$$

2.2. *A Compactness Theorem in Banach Spaces*

Let X_0, X, X_1, be three Banach spaces such that

$$X_0 \subset X \subset X_1, \tag{2.6}$$

where the injections are continuous and:

$$X_i \text{ is reflexive}, \quad i = 0, 1, \tag{2.7}$$

$$\text{the injection } X_0 \to X \text{ is compact.} \tag{2.8}$$

Let $T > 0$ be a fixed finite number, and let α_0, α_1, be two finite numbers such that $\alpha_i > 1$, $i = 0, 1$.

We consider the space

$$\mathscr{Y} = \mathscr{Y}(0, T; \alpha_0, \alpha_1; X_0, X_1) \tag{2.9}$$

$$\mathscr{Y} = \left\{ v \in L^{\alpha_0}(0, T; X_0),\ v' = \frac{\mathrm{d}v}{\mathrm{d}t} \in L^{\alpha_1}(0, T; X_1) \right\}. \tag{2.10}$$

The space $\mathscr{Y}$ is provided with the norm

$$\|v\|_{\mathscr{Y}} = \|v\|_{L^{\alpha_0}(0, T; X_0)} + \|v'\|_{L^{\alpha_1}(0, T; X_1)}, \tag{2.11}$$

which makes it a Banach space. It is evident that

$$\mathscr{Y} \subset L^{\alpha_0}(0, T; X),$$

with a continuous injection. Actually we shall prove that this injection is compact.

Theorem 2.1. *Under the assumptions* (2.6) *to* (2.9) *the injection of* $\mathscr{Y}$ *into* $L^{\alpha_0}(0, T; X)$ *is compact.*

Proof. (i) Let $\boldsymbol{u}_m$ be some sequence which is bounded in $\mathscr{Y}$. We must prove that this sequence contains a subsequence $\boldsymbol{u}_\mu$ strongly convergent in $L^{\alpha_0}(0, T; X)$.

Since the spaces X_i are reflexive spaces and $1 < \alpha_i < +\infty$, the spaces $L^{\alpha_i}(0, T; X_i)$, $i = 0, 1$, are likewise reflexive and hence $\mathscr{Y}$ is reflexive. Therefore, there exists some $\boldsymbol{u}$ in $\mathscr{Y}$ and some subsequence $\boldsymbol{u}_\mu$ with

$$\boldsymbol{u}_\mu \to \boldsymbol{u} \text{ in } \mathscr{Y} \text{ weakly, as } \mu \to \infty, \tag{2.12}$$

which means

$$u_\mu \to u \text{ in } L^{\alpha_0}(0, T; X_0) \text{ weakly}$$
$$u'_\mu \to u' \text{ in } L^{\alpha_1}(0, T; X_1) \text{ weakly}. \tag{2.13}$$

It suffices to prove that

$$v_\mu = u_\mu - u \text{ converges to 0 in } L^{\alpha_0}(0, T; X) \text{ strongly}. \tag{2.14}$$

(ii) The theorem will be proved if we show that

$$v_\mu \to 0 \text{ in } L^{\alpha_0}(0, T; X_1) \text{ strongly}. \tag{2.15}$$

In fact, due to Lemma 2.1, we have

$$\|v_\mu\|_{L^{\alpha_0}(0,T;X)} \leqslant \eta \|v_\mu\|_{L^{\alpha_0}(0,T;X_0)} + c_\eta \|v_\mu\|_{L^{\alpha_0}(0,T;X_1)}$$

and since the sequence v_μ is bounded in $\mathscr{Y}$:

$$\|v_\mu\|_{L^{\alpha_0}(0,T;X)} \leqslant c\eta + c_\eta \|v_\mu\|_{L^{\alpha_0}(0,T;X_1)}. \tag{2.16}$$

Passing to the limit in (2.16) we see by (2.15) that

$$\overline{\lim_{\mu \to \infty}} \|v_\mu\|_{L^{\alpha_0}(0,T;X)} \leqslant c\eta; \tag{2.17}$$

since $\eta > 0$ is arbitrarily small in Lemma 2.1, this upper limit is 0 and thus (2.14) is proved.

(iii) To prove (2.15) we observe that

$$\mathscr{Y} \subset \mathscr{C}([0, T]; X_1), \tag{2.18}$$

with a continuous injection; the inclusion (2.18) results from Lemma 1.1, and the continuity of the injection is very easy to check.

We infer from this, the majoration

$$\|v_\mu(t)\|_{X_1} \leqslant c, \ \forall\, t \in [0, T], \ \forall\, \mu. \tag{2.19}$$

According to Lebesgue's Theorem, (2.15) is now proved if we show that, for almost every t in $[0, T]$,

$$v_\mu(t) \to 0 \text{ in } X_1 \text{ strongly, as } \mu \to \infty. \tag{2.20}$$

We shall prove (2.20) for $t = 0$; the proof would be similar for any other t. We write

$$v_\mu(0) = v_\mu(t) - \int_0^t v'_\mu(\tau)\mathrm{d}\tau$$

and by integration

$$v_\mu(0) = \frac{1}{s}\left\{\int_0^s v_\mu(t)\mathrm{d}t - \int_0^s\int_0^t v'_\mu(\tau)\mathrm{d}\tau\mathrm{d}t\right\}.$$

Hence

$$v_\mu(0) = a_\mu + b_\mu \tag{2.21}$$

with

$$a_\mu = \frac{1}{s}\int_0^s v_\mu(t)\mathrm{d}t,\quad b_\mu = -\frac{1}{s}\int_0^s (s-t)v'_\mu(t)\mathrm{d}t. \tag{2.22}$$

For a given $\epsilon > 0$, we choose s so that

$$\|b_\mu\|_{X_1} \leqslant \int_0^s \|v'_\mu(t)\|_{X_1}\,\mathrm{d}t \leqslant \frac{\epsilon}{2}.$$

Then, for this fixed s, we observe that, as $\mu \to \infty$, $a_\mu \to 0$ in X_0 weakly and thus in X_1 strongly; for μ sufficiently large

$$\|a_\mu\|_{X_1} \leqslant \frac{\epsilon}{2},$$

and (2.20), for $t = 0$, follows.

2.3. *A Compactness Theorem Involving Fractional Derivatives*

The next compactness theorem is in the frame of Hilbert spaces and is based on the notion of fractional derivatives of a function.

Let us assume that X_0, X, X_1, are Hilbert spaces with

$$X_0 \subset X \subset X_1, \tag{2.23}$$

the injections being continuous and

$$\text{the injection of } X_0 \text{ into } X \text{ is compact.} \tag{2.24}$$

If v is a function from $\mathcal{R}$ into X_1, we denote by $\hat{v}$ its Fourier transform

$$\hat{v}(\tau) = \int_{-\infty}^{+\infty} \mathrm{e}^{-2i\pi t\tau}\, v(t)\mathrm{d}t. \tag{2.25}$$

The derivative in t of order γ of v is the inverse Fourier transform of $(2i\pi\tau)^\gamma \hat{v}$ or

$$\widehat{\mathrm{D}_t^\gamma v(\tau)} = (2i\pi\tau)^\gamma\, \hat{v}\,(\tau).\text{[1]} \tag{2.26}$$

For given $\gamma > 0$ [2], we define the space

$$\mathcal{H}^\gamma(\mathcal{R}; X_0, X_1) = \{v \in L^2(\mathcal{R}; X_0),\ \mathrm{D}_t^\gamma v \in L^2(\mathcal{R}; X_1)\}. \tag{2.27}$$

This is a Hilbert space for the norm.

$$\|v\|_{\mathcal{H}^\gamma(\mathcal{R}, X_0, X_1)} = \{\|v\|^2_{L^2(\mathcal{R}; X_0)} + \||\tau|^\gamma\, \hat{v}\|^2_{L^2(\mathcal{R}; X_1)}\}^{1/2}.$$

We associate with any set $K \subset \mathcal{R}$, the subspace $\mathcal{H}_K^\gamma$ of $\mathcal{H}^\gamma$ defined as the set of functions u in $\mathcal{H}^\gamma$ with support contained in K:

$$\mathcal{H}_K^\gamma(\mathcal{R}; X_0, X_1) = \{u \in \mathcal{H}^\gamma(\mathcal{R}; X_0, X_1),\ \text{support}\, u \subset K\}. \tag{2.28}$$

The compactness theorem may now be stated:

Theorem 2.2. *Let us assume that X_0, X, X_1 are Hilbert spaces which satisfy* (2.23) *and* (2.24).

Then for any bounded set K and any $\gamma > 0$, the injection of $\mathcal{H}_K^\gamma(\mathcal{R}; X_0, X_1)$ into $L^2(\mathcal{R}, X)$ is compact.

Proof. (i) Let γ and K be fixed, and let u_m be a bounded sequence in $\mathcal{H}_K^\gamma(\mathcal{R}; X_0, X_1)$. We must show that u_m contains a subsequence strongly convergent in $L^2(\mathcal{R}; X)$.

Since $\mathcal{H}^\gamma(\mathcal{R}; X_0, X_1)$ is a Hilbert space, the sequence u_μ contains a subsequence weakly convergent in this space to some element u. It is clear that u must also belong to $\mathcal{H}_K^\gamma$; therefore, setting

$$v_\mu = u_\mu - u,$$

the sequence v_μ appears as a bounded sequence of $\mathcal{H}_K^\gamma(\mathcal{R}; X_0, X_1)$, which converges weakly to 0 in $\mathcal{H}^\gamma$; this means

$$v_\mu \to 0 \text{ in } L^2(\mathcal{R}; X_0) \text{ weakly} \tag{2.29}$$

$$|\tau|^\gamma \hat{v}_\mu \to 0 \text{ in } L^2(\mathcal{R}; X_1) \text{ weakly.} \tag{2.30}$$

The theorem is proved if we show that u_μ converges strongly to u in $L^2(\mathcal{R}; X)$, which is the same as

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[1] The definition (2.26) is consistent with the usual definition for γ an integer.

[2] In the applications, $0 < \gamma \leqslant 1$ in general.

$$\nu_\mu \to 0 \text{ in } L^2(\mathbb{R}; X) \text{ strongly.} \tag{2.31}$$

(ii) The second point of the proof is to show that (2.31) is proved if we prove that

$$\nu_\mu \to 0 \text{ in } L^2(\mathbb{R}; X_1) \text{ strongly.} \tag{2.32}$$

Due to Lemma 2.1,

$$\|\nu_\mu\|_{L^2(\mathbb{R}; X)} \leqslant \eta \|\nu_\mu\|_{L^2(\mathbb{R}; X_0)} + c_\eta \|\nu_\mu\|_{L^2(\mathbb{R}; X_1)} \tag{2.33}$$

and since ν_μ is bounded in $L^2(\mathbb{R}; X_0)$,

$$\|\nu_\mu\|_{L^2(\mathbb{R}; X)} \leqslant c\eta + c_\eta \|\nu_\mu\|_{L^2(\mathbb{R}; X_1)}. \tag{2.34}$$

If we assume (2.32), then letting $\mu \to \infty$ in (2.34), we obtain

$$\overline{\lim_{\mu \to \infty}} \|\nu_\mu\|_{L^2(\mathbb{R}; X)} \leqslant c\eta.$$

Since η is arbitrarily small in Lemma 2.1, this upper limit must be 0 and (2.31) follows.

(iii) Finally let us prove (2.32). According to the Parseval theorem,

$$I_\mu = \int_{-\infty}^{+\infty} \|\nu_\mu(t)\|_{X_1}^2 \, dt = \int_{-\infty}^{+\infty} \|\hat{\nu}_\mu(\tau)\|_{X_1}^2 \, d\tau, \tag{2.35}$$

where $\hat{\nu}_\mu$ denotes the Fourier transform of ν_μ. We must show that

$$I_\mu \to 0 \text{ as } \mu \to \infty. \tag{2.36}$$

For this, we write

$$I_\mu = \int_{|\tau| \leqslant M} \|\hat{\nu}_\mu(\tau)\|_{X_1}^2 \, d\tau + \int_{|\tau| > M} (1 + |\tau|^{2\gamma}) \|\hat{\nu}_\mu(\tau)\|_{X_1}^2 \cdot \frac{d\tau}{(1 + |\tau|^{2\gamma})} \leqslant \frac{c}{1 + M^{2\gamma}} + \int_{|\tau| \leqslant M} \|\hat{\nu}_\mu(\tau)\|_{X_1}^2 \, d\tau,$$

since ν_μ is bounded in $\mathcal{H}^\gamma$.

For a given $\epsilon > 0$, we choose M such that

$$\frac{c}{1 + M^{2\gamma}} \leqslant \frac{\epsilon}{2}.$$

Hence

$$I_\mu \leqslant \int_{|\tau| \leqslant M} \|\hat{v}_\mu(\tau)\|_{X_1}^2 \, d\tau + \frac{\epsilon}{2},$$

and (2.36) is proved if we show that, for this fixed M,

$$J_\mu = \int_{|\tau| \leqslant M} \|\hat{v}_\mu(\tau)\|_{X_1}^2 \, d\tau \to 0, \text{ as } \mu \to \infty. \tag{2.37}$$

This is proved via the Lebesgue theorem. If χ denotes the characteristic function of K, then $v_\mu \chi = v_\mu$ and

$$\hat{v}_\mu(\tau) = \int_{-\infty}^{+\infty} e^{-2i\pi t\tau} \chi(t) v_\mu(t) dt.$$

Thus

$$\|\hat{v}_\mu(\tau)\|_{X_1} \leqslant \|v_\mu\|_{L^2(\mathbb{R};\, X_1)} \|e^{-2i\pi t\tau} \chi\|_{L^2(\mathbb{R})},$$

$$\|\hat{v}_\mu(\tau)\|_{X_1} \leqslant \text{Const.} \tag{2.38}$$

On the other hand for each σ in X_0, and each fixed τ,

$$((\hat{v}_\mu(\tau), \sigma))_{X_0} = \int_{-\infty}^{+\infty} ((v_\mu(t), e^{-2i\pi t\tau} \chi(t)\sigma))_{X_0} dt,$$

which goes to 0 as $\mu \to \infty$ because of (2.29). The sequence $\hat{v}_\mu(\tau)$ converges to 0 weakly in X_0 and therefore strongly in X and X_1.

With this last remark and (2.38), the Lebesgue theorem implies (2.37). □

Using the methods of the last theorem, we can prove another compactness theorem similar to Theorem 2.1. Nevertheless, this theorem is not contained in, nor itself contains, Theorem 2.2.

Theorem 2.3. *Under the hypotheses* (2.23) *and* (2.24), *the injection of* $\mathcal{Y}(0, T; 2, 1; X_0, X_1)$[1] *into* $L^2(0, T; X)$ *is compact.*

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[1] For the definition of this space see (2.9)–(2.10).

Proof. Let u_m be a bounded sequence in the space $\mathscr{Y}$; we denote by $\tilde{u}_m$ the function defined on the whole line $\mathcal{R}$, which is equal to u_m on $[0, T]$, and to 0 outside this interval. By Theorem 2.2, the result is proved if we show that the sequence $\tilde{u}_m$ remains bounded in the space $\mathcal{H}^\gamma(\mathcal{R}; X_0, X_1)$, for some $\gamma > 0$.

Because of Lemma 1.1, each function u_m is, after modification on a set of measure 0, continuous from $[0, T]$ into X_1; more precisely the injection of $\mathscr{Y}$ into $\mathscr{C}([0, T]; X_1)$ is continuous.

It is classical that since $\tilde{u}_m$ has two discontinuities, at 0 and T, the distribution derivative of $\tilde{u}_m$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{u}_m = \tilde{g}_m + u_m(0)\delta_0 - u_m(T)\delta_T, \tag{2.39}$$

where δ_0 and δ_T are the Dirac distributions at 0 and T, and

$$g_m = u_m' = \text{the derivative of } u_m \text{ on } [0, T]. \tag{2.40}$$

After a Fourier transformation, (2.39) gives

$$2i\pi\tau\hat{u}_m(\tau) = \hat{g}_m(\tau) + u_m(0) - u_m(T)\exp(-2i\pi\tau T), \tag{2.41}$$

where $\hat{g}_m$ and $\hat{u}_m$ denote the Fourier transforms of $\hat{g}_m$ and $\hat{u}_m$ respectively.

Since the functions g_m remain bounded in $L^1(0, T; X_1)$, the functions $\tilde{g}_m$ remain bounded in $L^1(\mathcal{R}; X_1)$ and the functions $\hat{g}_m$ are bounded in $L^\infty(\mathcal{R}; X_1)$:

$$\|\hat{g}_m(\tau)\|_{X_1} \leqslant \text{Const.}, \forall\, m, \forall\, \tau \in \mathcal{R}. \tag{2.42}$$

We have noted that the injection of $\mathscr{Y}$ into $\mathscr{C}([0, T]; X_1)$ is continuous; thus

$$\|u_m(0)\|_{X_1} \leqslant \text{Const.}, \quad \|u_m(T)\|_{X_1} \leqslant \text{Const.},$$

and (2.41) shows us that

$$|\tau|^2 \; \|\hat{u}_m(\tau)\|_{X_1}^2 \leqslant c, \; \forall\, m, \; \forall\, \tau \in \mathcal{R}. \tag{2.43}$$

For a fixed $\gamma < 1/2$, we observe that

$$|\tau|^{2\gamma} \leqslant c_0(\gamma)\,\frac{1 + \tau^2}{1 + |\tau|^{2(1-\gamma)}}, \; \forall\, \tau \in \mathcal{R}.$$

Therefore

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\hat{u}_m(\tau)\|_{X_1}^2 \,\mathrm{d}\tau \leqslant c_0(\gamma) \int_{-\infty}^{+\infty} \frac{1+\tau^2}{1+|\tau|^{2(1-\gamma)}} \|\hat{u}_m(\tau)\|_{X_1}^2 \,\mathrm{d}\tau$$

$$\leqslant c_1 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau}{1+|\tau|^{2(1-\gamma)}} + c_0(\gamma) \int_{-\infty}^{+\infty} \|\hat{u}_m(\tau)\|_{X_1}^2 \,\mathrm{d}\tau$$

by (2.43).

Since $\gamma < 1/2$, the integral

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau}{1+|\tau|^{2(1-\gamma)}}$$

is convergent; on the other hand, by the Parseval equality, we see that

$$\int_{-\infty}^{+\infty} \|\hat{u}_m(\tau)\|_{X_1}^2 \,\mathrm{d}\tau = \int_0^T \|u_m(t)\|_{X_1}^2 \,\mathrm{d}t,$$

and these integrals are bounded.

We conclude that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\hat{u}_m(\tau)\|_{X_1}^2 \,\mathrm{d}\tau \leqslant c_2, \tag{2.44}$$

where c_2 depends on γ.

It is now clear that the sequence u_m is bounded in $\mathcal{H}^\gamma(\mathcal{R}; X_0, X_1)$ and the proof is achieved. □

Remark 2.1. Assuming only that X_1 is a Hilbert space, X_0, X being Banach spaces, it can be proved in a similar way that the injection of

$$\mathcal{Y}(0, T; \alpha_0, 1; X_0, X_1),$$

into $L^{\alpha_0}(0, T; X)$ is compact for any finite number $\alpha_0 > 1$. □

§3. Existence and Uniqueness Theorems ($n \leqslant 4$)

This section is concerned with existence and uniqueness theorems for weak solutions of the full Navier–Stokes equations ($n \leqslant 4$). In Section

3.1 we give the variational formulation of these equations, following J. Leray, and state an existence theorem for such solutions for $n \leqslant 4$. The proof of this theorem, due to J. L. Lions, is given in Section 3.2. It is based on the construction of an approximate solution by the Galerkin method; then a passage to the limit using, in particular, an *a priori* estimate on a fractional derivative in time of the approximate solution, and a compactness theorem contained in Section 2. An alternate proof based on a semi-discretization in time and valid in all dimensions is discussed in Section 4.

In Section 3.3 we develop the uniqueness theorem of weak solutions ($n = 2$). In the three-dimensional case there is a gap between the class of functions where existence is known, and the smaller classes where uniqueness is proved; an example of such a uniqueness theorem is developed in Section 3.4 ($n = 3$). In Section 3.5 we show in the two-dimensional case the existence of more regular solutions, assuming more regularity on the data; a similar result holds in the three-dimensional case for local solutions, that is solutions which are defined on some "small" interval of time, assuming that the data is sufficiently small.

3.1. *An Existence Theorem in* $\mathbb{R}^n$ ($n \leqslant 4$)

Notations are as above, in particular that recalled at the beginning of Section 1.1; Ω is an open Lipschitz set which for simplicity, we suppose bounded; the unbounded case is discussed in Remarks 3.1 and 3.2.

We recall [1] that since the dimension is less than or equal to 4, one can define on $H_0^1(\Omega)$, and in particular on V, a trilinear continuous form $\boldsymbol{b}$ by setting

$$\boldsymbol{b}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i,j=1}^{n} \int_{\Omega} u_i(D_i v_j) w_j \mathrm{d}x. \tag{3.1}$$

If $\boldsymbol{u} \in V$, then

$$\boldsymbol{b}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0, \forall\, \boldsymbol{v} \in H_0^1(\Omega). \tag{3.2}$$

For $\boldsymbol{u}, \boldsymbol{v}$ in V, we denote by $B(\boldsymbol{u}, \boldsymbol{v})$ the element of V' defined by

$$\langle B(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\rangle = \boldsymbol{b}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \forall\, \boldsymbol{w} \in V, \tag{3.3}$$

and we set

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[1] cf. Section 1.1, Chapter II.

$$B(u) = B(u, u) \in V', \forall\, u \in V. \tag{3.4}$$

In its classical formulation, the initial boundary value problem of the full Navier–Stokes equations is the following:

To find a vector function

$$u: \Omega \times [0, T] \to \mathbb{R}^n$$

and a scalar function

$$p: \Omega \times [0, T] \to \mathbb{R},$$

such that

$$\frac{\partial u}{\partial t} - \nu\Delta u + \sum_{i=1}^{n} u_i D_i u + \operatorname{grad} p = f \text{ in } Q = \Omega \times (0, T), \tag{3.5}$$

$$\operatorname{div} u = 0 \text{ in } Q, \tag{3.6}$$

$$u = 0 \text{ on } \partial\Omega \times (0, T), \tag{3.7}$$

$$u(x, 0) = u_0(x), \text{ in } \Omega. \tag{3.8}$$

As before, the functions f and u_0 are given, defined on $\Omega \times [0, T]$ and Ω respectively.

Let us assume that u and p are classical solutions of (3.5)–(3.8), say $u \in \mathscr{C}^2(\bar{Q})$, $p \in \mathscr{C}^1(\bar{Q})$. Obviously $u \in L^2(0, T; V)$, and if v is an element of $\mathscr{V}$, one can check easily that

$$\frac{\mathrm{d}}{\mathrm{d}t}(u, v) + \nu((u, v)) + b(u, u, v) = \langle f, v\rangle. \tag{3.9}$$

By continuity, equation (3.9) will hold for each $v \in V$.

This suggests the following weak formulation of the problem (3.5)–(3.9) (cf. J. Leray [1], [2], [3]):

Problem 3.1. *For f and u given with*

$$f \in L^2(0, T; V') \tag{3.10}$$

$$u_0 \in H, \tag{3.11}$$

to find u satisfying

$$u \in L^2(0, T; V) \tag{3.12}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}, \boldsymbol{v}) + \nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \ \forall\, \boldsymbol{v} \in V \tag{3.13}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{3.14}$$

If $\boldsymbol{u}$ merely belongs to $L^2(0, T; V)$, the condition (3.14) need not make sense. But if $\boldsymbol{u}$ belongs to $L^2(0, T; V)$ and satisfies (3.13), then we will show as in the linear case (using Lemma 1.1) that $\boldsymbol{u}$ is almost everywhere equal to some continuous function, so that (3.14) is meaningful.

Before showing this, we recall that we are considering the case $n \leqslant 4$; we will modify slightly the preceding formulation in higher dimensions (see Section 4.1).

Lemma 3.1. *We assume that the dimension of the space is $n \leqslant 4$ and that $\boldsymbol{u}$ belongs to $L^2(0, T; V)$.*

Then the function $B\boldsymbol{u}$ defined by

$$\langle B\boldsymbol{u}(t), \boldsymbol{v}\rangle = b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}), \ \forall\, \boldsymbol{v} \in V, \text{ a.e. in } t \in [0, T],$$

belongs to $L^1(0, T; V')$.

Proof. For almost all t, $B\boldsymbol{u}(t)$ is an element of V', and the measurability of the function

$$t \in [0, T] \to B\boldsymbol{u}(t) \in V'$$

is easy to check. Moreover, since b is trilinear continuous on V,

$$\|B\boldsymbol{w}\|_{V'} \leqslant c\|\boldsymbol{w}\|^2, \ \forall\, \boldsymbol{w} \in V, \tag{3.15}$$

so that

$$\int_0^T \|B\boldsymbol{u}(t)\|_{V'}\, \mathrm{d}t \leqslant c \int_0^T \|\boldsymbol{u}(t)\|^2\, \mathrm{d}t < +\infty,$$

and the lemma is proved. $\square$

Now if $\boldsymbol{u}$ satisfies (3.12)–(3.13), then according to (1.6), (1.8), and the above lemma, one can write (3.13) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{f} - \nu A\boldsymbol{u} - B\boldsymbol{u}, \boldsymbol{v}\rangle, \ \forall\, \boldsymbol{v} \in V.$$

Since $A\boldsymbol{u}$ belongs to $L^2(0, T; V')$, as in the linear case, the function

$f - \nu Au - Bu$ belongs to $L^1(0, T; V')$. Lemma 1.1 implies then that

$$\left.\begin{aligned} &u' \in L^1(0, T; V') \\ &u' = f - \nu Au - Bu, \end{aligned}\right\} \tag{3.16}$$

and that u is almost everywhere equal to a function continuous from $[0, T]$ into V'. This remark makes (3.14) meaningful.

An alternate formulation of the problem (3.12)–(3.14) is:

Problem 3.2. *Given f and u_0 satisfying* (3.10)–(3.11), *to find u satisfying*

$$u \in L^2(0, T; V),\ u' \in L^1(0, T; V'), \tag{3.17}$$

$$u' + \nu Au + Bu = f \text{ on } (0, T), \tag{3.18}$$

$$u(0) = u_0. \tag{3.19}$$

We showed that any solution of Problem 3.1 is a solution of Problem 3.2; the converse is very easily checked and these problems are thus equivalent.

The existence of solutions of these problems is ensured by the following theorem.

Theorem 3.1. *The dimension is $n \leqslant 4$. Let there be given f and u_0 which satisfy* (3.10)–(3.11). *Then there exists at least one function u which satisfies* (3.17)–(3.19). *Moreover,*

$$u \in L^\infty(0, T; H) \tag{3.20}$$

and u is weakly continuous from $[0, T]$ *into H.*(1)

The proof of the existence of a u satisfying (3.20) is developed in Section 3.2; the continuity result is a direct consequence of (3.20), the continuity of u in V', and of Lemma 1.4.

Remark 3.1. (i) Theorem 3.1 also holds if we assume that

$$f = f_1 + f_2;\ f_1 \in L^2(0, T; V'),\ f_2 \in L^1(0, T; H).$$

For the corresponding modifications of the proof of the theorem, the reader is referred to Section 1.5.

(ii) Theorem 3.1 is also valid if Ω is unbounded; for details, see Remark 3.2. □

(1) i.e., $\forall\, v \in H$, $t \to (u(t), v)$ is a continuous scalar function.

3.2. *Proof of Theorem 3.1*

(i) We apply the Galerkin procedure as in the linear case. Since V is separable and $\mathscr{V}$ is dense in V, there exists a sequence $w_1, \ldots, w_m, \ldots$ of elements of $\mathscr{V}$, which is free and total in V [1]. For each m we define an approximate solution u_m of (3.13) as follows:

$$u_m = \sum_{i=1}^{m} g_{im}(t) w_i \tag{3.21}$$

and

$$\begin{aligned}(u_m'(t), w_j) + \nu((u_m(t), w_j)) + b(u_m(t), u_m(t), w_j)& \\ = \langle f(t), w_j \rangle, \quad t \in [0, T], \; j = 1, \ldots, m,&\end{aligned} \tag{3.22}$$

$$u_m(0) = u_{0m}, \tag{3.23}$$

where u_{0m} is the orthogonal projection in H of u_0 onto the space spanned by $w_1, \ldots, w_m$ [2].

The equations (3.22) form a nonlinear differential system for the functions $g_{1m}, \ldots, g_{mm}$:

$$\begin{aligned}\sum_{i=1}^{m} (w_i, w_j) g_{im}'(t) + \nu \sum_{i=1}^{m} ((w_i, w_j)) g_{im}(t)& \\ + \sum_{i, \ell=1}^{m} b(w_i, w_\ell, w_j) g_{im}(t) g_{\ell m}(t)& \\ = \langle f(t), w_j \rangle.&\end{aligned} \tag{3.24}$$

Inverting the nonsingular matrix with elements (w_i, w_j), $1 \leqslant i, j \leqslant m$, we can write the differential equations in the usual form

$$\begin{aligned}g_{im}'(t) + \sum_{j=1}^{m} \alpha_{ij} g_{jm}(t) + \sum_{j,k=1}^{m} \alpha_{ijk} g_{jm}(t) g_{km}(t)& \\ = \sum_{j=1}^{m} \beta_{ij} \langle f(t), w_j \rangle&\end{aligned} \tag{3.25}$$

---

[1] The w_j are chosen in $\mathscr{V}$ for simplicity. With some technical modifications we could take the w_j in V.

[2] We could take for u_{0m} any element of that space such that

$$u_{0m} \to u_0, \text{ in } H, \text{ as } m \to \infty.$$

where $\alpha_{ij}, \alpha_{ijk}, \beta_{ij} \in \mathcal{R}$.

The condition (3.23) is equivalent to the m scalar initial conditions

$$g_{im}(0) = \text{the } i^{\text{th}} \text{ component of } \boldsymbol{u}_{0m}. \tag{3.26}$$

The nonlinear differential system (3.25) with the initial condition (3.26) has a maximal solution defined on some interval $[0, t_m]$. If $t_m < T$, then $|\boldsymbol{u}_m(t)|$ must tend to $+\infty$ as $t \to t_m$; the *a priori* estimates we shall prove later show that this does not happen and therefore $t_m = T$.

(ii) The first *a priori* estimates are obtained as in the linear case. We multiply (3.22) by $g_{jm}(t)$ and add these equations for $j = 1, \ldots, m$. Taking (3.2) into account, we get

$$(\boldsymbol{u}_m'(t), \boldsymbol{u}_m(t)) + \nu \|\boldsymbol{u}_m(t)\|^2 = \langle \boldsymbol{f}(t), \boldsymbol{u}_m(t)\rangle, \tag{3.27}$$

Then we write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} |\boldsymbol{u}_m(t)|^2 + 2\nu \|\boldsymbol{u}_m(t)\|^2 &= 2\langle \boldsymbol{f}(t), \boldsymbol{u}_m(t)\rangle \\ &\leqslant 2\|\boldsymbol{f}(t)\|_{V'} \|\boldsymbol{u}_m(t)\| \\ &\leqslant \nu \|\boldsymbol{u}_m(t)\|^2 + \frac{1}{\nu} \|\boldsymbol{f}(t)\|_{V'}^2\end{aligned}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t} |\boldsymbol{u}_m(t)|^2 + \nu \|\boldsymbol{u}_m(t)\|^2 \leqslant \frac{1}{\nu} \|\boldsymbol{f}(t)\|_{V'}^2. \tag{3.28}$$

Integrating (3.28) from 0 to s we obtain, in particular,

$$\begin{aligned}|u_m(s)|^2 &\leqslant |\boldsymbol{u}_{0m}|^2 + \frac{1}{\nu} \int_0^s \|\boldsymbol{f}(t)\|_{V'}^2 \mathrm{d}t \\ &\leqslant |u_0|^2 + \frac{1}{\nu} \int_0^T \|\boldsymbol{f}(t)\|_{V'}^2 \mathrm{d}t.\end{aligned}$$

Hence

$$\operatorname*{Sup}_{s \in [0, T]} |\boldsymbol{u}_m(s)|^2 \leqslant |\boldsymbol{u}_0|^2 + \frac{1}{\nu} \int_0^T \|\boldsymbol{f}(t)\|_{V'}^2 \mathrm{d}t \tag{3.29}$$

which implies that

The sequence u_m remains in a bounded set of $L^\infty(0, T; H)$ (3.30)

We then integrate (3.28) from 0 to T to get

$$|u_m(T)|^2 + \nu \int_0^T \|u_m(t)\|^2 \mathrm{d}t \leqslant |u_{0m}|^2 + \frac{1}{\nu} \int_0^T \|f(t)\|_{V'}^2 \mathrm{d}t$$

$$\leqslant |u_0|^2 + \frac{1}{\nu} \int_0^T \|f(t)\|_{V'}^2 \mathrm{d}t.$$

This estimate enables us to say that

The sequence u_m remains in a bounded set of $L^2(0, T; V)$. (3.31)

(iii) Let $\tilde{u}_m$ denote the function from $\mathcal{R}$ into V, which is equal to u_m on $[0, T]$ and to 0 on the complement of this interval. The Fourier transform of $\tilde{u}_m$ is denoted by $\hat{u}_m$.

In addition to the previous inequalities, which are similar to the estimates in the linear case, we want to show that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{u}_m(\tau)|^2 \mathrm{d}t \leqslant \text{Const., for some } \gamma > 0. \tag{3.32}$$

Along with (3.31), this will imply that

$\tilde{u}_m$ belongs to a bounded set of $\mathcal{H}^\gamma(\mathcal{R}; V, H)$ (3.33)

and will enable us to apply the compactness result of Theorem 2.2.

In order to prove (3.32) we observe that (3.22) can be written [1]

$$\frac{\mathrm{d}}{\mathrm{d}t} (\tilde{u}_m, w_j) = \langle \tilde{f}_m, w_j \rangle + (u_{0m}, w_j)\delta_0$$

$$- (u_m(T), w_j)\delta_T, \; j = 1, \ldots, m \tag{3.34}$$

where δ_0, δ_T are Dirac distributions at 0 and T and

$$f_m = f - \nu A u_m - B u_m$$

$$\tilde{f}_m = f_m \text{ on } [0, T], \text{ 0 outside this interval.} \tag{3.35}$$

---

[1] Compare this with the proof of Theorem 2.3.

By the Fourier transform, (3.34) gives

$$2i\pi\tau(\hat{u}_m, w_j) = \langle \hat{f}_m, w_j\rangle + (u_{0m}, w_j) - (u_m(T), w_j)\exp(-2i\pi T\tau), \tag{3.36}$$

$\hat{u}_m$ and $\hat{f}_m$ denoting the Fourier transforms of $\tilde{u}_m$ and $\tilde{f}_m$ respectively.

We multiply (3.35) by $\hat{g}_{jm}(\tau)$ (= Fourier transform of $\tilde{g}_{jm}$) and add the resulting equations for $j = 1, \ldots, m$; we get:

$$2i\pi\tau|\hat{u}_m(\tau)|^2 = \langle \hat{f}_m(\tau), \hat{u}_m(\tau)\rangle + (u_{0m}, \hat{u}_m(\tau)) - (u_m(T), \hat{u}_m(\tau))\exp(-2i\pi T\tau). \tag{3.37}$$

Because of inequality (3.15),

$$\int_0^T \|f_m(t)\|_{V'}\,\mathrm{d}t \leqslant \int_0^T (\|f(t)\|_{V'} + \nu\|u_m(t)\| + c\,\|u_m(t)\|^2)\mathrm{d}t,$$

and this remains bounded according to (3.31). Therefore

$$\sup_{\tau\in\mathbb{R}} \|\hat{f}_m(\tau)\|_{V'} \leqslant \text{Const.}, \ \forall\, m.$$

Due to (3.29),

$$|u_m(0)| \leqslant \text{const.}, \quad |u_m(T)| \leqslant \text{const.},$$

and we deduce from (3.37) that

$$|\tau|\,|\hat{u}_m(\tau)|^2 \leqslant c_2\,\|\hat{u}_m(\tau)\| + c_3\,|\hat{u}_m(\tau)|$$

or

$$|\tau|\,|\hat{u}_m(\tau)|^2 \leqslant c_4\,\|\hat{u}_m(\tau)\|. \tag{3.38}$$

For γ fixed, $\gamma < 1/4$, we observe that

$$|\tau|^{2\gamma} \leqslant c_5(\gamma)\,\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \ \forall\ \tau\in\mathbb{R}.$$

Thus

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\,|\hat{u}_m(\tau)|^2\mathrm{d}\tau \leqslant c_5(\gamma)\int_{-\infty}^{+\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}\,|\hat{u}_m(\tau)|^2\mathrm{d}\tau$$

$$\leqslant (\text{by (3.38)})$$

$$\leqslant c_6 \int_{-\infty}^{+\infty} \frac{\|\hat{u}_m(\tau)\| \mathrm{d}\tau}{1 + |\tau|^{1-2\gamma}} + c_7 \int_{-\infty}^{+\infty} \|\hat{u}_m(\tau)\|^2 \mathrm{d}\tau.$$

Because of the Parseval equality and (3.31), the last integral is bounded as $m \to \infty$; thus (3.32) will be proved if we show that

$$\int_{-\infty}^{+\infty} \frac{\|\hat{u}_m(\tau)\|}{1 + |\tau|^{1-2\gamma}} \mathrm{d}\tau \leqslant \text{const.} \tag{3.39}$$

By the Schwarz inequality and the Parseval equality we can bound these integrals by

$$\left(\int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau}{(1 + |\tau|^{1-2\gamma})^2} \right)^{1/2} \left(\int_0^T \|u_m(t)\|^2 \mathrm{d}t \right)^{1/2},$$

which is finite since $\gamma < 1/4$, and bounded as $m \to \infty$ by (3.31).

The proof of (3.32) and (3.33) is achieved.

(iv) The estimates (3.30) and (3.31) enable us to assert the existence of an element $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$ and a sub-sequence $u_{m'}$ such that

$$\left. \begin{array}{l} u_{m'} \to u \text{ in } L^2(0, T; V) \text{ weakly, and in} \\ L^\infty(0, T; H) \text{ weak-star, as } m' \to \infty. \end{array} \right\} \tag{3.40}$$

Due to (3.33) and Theorem 2.2, we also have

$$u_{m'} \to u \text{ in } L^2(0, T; H) \text{ strongly.} \tag{3.41}$$

The convergence results (3.40) and (3.41) enable us to pass to the limit. We proceed essentially as in the linear case.

Let ψ be a continuously differentiable function on $[0, T]$ with $\psi(T) = 0$. We multiply (3.22) by $\psi(t)$, and then integrate by parts. This leads to the equation

$$-\int_0^T (u_m(t), \psi'(t) w_j) \mathrm{d}t + \nu \int_0^T ((u_m(t), w_j \psi(t))) \mathrm{d}t$$

$$+\int_0^T b(u_m(t), u_m(t), w_j\psi(t))\mathrm{d}t = (u_{0m}, w_j)\psi(0)$$

$$+\int_0^T \langle f(t), w_j\psi(t)\rangle \mathrm{d}t. \tag{3.42}$$

Passing to the limit with the sequence m' is easy for the linear terms; for the nonlinear term we apply the next lemma, Lemma 3.2. In the limit we find that the equation

$$-\int_0^T (u(t), v\psi'(t))\mathrm{d}t + \nu\int_0^T ((u(t), v\psi(t)))\mathrm{d}t$$

$$+\int_0^T b(u(t), u(t), v\psi(t))\mathrm{d}t = (u_0, v)\psi(0)$$

$$+\int_0^T \langle f(t), v\psi(t)\rangle \mathrm{d}t, \tag{3.43}$$

holds for $v = w_1, w_2, \ldots$; by linearity this equation holds for $v =$ any finite linear combination of the w_j, and by a continuity argument (3.43) is still true for any $v \in V$.

Now writing, in particular, (3.43) with $\psi = \phi \in \mathscr{D}((0, T))$, we see that u satisfies (3.13) in the distribution sense.

Finally, it remains to prove that u satisfies (3.14). For this we multiply (3.13) by ψ, and integrate. After integrating the first term by parts, we get

$$-\int_0^T (u(t), v\psi(t))\mathrm{d}t + \nu\int_0^T ((u(t), v\psi(t)))\mathrm{d}t$$

$$+\int_0^T b(u(t), u(t), v\psi(t))\mathrm{d}t = (u(0), v)\psi(0) +$$

$$+\int_0^T \langle f(t), v\psi(t)\rangle \mathrm{d}t. \tag{3.44}$$

By comparison with (3.43),

$$(\boldsymbol{u}(0) - \boldsymbol{u}_0, v)\psi(0) = 0.$$

We can choose ψ with $\psi(0) = 1$; thus

$$(\boldsymbol{u}(0) - \boldsymbol{u}_0, v) = 0, \forall\ v \in V,$$

and (3.14) follows.

The proof the Theorem 3.1 will be complete once we prove the following lemma.

Lemma 3.2. *If $\boldsymbol{u}_\mu$ converges to $\boldsymbol{u}$ in $L^2(0, T; V)$ weakly and $L^2(0, T; H)$ strongly, then for any vector function $\boldsymbol{w}$ with components in $\mathscr{C}^1(\overline{Q})$,*

$$\int_0^T b(\boldsymbol{u}_\mu(t), \boldsymbol{u}_\mu(t), \boldsymbol{w}(t))\mathrm{d}t \to \int_0^T b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{w}(t))\mathrm{d}t. \tag{3.45}$$

Proof. We write

$$\int_0^T b(\boldsymbol{u}_\mu, \boldsymbol{u}_\mu, \boldsymbol{w})\mathrm{d}t = -\int_0^T b(\boldsymbol{u}_\mu, \boldsymbol{w}, \boldsymbol{u}_\mu) =$$

$$-\sum_{i,j=1}^n \int_0^T \int_\Omega (\boldsymbol{u}_\mu)_i (D_i \boldsymbol{w}_j)(\boldsymbol{u}_\mu)_j \mathrm{d}x\mathrm{d}t.$$

These integrals converge to

$$-\sum_{i,j=1}^n \int_0^T \int_\Omega \boldsymbol{u}_i (D_i \boldsymbol{w}_j)\boldsymbol{u}_j \mathrm{d}x\mathrm{d}t =$$

$$-\int_0^T b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{u})\mathrm{d}t = \int_0^T b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w})\mathrm{d}t,$$

and the lemma is proved. □

Remark 3.2. (i) *The unbounded domain.*

When Ω is unbounded, we prove (3.30) and (3.31) as we did in Section 1.5 for the linear case. Then (3.32) and (3.33) follow as before. The main difference lies in the fact that the injection of V into H is no longer compact.

Nevertheless we can extract a sub-sequence $u_{m'}$ which satisfies (3.40). Then, for any ball $\mathcal{O}$ included in Ω, the injection of $H^1(\mathcal{O})$ into $L^2(\mathcal{O})$ is compact and (3.33) shows that:

$$u_m|_{\mathcal{O}} \text{ belongs to a bounded set of } \mathcal{H}^\gamma(\mathcal{R}; H^1(\mathcal{O}), L^2(\mathcal{O})),\ \forall \mathcal{O}. \tag{3.46}$$

Theorem 2.2 then implies that

$$u_{m'}|_{\mathcal{O}} \to u|_{\mathcal{O}} \text{ in } L^2(0, T; L^2(\mathcal{O})) \text{ strongly, } \forall \mathcal{O}.$$

which means

$$u_{m'} \to u \text{ in } L^2(0, T; L^2_{\mathrm{loc}}(\Omega)) \text{ strongly.}$$

In particular, for a fixed j,

$$u_{m'|\Omega'} \to u_{|\Omega'} \text{ in } L^2(0, T; L^2(\Omega')) \text{ strongly}$$

where Ω' denotes the support of w_j, and this suffices to pass to the limit in (3.42).

(ii) *Energy inequality*

By integration of (3.27) we see that

$$|u_m(t)|^2 + 2\nu \int_0^t \|u_m(s)\|^2 \mathrm{d}s = |u_{0m}|^2 + 2\int_0^t \langle f(s), u_m(s)\rangle \mathrm{d}s.$$

We multiply this equality by $\phi(t)$, where $\phi \in \mathcal{D}(]0, T[)$, $\phi(t) \geqslant 0$, and integrate in t:

$$\int_0^T \left\{ |u_m(t)|^2 + 2\nu \int_0^t \|u_m(s)\|^2 \mathrm{d}s \right\} \phi(t)\, \mathrm{d}t = \int_0^T \left\{ |u_{0m}|^2 + 2\int_0^t \langle f(s), u_m(s)\rangle\, \mathrm{d}s \right\} \phi(t)\, \mathrm{d}t.$$

Using (3.40) we now pass to the lower limit in this relation and obtain

$$\int_0^T \left\{ |\boldsymbol{u}(t)|^2 + 2\nu \int_0^t \|\boldsymbol{u}(s)\|^2 \, \mathrm{d}s \right\} \phi(t) \, \mathrm{d}t$$

$$\leqslant \int_0^T \left\{ |\boldsymbol{u}_0|^2 + 2 \int_0^t \langle \boldsymbol{f}(s), \boldsymbol{u}(s) \rangle \mathrm{d}s \right\} \phi(t) \, \mathrm{d}t,$$

for all $\phi \in \mathscr{D}(]0, T[)$, $\phi \geqslant 0$. This amounts to saying that

$$|\boldsymbol{u}(t)|^2 + 2\nu \int_0^t \|\boldsymbol{u}(s)\|^2 \, \mathrm{d}s \leqslant |\boldsymbol{u}_0|^2 + 2 \int_0^t \langle \boldsymbol{f}(s), \boldsymbol{u}(s) \rangle \, \mathrm{d}s, \quad (3.47)$$

for almost every $t \in [0, T]$.

This is an energy inequality satisfied by the function $\boldsymbol{u}$ given by Theorem 3.1. We shall show later that if $n = 2$, the corresponding equality holds (see Theorem 3.2 and Lemma 2.1). We do not know if the corresponding equality is satisfied in general (cf. Theorem 3.9).

3.3. *Regularity and Uniqueness* ($n = 2$)

When the dimension of the space is $n = 2$, the solution of (3.17)–(3.19) whose existence is ensured by Theorem 3.1 satisfies some further regularity property and is actually unique.

The proof of these results is based on the following lemmas.

Lemma 3.3. *If $n = 2$, for any open set Ω,*

$$\|v\|_{L^4(\Omega)} \leqslant 2^{1/4} \|v\|_{L^2(\Omega)}^{1/2} \|\operatorname{grad} v\|_{L^2(\Omega)}^{1/2}, \quad \forall v \in H_0^1(\Omega). \quad (3.48)$$

Proof. It suffices to prove (3.48) for $v \in D(\Omega)$. For such a v, we write

$$v^2(x) = 2 \int_{-\infty}^{x_1} v(\xi_1, x_2) D_1 v(\xi_1, x_2) \mathrm{d}\xi_1,$$

and therefore

$$v^2(x) \leqslant 2v_1(x_2), \quad (3.49)$$

where

$$\nu_1(x_2) = \int_{-\infty}^{+\infty} |v(\xi_1, x_2)|\,|D_1 v(\xi_1, x_2)|\,\mathrm{d}\xi_1 . \tag{3.50}$$

Similarly

$$v^2(x) \leqslant 2\nu_2(x_1), \tag{3.51}$$

where

$$\nu_2(x_1) = \int_{-\infty}^{+\infty} |v(x_1, \xi_2)|\,|D_2 v(x_1, \xi_2)|\,\mathrm{d}\xi_2 \tag{3.52}$$

and thus

$$\begin{aligned}\int_{\mathbb{R}^2} v^4(x)\mathrm{d}x &\leqslant 4 \int_{\mathbb{R}^2} \nu_1(x_2)\nu_2(x_1)\mathrm{d}x \\ &\leqslant 4 \left(\int_{-\infty}^{+\infty} \nu_1(x_2)\mathrm{d}x_2\right)\left(\int_{-\infty}^{+\infty} \nu_2(x_1)\mathrm{d}x_1\right) \\ &\leqslant 4\, \|v\|^2_{L^2(\mathbb{R}^2)} \|D_1 v\|_{L^2(\mathbb{R}^2)} \|D_2 v\|_{L^2(\mathbb{R}^2)} \\ &\leqslant 2\|v\|^2_{L^2(\mathbb{R}^2)} \|\operatorname{grad} v\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Lemma 3.4. *If $n = 2$,*

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant 2^{1/2}|\boldsymbol{u}|^{1/2}\,\|\boldsymbol{u}\|^{1/2}\,\|\boldsymbol{v}\|\,|\boldsymbol{w}|^{1/2}\,\|\boldsymbol{w}\|^{1/2}, \ \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H_0^1(\Omega) \tag{3.53}$$

If $\boldsymbol{u}$ belongs to $L^2(0, T; V) \cap L^\infty(0, T; H)$, then $B\boldsymbol{u}$ belongs to $L^2(0, T; V')$ and

$$\|B\boldsymbol{u}\|_{L^2(0,T;V')} \leqslant 2^{1/2}|\boldsymbol{u}|_{L^\infty(0,T;H)}\,\|\boldsymbol{u}\|_{L^2(0,T;V)}. \tag{3.54}$$

Proof. By repeated application of the Schwarz and Hölder inequalities we find:

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant \sum_{i,j=1}^{2} \int_\Omega |u_i(D_i v_j) w_j|\,\mathrm{d}x$$

$$\leq \sum_{i,j=1}^{2} \|u_i\|_{L^4(\Omega)} \|D_i v_j\|_{L^2(\Omega)} \|w_j\|_{L^4(\Omega)}$$

$$\leq \left(\sum_{i,j=1}^{2} \|D_i v_j\|^2_{L^2(\Omega)} \right)^{1/2} \cdot \left(\sum_{i=1}^{2} \|u_i\|^2_{L^4(\Omega)} \right)^{1/2} \cdot \left(\sum_{j=1}^{2} \|w_j\|^2_{L^4(\Omega)} \right)^{1/2}$$

Due to (3.48),

$$\sum_{i=1}^{2} \|u_i\|^2_{L^4(\Omega)} \leq 2^{1/2} \sum_{i=1}^{2} \left(\|u_i\|_{L^2(\Omega)} \|\operatorname{grad} u_i\|_{L^2(\Omega)} \right)$$
$$\leq 2^{1/2} |u| \, \|u\|.$$

With a similar inequality for w, we finally get (3.53).

If u, v, w belong to V, the relation

$$b(u, v, w) = -b(u, w, v)$$

gives another estimate of b:

$$|b(u, v, w| \leq 2^{1/2} \, |u|^{1/2} \, \|u\|^{1/2} \, |v|^{1/2} \, \|v\|^{1/2} \, \|w\|, \; \forall \, u, v, w \in V. \tag{3.55}$$

In particular,

$$|b(u, u, v)| \leq 2^{1/2} \, |u| \, \|u\| \, \|v\|, \forall \, u, v \in V, \tag{3.56}$$

and hence

$$\|Bu\|_{V'} \leq 2^{1/2} \, |u| \, \|u\|, \forall \, u \in V. \tag{3.57}$$

If now $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$, then $Bu(t)$ belongs to V' for almost every t and the estimate

$$\|Bu(t)\|_{V'} \leq 2^{1/2} \, |u(t)| \, \|u(t)\| \tag{3.58}$$

shows that Bu belongs to $L^2(0, T; V')$ and implies (3.54). □

We can now state and prove the main result (cf.. J. L. Lions and G. Prodi [1]).

Theorem 3.2. *In the two-dimensional case, the solution* $\boldsymbol{u}$ *of Problems* 3.1–3.2 *given by Theorem* 3.1 *is unique. Moreover* $\boldsymbol{u}$ *is almost everywhere equal to a function continuous from* $[0, T]$ *into* H *and*

$$u(t) \to u_0, \text{ in } H, \text{ as } t \to 0. \tag{3.59}$$

Proof. (i) We first prove the result of regularity.

According to (3.18) and Lemma 3.4,

$$u' = f - \nu Au - Bu,$$

and since each term in the right-hand side of this equation belongs to $L^2(0, T; V')$, u' also belongs to $L^2(0, T; V')$; this remark improves (3.17):

$$u' \in L^2(0, T; V'). \tag{3.60}$$

This improvement of (3.17) enables us to apply Lemma 1.2, which states exactly that u is almost everywhere equal to a function continuous from $[0, T]$ into H. Thus

$$u \in \mathscr{C}([0, T]; H) \tag{3.61}$$

and (3.59) follows easily.

We also recall that Lemma 1.2 asserts that for any function u in $L^2(0, T; V)$ which satisfies (3.60), the equation below holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} |u(t)|^2 = 2\langle u'(t), u(t)\rangle. \tag{3.62}$$

This result will be used in the following proof of uniqueness which we will start now.

(ii) Let us assume that u_1 and u_2 are two solutions of (3.17)–(3.19), and let $u = u_1 - u_2$. As shown before, u_1, u_2, and thus u, satisfy (3.60).

The difference $u = u_1 - u_2$ satisfies

$$u' + \nu Au = -Bu_1 + Bu_2 \tag{3.63}$$

$$u(0) = 0 \tag{3.64}$$

We take a.e. in t the scalar product of (3.63) with $u(t)$ in the duality between V and V'. Using (3.62), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} |u(t)|^2 + 2\nu \|u(t)\|^2 = 2b(u_2(t), u_2(t), u(t)) - 2b(u_1(t), u_1(t), u(t)). \tag{3.65}$$

Because of (3.2) the right-hand side of this equality is equal to

$$-2b(\boldsymbol{u}(t), \boldsymbol{u}_2(t), \boldsymbol{u}(t)).$$

With (3.53) we can majorize this expression by

$$2^{3/2}|\boldsymbol{u}(t)|\ \|\boldsymbol{u}(t)\|\ \|\boldsymbol{u}_2(t)\| \leqslant 2\nu\|\boldsymbol{u}(t)\|^2 + \frac{1}{\nu}|\boldsymbol{u}(t)|^2\ \|\boldsymbol{u}_2(t)\|^2.$$

Putting this in (3.65) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}(t)|^2 \leqslant \frac{1}{\nu}|\boldsymbol{u}(t)|^2\ \|\boldsymbol{u}_2(t)\|^2.$$

Since the function $t \to \|\boldsymbol{u}_2(t)\|^2$ is integrable, this shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\exp\left(-\frac{1}{\nu}\int_0^t \|\boldsymbol{u}_2(s)\|^2\mathrm{d}s\right)\cdot|\boldsymbol{u}(t)|^2\right\} \leqslant 0.$$

Integrating and applying (3.64), we find

$$|\boldsymbol{u}(t)|^2 \leqslant 0,\ \forall\ t \in [0, T].$$

Thus

$$\boldsymbol{u}_1 = \boldsymbol{u}_2.$$

and the solution is unique.

Remark 3.3. As a consequence of (3.48) the (unique) solution of the Navier–Stokes equations satisfies

$$\boldsymbol{u} \in L^4(Q) \quad (n = 2). \tag{3.66}$$

Remark 3.4. Theorem 3.2 covers both the bounded and unbounded cases; there are no differences in the proofs.

3.4. *On Regularity and Uniqueness* ($n = 3$)

The results of Section 3.3 cannot be extended to higher dimensions due to the lack of information concerning the regularity of the weak solutions given by Theorem 3.1.

Nevertheless, we shall prove some further regularity properties of a solution, which are weaker than those of the two-dimensional case. We then give a uniqueness theorem in a class of functions where the existence is not known; this result shows, however, how the information concerning the regularity of weak solutions are related to uniqueness.

The result similar to Lemma 3.3 is

Lemma 3.5. *If $n = 3$, for any open set Ω:*

$$\|v\|_{L^4(\Omega)} \leqslant 2^{1/2} \|v\|^{1/4}_{L^2(\Omega)} \|\text{grad } v\|^{3/4}_{L^2(\Omega)}, \ \forall\, v \in H^1_0(\Omega). \tag{3.67}$$

Proof. We only have to prove (3.67) for $v \in \mathscr{D}(\Omega)$. For such a v, by application of (3.48), we write

$$\int_{\mathbb{R}^3} v^4(x)\mathrm{d}x \leqslant 2 \int_{-\infty}^{+\infty} \left\{ \left(\int_{\mathbb{R}^2} v^2 \mathrm{d}x_1 \mathrm{d}x_2 \right) \cdot \left(\int_{\mathbb{R}^2} \sum_{i=1}^{2} (D_i v)^2 \mathrm{d}x_1 \mathrm{d}x_2 \right) \right\} \mathrm{d}x_3$$

$$\leqslant 2 \left(\operatorname*{Sup}_{x_3} \int_{\mathbb{R}^2} v^2 \mathrm{d}x_1 \mathrm{d}x_2 \right) \left(\sum_{i=1}^{2} \|D_i v\|^2_{L^2(\mathbb{R}^3)} \right). \tag{3.68}$$

But

$$v^2(x) = 2 \int_{-\infty}^{x_3} v(x_1, x_2, \xi_3) D_3 v(x_1, x_2, \xi_3) \mathrm{d}\xi_3$$

$$\leqslant 2 \int_{-\infty}^{+\infty} |v(x_1, x_2, \xi_3)|\, |D_3 v(x_1, x_2, \xi_3)| \mathrm{d}\xi_3$$

and hence

$$\operatorname*{Sup}_{x_3} \int_{\mathbb{R}^2} v^2 \mathrm{d}x_1 \mathrm{d}x_2 \leqslant 2 \int_{\mathbb{R}^3} |v|\, |D_3 v| \mathrm{d}x \leqslant 2 \|v\|_{L^2(\mathbb{R}^3)} \|D_3 v\|_{L^2(\mathbb{R}^3)}.$$

With this inequality we deduce from (3.68) that

$$\int_{\mathbb{R}^3} v^4(x)\mathrm{d}x \leqslant 4 \|v\|_{L^2(\mathbb{R}^3)} \|D_3 v\|_{L^2(\mathbb{R}^3)} \left(\sum_{i=1}^{2} \|D_i v\|^2_{L^2(\mathbb{R}^3)} \right)$$

$$\leqslant 4 \|v\|_{L^2(\mathbb{R}^3)} \left(\sum_{i=1}^{3} \|D_i v\|_{L^2(\mathbb{R}^3)} \right)^{3/2}$$

and (3.67) follows.

Theorem 3.3. *If $n = 3$, the solution $\boldsymbol{u}$ of* (3.17)–(3.19) *given by Theorem* 3.1 *satisfies*

$$\boldsymbol{u} \in L^{8/3}(0, T; L^4(\Omega)) \tag{3.69}$$

$$\boldsymbol{u}' \in L^{4/3}(0, T; V'). \tag{3.70}$$

Proof. For almost every t, according to (3.67),

$$\|\boldsymbol{u}(t)\|_{L^4(\Omega)} \leqslant c_0 |\boldsymbol{u}(t)|^{1/4} \|\boldsymbol{u}(t)\|^{3/4}. \tag{3.71}$$

The function on the right-hand side belongs to $L^{8/3}(0, T)$, and thus also the function on the left-hand side.

Using the Hölder inequality, we derived in Chapter II the inequality

$$|b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})| = |b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u})| \leqslant c_1 \|\boldsymbol{u}\|^2_{L^4(\Omega)} \|\boldsymbol{v}\|, \ \forall\, \boldsymbol{u}, \boldsymbol{v} \in V^{(1)}. \tag{3.72}$$

Therefore, if $\boldsymbol{u} \in L^2(0, T; V) \cap L^\infty(0, T; H)$, $B\boldsymbol{u}$ belongs to $L^{4/3}(0, T; V')$ since

$$\|B\boldsymbol{u}(t)\|_{V'} \leqslant c_1 \|\boldsymbol{u}(t)\|^2_{L^4(\Omega)} \tag{3.73}$$

$$\|B\boldsymbol{u}(t)\|_{V'} \leqslant c_2 |\boldsymbol{u}(t)|^{1/2} \|\boldsymbol{u}(t)\|^{3/2}, \text{ a.e. } \square \tag{3.74}$$

In the two-dimensional case we established that any solution of (3.17)–(3.19) satisfies (3.60) and (3.66) and this was the property which essentially enabled us to prove uniqueness. For $n = 3$, (3.60) and (3.66) are replaced by the weaker results (3.69)–(3.70).

Now we show that there is at most one solution in a class of functions smaller than that in which we obtained existence.

Theorem 3.4. *If $n = 3$, there is at most one solution of Problem* 3.2 *such that*

$$\boldsymbol{u} \in L^2(0, T; V) \cap L^\infty(0, T; H) \tag{3.75}$$

$$\boldsymbol{u} \in L^8(0, T; L^4(\Omega)). \tag{3.76}$$

Such a solution would be continuous from $[0, T]$ *into* H.

---

(1) This inequality with c depending on n holds for any dimension of space.

Proof.

(i) The inequalities (3.72) – (3.73) imply that if $\boldsymbol{u}$ satisfies (3.76) then

$$B\boldsymbol{u} \in L^2(0, T; V') \text{ (at least).} \tag{3.77}$$

Therefore if $\boldsymbol{u}$ satisfies (3.75) – (3.76) and (3.18), then

$$\boldsymbol{u}' \in L^2(0, T; V') \tag{3.78}$$

and according to Lemma 1.2, $\boldsymbol{u}$ is almost everywhere equal to a continuous function from $[0, T]$ into H.

(ii) By the Hölder inequality and (3.67),

$$|b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})| \leqslant c_0 \|\boldsymbol{u}\|_{L^4(\Omega)} \|\boldsymbol{v}\|_{L^4(\Omega)} \|\boldsymbol{u}\|,$$

$$|b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})| \leqslant c_1 |\boldsymbol{u}|^{1/4} \|\boldsymbol{u}\|^{7/4} \|\boldsymbol{v}\|_{L^4(\Omega)}. \tag{3.79}$$

(iii) Let us assume that $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of (3.17) – (3.19) which satisfy (3.75) – (3.76) and let $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$.

As in the proof of Theorem 3.2 one can show that

$$\frac{\mathrm{d}}{\mathrm{d}t} |\boldsymbol{u}(t)|^2 + 2\nu \|\boldsymbol{u}(t)\|^2 = 2b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{u}_2(t)). \tag{3.80}$$

We then bound the right-hand side, according to (3.79), by

$$2c_1 |\boldsymbol{u}(t)|^{1/4} \|\boldsymbol{u}(t)\|^{7/4} \|\boldsymbol{u}_2(t)\|_{L^4(\Omega)} \leqslant$$
$$\nu \|\boldsymbol{u}(t)\|^2 + c_2 |\boldsymbol{u}(t)|^2 \|\boldsymbol{u}_2(t)\|^8_{L^4(\Omega)}.$$

We get

$$\frac{\mathrm{d}}{\mathrm{d}t} |\boldsymbol{u}(t)|^2 \leqslant c_2 \|\boldsymbol{u}_2(t)\|^4_{L^4(\Omega)} |\boldsymbol{u}(t)|^2.$$

Since the function $t \to |\boldsymbol{u}_2(t)|^8_{L^4(\Omega)}$ is integrable, we may complete the proof as for Theorem 3.2.

Remark 3.5. The preceding proof is valid for Ω bounded or unbounded.

Remark 3.6. There are many similar results of uniqueness which can be proved by assuming some other properties of regularity. For example (c.f. Lions [2] p. 84), there is uniqueness in any dimension if, in place of (3.76), $\boldsymbol{u}$ satisfies

$$\boldsymbol{u} \in L^s(0, T; L^r(\Omega)) \tag{3.81}$$

with

$$\frac{2}{s} + \frac{n}{r} \leqslant 1 \text{ if } \Omega \text{ is bounded,}$$

$$\frac{2}{s} + \frac{n}{r} = 1 \text{ if } \Omega \text{ is unbounded.} \tag{3.82}$$

3.5. *More Regular Solutions*

Our purpose in this section is to prove that by assuming more regularity on the data, we can obtain more regular solutions in the two-dimensional case. In the three-dimensional case the existence of such more regular solutions is proved on arbitrary intervals of time provided we assume that the given data $\boldsymbol{u}_0, f$, are "small enough" or that ν is large enough.

3.5.1. *The Two-Dimensional Case*

Theorem 3.5. *We assume that $n = 2$ and that*

$$f \text{ and } f' \in L^2(0, T; V'), f(0) \in H \tag{3.83}$$

$$\boldsymbol{u}_0 \in H^2(\Omega) \cap V. \tag{3.84}$$

Then the unique solution of Problem 3.2 *given by Theorems* 3.1 *and* 3.2 *satisfies*

$$\boldsymbol{u}' \in L^2(0, T; V) \cap L^\infty(0, T; H). \tag{3.85}$$

Proof.

(i) We return to the Galerkin approximation used in the proof of Theorem 3.1. We need only to show that this approximate solution also satisfies the two *a priori* estimates:

$$\boldsymbol{u}'_m \textit{ remains in a bounded set of } L^2(0, T; V) \cap L^\infty(0, T, H). \tag{3.86}$$

In the limit (3.86) implies (3.85).

Since $\boldsymbol{u}_0 \in V \cap H^2(\Omega)$, we can choose $\boldsymbol{u}_{0m}$ as the orthogonal projection in $V \cap H^2(\Omega)$ of $\boldsymbol{u}_0$ onto the space spanned by $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m$; then

$$\left.\begin{aligned} &\boldsymbol{u}_{0m} \to \boldsymbol{u}_0 \text{ in } H^2(\Omega), \text{ as } m \to \infty, \\ &\|\boldsymbol{u}_{0m}\|_{H^2(\Omega)} \leqslant \|\boldsymbol{u}_0\|_{H^2(\Omega)}. \end{aligned}\right\} \tag{3.87}$$

(ii) We multiply (3.22) by $g'_{jm}(t)$ and add the resulting equations for $j = 1, \ldots, m$; this gives

$$|u'_m(t)|^2 + \nu((u_m(t), u'_m(t))) + b(u_m(t), u_m(t), u'_m(t))$$
$$= \langle f(t), u'_m(t)\rangle.$$

In particular, at time $t = 0$,

$$|u'_m(0)|^2 = (f(0), u'_m(0)) + \nu(\Delta u_{0m}, u'_m(0)) - b(u_{0m}, u_{0m}, u'_m(0)) \tag{3.89}$$

so that

$$|u'_m(0)| \leqslant |f(0)| + \nu|\Delta u_{0m}| + |Bu_{0m}|. \tag{3.90}$$

It is clear from (3.87) that

$$|\Delta u_{0m}| \leqslant c_0 \|u_{0m}\|_{H^2(\Omega)} \leqslant c_0 \|u_0\|_{H^2(\Omega)}.$$

For Bu_{0m} we have, by the Hölder inequality,

$$|b(u, u, v)| \leqslant c_1 \|u\|_{L^4(\Omega)} |\text{grad } u|_{L^4(\Omega)} |v|$$
$$\leqslant \text{(by (3.48) and the Sovolev inequality)}$$
$$\leqslant c_2 \|u\| \, \|u\|_{H^2(\Omega)} |v|, \ \forall\, u \in H^2(\Omega), \ \forall\, v \in L^2(\Omega)$$

and hence

$$|Bu_{0m}| \leqslant c_2 \|u_{0m}\| \, \|u_{0m}\|_{H^2(\Omega)} \leqslant \text{(by 3.87))}$$
$$\leqslant c_2 \|u_0\|^2_{H^2(\Omega)} \tag{3.91}$$

Finally (3.90) and the above estimates show that

$$u'_m(0) \textit{ belongs to a bounded set of } H. \tag{3.92}$$

(iii) We are allowed to differentiate (3.22) in the t variable, and since f satisfies (3.83), we get

$$(u''_m, w_j) + \nu((u'_m, w_j)) + b(u'_m, u_m, w_j)$$
$$+ b(u_m, u'_m, w_j) = \langle f', w_j\rangle, \ j = 1, \ldots, m. \tag{3.93}$$

We multiply (3.93) by $g'_{jm}(t)$ and add the resulting equations for $j = 1, \ldots, m$; we find (taking (3.2) into account):

$$\frac{\mathrm{d}}{\mathrm{d}t} |u'_m(t)|^2 + 2\nu \|u'_m(t)\|^2 + 2b(u'_m(t), u_m(t), u'_m(t)) =$$

$$= 2\langle f'(t), u'_m(t)\rangle. \tag{3.94}$$

By Lemma 3.4,

$$2\,|b(u'_m(t), u_m(t), u'_m(t))| \leqslant 2^{3/2}|u'_m(t)|\,\|u'_m(t)\|\,\|u_m(t)\|$$

$$\leqslant \nu\|u'_m(t)\|^2 + \frac{2}{\nu}\,\|u_m(t)\|^2\,|u'_m(t)|^2.$$

Thus, we deduce from (3.94) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|u'_m(t)|^2 + \frac{\nu}{2}\,\|u'_m(t)\|^2 \leqslant \frac{2}{\nu}\,|f'(t)|^2_{V'} + \phi_m(t)|u'_m(t)|^2 \tag{3.94}$$

where

$$\phi_m(t) = \frac{2}{\nu}\,\|u_m(t)\|^2.$$

Then, by the usual method of the Gronwall inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\{|u'_m(t)|^2\,\exp(-\int_0^t \phi_m(s)\mathrm{d}s\} \leqslant \frac{2}{\nu}\,|f'(t)|^2_{V'}$$

whence

$$|u'_m(t)|^2 \leqslant \{|u'_m(0)|^2 + \frac{2}{\nu}\int_0^t |f'(s)|^2_{V'}\mathrm{d}s\}\,\exp\int_0^t \phi_m(s)\mathrm{d}s. \tag{3.96}$$

Since the functions u_m remain in a bounded set of $L^2(0, T; V)$ (cf. (3.31)) and because of (3.92), the right-hand side of (3.96) is uniformly bounded for $s \in [0, T]$ and m:

$$u'_m \textit{ belongs to a bounded set of } L^\infty(0, T; H). \tag{3.97}$$

With (3.97) we infer easily from (3.95) that the u'_m remain in a bounded set of $L^2(0, T; V)$.

The proof is achieved. □

Theorem 3.6. *The assumptions are those of Theorem 3.5. and we assume moreover that Ω is a bounded set of class $\mathscr{C}^2$ and that*

$$f \in L^\infty(0, T; H). \tag{3.98}$$

Then the function u satisfies

$$u \in L^\infty(0, T; H^2(\Omega)). \tag{3.99}$$

Proof.

(i) We write (3.18) in the form

$$\nu\,((\boldsymbol{u}(t), \boldsymbol{v})) = (\boldsymbol{g}(t), \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in V, \tag{3.100}$$

where

$$\boldsymbol{g}(t) = \boldsymbol{f}(t) - \boldsymbol{u}'(t) - B\boldsymbol{u}(t). \tag{3.101}$$

The proof is now based on two successive applications of Proposition I.2.2.

(ii) Since $\boldsymbol{u} \in L^\infty(0, T; V)$ and

$$\begin{aligned} |b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v})| &\leqslant c_0 \|\boldsymbol{u}(t)\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{u}(t)\| \, \|\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)} \\ &\leqslant c_1 \|\boldsymbol{u}(t)\|^2 \, \|\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)}, \end{aligned} \tag{3.102}$$

we have

$$B\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{L}^{4/3}(\Omega)).$$

Thus $(\boldsymbol{f} - \boldsymbol{u}' \in L^\infty(0, T; H))$,

$$\boldsymbol{g} \in L^\infty(0, T; \boldsymbol{L}^{4/3}(\Omega)). \tag{3.103}$$

Proposition I.2.2. then implies that

$$\boldsymbol{u} \in L^\infty(0, T, W^{2,4/3}(\Omega)).$$

By the Sobolev theorem, $W^{2,4/3}(\Omega) \subset L^\infty(\Omega)$, and hence

$$\boldsymbol{u} \in \boldsymbol{L}^\infty(Q).$$

(iii) We can now improve (3.103). We replace (3.102) by the inequality

$$|b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v})| \leqslant c_2 \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(Q)} \|\boldsymbol{u}(t)\| \, \|\boldsymbol{v}\|$$

which shows that

$$B\boldsymbol{u} \in L^\infty(0, T; H).$$

This implies that

$$\boldsymbol{g} \in L^\infty(0, T; H)$$

and another application of Proposition I.2.2. gives

$$\boldsymbol{u} \in L^\infty(0, T; H^2(\Omega)).$$

Remark 3.7. By repeated application of Proposition I.2.2 it is now easy to prove, exactly as in Proposition II.1.1, that if Ω is of class $\mathscr{C}^\infty$, u_0 is given in $\mathscr{C}^\infty(\bar{\Omega})$, and f is given in $\mathscr{C}^\infty(\bar{Q})$, then the solution u is in $\mathscr{C}^\infty(\bar{Q})$. By the same method, intermediate regularity properties can be obtained with suitable hypotheses on the data. □

3.5.2. *The Three-Dimensional Case.*

We will prove for $n = 3$ some regularity properties similar to those obtained for $n = 2$, but in the present case this will be done only by assuming that the data are "small".

In the next theorem we denote by c some constant such that

$$|b(u, v, w)| \leqslant c\|u\|\,\|v\|\,\|w\|, \quad \forall\, u, v, w \in V. \tag{3.104}$$

Theorem 3.7. *We assume that* $n = 3$ *and that there are given* f *and* u_0 *satisfying*

$$u_0 \in H^2(\Omega) \cap V. \tag{3.105}$$

$$f \in L^\infty(0, T; H), f' \in L^1(0, T; H) \tag{3.106}$$

and a further condition given in the course of the proof which is satisfied if ν *is large enough or if* f *and* u_0 *are "small enough"* (1).

Then there exists a unique solution of Problem 3.2 *which satisfies moreover*

$$u' \in L^2(0, T; V) \cap L^\infty(0, T; H). \tag{3.107}$$

Proof.

(i) To begin with, we observe that uniqueness is merely a consequence of Theorem 3.4, because such a solution will satisfy

$$u \in L^\infty(0, T; V) \tag{3.108}$$

and then $V \subset L^4(\Omega)$ implies (see (3.76)) that

$$u \in L^\infty(0, T; L^4(\Omega)). \tag{3.109}$$

(ii) Some of the steps of the existence proof are the same as in Theorem 3.4: we use the Galerkin method of Theorem 3.1, and we choose the basis and u_m so that (3.87) holds. The estimates (3.90), (3.91), and thus (3.92) still hold:

---

(1) See (3.115).

$$|u_m'(0)| \leqslant d_1 = |f(0)| + \nu c_0 \|u_0\|_{H^2(\Omega)} + c_1 \|u_0\|^2_{H^2(\Omega)}. \tag{3.110}$$

We derive in the same fashion equation (3.94) and we deduce from it using (3.104):

$$\frac{\mathrm{d}}{\mathrm{d}t} |u_m'(t)|^2 + 2(\nu - c\|u_m(t)\|)\, \|u_m'(t)\|^2 \leqslant 2|f'(t)|\, |u_m'(t)|. \tag{3.111}$$

(iii) There results from (3.28) and (3.29) that

$$\begin{aligned} \nu\|u_m(t)\|^2 &\leqslant \frac{1}{\nu} \|f(t)\|^2_{V'} - 2(u_m(t), u_m'(t)) \\ &\leqslant \frac{1}{\nu} \|f(t)\|^2_{V'} + 2|u_m(t)|\, |u_m'(t)| \\ &\leqslant \frac{d_2}{\nu} + 2(|u_0|^2 + \frac{Td_2}{\nu})^{1/2} |u_m'(t)| \end{aligned} \tag{3.112}$$

where

$$d_2 = \|f\|^2_{L^\infty(0,T;V')}. \tag{3.113}$$

Using (3.110), we infer from (3.112) that, at time $t = 0$,

$$\nu\|u_m(0)\|^2 \leqslant \frac{d_2}{\nu} + 2d_1(|u_0|^2 + \frac{Td_2}{\nu})^{1/2} = d_3. \tag{3.114}$$

The hypothesis mentioned in the statement of the theorem is that

$$d_4 = \frac{d_2}{\nu} + (1 + d_1^2)(|u_0|^2 + \frac{Td_2}{\nu})^{1/2} \cdot$$

$$\cdot \exp\left(\int_0^T |f'(s)|\mathrm{d}s\right) < \frac{\nu^3}{c^2}. \tag{3.115}$$

Since $d_3 \leqslant d_4$, we get as a consequence of (3.114) – (3.115)

$$\nu\|u_m(0)\|^2 \leqslant d_3 \leqslant d_4 < \frac{\nu^3}{c^2}$$

and then

$$\nu - c\|u_m(0)\| > 0.$$

We deduce from this inequality that $\nu - c\|u_m(t)\|$ remains positive on some interval with origin 0. We denote by T_m the first time $t \leqslant T$ such that

$$\nu - c\|u_m(T_m)\| = 0$$

or, if this does not happen, $T_m = T$.

Then

$$\nu - c\|u_m(t)\| \geqslant 0, \quad 0 \leqslant t \leqslant T_m. \tag{3.116}$$

(iv) With (3.116) we deduce from (3.111) that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_m'(t)|^2 \leqslant 2|f'(t)|\,|u_m'(t)|,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(1+|u_m'(t)|^2) \leqslant |f'(t)|\,(1+|u_m'(t)|^2).$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\{(1+|u_m'(t)|^2)\exp(-\int_0^t |f'(s)|\,\mathrm{d}s)\} \leqslant 0,$$

$$1+|u_m'(t)|^2 \leqslant (1+|u_m'(0)|^2)\exp\left(\int_0^t |f'(s)|\mathrm{d}s)\right),$$

and, by (3.110),

$$1+|u_m'(t)|^2 \leqslant (1+d_1^2)\exp\left(\int_0^T |f'(s)|\mathrm{d}s\right), \; 0 \leqslant t \leqslant T_m. \tag{3.117}$$

From (3.112), (3.115), and (3.117) we get

$$\nu\|u_m(t)\|^2 \leqslant d_4, \quad 0 \leqslant t \leqslant T_m,$$

$$\nu - c\|u_m(t)\| \geqslant \nu - c\sqrt{\frac{d_4}{\nu}} > 0, \quad 0 \leqslant t \leqslant T_m \tag{3.118}$$

Then $T_m = T$, and (3.111) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_m'(t)|^2 + 2(\nu - c\sqrt{\frac{d_4}{\nu}})\,\|u_m'(t)\|^2 \leqslant 2|f'(t)|\,|u_m'(t)|, \; 0 \leqslant t \leqslant T,$$

and we easily deduce from this relation that

u'_m *remains in a bounded set of* $L^2(0, T; V) \cap L^\infty(0, T; H)$. (3.119)

The existence is proved. $\square$

As in the two-dimensional case, we also have

Theorem 3.8. *With the assumptions of Theorem 3.7, and if we assume moreover that Ω is of class $\mathscr{C}^\infty$, the function $\boldsymbol{u}$ satisfies*

$$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}^2(\Omega)). \tag{3.120}$$

Proof. We write (3.18) in the form,

$$\nu((\boldsymbol{u}(t), \boldsymbol{v})) = (\boldsymbol{g}(t), \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in V,$$

with

$$\boldsymbol{g}(t) = \boldsymbol{f}(t) - \boldsymbol{u}'(t) - B\boldsymbol{u}(t).$$

Since $\boldsymbol{f} - \boldsymbol{u}' \in L^\infty(0, T; H)$, Proposition I.2.2 gives (3.120) provided we show that

$$B\boldsymbol{u} \in L^\infty(0, T; H) \text{ (and hence } \boldsymbol{g} \in L^\infty(0, T; H)). \tag{3.121}$$

This result is also obtained by repeated application of Proposition I.2.2 and various estimates on the form b.

By the Hölder inequality, we have:

$$\begin{aligned} |b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v})| &\leqslant c_0 \|\boldsymbol{u}(t)\|_{L^6(\Omega)} \|\boldsymbol{u}(t)\|\, |\boldsymbol{v}|_{L^3(\Omega)} \\ |b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}| &\leqslant c_1 \|\boldsymbol{u}(t)\|^2\, \|\boldsymbol{v}\|_{L^3(\Omega)}. \end{aligned} \tag{3.122}$$

We deduce from (3.122) (and $\boldsymbol{u} \in L^\infty(0, T; V)$), that

$$B\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{L}^{3/2}(\Omega)), \boldsymbol{g} \in L^\infty(0, T; \boldsymbol{L}^{3/2}(\Omega)).$$

Proposition I.2.2 implies that

$$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{W}^{2, 3/2}(\Omega))$$

and, in particular (since $\boldsymbol{W}^{2, 3/2}(\Omega) \subset \boldsymbol{L}^8(\Omega)$ for $n = 3$),

$$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{L}^8(\Omega)).$$

Using again the Hölder inequality we estimate b by

$$b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}) \leqslant c_0 \|\boldsymbol{u}(t)\|_{L^8(\Omega)} \|\boldsymbol{u}(t)\|\, \|\boldsymbol{v}\|_{L^{8/3}(\Omega)}.$$

Hence

$$Bu, g \in L^\infty(0, T; L^{8/5}(\Omega))$$

and by Proposition I.2.2,

$$u \in L^\infty(0, T; W^{2,8/5}(\Omega)) \subset L^\infty(\Omega \times [0, T]). \tag{3.123}$$

With (3.123) and $u \in L^\infty(0, T; V)$, the proof of (3.121) is easy, and thus Theorem 3.8 is proved.

Remark 3.8. The same remark about regularity as Remark 3.7 holds.

Introduction of the Pressure ($n \leqslant 4$).

To introduce the pressure, let us set

$$U(t) = \int_0^t u(s)\mathrm{d}s,\ \beta(t) = \int_0^t Bu(s)\mathrm{d}s,\ F(t) = \int_0^t f(s)\mathrm{d}s.$$

If u is a solution of (3.17) – (3.19) then, for any $n \leqslant 4$,

$$U, \beta, F \in \mathscr{C}([0, T]; V'). \tag{3.124}$$

Integrating (3.18), we see that

$$\nu((U(t), v)) = \langle g(t), v\rangle, \quad \forall\, v \in V, \quad \forall\, t \in [0, T], \tag{3.125}$$

with

$$g(t) = F(t) - \beta(t) - u(t) + u_0, \quad g \in \mathscr{C}([0, T]; V').$$

By application of Proposition I.1.1 and Proposition I.1.2, we get for each $t \in [0, T]$, the existence of some function $p(t)$,

$$p(t) \in L^2(\Omega),$$

such that

$$-\nu\Delta U(t) + \operatorname{grad} p(t) = g(t)$$

or

$$u(t) - u_0 - \nu\Delta U(t) + \beta(t) + \operatorname{grad} p(t) = F(t). \tag{3.126}$$

According to Remark I.1.4, the gradient operator is an isomorphism from $L^2(\Omega)/\mathcal{R}$ into $H^{-1}(\Omega)$. Observing that

$$\operatorname{grad} p = g - \nu\Delta U,$$

we conclude that grad p belongs to $\mathscr{C}[0, T]; H^{-1}(\Omega))$ and therefore

$$p \in \mathscr{C}([0, T]; L^2(\Omega)). \tag{3.127}$$

This enables us to differentiate (3.126) in the distribution sense in $Q = \Omega \times (0, T)$; setting

$$p = \frac{\partial p}{\partial t}, \tag{3.128}$$

we obtain

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta \boldsymbol{u} + \sum_{i=1}^{n} u_i D_i \boldsymbol{u} + \operatorname{grad} p = \boldsymbol{f}, \text{ in } Q. \tag{3.129}$$

The pressure appears in general as a distribution on Q defined by (3.127) – (3.128). Under the assumptions of Theorems 3.6 ($n = 2$) or 3.8 ($n = 3$), the application of Proposition I.2.2 shows also that

$$p \in L^\infty(0, T; H^1(\Omega)). \tag{3.130}$$

3.6. *Relations between the problems of existence and uniqueness* ($n = 3$)

The above sections pointed out the two important problems in the three-dimensional case, for the theory of Navier–Stokes equations:

– uniqueness of "very weak solutions", i.e. of solutions whose existence is guaranteed by Theorem 3.1.

– existence in the large and for any data of "more regular solutions", for instance solutions whose uniqueness is guaranteed by Theorem 3.4, or the solutions whose existence is proved in a restrictive case in Theorem 3.7.

For convenience, until the end of this Section we will call,

– *weak solutions,* the solutions $\boldsymbol{u}$ of (3.13), (3.14), such that

$$\boldsymbol{u} \in L^2(0, T; V) \cap L^\infty(0, T; H) \tag{3.131}$$

and thus (Theorem 3.2)

$$\boldsymbol{u} \in L^{8/3}(0, T; L^4(\Omega)), \ \ \boldsymbol{u}' \in L^{4/3}(0, T; V') \tag{3.132}$$

– *strong solutions,* the solutions $\boldsymbol{v}$ of (3.13), (3.14), such that (3.131), (3.132) hold, and moreover

$$\boldsymbol{v} \in L^8(0, T; L^4(\Omega)). \tag{3.133}$$

According to Theorem 3.1, 3.2, 3.4 and 3.7, we know the existence but not the uniqueness of *weak solutions,* and we know the uniqueness but not the existence of *strong solutions* (except in some very restrictive cases).

We observe that the weak solutions given by Theorem 3.1 satisfy the energy inequality (3.47) (see remark 3.2):

$$|u(t)|^2 + 2\nu \int_0^t \|u(s)\|^2 \, ds$$

$$\leqslant |u_0|^2 + 2 \int_0^t \langle f(s), u(s) \rangle \, ds, \; \forall \, t \in [0, T]. \tag{3.134}$$

According to Theorem 3.4, (3.78), and Lemma 1.2, the strong solutions (if they exist) satisfy an energy equality instead of (3.134):

$$|v(t)|^2 + 2\nu \int_0^t \|v(s)\|^2 \, ds$$

$$= |u_0|^2 + 2 \int_0^t \langle f(s), v(s) \rangle \, ds, \; \forall \, t \in [0, T]. \tag{3.135}$$

The problems of the uniqueness of weak solutions and of the existence of strong solutions are related as follows:

Theorem 3.9. *We assume that* $n = 3$ *and that* f *and* u_0 *are arbitrarily given,*

$$f \in L^2(0, T; H), \; u_0 \in H. \tag{3.136}$$

If there exists a solution v *of* (3.13), (3.14) *satisfying* (3.131), (3.134) (*i.e. a strong solution*), *then there does not exist any other solution* u *of* (3.13) (3.14) *satisfying* (3.131), (3.132) *and* (3.134) (*i.e. a weak solution satisfying the energy inequality*).

This result was proved by J. Sather and J. Serrin (see Serrin [3]).

Proof. Let u and v be the two solutions mentioned in the statement of Theorem 3.9; u satisfies (3.134), v satisfies (3.135). We show in the next Lemma that

$$(u(t), v)) + 2\nu \int_0^t ((u(s), v(s)))\mathrm{d}s = |u_0|^2$$

$$+ \int_0^t \langle f(s), u(s) + v(s)\rangle \mathrm{d}s$$

$$- \int_0^t b(w(s), w(s), v(s))\mathrm{d}s \tag{3.137}$$

where $w = u - v$.

We now add (3.134) to (3.135) and subtract two times (3.137) from the corresponding inequality. After expanding we get

$$|w(t)|^2 + 2\nu \int_0^t \|w(s)\|^2 \mathrm{d}s$$

$$\leqslant 2 \int_0^t b(w(s), w(s), v(s))\, \mathrm{d}s. \tag{3.138}$$

Using the Hölder inequality, we derived in Chapter II the bound

$$|b(w(s), w(s), v(s))| \leqslant c_1 \|w(s)\|_{L^4(\Omega)} \|w(s)\| \, \|v(s)\|_{L^4(\Omega)}.$$

Because of Lemma 3.5 we can majorize this expression by

$$c_2 |w(s)|^{1/4} \|w(s)\|^{7/4} \|v(s)\|_{L^4(\Omega)} \leqslant \nu \|w(s)\|^2$$

$$+ c_3 |w(s)|^2 \|v(s)\|_{L^4(\Omega)}^8.$$

Using this majoration, we deduce from (3.138) that

$$|w(t)|^2 \leqslant 2c_3 \int_0^t |w(s)|^2 \|v(s)\|_{L^4(\Omega)}^8 \mathrm{d}x$$

Since the function $t \to \sigma(t) = 2c_3 \|v(t)\|_{L^4(\Omega)}^8$ is integrable (v satisfies (3.133)), the Gronwall inequality gives

$$|w(t)|^2 \leqslant \int_0^t \sigma(s)\,|w(s)|^2\,\mathrm{d}s \leqslant 0,$$

and thus $w = u, - v = 0$.

It remains to prove (3.137).

Lemma 3.6. *With* u *and* v *as in Theorem* 3.9, *the relation* (3.137) *holds.*

Proof. Let $\rho \in \mathscr{D}(\mathcal{R})$ be a regularizing function, $\rho \geqslant 0$, $\rho(-t) = \rho(t)$, $\rho(t) = 0$ for $|t| \geqslant 1$, and

$$\int_{\mathcal{R}} \rho(t)\,\mathrm{d}t = 1;$$

let ρ_ϵ be defined by $\rho_\epsilon(t) = 1/\epsilon\ \rho(t/\epsilon)$.

We associate with any function w defined on $[0, T]$ the function $\tilde{w}$ defined on $\mathcal{R}$, equal to w on $[0, T]$ and to 0 outside this integral.

Using (3.18) we write,

$$u' + \nu A u + B u = f \text{ on }]0, T[, \tag{3.139}$$

$$v' + \nu A v + B v = f \text{ on }]0, T[, \tag{3.140}$$

and after regularization, (3.139) gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}*\rho_\epsilon*\rho_\epsilon) = (\tilde{f} - B\tilde{u} - \nu A\tilde{u})*\rho_\epsilon*\rho_\epsilon, \text{ on }]\epsilon, T-\epsilon[. \tag{3.141}$$

Due to (3.139) and (3.141), the following relations hold on $]\epsilon, T - \epsilon[$

$$\frac{\mathrm{d}}{\mathrm{d}t}(v, \tilde{u}*\rho_\epsilon*\rho_\epsilon) = \langle v', \tilde{u}*\rho_\epsilon*\rho_\epsilon\rangle + \langle v, \tilde{u}'*\rho_\epsilon*\rho_\epsilon\rangle$$

$$= \langle f - \nu A v - B v, \tilde{u}*\rho_\epsilon*\rho_\epsilon$$

$$+ \langle v, (\tilde{f} - B\tilde{u} - \nu A\tilde{u})*\rho_\epsilon*\rho_\epsilon\rangle,$$

Since the function ρ_ϵ is odd, we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}(v, u_\epsilon) = \langle f - \nu A v - B v, u_\epsilon\rangle + \langle v_\epsilon, f - Bu - \nu A u\rangle, \tag{3.142}$$

where $u_\epsilon = \tilde{u}*\rho_\epsilon*\rho_\epsilon$, $v_\epsilon = \tilde{v}*\rho_\epsilon*\rho_\epsilon$.

We now integrate (3.142) from s to t, $\epsilon < s < t < T - \epsilon$; we get

$$(\boldsymbol{v}(t), \boldsymbol{u}_\epsilon(t)) - (\boldsymbol{v}(s), \boldsymbol{u}_\epsilon(s)) = \int_0^t \langle \boldsymbol{f}(\sigma), \boldsymbol{u}_\epsilon(\sigma) + \boldsymbol{v}_\epsilon(\sigma)\rangle$$

$$- \nu \int_s^t \{((\boldsymbol{v}(\sigma), \boldsymbol{u}_\epsilon(\sigma))) + ((\boldsymbol{v}_\epsilon(\sigma), \boldsymbol{u}(\sigma)))\} \, \mathrm{d}\sigma$$

$$- \int_s^t \{b(\boldsymbol{v}(\sigma), \boldsymbol{v}(\sigma), \boldsymbol{u}_\epsilon(\sigma)) + b(\boldsymbol{u}(\sigma), \boldsymbol{u}(\sigma), \boldsymbol{v}_\epsilon(\sigma))\} \, \mathrm{d}\sigma. \tag{3.143}$$

Due to (3.131), (3.132) and (3.133),

$$\left.\begin{aligned} &\boldsymbol{u}_\epsilon \to \boldsymbol{u} \text{ in } L^2_{\mathrm{loc}}(]0, T[; V) \text{ and } L^{8/3}_{\mathrm{loc}}(]0, T[; \boldsymbol{L}^4(\Omega)), \\ &\boldsymbol{v}_\epsilon \to \boldsymbol{v} \text{ in } L^2_{\mathrm{loc}}(]0, T[; V) \text{ and } L^{8}_{\mathrm{loc}}(]0, T[; \boldsymbol{L}^4(\Omega)). \end{aligned}\right\} \tag{3.144}$$

The passage to the limit in (3.143) is then legitimate; we see that for almost all s and t, $0 < s < t < T$:

$$(\boldsymbol{v}(t), \boldsymbol{u}(t)) - (\boldsymbol{v}(s), \boldsymbol{u}(s)) + 2\nu \int_s^t ((\boldsymbol{u}(\sigma), \boldsymbol{v}(\sigma))) \, \mathrm{d}\sigma$$

$$= \int_s^t \langle \boldsymbol{f}(\sigma), \boldsymbol{u}(\sigma) + \boldsymbol{v}(\sigma)\rangle \, \mathrm{d}\sigma - \int_s^t \{b(\boldsymbol{v}(\sigma), \boldsymbol{v}(\sigma), \boldsymbol{u}(\sigma))$$

$$+ b(\boldsymbol{u}(\sigma), \boldsymbol{u}(\sigma), \boldsymbol{v}(\sigma)\} \, \mathrm{d}\sigma. \tag{3.145}$$

As the function $\boldsymbol{u}$ is weakly continuous in H (Theorem 3.1) and the function $\boldsymbol{v}$ is strongly continuous in H (Theorem 3.4), the function

$$t \to (\boldsymbol{u}(t), \boldsymbol{v}(t))$$

is continuous and therefore the relation (3.145) holds for all s and t, $0 \leqslant s < t \leqslant T$. Setting $s = 0$ in (3.145) we obtain precisely (3.137) if we moreover observe that

$$b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{u}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = b(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{v}) = b(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{v}).$$

3.7. *Utilization of a special basis.*

If Ω is a bounded Lipschitz open set in $\mathbb{R}^n$ ($n = 2, 3$), one can use as a special basis for the Galerkin method (§3.2), the basis of eigenfunctions w_j introduced in Chapter I, §2.6. This will enable us to obtain further *a priori* estimates on the solution and existence results of regular solutions, slightly different from that of Section 3.5.

3.7.1. *Preliminary Results.*

The following Lemmas will be useful.

Lemma 3.7. *Let Ω be a bounded open set of class $\mathscr{C}^2$ in $\mathbb{R}^n$ (arbitrary n). Then $|Au|$ is a norm on $V \cap H^2(\Omega)$ which is equivalent to the norm induced by $H^2(\Omega)$.*

Proof. For $u \in V, f \in L^2(\Omega)$, $Au = f$ is equivalent to

$$((u, v)) = (f, v), \quad \forall\, v \in V. \tag{3.146}$$

The interpretation of (3.146) as a linear Stokes problem and Proposition I.2.2. give that

$$\|u\|_{H^2(\Omega)} \leqslant c_0\, |f| = c_0\, |Au|.$$

The inverse inequality is easy and the Lemma is proved.

Lemma 3.8. *Assume that $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) is bounded and of class $\mathscr{C}^2$. If $u \in V \cap H^2(\Omega)$, then $B\,u \in H \subset L^2(\Omega)$ and*

$$|Bu| \leqslant c_1\, |u|^{1/2}\, \|u\|\, |Au|^{1/2} \ \text{if } n = 2 \tag{3.147}$$

$$|Bu| \leqslant c_2\, \|u\|^{3/2}\, |Au|^{1/2} \ \text{if } n = 3. \tag{3.148}$$

Proof. If $n = 2$, we apply Hölder's inequality with exponents 4, 4, 2, as follows:

$$\left| \int_\Omega u_i (D_i u_j)\, v_j \,\mathrm{d}x \right| \leqslant |u_i|_{L^4(\Omega)}\, |D_i u_j|_{L^4(\Omega)}\, |v_j|_{L^2(\Omega)}.$$

By (3.48)[1], the right hand side is bounded by

---

[1] Relation (3.48) established for $v \in H_0^1(\Omega)$ is also valid if $v \in H^1(\Omega)$, the coefficient $2^{1/4}$ being replaced by some other constant $c = c(\Omega)$. See Lions & Magenes [1].

$$c_3\, |u_i|^{1/2}\ |\text{grad}\, u_j|\ |\text{grad}\, D_i u_j|^{1/2}\, |v_j|$$

Whence

$$|b(u, u, v)| \leqslant c_4\, |u|^{1/2}\, \|u\|\ \|u\|^{1/2}_{H^2(\Omega)}\, |v|$$

which implies (3.147) after application of Lemma 3.7.

If $n = 3$, we apply Hölder's inequality with exponents 6, 4, 12 and 2

$$\left|\int_\Omega u_i(D_i u_j)\, v_j\, \mathrm{d}x\right| \leqslant \int_\Omega |u_i|\ |D_i u_j|^{1/2}\, |D_i u_j|^{1/2}\, |v_j|\, \mathrm{d}x$$

$$\leqslant |u_i|_{L^6(\Omega)} |D_i u_j|^{1/2}_{L^2(\Omega)} |D_i u_j|^{1/2}_{L^6(\Omega)} |v_j|_{L^2(\Omega)}.$$

By the Sobolev imbedding Theorems, this is less than

$$c_5\, |u_i|^{3/2}_{H^1(\Omega)} |D_i u_j|^{1/2}_{H^1(\Omega)}\, |v_j|_{L^2(\Omega)}$$

whence

$$|b(u, u, v)| \leqslant c_6\, \|u\|^{3/2}\ \|u\|^{1/2}_{H^2(\Omega)}\, |v|,$$

and (3.148) is proved.

3.7.2. *The Two-Dimensional case*

Theorem 3.10. *We assume that Ω is a bounded open set of class $\mathscr{C}^2$ in $\mathbb{R}^2$.*

Let f and u_0 be given such that

$$u_0 \in H, \tag{3.149}$$

$$f \in L^2(0, T; H). \tag{3.150}$$

Then there exists a unique solution to Problem 3.2., which satisfies moreover

$$\sqrt{t}\, u \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; V), \sqrt{t}\, u' \in L^2(0, T; H) \tag{3.151}$$

If $u_0 \in V$, then

$$u \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; V),\ u' \in L^2(0, T; H) \tag{3.152}$$

Proof. We begin with the case $u_0 \in V$.

We consider again the Galerkin approximation used in the proof of Theorem 3.1. This time the w_j's are those given in Chapter I, §2.6, and we assume that $u_{0m} \in \mathrm{Sp}\,[w_1, \ldots, w_n]$ is chosen so that

$$u_{0m} \to u_0, \text{ strongly in } V, \text{ as } m \to \infty.$$

Relation (3.22) can be written

$$(u_m', w_j) + (\nu A u_m + B u_m, w_j) = (f, w_j),\ 1 \leqslant j \leqslant m. \tag{3.153}$$

By I (2.64),

$$((w_j, v)) = (A w_j, v) = \lambda_j (w_j, v),\quad \forall v \in V. \tag{3.154}$$

Hence after multiplication by λ_j, we can write (3.153) as follows

$$((u_m', w_j)) + \nu (A u_m, A w_j) + (B u_m, A w_j) = (f, A w_j).$$

We multiply the relation by g_j (see (3.21)) and add for $j = 1, \ldots, m$; we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|^2 + \nu |A u_m|^2 + (B u_m, A u_m) = (f, A u_m). \tag{3.155}$$

We deduce from Lemma 3.8 that,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|u_m\|^2 + \nu |A u_m|^2 \\ &\leqslant |f|\,|A u_m| + c_1 |u_m|^{1/2} \|u_m\|\,|A u_m|^{3/2} \\ &\leqslant \frac{\nu}{4}|A u_m|^2 + \frac{1}{\nu}|f|^2 + \frac{\nu}{4}|A u_m|^2 + c_7 |u_m|^2 \|u_m\|^4\end{aligned}$$

Whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|^2 + \nu |A u_m|^2 \leqslant \frac{2}{\nu}|f|^2 + 2c_7 |u_m|^2 \|u_m\|^4. \tag{3.156}$$

In particular, setting $\sigma_m(t) = 2c_7 |u_m(t)|^2 \|u_m(t)\|^2$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|^2 \leqslant \frac{2}{\nu}|f|^2 + \sigma_m \|u_m\|^2.$$

From the estimates (3.30), (3.31),

$$\int_0^T \sigma_m(t)\,\mathrm{d}t \leqslant \text{const} = c_8,$$

and by Gronwall's method and (3.153), we find that

$$\textit{the sequence } \boldsymbol{u}_m \textit{ remains bounded in } L^\infty(0, T; V). \tag{3.157}$$

Back to (3.156) we get now,

$$\|\boldsymbol{u}_m(T)\|^2 + \nu \int_0^T |A\boldsymbol{u}_m|^2 \, \mathrm{d}t \leq \|\boldsymbol{u}_{0m}\|^2 + \frac{2}{\nu} \int_0^T |\boldsymbol{f}|^2 \, \mathrm{d}t$$

$$+ 2c_7 \int_0^T |\boldsymbol{u}_m|^2 \|\boldsymbol{u}_m\|^4 \, \mathrm{d}t$$

and from (3.157) and Lemma 3.7,

$$\textit{the sequence } u_m \textit{ remains bounded in } L^2(0, T; \boldsymbol{H}^2(\Omega)). \tag{3.158}$$

It is easy to conclude that $\boldsymbol{u}_m$ converges to $\boldsymbol{u}$, with $\boldsymbol{u}$ in $L^\infty(0, T; V) \cap L^2(0, T; \boldsymbol{H}^2(\Omega))$. Lemma 3.8 implies that $B\boldsymbol{u} \in L^4(0, T; H)$; on the other hand, $A\boldsymbol{u} \in L^2(0, T; H)$, and by (3.18), $\boldsymbol{u}' = \boldsymbol{f} - B\boldsymbol{u} - \nu A\boldsymbol{u} \in L^2(0, T; H)$.

The theorem is proved in the case $\boldsymbol{u}_0 \in V$.

If $\boldsymbol{u}_0 \in H$, we multiply (3.156) by t and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(t\|\boldsymbol{u}_m\|^2) + \nu t|A\boldsymbol{u}_m|^2 \leq \|\boldsymbol{u}_m\|^2 + \frac{2t}{\nu}|\boldsymbol{f}|^2 + t\sigma_m\|\boldsymbol{u}_m\|^2$$

and we obtain the result that $\sqrt{t}\,\boldsymbol{u}_m$ remains bounded in $L^\infty(0, T; V)$ and $L^2(0, T; \boldsymbol{H}^2(\Omega))$. At the limit $m \to \infty$, we obtain the first part of (3.151); the result $\sqrt{t}\,\boldsymbol{u}' \in L^2(0, T; H)$ is obtained with a proof similar to the preceding one: by (3.147),

$$\sqrt{t}\,\boldsymbol{u}' = \sqrt{t}\,(\boldsymbol{f} - B\boldsymbol{u} - \nu A\boldsymbol{u})$$

clearly belongs to $L^2(0, T; H)$.

3.7.3. *The three-dimensional case.*

Theorem 3.11. *We assume that Ω is a bounded open set of class $\mathscr{C}^2$ in $\mathbb{R}^3$.*

Let there be given $\boldsymbol{u}_0$ and $\boldsymbol{f}$ such that

$$\boldsymbol{u}_0 \in V, \boldsymbol{f} \in L^\infty(0, T; H). \tag{3.159}$$

Then there exists $T_* = \min(T, T_1)$,

$$T_1 = \frac{3}{4 c_9 \mu^2} \tag{3.160}$$

$$\mu = 4 \max\left(\|u_0\|^2, \frac{2}{c_{10}\nu^2} N(f)^2\right), \quad N(f) = |f|_{L^\infty(0, T; H)}, \tag{3.161}$$

such that there exists a unique solution u *of Problem 3.2 on* $(0, T_*)$; *moreover* u *satisfies*

$$u \in L^\infty(0, T_*; V) \cap L^2(0, T_*; H^2(\Omega)) \tag{3.162}$$

$$u' \in L^2(0, T_*; H). \tag{3.163}$$

Proof. We proceed as in Theorem 3.10, until the equation (3.155). Then Lemma 3.8 gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|^2 + \nu|Au_m|^2 \leqslant |f|\,|Au_m| + c_2\|u_m\|^{3/2}|Au_m|^{3/2}$$

$$\leqslant \frac{\nu}{4}|Au_m|^2 + \frac{1}{\nu}|f|^2 + \frac{\nu}{4}|Au_m|^2 + c_9\|u_m\|^6 .$$

Whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|^2 + \nu|Au_m|^2 \leqslant \frac{2}{\nu}|f|^2 + 2c_9\|u_m\|^6 . \tag{3.164}$$

By Lemma 3.8, $|Av| \leqslant c_{10}\|v\|$, and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|^2 + c_{10}\nu\|u_m\|^2 \leqslant \frac{2}{\nu}N(f) + 2c_9\|u_m\|^6 .$$

Now we claim that

$$\|u_m(t)\|^2 \leqslant \mu, \; 0 \leqslant t \leqslant T_1, \tag{3.166}$$

provided $\|u_{0m}\| \leqslant \|u_0\|$ and this is satisfied for instance if $u_{0m} = P_m u_0$ is the projection of u_0 (either in V or H) on the space spanned by $w_1, \ldots, w_m$.

Indeed let $z = \max(\mu/2, \|u_m\|^2)$. Then, by a well known result of G. Stampacchia [1], the function z is almost everywhere differentiable and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 0 \text{ if } \|u_m\|^2 \leqslant \mu/2$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\ \|u_m\|^2 \text{ otherwise.}$$

Whence

$$\frac{\mathrm{d}z}{\mathrm{d}t} \leqslant 2\ c_9\ z^3,\ z(0) = \mu/2,$$

so that

$$\frac{1}{\mu^2} \leqslant \frac{4(1 - c_9\ t\ \mu^2)}{\mu^2} \leqslant \frac{1}{z^2}$$

for $t \leqslant T_1$.

We deduce that the sequence u_m remains bounded in $L^\infty(0, T_*; V)$ and then, as before, in $L^2(0, T_*; H^2(\Omega))$. Also u'_m remains bounded in $L^2(0, T_*; H)$. The limit u is the unique solution to Problem 3.2 on $[0, T_*]$. Whence the result.

Remark 3.9. Either directly or by passing to the lower limit in (3.156), (3.164), we see that u satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\ \|u\|^2 + \nu|Au|^2 \leqslant \frac{2}{\nu}\,|f|^2 + 2c_7\,|u|^2\ \|u\|^4$$

$$(n = 2,\ 0 < t < T) \tag{3.167}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\ \|u\|^2 + \nu|Au|^2 \leqslant \frac{2}{\nu}\,|f|^2 + 2c_9\,\|u\|^6$$

$$(n = 3,\ 0 < t < T_*). \tag{3.168}$$

3.8. *The special case $f = 0$*

We are going to show that for $f = 0$, the fluid smoothly tends to the equilibrium, as $t \to \infty$.

Theorem 3.12. *We assume that Ω is a $\mathscr{C}^2$ open bounded set in $\mathbb{R}^n$ $(n = 2, 3)$, and that $u_0 \in V$ and $f = 0$.*

If $n = 2$, then $u \in L^\infty(0, \infty; V)$ and tends to 0 in V as $t \to +\infty$.

If $n = 3$, then

$$u \in L^\infty(0, T_2; V),\ u \in L^\infty(T_3, \infty; V)$$

for some T_2 and T_3 estimated below, $(0 < T_2 \leqslant T_3)$ and u tends to 0 in V as $t \to +\infty$.

Proof. Let u be some solution to Problem 3.2 given by Theorem 3.1 (*the solution* if $n = 2$). From (3.47)

$$|u(t)|^2 + 2\nu \int_0^t \|u(s)\|^2 \, ds \leqslant |u_0|^2, \quad \forall t > 0, \tag{3.169}$$

and (3.167), (3.168) give

$$\frac{d}{dt} \|u\|^2 + c_{10} \nu \|u\|^2 \leqslant c_{11} \|u\|^{2n}, \tag{3.170}$$

$$c_{11} = 2c_7 |u_0|^2 \text{ if } n = 2, \; c_{11} = 2c_9 \text{ if } n = 3.$$

If for some t_1, $u(t_1) \in V$ and $\|u(t_1)\|$ is sufficiently small so that:

$$\|u(t_1)\|^{2(n-1)} \leqslant \frac{7c_{11}}{2c_{11}} \tag{3.171}$$

then

$$\frac{d}{dt} \|u(t_1)\|^2 + \frac{\nu c_{10}}{2} \|u(t_1)\|^2 \leqslant 0,$$

so that $\|u(t)\|$ will decay after t_1; more precisely (3.171) will remain valid for any $t \geqslant t_1$, $u \in L^\infty(t_1, \infty; V)$, and $\|u(t)\| \to 0$ exponentially as $t \to +\infty$.

The existence of some t_1 for which (3.171) holds is guaranteed by (3.169). Otherwise we must have

$$|u_0|^2 \geqslant 2\nu \, t_1 \left(\frac{\nu c_{10}}{2c_{11}}\right)^{1/(n-1)},$$

and this is impossible for t_1 sufficiently large.

If $n = 3$ we can take $T_2 = T_1$ given by Theorem 3.11. Or more directly

$$\frac{d}{dt} \|u\|^2 \leqslant c_{11} \|u\|^6,$$

gives

$$\|u(t)\| \leqslant \frac{\|u_0\|}{(1 - 2c_{11} \, t \, \|u_0\|^4)^{1/4}},$$

for $t < T_2 = (2c_{11} \|u_0\|^4)^{-1}$.

§4. Alternate Proof of Existence by Semi-Discretization

Our goal now is to give an alternate proof of the existence of weak solutions of the Navier–Stokes equations which will be valid in any number of space dimensions. An approximate solution is constructed by semi-discretization in t, and we then pass to the limit using compactness arguments.

In Section 4.1 we reformulate the problem in a way which is appropriate in any dimension and we state the existence results; Section 4.2 describes the construction of the approximate solution; Sections 4.3 and 4.4 deal with the *a priori* estimates and the passage to the limit.

4.1. *Statement of the Problem*

Before giving the existence theorem in higher dimensions we must reformulate the problem of weak solutions. As in the stationary case, if $n > 4$, the form b is not trilinear continuous on V and a statement such as (3.10)–(3.14) does not make sense since the $b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v})$ term in (3.13) may not be defined.

For this purpose we introduce again (see Chapter II, Section 1.2) the spaces V_s:

$$V_s = \text{the closure of } \mathscr{V} \text{ in } H_0^1(\Omega) \cap H^s(\Omega),\ s \geqslant 1. \tag{4.1}$$

The spaces $H_0^1(\Omega) \cap H^s(\Omega)$ and V_s are endowed with the usual Hilbert norm of $H^s(\Omega)$:

$$\|\boldsymbol{u}\|_{H^s(\Omega)} = \left\{ \sum_{[j] \leqslant s} |D^j \boldsymbol{u}|^2 \right\}^{1/2} \quad (s \text{ integer}). \tag{4.2}$$

Obviously ($s \geqslant 1$),

$$V_s \subset V \tag{4.3}$$

with a continuous injection and V_s is dense in V.

The form b is defined on $V \times V \times V_s$, provided $s \geqslant n/2$; more precisely:

Lemma 4.1. *The form b is trilinear continuous on $V \times V \times V_s$ if $s \geqslant n/2$ and*

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant c|\boldsymbol{u}|\ \|\boldsymbol{v}\|\ \|\boldsymbol{w}\|_{V_s}\ ^{(1)}. \tag{4.4}$$

---

(1) Any dimension, Ω bounded or not.

Proof. For $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathscr{V}$, the Hölder inequality gives

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| = |b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})| \leqslant \sum_{i,j=1}^{n} \|u_i\|_{L^2(\Omega)} \|D_i w_j\|_{L^n(\Omega)} \cdot \|v_j\|_{L^{2n/n-2}(\Omega)}$$

$$\leqslant \text{(by the Sobolev inequality } H_0^1(\Omega) \subset L^{2n/n-2}(\Omega))$$

$$\leqslant c_0 |\boldsymbol{u}|\ \|\boldsymbol{v}\| \sum_{i,j=1}^{n} \|D_i w_j\|_{L^n(\Omega)}.$$

Since $s \geqslant n/2$, $H^{s-1}(\Omega)$ is included in $L^q(\Omega)$ where

$$\frac{1}{q} = \frac{1}{2} - \frac{s-1}{n}, \ q \geqslant n. \tag{4.5}$$

If $w \in V_s$ then $D_i w_j$ belongs to $H^{s-1}(\Omega)$ and to $L^q(\Omega)$; $D_i w_j$ belonging to $L^q(\Omega) \cap L^2(\Omega)$ then $D_i w_j \in L^n(\Omega)$ also and

$$\|D_i w_j\|_{L^n(\Omega)} \leqslant c_1 \|\boldsymbol{w}\|_{V_s}$$

so that

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant c_2 |\boldsymbol{u}|\ \|\boldsymbol{v}\|\ \|\boldsymbol{w}\|_{V_s}. \tag{4.6}$$

This estimate shows that we can extend by continuity the form b from $\mathscr{V} \times \mathscr{V} \times \mathscr{V}$ onto $V \times V \times V_s$, and even $H \times V \times V_s$, by (4.6).

Lemma 4.2. *If* $\boldsymbol{u}$ *belongs to* $L^2(0, T; V) \cap L^\infty(0, T; H)$ *then* $B\boldsymbol{u}$ *belongs to* $L^2(0, T; V_s')$ *for* $s \geqslant n/2$.

Proof. By the definition of B and because of (4.4),

$$|\langle B\boldsymbol{u}(t), \boldsymbol{v}\rangle| = |b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v})|$$

$$\leqslant c|\boldsymbol{u}(t)|\ \|\boldsymbol{u}(t)\|\ \|\boldsymbol{v}\|_{V_s}, \ \forall\ \boldsymbol{v} \in V_s;$$

hence

$$\|B\boldsymbol{u}(t)\|_{V_s'} \leqslant c|\boldsymbol{u}(t)|\ \|\boldsymbol{u}(t)\| \text{ for a.e. } t \in [0, T] \tag{4.7}$$

and the lemma is proved. □

In all dimensions of space, we can give the following weak formulation of the Navier–Stokes problem:

Problem 4.1. *For f and $\boldsymbol{u}_0$ given such that*

$$f \in L^2(0, T; V'), \tag{4.7}$$

$$\boldsymbol{u}_0 \in H, \tag{4.8}$$

to find $\boldsymbol{u}$ satisfying

$$\boldsymbol{u} \in L^2(0, T; V) \cap L^\infty(0, T; H) \tag{4.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}, \boldsymbol{v}) + \nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})$$

$$= \langle f, \boldsymbol{v}\rangle, \forall\ \boldsymbol{v} \in V_s \left(s \geqslant \frac{n}{2} \right) \tag{4.10}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{4.11}$$

If $\boldsymbol{u}$ satisfies (4.9) and (4.10), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{g}, \boldsymbol{v}\rangle, \forall\ \boldsymbol{v} \in V_s$$

with

$$\boldsymbol{g} = f - B\boldsymbol{u} - \nu A\boldsymbol{u}.$$

Due to Lemma 4.2, $B\boldsymbol{u}$ belongs to $L^2(0, T; V_s')$ and since $f - \nu A\boldsymbol{u}$ belongs to $L^2(0, T; V')$,

$$\boldsymbol{g} \in L^2(0, T; V_s'). \tag{4.12}$$

Lemma 1.1 then implies that

$$\left.\begin{aligned} &\boldsymbol{u}' \in L^2(0, T; V_s') \\ &\boldsymbol{u}' = f - \nu A\boldsymbol{u} - B\boldsymbol{u}; \end{aligned}\right\} \tag{4.13}$$

therefore $\boldsymbol{u}$ is almost everywhere equal to a continuous function from $[0, T]$ into V_s' and (4.11) makes sense.

An alternate formulation of Problem 4.1 is the following one.

Problem 4.2. *Given f and $\boldsymbol{u}_0$, satisfying* (4.7)–(4.8), *to find $\boldsymbol{u}$ satisfying*

$$\boldsymbol{u} \in L^2(0, T; V) \cap L^\infty(0, T; H),\ \boldsymbol{u}' \in L^2(0, T; V_s') \left(s \geqslant \frac{n}{2} \right), \tag{4.14}$$

$$u' + \nu Au + Bu = f \text{ on } (0, T), \tag{4.15}$$

$$u(0) = u_0, \tag{4.16}$$

The formulations (4.9)–(4.11) and (4.14)–(4.16) are equivalent.

The existence of solutions of these problems is given by the following theorem which implies Theorem 3.1:

Theorem 4.1. *Let there be given f and u_0 which satisfy* (4.7)–(4.8). *Then, there exists at least one solution u of Problem* 4.2. *Moreover u is weakly continuous from* $[0, T]$ *into H.*

This theorem is proved in Sections 4.2 and 4.3; the weak continuity in H is a direct consequence of (4.14) and Lemma 1.4.

4.2. *The Approximate Solutions*

Let N be an integer which will later go to infinity and set

$$k = T/N. \tag{4.17}$$

We will define recursively a family of elements of V, say $u^0, u^1, \ldots, u^N$, where u^m will be in some sense an approximation of the function u we are looking for, on the interval $mk < t < (m+1)k$.

We define first the elements $f^1, \ldots, f^N$ of V':

$$f^m = \frac{1}{k}\int_{(m-1)k}^{mk} f(t)\mathrm{d}t,\ m = 1, \ldots, N;\ f^m \in V'. \tag{4.18}$$

We begin with

$$u^0 = u_0, \text{ the given initial data;} \tag{4.19}$$

then when $u^0, \ldots, u^{m-1}$ are known, we define u^m as an element of V which satisfies

$$\frac{u^m - u^{m-1}}{k} + \nu Au^m + Bu^m = f^m; \tag{4.20}$$

u^m depends on k; for simplification we denote it u^m in lieu of u_k^m.

The existence of such a u^m is asserted by Lemma 4.3, whose proof is postponed to the end of this Section.

Lemma 4.3. *For each fixed k and each $m \geqslant 1$, there exists at least one u^m satisfying* (4.20) *and moreover*

$$|u^m|^2 - |u^{m-1}|^2 + |u^m - u^{m-1}|^2 + 2k\nu\|u^m\|^2 \leqslant 2k\langle f^m, u^m\rangle. \tag{4.21}$$

For each fixed k (or N), we associate to the elements $u^1, \ldots, u^N$, the following approximate functions:

$$u_k\colon [0,T] \to V,\ u_k(t) = u^m,\ t \in [(m-1)k, mk],\quad m = 1, \ldots, N \tag{4.22}$$

$$w_k\colon [0,T] \to H,\ w_k \textit{ is continuous, linear on each interval } [(m-1)k, mk] \textit{ and } w_k(mk) = u^m,\ m = 0, \ldots, N. \tag{4.23}$$

In Section 4.3 we will give *a priori* estimates of these functions; we will then pass to the limit $k \to 0$ (Section 4.3). □

Proof of Lemma 4.3. The equation (4.20) must be understood in a space larger than V', for example in a space V_s', $s \geqslant n/2$. It is equivalent to

$$(u^m, v) + k\nu((u^m, v)) + kb(u^m, u^m, v) = \langle u^{m-1} + kf^m, v\rangle,\ \forall v \in V_s. \tag{4.24}$$

We proceed by the Galerkin method, essentially as for Theorem II.1.2 We choose a sequence of elements $w_1, \ldots, w_i, \ldots$, of V_s which is free and total in V_s and thus in V. For each r, by application of Lemma 1.4, we prove the existence of an element ϕ_r (depending on r, k, m):

$$\phi_r = \sum_{i=1}^{r} \xi_{i,r} w_i, \tag{4.25}$$

$$(\phi_r, v) + k\nu((\phi_r, v)) + kb(\phi_r, \phi_r, v) = \langle u^{m-1} + kf^m, v\rangle,\ \forall v \in Sp(w_1, \ldots, w_r).^{(1)} \tag{4.26}$$

We must then get an *a priori* estimate independent of r, and pass to the limit $r \to \infty$ (k and m are fixed in this proof).

Taking $v = \phi_r$ in (4.26) we get

$$(\phi_r - u^{m-1}, \phi_r) + k\nu\|\phi_r\|^2 = k\langle f^m, \phi^r\rangle. \tag{4.27}$$

Now

$$2(a - b, a) = |a|^2 - |b|^2 + |a - b|^2,\ \forall a, b \in H, \tag{4.28}$$

---

(1) $Sp(w_1, \ldots, w_r)$ = the space spanned by $w_1, \ldots, w_r$.

so that (4.27) gives

$$|\phi_r|^2 + |\phi_r - u^{m-1}|^2 + 2k\nu\|\phi_r\|^2 = |u^{m-1}|^2 + 2k\langle f^m, \phi_r\rangle \leqslant |u^{m-1}|^2 + 2k\|f^m\|_{V'}|\phi_r| \leqslant |u^{m-1}|^2 + \nu k\|\phi_r\|^2 + \frac{k}{\nu}\|f^m\|_{V'}^2. \tag{4.29}$$

Hence

$$|\phi_r|^2 + |\phi_r - u^{m-1}|^2 + k\nu\|\phi_r\|^2 \leqslant |u^{m-1}|^2 + \frac{k}{\nu}\|f^m\|_{V'}^2. \tag{4.30}$$

The inequality (4.30) shows that the sequence ϕ_r remains bounded in V as $r \to \infty$. Therefore we can extract from ϕ_r a subsequence $\phi_{r'}$ such that

$$\phi_{r'} \to \phi \text{ in } V \text{ weakly, as } r' \to \infty. \tag{4.31}$$

By standard arguments we then pass to the limit in (4.26) and prove that $\phi = u^m$ satisfies (4.24).

It remains to establish (4.21). This would be obvious if we could take $v = u^m$ in (4.24); since $u^m \notin V_s$ in general we proceed instead by passage to the limit. We pass to the lower limit in (4.29), noting that the norm is lower semi-continuous for the weak topology:

$$|\phi|^2 \leqslant \varliminf_{r'\to\infty} |\phi_{r'}|^2, \quad \|\phi\|^2 \leqslant \varliminf_{r'\to\infty} \|\phi_{r'}\|^2.$$

The proof is complete. □

4.3. *A Priori Estimates*

Lemma 4.4.

$$|u^m|^2 \leqslant d_1, \quad m = 1, \ldots, N, \tag{4.32}$$

$$k \sum_{m=1}^{N} \|u^m\|^2 \leqslant \frac{1}{\nu} d_1, \tag{4.33}$$

$$\sum_{m=1}^{N} |u^m - u^{m-1}|^2 \leqslant d_1, \tag{4.34}$$

where d_1 depends only on the data:

$$d_1 = |u_0|^2 + \frac{1}{\nu}\int_0^T \|f(s)\|_{V'}^2 \mathrm{d}s. \tag{4.35}$$

Proof. As mentioned in the proof of Lemma 4.3, we cannot take the scalar product of (4.20) by u^m, at least for $n > 4$ ($Bu^m \notin V'$). But (4.21) will play the same role as the equation we would obtain by this procedure.

We majorize the right-hand side of (4.21) by

$$2k\|f^m\|_{V'}\|u^m\| \leqslant k\nu\|u^m\|^2 + \frac{k}{\nu}\|f^m\|_{V'}^2,$$

and we obtain

$$|u^m|^2 - |u^{m-1}|^2 + |u^m - u^{m-1}|^2 + k\nu\|u^m\|^2 \leqslant \frac{k}{\nu}\|f^m\|_{V'}^2, \; m = 1, \dots, N. \tag{4.36}$$

Summing the equalities (4.36) for $m = 1, \dots, N$, we find

$$|u^N|^2 + \sum_{m=1}^{N} |u^m - u^{m-1}|^2 + \nu k \sum_{m=1}^{N} \|u^m\|^2 \leqslant |u_0|^2 + \frac{k}{\nu}\sum_{m=1}^{N} \|f^m\|_{V'}^2. \tag{4.37}$$

Summing the inequalities (4.36) for $m = 1, \dots, r$, and dropping the term $\|u^m\|^2$, we get

$$|u^r|^2 \leqslant |u_0|^2 + \frac{k}{\nu}\sum_{m=1}^{r} \|f^m\|_{V'}^2 \leqslant |u_0|^2 + \frac{k}{\nu}\sum_{m=1}^{N} \|f^m\|_{V'}^2, \; r = 1, \dots, N. \tag{4.38}$$

The lemma is now a consequence of (4.37) – (4.38) and of a majoration of the right-hand side of these inequalities given in the next lemma.

Lemma 4.5. *Let f^m be defined by* (4.18). *Then*

$$k \sum_{m=1}^{N} \|f^m\|_{V'}^2 \leqslant \int_0^T \|f(t)\|_{V'}^2 \mathrm{d}t. \tag{4.39}$$

Proof. Due to the Schwarz inequality,

$$\|f^m\|_{V'}^2 = \frac{1}{k^2} \left\| \int_{(m-1)k}^{mk} f(t)\mathrm{d}t \right\|_{V'}^2 \leqslant \frac{1}{k} \int_{(m-1)k}^{mk} \|f(t)\|^2 \mathrm{d}t.$$

Then (4.39) follows by summation of these inequalities for $m = 1, \ldots, N$. □

The last *a priori* estimate is the following:

Lemma 4.6.

The sum $k \sum_{m=1}^{N} \left\| \frac{u^m - u^{m-1}}{k} \right\|_{V'}^2$ *is bounded independently of* k.

Proof. Taking the norm in (4.20) we obtain

$$\left\| \frac{u^m - u^{m-1}}{k} \right\|_{V_s'} \leqslant \|f^m\|_{V_s'} + \nu\|Au^m\|_{V_s'} + \|Bu^m\|_{V_s'} \leqslant$$

$$c_1 \{\|f^m\|_{V'} + \|u^m\|_V\} + \|Bu^m\|_{V_s'},$$

$$\left\| \frac{u^m - u^{m-1}}{k} \right\|_{V_s'}^2 \leqslant c_2 \{\|f^m\|_{V'}^2 + \|u^m\|_V^2 + \|Bu^m\|_{V_s'}^2\}.$$

From (4.4) and (4.32) we get

$$\|Bu^m\|_{V_s'}^2 \leqslant c_3 |u^m|^2 \|u^m\|^2 \leqslant c_4 \|u^m\|^2.$$

We finally have

$$k \sum_{m=1}^{N} \left\| \frac{u^m - u^{m-1}}{k} \right\|_{V_s'}^2 \leqslant c_5 k \sum_{m=1}^{N} (\|f^m\|_{V'}^2 + \|u^m\|^2),$$

and we finish the proof using (4.33) and Lemma 4.5. □

It is interesting now to interpret the above in terms of the approximate functions:

Lemma 4.7. *The functions u_k and w_k remain in a bounded set of $L^2(0, T; V) \cap L^\infty(0, T; H)$; w'_k is bounded in $L^2(0, T; V'_s)$ and*

$$u_k - w_k \to 0 \text{ in } L^2(0, T; H) \text{ as } k \to \infty. \tag{4.40}$$

Proof. The estimations on u_k and w_k are just interpretations of (4.32)–(4.33) and Lemma 4.6; (4.40) is a consequence of (4.34) and the next lemma.

Lemma 4.8.

$$|u_k - w_k|_{L^2(0,T;H)} = \sqrt{\frac{k}{3}}\left(\sum_{m=1}^{N} |u^m - u^{m-1}|^2\right)^{1/2}. \tag{4.41}$$

Proof.

$$w_k(t) - u_k(t) = \frac{(t-mk)}{k}(u^m - u^{m-1}) \text{ for } (m-1)k \leqslant t \leqslant mk,$$

$$\int_{(m-1)k}^{mk} |w_k(t) - u_k(t)|^2 \mathrm{d}t = \frac{k}{3}|u^m - u^{m-1}|^2,$$

and we find (4.41) by summation.

4.4. *Passage to the Limit*

Due to Lemma 4.7, we can extract from u_k a subsequence $u_{k'}$ such that

$$\begin{aligned} u_{k'} \to u \text{ in } & L^2(0, T; V) \text{ weakly,} \\ \text{in } & L^\infty(0, T; H) \text{ weak-star} \end{aligned} \tag{4.42}$$

We want to prove that u is a solution of (4.14) – (4.16); we need a strong convergence result for the u_k, in order to pass to the limit in (4.20). The functions w_k will play, for this, a useful auxiliary role.

We can choose the subsequence k', so that

$$\begin{aligned} w_{k'} \to u_* \text{ in } & L^2(0, T; V) \text{ weakly,} \\ \text{in } & L^\infty(0, T; H) \text{ weak-star,} \end{aligned} \tag{4.43}$$

$$\frac{\mathrm{d}w_{k'}}{\mathrm{d}t} \to u'_* \text{ in } L^2(0, T; V'_s) \text{ weakly.} \tag{4.44}$$

Because of (4.40), $u = u_*$.

Theorem 2.1 shows us that

$$w_{k'} \to u \text{ in } L^2(0, T; H); \tag{4.45}$$

thus by (4.40),

$$u_{k'} \to u \text{ in } L^2(0, T; H). \tag{4.46}$$

The equations (4.20) can be interpreted as

$$\frac{\mathrm{d}w_k}{\mathrm{d}t} + Au_k + Bu_k = f_k, \tag{4.47}$$

with f_k defined by

$$f_k(t) = f^m, (m-1)k \leqslant t < mk, m = 1, \ldots, N.$$

Because of (4.42), (4.46) and Lemmas 3.2 and 4.2,

$$Bu_{k'} \to Bu \text{ in } L^2 \ (0, T; V'_s) \text{ weakly.}$$

By Lemma 4.9 below,

$$f_k \to f \text{ in } L^2(0, T; V');$$

therefore we can pass to the limit in (4.47), and we find

$$u' + \nu Au + Bu = f.$$

Due to (4.43), (4.44) and Lemma 4.1,

$$\langle w_{k'}(t), \sigma\rangle \to \langle u(t), \sigma\rangle, \ \ \forall \sigma \in V'_s, \ \ \forall t \in [0, T];$$

since $w_{k'}(0) = u_0$, we get

$$u(0) = u_0.$$

We have proved that u satisfies (4.14) – (4.16); the proof of Theorem 4.1 will be complete once we prove

Lemma 4.9.

$$f_k \to f \textit{ in } L^2(0, T; V'), \textit{ as } k \to 0. \tag{4.48}$$

---

(1) Strictly speaking if $u_0 \in V$, $w_k(t) \in V$ for $0 \leqslant t \leqslant k$; in this case we simply replace $L^2(0, T; V)$ by $L^2_{\text{loc}}([0, T]; V)$ whenever we are considering the functions w_k.

Proof. We observe that the transformation

$$f \to f_k$$

is a linear averaging mapping in $L^2(0, T; V')$; this mapping is continuous by Lemma 4.5 which enables us to assert that:

$$\|f_k\|_{L^2(0,T;V')} \leqslant \|f\|_{L^2(0,T;V)}. \tag{4.49}$$

Therefore, instead of proving (4.48) for any f in $L^2(0, T; V')$ we need only to prove it for f in a dense subspace of $L^2(0, T; V')$; for an f in $\mathscr{C}([0, T]; V')$ the result is elementary and we skip its proof.

Remark 4.1. Summing the equations (4.21) for $m = 1, \ldots, r$, and dropping the terms $|\boldsymbol{u}^m - \boldsymbol{u}^{m-1}|^2$, we get

$$|\boldsymbol{u}^r|^2 + 2k\nu \sum_{m=1}^{r} \|\boldsymbol{u}^m\|^2 \leqslant |\boldsymbol{u}_0|^2 + 2k \sum_{m=1}^{r} \langle \boldsymbol{f}^m, \boldsymbol{u}^m \rangle. \tag{4.50}$$

The relation (4.50) can be interpreted as

$$|\boldsymbol{u}_k(t)|^2 + 2\nu \int_0^{t_k} \|\boldsymbol{u}_k(s)\|^2 \mathrm{d}s \leqslant |\boldsymbol{u}_0|^2 + \int_0^{t_k} \langle \boldsymbol{f}_k(s), \boldsymbol{u}_k(s) \rangle \,\mathrm{d}s, \tag{4.51}$$

where

$$t_k = (m+1)k, \text{ for } mk \leqslant t < (m+1)k. \tag{4.52}$$

For each fixed t, $\boldsymbol{u}_k \cdot (t)$ is bounded in H independently of k and t; as $k' \to 0$, $\boldsymbol{u}_{k'}(t)$ converges to $\boldsymbol{u}(t)$ in V'_s weakly; therefore [(1)]

$$\boldsymbol{u}_{k'}(t) \to \boldsymbol{u}(t) \text{ in } H \text{ weakly, as } k' \to 0, \ \forall\, t \in [0, T]. \tag{4.53}$$

We then pass to the lower limit in (4.51) (t fixed, $k' \to 0$), using (4.42) and (4.53). This leads to the *energy inequality*:

$$\begin{aligned} &|\boldsymbol{u}(t)|^2 + 2\nu \int_0^t \|\boldsymbol{u}(s)\|^2 \,\mathrm{d}s \leqslant \\ &\qquad |\boldsymbol{u}_0|^2 + 2 \int_0^t \langle \boldsymbol{f}(s), \boldsymbol{u}(s) \rangle \,\mathrm{d}s, \quad \forall\, t \in [0, T]. \end{aligned} \tag{4.54}$$

---

(1) Proof by contradiction.

If $n = 2$, using (3.62), it is easy to prove directly the *energy equality:*

$$|\boldsymbol{u}(t)|^2 + 2\nu \int_0^t \|\boldsymbol{u}(s)\|^2 \,\mathrm{d}s = |\boldsymbol{u}_0|^2 + 2\int_0^t \langle \boldsymbol{f}(s), \boldsymbol{u}(s)\rangle \,\mathrm{d}s, \quad \forall\, t \in [0, T]. \tag{4.55}$$

§5. Discretization of the Navier–Stokes Equations: General Stability and Convergence Theorems

This section is concerned with a general discussion of the discretization of the evolution Navier–Stokes equations. We study here a full discretization of the equations, both in the space and time variables:

1) The discretization in the space variables appears through the introduction of an external approximation of the space V; for example, one of the approximations (APX1) to (APX4), corresponding either to finite differences or finite elements. Actually these particular examples will be discussed in more detail in reference [9].

2) For the discretization in the time variables, we propose, among many natural and classical schemes, four schemes with two levels in time (fully implicit scheme, Crank–Nicholson scheme, a scheme implicit in the linear part and explicit in its nonlinear part, and a scheme of explicit type).

After the description of the scheme under consideration we proceed to study the stability of these schemes. The problem of stability is the terminology in Numerical Analysis for the problem of getting *a priori* estimates on the approximate solutions. Classically, discretization in both space and time of evolution equations can lead to unstable or conditionally stable schemes: the approximate solutions are unbounded unless the discretization parameters satisfy some restriction. We discuss in full detail the numerical stability of the four schemes considered. To our knowledge the methods used here are non-classical methods for studying the stability of nonlinear equations. The study of nonlinear instability is a difficult problem; our study here, based on the energy method, leads only to sufficient conditions for stability; the stability conditions which are obtained seem close to being necessary, but the problem of necessary conditions of stability is not studied at all in the text.

The last subject treated in this section is the convergence of the schemes. Two general convergence theorems in suitable spaces are proved for the different schemes. The proof of convergence depends on discrete compactness methods. Owing to the lack of uniqueness of weak solutions in the three-dimensional case, the convergence results obtained in the two- and three-dimensional cases are different, and better, of course, if $n = 2$.

The division of this material throughout subsequent subsections is as follows: In Subsection 5.1 we describe the general type of discretization and the numerical schemes which will be studied. In Subsections 5.2, 5.3, and 5.4, we successively study the stability of Schemes 5.1 and 5.2 (Subsection 5.2), 5.3 (Subsection 5.3) and 5.4 (Subsection 5.4). Subsection 5.5 deals with auxiliary *a priori* estimates of a rather technical character (involving fractional derivatives in time of the approximate functions). Subsection 5.6 contains the description of the consistency hypotheses, the statement of the general convergence theorems, and the proofs of these theorems.

The application of these results to specific approximations of the space V will be treated in Section 6. There we will also study practical methods for the resolution of the discrete problems, and the Appendix contains the description of practical examples. Other methods of approximation of the nonlinear evolution Navier–Stokes equations are given in Sections 7 and 8, including the fractional step or projection method and the artificial compressibility method.

From now on we restrict ourselves to the "concrete" dimensions of space, $n = 2$ and $n = 3$.

5.1. *Description of the Approximation Schemes*

From now on we will be concerned with the approximation of the solutions of the Navier–Stokes equations in the two- and three-dimensional cases exclusively, Ω *being bounded.* For simplicity we suppose that the given data, $\boldsymbol{u}_0, \boldsymbol{f}$, satisfy

$$\boldsymbol{f} \in L^2(0, T; H), \tag{5.1}$$

and, as before,

$$\boldsymbol{u}_0 \in H. \tag{5.2}$$

Theorems 3.1 and 3.2 tell us that there exists a unique solution of Problem 3.2 if $n = 2$, and that there exists at least one such solution if $n = 3$.

Let there be given a stable and convergent external approximation of the space V, say $\{(V_h, p_h, r_h)_{h \in \mathcal{H}}, (\overline{\omega}, F)\}$; the V_h are assumed to be finite dimensional spaces. This approximation could be any of the approximations (APX1), ... , (APX5), that were described in Chapter I. For simplicity we assume that

$$V_h \subset L^2(\Omega),\ \forall\, h \in \mathcal{H}, \tag{5.3}$$

a condition which is realized by all the previous approximations. The space V_h is therefore equipped with two norms: the norm $|\cdot|$ induced by $L^2(\Omega)$ and its own norm $\|\cdot\|_h$. Since V_h is finite dimensional, these norms must be equivalent; the quotient of the two norms is bounded by a constant which may depend on h. Therefore we assume more precisely that

$$|u_h| \leqslant d_0\, \|u\|_h,\ \forall\, u_h \in V_h, \tag{5.4}$$

d_0 independent of h, and

$$\|u_h\|_h \leqslant S(h)\, |u_h|,\ \ \forall\, u_h \in V_h. \tag{5.5}$$

The constant $S(h)$, which usually depends on h, plays an important role in the study of the stability of the numerical approximation; for this reason $S(h)$ is sometimes called the *stability constant.* Usually $S(h) \to +\infty$, as $h \to 0$.

Let there be given a trilinear continuous form on V_h, say $b_h(u_h, v_h, w_h)$, which satisfies

$$b_h(u_h, v_h, v_h) = 0\ \ \forall\, u_h, v_h \in V_h, \tag{5.6}$$

$$|b_h(u_h, v_h, w_h)| \leqslant d_1\, \|u_h\|_h\, \|v_h\|_h\, \|w_h\|_h, \tag{5.7}$$

$$\forall\, u_h, v_h, w_h \in V_h,\ (d_1 \text{ independent of } h),$$

and some further properties which will be announced when needed (i.e., when discussing the stability and the convergence of the schemes).

Let us divide the interval $[0, T]$ into N intervals of equal length k:

$$k = T/N. \tag{5.8}$$

As in Section 4, we associate with k and the functions f, the elements $f^1, \ldots, f^N$:

$$f^m = \frac{1}{k}\int_{(m-1)k}^{mk} f(t)\,\mathrm{d}t,\ m = 1, \dots, N;\ f^m \in L^2(\Omega). \tag{5.9}$$

We will describe and study four basic schemes chosen from among a large class of interesting and sometimes classical schemes which have been proposed for the Navier–Stokes equations.

For all the four schemes we define recursively for each h and k a family of elements $u_h^0, \dots, u_h^N$, of V_h. Actually these elements depend on h, k (and the data), and should be denoted u_{hk}^m; nevertheless, for simplicity we do not emphasize this double dependence.

In each of the four schemes, we start the recurrence with

$$u_h^0 = \textit{the orthogonal projection of } u_0 \textit{ onto } V_h, \textit{ in } L^2(\Omega); \tag{5.10}$$

this definition makes sense by (5.3) and we immediately observe that

$$|u_h^0| \leqslant |u_0|,\ \ \forall\, h. \tag{5.11}$$

Scheme 5.1. *When $u_h^0, \dots, u_h^{m-1}$, are known, u_h^m is the solution in V_h of*

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \nu((u_h^m, v_h))_h + b_h(u_h^{m-1}, u_h^m, v_h)$$
$$= (f^m, v_h),\ \ \forall\, v_h \in V_h. \tag{5.12}$$

Scheme 5.2. *When $u_h^0, \dots, u_h^{m-1}$, are known, u_h^m is the solution in V_h of*

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \frac{\nu}{2}((u_h^{m-1} + u_h^m, v_h))_h$$
$$+ \frac{1}{2} b_h(u_h^{m-1}, u_h^{m-1} + u_h^m, v_h)$$
$$= (f^m, v_h),\ \ \forall\, v_h \in V_h. \tag{5.13}$$

Scheme 5.3. *When $u_h^0, \dots, u_h^{m-1}$, are known, u_h^m is the solution in V_h of*

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \nu((u_h^m, v_h))_h + b_h(u_h^{m-1}, u_h^{m-1}, v_h)$$
$$= (f^m, v_h),\ \ \forall\, v_h \in V_h. \tag{5.14}$$

Scheme 5.4. *When $u_h^0, \dots, u_h^{m-1}$, are known, u_h^m is the solution in V_h of*

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \nu((u_h^{m-1}, v_h))_h + b_h(u_h^{m-1}, u_h^{m-1}, v_h)$$

$$= (f_h^m, v_h), \quad \forall\, v_h \in V_h. \tag{5.15}$$

For all the schemes the equation defining u_h^m is equivalent to a linear equation of the form

$$a_h(u_h^m, v_h) = L_h(v_h), \quad \forall\, v_h \in V_h, \tag{5.16}$$

L_h depends on m, a_h depends on m for Schemes 5.1 and 5.2, but not in the case of Schemes 5.3 and 5.4.

We observe, in all cases, that

$$a_h(v_h, v_h) \geqslant \frac{1}{k}\,|v_h|^2, \tag{5.17}$$

and therefore the existence and uniqueness of the solution of (5.16) is a consequence of the Projection Theorem (Theorem I.2.2).

Remark 5.1.

(i) The computation of u_h^m requires the inversion of a matrix;

- the matrix is positive definite, nonsymmetric and depends on m for Schemes 5.1 and 5.2,
- the matrix is positive definite, symmetric, and does not depend on m for Schemes 5.3 and 5.4.

(ii) Scheme 5.1 is the standard fully implicit scheme; Scheme 5.2 is an interpretation of the classical Crank–Nicholson scheme. Scheme 5.3 is a partially implicit scheme, implicit only in the linear part of the operator.

(iii) Scheme 5.4 is an explicit scheme or more precisely an interpretation of the so-called explicit schemes; this terminology is justified by the fact that this type of scheme usually gives u_h^m explicitly, that is to say without inverting any matrix. In the present case, due to the discrete condition $\operatorname{div} u = 0$ built in the space V_h, the determination of u_h^m necessitates the inversion of a matrix. This restricts considerably the interest of this scheme, but we considered it of interest nevertheless

(iv) Besides this discussion on the type of scheme, the reader is referred to Section 6 for practical methods of computation of the u_h^m.

Remark 5.2: Related Schemes.

(i) A related form of Schemes 5.1 and 5.2 is a nonlinear form of these schemes:

Scheme 5.1'.

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \nu((u_h^m, v_h))_h + b_h(u_h^m, u_h^m, v_h) = (f^m, v_h), \quad \forall v_h \in V_h. \tag{5.18}$$

Scheme 5.2'.

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \frac{\nu}{2}((u_h^{m-1} + u_h^m, v_h))_h + \frac{1}{4} b_h(u_h^{m-1} + u_h^m, u_h^{m-1} + u_h^m, v_h) = (f^m, v_h), \quad \forall v_h \in V_h. \tag{5.19}$$

(ii) A related form of Scheme 5.3 is a Crank–Nicholson scheme, implicit in its linear part:

Scheme 5.3'.

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \frac{\nu}{2}((u_h^{m-1} + u_h^m, v_h))_h + b_h(u^{m-1}, u^{m-1}, v_h) = (f^m, v_h), \quad \forall v_h \in V_h. \tag{5.20}$$

(iii) These Schemes could be studied by exactly the same methods as Schemes 5.1 – 5.4.

5.2. *Stability of Schemes 5.1 and 5.2*

The problem is to prove some *a priori* estimates on the approximate solution.

5.2.1. *Scheme 5.1*

Lemma 5.1. *The solution u_h^m of* (5.12) *remain bounded in the following sense:*

$$|u_h^m|^2 \leqslant d_2, \; m = 0, \ldots, N, \tag{5.21}$$

$$\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 \leqslant d_2, \tag{5.22}$$

$$k \sum_{m=1}^{N} \|u_h^m\|^2 \leqslant \frac{1}{\nu} d_2, \tag{5.23}$$

where

$$d_2 = |u_0|^2 + \frac{d_0^2}{\nu} \int_0^T |f(s)|^2 \, ds. \tag{5.24}$$

Proof. We take $v_h = u_h^m$ in (5.12). Due to (5.6) and the identity

$$2(a-b, a) = |a|^2 - |b|^2 + |a-b|^2, \quad \forall\, a, b \in L^2(\Omega), \tag{5.25}$$

we obtain

$$\begin{aligned} |u_h^m|^2 - |u_h^{m-1}|^2 + |u_h^m - u_h^{m-1}|^2 + 2k\nu \|u_h^m\|_h^2 \\ &= 2k\,(f^m, u_h^m) \\ &\leqslant 2k\,|f^m|\,|u_h^m| \leqslant \text{(by (5.4))} \\ &\leqslant 2kd_0 |f^m|\,\|u_h^m\|_h \\ &\leqslant k\nu \|u_h^m\|_h^2 + \frac{kd_0^2}{\nu} |f^m|^2. \end{aligned} \tag{5.26}$$

Hence

$$\begin{aligned} |u_h^m|^2 - |u_h^{m-1}|^2 + |u_h^m - u_h^{m-1}|^2 + k\nu \|u_h^m\|_h^2 \\ \leqslant \frac{kd_0^2}{\nu} |f^m|^2, \; m = 1, \ldots, N. \end{aligned} \tag{5.27}$$

Adding these inequalities for $m = 1, \ldots, N$, we get

$$\begin{aligned} |u_h^N|^2 + \sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 + k\nu \sum_{m=1}^{N} \|u_h^m\|_h^2 \\ \leqslant |u_h^0|^2 + \frac{kd_0^2}{\nu} \sum_{m=1}^{N} |f^m|^2. \end{aligned} \tag{5.28}$$

One can check as in Lemma 4.5 that

$$k \sum_{m=1}^{N} |f^m|^2 \leqslant \int_0^T |f(s)|^2 \, ds; \tag{5.29}$$

thus, by (5.11) and (5.29), it follows that the right-hand side of (5.28) is bounded by

$$d_2 = |u_0|^2 + \frac{d_0^2}{\nu} \int_0^T |f(s)|^2 \, ds. \tag{5.30}$$

This proves (5.22) and (5.23).

We then add the inequalities (5.27) for $m = 1, \ldots, r$; dropping some positive terms, we get

$$|u_h^r|^2 \leqslant |u_h^0|^2 + \frac{kd_0^2}{\nu} \sum_{m=1}^{r} |f^m|^2$$
$$\leqslant \text{(due to the above)} \leqslant d_2;$$

(5.21) is proved too.

5.2.2. *Scheme 5.2*

Lemma 5.2. *The solutions* u_h^m *of* (5.13) *remain bounded in the following sense:*

$$|u_h^m|^2 \leqslant d_2, \; m = 1, \ldots, N, \tag{5.31}$$

$$k \sum_{m=1}^{N} \left\| \frac{u_h^m + u_h^{m-1}}{2} \right\|_h^2 \leqslant \frac{d_2}{\nu}, \tag{5.32}$$

with the same d_2 *as* (5.24).

Proof. We take $v_h = u_h^m + u_h^{m-1}$ (in 5.13). Due to (5.6) we find

$$\begin{aligned} |u_h^m|^2 - |u_h^{m-1}|^2 + 2k\nu \left\| \frac{u_h^m + u_h^{m-1}}{2} \right\|_h^2 & \\ = k\,(f^m, u_h^m + u_h^{m-1}) & \\ \leqslant 2kd_0 |f^m| \left\| \frac{u_h^m + u_h^{m-1}}{2} \right\|_h & \\ \leqslant k\nu \left\| \frac{u_h^m + u_h^{m-1}}{2} \right\|_h^2 + k \frac{d_0^2}{\nu} |f^m|^2. & \end{aligned} \tag{5.33}$$

Therefore

$$|u_h^m|^2 + k\nu \left\| \frac{u_h^m + u_h^{m-1}}{2} \right\|_h^2 \leqslant |u_h^{m-1}|^2 + \frac{kd_0^2}{\nu} |f^m|^2. \tag{5.34}$$

We add these relations for $m = 1, \ldots, N$ and get

$$|u_h^N|^2 + k\nu \sum_{m=1}^{N} \left\| \frac{u_h^m + u_h^{m-1}}{2} \right\|_h^2$$

$$\leqslant |u_h^0|^2 + \frac{kd_0^2}{\nu} \sum_{m=1}^{N} |f^m|^2$$

$$\leqslant \text{(as before)} \leqslant d_2.$$

This proves (5.32); adding then the relations (5.34) for $m = 1, \ldots, r$, and dropping the unnecessary terms, we find

$$|u_h^r|^2 \leqslant |u_h^0|^2 + \frac{kd_0^2}{\nu} \sum_{m=1}^{r} |f^m|^2 \leqslant d_2;$$

this implies (5.31).

5.2.3. *Stability Theorems*

We recall first a definition:

Definition 5.1. *An infinite set of functions* E *is called* $L^p(0, T; X)$ *stable if and only if* E *is a bounded subset of* $L^p(0, T; X)$.

It is interesting to deduce from the previous estimations some stability results.

In order to state these results, we introduce the approximate functions u_h

$$u_h : [0, T] \mapsto V_h, \tag{5.35}$$

$$u_h(t) = u_h^m, \ (m-1)\,k \leqslant t < mk \ \text{(Scheme 5.1)}$$

$$u_h(t) = \frac{u_h^m + u_h^{m-1}}{2}, \ (m-1)\,k \leqslant t < mk \ \text{(Scheme 5.2)},$$

$$m = 1, \ldots, N. \tag{5.36}$$

Due to Lemmas 5.1 and 5.2,

$$\operatorname*{Sup}_{t\in[0,T]} |u_h(t)| \leqslant \sqrt{d_2}, \tag{5.37}$$

$$\int_0^T \|u_h(t)\|_h^2 \, \mathrm{d}t \leqslant \frac{d_2}{\nu}. \tag{5.38}$$

Since the prolongation operators $p_h \in \mathcal{L}(V_h, F)$ are stable, we have

$$\|p_h u_h\|_F \leqslant d_3 \|u_h\|_h, \ \forall u_h \in V_h, \ (d_3 \text{ independent of } h). \tag{5.39}$$

We infer from (5.38) that

$$\int_0^T \|p_h u_h(t)\|_F^2 \mathrm{d}t \leqslant \frac{d_3^2 d_2}{\nu} .$$

These remarks enable us to state the stability theorem:

Theorem 5.1. *The functions u_h, $h \in \mathcal{H}$, corresponding to Schemes* 5.1 *and* 5.2 *are unconditionally $L^\infty(0, T; L^2(\Omega))$ stable; the functions $p_h u_h$ are unconditionally $L^2(0, T; F)$ stable.*

Remark 5.2. The majoration (5.22), and similar majorations for the other schemes which we will give later on, does not correspond to stability results but will be technically useful for the proof of the convergence of the scheme.

For the same majoration for Scheme 5.2, see Subsection 5.4.3.

5.3. *Stability of Scheme 5.3*

We infer from (5.5) and (5.7) that

$$\begin{aligned} |b_h(u_h, u_h, v_h)| &\leqslant d_1 \|u_h\|_h^2 \|v_h\|_h \\ &\leqslant d_1 S^2(h) |u_h| \ \|u_h\|_h \ |v_h|, \ \forall \ u_h, v_h \in V_h. \end{aligned} \tag{5.40}$$

Sometimes this relation can be improved and this means an important improvement of some restrictive conditions of stability which will appear later on in this section; for this reason we will assume that

$$|b_h(u_h, u_h, v_h)| \leqslant S_1(h) \ |u_h| \ \|u_h\|_h \ |v_h|, \ \forall u_h, v_h \in V_h, \tag{5.41}$$

where at least

$$S_1(h) \leqslant d_1 S^2(h). \tag{5.42}$$

5.3.1. *A Priori Estimates*

Lemma 5.3. *We assume that k and h satisfy*

$$kS_1^2(h) \leqslant d', \quad kS^2(h) \leqslant d''\ {}^{(1)}, \tag{5.43}$$

where d' and d'' are some constants depending on the data and are estimated in the course of the proof.

Then, the $\boldsymbol{u}_h^m$ given by (5.14) *remain bounded in the following sense:*

$$|\boldsymbol{u}_h^m| \leqslant d_4, \; m = 0, \dots, N, \tag{5.44}$$

$$\sum_{m=1}^{N} |\boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1}|^2 \leqslant d_4, \tag{5.45}$$

$$k \sum_{m=1}^{N} \|\boldsymbol{u}_h^m\|_h^2 \leqslant d_4, \tag{5.46}$$

where d_4 is some constant depending only on the data, d', and d''.

Proof. We write (5.14) with $\boldsymbol{v}_h = \boldsymbol{u}_h^m$. Using again (5.25), we obtain the relation

$$\begin{aligned}|\boldsymbol{u}_h^m|^2 - |\boldsymbol{u}_h^{m-1}|^2 + |\boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1}|^2 + 2k\nu\|\boldsymbol{u}_h^m\|_h^2 \\ = -2kb_h(\boldsymbol{u}_h^{m-1}, \boldsymbol{u}_h^{m-1}, \boldsymbol{u}_h^m) + 2k(\boldsymbol{f}^m, \boldsymbol{u}_h^m).\end{aligned} \tag{5.47}$$

Due to (5.6) the right-hand side of (5.47) is equal to

$$-2kb_h(\boldsymbol{u}_h^{m-1}, \boldsymbol{u}_h^{m-1}, \boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1}) + 2k(\boldsymbol{f}^m, \boldsymbol{u}_h^m);$$

this expression is less than (cf. (5.4) and (5.41)):

$$\begin{aligned}&2kS_1(h)|\boldsymbol{u}_h^{m-1}|\;\|\boldsymbol{u}_h^{m-1}\|_h\;|\boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1}| \\ &\qquad\qquad + 2kd_0|\boldsymbol{f}^m|\;\|\boldsymbol{u}_h^m\|_h, \\ &k\nu\|\boldsymbol{u}_h^m\|_h^2 + 2k^2S_1^2(h)|\boldsymbol{u}_h^{m-1}|^2\;\|\boldsymbol{u}_h^{m-1}\|_h^2 \\ &\qquad + \frac{1}{2}|\boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1}|^2 + \frac{kd_0^2}{\nu}|\boldsymbol{f}^m|^2.\end{aligned}$$

Therefore

$$|\boldsymbol{u}_h^m|^2 - |\boldsymbol{u}_h^{m-1}|^2 + \frac{1}{2}|\boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1}|^2 + k\nu\|\boldsymbol{u}_h^m\|_h^2$$

---

(1) In practice, one of these relations should be a consequence of the other (this depends on the explicit values of S and S_1).

$$-2k^2 S_1^2(h)|u_h^{m-1}|^2\ \|u_h^{m-1}\|_h^2 \leqslant \frac{kd_0^2}{\nu}\ |f^m|^2. \tag{5.48}$$

We add these inequalities for $m = 1, \ldots, r$;

$$|u_h^r|^2 + \frac{1}{2} \sum_{m=1}^{r} |u_h^m - u_h^{m-1}|^2 + k\nu \sum_{m=1}^{r} \|u_h^m\|_h^2$$

$$-2k^2 S_1^2(h) \sum_{m=2}^{r} |u_h^{m-1}|^2\ \|u_h^{m-1}\|_h^2 \leqslant \lambda_m, \tag{5.49}$$

$$\lambda_m = |u_h^0|^2 + \frac{kd_0^2}{\nu} \sum_{m=1}^{r} |f^m|^2 + 2k^2 S_1^2(h)|u_h^0|^2\ \|u_h^0\|_h^2. \tag{5.50}$$

Let us assume that

$$2kS_1^2(h)\lambda_N \leqslant \nu - \delta,\ \text{ for some fixed } \delta,\ 0 < \delta < \nu. \tag{5.51}$$

If this inequality holds, it is easy to show recursively that

$$|u_h^r|^2 + \frac{1}{2} \sum_{m=1}^{r} |u_h^m - u_h^{m-1}|^2$$

$$+ k\delta \sum_{m=1}^{r} \|u_h^m\|_h^2 \leqslant \lambda_r,\ \ r = 1, \ldots, N. \tag{5.52}$$

Indeed the relation (5.48) written with $m = 1$, shows us that (5.52) is true for $r = 1$. Let us assume then that (5.52) is valid up to the order $r - 1$, and let us show this relation for the integer r.

We observe that, by assumption,

$$|u_h^m|^2 \leqslant \lambda_m \leqslant \lambda_N,\ \ m = 1, \ldots, r-1; \tag{5.53}$$

therefore, by (5.51),

$$2k^2 S_1^2(h) \sum_{m=2}^{r} |u_h^{m-1}|^2\ \|u_h^{m-1}\|_h^2$$

$$\leqslant 2k^2 S_1^2(h)\lambda_N \sum_{m=2}^{r} \|u_h^{m-1}\|_h^2$$

$$\leqslant 2k^2 S_1^2(h)\lambda_N \sum_{m=1}^{r} \|u_h^m\|_h^2$$

$$\leqslant k(\nu - \delta) \sum_{m=1}^{r} \|u_h^m\|_h^2.$$

Putting this majoration into (5.49), we get (5.52) for the integer r.

The proof is complete if we show that a condition of the type (5.43) ensures (5.51).

According to a majoration used in Lemmas 5.1 and 5.2 (see (5.11), (5.29))

$$\lambda_N \leqslant |u_0|^2 + \frac{d_0^2}{\nu}\int_0^T |f(s)|^2\,\mathrm{d}s + 2k^2 S_1^2(h)|u_0|^2 \ \|u_h^0\|_h^2$$

$$\leqslant \text{(see (5.5) and (5.24))}$$

$$\leqslant d_2 + 2k^2 S_1^2(h) S^2(h)|u_0|^2.$$

Hence, if (5.43) is satisfied,

$$2kS_1^2(h)\lambda_N \leqslant 2d'(d_2 + 2d'\,d''|u_0|^2).$$

and this is certainly bounded by $\nu - \delta$ if d' and d'' are sufficiently small:

$$2d'(d_2 + 2d'd''|u_0|^2) \leqslant \nu - \delta. \tag{5.54}$$

The proof is complete.

5.3.2. *The Stability Theorem*

We define for the Scheme 5.3 the approximate functions u_h by:

$$u_h\colon [0, T) \to V_h$$

$$u_h(t) = u_h^m,\ (m-1)k \leqslant t < mk,\ m = 1, \ldots, N. \tag{5.55}$$

We infer from (5.39), (5.44), (5.45) that if (5.43) holds then

$$\operatorname*{Sup}_{t\in[0,T]} |u_h(t)| \leqslant \sqrt{d_4},$$

$$\int_0^T \|p_h u_h(t)\|_F^2\,\mathrm{d}t \leqslant d_3 d_4,$$

and thus

Theorem 5.2. *The functions $\boldsymbol{u}_h$ and $p_h \boldsymbol{u}_h$, $h \in \mathcal{H}$, corresponding to the Scheme* 5.3 *are respectively $L^\infty(0, T; L^2(\Omega))$ and $L^2(0, T; F)$ stable, provided k and h remain connected by* (5.43).

Definition 5.2. *Conditions such as* (5.43) *are called stability conditions. They are sufficient conditions ensuring the stability of the scheme. A scheme is called conditionally or unconditionally stable according to whether such a condition occurs or not in proving stability.*

5.4. *Stability of Scheme 5.4*

5.4.1. *A Priori Estimates*

Lemma 5.4. *We assume that k and h satisfy*

$$kS^2(h) \leqslant \frac{1-\delta}{4\nu}, \text{ for some } \delta,\ 0 < \delta < 1, \tag{5.56}$$

and

$$kS_1^2(h) \leqslant \frac{\nu\delta}{8d_5} \tag{5.57}$$

where

$$d_5 = |\boldsymbol{u}_0|^2 + \left(\frac{d_0^2}{\nu} + 4T\right) \int_0^T |f(s)|^2 \,\mathrm{d}s. \tag{5.58}$$

Then the $\boldsymbol{u}_h^m$ given by (5.15) *remain bounded in the following sense:*

$$|\boldsymbol{u}_h^m|^2 \leqslant d_5,\ m = 1, \ldots, N, \tag{5.59}$$

$$k \sum_{m=1}^N \|\boldsymbol{u}_h^m\|_h^2 \leqslant \frac{2d_5}{\delta\nu}, \tag{5.60}$$

$$\sum_{m=1}^N |\boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1}|^2 \leqslant \frac{d_5}{\delta}(2-\delta) + 4T \int_0^T |f(s)|^2 \,\mathrm{d}s. \tag{5.61}$$

Proof. We replace v_h by u_h^{m-1} in (5.15); due to the identity

$$2(a - b, b) = |a|^2 - |b|^2 - |a - b|^2, \forall\, a, b \in L^2(\Omega), \tag{5.62}$$

we find

$$\begin{aligned}|u_h^m|^2 - |u_h^{m-1}|^2 - |u_h^m - u_h^{m-1}|^2 + 2k\nu \|u_h^{m-1}\|_h^2 \\ = 2k(f^m, u_h^m) \\ \leqslant 2kd_0 |f^m|\, \|u_h^m\|_h \\ \leqslant \nu k \|u_h^m\|_h^2 + \frac{kd_0^2}{\nu} |f^m|^2,\end{aligned}$$

$$\begin{aligned}|u_h^m|^2 - |u_h^{m-1}|^2 - |u_h^m - u_h^{m-1}|^2 + k\nu \|u_h^{m-1}\|_h^2 \\ \leqslant \frac{kd_0^2}{\nu} |f^m|.\end{aligned} \tag{5.63}$$

The above differs from (5.26) in that the term $|u_h^m - u_h^{m-1}|^2$ on the left-hand side is affected with a minus sign and so, we must majorize it.

In order to majorize $|u_h^m - u_h^{m-1}|^2$, we write (5.15) with $v_h = u_h^m - u_h^{m-1}$. This gives

$$\begin{aligned}2|u_h^m - u_h^{m-1}|^2 = -2k\nu((u_h^{m-1}, u_h^m - u_h^{m-1}))_h \\ -2kb_h(u_h^{m-1}, u_h^{m-1}, u_h^m - u_h^{m-1}) + 2k(f^m, u_h^m - u_h^{m-1})\end{aligned} \tag{5.64}$$

We successively majorize all the terms on the right-hand side, using repeatedly (5.5), (5.41), and the Schwarz inequality:

$$\begin{aligned}&-2k\nu((u_h^{m-1}, u_h^m - u_h^{m-1}))_h \leqslant 2k\nu \|u_h^{m-1}\|_h \|u_h^m - u_h^{m-1}\|_h \\ &\leqslant 2k\nu S(h) \|u_h^{m-1}\|_h |u_h^m - u_h^{m-1}|_h \\ &\leqslant \frac{1}{4} |u_h^m - u_h^{m-1}|_h^2 + 4k^2\nu^2 S^2(h) \|u_h^{m-1}\|_h^2;\end{aligned}$$

$$\begin{aligned}&-2kb_h(u_h^{m-1}, u_h^{m-1}, u_h^{m-1}) \\ &\leqslant 2kS_1(h)|u_h^{m-1}|\, \|u_h^{m-1}\|_h |u_h^m - u_h^{m-1}| \\ &\leqslant \frac{1}{4} |u_h^m - u_h^{m-1}|^2 + 4k^2 S_1^2(h) |u_h^{m-1}|^2 \|u_h^{m-1}\|_h^2;\end{aligned}$$

$$\begin{aligned}&2k(f^m, u_h^m - u_h^{m-1}) \leqslant 2k|f^m|\, |u_h^m - u_h^{m-1}| \\ &\leqslant \frac{1}{4} |u_h^m - u_h^{m-1}|^2 + 4k^2 |f^m|^2.\end{aligned}$$

Therefore (5.64) becomes

$$
\begin{aligned}
|u_h^m - u_h^{m-1}|^2 &\leqslant 4k^2\nu^2 S^2(h)\,\|u_h^{m-1}\|_h^2 \\
&\quad + 4k^2 S_1^2(h)|u_h^{m-1}|^2\,\|u_h^{m-1}\|_h^2 + 4k^2|f^m|^2 \\
&\leqslant \text{(by (5.56))} \\
&\leqslant k\nu(1-\delta)\|u_h^{m-1}\|_h^2 + 4k^2 S_1^2(h)|u_h^{m-1}|^2\,\|u_h^{m-1}\|_h^2 \\
&\quad + 4k^2|f^m|^2\,; \qquad (5.65)
\end{aligned}
$$

for (5.63) we then have

$$
\begin{aligned}
|u_h^m|^2 - |u_h^{m-1}|^2 &+ k(\nu\delta - 4kS_1^2(h)|u_h^{m-1}|^2)\|u_h^{m-1}\|_h^2 \\
&\leqslant k\left(\frac{d_0^2}{\nu} + 4k\right)|f^m|^2 \\
&\leqslant \text{(since } k \leqslant T) \\
&\leqslant k\left(\frac{d_0^2}{\nu} + 4T\right)|f^m|^2. \qquad (5.66)
\end{aligned}
$$

Summing these inequalities for $m = 1, \ldots, r$, we arrive at

$$
|u_h^r|^2 + k\sum_{m=1}^{r}(\nu\delta - 4kS_1^2(h)|u_h^{m-1}|^2)\|u_h^{m-1}\|_h^2 \leqslant \mu_r, \quad (5.67)
$$

where

$$
\mu_r = |u_h^0|^2 + k\left(\frac{d_0^2}{\nu} + 4T\right)\sum_{m=1}^{r}|f^m|^2. \qquad (5.68)
$$

Using (5.57) we will now prove recursively that

$$
|u_h^r|^2 + \frac{k\nu\delta}{2}\sum_{m=1}^{r}\|u_h^{m-1}\|_h^2 \leqslant \mu_r, \quad r = 1, \ldots, N. \qquad (5.69)
$$

We observe first that

$$
\begin{aligned}
\mu_r \leqslant \mu_N &= |u_h^0|^2 + k\left(\frac{d_0^2}{\nu} + 4T\right)\sum_{m=1}^{N}|f^m|^2 \\
&\leqslant |u_0|^2 + \left(\frac{d_0^2}{\nu} + 4T\right)\int_0^T |f(s)|^2\,\mathrm{d}s = d_5. \qquad (5.70)
\end{aligned}
$$

The relation (5.69) is obvious for $r = 1$; writing (5.66) for $m = 1$ and using (5.57) we get

$$|u_h^1|^2 + k\nu\delta \|u_h^0\|_h^2 \leqslant |u_h^0|^2 + k\left(\frac{d_0^2}{\nu} + 4T\right)|f^1|^2$$

$$+ 4k^2 S_1^2(h)|u_h^0|^2 \|u_h^0\|_h^2 \leqslant \mu_1 + \frac{\nu\delta}{2} \|u_h^0\|_h^2,$$

which is (5.69) for $r = 1$.

Assuming now that the relation (5.69) holds up to the order $r - 1$, we will prove it at the order r. In fact by the recurrence hypothesis,

$$|u_h^{r-1}|^2 \leqslant \mu_{r-1} \leqslant \mu_N \leqslant \text{(by (5.70))} \leqslant d_5. \tag{5.71}$$

Hence (5.67) gives

$$|u_h^r|^2 + k\nu\delta \sum_{m=1}^{r} \|u_h^{m-1}\|_h^2$$

$$\leqslant \mu_r + 4k^2 S_1^2(h) d_5 \sum_{m=1}^{r} \|u_h^{m-1}\|_h^2$$

$$\leqslant \mu_r + \frac{k\nu\delta}{2} \sum_{m=1}^{r} \|u_h^{m-1}\|_h^2, \tag{5.72}$$

and (5.69) at the order r follows.

It remains to prove (5.61). For this we return to (5.65); using (5.56), (5.57), we get

$$|u_h^m - u_h^{m-1}|^2 \leqslant k\nu\left(1 - \frac{\delta}{2}\right)\|u_h^{m-1}\|_h^2 + 4kT|f^m|^2. \tag{5.73}$$

By summation and using (5.29), we find (5.61).

5.4.2. *The Stability Theorem*

We now set

$$u_h\colon [0, T] \to V_h \tag{5.74}$$

$$u_h(t) = u_h^{m-1}, \quad (m-1)k \leqslant t < mk, \quad m = 1, \ldots, N, \tag{5.75}$$

and we have

Theorem 5.3. *The functions u_h and $p_h u_h$, $h \in \mathcal{H}$, corresponding to Scheme 5.4 are respectively $L^\infty(0, T; L^2(\Omega))$ and $L^2(0, T; F)$ stable. provided k and h remain connected by* (5.56)–(5.58).

5.5. *A Complementary Estimate for Scheme 5.2*

Using the techniques extensively applied in Sections 5.3 and 5.4, we can complete Section 5.2 by giving, in the case of Scheme 5.2, an estimation similar to the estimation

$$\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 \leqslant \text{Const.} \tag{5.76}$$

that we proved for Schemes 5.1, 5.3, and 5.4. As mentioned in Remark 5.2, these estimations will be useful for the proof of the convergence.

Lemma 5.5. *The u_h^m defined by* (5.13) (*Scheme* 5.2) *satisfy:*

$$\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 \leqslant d(1 + kS^4(h)), \tag{5.77}$$

where d denotes a constant depending only on the data.

Proof. We take $v_h = 2k(u_h^m - u_h^{m-1})$ in (5.13) and obtain

$$\begin{aligned}
2|u_h^m - u_h^{m-1}|^2 &= -k\nu\|u_h^m\|_h^2 + k\nu\|u_h^{m-1}\|_h^2 \\
&\quad - kb_h(u_h^{m-1}, u_h^m + u_h^{m-1}, u_h^m - u_h^{m-1}) \\
&\quad + 2k(f^m, u_h^m - u_h^{m-1}) \\
&\leqslant (\text{by (5.40) and (5.5)}) \\
&\leqslant -k\nu\|u_h^m\|_h^2 + k\nu\|u_h^{m-1}\|_h^2 \\
&\quad + kd_1 S^2(h)|u_h^{m-1}|\ \|u_h^m + u_h^{m-1}\|_h\ |u_h^m - u_h^{m-1}| \\
&\quad + 2k|f^m|\ |u_h^m - u_h^{m-1}| \\
&\leqslant -k\nu\|u_h^m\|_h^2 + k\nu\|u_h^{m-1}\|_h^2 + \frac{1}{2}|u_h^m - u_h^{m-1}|^2 \\
&\quad + \frac{k^2}{2} d_1^2 S^4(h)|u_h^{m-1}|^2\ \|u_h^m + u_h^{m-1}\|_h^2 \\
&\quad + \frac{1}{2}|u_h^m - u_h^{m-1}|^2 + 2kT|f^m|^2.
\end{aligned}$$

Thus

$$\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 \leqslant k\nu \|u_h^0\|_h^2 + 2kT \sum_{m=1}^{N} |f^m|^2$$

$$+ \frac{k^2}{2} d_1^2 S^4(h) \sum_{m=1}^{N} |u_h^{m-1}|^2 \, \|u_h^m + u_h^{m-1}\|_h^2$$

$$\leqslant \text{(by (5.5), (5.11), (5.29), (5.31), (5.32))}$$

$$\leqslant k\nu S^2(h)|u_0|^2 + 2T \int_0^T |f(s)|^2 \, \mathrm{d}s$$

$$+ \frac{2}{\nu} d_1^2 d_2^2 k S^4(h).$$

The proof is complete.

5.6. *Other A Priori Estimates*

In order to prove strong convergence results we will establish some further *a priori* estimates concerning the fractional derivatives in t of approximate functions. This Section 5.6 is essentially a technical section which is used in Section 5.7 where the convergence of the schemes is proved.

For all the four schemes we define w_h, a function from $\mathcal{R}$ into V_h, by:

$$\begin{aligned}&w_h \textit{ is a continuous function from } \mathcal{R} \textit{ into } V_h, \textit{ linear on}\\ &\textit{each interval } [mk, (m+1)k], \textit{ and } w_h(mk) = u_h^m,\\ &m = 0, \ldots, N-1;\ w_h = 0 \textit{ outside the interval } [0, T].\end{aligned} \tag{5.78}$$

Lemma 5.6. *Assuming the same stability conditions as in Theorems 5.1, 5.2, 5.3,*[(1)] *the Fourier transform $\hat{w}_h$ of w_h satisfies*

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{w}_h(\tau)|^2 \, \mathrm{d}\tau \leqslant \text{Const.}, \quad \text{for } 0 < \gamma < \frac{1}{4}. \tag{5.79}$$

where the constant depends on γ and on the data.

[(1)]No condition for Schemes 5.1, 5.2; conditions (5.43) for Scheme 5.3; conditions (5.56)–(5.57) for Scheme 5.4.

Proof. The four equations (5.12)–(5.15) can be interpreted as

$$\frac{\mathrm{d}}{\mathrm{d}t}(w_h(t), v_h) = ((g_h(t), v_h))_h, \quad \forall v_h \in V_h, \ t \in (0, T), \tag{5.80}$$

where the function g_h satisfies

$$\int_0^T \|g_h(t)\|_h \,\mathrm{d}t \leqslant \text{Const.} \tag{5.81}$$

For example, for Scheme 5.1, g_h is defined by

$$((g_h(t), v_h))_h = (f^m, v_h) - b_h(u_h^{m-1}, u_h^m, v_h) - \nu((u_h^m, v_h))_h,$$
$$\forall v_h \in V_h, \ (m-1)k \leqslant t < mk.$$

Inequality (5.81) follows from (5.4), (5.7), and the previous a priori estimates:

$$\|g_h(t)\|_h \leqslant d_0|f^m| + d_1\|u_h^{m-1}\|_h\|u_h^m\|_h + \nu\|u_h^m\|_h,$$

$$\int_0^T \|g_h(t)\|_h \,\mathrm{d}t \leqslant k \sum_{m=1}^{N} (d_0|f^m| + d_1\|u_h^{m-1}\|_h\|u_h^m\|_h + \nu\|u_h^m\|_h);$$

the right-hand side of this relation is bounded according to Lemma 5.1.

Let us infer (5.79) from (5.80)–(5.81). Extending g_h by 0 outside $[0, T]$ we get a function $\tilde{g}_h$ such that the following equality holds on the whole t line:

$$\frac{\mathrm{d}}{\mathrm{d}t}(w_h(t), v_h) = ((\tilde{g}_h(t), v_h))_h + (u_h^0, v_h)\delta_0 - (u_h^N, v_h)\delta_T, \quad \forall v_h \in V_h, \tag{5.82}$$

where δ_0, δ_T denote the Dirac distribution at 0 and T.

By taking the Fourier transform, we then have

$$-2i\pi\tau(\hat{w}_h(\tau), v_h) = ((\hat{g}_h(\tau), v_h) + (u_h^0, v_h) - (u_h^N, v_h)\exp(-2i\pi\tau T);$$

($\hat{g}_h$ = Fourier transform of $\tilde{g}_h$).

Putting $v_h = \hat{w}_h(\tau)$ and then taking absolute values we get

$$2\pi|\tau|\,|\hat{w}_h(\tau)|^2 \leqslant \|\hat{g}_h(\tau)\|_h\|\hat{w}_h(\tau)\|_h + c_1|\hat{w}_h(\tau)|,$$

since u_h^0 and u_h^N remain bounded.

Due to (5.81) we also have

$$\|\hat{g}_h(\tau)\|_h \leqslant \int_0^T \|g_h(t)\|_h \, \mathrm{d}t \leqslant \text{Const.} = c_2,$$

and, finally,

$$|\tau|\, |\hat{w}_h(\tau)|^2 \leqslant c_3 \|\hat{w}_h(\tau)\|_h. \tag{5.83}$$

For fixed γ, $\gamma < 1/4$, we observe that

$$|\tau|^{2\gamma} \leqslant c_4(\gamma) \frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}}, \ \forall \tau \in \mathcal{R}.$$

Hence

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{w}_h(\tau)|^2 \, \mathrm{d}\tau &\leqslant c_4(\gamma) \int_{-\infty}^{+\infty} \frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}} |\hat{w}_h(\tau)|^2 \, \mathrm{d}\tau \\
&\leqslant \text{(by (5.83))} \\
&\leqslant c_4(\gamma) \int_{-\infty}^{+\infty} |\hat{w}_h(\tau)|^2 \, \mathrm{d}\tau \\
&\quad + c_5 \int_{-\infty}^{+\infty} \frac{\|\hat{w}_h(\tau)\|_h}{1 + |\tau|^{1-2\gamma}} \, \mathrm{d}\tau \\
&\leqslant \text{(by the Schwarz inequality)} \\
&\leqslant c_4 \int_{-\infty}^{+\infty} |\hat{w}_h(\tau)|^2 \, \mathrm{d}\tau \\
&\quad + c_5 \left(\int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau}{(1 + |\tau|^{1-2\gamma})^2} \right)^{1/2} \\
&\quad \cdot \left(\int_{-\infty}^{+\infty} \|\hat{w}_h(\tau)\|_h^2 \, \mathrm{d}\tau \right)^{1/2}.
\end{aligned} \tag{5.84}$$

The integral

$$\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\gamma})^2}$$

is finite for $\gamma < 1/4$. Therefore the right-hand side of the last inequality is finite and bounded due to the Parseval relation and the previous estimations:

$$\int_{-\infty}^{+\infty} |\hat{w}_h(\tau)|^2\, d\tau \leqslant d_0^2 \int_{-\infty}^{+\infty} \|\hat{w}_h(\tau)\|_h^2\, d\tau$$

$$= d_0^2 \int_0^T \|w_h(t)\|_h^2\, dt \leqslant \text{Const.} \tag{5.85}$$

The lemma follows.

5.7. *Convergence of the Numerical Schemes*

Our aim is to prove the convergence of Schemes 5.1 to 5.4, in some sense which will be described later on. We first state the consistency and compactness properties on the discretized data which are required to ensure the convergence. We then state and prove the convergence results.

5.7.1. *Consistency and Compactness Hypotheses*

The subsequent hypotheses will be easier to state after this lemma:

Lemma 5.7. *Let* $\{(V_h, p_h, r_h)_{h\in\mathcal{H}}, (\bar{\omega}, F)\}$ *denote a stable and convergent external approximation of* V. *Let us assume that for some sequence* $h' \to 0$, *a family of functions*

$$u_{h'}: [0, T] \to V_{h'},$$

satisfies

$$p_{h'}u_{h'} \to \phi \text{ in } L^2(0, T; F) \text{ weakly, as } h' \to 0. \tag{5.86}$$

Then for almost every t, $\phi(t) = \bar{\omega}u(t)$, *and*

$$u = \bar{\omega}^{-1}\phi \in L^2(0, T; V). \tag{5.87}$$

Proof. Let us denote by θ some function in $\mathscr{D}((0, T))$. It is easily checked that

$$\int_0^T p_{h'}u_{h'}(t)\theta(t)\mathrm{d}t \to \int_0^T \phi(t)\theta(t)\mathrm{d}t,$$

as h' goes to 0. But condition (C2) of Definition I.3.6 [(1)] shows us that, under these circumstances,

$$\int_0^T \phi(t)\theta(t)\mathrm{d}t \in \bar{\omega}V.$$

Since by definition, $\bar{\omega}V$ is isomorphic to V, $\bar{\omega}V$ is a closed subspace of F; taking now a sequence of functions θ_ϵ converging to the Dirac distribution at the point s, $s \in (0, T)$, we see that for almost every s in $[0, T]$,

$$\int_0^T \phi(t)\theta_\epsilon(t)\mathrm{d}t \to \phi(s) \text{ in } F,$$

and hence

$$\phi(s) \in \bar{\omega}V \text{ a.e.}$$

Then, as $\bar{\omega}$ is an isomorphism, $\bar{\omega}^{-1}\phi$ is defined and belongs to $L^2(0, T; V)$. □

The preceding lemma was quite general, but, in the present situation we assumed that

$$V_h \subset L^2(\Omega),\ \forall\, h; \tag{5.88}$$

therefore it can happen that for some sequence $h' \to 0$,

$$u_{h'} \to u \text{ in } L^2(0, T; L^2(\Omega)) \text{ weakly,}$$

$$p_{h'}u_{h'} \to \phi \text{ in } L^2(0, T; F) \text{ weakly.}$$

By Lemma 5.6, $\phi = \bar{\omega}u_*$, $u_* \in L^2(0, T; V)$. Without further information we cannot assert that $u = u_*$. However, this will be proved for each approximation considered:

---

(1) Definition of the approximation of a normed space in the general frame.

Let $u_{h'}$ be a sequence of functions from $[0, T]$ into $V_{h'}$ such that, as $h' \to 0$.

$$u_{h'} \to u \text{ in } L^2(0, T; \boldsymbol{L}^2(\Omega)) \text{ weakly,}$$

$$p_{h'}u_{h'} \to \phi \text{ in } L^2(0, T; F) \text{ weakly.}$$

Then

$$u \in L^2(0, T; V) \text{ and } \phi = \bar{\omega}u. \tag{5.89}$$

Besides (5.89), the consistency hypotheses are now the following [1]:

Let $v_{h'}$, $w_{h'}$, be two sequences of functions from $[0, T]$ into $V_{h'}$, such that, as $h' \to 0$,

$$p_{h'}v_{h'} \to \bar{\omega}v \text{ in } L^2(0, T; F) \text{ weakly,}$$

$$p_{h'}w_{h'} \to \bar{\omega}w \text{ in } L^2(0, T; F) \text{ (strongly).}$$

Then, as $h' \to 0$,

$$\int_0^T ((v_{h'}(t), w_{h'}(t)))_{h'}\,\mathrm{d}t \to \int_0^T ((v(t), w(t)))\,\mathrm{d}t. \tag{5.90}$$

Let $u_{h'}$, $v_{h'}$, be two sequences of functions from $[0, T]$ into $V_{h'}$ such that, as $h' \to 0$,

$$p_{h'}u_{h'} \to \bar{\omega}u \text{ in } L^2(0, T; F) \text{ weakly,}$$

$$u_{h'} \to u \text{ in } \boldsymbol{L}^2(Q) \text{ strongly, } Q = \Omega \times (0, T),$$

and

$$p_{h'}v_{h'} \to \bar{\omega}v \text{ in } L^2(0, T; F) \text{ weakly.}$$

Then as $h' \to 0$,

$$\int_0^T b_{h'}(u_{h'}(t), v_{h'}(t), \psi(t)r_{h'}w_{h'})\,\mathrm{d}t \to$$

$$\int_0^T b(u(t), v(t), \psi(t)w)\,\mathrm{d}t,$$

---

[1] Compare with the stationary case, (3.7), (3.8), Chapter I; (3.4), (3.5), (3.7), Chapter II.

for each scalar valued function $\psi \in L^\infty(0, T)$ *and each* $\mathbf{w} \in \mathscr{V}$. *If moreover a sequence of functions* $\psi_{k'}$ *is given with*

$$\psi_{k'} \to \psi \text{ in } L^\infty(0, T), \text{ as } k' \to 0,$$

then, as $h' \to 0$, $k' \to 0$,

$$\int_0^T b_{h'}(\mathbf{u}_{h'}(t), \mathbf{v}_{h'}(t), \psi_{k'}(t) r_{h'} \mathbf{w}_{h'}) \mathrm{d}t \to$$

$$\int_0^T b(\mathbf{u}(t), \mathbf{v}(t), \psi(t)\mathbf{w}) \mathrm{d}t. \tag{5.91}$$

In order to prove strong convergence results as required by (5.91) ($\mathbf{u}_{h'} \to \mathbf{u}$ in $L^2(Q)$ strongly) we will assume the following:

Let $\mathbf{v}_{h'}$ *denote a sequence of functions from* $\mathbb{R}$ *into* $V_{h'}$, *with support in* $[0, T]$ *and such that*

$$\int_0^T \|\mathbf{v}_{h'}(t)\|_{h'}^2 \mathrm{d}t \leqslant \text{Const.},$$

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{\mathbf{v}}_{h'}(\tau)|^2 \mathrm{d}\tau \leqslant \text{Const.}, \text{ for some } 0 < \gamma,$$

where $\hat{\mathbf{v}}_{h'}$ *is the Fourier transform of* $\mathbf{v}_{h'}$.

Then such a sequence $\mathbf{v}_{h'}$ *is relatively compact in* $L^2(Q)$.

In particular, one can extract from $\mathbf{v}_{h'}$ *a subsequence (still denoted* $\mathbf{v}_{h'}$*) with*

$$p_{h'}\mathbf{v}_{h'} \to \bar{\omega}\mathbf{v} \text{ in } L^2(0, T; F) \text{ weakly},$$

$$\mathbf{v}_{h'} \to \mathbf{v} \text{ in } L^2(Q) \text{ strongly}. \tag{5.92}$$

5.7.2. *The Convergence Theorems*

The convergence theorems are stated differently according to the dimension of the space ($n = 2$ or 3) and to the scheme considered.

We recall that we associated with the elements $\mathbf{u}_h^m$ a function $\mathbf{u}_h$

$$\mathbf{u}_h : [0, T] \to V_h,$$

defined slightly differently for the four schemes (see (5.36), (5.55), (5.75)) [1]

for $(m-1)k \leqslant t < mk$ $(m = 1, \ldots, N)$

$$u_h(t) = \begin{cases} u_h^m & \text{(Schemes 5.1 and 5.3)} \\ \dfrac{1}{2}(u_h^m + u_h^{m-1}) & \text{(Scheme 5.2)} \\ u_h^{m-1} & \text{(Scheme 5.4)} \end{cases} \tag{5.93}$$

We have:

Theorem 5.4. *The dimension of the space is $n = 2$ and the assumptions are* (5.1) *to* (5.7), (5.9), (5.10), (5.41), *and* (5.89) *to* (5.92). *We denote by $\boldsymbol{u}$ the unique solution of Problem 3.1.*

The following convergence results hold, as h and $k \to 0$,

$$u_h \to u \text{ in } L^2(Q) \text{ strongly, } L^\infty(0, T; L^2(\Omega)) \text{ weak-star,} \tag{5.94}$$

$$p_h u_h \to \bar{\omega} u \text{ in } L^2(0, T; F) \text{ weakly,} \tag{5.95}$$

provided:

(i) *Scheme* 5.1: *no condition,*
(ii) *Scheme* 5.2,

$$kS^2(h) \to 0, \tag{5.96}$$

(iii) *Scheme* 5.3: (5.43) *is satisfied,*
(iv) *Scheme* 5.4: (5.56)–(5.57) *are satisfied.*

Remark 5.3.

(i) For Schemes 5.1 and 5.2 it can be proved, without any further hypotheses, that

$$p_h u_h \to \bar{\omega} u \text{ in } L^2(0, T; F) \text{ strongly, as } h, k \to 0. \tag{5.97}$$

(ii) The same results hold for the other schemes provided we also assume that

$$kS_1^2(h) \to 0 \text{ and } kS^2(h) \to 0 \text{ (Schemes 5.3 and 5.4).} \tag{5.98}$$

(iii) The hypotheses (5.96) used in the proof of the convergence of Scheme 5.2 is probably unnecessary since the scheme is unconditionally

---

[1] We emphasize that u_h depends on h and k; only for reasons of simplicity have we denoted this function by u_h instead of u_{hk}.

$L^2(0, T; F)$ and $L^\infty(0, T; L^2(\Omega))$ stable.

Theorem 5.5. *The dimension of the space is* $n = 3$ *and, otherwise, the assumptions are the same as in Theorem* 5.4.

Then, there exists some sequence $h', k' \to 0$, *such that*

$$u_{h'} \to u \text{ in } L^2(Q) \text{ strongly,} \tag{5.99}$$

$$u_{h'} \to u \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak-star,} \tag{5.100}$$

$$p_{h'} u_{h'} \to \bar{\omega} u \text{ in } L^2(0, T; F) \text{ weakly,} \tag{5.101}$$

where u *is some solution of Problem* 3.1.

For any other sequence $h', k' \to 0$, *such that the convergences* (5.99) *to* (5.101) *hold,* u *must be some solution of Problem* 3.1.

Remark 5.4. We are not able to prove that the whole sequence converges due to lack of uniqueness of solution for Problem 3.1.

We also cannot prove strong convergence in $L^2(0, T; F)$ due to the lack of an energy equality for the exact problem (Problem 3.1) (for $n = 3$ we only have an energy inequality; see Remark 4.1). □

The two theorems are proved in the remainder of this Section 5.7; we will prove Theorem 5.4 with full details for Scheme 5.1 (including (5.97)) and in the other cases we will only sketch the proofs which are actually very similar.

5.7.3. *Proof of Theorem 5.4 (Scheme* 5.1*)*

According to the stability theorem (Theorem 5.1), and to (5.89), there exists a sequence $h', k' \to 0$, such that

$$\begin{aligned} &u_{h'} \to u \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak star,} \\ &p_{h'} u_{h'} \to \bar{\omega} u \text{ in } L^2(0, T; F) \text{ weakly,} \end{aligned} \tag{5.102}$$

for some u in $L^2(0, T; V) \cap L^\infty(0, T; H)$.

Let us consider the piecewise linear function w_h introduced in Section 5.6 (see (5.78)). By Lemma 5.6 and the estimations on the u_h^m, we have

$$|w_h|_{L^\infty(0,T;L^2(\Omega))} \leqslant \text{Const.},$$

$$\|p_h w_h\|_{L^2(0,T;F)} \leqslant \text{Const.},$$

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{w}_h(\tau)|^2 \,\mathrm{d}\tau \leqslant \text{Const.}$$

Hence, according to (5.92), the sequence $h', k' \to 0$ can be chosen so that

$$\begin{aligned} &w_{h'} \to w \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak-star,} \\ &w_{h'} \to w \text{ in } L^2(0, T; L^2(\Omega)) \text{ strongly,} \\ &p_{h'} w_{h'} \to \overline{\omega} w \text{ in } L^2(0, T; F) \text{ weakly,} \end{aligned} \tag{5.103}$$

where $w \in L^2(0, T; V) \cap L^\infty(0, T; H)$.

We now observe that:

Lemma 5.8.

$$u_h - w_h \to 0 \text{ in } L^2(0, T; L^2(\Omega)) \textit{ strongly as } h \textit{ and } k \to 0. \tag{5.104}$$

Thus

$$w = u \tag{5.105}$$

and

$$u_{h'} \to u \textit{ in } L^2(0, T; L^2(\Omega)) \textit{ strongly,} \tag{5.106}$$

as h' *and* $k' \to 0$.

Proof. Exactly as in Lemma 4.8, we check that

$$|u_h - w_h|_{L^2(0, T; L^2(\Omega))} =$$

$$\sqrt{\frac{k}{3}} \left(\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 \right)^{1/2}. \tag{5.107}$$

Then (5.104) follows from the majoration (5.22); (5.105), (5.106) are obvious consequences of (5.102), (5.103), and (5.104).

The next point is to prove that u is a solution of Problem 3.1.

Lemma 5.9. *The function* u *appearing in* (5.102), (5.103), (5.105) *is a solution of Problem* 3.1.

Proof. We easily interpret (5.12) in the following way:

$$\frac{\mathrm{d}}{\mathrm{d}t}(w_h(t), v_h) + \nu((u_h(t), v_h))_h$$
$$+ b_h(u_h(t-k), u_h(t), v_h)$$
$$= (f_k(t), v_h), \forall\ t \in [0, T],\ \forall v_h \in V_h. \tag{5.108}$$

where

$$f_k(t) = f^m,\ (m-1)k \leqslant t < mk. \tag{5.109}$$

Let v be any element in $\mathscr{V}$ and let us take $v_h = r_h v$ in (5.108). Let ψ be a continuously differentiable scalar function on $[0, T]$, with

$$\psi(T) = 0. \tag{5.110}$$

We multiply (5.108) (where $v_h = r_h v$) by $\psi(t)$, integrate in t, and integrate the first term by parts to get:

$$-\int_0^T (w_h(t), \psi'(t) r_h v)\mathrm{d}t + \nu \int_0^T ((u_h(t), \psi(t) r_h v))_h \mathrm{d}t$$
$$+ \int_0^T b_h(u_h(t-k), u_h(t), \psi(t) r_h v)\mathrm{d}t = (u_h^0, r_h v)\psi(0)$$
$$+ \int_0^T (f_k(t), \psi(t) r_h v)\mathrm{d}t. \tag{5.111}$$

We now pass to the limit in (5.111) with the sequence $h', k' \to 0$ using essentially (5.90), (5.91), (5.102), (5.103), and Lemma 5.8; we recall also that

$$r_h v \to \bar{\omega} v \text{ in } F \text{ (strongly)}, \tag{5.112}$$
$$u_h^0 \to u_0 \text{ in } L^2(\Omega) \text{ (strongly)}^{(1)}, \tag{5.113}$$
$$f_k \to f \text{ in } L^2(0, T; L^2(\Omega)) \text{ (see Lemma 4.9)}. \tag{5.114}$$

We find in the limit

---

(1) We recall the proof of (5.113); due to (5.11) it suffices to prove this for $u_0 \in \mathscr{V}$ and in this case

$$|u_h^0 - u_0| \leqslant |r_h u_0 - u_0| \leqslant \|p_h r_h u_0 - \bar{\omega} u_0\|_F \to 0.$$

$$-\int_0^T (\boldsymbol{u}(t), \psi'(t)\boldsymbol{v})\mathrm{d}t + \nu \int_0^T ((\boldsymbol{u}(t), \psi(t)\boldsymbol{v}))\mathrm{d}t$$

$$+\int_0^T b(\boldsymbol{u}(t), \boldsymbol{u}(t), \psi\boldsymbol{v})\mathrm{d}t = (\boldsymbol{u}_0, \boldsymbol{v})\psi(0)$$

$$+\int_0^T (\boldsymbol{f}(t), \boldsymbol{v})\psi(t)\mathrm{d}t. \tag{5.115}$$

We infer from this equality that $\boldsymbol{u}$ is a solution of Problem 3.1, exactly as we did in the proof of Theorem 3.1 after (3.43).

Since the solution of Problem 3.1 is unique (see Theorem 3.2), a contradiction argument that we have already used very often shows that

The convergences (5.102), (5.103) *hold for the whole family* $h, k \to 0$. (5.116)

This completes the proof of Theorem 5.4.

5.7.4. *Proof of (5.97)*

For the sake of completeness we will also prove (5.97). In order to prove this point we need a preliminary result which is quite general and interesting by itself.

Lemma 5.10. *Let* $\{(V_h, p_h, r_h)_{\mathcal{H}}, (\bar{\omega}, F)\}$ *be a stable and convergent external approximation of* V. *For a given element* $\boldsymbol{v}$ *of* $L^2(0, T; V)$, *one can define for each* $h \in \mathcal{H}$ *a function* $\boldsymbol{v}_h^+ \in L^2(0, T; V_h)$ *such that*

$$p_h \boldsymbol{v}_h^+ \to \bar{\omega}\boldsymbol{v} \text{ in } L^2(0, T; F) \text{ as } h \to 0.$$

Proof. The proof is essentially the same as that of Proposition I.3.1.

The result is obvious if $\boldsymbol{v}$ is a step function; since the step functions are dense in $L^2(0, T; V)$, the result follows in the general case by an argument of double passage to the limit as in Proposition 1.3.1.

Lemma 5.11. *The dimension of the space is* $n = 2$; *then for Scheme* 5.1

$$p_h \boldsymbol{u}_h \to \bar{\omega}\boldsymbol{u} \text{ in } L^2(0, T; F) \text{ (strongly)}, \tag{5.117}$$

as h and k $\to 0$.

Proof. We consider the expression

$$X_h = |u_h^N - u(T)|^2 + \sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2$$

$$+ 2\nu \int_0^T \|u_h(t) - u_h^+(t)\|_h^2 \, \mathrm{d}t,$$

with u_h^+ defined as in Lemma 5.10.

According to (5.21) (Lemma 5.1),

$$|u_h^N| \leqslant \text{Const.};$$

hence there exists a sequence $h', k' \to 0$, with

$$u_{h'}^N \to \chi \text{ in } L^2(\Omega) \text{ weakly.} \tag{5.118}$$

We temporarily assume that

$$(\chi - u(T), v) = 0, \forall\, v \in H. \tag{5.119}$$

and we prove that

$$X_{h'} \to 0.$$

We set

$$X_h = X_h^1 + X_h^2 + X_h^3,$$

where

$$X_h^1 = |u(T)|^2 + 2\nu \int_0^T \|u_h^+(t)\|_h^2 \, \mathrm{d}t \to |u(T)|^2 + 2\nu \int_0^T \|u(t)\|^2 \, \mathrm{d}t$$

(by Lemma 5.10 and (5.90)),

$$X_h^2 = -2(u_h^N, u(T)) - 4\nu \int_0^T ((u_h(t), u_h^+(t)))_h \to -2|u(T)|^2$$

$$- 4\nu \int_0^T \|u(t)\|^2 \, \mathrm{d}t$$

(by Lemma 5.10, (5.90) and (5.118)–(5.119), we recall that $u(T) \in H$), and

$$X_h^3 = |u_h^N|^2 + \sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 + 2\nu \int_0^T \|u_h(t)\|_h^2 \mathrm{d}t \tag{5.120}$$

$$= |u_h^N|^2 + \sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 + 2k \sum_{m=1}^{N} \|u_h^m\|_h^2.$$

By summation of the equalities (5.26) for $m = 1, \ldots, N$, we get

$$X_h^3 = |u_h^0|^2 + 2k \sum_{m=1}^{N} (f^m, u_h^m)$$

$$= |u_h^0|^2 + 2 \int_0^T (f_k(t), u_h(t)\mathrm{d}t.$$

It is then clear that

$$X_h^3 \to |u_0|^2 + 2 \int_0^T (f(t), u(t))\mathrm{d}t, \text{ as } h, k \to 0.$$

Hence

$$X_{h'} \to |u_0|^2 + 2 \int_0^T (f(t), u(t))\mathrm{d}t - |u(T)|^2 - 2\nu \int_0^T \|u(t)\|^2 \mathrm{d}t, \tag{5.122}$$

and this limit is 0 due to (4.55).

By a contradiction argument we show as well that the whole family X_h converges to 0:

$$X_h \to 0, \text{ as } h, k \to 0.$$

In particular

$$\int_0^T \|u_h(t) - u_h^+(t)\|_h^2 \mathrm{d}t \to 0,$$

and

$$\int_0^T \|p_h u_h(t) - \bar{\omega} u(t)\|_F^2 \mathrm{d}t \leqslant c \left\{ \int_0^T \|u_h(t) - u_h^+(t)\|_h^2 \mathrm{d}t \right.$$

$$\left. + \int_0^T \|p_h u_h^+(t) - \bar{\omega} u(t)\|_F^2 \mathrm{d}t \right\} \to 0$$

and (5.117) follows.

It remains to prove (5.119).

By summation of (5.12) for $m = 1, \ldots, N$, we get

$$(u_h^N - u_h^0, v_h) + k\nu \sum_{m=1}^{N} ((u_h^m, v_h))_h$$

$$+ k \sum_{m=1}^{N} b_h(u_h^{m-1}, u_h^m, v_h) = k \sum_{m=1}^{N} (f^m, v_h).$$

Taking $v_h = r_h v$, $v \in \mathscr{V}$, we easily pass to the limit and get

$$(\chi - u_0, v) + \nu \int_0^T ((u(t), v))\mathrm{d}t + \int_0^T b(u(t), u(t), v)\mathrm{d}t$$

$$= \int_0^T (f(t), v)\mathrm{d}t, \ \forall v \in \mathscr{V}.$$

But since we deduce by integration of (3.13) a similar equation with χ replaced by $u(T)$, we conclude that

$$(\chi - u(T), v) = 0, \ \forall\, v \in \mathscr{V},$$

which implies (5.119) by density.

The proof of Lemma 5.11 is complete.

5.7.5. *Proof of Theorems 5.4 and 5.5 (other cases)*

For Schemes 5.2, 5.3, 5.4 and in the case $n = 2$, the proof is very similar to the above, using the corresponding *a priori* estimates.

For Scheme 5.2, we introduced the condition (5.96) as a sufficient condition to prove (5.104); more precisely, in this case, the analogue of (5.104) is a consequence of (5.77), (5.107) and (5.96). For Schemes 5.3 5.4, the stability conditions (5.43), (5.56), (5.57) merely ensure that u_h and $p_h u_h$ remain bounded in the suitable spaces.

For the proof of (5.97), the condition (5.98) appears as follows:

– For Scheme 5.4 the "natural" expression similar to X_h in Lemma 5.11 is

$$Y_h = |u_h^N - u(T)|^2 - \sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 + 2\nu \int_0^T \|u_h(t) - u_h^+(t)\|_h^2 \, dt;$$

in order to deduce (5.97) from the fact that $Y_h \to 0$, it suffices that

$$\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 \to 0,$$

and this is a consequence of (5.98).

– For Scheme 5.3, we consider the same expression X_h, but due to some terms involving b_h (see (5.47)), the expression X_h^3 is not as simple as (5.121); (5.98) shows that the supplementary terms of X_h^3 involving b_h converge to 0.

If $n = 3$, we observe first that the stability conditions are not explicitly mentioned in the statement of Theorem 5.5. Actually it is with the help of these conditions, and the *a priori* estimates that they imply, that we prove the existence of a subsequence $h', k' \to 0$, such that (5.99)–(5.101) hold; that u is a solution of Problem 3.1 is proved exactly as before.

§ 6. Discretization of the Navier–Stokes Equations: Application of the General Results.

We want to state explicitly all the hypotheses and conclusions of the stability and convergence theorems (Theorems 5.1 to 5.5) for specific approximations of the space V. We recall that the final form and effectiveness of the Schemes studied in Section 5 are also based on the choice of the approximation of V, i.e., the discretization in the space variables.

In Section 6.1 we consider the approximation of V by finite differences (approximation APX1). Section 6.2 deals with conforming finite element methods: the approximation of V being one of the approximations (APX2) to (APX4). Section 6.3 deals with non-conforming finite element methods (approximation APX5). Then in Section 6.4 we study some algorithms adapted to the practical resolution of the finite dimensional problems (i.e., practical computation of u_h^m for one of Schemes 5.1 to 5.4).

6.1. *Finite Differences* (APX1)

The general approximation of V considered at the beginning of Section 5.1 is at present taken to be the specific approximation (APX1). We will successively check and interpret in this case the hypotheses of Theorems 5.1 to 5.5.

6.1.1. *Computation of* $S(h)$.

The conditions (5.3) and (5.4) are obviously satisfied; according to Proposition I.3.3,

$$|u_h| \leqslant d_0 \|u_h\|_h, \quad d_0 = 2\ell, \tag{6.1}$$

where ℓ is the smallest of the widths of Ω in the directions $x_1, \ldots, x_n$. The purpose of the next proposition is to verify (5.5) and to give an explicit value of $S(h)$.

Proposition 6.1. *For the approximation* (APX1)

$$S(h) = 2\left(\sum_{i=1}^{n} \frac{1}{h_i^2}\right)^{1/2}, \quad \forall h = (h_1, \ldots, h_n). \tag{6.2}$$

Proof. We have

$$\|u_h\|_h^2 = \sum_{i,j=1}^{n} \left\{ \frac{1}{h_j^2} \int_\Omega \left| u_{ih}\left(x + \frac{\vec{h}_j}{2}\right) - u_{ih}\left(x - \frac{\vec{h}_j}{2}\right) \right|^2 dx \right\}$$

$$\leqslant 2 \sum_{i,j=1}^{n} \left\{ \frac{1}{h_j^2} \int_{\mathbb{R}^n} \left(\left| u_{ih}\left(x + \frac{\vec{h}_j}{2}\right)\right|^2 + \left| u_{ih}\left(x - \frac{\vec{h}_j}{2}\right)\right|^2 \right) dx \right.$$

$$\leqslant 4 \sum_{i,j=1}^{n} \left\{ \frac{1}{h_j^2} \int_{\mathbb{R}^n} |u_{ih}(x)|^2 \, dx \right\}$$

$$\leqslant 4 \left(\sum_{j=1}^{n} \frac{1}{h_j^2} \right) |u_h|^2,$$

and (6.2) follows.

6.1.2. *The form b_h and $S_1(h)$*

For the approximation (APX1), we choose again the form b_h defined in Section 3, chapter II (see (3.27) to (3.29)).

$$b_h(u_h, v_h, w_h) = b_h'(u_h, v_h, w_h) + b_h''(u_h, v_h, w_h) \tag{6.3}$$

$$b_h'(u_h, v_h, w_h) = \frac{1}{2} \sum_{i,j=1}^{n} \int_{\Omega} u_{ih}(\delta_{ih} v_{jh}) \, w_{jh} \, dx \tag{6.4}$$

$$b_h''(u_h, v_h, w_h) = b_h'(u_h, w_h, v_h). \tag{6.5}$$

It is clear that b_h is a trilinear continuous form on $V_h \times V_h \times V_h$ and that (5.6) holds; in order to get some estimate of the constant d_1 in (5.7) we will prove

Lemma 6.1. *For $n = 2$ or 3*

$$\begin{aligned} &|b_h'(u_h, v_h, w_h)| \\ &\qquad \leqslant \frac{3}{\sqrt{2}} |u_h|^{1/2} \|u_h\|_h^{1/2} \|v_h\|_h |w_h|^{1/2} \|w_h\|_h^{1/2} \ (n = 2) \\ &\qquad \leqslant 3^{3/2} |u_h|^{1/4} \|u_h\|_h^{3/4} \|v_h\|_h |w_h|^{1/4} \|w_h\|_h^{3/4} \ (n = 3), \end{aligned} \tag{6.6}$$

$$\begin{aligned} &|b_h''(u_h, v_h, w_h)| \\ &\qquad \leqslant \frac{3}{\sqrt{2}} |u_h|^{1/2} \|u_h\|_h^{1/2} |v_h|^{1/2} \|v_h\|_h^{1/2} \|w_h\|_h \ (n = 2) \\ &\qquad \leqslant 3^{3/2} |u_h|^{1/4} \|u_h\|_h^{3/4} |v_h|^{1/4} \|v_h\|_h^{3/4} \|w_h\|_h \ (n = 3). \end{aligned} \tag{6.7}$$

Proof. This proof is based on Proposition II.2.1. Due to II.(2.12) and II.(2.13),

$$\sum_{i=1}^{n} \|u_{ih}\|^2_{L^4(\Omega)} \leqslant 3\sqrt{2} \sum_{i=1}^{n} \left\{ |u_{ih}| \left(\sum_{j=1}^{n} |\delta_{jh} u_{ih}|^2 \right)^{1/2} \right\}$$

$$\leqslant 3\sqrt{2}\, |u_h|\, \|u_h\|_h, \tag{6.8}$$

= 2, and, if $n = 3$:

$$\sum_{i=1}^{n} \|u_{ih}\|^2_{L^4(\Omega)} \leqslant 2.3^{3/2} \sum_{i=1}^{n} \left\{ |u_{ih}|^{1/2}_{L^2(\Omega)} \left(\sum_{j=1}^{n} |\delta_{jh} u_{ih}|^2_{L^2(\Omega)} \right)^{3/4} \right\}$$

$$\leqslant \text{(by Hölder's inequality)}$$

$$\leqslant 2.3^{3/2} \left(\sum_{i=1}^{n} |u_{ih}|^2_{L^2(\Omega)} \right)^{1/4} .$$

$$\left(\sum_{i,j=1}^{n} |\delta_{jh} u_{ih}|^2_{L^2(\Omega)} \right)^{3/4} \leqslant 2.3^{3/2} |u_h|^{1/4}\, \|u_h\|_h^{3/4}. \tag{6.9}$$

Using Hölder's inequality again we now have

$$|b_h'(u_h, v_h, w_h)| \leqslant \frac{1}{2} \sum_{i,j=1}^{n} \|u_{ih}\|_{L^4(\Omega)} \|\delta_{ih} v_{jh}\|_{L^2(\Omega)} \|w_{jh}\|_{L^4(\Omega)}$$

$$\leqslant \frac{1}{2} \left(\sum_{i=1}^{n} \|u_{ih}\|^2_{L^4(\Omega)} \right)^{1/2} \left(\sum_{j=1}^{n} \|w_{jh}\|^2_{L^4(\Omega)} \right)^{1/2} \|v_h\|_h,$$

$$|b''(u_h, v_h, w_h)| \leqslant \frac{1}{2} \sum_{i,j=1}^{n} \|u_{ih}\|_{L^4(\Omega)} \|v_{jh}\|_{L^4(\Omega)} \|\delta_{ih} w_{jh}\|_{L^2(\Omega)}$$

$$\leqslant \frac{1}{2} \left(\sum_{i=1}^{n} \|u_{ih}\|^2_{L^4(\Omega)} \right)^{1/2} \left(\sum_{j=1}^{n} \|v_{jh}\|^2_{L^4(\Omega)} \right)^{1/2} \|w_h\|_h.$$

Then, if $n = 2$, we apply (6.8) to get:

$$|b_h'(u_h, v_h, w_h)| \leqslant \frac{3}{\sqrt{2}}\, |u_h|^{1/2} \|u_h\|_h^{1/2} \|v_h\|_h |w_h|_h^{1/2}\, \|w_h\|_h^{1/2},$$

$$|b''_h(u_h, v_h, w_h)| \leqslant \frac{3}{\sqrt{2}} |u_h|^{1/2} \|u_h\|_h^{1/2} |v_h|^{1/2} \|v_h\|_h^{1/2} \|w_h\|_h .$$

For $n = 3$, using (6.9) we also obtain the results stated in (6.6) and (6.7).

Lemma 6.2.

$$|b_h(u_h, v_h, w_h)| \leqslant d_1 \|u_h\|_h \|v_h\|_h \|w_h\|_h, \ \forall\, u_h, v_h, w_h \in V_h, \tag{6.10}$$

with

$$d_1 = 6\sqrt{2}\ell \text{ if } n = 2, \ d_1 = 6^{3/2}\sqrt{\ell} \text{ if } n = 3. \tag{6.11}$$

Proof. An immediate consequence of (6.1), (6.3), (6.6) and (6.7).

Proposition 6.2. *For the approximation* (APX1), *the inequality* (5.41) *holds with*

$$\begin{aligned} &S_1(h) = 3\sqrt{2}\, S(h) \ \text{ if } n = 2,\\ &S_1(h) = 2.3^{3/2} S^{3/2}(h) \ \text{ if } n = 3, \end{aligned} \tag{6.12}$$

where $S(h)$ is given by (6.2).

Proof. Using (6.1), (6.6) and (6.7) we can write:

$$\begin{aligned} |b'_h(u_h, u_h, w_h)| &\leqslant \frac{3}{\sqrt{2}} |u_h|^{1/2} \|u_h\|_h^{3/2} |w_h|^{1/2} \|w_h\|_h^{1/2} \\ &\leqslant \frac{3}{\sqrt{2}} S(h) |u_h| \|u_h\|_h |w_h| \quad (n = 2), \\ |b'_h(u_h, u_h, w_h)| &\leqslant 3^{3/2} |u_h|^{1/4} \|u_h\|_h^{3/4} \|v_h\|_h |w_h|^{1/4} \|w_h\|_h^{3/4} \\ &\leqslant 3^{3/2} S^{3/2}(h) |u_h| \|u_h\|_h |w_h| \quad (n = 3), \\ |b''_h(u_h, u_h, w_h)| &\leqslant \frac{3}{\sqrt{2}} |u_h| \|u_h\|_h \|w_h\|_h \\ &\leqslant \frac{3}{\sqrt{2}} S(h) |u_h| \|u_h\|_h |w_h| \quad (n = 2), \\ |b''_h(u_h, u_h, w_h)| &\leqslant 3^{3/2} |u_h|^{1/2} \|u_h\|_h^{3/2} \|w_h\|_h \\ &\leqslant 3^{3/2} S^{3/2}(h) |u_h| \|u_h\|_h |w_h| \quad (n = 3). \end{aligned}$$

Then, we easily obtain (6.12).

6.1.3. *Application of the stability and convergence theorems*

Schemes 5.1 and 5.2 are unconditionally stable. The stability conditions of Scheme 5.3 are given in Theorem 5.2 and Lemma 5.3:

$$k\left(\sum_{i=1}^{2}\frac{1}{h_i^2}\right)\leqslant\min\left(\frac{d'}{2^3 3^2},\frac{d''}{2^2}\right)\quad \text{if } n=2, \tag{6.13}$$

$$k\left(\sum_{i=1}^{3}\frac{1}{h_i^2}\right)^{3/2}\leqslant\frac{d'}{2^5 3^3},\ k\left(\sum_{i=1}^{3}\frac{1}{h_i^2}\right)\leqslant\frac{d''}{4}$$

$$\text{if } n=3,\qquad {}^{(1)} \tag{6.14}$$

where d' and d'' are defined in the proof of Lemma 5.3.

For Scheme 5.4 the stability conditions given in Theorem 5.3 and 5.4 are

$$k\left(\sum_{i=1}^{2}\frac{1}{h_i^2}\right)\leqslant\frac{1-\delta}{2^4\nu},\ k\left(\sum_{i=1}^{2}\frac{1}{h_i^2}\right)\leqslant\frac{\nu\delta}{3^2 2^6 d_5},$$

$$\text{if } n=2, \tag{6.15}$$

$$k\left(\sum_{i=1}^{3}\frac{1}{h_i^2}\right)\leqslant\frac{1-\delta}{2^4\nu};\ k\left(\sum_{i=1}^{3}\frac{1}{h_i^2}\right)^{3/2}\leqslant\frac{\nu\delta}{2^8 3^3 d_5},$$

$$\text{if } n=3, \tag{6.16}$$

for some δ, $0<\delta<1$, and d_5 is given by (5.58).

When applicable, the convergence Theorems 5.4 and 5.5 give

$$\boldsymbol{u}_h\to\boldsymbol{u} \text{ in } L^2(Q) \text{ strongly, } L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \text{ weak star,} \tag{6.17}$$

$$\delta_{ih}\boldsymbol{u}_h\to D_i\boldsymbol{u} \text{ in } L^2(Q) \text{ strongly or weakly} \tag{6.18}$$

(depending on whether convergence in $L^2(0,T;F)$ is strong or weak).

Theorems 5.4 and 5.5 are based on the hypotheses (5.89) to (5.92). We must show that these conditions are met; this technical point will be considered now.

6.1.4. *Proof of* (5.89)–(5.92)

Verification of (5.89). If

$$\boldsymbol{u}_{h'}\to\boldsymbol{u} \text{ in } L^2(0,T;\boldsymbol{L}^2(\Omega)),\ \text{weakly,} \tag{6.19}$$

---

(1) The second condition in (6.14) is consequence of the first one for h sufficiently small.

$$p_{h'}u_{h'} \to \phi = (\phi_0, \ldots, \phi_n) \text{ in } L^2(0, T; F) \text{ weakly,} \tag{6.20}$$

then, by Lemma 5.7, ϕ must be equal to $\bar{\omega}v$, where $v \in L^2(0, T; V)$. On the other hand, by the definition of p_h,

$$p_{h'}u_{h'} \to \phi_0 = v \text{ in } L^2(Q) \text{ weakly,} \tag{6.21}$$

and the comparison of (6.19) and (6.21) shows that $v = u$, as elements of $L^2(Q)$. Hence

$$\phi(t) = \bar{\omega}\, u(t) \text{ a.e.,}$$

and $u \in L^2(0, T; V)$.

Verification of (5.90). If

$$p_{h'}\, v_{h'} \to \bar{\omega}\, v \text{ in } L^2(0, T; F) \text{ weakly,}$$

$$p_{h'}\, w_{h'} \to \bar{\omega}\, w \text{ in } L^2(0, T; F) \text{ strongly,}$$

then

$$\delta_{ih'}v_{h'} \to D_i v \text{ in } L^2(Q) \text{ weakly,}$$

$$\delta_{ih'}w_{h'} \to D_i w \text{ in } L^2(Q) \text{ strongly,}$$

and it is clear that

$$\int_0^T ((v_{h'}(t), w_{h'}(t)))_{h'}\mathrm{d}t = \int_0^T \int_\Omega \sum_{i=1}^n (\delta_{ih'}\, v_{h'})(\delta_{ih'}\, w_{h'})\mathrm{d}x\, \mathrm{d}t$$

$$\to \int_0^T \int_\Omega \left(\sum_{i=1}^n D_i v . D_i w \right) \mathrm{d}x\, \mathrm{d}t = \int_0^T ((v(t), w(t)))\mathrm{d}t.$$

Verification of (5.91). The proof is similar to the proof of Lemma 3.1, chapter II. Let us assume that

$$u_{h'} \to u \text{ in } L^2(Q) \text{ strongly,} \tag{6.22}$$

$$p_{h'}\, v_{h'} \to \bar{\omega}\, v \text{ in } L^2(0, T; F) \text{ weakly,} \tag{6.23}$$

$$\psi_{h'} \to \psi \text{ in } L^\infty(0, T), \tag{6.24}$$

and that w is a fixed element of $\mathcal{V}$. It follows from (6.23) that

$$v_{h'} \to v \text{ in } L^2(Q) \text{ weakly,} \tag{6.25}$$

$$\delta_{ih'}v_{h'} \to D_i v \text{ in } L^2(Q) \text{ weakly, } 1 \leqslant i,\ j \leqslant n. \tag{6.26}$$

On the other hand, as observed in the proof of Lemma II.3.

$$r_h w \to w \text{ in the norm of } L^\infty(\Omega), \tag{6.27}$$

$$\delta_{ih} r_h w \to D_i w \text{ in the norm of } L^\infty(\Omega),\ 1 \leqslant i \leqslant n. \tag{6.28}$$

We consider first the form b_h':

$$\int_0^T b_h'(u_h(t), v_h(t), \psi_h(t) r_h w)\,dt$$

$$= \frac{1}{2} \sum_{i,j=1}^n \int_0^T \int_\Omega u_{ih}(\delta_{ih} v_{jh})\,\psi_h\, w_{jh}\,dx\,dt.$$

It is easy to see from (6.22), (6.24), (6.26) and (6.27) that, for each i and j

$$\int_0^T \int_\Omega u_{ih'}(\delta_{ih'} v_{jh'})\,\psi_k\, w_{jh'}\,dx\,dt \to \int_0^T \int_\Omega u_i(D_i v_j)\psi w_j\,dx\,dt,$$

and hence

$$\int_0^T b_{h'}'(u_{h'}(t), v_{h'}(t), \psi_{k'}(t) r_{h'} w)\,dt \to \frac{1}{2}\int_0^T b(u(t), v(t), \psi(t) w)\,dt$$

Similarly,

$$\int_0^T b_{h'}''(u_{h'}(t), v_{h'}(t), \psi_{k'}(t) r_{h'} w)\,dt \to$$

$$-\frac{1}{2}\int_0^T b(u(t), \psi(t) w, v(t))\,dt = \frac{1}{2}\int_0^T b(u(t), v(t), \psi(t) w)\,dt,$$

and (5.91) is proved.

Verification of (5.92). Let $\{v_{h'}\}$ denote a sequence of functions from $\mathcal{R}$ into V_h with support in $[0, T]$ or any fixed compact subset of $\mathcal{R}$ and such that

$$\int_0^T \|v_{h'}(t)\|_{h'}^2 \, dt \leqslant \text{Const.}, \tag{6.29}$$

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{v}_{h'}(\tau)|^2 \, d\tau \leqslant \text{Const.}, \quad (0 < \gamma). \tag{6.30}$$

We must prove that this sequence is relatively compact in $L^2(Q)$. Since the p_h are stable, we have

$$\int_0^T \|p_{h'} \, v_{h'}(t)\|_F^2 \, dt \leqslant \text{Const.}, \tag{6.31}$$

and hence, the sequence h' contains a subsequence (still denoted h'), such that

$$p_{h'} \, v_{h'} \to \overline{\omega} \, v \text{ in } L^2(\mathcal{R}; F) \text{ weakly}, \tag{6.32}$$

$$v_{h'} \to v \text{ in } L^2(\mathcal{R}; L^2(\Omega)) \text{ weakly}, \tag{6.33}$$

v being some element of $L^2(\mathcal{R}; V)$ which vanishes outside the interval $[0, T]$ (here we have used (5.89)). It suffices to prove that the convergence (6.33) holds in $L^2(\mathcal{R}; L^2(\Omega))$ strongly, which amounts proving that one of the following expressions converges to zero:

$$I_{h'} = \int_{-\infty}^{+\infty} |v_{h'}(t) - v(t)|^2 \, dt = \text{(by the Parseval equality)}$$

$$= \int_{-\infty}^{+\infty} |\hat{v}_{h'}(\tau) - \hat{v}(\tau)|^2 \, d\tau. \tag{6.34}$$

Following the proof of Theorem 2.2, we write

$$I_{h'} = \int_{|\tau| \leqslant M} |\hat{v}_{h'}(\tau) - \hat{v}(\tau)|^2 \, d\tau$$

$$+ \int_{|\tau| > M} (1 + |\tau|^{2\gamma}) |\hat{v}_{h'}(\tau) - \hat{v}(\tau)|^2 \, \frac{d\tau}{(1 + |\tau|^{2\gamma})}$$

$$\leqslant \int_{|\tau| \leqslant M} |\hat{v}_{h'}(\tau) - \hat{v}(\tau)|^2 \, d\tau$$

$$+ \frac{1}{1 + M^{2\gamma}} \int_{-\infty}^{+\infty} (1 + |\tau|^{2\gamma}) |\hat{v}_{h'}(\tau) - \hat{v}(\tau)|^2 \, d\tau$$

$\leqslant$ (by (6.1), (6.29), and (6.30))

$$\leqslant \int_{|\tau| \leqslant M} |\hat{v}_{h'}(\tau) \hat{v}(\tau)|^2 \, d\tau + \frac{C}{1 + M^{2\gamma}} .$$

For a given $\epsilon > 0$, we choose M such that

$$\frac{C}{1 + M^{2\gamma}} \leqslant \frac{\epsilon}{2} .$$

Hence

$$I_{h'} \leqslant J_{h'} + \frac{\epsilon}{2} , \tag{6.35}$$

where

$$J_{h'} = \int_{|\tau| \leqslant M} |\hat{v}_{h'}(\tau) - \hat{v}(\tau)|^2 \, d\tau \tag{6.36}$$

and the convergence of $I_{h'}$ to zero will be proved if we show that

$$J_{h'} \to 0, \quad \text{as } h', k' \to 0. \tag{6.37}$$

This will follow from Lebesgue's Theorem as we now show.

For every τ, $\hat{v}_h(\tau)$ is equal to

$$\hat{v}_{h'}(\tau) = \int_{-\infty}^{+\infty} v_{h'}(t) \, e^{-2i\pi t\tau} \, dt$$

or

$$\int_{-\infty}^{+\infty} v_{h'}(t) \, \chi(t) \, e^{-2i\pi t\tau} \, dt. \tag{6.38}$$

where χ is a smooth function with compact support, equal to 1 on $[0, T]$ (so that $v_{h'} = \chi v_{h'}$, $\forall\, h'$).

Then

$$|\hat{v}_{h'}(\tau)| \leqslant |v_{h'}|_{L^2(\mathbb{R};L^2(\Omega))} \, \|e^{-2i\pi t\tau}\chi\|_{L^2(\mathbb{R})} \leqslant \text{Const.} = C_1 \tag{6.39}$$

$$|\hat{v}_{h'}(\tau) - \hat{v}(\tau)|^2 \leqslant 2(C_1^2 + |\hat{v}(\tau)|^2), \ \forall\, \tau, \tag{6.40}$$

and, according to (6.33) and (6.38):

$$\hat{v}_{h'}(\tau) \to \hat{v}(\tau) = \int_{-\infty}^{+\infty} v(t)\,\chi(t)\, e^{-2i\pi t\tau}\,\mathrm{d}t, \quad \text{in } L^2(\Omega) \text{ weakly,} \tag{6.41}$$

for every τ.

We also have, due to (6.38),

$$\|v_{h'}(\tau)\|_{h'} \leqslant \left(\int_{-\infty}^{+\infty} \|v_{h'}(t)\|^2 \mathrm{d}t \right)^{1/2} \|e^{-2i\pi t\tau}\chi\|_{L^2(\mathbb{R})} \leqslant \text{Const.}$$

This last estimate and the discrete compactness theorem (Theorem II.2.2 and Remark II.2.4) show that the convergence result (6.41) holds also in the norm of $L^2(\Omega)$:

$$|\hat{v}_{h'}(\tau) - \hat{v}(\tau)|^2 \to 0, \ \forall\, \tau.$$

An application of Lebesgue's Theorem gives (6.37) as a consequence of (6.40) and the preceding convergence results.

6.2. *Conforming Finite Elements* (APX2) (APX3) (APX4)

The general approximation of V considered at the beginning of Section 5.1 is now taken as one of the approximations (APX2) (APX3) (APX4). We will check and interpret the hypotheses of Theorems 5.1 to 5.5.

For all these methods,

$$V_h \subset H_0^1(\Omega), \qquad \|u_h\|_h = \|u_h\|, \ \forall\, u_h \in V_h, \tag{6.42}$$

and p_h is the identity, $\forall\, h$.

6.2.1. *Computation of $S(h)$*

The conditions (5.3) and (5.4) are obviously satisfied; according to the Poincaré Inequality (see Chapter I (1.9))

$$|u_h| \leqslant d_0 \|u_h\|_h, \quad d_0 = 2\ell, \tag{6.43}$$

where ℓ is the smallest of the widths of Ω in the directions $x_1, \ldots, x_n$.
For (5.5) we have

Proposition 6.3. *For the approximations* (APX2) *to* (APX4),

$$S(h) = \frac{c_q}{\rho'(h)}, \tag{6.44}$$

where c_q is a constant depending on the degree q of the elements [1].

Proof. The space V_h is a space of polynomials of degree less than or equal to $q = 2, 3$ or 4 for the approximations (APX2), (APX3), or (APX4). The stability constant $S(h)$ is a bound of the square root of the supremum

$$\operatorname*{Sup}_{u_h \in V_h} \left\{ \frac{\sum_{i=1}^{n} \int_\Omega |\operatorname{grad} u_{ih}(x)|^2 \, \mathrm{d}x}{\sum_{i=1}^{n} \int_\Omega |u_{ih}(x)|^2 \, \mathrm{d}x} \right\}. \tag{6.45}$$

This supremum is related to the supremum of

$$\left\{ \frac{\int_{\mathscr{S}} |\operatorname{grad} \phi(x)|^2 \, \mathrm{d}x}{\int_{\mathscr{S}} |\phi(x)|^2 \, \mathrm{d}x} \right\}, \tag{6.46}$$

among all functions ϕ which are polynomial of degree less than or equal to q, and all $\mathscr{S} \in \mathscr{T}_h$. Actually, let μ_q be a bound for this supremum; we then write for each $i = 1, \ldots, n$, and such $\mathscr{S} \in \mathscr{T}_h$

$$\int_{\mathscr{S}} |\operatorname{grad} u_{ih}(x)|^2 \, \mathrm{d}x \leqslant \mu_q \int_{\mathscr{S}} |u_{ih}(x)|^2 \, \mathrm{d}x,$$

---

(1) $\rho'(h)$ is defined in Chapter 1, (4.19).

where u_h belongs to V_h and q has the appropriate value corresponding to the actual space V_h.

Then by summation in i and $\mathscr{S}$:

$$\|u_h\|_h^2 = \sum_{i=1}^{n} \int_\Omega |\text{grad } u_{ih}(x)|^2 \, dx \leqslant \mu_q \sum_{i=1}^{n} \int_\Omega |u_{ih}(x)|^2 \, dx$$

$$= \mu_q |u_h|^2 \tag{6.47}$$

and we can take

$$S(h) = \sqrt{\mu_q}. \tag{6.48}$$

In estimating (6.46) we use a linear mapping Λ,

$$x = \Lambda \bar{x},$$

which maps $\mathscr{S}$ onto a reference n-simplex $\bar{\mathscr{S}}$:

$$0 \leqslant \bar{x}_i \leqslant 1, \ \sum_{i=1}^{n} \bar{x}_i \leqslant 1.$$

The function $\phi(\Lambda \bar{x})$ is a polynomial of degree less than or equal to q on $\bar{S}$. We observe that

$$\left\{ \int_{\bar{\mathscr{S}}} |\text{grad } \psi(\bar{x})|^2 \, d\bar{x} \right\} \text{ and } \left\{ \int_{\bar{\mathscr{S}}} |\psi(\bar{x})|^2 \, d\bar{x} \right\},$$

are a semi-norm and a norm on the finite-dimensional space of polynomial functions of degree less than or equal to q on $\bar{\mathscr{S}}$. Hence, there exists a constant $\bar{\mu}_q$ depending only on q (and $\bar{\mathscr{S}}$) such that,

$$\int_{\bar{\mathscr{S}}} |\text{grad } \psi(\bar{x})|^2 \, d\bar{x} \leqslant \bar{\mu}_q \int_{\bar{\mathscr{S}}} |\psi(\bar{x})|^2 \, d\bar{x},$$

for all such polynomials ψ. In particular,

$$\sum_{j=1}^{n} \int_{\bar{\mathscr{S}}} \left| \frac{\partial}{\partial x_j} \phi(\Lambda \bar{x}) \right|^2 d\bar{x} \leqslant \bar{\mu}_q \int_{\bar{\mathscr{S}}} |\phi(\Lambda \bar{x})|^2 \, d\bar{x}, \tag{6.49}$$

for the considered polynomials ϕ.

But

$$\frac{\partial \phi}{\partial x_k}(x) = \frac{\partial}{\partial x_k}\phi(\Lambda \bar{x}) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\phi(\Lambda \bar{x}) \cdot \Lambda_{jk},$$

$$\sum_{k=1}^{n} \left| \frac{\partial \phi}{\partial x_k}(x)\right|^2 = \sum_{k=1}^{n} \left\{ \left| \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\phi(\Lambda \bar{x})\,\Lambda_{jk}\right|^2 \right\}$$

$$\leqslant \|\Lambda\|^2 \sum_{j=1}^{n} \left\{ \left| \frac{\partial}{\partial x_j}\phi(\Lambda \bar{x})\right|^2 \right\} \text{ (1)}$$

We infer from this and (6.49) that

$$\sum_{k=1}^{n} \int_{\bar{\mathscr{S}}} \left| \frac{\partial \phi}{\partial x_k}(\Lambda \bar{x})\right|^2 \mathrm{d}x \leqslant \bar{\mu}_q \|\Lambda\|^2 \int_{\bar{\mathscr{S}}} |\phi(\Lambda \bar{x})|^2 \,\mathrm{d}\bar{x}.$$

and coming back to the simplex $\mathscr{S}$ by a new change of variable $\bar{x} = \Lambda^{-1}x$, we find

$$\int_{\mathscr{S}} |\mathrm{grad}\ \phi(x)|^2 \,\mathrm{d}x \leqslant \bar{\mu}_q \|\Lambda\|^2 \int_{\mathscr{S}} |\phi(x)|^2 \,\mathrm{d}x.$$

According to Lemma 4.3, Chapter 1,

$$\|\Lambda\| \leqslant \frac{\rho_{\bar{\mathscr{S}}}}{\rho'_{\mathscr{S}}}.$$

For the triangulation considered

$$\rho'_{\mathscr{S}} \geqslant \rho'(h);$$

hence

$$\|\Lambda\| \leqslant \frac{\rho_{\bar{\mathscr{S}}}}{\rho'(h)}$$

and we can take

$$\mu_q = \bar{\mu}_q \left(\frac{\rho_{\bar{\mathscr{S}}}}{\rho'(h)} \right)^2.$$

---

(1) $\|\Lambda\|$ is the norm of the linear operator Λ associated with the euclidian norm in $\mathbb{R}^n$.

Thus (6.44) holds with

$$c_q = \sqrt{\bar{\mu}_q}\, \rho_{\,\!/} . \tag{6.50}$$

6.2.2. *The form b_h and $S_1(h)$*

We take again for the approximations (APX2) to (APX4) the form b_h defined in Section 3, Chapter II (see (3.55)):

$$b_h(u_h, v_h, w_h) = b'(u_h, v_h, w_h) + b''(u_h, v_h, w_h) \tag{6.51}$$

$$b'(u, v, w) = \frac{1}{2} \sum_{i,j=1}^{n} \int_\Omega u_i (D_i v_j) w_j \, dx \tag{6.52}$$

$$b''(u, v, w) = - b'(u, w, v). \tag{6.53}$$

The form b_h is trilinear continuous on $V_h \times V_h \times V_h$ and (5.6) holds; an estimate of the constants d_1 and $S_1(h)$ will follow from the next lemma.

Lemma 6.3. *For $n = 2$ or 3, and for u, v, w in $H_0^1(\Omega)$:*

$$\begin{aligned} |b'(u, v, w)| &\leqslant \frac{1}{\sqrt{2}} |u|^{1/2} \|u\|^{1/2} \|v\| \, |w|^{1/2} \|w\|^{1/2} \quad (n = 2) \\ &\leqslant |u|^{1/4} \|u\|^{3/4} \|v\| \, |w|^{1/4} \|w\|^{3/4} \quad (n = 3) \end{aligned} \tag{6.54}$$

$$\begin{aligned} |b''(u, v, w)| &\leqslant \frac{1}{\sqrt{2}} |u|^{1/2} \|u\|^{1/2} |v|^{1/2} \|v\|^{1/2} \|w\| \quad (n = 2) \\ &\leqslant |u|^{1/4} \|u\|^{3/4} |v|^{1/4} \|v\|^{3/4} \|w\| \quad (n = 3) \end{aligned} \tag{6.55}$$

Proof. For $n = 2$ the result is proved in Lemma 3.4, observing that

$$b'(u, v, w) = \frac{1}{2} b(u, v, w)$$

$$b''(u, v, w) = - \frac{1}{2} b(u, w, v).$$

For $n = 3$, the proof is based on Lemma 3.5:

$$|b'(u, v, w)| \leqslant \frac{1}{2} \sum_{i,j=1}^{3} \int_\Omega |u_i (D_i v_j) w_j| \, dx \tag{6.56}$$

$$\leqslant \frac{1}{2} \sum_{i,j=1}^{3} \|u_i\|_{L^4(\Omega)} \|D_i v_j\|_{L^2(\Omega)} \|w_j\|_{L^4(\Omega)}$$

$$\leqslant \frac{1}{2} \left(\sum_{i,j=1}^{3} \|D_i v_j\|^2_{L^2(\Omega)} \right)^{1/2} \left(\sum_{i=1}^{3} \|u_i\|^2_{L^4(\Omega)} \right)^{1/2} \left(\sum_{j=1}^{3} \|w_j\|^2_{L^4(\Omega)} \right)^{1/2}.$$

Due to (3.67),

$$\sum_{i=1}^{3} \|u_i\|^2_{L^4(\Omega)} \leqslant 2 \sum_{i=1}^{3} (\|u_i\|^{1/2}_{L^2(\Omega)} \|\text{grad } u_i\|^{3/2}_{L^2(\Omega)})$$

$$\leqslant 2 \left(\sum_{i=1}^{3} \|u_i\|^2_{L^2(\Omega)} \right)^{1/4} \left(\sum_{i=1}^{3} \|\text{grad } u_i\|^2_{L^2(\Omega)} \right)^{3/4}$$

$$\leqslant 2\, |\boldsymbol{u}|^{1/2} \|\boldsymbol{u}\|^{3/2}. \tag{6.57}$$

Similarly,

$$\sum_{j=1}^{3} \|w_j\|^2_{L^4(\Omega)} \leqslant 2\, |\boldsymbol{w}|^{1/2} \|\boldsymbol{w}\|^{3/2} \tag{6.58}$$

and then (6.54) follows from (6.56).

For b'' we simply observe that

$$b''(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b'(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}),$$

and apply (6.54).

Lemma 6.4.

$$|b_h(\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h)| \leqslant d_1 \|\boldsymbol{u}_h\|\, \|\boldsymbol{v}_h\|\, \|\boldsymbol{w}_h\|, \ \forall\, \boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h \in V_h, \tag{6.59}$$

with

$$d_1 = 2\sqrt{2\ell} \ \textit{if } n = 2,\ d_1 = 2^{3/2}\sqrt{\ell} \ \textit{if } n = 3. \tag{6.60}$$

Proof. An immediate consequence of (6.43), (6.51), (6.54) and (6.55).

Proposition 6.4. *For the approximation* (APX2) *to* (APX4), *inequality* (5.41) *holds with*

$$S_1(h) = \sqrt{2}\ S(h) \ \textit{if } n = 2,\ S_1(h) = 2S^{3/2}(h) \ \textit{if } n = 3, \tag{6.61}$$

where $S(h)$ *is given by* (6.44).

Proof. Using (6.43), (6.54) and (6.55) we write:

$$|b_h'(u_h, v_h, w_h)| \leqslant \frac{1}{\sqrt{2}} S(h)|u_h| \, \|u_h\| \, |w_h| \quad (n = 2)$$

$$|b_h'(u_h, v_h, w_h)| \leqslant S^{3/2}(h)|u_h| \, \|u_h\| \, |w_h| \quad (n = 3)$$

$$|b_h''(u_h, v_h, w_h)| \leqslant \frac{1}{\sqrt{2}} S(h)|u_h| \, \|u_h\| \, |w_h| \quad (n = 2)$$

$$|b_h''(u_h, v_h, w_h)| \leqslant S^{3/2}(h)|u_h| \, \|u_h\| \, |w_h| \quad (n = 3)$$

6.2.3. *Application of the stability and convergence theorems*

Schemes 5.1 and 5.2 are unconditionally stable. The stability conditions of Scheme 5.3 are given in Theorem 5.2 and Lemma 5.3:

$$\frac{k}{[\rho'(h)]^2} \leqslant \min\left(\frac{d'}{4c_q^2}, \frac{d''}{c_q}\right) \text{ if } n = 2, \tag{6.62}$$

$$\frac{k}{[\rho'(h)]^3} \leqslant \frac{d'}{4c_q^3}, \; \frac{k}{[\rho'(h)]^2} \leqslant \frac{d''}{c_q^2}, \text{ if } n = 3, \tag{6.63}$$

where d' and d'' are defined in the proof of Lemma 5.3, and c_q is defined in the proof of Proposition 6.3.

For Scheme 5.4, the stability conditions given in Theorem 5.3 and Lemma 5.4 are.

$$\frac{k}{[\rho'(h)]^2} \leqslant \frac{1-\delta}{4\nu c_q^2}, \; \frac{k}{[\rho'(h)]^2} \leqslant \frac{\nu\delta}{16 c_q^2 d_5} \text{ if } n = 2, \tag{6.64}$$

$$\frac{k}{[\rho'(h)]^2} \leqslant \frac{1-\delta}{4\nu c_q^2}, \; \frac{k}{[\rho'(h)]^3} \leqslant \frac{\nu\delta}{32 c_q^3 d_5} \text{ if } n = 3, \tag{6.65}$$

for some δ, $0 < \delta < 1$, and d_5 is given by (5.58).

The interpretation of the convergence theorems (Theorem 5.4 and 5.5) is very simple in the present case since $F = H_0^1(\Omega)$; we have:

$$u_h \to u \text{ in } L^2(Q) \text{ strongly, } L^\infty(0, T; L^2(\Omega)) \text{ weak star,} \tag{6.66}$$

$u_h \to u$ in $L^2(0, T; H_0^1(\Omega))$ strongly or weakly (depending on whether convergence in $L^2(0, T; F)$ is strong or weak). (6.67)

The hypotheses (5.89) – (5.92) on which the proof of these theorems is based, are very easy to check and we will deal with this point very rapidly.

Verification of the hypotheses (5.89) *to* (5.91).

The condition (5.89) is evident as $F = H_0^1(\Omega)$, and the operators $\overline{\omega}$ and p_h are identity operators.

The condition (5.90) amounts to saying that if

$$v_{h'} \to v \text{ in } L^2(0, T; H_0^1(\Omega)) \text{ weakly,}$$

$$w_{h'} \to w \text{ in } L^2(0, T; H_0^1(\Omega)) \text{ strongly,}$$

then

$$\int_0^T ((v_{h'}(t), w_{h'}(t)))\mathrm{d}t \to \int_0^T ((v(t), w(t)))\mathrm{d}t;$$

this is obvious.

One can prove (5.91) as done several times before in similar situations[1]. Finally, in the present cases, the condition (5.92) is contained in Theorem 2.2.

6.3. *Non-conforming finite elements* (APX5)

The general approximation of V considered at the beginning of Section 5.1 is now chosen as the approximation (APX5) (piecewise linear nonconforming finite elements). We will successively check and interpret in this case the hypotheses of Theorems 5.1 to 5.5.

6.3.1. *Computation of* $S(h)$

The conditions (5.3) and (5.4) are obviously satisfied; according to Proposition I.4.13.

$$|u_h| \leqslant d_0 \|u_h\|_h \tag{6.68}$$

where $d_0 = c'(\Omega, \alpha)$ is a constant rather difficult to express explicitly and depending on Ω and α (see (4.179) – α has the same signification as in I.(4.21)).

---

(1) Recall that $w \in \mathscr{V}$, $r_h w \to w$ in $L^\infty(Q)$, $D_i r_h w \to D_i w$ in $L^\infty(Q)$, $1 \leqslant i \leqslant n$, as $h \to 0$, for all finite element methods considered.

Proposition 6.5. *For the approximation* (APX5),

$$S(h) = \frac{c_0(n)}{\rho'(h)}, \tag{6.69}$$

where $c_0(n)$ is a constant depending only on n, and $\rho'(h)$ is defined as in I.(4.20).

Proof. The proof of Proposition 6.3 is valid. The constant $S(h)$ is a bound of the square root of the supremum

$$\operatorname*{Sup}_{u_h \in V_h} \left\{ \frac{\sum_{i=1}^{n} \int_{\Omega} |\operatorname{grad} u_{ih}(x)|^2 \, dx}{\sum_{i=1}^{n} \int_{\Omega} |u_{ih}(x)|^2 \, dx} \right\} \tag{6.70}$$

It can be shown as in (6.48) that $S(h)^2$ is bounded by the supremum μ_1 of the expressions

$$\left\{ \frac{\int_{\mathscr{S}} |\operatorname{grad} \phi(x)|^2 \, dx}{\int_{\mathscr{S}} |\phi(x)|^2 \, dx} \right\}$$

among all linear functions ϕ, and all $\mathscr{S} \in \mathscr{T}_h$.

Using a linear affine mapping Λ which maps $\mathscr{S}$ on a reference n-simplex $\overline{\mathscr{S}}$ of $\mathbb{R}^n$:

$$0 \leqslant \bar{x}_i \leqslant 1, \quad \sum_{i=1}^{n} \bar{x}_i \leqslant 1,$$

we see that

$$\mu_1 = \bar{\mu}_1 \left(\frac{\rho_{\mathscr{S}}}{\rho'(h)} \right)^2$$

where $\bar{\mu}_1$ is the supremum among all linear functions ψ of the ratio

$$\left\{\frac{\displaystyle\int_{\bar{\mathscr{S}}} |\mathrm{grad}\ \phi(\bar{x})|^2\,\mathrm{d}\bar{x}}{\displaystyle\int_{\bar{\mathscr{S}}} |\psi(x)|^2\,\mathrm{d}\bar{x}}\right\}.$$

Hence (6.69) holds with

$$c_0(n) = \sqrt{\bar{\mu}_1}\ \rho_{\bar{\mathscr{S}}} \tag{6.71}$$

6.3.2. *The form b_h and $S_1(h)$*

For the approximation (APX5) we choose again the form b_h defined in Section 3.1, Chapter II (see (3.80) to (3.82)):

$$b_h(u_h, v_h, w_h) = b_h'(u_h, v_h, w_h) + b_h''(u_h, v_h, w_h), \tag{6.72}$$

$$b_h'(u_h, v_h, w_h) = \frac{1}{2}\sum_{i,j=1}^{n}\int_\Omega u_{ih}(D_{ih}v_{jh})w_{jh}\,\mathrm{d}x, \tag{6.73}$$

$$b_h''(u_h, v_h, w_h) = -\,b_h'(u_h, w_h, v_h). \tag{6.74}$$

It is clear that b_h', b_h'' and b_h are trilinear continuous forms on $V_h \times V_h \times V_h$ and that (5.6) holds. An indication on the value of the constant d_1 in (5.7) will follow from next Lemma:

Lemma 6.5. *For $n = 2$*

$$\begin{aligned}&|b_h'(u_h, v_h, w_h)|\\ &\qquad\leqslant c_1(\Omega, \alpha, \epsilon)|u_h|^{1-\epsilon/2}\,\|u_h\|_h^{1+\epsilon/2}\,\|v_h\|_h\,|w_h|^{1-\epsilon/2}\,\|w_h\|_h^{1+\epsilon/2}\\ &|b_h''(u_h, v_h, w_h)|\\ &\qquad\leqslant c_1(\Omega, \alpha, \epsilon)|u_h|^{1-\epsilon/2}\,\|u_h\|_h^{1+\epsilon/2}\,|v_h|^{1-\epsilon/2}\,\|v_h\|_h^{1+\epsilon/2}\,\|w_h\|_h\end{aligned} \tag{6.75}$$

for any $0 < \epsilon < 1$, where $c_1(\Omega, \alpha)$ depends only on Ω, α and ϵ.

For $n = 3$

$$\begin{aligned}&b_h'(u_h, v_h, w_h)|\\ &\qquad\leqslant c_2(\Omega, \alpha)|u_h|^{1/4}\,\|u_h\|_h^{3/4}\,\|v_h\|_h\,|w_h|^{1/4}\,\|w_h\|_h^{3/4}\\ &b_h''(u_h, v_h, w_h)|\\ &\qquad\leqslant c_2(\Omega, \alpha)|u_h|^{1/4}\,\|u_h\|_h^{3/4}\,|v_h|^{1/4}\,\|v_h\|_h^{3/4}\,\|w_h\|_h\,.\end{aligned} \tag{6.76}$$

Proof. The estimates for b_h'' are deduced from the estimates for b_h', by a simple permutation of v_h and w_h.

Using Hölder inequality as for Lemma 6.1, it appears that

$$|b_h'(u_h, v_h, w_h)| \leqslant \frac{1}{2}\left(\sum_{i=1}^{n} \|u_{ih}\|_{L^4(\Omega)}^2\right)^{1/2}$$

$$\left(\sum_{j=1}^{n} \|w_{jh}\|_{L^4(\Omega)}^2\right) \|v_h\|_h, \tag{6.77}$$

In order to estimate the L^4-norms of u_{ih} and v_{ih}, we apply Theorem 11.2.3 and Remark 11.2.6. For $n = 3$, Schwarz inequality allows us to write

$$\begin{aligned}\int_\Omega u_{ih}^4 \,\mathrm{d}x &\leqslant \left(\int_\Omega u_{ih}^2 \,\mathrm{d}x\right)^{1/2} \left(\int_\Omega u_{ih}^6 \,\mathrm{d}x\right)^{1/2}\\ &\leqslant \text{(because of II.(2.42), } n = 2, p = 2)\\ &\leqslant |u_{ih}|\; c(n, p, \Omega, \alpha)\|u_{ih}\|_h^3.\end{aligned}$$

Hence

$$\sum_{i=1}^{3} \|u_{ih}\|_{L^4(\Omega)}^2 \leqslant c_3(\Omega, \alpha)|u_h|^{1/2}\|u_h\|_h^{3/2}, \tag{6.78}$$

and the relation (6.67) for b_h' follows from this majoration and (6.77).

For $n = 2$, due to lack of majorations of the type found in Proposition II.2.1, we proceed differently. By the Schwarz inequality,

$$\begin{aligned}\int_\Omega u_{ih}^4 \,\mathrm{d}x &= \int_\Omega |u_{ih}|^{1-\epsilon}|u_{ih}|^{3+\epsilon} \,\mathrm{d}x\\ &\leqslant \left(\int_\Omega u_{ih}^2 \,\mathrm{d}x\right)^{1-\epsilon/2} \left(\int_\Omega |u_{ih}|^{2(3+\epsilon)/1+\epsilon} \mathrm{d}x\right)^{1+\epsilon/2}\\ &\leqslant \text{(because of II. (2.48))}\\ &\leqslant c(\Omega, \alpha, \epsilon)|u_{ih}|^{1-\epsilon}\|u_{ih}\|_h^{3+\epsilon}.\end{aligned}$$

Hence

$$\sum_{i=1}^{2} \|u_{ih}\|^2_{L^4(\Omega)} \leqslant c_4(\Omega, \alpha, \epsilon)|u_h|^{1-\epsilon} \|u_h\|_h^{3+\epsilon} \tag{6.79}$$

and the relation (6.75) for b_h' follows.

The combination of (6.68) and Lemma 6.5 gives easily (5.7):

$$b_h(u_h, v_h, w_h)| \leqslant d_1 \|u_h\|_h \|v_h\|_h \|w_h\|_h, \ \forall u_h, v_h, w_h \in V_h; \tag{6.80}$$

the constant d_1 depends on Ω and α.

Proposition 6.6. *For the approximation* (APX5), *the inequality* (5.41) *holds with*

$$S_1(h) = c_1(\Omega, \alpha, \epsilon) S(h)^{1+\epsilon}, \ 0 < \epsilon < 1 \text{ arbitrary, if } n = 2, \tag{6.81}$$

$$S_1(h) = c_2(\Omega, \alpha)\, S(h)^{3/2}, \text{ if } n = 3 \tag{6.82}$$

where $S(h)$, c_1, c_2 *are given in* (6.69), (6.75), (6.76).

Proof. Using (6.69), (6.75) and (6.76) we can write

$$|b_h'(u_h, v_h, w_h)| \leqslant c_1(\Omega, \alpha, \epsilon) S(h)^{1+\epsilon} |u_h| \ \|u_h\|_h |w_h|$$

$$|b_h''(u_h, u_h, v_h)| \leqslant c_1(\Omega, \alpha, \epsilon) S(h)^{1+\epsilon} |u_h| \ \|u_h\|_h |w_h|$$

for $n = 2$, and for $n = 3$,

$$|b_h'(u_h, u_h, w_h)| \leqslant c_2(\Omega, \alpha) S(h)^{3/2} |u_h| \ \|u_h\|_h |w_h|$$

$$|b_h''(u_h, u_h, w_h)| \leqslant c_2(\Omega, \alpha) S(h)^{3/2} |u_h| \ \|u_h\|_h |w_h|.$$

6.3.3. *Application of the stability and convergence theorems.*

Schemes 5.1 and 5.2 are unconditionally stable. The stability conditions of Scheme 5.3 are given in Theorem 5.2 and Lemma 5.3

$$\frac{k}{[\rho'(h)]^{2(1+\epsilon)}} \leqslant \frac{d'}{c_0(n)^{2(1+\epsilon)} c_1(\Omega, \alpha, \epsilon)^2},$$

$$\frac{k}{[\rho'(h)]^2} \leqslant \frac{d''}{c_0(n)^2}, \text{ if } n = 2, \tag{6.83}$$

$$\frac{k}{\rho'(h)^3} \leqslant \frac{d'}{c_0(n)^3 c_2(\Omega, \alpha)^2},$$

$$\frac{k}{\rho'(h)^2} \leqslant \frac{d''}{c_0(n)^2}, \text{ if } n = 3, \tag{6.84}$$

where d', d'' are some constants defined in the proof of Lemma 5.3 (ϵ arbitrarily fixed, $0 < \epsilon < 1$).

For Scheme 5.4 the stability conditions given in Theorem 5.3 and Lemma 5.4 are

$$\frac{k}{[\rho'(h)]^2} \leqslant \frac{1-\delta}{4\nu c_0^2}, \quad \frac{k}{[\rho'(h)]^{2(1+\epsilon)}} \leqslant \frac{\gamma\delta}{8d_5 c_1^2 c_0^{2(1+\epsilon)}} \text{ if } n = 2, \tag{6.85}$$

$$\frac{k}{[\rho'(h)]^2} \leqslant \frac{1-\delta}{4\nu c_0^2}, \quad \frac{k}{[\rho'(h)]^3} \leqslant \frac{\nu\delta}{8d_5 c_1^2 c_0^3}, \text{ if } n = 3. \tag{6.86}$$

for some δ, $0 < \delta < 1$, some ϵ, $0 < \epsilon < 1$, and d_5 given by (5.58).

Of course the stability conditions (6.83) - (6.86) *are not completely satisfying since we only have imprecise information on the constants in the right-hand side of these relations.*

The interpretation of the convergence theorems (Theorem 5.4 and 5.5) is very simple:

$$\begin{aligned} &\boldsymbol{u}_h \to \boldsymbol{u} \text{ in } \boldsymbol{L}^2(Q) \text{ strongly, } L^\infty(0, T; \boldsymbol{L}^2(\Omega)) \text{ weak star,} \\ &D_{ih}\, \boldsymbol{u}_h \to D_i \boldsymbol{u} \text{ in } \boldsymbol{L}^2(Q) \text{ strongly or weakly (depending on} \\ &\text{whether convergence in } L^2(0, T; F) \text{ is strong or weak).} \end{aligned} \tag{6.87}$$

Theorems 5.4 and 5.5 are based on the hypotheses (5.89) to (5.92). We must show that these conditions are met: the verification of these properties is exactly the same as for finite differences, we just have to repeat the arguments of Section 6.1.4, replacing everywhere δ_{ih} by D_{ih}.

6.4. *Numerical algorithms. Approximation of the pressure*

The practical computation of the element $\boldsymbol{u}_h^m$ of V_h, defined by one of the considered schemes, is not easy. The difficulty is connected with the constraint "div $\boldsymbol{u} = 0$" built into the definition of the space V_h and, actually, the situation is exactly the same as in Chapter I for the Stokes problem. In this subsection we will show how the algorithms studied in Section 5 of Chapter I (resolution of Stokes Problem) can be extended to the resolution of the problems (5.12) to (5.15). At the same time we will introduce the discrete approximation of the pressure.

6.4.1. *Approximation of the pressure*

For each type of approximation V_h of V, we have also defined an approximation W_h of $H_0^1(\Omega)$ (see Chapter I, Section 3 and 4). Essentially the elements u_h of W_h are exactly of the same type as the elements u_h of V_h, but no divergence condition is imposed. Later in this section we will show how the element u_h^m of V_h can be approximated by a sequence of elements of W_h, $u_h^{m,r}$, $r = 1, \ldots \infty$.

Approximation (*APX1*)

The space W_h is the space of step functions

$$u_h = \sum_{M \in \mathring{\Omega}_h^1} \xi_M \, w_{hM}, \; \xi_M \in \mathbb{R}^n. \tag{6.88}$$

The discrete pressure is an element of the space X_h of step functions of the type

$$\pi_h = \sum_{M \in \mathring{\Omega}_h^1} \eta_M \, w_{hM}, \; \eta_M \in \mathbb{R}. \tag{6.89}$$

For $u_h \in W_h$, we defined the discrete divergence $D_h u_h$ as the step function of X_h given by

$$D_h u_h = \sum_{M \in \mathring{\Omega}_h^1} (D_h u_h(M)) \, w_{hM},$$

$$D_h u_h(M) = \sum_{i=1}^{n} \nabla_{ih} u_{ih}(M), \quad \forall M \in \mathring{\Omega}_h^1. \tag{6.90}$$

An element u_h of W_h belongs to V_h if and only if

$$D_h u_h = 0 \tag{6.91}$$

Exactly as in Subsection 3.3, Chapter I (see I.(3.72)) we prove that if u_h^m is solution of (5.12) (i.e., Scheme 5.1), there exists some step function π_h^m of type (6.89), such that

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \nu((u_h^m, v_h))_h + b_h(u_h^{m-1}, u_h^m, v_h)$$
$$- (\pi_h^m, D_h v_h) = (f^m, v_h), \ \forall\, v_h \in W_h. \tag{6.92}$$

The main differences between (6.92) and (5.12) are the appearance of the term $-(\pi_h^m, D_h v_h)$ and the fact that the equation is satisfied for all v_h in W_h, i.e., even if v_h is not in some sense divergence free.

For Schemes 5.2 to 5.4, the same results are valid. In these three cases there exists a π_h^m of type (6.69) such that, respectively:

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \frac{\nu}{2}((u_h^m + u_h^{m-1}, v_h))_h + \frac{1}{2} b_h(u_h^{m-1}, u_h^m$$
$$+ u_h^{m-1}, v_h) - (\pi_h^m, D_h v_h) = (f^m, v_h), \forall\, v_h \in W_h. \tag{6.93}$$

(Scheme 5.2)

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \nu((u_h^m, v_h))_h + b_h(u_h^{m-1}, u_h^{m-1}, v_h)$$
$$- (\pi_h^m, D_h v_h) = (f^m, v_h), \ \forall\, v_h \in W_h. \tag{6.94}$$

(Scheme 5.3)

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \nu((u_h^{m-1}, v_h))_h + b_h(u_h^{m-1}, u_h^{m-1}, v_h)$$
$$- (\pi_h^m, D_h v_h) = (f^m, v_h), \ \forall\, v_h \in W_h. \tag{6.95}$$

(Scheme 5.4)

Other Approximations.

For the other approximations the same result holds; the differences only arise in the definition of D_h, the space X_h to which π_h^m belongs (and of course the definition of V_h and W_h).

For the approximations (APX2), (APX3) and (APX5), X_h is the space of step functions

$$\pi_h = \sum_{\mathscr{S} \in \mathscr{T}_h} \eta_{\mathscr{S}} \chi_{\mathscr{S}}, \eta_{\mathscr{S}} \in \mathbb{R}, ^{(1)} \tag{6.96}$$

where $\chi_{\mathscr{S}}$ is the characteristic function of the simplex $\mathscr{S}$. The discrete divergence is understood as the step function of type (6.96) defined by

$$\left.\begin{aligned} D_h u_h &= \sum_{\mathscr{S} \in \mathscr{T}_h} \eta_{\mathscr{S}} \chi_{\mathscr{S}} \\ \eta_{\mathscr{S}} &= \frac{1}{\operatorname{meas} \mathscr{S}} \int_{\mathscr{S}} \operatorname{div} u_h \, dx. \end{aligned}\right\} \tag{6.97}$$

(1) $\eta_{\mathscr{S}} = \pi_h(x), \quad x \in \mathscr{S}$.

A function u_h of W_h belongs to V_h if and only if the step function $D_h\, u_h$ vanishes.

With this notation we also get the existence of a π_h^m satisfying respectively (6.92), (6.93) or (6.95) (Schemes 5.1 to 5.4), in the case of approximations (APX2), (APX3) or (APX5).

A similar result can be proved for the approximation (APX4) but then the characterization of the space X_h ($\pi_h^m \in X_h$) is technically more complicated.

6.4.2. *Uzawa Algorithm*

We want to study an adaptation of the Uzawa algorithm for the resolution of the problems (6.92) to (6.95). If the elements of the step $m - 1$ have been computed, we must compute the unknowns

$$u_h^m \in V_h \text{ and } \pi_h^m \in X_h. \tag{6.98}$$

We will obtain them as the limits of two sequences of elements

$$u_h^{m,r} \in W_h \text{ and } \pi_h^{m,r} \in X_h,\ r = 0, 1, \dots \infty. \tag{6.99}$$

As in Subsection 5.1 and 5.3 of Chapter 1, we start the algorithm with any

$$\pi_h^{m,0} \in X_h. \tag{6.100}$$

When $\pi_h^{m,r}$ is known we define $u_h^{m,r+1}$ and $\pi_h^{m,r+1}$ ($r \geqslant 0$) by

(Scheme 5.1 or (6.92)) $u_h^{m,r+1} \in W_h$ *and*

$$\begin{aligned}(u_h^{m,r+1}, v_h) + k\nu((u_h^{m,r+1}, v_h))_h + kb_h(u_h^{m-1}, u_h^{m,r+1}, v_h)&\\ - k\,(\pi_h^{m,r}, D_h\, v_h) = (u_h^{m-1}, v_h)&\\ + k\,(f^m, v_h),\ \forall\, v_h \in W_h.&\end{aligned} \tag{6.101}$$

$\pi_h^{m,r+1} \in X_h$ *and*

$$(\pi_h^{m,r+1} - \pi_h^{m,r}, q_h) + \rho\,(D_h\, u_h^{m,r+1}, q_h) = 0\ \ \forall\, q_h \in X_h. \tag{6.102}$$

The existence and uniqueness of the solution $u_h^{m,r+1}$ of (6.101) is a consequence of the Projection Theorem (the problem is linear with respect to $u_h^{m,r+1}$). The same argument is also valid for (6.102) but actually (6.102) explicitly gives $\pi_h^{m,r+1}$ without inverting any matrix. The determination of $u_h^{m,r+1}$ amounts to solving a discrete Dirichlet problem (see Chapter I, Section 5.1).

For Schemes 5.2 to 5.4 (i.e., (6.93) to (6.95)) we leave (6.102) unchanged and replace (6.101) by, respectively,

(Scheme 5.2 or (6.93)) $u_h^{m,r+1} \in W_h$ *and*

$$(u_h^{m,r+1}, v_h) + \frac{k\nu}{2}((u_h^{m-1} + u_h^{m,r+1}, v_h))_h + \frac{k}{2} b_h(u_h^{m-1}, u_h^{m-1} + u_h^{m,r+1}, v_h) - k(\pi_h^{m,r}, D_h v_h) = (u_h^{m-1}, v_h) + k(f^m, v_h), \ \forall v_h \in W_h. \tag{6.103}$$

(Scheme 5.3 or (6.94)) $u_h^{m,r+1} \in W_h$ *and*

$$(u_h^{m,r+1}, v_h) + k\nu((u_h^{m,r+1}, v_h))_h + kb_h(u_h^{m-1}, u_h^{m-1}, v_h) - k(\pi_h^{m,r}, D_h, v_h) = (u_h^{m-1}, v_h) + k(f^m, v_h), \ \forall v_h \in W_h. \tag{6.104}$$

(Scheme 5.4 or (6.95)) $u_h^{m,r+1} \in W_h$ *and*

$$(u_h^{m,r+1}, v_h) + k\nu((u_h^{m-1}, v_h))_h + kb_h(u_h^{m-1}, u_h^{m-1}, v_h) - k(\pi_h^{m,r}, D_h v_h) = (u_h^{m-1}, v_h) + k(f^m, v_h), \ \forall v_h \in W_h. \tag{6.105}$$

We emphasize that all the elements of step $m-1$ are known, and to compute u_h^m, π_h^m, we reiterate on the index r computing $u_h^{m,r+1}$, $\pi_h^{m,r+1}$.

Regarding the convergence of this algorithm, the interesting point is that the convergence criteria are almost the same as for the linear problem.

Proposition 6.7. *If the number ρ satisfies*

$$0 < \rho < \frac{2\nu}{\mathrm{n}} \text{ [1]} \tag{6.106}$$

then as $r \to \infty$, $u_h^{m,r}$ converges to u_h^m in W_h and $\pi_h^{m,r}$ converges to π_h^m in $X_h/\mathbb{R}$.

Proof. We only give the proof for Scheme 5.1 ((6.101) and (6.102) are associated with (6.92)). We drop the indices m and h and set

---

(1) n = 2 or 3, the dimension of the space.

$$v^r = u_h^{m,r} - u_h^m, \tag{6.107}$$

$$\pi^r = \pi_h^{m,r} - \pi_h^m. \tag{6.108}$$

Subtracting (6.92) from (6.101) we obtain

$$(v^{r+1}, v_h) + k\nu((v^{r+1}, v_h))_h + kb_h(u_h^{m-1}, v^{r+1}, v_h)$$
$$= k(\pi^r, D_h v_h), \quad \forall v_h \in W_h;$$

and with $v_h = v^{r+1}$ we get (recall (5.6))

$$|v^{r+1}|^2 + k\nu\|v^{r+1}\|_h^2 = k(\pi^r, D_h v^{r+1}). \tag{6.109}$$

Recalling that $D_h u_h^m = 0$ since $u_h^m \dot{\in} V_h$, we write (6.102) as

$$(\pi^{r-1} - \pi^r, q_h) = -\rho(D_h v^{r+1}, q_h), \quad \forall q_h \in X_h;$$

setting $q_h = \pi^{r+1}$ we obtain

$$|\pi^{r+1}|^2 - |\pi^r|^2 - |\pi^r|^2 + |\pi^{r+1} - \pi^r|^2 = -2\rho(D_h v^{r+1}, \pi^{r+1}). \tag{6.110}$$

We continue as in the proof of Theorem I.5.1. The right-hand side of (6.110) is equal to

$$-2\rho(D_h v^{r+1}, \pi^{r+1} - \pi^r) - 2\rho(D_h v^{r+1}, \pi^r)$$

and, as we prove below,

$$|(D_h v_h, \pi_h)| \leqslant \sqrt{n}\, |\pi_h|\, \|v_h\|_h, \quad \forall \pi_h \in X_h, v_h \in W_h. \tag{6.111}$$

Admitting this point temporarily, we majorize the right-hand side of (6.110) by

$$- 2\rho(D_h v^{r+1}, \pi^r) + 2\rho\sqrt{n}\, \|v^{r+1}\|_h |\pi^{r+1} \ldots \pi^r|$$
$$\leqslant \delta|\pi^{r+1} - \pi^r|^2 + \frac{\rho^2 n}{\delta}\|v^{r+1}\|_h^2 + 2\rho(D_h v^{r+1}, \pi^r).$$

We now add (6.110) multiplied by k, to the equation (6.109) multiplied by 2ρ. Dropping two opposite terms and simplifying we find

$$\begin{aligned} &k|\pi^{r+1}|^2 - k|\pi^r|^2 + (1-\delta)k|\pi^{r+1} - \pi^r|^2 \\ &\qquad + |v^{r+1}|^2 + k\rho(2\nu - \frac{\rho n}{\delta})\|v^{r+1}\|^2 \leqslant 0. \end{aligned} \tag{6.112}$$

If ρ satisfies (6.106) then there exists some $0 < \delta < 1$, such that

$$2\nu - \frac{\rho n}{\delta} > 0.$$

By summation of the inequalities (6.112) for $r = 0, \ldots, s$, we get

$$k|\pi^{s+1}|^2 + \sum_{r=0}^{s} \{k(1-\delta)|\pi^{r+1} - \pi^r|^2 + |v^{r+1}|^2\}$$
$$+ k\rho(2\nu - \frac{\rho n}{\delta}) \sum_{r=0}^{s} \|v^{r+1}\|^2 \leqslant k|\pi^1|^2. \tag{6.113}$$

This shows that the series

$$\sum_{r=0}^{\infty} \|v^r\|^2, \quad \sum_{r=0}^{\infty} |v^r|^2$$

are convergent and hence

$$v^r \to 0 \text{ in } W_h, \text{ as } r \to \infty. \tag{6.114}$$

The relation (6.113) shows also that the sequence π^s is bounded. According to the preceding and (6.102), any convergent subsequence extracted from π^s must converge to 0 in $X_h/\mathcal{R}$. Thus the sequence π^s converges as a whole to 0 as $s \to \infty$, in $X_h/\mathcal{R}$ (i.e. in X_h up to an additive constant).

The proof is completed. □

It remains to prove (6.111).

Lemma 6.6. *For the approximation* (APX1) *to* (APX3) *and* (APX5),

$$|(D_h v_h, \pi_h)| \leqslant \sqrt{n}\, |\pi_h|\, \|v_h\|_h, \quad \forall \pi_h \in X_h, \quad \forall v_h \in W_h. \tag{6.115}$$

Proof. For finite differences (APX1), since the function π_h is constant on the blocks $\sigma_h(M)$, $M \in \mathring{\Omega}_h^1$,

$$(\pi_h, D_h v_h) = \int_\Omega \pi_h(x) \left(\sum_{i=1}^{n} \nabla_{ih} v_{ih}(x) \right) \mathrm{d}x.$$

By Schwarz's inequality, we estimate this by

$$|\pi_h| \cdot \left\{ \int_\Omega \left(\sum_{i=1}^{n} \nabla_{ih} v_{ih}(x) \right)^2 \mathrm{d}x \right\}^{1/2}$$

$$\leqslant |\pi_h| \sqrt{n} \left(\sum_{i=1}^{n} \int_\Omega |\nabla_{ih} v_{ih}(x)|^2 \mathrm{d}x \right)^{1/2}.$$

But

$$\int_\Omega |\nabla_{ih} v_{ih}(x)|^2 \mathrm{d}x = \frac{1}{h_i^2} \int_{\mathbb{R}^n} |v_{ih}(x + \vec{h_i}) - v_{ih}(x)|^2 \,\mathrm{d}x$$

$$= \frac{1}{h_i^2} \int_{\mathbb{R}^n} |v_{ih}(x + \tfrac{1}{2}\vec{h_i}) - v_{ih}(x - \tfrac{1}{2}\vec{h_i})|^2 \,\mathrm{d}x$$

$$= \int_\Omega |\delta_{ih} v_{ih}(x)|^2 \,\mathrm{d}x,$$

and then

$$|(\pi_h, D_h v_h)| \leqslant \sqrt{n}\, |\pi_h| \left(\sum_{i=1}^{n} |\delta_{ih} v_{ih}|^2 \right)^{1/2} = \sqrt{n}\, |\pi_h|\, \|v_h\|_h.$$

For finite elements, we observe again that π_h is constant on a simplex $\mathscr{S} \in \mathscr{T}_h$ so that

$$(\pi_h, D_h v_h) = \int_\Omega \pi_h(x) \operatorname{div} v_h(x) \,\mathrm{d}x.$$

Then, by Schwarz's inequality,

$$|(\pi_h, D_h v_h)| \leqslant |\pi_h|\, |\operatorname{div} v_h| \leqslant \sqrt{n}\, |\pi_h|\, \|v_h\|. \quad \square$$

6.4.3. *Arrow-Hurwicz Algorithm*

We compute $\boldsymbol{u}_h^m$, π_h^m, as limit of two sequences of elements

$$\boldsymbol{u}_h^{m,r} \in W_h,\ \pi_h^{m,r} \in X_h,\ r \geqslant 0,$$

defined in a slightly different way from before.

There are two positive parameters ρ and α.

We start the recurrence with any

$$\boldsymbol{u}_h^{m,0} \in W_h,\ \pi_h^{m,0} \in X_h. \tag{6.116}$$

When $\pi_h^{m,r}$ and $u_h^{m,r}$ are known, we define $\pi_h^{m,r+1}$ and $u_h^{m,r+1}$ as the solutions of

(Scheme 5.1 or (6.92)) $u_h^{m,r+1} \in W_h$ *and*

$$k((u_h^{m,r+1} - u_h^{m,r}, v_h))_h + (u_h^{m,r}, v_h) + k\nu((u_h^{m,r}, v_h))_h + kb_h(u_h^{m-1}, u_h^{m,r}, v_h) - k(\pi_h^{m,r}, D_h v_h) = (u_h^{m-1}, v_h) + k(f^m, v_h), \quad \forall v_h \in W_h, \tag{6.117}$$

$\pi_h^{m,r+1} \in X_h$ *and*

$$\alpha(\pi_h^{m,r+1} - \pi_h^{m,r}, q_h) + \rho(D_h u_h^{m,r+1}, q_h) = 0, \quad \forall q_h \in X_h. \tag{6.118}$$

The existence and uniqueness of $u_h^{m,r+1}$ follows from the Projection Theorem, those of $\pi_h^{m,r+1}$ too, but in practice it is simpler to observe that (6.118) defines $\pi_h^{m,r+1}$ explicitly.

For the other schemes (Schemes 5.2 to 5.4, or (6.93) to (6.95), we leave (6.118) unchanged and we respectively replace (6.117) by

(Scheme 5.2 or (6.93)) $u_h^{m,r+1} \in W_h$ *and*

$$k((u_h^{m,r+1} - u_h^{m,r}, v_h))_h + (u_h^{m,r}, v_h) + \frac{k\nu}{2}((u_h^{m,r} + u_h^{m-1}, v_h))_h + \frac{k}{2} b_h(u_h^{m-1}, u_h^{m-1} + u_h^{m,r}, v_h) - k(\pi_h^{m,r}, D_h v_h) = (u_h^{m-1}, v_h) + k(f^m, v_h), \quad \forall v_h \in W_h. \tag{6.119}$$

(Scheme 5.3 or (6.94)), $u_h^{m,r+1} \in W_h$ *and*

$$k((u_h^{m,r+1} - u_h^{m,r}, v_h))_h + (u_h^{m,r}, v_h) + k\nu((u_h^{m,r}, v_h))_h + kb_h(u_h^{m-1}, u_h^{m-1}, v_h) - k(\pi_h^{m,r}, D_h v_h) = (u_h^{m-1}, v_h) + k(f^m, v_h), \quad \forall v_h \in W_h. \tag{6.120}$$

(Scheme 5.4 or (6.95)), $u_h^{m,r+1} \in W_h$ *and*

$$k((u_h^{m,r+1} - u_h^{m,r}, v_h))_h + (u_h^{m,r}, v_h) + k\nu((u_h^{m-1}, v_h))_h + kb_h(u_h^{m-1}, u_h^{m-1}, v_h) - k(\pi_h^{m,r}, D_h v_h) = (u_h^{m-1}, v_h) + k(f^m, v_h), \quad \forall v_h \in W_h. \tag{6.121}$$

Regarding convergence, we have

Proposition 6.8. *We assume that ρ and α satisfy*

$$0 < \rho < \frac{2\alpha\nu}{\alpha\nu^2 + n}\text{ [1]} \tag{6.122}$$

Then, as $r \to \infty$, $u_h^{m,r}$ converges to u_h^m in W_h, and $\pi_h^{m,r}$ converges to π_h^m in $X_h/\mathbb{R}$.

We omit the details of the proof which are similar to those of Theorem I.5.2. and Proposition 6.7. $\square$

§7. Approximation of the Navier–Stokes Equations by the Fractional Step Method.

This section deals with the approximation of the Navier–Stokes equations by a fractional step method.

The fractional step, or splitting-up, method is a method of approximation of evolution equations based on a decomposition of the operators We first present the idea of the method in the following very simple situation. Let us assume that we are approximating a linear evolution equation

$$u' + \mathscr{A}u = 0, \quad 0 < t < T \tag{7.1}$$

$$u(0) = u_0, \tag{7.2}$$

where $u(t)$ is a finite dimensional vector, $u(t) \in \mathbb{R}^h$, and $\mathscr{A}$ is a square matrix of the order m. With a standard implicit scheme (similar to Scheme 5.1), we define a sequence of vectors u^m, $m = 0, \ldots, N$ as follows ($T = kN$, k = the mesh size, N is an integer):

$$u^0 = u_0, \tag{7.3}$$

$$\frac{u^{m+1} - u^m}{k} + \mathscr{A}u^{m+1} = 0, \quad m = 0, \ldots, N-1. \tag{7.4}$$

---

[1] n = 2 or 3, the dimension of the space.

A splitting-up method is based on the existence of a decomposition of $\mathscr{A}$ as a sum

$$\mathscr{A} = \sum_{i=1}^{q} \mathscr{A}_i. \tag{7.5}$$

Starting again with

$$\boldsymbol{u}^0 = \boldsymbol{u}_0, \tag{7.6}$$

we recursively define a family of elements $\boldsymbol{u}^{m+i/q}$, $m = 0, \ldots, N-1$, $i = 1, \ldots, q$, by setting

$$\frac{\boldsymbol{u}^{m+i/q} - \boldsymbol{u}^{m+(i-1)/q}}{k} + \mathscr{A}_i \boldsymbol{u}^{m+i/q} = 0, \quad i = 1, \ldots, q,$$

$$m = 0, \ldots, N-1. \tag{7.7}$$

When $\boldsymbol{u}^m$ is known, $\boldsymbol{u}^{m+1}$ can be computed in the case of an ordinary implicit scheme (i.e., (7.4)) by the inversion of the matrix $I + k\mathscr{A}$; in case of the fractional step method (i.e., (7.7)) the computation of $\boldsymbol{u}^{m+1}$ necessitates the inversion of the q matrices $(I + k\mathscr{A}_1), \ldots, (I + k\mathscr{A}_q)$; the algorithm is useful if all these q matrices are simpler to invert than $I + k\mathscr{A}$.

This method can be adapted to the Navier–Stokes equations in many ways corresponding to the many possible decompositions of the operators. We will consider two of them. The first one corresponds to $q = 2$, and

$$\mathscr{A}_1 \boldsymbol{u} = -\nu \Delta \boldsymbol{u} + \sum_{i=1}^{n} u_i D_i \boldsymbol{u}, \tag{7.8}$$

while $\mathscr{A}_2$ is an heuristic operator taking into account the term grad p and the condition div $\boldsymbol{u} = 0$. We will be more precise in the course of the section, but we emphasise now that, in case of the Navier–Stokes equations, the method we will study is an interpretation of the fractional step method described in (7.5)–(7.7) and not merely a particular case.

For the second decomposition of $\mathscr{A}$, we have $q = n + 1$, the "operator" $\mathscr{A}_{n+1}$ being like the preceding operator $\mathscr{A}_2$, and $\mathscr{A}_1, \ldots, \mathscr{A}_n$, being defined by

$$\mathscr{A}_i \boldsymbol{u} = -\nu D_i^2 \boldsymbol{u} + \boldsymbol{u}_i D_i \boldsymbol{u} \quad i = 1, \ldots, n. \tag{7.9}$$

We can also associate this method with a discretization in the space variables. It would be ineffectual and tedious to study all the possible combinations. Consequently we will restrict our attention to two characteristic cases: the first decomposition above without any discretization in the space variables (subsection 7.1); then the second decomposition with a discretization by finite differences (subsections 7.2 and 7.3).

7.1. *A scheme with two intermediate steps*

We describe here an approximation of the Navier–Stokes equation by a fractional step method without discretization in the space variables.

7.1.1. *Description of the scheme*

The dimension of the space is $n = 2$ or 3 and we want to approximate the solution of Problem 3.1 (or 3.2). For simplicity we assume that f is given in $L^2(0, T; H)$

$$f \in L^2(0, T; H), \tag{7.10}$$

and, as before

$$u_0 \in H. \tag{7.11}$$

The interval $[0, T]$ is split into N intervals of length k ($T = kN$). We set

$$f^m = \frac{1}{k} \int_{(m-1)k}^{mk} f(t)\,\mathrm{d}t, \qquad m = 1, \ldots, N. \tag{7.12}$$

We will define a family of elements of $L^2(\Omega)$, denoted $u^{m+i/2}$, $i = 0, 1$, $m = 0, \ldots, N-1$. These elements are computed successively in the order of increasing values of the fractional index $m + i/2$.

We start with

$$u^0 = u_0. \tag{7.13}$$

When u^m is known ($m \geqslant 0$), we successively define $u^{m+1/2}$ and u^{m+1}:

$$u^{m+1/2} \in H_0^1(\Omega) \textit{ and,}$$

$$\frac{1}{k}(u^{m+1/2} - u^m, v) + \nu((u^{m+1/2}, v)) + \hat{b}(u^{m+1/2}, u^{m+1/2}, v)$$

$$= (f^m, v), \forall\, v \in H_0^1(\Omega). \tag{7.14}$$

$$u^{m+1} \in H \quad and,$$

$$(u^{m+1}, v) = (u^{m+1/2}, v), \ \forall v \in H. \tag{7.15}$$

The form $\hat{b}$ introduced in Chapter II is the skew component of b:

$$\hat{b}(u, v, w) = \frac{1}{2} \{b(u, v, w) - b(u, w, v)\}. \tag{7.16}$$

Since $n \leqslant 3$ it is clear that the form $\hat{b}$, like b, is a trilinear continuous form on $H_0^1(\Omega)$, and that

$$\hat{b}(u, v, v) = 0, \ \forall u, v \in H_0^1(\Omega). \tag{7.17}$$

The equation (7.14) defining $u^{m+1/2}$ is a nonlinear equation very similar to the stationary Navier–Stokes equation, and the proof of existence of at least one element $u^{m+1/2}$ satisfying (7.14) is carried out exactly as the proof of Theorem 1.2, Chapter II, using the Galerkin procedure and (7.16).

The relation (7.15) amounts to saying that u^{m+1} is the orthogonal projection of $u^{m+1/2}$ on H in $L^2(\Omega)$. Thus, the element u^{m+1} is well defined by (7.15); we write

$$u^{m+1} = P_H u^{m+1/2}, \tag{7.18}$$

where P_H denotes the orthogonal projector in $L^2(\Omega)$ on the space H. Due to the characterization of H and $H^\perp$ given by Theorem 1.4, Chapter I, the difference $u^{m+1/2} - u^{m+1}$ is the gradient of some function of $H^1(\Omega)$ and it is convenient to denote this function by kp^{m+1}:

$$\frac{u^{m+1} - u^{m+1/2}}{k} + \operatorname{grad} p^{m+1} = 0, \quad p^{m+1} \in H^1(\Omega). \tag{7.19}$$

The relation (7.15) is equivalent to two conditions, namely (7.19) and

$$u^{m+1} \in L^2(\Omega), \quad \operatorname{div} u^{m+1} = 0, \quad \gamma_\nu u^{m+1} = 0. \tag{7.20}$$

Remark 7.1. The equation (7.14) which defines $u^{m+1/2}$ is essentially a nonlinear Dirichlet problem; writing (7.14) with $v \in \mathscr{D}(\Omega)$, we get

$$\frac{1}{k}(u^{m+1/2} - u^m) - \nu\Delta u^{m+1/2}$$

$$+ \sum_{i=1}^{n} u_i^{m+1/2} D_i u^{m+1/2} + \frac{1}{2}(\operatorname{div} u^{m+1/2}) u^{m+1/2} = f^m. \tag{7.21}$$

Remark 7.2. The relations (7.19) (7.20) defining $u^{m+1/2}$ and p^{m+1} are equivalent to a Neumann problem for p^{m+1}.(1) Application of the operator "div" to both sides of (7.19) leads to

$$\Delta p^{m+1} = \frac{1}{k} \operatorname{div} u^{m+1/2} \text{ (since } \operatorname{div} u^{m+1} = 0), \tag{7.22}$$

and application of the operator γ_ν(= the normal component on $\partial\Omega$) leads to

$$\frac{\partial p^{m+1}}{\partial \nu} = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{7.23}$$

since $\gamma_\nu u^{m+1} = \gamma_\nu u^{m+1/2} = 0$.

It is interesting to observe that this boundary condition "$\partial p/\partial\nu = 0$ on Γ", is not satisfied by the exact pressure; this affects the accuracy of the p^{m+1} as approximations of p; nevertheless, as will be proved later, this does not affect the convergence of the scheme.

This peculiarity does not arise in the other fractional step method of subsection 7.2. □

Our purpose now is to prove *a priori* estimates for the $u^{m+i/2}$, and then study the convergence of the scheme.

7.1.2. *A priori estimates* (I)

Lemma 7.1. *The elements $u^{m+i/2}$ remain bounded in the following sense:*

$$|u^{m+i/2}|^2 \leqslant d_2, \quad m = 0, \ldots, N-1, \quad i = 1, 2, \tag{7.24}$$

$$k \sum_{m=0}^{N-1} \|u^{m+1}\|^2 \leqslant \frac{d_2}{\nu}, \tag{7.25}$$

$$\sum_{m=0}^{N-1} |u^{m+1} - u^{m+1/2}|^2 \leqslant d_2, \tag{7.26}$$

$$\sum_{m=0}^{N-1} |u^{m+1/2} - u^m|^2 \leqslant d_2 \tag{7.27}$$

(1)When p^{m+1} is known, u^{m+1} is directly given by (7.19).

where

$$d_2 = |u_0|^2 + \frac{d_0^2}{\nu} \int_0^T |f(s)|^2 \, ds. \tag{7.28}$$

Proof. If we write (7.14) with $v = u^{m+1/2}$ and take into account (7.17), we get

$$\begin{aligned} |u^{m+1/2}|^2 - |u^m|^2 + |u^{m+1/2} - u^m|^2 \\ + 2k\nu \|u^{m+1/2}\|^2 &= 2k(f^m, u^{m+1/2}) \\ &\leqslant 2k|f^m|\,|u^{m+1/2}| \leqslant 2k\, d_0 |f^m|\, \|u^{m+1/2}\| \text{ (1)} \\ &\leqslant k\nu \|u^{m+1/2}\|^2 + \frac{kd_0^2}{\nu} |f^m|^2. \end{aligned} \tag{7.29}$$

Hence

$$\begin{aligned} |u^{m+1/2}|^2 - |u^m|^2 + |u^{m+1/2} - u^m|^2 + k\nu \|u^{m+1}\|^2 \\ \leqslant k \frac{d_0^2}{\nu} |f^m|^2. \end{aligned} \tag{7.30}$$

We are permitted to write (7.15) with $v = u^{m+1}$; this gives

$$(u^{m+1} - u^{m+1/2}, u^{m+1}) = 0$$

or

$$|u^{m+1}|^2 - |u^{m+1/2}|^2 + |u^{m+1} - u^{m+1/2}|^2 = 0. \tag{7.31}$$

We add relations (7.29) and (7.30) for $m = 0, \ldots, N-1$, obtaining

$$\begin{aligned} |u^N|^2 + \sum_{m=0}^{N-1} \{ |u^{m+1} - u^{m+1/2}|^2 + |u^{m+1/2} - u^m|^2 \} \\ + k\nu \sum_{m=0}^{N-1} \|u^{m+1/2}\|^2 \leqslant |u_0|^2 \\ + \frac{kd_0^2}{\nu} \sum_{m=1}^{N} |f^m|^2 \leqslant \text{(by (5.29))} \end{aligned}$$

(1) $|v| \leqslant d_0 \|v\|, \; \forall\, v \in H_0^1(\Omega)$.

$$\leqslant |u_0|^2 + \frac{d_0^2}{\nu}\int_0^T |f(s)|^2\,ds = d_2. \tag{7.32}$$

This proves the estimates (7.25) to (7.27).

Next we add relation (7.30), for $m = 0, \ldots, r$, and relations (7.31) for $m = 0, \ldots, r - 1$. Dropping some positive terms, we find

$$|u^{r+1/2}|^2 \leqslant |u^0|^2 + \frac{kd_0^2}{\nu}\sum_{m=0}^{r-1} |f^m|^2 \leqslant d_2.$$

Similarly adding relations (7.30) and (7.31) for $m = 0, \ldots, r$, we get after some simplification,

$$|u^{r+1}|^2 \leqslant |u^0|^2 + \frac{kd_0^2}{\nu}\sum_{m=1}^{r} |f^r|^2 \leqslant d_2;$$

thus (7.24) is proved.

The approximate functions

We introduce the "approximate" functions, $u_k^{(i)}$, $i = 1, 2$, and u_k:

$$u_k^{(i)} : [0, T] \to L^2(\Omega),$$
$$u_k^{(i)}(t) = u^{m+i/2} \quad \text{for} \quad mk \leqslant t < (m+1)k, \quad i = 1, 2 \tag{7.33}$$

u_k *is a continuous function from* $[0, T]$ *into* $L^2(\Omega)$, *linear on each interval* $[mk, (m+1)k]$, $m = 0, \ldots, N-1$, *and* $u_k(mk) = u^m$, $m = 0, \ldots, N$. (7.34)

Lemma 7.1 implies:

Lemma 7.2. *The functions* u_k, $u_k^{(i)}$, $i = 1, 2$, *remain bounded in* $L^\infty(0, T; L^2(\Omega))$, *as* $k \to 0$. *The functions* $u_k^{(1)}$ *remain bounded in* $L^2(0, T; H_0^1(\Omega))$.

Lemma 7.3.

$$|u_k^{(2)} - u_k^{(1)}|_{L^2(0,T;L^2(\Omega))} \leqslant \sqrt{k\,d_2}, \tag{7.35}$$

$$|u_k - u_k^{(2)}|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{\sqrt{4k\,d_2}}{3}. \tag{7.36}$$

Proof. Lemma 7.2 and (7.35) are direct consequences of Lemma 7.1. For (7.36), the computations of Lemma 4.8 give

$$|u_k - u_k^{(2)}|^2_{L^2(0,T,L^2(\Omega))} = \frac{k}{3} \sum_{m=0}^{N-1} |u^{m+1} - u^m|^2 .$$

But

$$|u^{m+1} - u^m| \leqslant |u^{m+1} - u^{m+1/2}| + |u^{m+1/2} - u^m|$$

$$|u^{m+1} - u^m|^2 \leqslant 2|u^{m+1} - u^{m+1/2}|^2 + 2|u^{m+1/2} - u^m|^2$$

and therefore, by (7.26)–(7.27)

$$|u_k - u_k^{(2)}|^2_{L^2(0,T;L^2(\Omega))} \leqslant \frac{4k}{3} d_2 .$$

7.1.3. *A priori estimates* (II)

In order to apply the compactness tools, we need some estimate of the time derivatives of u_k.

We extend the function u_k by 0 outside the interval $[0, T]$ and denote by $\hat{u}_k$ the Fourier transform of the extended function.

Lemma 7.4. *The Fourier transform $\hat{u}_k$ of u_k satisfies*

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\hat{u}_k(\tau)\|^2_{V'} \, \mathrm{d}\tau \leqslant \text{Const}, \qquad 0 < \gamma < \frac{1}{4}, \tag{7.37}$$

where the constant depends on γ and the data.

Proof. According to Theorem I.1.4 and I.1.6, $V = H \cap H_0^1(\Omega)$, and the relations (7.14) and (7.15) hold for any $v \in V$. By summation we get

$$\frac{1}{k}(u^{m+1} - u^m, v) + \nu((u^{m+1/2}, v)) + \hat{b}(u^{m+1/2}, u^{m+1}, v)$$
$$= (f^m, v), \ \forall\, v \in V, \ m = 0, \ldots, N-1 \tag{7.38}$$

and this equation is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_k(t), v) + \nu((u_k^{(1)}(t), v)) + \hat{b}(u_k^{(1)}(t), u_k^{(1)}(t), v)$$
$$= (f_k(t), v), \ \forall\, t \in (0, T), \ \forall\, v \in V, ^{(1)} \tag{7.39}$$

(1) The symbol ^ on b has nothing to do with the symbol ^ denoting the Fourier transform.

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_k(t), v) = \langle g_k(t), v\rangle, \ \forall\, t \in (0, T), \ \forall\, v \in V, \tag{7.40}$$

where

$$f_k(t) = f^m, \quad mk \leqslant t < (m+1)k, \quad m = 0, \ldots, N-1 \tag{7.41}$$

and $g_k(t) \in V'$ is defined by

$$\begin{aligned}\langle g_k(t), v\rangle = &- \nu((u_k^{(1)}(t), v)) - \hat{b}(u_k^{(1)}(t), u_k^{(1)}(t), v)\\ &+ (f_k(t), v), \ \forall\, v \in V.\end{aligned} \tag{7.42}$$

It is clear from the properties of $\hat{b}$ that

$$\|g_k(t)\|_{V'} \leqslant \nu\|u_k^{(1)}(t)\| + c\,\|u_k^{(1)}(t)\|^2 + |f_k(t)|, \tag{7.43}$$

and due to Lemma 7.2,

$$g_k \text{ remains bounded in } L^2(0, T; V'). \tag{7.44}$$

After this, we can repeat exactly the arguments of the proof of Theorem 2.2 and arrive at (7.37).

7.1.4. *Convergence of the Scheme*

The behaviour of the approximate functions $u_k^{(i)}$, u_k, as $k \to 0$, is described by the following theorems.

Theorem 7.1. *The dimension of the space is $n = 2$, and f and u_0 are given satisfying* (7.10)–(7.11); *u denotes the unique solution of Problem* 3.1, *$u_k^{(i)}$, u_k are the approximate functions defined by* (7.33) (7.34).

As $k \to 0$ the following convergence results hold:

$$u_k^{(i)}, u_k, \textit{ convergence to } u \textit{ in } L^2(Q) \textit{ strongly, } L^\infty(0, T; L^2(\Omega)) \textit{ weak-star} \tag{7.45}$$

$$u_k^{(1)} \textit{ converges to } u \textit{ in } L^2(0, T; H_0^1(\Omega)) \textit{ strongly.} \tag{7.46}$$

Theorem 7.2. *The dimension of the space is $n = 3$ and f and u_0 are given satisfying* (7.10)–(7.11).

Then there exists some sequence $k' \to 0$, such that

$$u_{k'}^{(i)}, u_{k'}, \textit{ convergence to } u \textit{ in } L^2(Q) \textit{ strongly, } L^\infty(0, T; L^2(\Omega)) \textit{ weak-star} \tag{7.47}$$

$$u_{k'}^{(1)} \textit{ converges to } u \textit{ in } L^2(0, T; H_0^1(\Omega)) \textit{ weakly, where } u \textit{ is some solution of Problem } 3.1. \tag{7.48}$$

For any other sequence $k' \to 0$, such that the convergence results (7.47)–(7.48) *hold, $\boldsymbol{u}$ must be a solution of Problem* 3.1.

The proof of these two theorems is the same except for the strong convergence in $L^2(0, T; \boldsymbol{H}_0^1(\Omega))$.

7.1.5. *Strong convergence in $\boldsymbol{L}^2(Q)$*

Due to Lemma 7.2, there exists a subsequence $k' \to 0$, such that

$$\boldsymbol{u}_{k'}^{(1)} \to \boldsymbol{u}^{(1)} \text{ in } L^\infty(0, T; \boldsymbol{L}^2(\Omega)) \text{ weak-star}, L^2(0, T; \boldsymbol{H}_0^1(\Omega)) \text{ weak}, \tag{7.49}$$

$$u_{k'}^2 \to \boldsymbol{u}^{(2)} \text{ in } L^\infty(0, T; \boldsymbol{L}^2(\Omega)) \text{ weak-star}, \tag{7.50}$$

$$\boldsymbol{u}_{k'} \to \boldsymbol{u}_* \text{ in } L^\infty(0, T; \boldsymbol{L}^2(\Omega)) \text{ weak-star}. \tag{7.51}$$

Due to Lemma 7.3, we have

$$\boldsymbol{u}^{(1)} = \boldsymbol{u}^{(2)} = \boldsymbol{u}_*\,. \tag{7.52}$$

The functions $\boldsymbol{u}_k^{(2)}$ and hence the function $\boldsymbol{u}^{(2)}$ belongs to $L^\infty(0, T; H)$. We infer from this the property:

$$\boldsymbol{u}_*(t) \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H} = V, \text{ a.e.},$$

which implies

$$\boldsymbol{u}_* \in L^2(0, T; V) \cap L^\infty(0, T; H)\,. \tag{7.53}$$

The results of strong convergence in $\boldsymbol{L}^2(Q)$, which are essential in order to pass to the limit, are not obtained merely by the application of a compactness theorem; they require some further arguments which will be developed now.

We recall that by definition of $\boldsymbol{u}_k^{(2)}$, $\boldsymbol{u}_k^{(2)}(t)$ is an element of H for all $t \in [0, T]$ and by virtue of Theorem I.1.4, $\boldsymbol{L}^2(\Omega)$ is the direct sum of H and its orthogonal complement $H^\perp$. Denoting by P_H and $P_{H^\perp}$ the orthogonal projection in $\boldsymbol{L}^2(\Omega)$ onto H and $H^\perp$, we have

$$\boldsymbol{u}_k^{(1)} = P_H \boldsymbol{u}_k^{(1)} + P_{H^\perp} \boldsymbol{u}_k^{(1)},$$

and hence

$$|\boldsymbol{u}_k^{(1)}(t) - \boldsymbol{u}_k^{(2)}(t)|^2 = |P_H\, \boldsymbol{u}_k^{(1)}(t) - \boldsymbol{u}_k^{(2)}(t)|^2 + |P_{H^\perp}\, \boldsymbol{u}_k^{(1)}(t)|^2. \tag{7.54}$$

From (7.54) and (7.35), we infer that

$$P_H\, \boldsymbol{u}_k^{(1)} - \boldsymbol{u}_k^{(2)} \to 0 \text{ in } L^2(0, T; H) \text{ strongly}, \tag{7.55}$$

$$P_{H^\perp} u_k^{(1)} \to 0 \text{ in } L^2(0, T; \boldsymbol{L}^2(\Omega)) \text{ strongly.} \tag{7.56}$$

It follows from Remark I.1.6 and Lemma 7.2 that $P_H\, u_k^{(1)}$ is a bounded sequence in $L^2(0, T; \boldsymbol{H}^1(\Omega) \cap H)$ and, therefore, we can choose the previous subsequence k' so that

$$P_H\, u_{k'}^{(1)} \to P_H\, u_* = u_* \text{ in } L^2(0, T; \boldsymbol{H}^1(\Omega) \cap H) \text{ weakly.} \tag{7.57}$$

We now apply Proposition 2.1 in the following way: $X_0 = \boldsymbol{H}^1(\Omega) \cap H$, $X_1 = V'$, the sequences $\{u_m\}$ and $\{v_m\}$ replaced by the sequences $\{u_{k'}\}$, $\{P_H\, u_{k'}^{(1)}\}$. These sequences possess the required properties; moreover,

$$H^1(\Omega) \cap H \subset H \subset V',$$

and since the injection of $H^1(\Omega) \cap H$ into H is compact, so is the injection of $H^1(\Omega) \cap H = X_0$ into $V' = X_1$. Proposition 2.1 enables us to assert that the sequence $P_H\, u_{k'}^{(1)}$ is relatively compact in $L^2(0, T; V')$ and therefore

$$P_H\, u_{k'}^{(1)} \to P_H\, u_* = u_* \text{ in } L^2(0, T; V') \text{ strongly.} \tag{7.58}$$

An application of Lemma 2.1 with $X_0 = \boldsymbol{H}^1(\Omega) \cap H$, $X = H$, $X_1 = V'$ leads to:

$$|P_H u_{k'}^{(1)} - u_*|_{L^2(0,T;H)} \leqslant \epsilon \|P_H u_{k'}^{(1)} - u_*\|_{L^2(0,T;\boldsymbol{H}^1(\Omega) \cap \boldsymbol{H})} + C(\epsilon)\|P_H u_{k'}^{(1)} - u_*\|_{L^2(0,T;V')}$$

and since $P_H u_{k'}^{(1)}$ is bounded in $L^2(0, T; \boldsymbol{H}^1(\Omega))$;

$$|P_H u_{k'}^{(1)} - u_*|_{L^2(0,T;H)} \leqslant C\,\epsilon + C(\epsilon)\|P_H u_{k'}^{(1)} - u_*\|_{L^2(0,T;V')}. \tag{7.59}$$

Taking the upper limit of (7.59) we find

$$\overline{\lim_{k' \to 0}}\, |P_H u_{k'}^{(1)} - u_*|_{L^2(0,T;H)} \leqslant C\,\epsilon;$$

since ϵ is arbitrarily small, this upper limit is zero, and therefore

$$P_H u_{k'}^{(1)} \to u_* \text{ in } L^2(0, T; H), \text{ as } k' \to 0 \tag{7.60}$$

By comparison of (7.56) and (7.60) we get:

$$u_{k'}^{(1)} \to u_* \text{ in } L^2(0, T; \boldsymbol{L}^2(\Omega)), \text{ (strongly) as } k' \to 0. \tag{7.61}$$

Finally (7.35) and (7.36) imply that

$$u_{k'}^{(2)} \to u_* \text{ in } L^2(0, T; \boldsymbol{L}^2(\Omega)) \text{ (strongly) as } k' \to 0, \tag{7.62}$$

$$u_{k'} \to u_* \text{ in } L^2(0, T; \boldsymbol{L}^2(\Omega)) \text{ (strongly) as } k' \to 0. \tag{7.63}$$

7.1.6. *Proof of Theorems* 7.1 *and* 7.2

Let ψ be any continuously differentiable function on $[0, T]$ with $\psi(T) = 0$. We multiply (7.39) by $\psi(t)$ and, integrate in t. Integrating the first term by parts, we obtain ($\boldsymbol{u}_k(0) = \boldsymbol{u}_0$):

$$-\int_0^T (\boldsymbol{u}_k(t), \psi'(t)\boldsymbol{v})\, \mathrm{d}t + \nu \int_0^T ((\boldsymbol{u}_k^{(1)}(t), \boldsymbol{v}\psi(t)))\, \mathrm{d}t$$

$$+ \int_0^T \hat{b}(\boldsymbol{u}_k^{(1)}(t), \boldsymbol{u}_k^{(1)}(t), \boldsymbol{v}\psi(t))\, \mathrm{d}t = (\boldsymbol{u}_0, \psi(0)\boldsymbol{v})$$

$$+ \int_0^T (\boldsymbol{f}_k(t), \boldsymbol{v}\psi(t))\, \mathrm{d}t, \ \forall\, \boldsymbol{v} \in V.$$

Due to (7.49), (7.51) and (7.52),

$$\int_0^T (\boldsymbol{u}_{k'}(t), \boldsymbol{v}\psi'(t))\, \mathrm{d}t \to \int_0^T (\boldsymbol{u}_*(t), \boldsymbol{v}\psi'(t))\, \mathrm{d}t,$$

$$\int_0^T ((\boldsymbol{u}_{k'}(t), \boldsymbol{v}\psi(t)))\, \mathrm{d}t \to \int_0^T ((\boldsymbol{u}_*(t), \boldsymbol{v}\psi(t)))\, \mathrm{d}t.$$

By (7.16), (7.61), and Lemma 3.2,

$$\int_0^T \hat{b}(\boldsymbol{u}_{k'}^{(1)}(t), \boldsymbol{u}_{k'}^{(1)}(t), \boldsymbol{v}\psi(t))\, \mathrm{d}t \to \int_0^T \hat{b}(\boldsymbol{u}_*(t), \boldsymbol{u}_*(t), \boldsymbol{v}\psi(t))\, \mathrm{d}t.$$

Since $\boldsymbol{u}_*(t) \in V$ a.e., Lemma II.1.3 and (7.16) imply that

$$\hat{b}(\boldsymbol{u}_*(t), \boldsymbol{u}_*(t), \boldsymbol{v}) = b(\boldsymbol{u}_*(t), \boldsymbol{u}_*(t), \boldsymbol{v}), \text{ a.e.}$$

We deduce easily from Lemma 4.9 that

$$\int_0^T (\boldsymbol{f}_k(t), \boldsymbol{v}\psi(t))\, \mathrm{d}t \to \int_0^T (\boldsymbol{f}(t), \boldsymbol{v}\psi(t))\, \mathrm{d}t.$$

Thus we obtain in the limit

$$-\int_0^T (u_*(t), v\psi'(t))\,\mathrm{d}t + \nu \int_0^T ((u_*(t), v\psi(t)))\,\mathrm{d}t$$

$$+\int_0^T b(u_*(t), u_*(t), v\psi(t))\,\mathrm{d}t = (u_0, v\psi(0))$$

$$+\int_0^T (f(t), v\psi(t))\,\mathrm{d}t,\ \forall\, v \in V. \tag{7.64}$$

This equation is the same as (3.43) and from it we conclude, as in Theorem 3.1, that u_* is solution of Problem 3.1.

The same argument can be repeated for any other convergent subsequence of u_k, $u_k^{(i)}$. This completes the proof of Theorem 7.2 ($n = 3$). If $n = 2$, there exists only one solution u of Problem 3.1; hence $u_* = u$ and the sequences $u_k^{(i)}$, u_k converge to u as a whole, in the sense of (7.49)–(7.51). It remains to prove the strong convergence in $L^2(0, T; H_0^1(\Omega))$; this is our goal in next subsection.

7.1.7. *Proof of Theorem* 7.1 *(strong convergence)*

Lemma 7.4. *If* $n = 2$, $u^N = u_k^N \to u(T)$ *in* $L^2(\Omega)$ *weakly, as* $k \to 0$.

Proof. According to (7.24), $|u_k^N| \leqslant$ Const, and thus, the subsequence k' can be chosen so that

$$u_{k'}^N \to \chi \text{ in } H \text{ weakly } (u_k^N \text{ and } \chi \in H). \tag{7.65}$$

By integration of (7.39) we see that

$$(u_k(T), v) = (u_k^N, v) = (u_0, v) - \nu \int_0^T ((u_k^{(1)}(t), v))\,\mathrm{d}t$$

$$-\int_0^T \hat{b}(u_k^{(1)}(t), u_k^{(1)}(t), v)\,\mathrm{d}t$$

$$+\int_0^T (f_k(t), v)\,\mathrm{d}t,\ \forall\, v \in V.$$

It is easy to pass to the limit in this relation with the sequence k'; we find

$$(\chi, v) = (u_0, v) - \nu \int_0^T ((u(t), v))\, dt - \int_0^T \hat{b}(u(t), u(t), v)\, dt + \int_0^T (f(t), v)\, dt.$$

By comparison with the relation (3.13) integrated from 0 to T, it follows that

$$(\chi, v) = (u(T), v), \forall\, v \in V,$$

and since $\chi \in H$,

$$u(T) = \chi.$$

Since the limit is independent of the choice of the subsequence k', the convergence result (7.65) is in fact true for the whole sequence k.

Lemma 7.5. *If $n = 2$, $u_k^{(1)} \to u$ in $L^2(0, T; H_0^1(\Omega))$ strongly.*

Proof. We consider the expression

$$X_k = |u^N - u(T)|^2 + 2\nu \int_0^T \|u_k^{(1)}(t) - u(t)\|^2\, dt.$$

We write

$$X_k = X^{(1)} + X_k^{(2)} + X_k^{(3)},$$

$$X^{(1)} = |u(T)|^2 + 2\nu \int_0^T \|u(t)\|^2\, dt,$$

$$X_k^{(2)} = -2(u^N, u(T)) - 4\nu \int_0^T ((u_k^{(1)}(t), u(t)))\, dt,$$

$$X_k^{(3)} = |u^N|^2 + 2\nu \int_0^T \|u_k^{(1)}(t)\|^2\, dt.$$

The term $X^{(1)}$ is independent of k; we can easily pass to the limit in $X_k^{(2)}$; by (7.49) and Lemma 7.4,

$$X_k^{(2)} \to -2|u(T)|^2 - 4\nu \int_0^T \|u(t)\|^2 \,\mathrm{d}t = -2X^{(1)}.$$

The weak convergence results already proved are not sufficient to pass to the limit in $X_k^{(3)}$. But, by summation of the relations (7.29) and (7.31) for $m = 0, \ldots, N-1$, we can write

$$|u^N|^2 + 2k\nu \sum_{m=1}^{N-1} \|u^{m+1/2}\|^2 +$$

$$\sum_{m=0}^{N-1} \{|u^{m+1} - u^{m+1/2}|^2 + |u^{m+1/2} - u^m|^2\}$$

$$= |u_0|^2 + 2k \sum_{m=0}^{N-1} (f^m, u^{m+1/2})$$

or

$$X_k^{(3)} \leqslant |u_0|^2 + 2\int_0^T (f_k(t), u_k^{(1)}(t))\,\mathrm{d}t.$$

Passing to the upper limit, there results

$$\overline{\lim_{k\to 0}}\ X_k^{(3)} \leqslant |u_0|^2 + 2\int_0^T (f(t), u(t))\,\mathrm{d}t.$$

Due to the energy equality of the exact problem (see (4.55)), the right-hand side of the last inequality is equal to $X^{(1)}$. Hence

$$\overline{\lim_{k\to 0}}\ X_k^{(3)} \leqslant X^{(1)}$$

and combining the different results,

$$\overline{\lim_{k\to 0}}\ X_k \leqslant 0.$$

This shows that $X_k \to 0$, as $k \to 0$, and the proof is complete.

7.2. *A scheme with $n + 1$ intermediate steps*

We describe this scheme in a discrete frame. We consider the approximation of $H_0^1(\Omega)$ by finite differences studied in Chapter I, and the corresponding approximation (APX1) of V. The following can be partially extended to other approximations of the space V but the interest of the Scheme would be considerably lessened since the decomposition method is most powerful in the frame of finite differences methods.

7.2.1. *The decomposition of the operators*

For each $h = (h_1, \dots, h_n)$, $h_i > 0$, we have defined in Chapter I, Subsection 3.3, an approximation W_h of $H_0^1(\Omega)$ and a corresponding approximation V_h of V (finite differences, approximation (APX1)). Both spaces are finite dimensional spaces, equipped with either the scalar product $(\boldsymbol{u}, \boldsymbol{v})$ induced by $\boldsymbol{L}^2(\Omega)$, or with the scalar product

$$((u_h, v_h))_h = \sum_{i=1}^{n} (\delta_{ih} u_h, \delta_{ih} v_h).$$

We define on W_h (and hence on V_h), n other scalar products

$$((u_h, v_h))_{ih} = (\delta_{ih} u_h, \delta_{ih} v_h), \quad 1 \leqslant i \leqslant n. \tag{7.66}$$

Due to the Discrete Poincaré inequality (see Proposition I.3.3),

$$|u_h| \leqslant d_0 \, \|u_h\|_{ih}, \; \forall\, h \in W_h, \; d_0 = 2\ell, \tag{7.67}$$

where ℓ denotes now the maximum of the widths of Ω in the x_i directions. This inequality implies that $\|.\|_{ih}$ is a norm on W_h, and $((.,.))_{ih}$ a Hilbert scalar product.

We now write the trilinear form b_h considered in (6.3)–(6.5) in the form

$$b_h(u_h, v_h, w_h) = \sum_{i=1}^{n} b_{ih}(u_h, v_h, w_h), \tag{7.68}$$

$$b_{ih}(u_h, v_h, w_h) = b'_{ih}(u_h, v_h, w_h) + b''_{ih}(u_h, v_h, w_h), \tag{7.69}$$

$$b'_{ih}(u_h, v_h, w_h) = \frac{1}{2} \sum_{j=1}^{n} \int_\Omega u_{ih} (\delta_{ih} v_{ih})\, w_{jh}\, \mathrm{d}x, \tag{7.70}$$

$$b''_{ih}(u_h, v_h, w_h) = -\frac{1}{2} \sum_{j=1}^{n} \int_\Omega u_{ih} v_{jh} (\delta_{ih} w_{jh})\, \mathrm{d}x. \tag{7.71}$$

All of these forms are obviously defined and trilinear continuous on $W_h \times W_h \times W_h$; it is also clear that

$$b_{ih}(\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{v}_h) = 0, \forall \boldsymbol{u}_h, \ \boldsymbol{v}_h \in W_h. \tag{7.72}$$

7.2.2. *The scheme*

The data f and $\boldsymbol{u}_0$ satisfy (7.10)–(7.11). We assume that we are given a decomposition of f as

$$f = \sum_{i=1}^{n} f_i, f_i \in L^2(0, T; H); \tag{7.73}$$

this decomposition can be quite arbitrary and the simplest choice of the f_i would be $f_1 = f, f_i = 0, i = 2, \ldots, n$.

The interval $[0, T]$ is divided into N intervals of length k $(T = kN)$ and we set

$$f^{m+i/q} = \frac{1}{k} \int_{mk}^{(m+1)k} f_i(t)\,\mathrm{d}t, \ i = 1, \ldots, n, \tag{7.74}$$

where

$$q = n + 1. \tag{7.75}$$

We will now define a family of elements $\boldsymbol{u}_h^{m+i/q}$ of W_h, $m = 0, \ldots, N-1, i = 1, \ldots, q$. These elements are defined successively in the order of increasing values of the fractional index $m + i/q$.

We start with

$$\boldsymbol{u}_h^0 = \textit{the orthogonal projection of } \boldsymbol{u}_0 \textit{ onto } V_h \textit{ in } \boldsymbol{L}^2(\Omega). \tag{7.76}$$

This definition makes sense since $W_h \subset \boldsymbol{L}^2(\Omega)$, and obviously

$$|\boldsymbol{u}_h^0| \leqslant |\boldsymbol{u}_0|, \forall h, \tag{7.77}$$

When $\boldsymbol{u}_k^m$ is known $(m \geqslant 0)$, we define the $\boldsymbol{u}^{m+i/q}$ as follows:

For $1 \leqslant i \leqslant n$, $\boldsymbol{u}_h^{m+i/q} \in W_h$ *and*

$$\begin{aligned}\frac{1}{k}(\boldsymbol{u}^{m+i/q} - \boldsymbol{u}_h^{m+i-1/q}, \boldsymbol{v}_h) &+ \nu((\boldsymbol{u}_h^{m+i/q}, \boldsymbol{v}_h))_{ih} \\ &+ b_{ih}(\boldsymbol{u}_h^{m+i-1/q}, \boldsymbol{u}_h^{m+i/q}, \boldsymbol{v}_h) \\ &= (f^{m+i/q}, \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in W_h\end{aligned} \tag{7.78}$$

For $i = n + 1$ $(= q)$, $u_h^{m+1} \in V_h$ *and,*

$$(u_h^{m+1}, v_h) = (u_h^{m+n/q}, v_h), \ \forall \, v_h \in V_h. \tag{7.79}$$

Relation (7.79) means that u_h^{m+1} is the orthogonal projection of $u_h^{m+n/q}$ on V_h in the space W_h equipped with the scalar product (.,.); u_h^{m+1} is well defined by (7.79).

Equation (7.78) is linear in $u_h^{m+i/q}$; the existence and uniqueness of $u_h^{m+i/q}$ is assured by the Projection Theorem (Theorem I.2.2). Indeed, the form

$$\{u_h, v_h\} \to \frac{1}{k}(u_h, v_h) + \nu((u_h, v_h))_{ih} + b_h(u_h^{m+i-1/q}, u_h, v_h) \tag{7.80}$$

is bilinear continuous on W_h, and the form

$$v_h \to \frac{1}{k}(u_h^{m+i-1/q}, v_h) + (f^{m+i/q}, v_h),$$

is linear and continuous; the coercivity of the form (7.80) is a consequence of (7.72).

Remark 7.3. (*Interpretation of* (7.79)). We give an interpretation of (7.79) which will enable us to introduce an approximation of the pressure.

We proceed as in Subsection 3.3; the element $u_h^{m+1} - u_h^{m+n/q}$ of W_h is orthogonal to V_h (scalar product (.,.)). This amounts to saying that

$$(u_h^{m+1} - u_h^{m+n/q}, v_h) = 0,$$

when $v_h \in W_h$ and (characterization of V_h)

$$\sum_{i=1}^{n} \nabla_i v_{ih}(M) = 0, \ \forall M \in \mathring{\Omega}_h^1. \tag{7.81}$$

By a classical result of linear algebra, there exists a family of numbers λ_M, $M \in \mathring{\Omega}_h^1$ such that

$$(u_h^{m+1} - u_h^{m+n/q}, v_h) = \sum_{M \in \mathring{\Omega}_h^1} \lambda_M \left\{ \sum_{i=1}^{n} \nabla_{ih} v_{ih}(M) \right\},$$

$$\forall \, v_h \in W_h. \tag{7.82}$$

If π_h^{m+1} denotes the step function of X_h given by

$$\pi_h^{m+1} = \sum_{M \in \overset{\sigma}{\Omega}{}_h^1} \frac{k\,\lambda_M}{h_1 \dots h_n}\, w_{hM},\text{[1]} \tag{7.83}$$

relation (7.82) can then be interpreted as,[2]

$$\frac{1}{k}(u_h^{m+1} - u_h^{m+n/q}, v_h) - (\pi_h^{m+1}, D_h v_h) = 0,$$

$$\forall v_h \in W_h,\text{[3]} \tag{7.84}$$

or

$$\frac{1}{k}(u_{ih}^{m+1}(M) - u_{ih}^{m+n/q}(M)) + \overline{\nabla}_{ih}\, \pi_h^{m+1}(M) = 0,$$

$$\forall M \in \mathring{\Omega}_h^1. \tag{7.85}$$

Remark 7.4. (*Resolution of* (7.78)). Relations (7.78) are equivalent to the relations

$$\frac{1}{k}(u_h^{m+i/q}(M) - u_h^{m+i-1/q}(M)) - \nu\, \delta_{ih}^2\, u_h^{m+i/q}(M)$$

$$+ \frac{1}{2} u_{ih}^{m+i-1/q}(M)\, \delta_{ih}\, u_h^{m+i/q}(M)$$

$$+ \frac{1}{2} \delta_{ih}(u_{ih}^{m+i-1/q}\, u_h^{m+i/q})(M)$$

$$= f_h^{m+i/q}(M) = \frac{1}{h_1 \dots h_n} \int_{\sigma_h(M)} f^{m+i/q}(x)\, dx,$$

$$\forall M \in \mathring{\Omega}_h^1.$$

---

(1) w_{hM} is the characteristic function of the block $\sigma_h(M)$.

(2) Compare to I (3.71)–(3.72).

(3) $D_h v_h(x) = \sum_{i=1}^{n} \nabla_{ih}\, v_{ih}(x)$.

The unknowns, when we compute $u_h^{m+i/q}$, are the components of $u_h^{m+i/q}(M)$; the crux of the fractional step method is that the above system is actually uncoupled into several subsystems involving only the unknowns $u_h^{m+i/q}(M)$ where the M are all on the same line parallel to the x_i direction. This makes the resolution of (7.78) very easy.

Remark 7.5. (*Resolution of* (7.79)). As in Remark 7.2, we can interpret (7.79) as the discretization of a Neumann problem for the pressure π^{m+1} (with a boundary condition different from (7.23)):

$$\left.\begin{aligned} \Delta\pi^{m+1} &= \frac{1}{k}\operatorname{div} u^{m+n/q}, \\ \frac{\partial\pi^{m+1}}{\partial\nu} &= \frac{1}{k}\gamma_\nu u^{m+n/q}. \end{aligned}\right\} \tag{7.86}$$

This allows us to solve (7.78) by instead solving a discrete Neumann problem for π_h^{m+1}. For this procedure, see A.J. Chorin [3], M. Fortin [1].

An alternate procedure for solving (7.79) would be an iterative algorithm of the types considered in Chapter I, Section 5, and in this Chapter, Subsection 6.3. The situation here is very similar and we omit the details. /

7.2.3. *Unconditional a priori estimates*

We will establish two types of *a priori* estimates: unconditional *a priori* estimates and *a priori* estimates which are obtained by assuming that k and h satisfy some conditions similar to a stability condition. This subsection deals with unconditional *a priori* estimates; the conditional ones will be studied in Subsection 7.3.3, after the development of some preliminary tools in Subsection 7.3.1 and 7.3.2.

Lemma 7.6. *The elements* $u_h^{m+i/q}$ *remain bounded in the following sense:*

$$|u_h^{m+i/q}|^2 \leqslant d_2,\quad m = 0,\dots,N-1,\ i = 1,\dots,q, \tag{7.87}$$

$$k\sum_{m=0}^{N-1} \|u_h^{m+i/q}\|^2 \leqslant \frac{d_2}{\nu},\quad i = 1,\dots,n(=q-1) \tag{7.88}$$

$$\sum_{m=0}^{N} |u_h^{m+i/q} - u_h^{m+i-1/q}|^2 \leqslant d_2,\ i = 1,\dots,q, \tag{7.89}$$

where

$$d_2 = |\boldsymbol{u}_0|^2 + \sum_{i=1}^{n} \int_0^T |f_i(s)|^2 \, \mathrm{d}s. \tag{7.90}$$

Proof. We write (7.78) with $\boldsymbol{v}_h = \boldsymbol{u}_h^{m+i/q}$; using (7.72) we find:

$$\begin{aligned} |\boldsymbol{u}_h^{m+i/q}|^2 & - |\boldsymbol{u}_h^{m+i-1/q}|^2 + |\boldsymbol{u}_h^{m+i/q} - \boldsymbol{u}_h^{m+i-1/q}|^2 \\ & + 2k\nu \, \|\boldsymbol{u}_h^{m+i/q}\|_{ih}^2 = 2k(\boldsymbol{f}^{m+i/q}, \boldsymbol{u}_h^{m+i/q}) \\ & \leqslant 2k|\boldsymbol{f}^{m+i/q}| \, |\boldsymbol{u}_h^{m+i/q}| \leqslant \text{(by (7.67))} \\ & \leqslant 2k \, d_0 |\boldsymbol{f}^{m+i/q}| \, \|\boldsymbol{u}_h^{m+i/q}\|_{ih} \\ & \leqslant k\nu \|\boldsymbol{u}_h^{m+i/q}\|_{ih}^2 + \frac{k \, d_0^2}{\nu} |\boldsymbol{f}^{m+i/q}|^2. \end{aligned} \tag{7.91}$$

Finally, we obtain

$$\begin{aligned} |\boldsymbol{u}_h^{m+i/q}|^2 & - |\boldsymbol{u}_h^{m+i-1/q}|^2 + |\boldsymbol{u}_h^{m+i/q} - \boldsymbol{u}_h^{m+i-1/q}|^2 \\ & + k\nu \|\boldsymbol{u}_h^{m+i/q}\|_{ih}^2 \leqslant k \frac{d_0^2}{\nu} |\boldsymbol{f}^{m+i/q}|^2, \quad 1 \leqslant i \leqslant 1 \end{aligned} \tag{7.92}$$

Writing (7.79) with $\boldsymbol{v} = \boldsymbol{u}_h^{m+1}$, we get

$$|\boldsymbol{u}_h^{m+1}|^2 + |\boldsymbol{u}_h^{m+n/q}|^2 + |\boldsymbol{u}_h^{m+1} - \boldsymbol{u}_h^{m+n/q}|^2 = 0. \tag{7.93}$$

We add all the relations (7.92) and (7.93) for $i = 1, \ldots, n$, $m = 0, \ldots, N-1$. We find after some simplification

$$\begin{aligned} |\boldsymbol{u}_h^N|^2 & + \sum_{i=1}^{q} \sum_{m=0}^{N-1} |\boldsymbol{u}_h^{m+i/q} - \boldsymbol{u}_h^{m+i-1/q}|^2 \\ & + k\nu \sum_{i=1}^{q-1} \sum_{m=0}^{N-1} \|\boldsymbol{u}_h^{m+i/q}\|_{ih}^2 \leqslant |\boldsymbol{u}_h^0|^2 \\ & + \frac{k \, d_0^2}{\nu} \sum_{i=1}^{q-1} \sum_{m=0}^{N-1} |\boldsymbol{f}^{m+i/q}|^2 \end{aligned} \tag{7.94}$$

By (7.77), $|\boldsymbol{u}_h^0| \leqslant |\boldsymbol{u}_0|$, and because of (5.29),

$$k \sum_{m=0}^{N-1} |\boldsymbol{f}^{m+i/q}|^2 \leqslant \int_0^T |f_i(s)|^2 \, \mathrm{d}s.$$

Therefore, the right-hand side of (7.94) is less than or equal to d_2 and (7.88) and (7.89) are proved.

For r and j fixed, $0 \leqslant r \leqslant N-1$, $1 \leqslant j \leqslant q$, we add the relations (7.92) and (7.93) for $m = 0, \ldots, r-1$, $i = 1, \ldots, q$ and for $m = r$, $1 \leqslant i \leqslant j$;[1] dropping several positive terms, we find

$$|u^{r+j/q}|^2 \leqslant |u_h^0|^2 + kd_0^2\nu^{-1} \sum_{\substack{m,i \\ 0 \leqslant m+i/q \leqslant r+j/q}} |f^{m+i/q}|^2$$

$$\leqslant |u_h^0|^2 + \frac{k\,d_0^2}{\nu} \sum_{i=1}^{q-1} \sum_{m=0}^{N-1} f^{m+i/q}|^2 \leqslant d_2,$$

$r = 0, \ldots, N-1,\ j = 1, \ldots, q.$

The proof of the lemma is complete.

7.2.4. *The Stability Theorem.*

We introduce the approximate functions $u_h^{(i)}$, $i = 1, \ldots, q$, and u_h:

$$u_h^{(i)}\colon\ [0, T] \to W_h,$$

$$u_h^{(i)}(t) = u_h^{m+i/q} \text{ for } mk \leqslant t < (m+1)k,\ i = 1, \ldots, q, \tag{7.95}$$

u_h *is a continuous function from* $[0, T]$ *into* W_h, *linear on each interval* $[mk, (m+1)k]$, $m = 0, \ldots, N-1$, *and* $u_h(mk) = u_h^m$, $m = 0, \ldots, N$. (7.96)

We infer from Lemma 7.6 the following stability theorem.

Theorem 7.3. *The functions* $u_h^{(i)}$, *and* u_h, *defined by* (7.95), (7.96) *are unconditionally* $L^\infty(0, T; L^2(\Omega))$ *stable* $(1 \leqslant i \leqslant q)$. *The functions* $\delta_{ih}\, u_h^{(i)}$ $(1 \leqslant i \leqslant n)$ *are unconditionally* $L^2(0, T; L^2(\Omega))$ *stable.*

Remark 7.6. (i) As a consequence of (7.89), we have

$$|u_h^{(i)} - u_h^{(i-1)}|_{L^2(0,T;L^2(\Omega))} \leqslant \sqrt{k\,d_2},\ i = 2, \ldots, n. \tag{7.97}$$

(ii) As in Lemma 7.3,

$$|u_h^{(q)} - u_h|_{L^2(0,T;L^2(\Omega))} \leqslant \sqrt{\frac{kd_2}{2}}\,. \tag{7.98}$$

[1] We are summing in m and i for $0 \leqslant m + i/q \leqslant r + j/q$.

Indeed, using the computations of Lemma 4.8, we see that

$$|u_h^{(q)} - u_h|^2_{L^2(0,T;L^2(\Omega))} = \frac{k}{2} \sum_{m=0}^{N-1} |u_h^{m+1} - u_h^m|^2$$

$$\leqslant \frac{k}{2} \sum_{m=0}^{N-1} \left(\sum_{i=1}^{q} |u_h^{m+i/q} - u_h^{m+i-1/q}| \right)^2$$

$$\leqslant \frac{kq}{2} \sum_{m=0}^{N-1} \sum_{i=1}^{q} |u_h^{m+i/q} - u_h^{m+i-1/q}|^2$$

$$\leqslant \frac{kq}{2} \mathrm{d}_2 .$$

7.3. *Convergence of the Scheme*

We want to prove the convergence of the last scheme, (7.78) and (7.79). First we must establish some further *a priori* estimates.

7.3.1. *Auxiliary Results.*

We denote by A_{ih} $(1 \leqslant i \leqslant n)$ the linear operator from W_h into W_h defined by

$$(A_{ih}\, u_h, v_h) = ((u_h, v_h))_{ih} \quad \forall u_h, v_h \in W_h . \tag{7.99}$$

Also we denote by B_{ih} the bilinear continuous operator from $W_h \times W_h$ into W_h defined by

$$(B_{ih}(u_h, v_h), w_h) = b_{ih}(u_h, v_h, w_h), \ \forall u_h, v_h, w_h \in W_h \tag{7.100}$$

In terms of these operators, relations (7.78) can be written as

$$\frac{1}{k}(u_h^{m+i/q} - u_h^{m+i-1/q}) + \nu A_{ih}\, u_h^{m+i/q}$$
$$+ B_{ih}(u_h^{m+i-1/q}, u_h^{m+i/q}) = f_h^{m+i/q} . \tag{7.101}$$

Lemma 7.7.

$$\|u_h\|_{ih} \leqslant S_i(h)|u_h|, \ \forall u_h \in W_h , \tag{7.102}$$

where

$$S_i(h) = \frac{2}{h_i} \; (1 \leqslant i \leqslant n). \tag{7.103}$$

Proof. This is essentially proved in Proposition 6.1.[1]

Lemma 7.8.

$$|A_{ih}\, u_h| \leqslant S_i(h) \|u_h\|_{ih}, \; \forall u_h \in W_h. \tag{7.104}$$

Proof. Due to (7.99) and (7.102),

$$\begin{aligned} |(A_{ih}\, u_h, v_h)| = |((u_h, v_h))_{ih}| &\leqslant \|u_h\|_{ih} \|v_h\|_{ih} \\ &\leqslant S_i(h) \|u_h\|_{ih} |v_h|, \; \forall u_h, v_h \in W_h, \end{aligned}$$

and (7.104) follows.

7.3.2. *Estimates for the form* b_{ih}.
Lemma 7.9. *If* $n = 2$,

$$|b'_{ih}(u_h, v_h, w_h)|$$
$$\leqslant \sqrt{3}\, |u_h|^{1/2} \|u_h\|_{jh}^{1/2} \|v_h\|_{ih} |w_h|^{1/2} \|w_h\|_{ih}^{1/2}, \tag{7.105}$$
$$|b''_{ih}(u_h, v_h, w_h)|$$
$$\leqslant \sqrt{3}\, |u_h|^{1/2} \|u_h\|_{jh}^{1/2} |v_h|^{1/2} \|v_h\|_{ih}^{1/2} \|w_h\|_{ih}, \tag{7.106}$$
$$|B_{ih}(u_h, v_h)|$$
$$\leqslant 2\sqrt{3} S_i(h) |u_h|^{1/2} \|u_h\|_{jh}^{1/2} |v_h|^{1/2} \|v_h\|_{ih}^{1/2}, \tag{7.107}$$

where $\{i, j\}$ *is a permutation of the set* $\{1, 2\}$.

Proof. We prove (7.105) for $i = 1, j = 2$. In order to simplify the notation, we drop the indices h and we set

$$u = \{u_1, u_2\}, \; v = \{v_1, v_2\}, \; w = \{w_1, w_2\}.$$

By the definition of b_{ih},

$$b_{1h}(u, v, w) = \frac{1}{2} \sum_{\ell=1}^{2} \int_\Omega u_1 (\delta_{1h} v_\ell) w_\ell \, dx,$$

---

[1] Consider a fixed value of j in the proof of Proposition 6.1.

By the Schwarz inequality

$$\left|\int_\Omega u_1(\delta_{1h}v_\ell)w_\ell\,\mathrm{d}x\right| = \left|\int_{\mathcal{R}^2} u_1(\delta_{1h}v_\ell)w_\ell\,\mathrm{d}x\right|$$

$$\leqslant \left\{\int_{\mathcal{R}^2} |\delta_{1h}v_\ell|^2\,\mathrm{d}x\right\}^{1/2}$$

$$\cdot\left\{\int_{\mathcal{R}^2} |u_1 w_\ell|^2\,\mathrm{d}x\right\}^{1/2};$$

$$\int_{\mathcal{R}^2} |u_1 w_\ell|^2\,\mathrm{d}x \leqslant \int_{\mathcal{R}^2} \{\operatorname*{Sup}_{\xi_2}|u_1(x_1,\xi_2)|^2\}\,\{\operatorname*{Sup}_{\xi_2}|w_\ell(\xi_1,x_2)|^2\}\,\mathrm{d}x.$$

The integrations in x_1 and x_2 are independent and thus

$$\int_{\mathcal{R}^2} |u_1 w_\ell|^2\,\mathrm{d}x \leqslant \left\{\int_{-\infty}^{+\infty} (\operatorname*{Sup}_{\xi_2}|u_1(x_1,\xi_2)|^2)\,\mathrm{d}x_1\right\}\cdot$$

$$\left\{\int_{-\infty}^{+\infty} (\operatorname*{Sup}_{\xi_1}|w_\ell(\xi_1,x_2)|^2)\,\mathrm{d}x_2\right\}.$$

Due to II.(2.15),

$$\operatorname*{Sup}_{\xi_2}|u_1(x_1,\xi_2)|^2$$

$$\leqslant 2\int_{-\infty}^{+\infty} |\delta_{2h}u_1(x_1,\xi_2)|\cdot\left(\sum_{\alpha=1}^{1}\left|u_1\left(x_1,\xi_2+\frac{\alpha h_2}{2}\right)\right|\right)\mathrm{d}\xi_2,$$

$$\int_{-\infty}^{+\infty} (\operatorname*{Sup}_{\xi_2}(u_1(x_1,\xi_2)|^2)\,\mathrm{d}x_1 \leqslant 2.3^{1/2}|\delta_{2h}u_1|\,|u_1|.$$

For the same reasons,

$$\int_{-\infty}^{+\infty} (\operatorname*{Sup}_{\xi_2}|w_\ell(\xi_1,x_2)|^2)\mathrm{d}x_2 \leqslant 2.3^{1/2}|\delta_{1h}w_\ell|\,|w_\ell|,$$

and we can write

$$\int_{\mathbb{R}^2} |u_1 w_\ell|^2 \, dx \leqslant 12|u_1| \, |\delta_{2h} u_1| \, |w_\ell| \, |\delta_{1h} w_\ell|,$$

$$|b'_{1h}(u, v, w)| \leqslant \sqrt{3} \sum_{\ell=1}^{2} |u_1|^{1/2} |\delta_{2h} u_1|^{1/2} |\delta_{1h} v_\ell| |w_\ell|^{1/2}$$
$$|\delta_{1h} w_\ell|^{1/2}.$$

Finally, using the Schwarz inequality again, we estimate the right-hand side of the last relation by

$$\sqrt{3} \, |u_1|^{1/2} |\delta_{2h} u_1|^{1/2} \left(\sum_{\ell=1}^{2} |\delta_{1h} v_\ell|^2 \right)^{1/2} \cdot$$
$$\left(\sum_{\ell=1}^{2} |w_\ell| \, |\delta_{1h} w_\ell| \right) \leqslant \sqrt{3} \, |u_1|^{1/2} |\delta_{2h} u_1|^{1/2} \, \|v\|_{1h} |w|^{1/2} \, \|w\|_{1h}^{1/2}$$
$$\leqslant \sqrt{3} \, |u|^{1/2} \, \|u\|_{2h}^{1/2} \, \|v\|_{1h} |w|^{1/2} \, \|w\|_{1h}^{1/2},$$

and (7.105) follows.

In order to establish (7.106) we simply observe that

$$b''_{ih}(u_h, v_h, w_h) = b'_{ih}(u_h, v_h, w_h)$$

and apply (7.105). For (7.107) we observe that

$$|b_{ih}(u_h, v_h, w_h)|$$
$$\leqslant 2\sqrt{3} \, S_i(h) |u_h|^{1/2} \, \|u_h\|_{jh}^{1/2} |v_h|^{1/2} \, \|v_h\|_{ih}^{1/2} |w_h|, \ \forall u_h, v_h, w_h \in W_h.$$

Lemma 7.10. *If $n = 3$,*

$$|b'_{ih}(u_h, v_h, w_h)| \leqslant 3^{3/2} |u_h|^{1/4} \, \|u_h\|_h^{3/4} \, \|v_h\|_{ih} \, |w_h|^{1/4} \, \|w_h\|_h^{3/4} \tag{7.108}$$

$$|b''_{ih}(u_h, v_h, w_h)| \leqslant 3^{3/2} |u_h|^{1/4} \, \|u_h\|_h^{3/4} |v_h|^{1/4} \, \|v_h\|_h^{3/4} \, \|w_h\|_{ih} \tag{7.109}$$

$$|B_{ih}(u_h, v_h)| \leqslant 3^{3/2} |u_h|^{1/4} \, \|u_h\|_h^{3/4} \, \{S^{3/4}(h) \, \|v_h\|_{ih} + S_i(h) |v_h|^{1/4} \, \|v_h\|_h^{3/4}\} . \tag{7.110}$$

Proof. The proof of (7.108) and (7.109) is the same as the proof of Lemma 6.1; (7.110) is an easy consequence of (7.108) and (7.109).

7.3.3. *Conditional a priori estimates.*

Lemma 7.11. *We assume that $n = 2$ and that k and h satisfy*

$$k\, S(h)^2 \leqslant M, \tag{7.111}$$

where M is fixed, and arbitrarily large.

Then, we have

$$k \sum_{m=0}^{N-1} \|u_h^{m+i/3}\|_h^2 \leqslant \text{Const.},\ i = 1, 2, 3, \tag{7.112}$$

with a constant depending on M and the data.

Proof. We start with the form (7.101) of the scheme. Here $q = 3$ and, for $i = 2$, (7.101) becomes

$$u_h^{m+1/3} = u_h^{m+2/3} + k\nu\, A_{2h} u_h^{m+2/3} + kB_{2h}(u_h^{m+1/3}, u_h^{m+2/3}) - k\, f_h^{m+2/3},$$

and, taking the norms $\|\cdot\|_{2h}$ of each side,

$$\|u_h^{m+1/3}\|_{2h} \leqslant \|u_h^{m+2/3}\|_{2h} + k\nu \|A_{2h} u_h^{m+2/3}\|_{2h} + k\|B_{2h}(u_h^{m+1/3}, u_h^{m+2/3})\|_{2h} + k\|f_h^{m+2/3}\|_h.$$

By virtue of (7.102), (7.104), and (7.107), the right-hand side of this inequality is bounded by

$$\|u_h^{m+1/3}\|_{2h} + k\nu\, S_2(h)^2 \|u_h^{m+2/3}\|_{2h} + 2k\sqrt{3}\, S_2(h)^2 |u_h^{m+1/3}|^{1/2} \|u_h^{m+1/3}\|_{1h}^{1/2} |u_h^{m+2/3}|^{1/2} \cdot \|u_h^{m+2/3}\|_{2h}^{1/2} + kS_2(h)|f^{m+2/3}|$$

By the Schwarz inequality,

$$\|u_h^{m+1/3}\|_{2h}^2 \leqslant 4\|u_h^{m+2/3}\|_{2h}^2 + 4k^2\, \nu^2\, S_2(h)^4\, \|u_h^{m+2/3}\|_{2h}^2 + 48\, k^2\, S_2(h)^4 |u_h^{m+1/3}|\, \|u_h^{m+1/3}\|_{1h} |u_h^{m+2/3}|\, \|u_h^{m+2/3}\|_{2h} + 4\, k^2\, S_2(h)^2 |f^{m+2/3}|^2.$$

Because of the previous estimates of Lemma 7.6 and (7.111), the sum from $m = 0$ to $N - 1$ of the right side of the last inequality is bounded by a constant time k^{-1} and (7.112) is proved for $i = 1$.

In order to prove the same result for $i = 2$, write (7.101) for $i = 2$ ($q = 3$) as

$$u_h^{m+2/3} = u_h^{m+1/3} - k\nu A_{2h}\, u_h^{m+2/3} \\ - k\, B_{2h}(u_h^{m+1/3}, u_h^{m+2/3}) - k\, f_h^{m+2/3},$$

take the $\|\cdot\|_{1h}$ norm of each side, and proceed as before.

To prove (7.112) when $i = 3$, we write (7.101) for $i = 1$ ($q = 3$) as

$$u_h^m = u_h^{m+1/3} - k\nu A_{1h}\, u_h^{m+1/3} - k\, B_{1h}(u_h^m, u_h^{m+1/3}) \\ - k\, f_h^{m+1/3},$$

and then take successively the norms $\|\cdot\|_{1h}$ and $\|\cdot\|_{2h}$ of each side, and proceed essentially as before.

Lemma 7.12. *We assume that $n = 3$ and that k and h satisfy*

$$k\, S(h)^{11/4} \leqslant M, \tag{7.113}$$

where M is fixed and arbitrarily large.

Then we have

$$k \sum_{m=0}^{N-1} \|u_h^{m+1/3}\|_h^2 \leqslant \text{Const.}, \quad i = 1, 2, 3, 4, \tag{7.114}$$

with a constant depending only on M and the data.

Proof. The same as the proof of Lemma 7.11. The estimates (7.107) of B_{ih} are replaced by the cruder estimates (7.110) and for this reason (7.111) is replaced by the stronger relation (7.113).

From these lemmas we infer a conditional stability theorem.

Theorem 7.4. *Assuming that k and h satisfy the stability condition* (7.111) (*if $n = 2$*) *or* (7.113) (*if $n = 3$*)*, all of the functions*

$$\delta_{jh} u_h^{(i)}, \ \delta_{jh} u_h, \ i = 1, \ldots, n+1, \ j = 1, \ldots, n,$$

are $L^2(0, T; L^2(\Omega))$ stable.

7.3.4. *The convergence theorems.*

Theorem 7.5. *Assuming that the dimension is $n = 2$ and that k and h re-*

main tied by (7.111) *the following convergence results hold as k and h go to zero:*

$$u_h^{(i)}, \boldsymbol{u}_h, \text{ converge to } \boldsymbol{u} \text{ in } L^2(Q) \text{ strongly, } L^\infty(0,T;L^2(\Omega)) \text{ weak-star, } i=1,2,3, \tag{7.115}$$

$$\delta_{jh}\boldsymbol{u}_h^{(i)}, \delta_{jh}\boldsymbol{u}_h, \text{ converge to } D_j\boldsymbol{u} \text{ in } L^2(Q) \text{ weakly, } i=1,2,3, j=1,2, \tag{7.116}$$

$$\delta_{ih}\boldsymbol{u}_h^{(i)} \text{ converge to } D_i\boldsymbol{u} \text{ in } L^2(Q) \text{ strongly, } i=1,2, \tag{7.117}$$

where $\boldsymbol{u}$ *is the unique solution of Problem* 3.1 *corresponding to the data* $f, \boldsymbol{u}_0$, *in* (7.10), (7.11).

Theorem 7.6. *Assuming that the dimension is* $n = 3$, *there exists a sequence* h', k' *converging to zero* [1], *such that*

$$\boldsymbol{u}_{h'}^{(i)}, \boldsymbol{u}_{h'}, \text{ converge to } \boldsymbol{u} \text{ in } L^2(Q) \text{ strongly, } L^\infty(0,T;L^2(\Omega)) \text{ weak-star, } \tag{7.118}$$

$$\delta_{jh'}\boldsymbol{u}_{h'}^{(i)}, \delta_{jh'}\boldsymbol{u}_{h'}, \text{ converge to } D_j\boldsymbol{u} \text{ in } L^2(Q) \text{ weakly, } i=1,\ldots,4,\ j=1,2,3, \text{ where } \boldsymbol{u} \text{ is some solution of Problem 3.1.} \tag{7.119}$$

The principle of the proof of these theorems is very similar to those of Theorems 7.1, 7.2, 5.4 and 5.5 and we will describe only the main lines of the proof.

7.3.5. *Proof of convergence.*

Lemma 7.13. *Under the conditions* (7.111) (*if* $n = 2$) *or* (7.113) (*if* $n = 3$),

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\,|\hat{\boldsymbol{u}}_h(\tau)|^2\, d\tau \leqslant \text{Const.}, \textit{ for some } 0<\gamma<\frac{1}{4},$$

where $\hat{\boldsymbol{u}}_h$ *is the Fourier transform in t of the function* $\boldsymbol{u}_h$ *extended by* 0 *outside the interval* $[0, T]$; *the constant depends on* γ, *M and the data.*

Proof. We add (7.79) and the relations (7.78) for $i = 1, \ldots, n$; this gives the result

---

[1] h' and k' satisfying (7.113).

$$\frac{1}{k}(u_h^{m+1} - u_h^m, v_h) + \sum_{i=1}^{n} ((u_h^{m+i/q}, v_h))_{ih}$$

$$+ \sum_{i=1}^{n} b_{ih}(u_h^{m+i-1/q}, u_h^{m+i/q}, v_h)$$

$$= \sum_{i=1}^{n} (f^{m+i/q}, v_h), \ \forall\, v_h \in V_h,$$

which is equivalent (see (7.95)–(7.96)) to

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_h(t), v_h) + \sum_{i=1}^{n} ((u_h^{(i)}(t), v_h))_{ih}$$

$$+ \sum_{i=1}^{n} b_{ih}(u_h^{(i-1)}(t), u_h^{(i)}(t), v_h)$$

$$= (f_{ih}(t), v_h), \ \forall\, t \in (0, T), \ \forall v_h \in V_h. \tag{7.120}$$

Using the previous *a priori* estimates, we then repeat the proof of Lemma 5.6.

Proof of Theorems 7.5 and 7.6. By virtue of Theorem 7.3, there exists a sequence $h', k' \to 0$, such that

$$u_{h'}^{(i)} \to u^{(i)} \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak star}, i = 1, \ldots, q, \tag{7.121}$$

$$u_{h'} \to u_* \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak-star}. \tag{7.122}$$

Due to Remark 7.6, $u_{h'}^{(i)} - u_{h'}^{(i-1)}$ converges to 0 in $L^2(Q)$ strongly, $i = 2, \ldots, q$, and $u_{h'} - u_{h'}^{(q)}$ also converges to zero; hence all the limits are the same:

$$u^{(1)} = \ldots = u^{(q)} = u_*. \tag{7.123}$$

Our goal is to show that u_* is a solution of Problem 3.1.

According to Theorem 7.4, the sequence $h', k' \to 0$, can be chosen in such a way that the following convergence results also hold:

$$\delta_{jh'} u_{h'}^{(i)}, \ \delta_{jh'} u_{h'} \to D_j u_*, \text{ in } L^2(Q) \text{ weakly}. \tag{7.124}$$

By Lemma 7.13 and the property (5.92), which was proved in Subsection 6.1.3 for finite differences,

$$u_{h'} \to u_* \text{ in } L^2(Q) \text{ strongly.} \tag{7.125}$$

Using again Remark 7.6, we find that

$$u_{h'}^{(i)} \to u_* \text{ in } L^2(Q) \text{ strongly}, i = 1, \dots, q. \tag{7.126}$$

The convergence results (7.121), (7.122) and (7.124) to (7.126) allow us to pass to the limit in (7.120), as we did in the proofs of Theorems 5.4, 5.5, 7.1 and 7.2. We find that u_* is a solution of Problem 3.1.

If $n = 2$, the solution of Problem 3.1 is unique; thus $u_* = u$, and the above convergence results hold for the whole sequence $k, h \to 0$.

The proof of the strong convergence (7.117) ($n = 2$) follows that of Lemmas 5.11 and 7.5, showing that the expression

$$X_h = |u_h^N - u(T)|^2 + 2\nu \sum_{i=1}^{2} \int_0^T |D_i u(t) - \delta_{ih} u_h^{(i)}(t)|^2 \, dt$$

converges to zero, as $h, k \to 0$.

Remark 7.7. The scheme (7.78)–(7.79) is the analogue, discretized in the space variables, of the following scheme:

$$\frac{1}{k}(u^{m+i/q} - u^{m+i-1/q}) - \nu D_i^2 u^{m+i/q} + \sum_{j=1}^{n} u_j^{m+i-1/q}(D_j u^{m+i/q}) + \frac{1}{2}(\operatorname{div} u^{m+i-1/q})u^{m+i/q} = f^{m+i/q}, \quad i = 1, \dots, n \tag{7.127}$$

$$u^{m+1} \in H \text{ and } (u^{m+1}, v) = (u^{m+n/q}, v), \ \forall v \in H.$$

Equations (7.127) are associated with appropriate boundary conditions: $u^{m+i/q}$ vanishes on some part of Γ which depends on i; more precisely (see Temam [1]):

$$u^{m+i/q} \cos(\nu, X_i) = 0 \text{ on } \Gamma.$$

The elements $u^{m+i/q}$ satisfy *a priori* estimates similar to those proved in Lemma 7.6 for the $u_h^{m+i/q}$; however, due to a lack of *a priori* estimates similar to those established in Lemmas 7.11 and 7.12, we are not able to prove the convergence of this semi-discretized scheme. In the frame of full discretization in both space and time variables, the difficulty is overcome by requiring that k and h satisfy some stability conditions [(7.111), (7.113)]. These stability conditions allow us to obtain enough additional *a priori* estimates to pass to the limit.

§8. Approximation of the Navier–Stokes equations by the Artificial Compressibility Method

In this section we study the numerical approximation of the Navier–Stokes equations by the Artificial Compressibility Method. This is another method to overcome the computational difficulties connected with the constraint "div $u = 0$". We introduce a family of perturbed systems (depending on a positive parameter ϵ) which approximates in the limit the Navier–Stokes equations and which do not contain this constraint. One of the most common perturbed systems is essentially the equations of a slightly compressible medium with an artificial state equation

$$\rho = \rho_0 + \epsilon p, \quad \text{"}\epsilon > 0 \text{ small"}, \tag{8.1}$$

where ρ is the density, p the pressure, and ρ_0 a constant which represents a first approximation of the density.[1] Linearizing the equations of motion with respect to ϵ, we obtain as a first approximation, the equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta \boldsymbol{u} + \sum_{i=1}^{n} u_i D_i \boldsymbol{u} + \operatorname{grad} p = f, \tag{8.2}$$

$$\epsilon \frac{\partial p}{\partial t} + \operatorname{div} \boldsymbol{u} = 0. \tag{8.3}$$

The equations (8.2)–(8.3) are the perturbed equations we shall study. They are easier to approximate than the original Navier–Stokes equations as the constraint "div $u = 0$" has been replaced by the evolution

(1) In all of the preceding sections we always took $\rho_0 = 1$ for simplification. If $\rho_0 \neq 1$, we arrive at the same result by dividing the equations of motion by ρ_0.

equation (8.3). The problems are now the following:

– existence and uniqueness of solutions of the perturbed equations (8.2)–(8.3) (associated with appropriate initial and boundary conditions)

– does the solution $\boldsymbol{u}_\epsilon$, p_ϵ, of (8.2)–(8.3) approximate the solution $\boldsymbol{u}$, p, of Navier–Stokes equations?

– discretization of the perturbed problem; convergence of the discrete approximations to the solution of the Navier–Stokes equations themselves.

In Subsection 8.1 we state in a more precise way a boundary value problem associated with (8.2)–8.3) and give existence and uniqueness theorems for these equations ($\epsilon > 0$ fixed). The situation is essentially the same as for the Navier–Stokes equations: existence and uniqueness of weak solutions in the two-dimensional case, and existence of weak solutions in the three dimensional case[(1)]. In subsection 8.2 we show how the solutions of the perturbed problems converge to the solutions of the Navier–Stokes equations as $\epsilon \to 0$. Subsection 8.3 deals with numerical approximation of the perturbed equations and many numerical methods are available for their approximation. We do not intend to give a systematic study of the different methods, but choose instead to study in detail the approximation of the perturbed equations by the fractional step method. We obtain an implicit scheme, unconditionally stable in some spaces. Finally, we study the convergence of the discrete approximation to the solution of the Navier–Stokes equations as ϵ, h and k tend to zero.

8.1. *Study of the perturbed problems*

8.1.1. *Description of the problem*

The dimension of the space is $n = 2$ or 3 and Ω is bounded. We assume that $\boldsymbol{u}_0$ is given as in Problem 3.1,

$$\boldsymbol{u}_0 \in H, \tag{8.4}$$

and, for simplicity, f is assumed to be in $L^2(0, T; H)$

$$f \in L^2(0, T; H). \tag{8.5}$$

For any given $\epsilon > 0$, we consider the following initial boundary value problem:

To find $\boldsymbol{u}_\epsilon = \{u_{1\epsilon}, \ldots, u_{n\epsilon}\}$, a vector function from $Q = \Omega \times (0, T)$ into $\mathbb{R}^n$ and p_ϵ, a scalar function from Q into $\mathbb{R}$, such that:

---

(1) We do not study existence and behaviour of strong solutions.

$$\frac{\partial u_\epsilon}{\partial t} - \nu\Delta u_\epsilon + \sum_{i=1}^{n} u_{i\epsilon} D_i u_\epsilon + \frac{1}{2}(\operatorname{div} u_\epsilon)u_\epsilon$$

$$+ \operatorname{grad} p_\epsilon = f \text{ in } Q, \tag{8.6}$$

$$\epsilon \frac{\partial p_\epsilon}{\partial t} + \operatorname{div} u_\epsilon = 0 \text{ in } Q, \tag{8.7}$$

$$u_\epsilon = 0, x \in \partial\Omega, t \in (0, T), \tag{8.8}$$

$$u_\epsilon = u_0 \text{ at } t = 0, \tag{8.9}$$

$$p_\epsilon = p_0 \text{ at } t = 0. \tag{8.10}$$

The function p_0, which is not given with Problem 3.1, is arbitrarily chosen (but independent of ϵ),

$$p_0 \in L^2(\Omega). \tag{8.11}$$

Equation (8.6) contains the term $1/2(\operatorname{div} u_\epsilon)u_\epsilon$ which does not appear in either (8.2) or the exact problem (see 3.5)). This is a stabilization term, already used several times in previous sections, which corresponds to the substitution of the form $\hat{b}$ for the form b [1].

Let us assume that u_ϵ and p_ϵ are classical solutions of (8.6) – (8. 0), say $u_\epsilon \in \mathscr{C}^2(\bar{Q})$, $p_\epsilon \in \mathscr{C}^1(\bar{Q})$. Then if $v \in \mathscr{D}(\Omega)$ and $q \in \mathscr{D}(\Omega)$, multiplying (8.6) by v, (8.7) by q, and integrating over Ω, we easily obtain the relations

$$\frac{d}{dt}(u_\epsilon, v) + \nu((u_\epsilon, v)) + \hat{b}(u_\epsilon, u_\epsilon, v) + (\operatorname{grad} p_\epsilon, v) = (f, v)$$

$$\epsilon \frac{d}{dt}(p_\epsilon, q) + (\operatorname{div} u_\epsilon, q) = 0.$$

These relations are still valid, by a continuity argument, for any v in $H_0^1(\Omega)$ and any q in $L^2(\Omega)$.

This remark leads to a first formulation of the problem:

Problem 8.1. *For $\epsilon > 0$ fixed and f, u_0, p_0 given, satisfying* (8.4), (8.5), *and* (8.11), *to find u_ϵ and p_ϵ such that*

$$u_\epsilon \in L^2(0, T; H_0^1(\Omega)), \ p_\epsilon \in L^2(0, T; L^2(\Omega)), \tag{8.12}$$

[1] We recall that $b(u, u, u) \neq 0$ if $\operatorname{div} u \neq 0$. For this reason we introduced the form $\hat{b}$: $\hat{b}(u, v, w) = \hat{b}(u, v, w)$ if $\operatorname{div} u = 0$, and $\hat{b}(u, v, v) = 0$, $\forall\, u, v$. This allows us to obtain perturbed problems which admit solutions for all time.

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_\epsilon, v) + \nu((u_\epsilon, v)) + \hat{b}(u_\epsilon, u_\epsilon, v) + (\operatorname{grad} p_\epsilon, v)$$
$$= (f, v),\ \forall\, v \in H_0^1(\Omega), \tag{8.13}$$

$$\epsilon \frac{\mathrm{d}}{\mathrm{d}t}(p_\epsilon, q) + (\operatorname{div} u_\epsilon, q) = 0,\ \ \forall\, q \in L^2(\Omega), \tag{8.14}$$

$$u_\epsilon(0) = u_0,\ \ p_\epsilon(0) = p_0. \tag{8.15}$$

Remark 8.1. If u_ϵ and p_ϵ only satisfy (8.12) the conditions (8.15) do not necessarily make sense. We will show, as for the exact Navier–Stokes equations, that if u_ϵ, p_ϵ satisfy (8.12) – (8.14), then u_ϵ and p_ϵ are continuous (into large enough spaces) so that (8.15) makes sense. □

For $u, v \in H_0^1(\Omega)$, we define $\hat{B}(u, v) \in H^{-1}(\Omega)$ and $\hat{B}(u) \in H^{-1}(\Omega)$ bu setting

$$\langle \hat{B}(u, v), w\rangle = \hat{b}(u, v, w),\ \forall\, u, v, w \in H_0^1(\Omega) \tag{8.16}$$
$$\hat{B}(u) = \hat{B}(u, u). \tag{8.17}$$

Lemma 8.1. *If u belongs to $L^2(0, T; H_0^1(\Omega))$, the function $t \mapsto \hat{B}(u(t))$, belongs to $L^1(0, T; H^{-1}(\Omega))$.*

Proof. We already observed that the form $\hat{b}$ given by

$$\hat{b}(u, v, w) = \frac{1}{2}\{b(u, v, w) - b(u, w, v)\} \tag{8.18}$$

is, like b, trilinear continuous on $H_0^1(\Omega)$; thus

$$|\hat{b}(u, v, w)| \leqslant d_1 \|u\|\, \|v\|\, \|w\|,\ \ \forall\, u, v, w \in H_0^1(\Omega)$$
$$\|\hat{B}(u, v)\|_{H^{-1}(\Omega)} \leqslant d_1 \|u\|\, \|v\|,$$
$$\|\hat{B}(u)\|_{H^{-1}(\Omega)} \leqslant d_1 \|u\|^2. \tag{8.19}$$

The lemma follows from (8.19). □

Now if u_ϵ satisfies (8.12) and (8.13) then, according to (1.6) and (1.8), one can write (8.13) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u_\epsilon, v\rangle = \langle f + \nu\Delta u_\epsilon - \hat{B}(u_\epsilon), v\rangle,\ \forall\, v \in V.$$

It is clear that $\Delta u_\epsilon \in L^2(0, T; H^{-1}(\Omega))$, $\hat{B}(u_\epsilon) \in L^1(0, T; H^{-1}(\Omega))$, so that $f + \nu\Delta u_\epsilon - \hat{B}(u_\epsilon)$ is in $L^1(0, T; H^{-1}(\Omega))$ and thus due to Lemma 1.1,

$$\left.\begin{aligned} & u_\epsilon' \in L^1(0, T; H^{-1}(\Omega)) \\ & u_\epsilon' = f + \nu\Delta u_\epsilon - \hat{B}(u_\epsilon). \end{aligned}\right\} \tag{8.20}$$

Similarly, (8.12), (8.14) and Lemma 1.1 imply that

$$\left.\begin{aligned} & p_\epsilon' \in L^2(0, T; H^{-1}(\Omega)), \\ & \epsilon \frac{\partial p_\epsilon}{\partial t} + \operatorname{div} u_\epsilon = 0. \end{aligned}\right\} \tag{8.21}$$

An alternative formulation of Problem 8.1 is now the following:

Problem 8.2. *For $\epsilon > 0$ fixed, f, u_0, p_0 given satisfying* (8.4), (8.5) *and* (8.11), *to find u_ϵ and p_ϵ,*

$$u_\epsilon \in L^2(0, T; H_0^1(\Omega)),\ u_\epsilon' \in L^1(0, T; H^{-1}(\Omega)), \tag{8.22}$$

$$p_\epsilon \in L^2(0, T; L^2(\Omega)),\ p_\epsilon' \in L^2(0, T; H^{-1}(\Omega)), \tag{8.23}$$

$$u_\epsilon' - \nu\Delta u_\epsilon + \hat{B}(u_\epsilon) + \operatorname{grad} p_\epsilon = f, \tag{8.24}$$

$$\epsilon p_\epsilon' + \operatorname{div} u_\epsilon = 0, \tag{8.25}$$

$$u_\epsilon(0) = u_0, \qquad p_\epsilon(0) = p_0. \tag{8.26}$$

We showed that any solution of Problem 8.1 is a solution of Problem 8.2; the converse is very easy to check (using Lemma 1.1) and thus these problems are equivalent.

Our goal now is to study the existence and uniqueness of solutions of these problems for $\epsilon > 0$ fixed; then we will see how the solutions of these problems approximate those of Problems 3.1 and 3.2.

8.1.2. *Existence of solutions of the perturbed problems*

Theorem 8.1. *For $\epsilon > 0$ fixed, for f, u_0, p_0, given satisfying* (8.4), (8.5) *and* (8.11), *there exists at least one solution $\{u_\epsilon, p_\epsilon\}$ of Problems* 8.1 *and* 8.2. *Moreover,*

$$u_\epsilon \in L^\infty(0, T; L^2)),\ p_\epsilon \in L^\infty(0, T; L^2(\Omega)), \tag{8.27}$$

and u_ϵ (resp. p_ϵ) is weakly continuous from $[0, T]$ *into $L^2(\Omega)$ (resp. $L^2(\Omega)$).*

The existence is established in the next subsection; the weak continuity follows from Lemma 8.1 and the weak continuity of $\boldsymbol{u}_\epsilon$ and p_ϵ with values in $\boldsymbol{H}^{-1}(\Omega)$ and $H^{-1}(\Omega)$ has already been proved.

8.1.3. *Proof of Theorem 8.1.*

(i) In order to apply the Galerkin procedure, we consider a basis of $\boldsymbol{H}_0^1(\Omega)$ constituted of elements $\boldsymbol{w}_i$ of $\mathscr{D}(\Omega)$, and a basis of $L^2(\Omega)$ constituted of elements r_i of $\mathscr{D}(\Omega)$.

For each m, we define an approximate solution $\boldsymbol{u}_{\epsilon m}, p_{\epsilon m}$, of Problem 8.1 by

$$\boldsymbol{u}_{\epsilon m}(t) = \sum_{i=1}^{m} g_{im}(t)\,\boldsymbol{w}_i,\ \ p_{\epsilon m}(t) = \sum_{j=1}^{m} \xi_{jm}(t)\,r_j, \tag{8.28}$$

and

$$(\boldsymbol{u}_{\epsilon m}'(t), \boldsymbol{w}_k) + \nu((\boldsymbol{u}_{\epsilon m}(t), \boldsymbol{w}_k)) + \hat{b}(\boldsymbol{u}_{\epsilon m}(t), \boldsymbol{u}_{\epsilon m}(t), \boldsymbol{w}_k)$$
$$+ (\operatorname{grad} p_{\epsilon m}(t), \boldsymbol{w}_k), = (\boldsymbol{f}(t), \boldsymbol{w}_k),\ k = 1, \ldots, m, \tag{8.29}$$

$$\epsilon(p_{\epsilon m}'(t), r_\ell) + (\operatorname{div} \boldsymbol{u}_{\epsilon m}(t), r_\ell) = 0,\ \ell = 1, \ldots, m. \tag{8.30}$$

Moreover, this differential system is required to satisfy the initial conditions

$$\boldsymbol{u}_{\epsilon m}(0) = \boldsymbol{u}_{0m},\ p_{\epsilon m}(0) = p_{0m}, \tag{8.31}$$

where $\boldsymbol{u}_{0m}$ (or p_{0m}) is the orthogonal projection of $\boldsymbol{u}_0$ (or p_0) onto the space spanned by $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ (or $r_1, \ldots, r_m$), in $\boldsymbol{L}^2(\Omega)$ (resp. $L^2(\Omega)$).

The equations (8.29) and (8.30) form a non-linear differential system for the functions $g_{1m}, \ldots, g_{mm}, \xi_{1m}, \ldots, \xi_{mm}$. As for Theorem 3.1, we have the existence of a solution defined at least on some interval $[0, t_m)$, $0 < t_m \leqslant T$, and the following *a priori* estimates show that in fact $t_m = T$.

(ii) If we multiply (8.29) by $g_{km}(t)$, multiply (8.30) by $\xi_{\ell m}(t)$, and then add all these equations for $k = 1, \ldots, m$, $\ell = 1, \ldots, m$, there results

$$(\boldsymbol{u}_{\epsilon m}', \boldsymbol{u}_{\epsilon m}) + \nu\|\boldsymbol{u}_{\epsilon m}\|^2 + \hat{b}(\boldsymbol{u}_{\epsilon m}, \boldsymbol{u}_{\epsilon m}, \boldsymbol{u}_{\epsilon m}) + (\operatorname{grad} p_{\epsilon m}, \boldsymbol{u}_{\epsilon m})$$
$$+ \epsilon(p_{\epsilon m}', p_{\epsilon m}) + (\operatorname{div} \boldsymbol{u}_{\epsilon m}, p_{\epsilon m}) = (\boldsymbol{f}, \boldsymbol{u}_{\epsilon m}).$$

Due to (8.18), $\hat{b}(\boldsymbol{u}_{\epsilon m}, \boldsymbol{u}_{\epsilon m}, \boldsymbol{u}_{\epsilon m}) = 0$, and since $\boldsymbol{u}_{\epsilon m}$ vanishes on $\partial\Omega$,

$$(\operatorname{grad} p_{\epsilon m}, \boldsymbol{u}_{\epsilon m}) + (p_{\epsilon m}, \operatorname{div} \boldsymbol{u}_{\epsilon m}) = 0.$$

Thus there remains

$$\frac{\mathrm{d}}{\mathrm{d}t}\{|u_{\epsilon m}|^2+\epsilon|p_{\epsilon m}|^2\}+2\nu\|u_{\epsilon m}\|^2=2(f,u_{\epsilon m}). \tag{8.32}$$

The right side is bounded by

$$2|f|\,|u_{\epsilon m}|\leqslant 2d_0|f|\,\|u_{\epsilon m}\|\leqslant \nu\|u_{\epsilon m}\|^2+\frac{d_0^2}{\nu}|f|^2$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\{|u_{\epsilon m}|^2+\epsilon|p_{\epsilon m}|^2\}+\nu\|u_{\epsilon m}\|^2\leqslant\frac{d_0^2}{\nu}|f|^2. \tag{8.33}$$

Integration of (8.33) form 0 to s shows that

$$|u_{\epsilon m}(s)|^2+\epsilon|p_{\epsilon m}(s)|^2\leqslant|u_{0m}|^2+\epsilon|p_{0m}|^2+\frac{d_0^2}{\nu}\int_0^s|f(t)|^2\,\mathrm{d}t$$

$$\leqslant|u_0|^2+\epsilon|p_0|^2+\frac{d_0^2}{\nu}\int_0^T|f(t)|^2\,\mathrm{d}t,\ 0<s<t_m.$$

Hence $t_m=T$, and

$$\operatorname*{Sup}_{s\in[0,T]}\{|u_{\epsilon m}(s)|^2+\epsilon|p_{\epsilon m}(s)|^2\}\leqslant d_3, \tag{8.34}$$

$$d_3=|u_0|^2+|p_0|^2+\frac{d_0^2}{\nu}\int_0^T|f(t)|^2\,\mathrm{d}t.\ ^{(1)} \tag{8.35}$$

Next, we integrate (8.33) from 0 to T:

$$|u_{\epsilon m}(T)|^2+\epsilon|p_{\epsilon m}(T)|^2+\nu\int_0^T\|u_{\epsilon m}(t)\|^2\,\mathrm{d}t\leqslant|u_{0m}|^2+\epsilon|p_{0m}|^2$$

$$+\frac{d_0^2}{\nu}\int_0^T|f(t)|^2\,\mathrm{d}t\leqslant|u_0|^2+\epsilon|p_0|^2+\frac{d_0^2}{\nu}\int_0^T|f(t)|^2\,\mathrm{d}t\leqslant d_3$$

(1) We are interested in small values of ϵ; thus we assume ϵ bounded from above, say $\epsilon \leqslant 1$.

Thus

$$\int_0^T \|u_{\epsilon m}(t)\|^2\, \mathrm{d}t \leqslant \frac{d_3}{\nu}. \tag{8.36}$$

(iii) In order to pass to the limit in the nonlinear term we need an estimate of the fractional derivative in time of $u_{\epsilon m}$.

Setting

$$\phi_m(t) = f(t) + \nu\Delta u_{\epsilon m} - \hat{B}(u_{\epsilon m}),$$

it follows from (8.36) and (8.19) that

$$\|\phi_m(t)\|_{H^{-1}(\Omega)} \leqslant |f(t)| + \nu\|u_{\epsilon m}(t)\| + d_1\|u_{\epsilon m}(t)\|^2 \tag{8.37}$$

and therefore

$$\phi_m \textit{ remains in a bounded set of } L^1(0, T; H^{-1}(\Omega)). \tag{8.38}$$

The relations (8.29) and (8.30) can be written

$$(u'_{\epsilon m}(t), w_k) + (\operatorname{grad} p_{\epsilon m}(t), w_k) = (\phi_m(t), w_k),\ k = 1, \ldots, m,$$

$$\epsilon(p'_{\epsilon m}(t), r_\ell) + (\operatorname{div} u_{\epsilon m}(t), r_\ell) = 0,\ \ell = 1, \ldots, m.$$

As done several times before, we extend all functions by 0 outside the interval $[0, T]$ and consider the Fourier transform of the different equations.

The following relations then hold on $\mathcal{R}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}_{\epsilon m}, w_k) + (\operatorname{grad}\tilde{p}_{\epsilon m}, w_k) = \langle\tilde{\phi}_m, w_k\rangle + (u_{0m}, w_k)\,\delta_{(0)}$$
$$- (u_{\epsilon m}(T), w_k)\,\delta_{(T)}$$

$$\epsilon\,\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{p}_{\epsilon m}, r_\ell) + (\operatorname{div}\tilde{u}_{\epsilon m}, r_\ell) = \epsilon(p_{0m}, r_\ell)\,\delta_{(0)}$$
$$- \epsilon(p_{\epsilon m}(T), r_\ell)\,\delta_{(T)}.$$

After taking Fourier transforms, there results

$$2i\pi\tau(\hat{u}_{\epsilon m}(\tau), w_k) + (\operatorname{grad}\hat{p}_{\epsilon m}(\tau), w_k) = \langle\hat{\phi}_m(\tau), w_k\rangle$$
$$+ (u_{0m}, w_k) - (u_{\epsilon m}(T), w_k)\exp(-2i\pi\tau T),$$

$$2i\pi\tau\epsilon(\hat{p}_{\epsilon m}(\tau), r_\ell) + (\operatorname{div}\hat{u}_{\epsilon m}(\tau), r_\ell) = \epsilon(p_{0m}, r_\ell)$$
$$- \epsilon(p_{\epsilon m}(T), r_\ell)\exp(-2i\pi\tau T).$$

We multiply the first of the last two equations by $\hat{g}_{km}(\tau)$ ($\hat{g}_{km}$ = Fourier transform of $\tilde{g}_{km}$), and the second by $\hat{\xi}_{km}(\tau)$ ($\hat{\xi}_{\ell m}$ = Fourier transform of $\tilde{\xi}_{\ell m}$), and then add these relations for $k = 1, \ldots, m$, $\ell = 1, \ldots, m$, obtaining

$$\begin{aligned}2i\pi\tau\{|\hat{u}_{\epsilon m}(\tau)|^2 + \epsilon|\hat{p}_{\epsilon m}(\tau)|^2\} &+ (\operatorname{grad}\hat{p}_{\epsilon m}(\tau), \hat{u}_{\epsilon m}(\tau))\\ &+ (\operatorname{div}\hat{u}_{\epsilon m}(\tau), \hat{p}_{\epsilon m}(\tau)) = \langle\hat{\phi}_m(\tau), \hat{u}_{\epsilon m}(\tau)\rangle\\ &+ (u_{0m}, \hat{u}_{\epsilon m}(\tau)) + \epsilon(p_{0m}, \hat{p}_{\epsilon m}(\tau))\\ &- \{(u_{\epsilon m}(T), u_{\epsilon m}(\tau)) + \epsilon(p_{\epsilon m}(T), p_{\epsilon m}(\tau))\}\\ &\quad \exp(-2i\pi T\tau). \end{aligned} \tag{8.39}$$

The term

$$(\operatorname{grad}\hat{p}_{\epsilon m}, \hat{u}_{\epsilon m}) + (\operatorname{div} u_{\epsilon m}, \hat{p}_{\epsilon m})$$

vanishes. With this simplification, we deduce from (8.39) that

$$\begin{aligned}2\pi|\tau|\{|\hat{u}_{\epsilon m}(\tau)|^2 + \epsilon|\hat{p}_{\epsilon m}(\tau)|^2\} &\leqslant |\langle\hat{\phi}_m(\tau), \hat{u}_{\epsilon m}(\tau)\rangle|\\ &+ |u_{0m}|\,|\hat{u}_{\epsilon m}(\tau)| + \epsilon|p_{0m}|\,|\hat{p}_{\epsilon m}(\tau)|\\ &+ |u_{\epsilon m}(T)|\,|\hat{u}_{\epsilon m}(\tau)| + \epsilon|p_{\epsilon m}(T)|\,|\hat{p}_{\epsilon m}(\tau)|\end{aligned}$$

Due to the previous estimates (8.34) and (8.36),

$$\begin{aligned}2\pi|\tau|\,|\hat{u}_{\epsilon m}(\tau)|^2 &\leqslant \|\hat{\phi}_m(\tau)\|_{H^{-1}(\Omega)}\|\hat{u}_{\epsilon m}(\tau)\|\\ &+ 2\sqrt{d_3}\,|\hat{u}_m(\tau)| + 2\sqrt{d_3}\,\epsilon|\hat{p}_{\epsilon m}(\tau)|.\end{aligned}$$

But

$$\begin{aligned}\|\hat{\phi}_m(\tau)\|_{H^{-1}(\Omega)} &\leqslant \int_{-\infty}^{+\infty} \|\tilde{\phi}_m(t)\|_{H^{-1}(\Omega)}\,dt \leqslant (\text{by } (8.37))\\ &\leqslant \int_0^T \{|f(t)| + \nu\|u_{\epsilon m}(t)\| + d_0\|u_{\epsilon m}(t)\|^2\}dt\\ &\leqslant (\text{by } (8.38)) \leqslant \text{Const.}\,,\end{aligned}$$

$$\begin{aligned}\epsilon|\hat{p}_{\epsilon m}(\tau)| &\leqslant \epsilon\int_{-\infty}^{+\infty}|\tilde{p}_{\epsilon m}(t)|\,dt = \epsilon\int_0^T |p_{\epsilon m}(t)|\,dt\\ &\leqslant (\text{by } (8.34)) \leqslant \sqrt{\epsilon}\,\sqrt{d_3}\,T \leqslant \text{Const.}\,;\end{aligned}$$

finally we have

$$2\pi|\tau|\,|\hat{u}_{\epsilon m}(\tau)|^2 \leqslant c_1 \|\hat{u}_{\epsilon m}(\tau)\| + c_2. \tag{8.40}$$

As in the proof of Theorem 2.2, this inequality implies that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{u}_{\epsilon m}(\tau)|^2 \,\mathrm{d}\tau \leqslant \text{Const., for some } \gamma,\ 0<\gamma<\frac{1}{4}. \tag{8.41}$$

(iv) We want to pass to the limit as $m \to \infty$ in (8.29) – (8.31) using the estimates (8.34), (8.36) and (8.41). We recall that at the present time $\epsilon > 0$ is fixed, and we are only concerned with a passage to the limit as $m \to \infty$.

There exists a sequence $m' \to \infty$, such that

$$u_{\epsilon m'} \to u_\epsilon \text{ in } L^2(0, T; H_0^1(\Omega)) \text{ weakly,} \tag{8.42}$$

$L^\infty(0, T; L^2(\Omega))$ weak-star, and (due to (8.41) and Theorem 2.1), $L^2(0, T; L^2(\Omega))$ strongly;

$$p_{\epsilon m'} \to p_\epsilon \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak-star.} \tag{8.43}$$

Let ψ be a continuously differentiable scalar function on $[0, T]$, with $\psi(T) = 0$. We multiply (8.29) (resp. (8.30)) by $\psi(t)$, integrate over $[0, T]$, and then integrate the first term by parts:

$$\begin{aligned} -\int_0^T (u_{\epsilon m}(t), w_k\psi'(t))\,\mathrm{d}t &+ \int_0^T \nu((u_{\epsilon m}(t), w_k\psi(t)))\,\mathrm{d}t \\ + \int_0^T \{\hat{b}\,(u_{\epsilon m}(t), &\,u_{\epsilon m}(t), w_k\psi(t)) \\ &+ (\operatorname{grad} p_{\epsilon m}(t), w_k\psi(t))\}\,\mathrm{d}t \\ &= (u_{0m}, w_k)\psi(0) + \int_0^T (f(t), w_k\psi(t))\,\mathrm{d}t, \end{aligned} \tag{8.44}$$

$$\begin{aligned} -\int_0^T \epsilon(p_{\epsilon m}(t), r_\ell\psi'(t))\,\mathrm{d}t &+ \int_0^T (\operatorname{div} u_{\epsilon m}(t), r_\ell\psi(t))\,\mathrm{d}t \\ = \epsilon(p_{0m}, r_\ell)\psi(0),\quad 1 \leqslant k,\ \ell \leqslant m. \end{aligned} \tag{8.45}$$

It is easy to pass to the limit in (8.44) and (8.45) with the sequence m' and we find

$$
\begin{aligned}
&-\int_0^T (u_\epsilon(t), w_k\psi'(t)) + \nu\int_0^T ((u_\epsilon(t), w_k\psi(t)))\,\mathrm{d}t \\
&+\int_0^T \{\hat{b}\,(u_\epsilon(t), u_\epsilon(t), w_k\psi(0)) + (\operatorname{grad} p_\epsilon(t), w_k\psi(t))\}\mathrm{d}t \\
&= (u_0, w_k)\psi(0) + \int_0^T (f(t), w_k\psi(t))\,\mathrm{d}t,
\end{aligned}
\tag{8.46}
$$

$$
\begin{aligned}
&-\epsilon\int_0^T (p_\epsilon(t), r_\ell\psi'(t))\,\mathrm{d}t + \int_0^T (\operatorname{div} u_\epsilon(t), r_\ell\psi(t))\,\mathrm{d}t \\
&\qquad = \epsilon(p_0, r_\ell)\psi(0),\quad 1 \leqslant k,\ \ell \leqslant m.
\end{aligned}
\tag{8.47}
$$

The relations (8.46) and (8.47) imply that $\{u_\epsilon, p_\epsilon\}$ are solutions of Problem 8.1: the proof is identical to the interpretation of (3.43).

The existence of solutions of Problem 8.1 and 8.2 is thereby proved and the proof of Theorem 8.1 is complete.

Remark 8.2. It is useful, in view of passing to the limit $\epsilon \to 0$, to establish some *a priori* estimates, *independent* of ϵ, satisfied by u_ϵ, p_ϵ.

Due to (8.34), (8.36), (8.42) and the lower semi-continuity of the norm for the weak topology:

$$
|u_\epsilon|_{L^\infty(0,T;L^2(\Omega))} \leqslant \varliminf_{m'\to\infty} |u_{\epsilon m'}|_{L^\infty(0,T;L^2(\Omega))} \leqslant \sqrt{d_3}, \tag{8.48}
$$

$$
\|u_\epsilon\|_{L^2(0,T;H_0^1(\Omega))} \leqslant \varliminf_{m'\to\infty} \|u_{\epsilon m'}\|_{L^2(0,T;H_0^1(\Omega))} \sqrt{\frac{d_3}{\nu}}. \tag{8.49}
$$

Due to (8.34) and (8.43)

$$
\sqrt{\epsilon}\ |p_\epsilon|_{L^\infty(0,T;L^2(\Omega))} \leqslant \varliminf_{m'\to\infty} \sqrt{\epsilon}\ |p_{\epsilon m'}|_{L^\infty(0,T;L^2(\Omega))} \leqslant \sqrt{d_3}. \tag{8.50}
$$

Similarly, by inspection of the proof of (8.40) and (8.41), we observe that the constants appearing in these relations are independent of ϵ (and of course m). Thus, by (8.42), we obtain

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{u}_\epsilon(\tau)|^2 \, dt \leqslant \varliminf_{m'\to\infty} \int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{u}_{\epsilon m'}(\tau)|^2 \, d\tau \leqslant \text{Const}; \tag{8.51}$$

with a constant independent of ϵ, and for $0 < \gamma < \dfrac{1}{4}$.

8.1.4. *Uniqueness of Solutions of the perturbed problems.*

Theorem 8.2. *We assume that $n = 2$ and otherwise the assumptions are the same as those of Theorem* 8.1.

There exists a unique solution $\{u_\epsilon, p_\epsilon\}$ of Problems 8.1 *and* 8.2 *which belongs to $L^\infty(0, T; L^2(\Omega)) \times L^\infty(0, T; L^2(\Omega))$, and u_ϵ is a continuous function from $[0, T]$ into $L^2(\Omega)$.*

The theorem follows from several lemmas, some of them giving additional regularity properties of u_ϵ.

Lemma 8.2. *If $n = 2$, $u \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, then*

$$u \in L^4(0, T; L^4(\Omega)), \tag{8.52}$$

$$\hat{B}\, u \in L^{4/3}(0, T; L^{4/3}(\Omega)). \tag{8.53}$$

Proof. Property (8.52) follows from Lemma 3.3 which gives the estimate

$$\|u(t)\|_{L^4(\Omega)}^4 \leqslant c\, |u(t)|^2 \, \|u(t)\|^2, \text{ a.e. in } t. \tag{8.54}$$

For (8.53) we observe first that

$$\hat{b}(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i (D_i v_j) w_j \, dx + \frac{1}{2} \sum_{j=1}^{2} \int_\Omega (\operatorname{div} u)\, v_j w_j \, dx. \tag{8.55}$$

The relation (8.55) is obvious if $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathscr{D}(\Omega)$; it is still valid by continuity for $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{H}_0^1(\Omega)$. In fact, due to the Hölder inequality,

$$|\hat{b}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant \sum_{i,j=1}^{2} \|u_i\|_{L^4(\Omega)} |D_i v_j|_{L^2(\Omega)} |w_j|_{L^4(\Omega)}$$

$$+ \frac{1}{2} \sum_{j=1}^{2} |\operatorname{div} \boldsymbol{u}|_{L^2(\Omega)} |v_j|_{L^4(\Omega)} |w_j|_{L^4(\Omega)}$$

and therefore

$$|\hat{b}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w})| \leqslant c \|\boldsymbol{u}\|_{L^4(\Omega)} \|\boldsymbol{u}\| \, \|\boldsymbol{w}\|_{L^4(\Omega)} \leqslant \text{(because of (8.54))}$$
$$\leqslant c |\boldsymbol{u}|^{1/2} \|\boldsymbol{u}\|^{3/2} \|\boldsymbol{w}\|_{L^4(\Omega)}, \ \forall \ \boldsymbol{u} \in H_0^1(\Omega), \ \forall \ \boldsymbol{w} \in L^4(\Omega)$$

Since the space $\boldsymbol{L}^{4/3}(\Omega)$ is the dual of $\boldsymbol{L}^4(\Omega)$, it follows that

$$\|\hat{\boldsymbol{B}}\boldsymbol{u}\|_{L^{4/3}(\Omega)} \leqslant c |\boldsymbol{u}|^{1/2} \|\boldsymbol{u}\|^{3/2}, \ \forall \ \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega) \tag{8.57}$$

and (8.53) is easily deduced from this estimate.

Lemma 8.3. *If* $\boldsymbol{u}_\epsilon$ *is a solution of Problem* 8.1 *belonging to* $L^\infty(0, T; \boldsymbol{L}^2(\Omega))$, *and* $n = 2$, *then:*

$$\boldsymbol{u}_\epsilon \in L^2(0, T; \boldsymbol{H}_0^1(\Omega) \cap L^4(0, T; \boldsymbol{L}^4(\Omega)) \tag{8.58}$$

$$\boldsymbol{u}_\epsilon' \in L^2(0, T; \boldsymbol{H}^{-1}(\Omega)) + L^{4/3}(0, T; \boldsymbol{L}^{4/3}(\Omega)). \tag{8.59}$$

Proof. This is merely a consequence of Lemma 8.2; (8.58) is equivalent to (8.52) and for (8.59) we use (8.24):

$$\boldsymbol{u}_\epsilon' = \boldsymbol{f} + \nu \Delta \boldsymbol{u}_\epsilon - \operatorname{grad} p_\epsilon - \hat{\boldsymbol{B}}\boldsymbol{u}_\epsilon.$$

The first three terms on the right are in $L^2(0, T; \boldsymbol{H}^{-1}(\Omega))$; the last one belongs to $L^{4/3}(0, T; \boldsymbol{L}^{4/3}(\Omega))$ (by (8.53)).

Lemma 8.4. *For any function* $\boldsymbol{u}_\epsilon$ *satisfying* (8.58) (8.59),

$$2 \langle \boldsymbol{u}_\epsilon'(t), \boldsymbol{u}_\epsilon(t) \rangle = \frac{\mathrm{d}}{\mathrm{d}t} |\boldsymbol{u}_\epsilon(t)|^2, \tag{8.60}$$

in the distribution sense on $(0, T)$.

Moreover, $\boldsymbol{u}_\epsilon$ *is almost everywhere equal to a continuous function from* $[0, T]$ *into* $\boldsymbol{L}^2(\Omega)$.

Proof. The same as the proof of Lemma 1.2, observing that the spaces in (8.58) and (8.59) are in duality.

Proof of Theorem 8.2. It remains to prove the uniqueness. For this we drop the indices ϵ and denote by $\{\boldsymbol{u}_1, p_1\}$, $\{\boldsymbol{u}_2, p_2\}$ two solutions of Problem 8.1, and then set

$$\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2, \ p = p_1 - p_2.$$

By subtracting the relations (8.24) (resp. (8.25)) satisfied by $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ (resp. p_1 and p_2), we see that

$$\boldsymbol{u}' - \nu\Delta\boldsymbol{u} + \operatorname{grad} p = \hat{B}\boldsymbol{u}_2 - \hat{B}\boldsymbol{u}_1$$

$$\epsilon p' + \operatorname{div} \boldsymbol{u} = 0. \tag{8.61}$$

Taking the scalar product of (8.60) with $\boldsymbol{u}(t)$ and (8.61) with $p(\mathrm{t})$ and adding these equations, we find

$$\begin{aligned}\langle \boldsymbol{u}'(t), \boldsymbol{u}(t)\rangle + \epsilon(p'(t), p(t)) + \nu\|\boldsymbol{u}(t)\|^2 \\ = -\hat{b}(\boldsymbol{u}(t), \boldsymbol{u}_2(t), \boldsymbol{u}(t));\end{aligned} \tag{8.62}$$

the term

$$(\operatorname{grad} p, \boldsymbol{u}) + (p, \operatorname{div} \boldsymbol{u})$$

has disappeared, and we have used the property of $\hat{b}$:

$$\hat{b}(v, w, w) = 0 \quad \forall\, v, w.$$

Thanks to Lemma 8.4 we can write (8.62) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\{|\boldsymbol{u}(t)|^2 + \epsilon|p(t)|^2\} + 2\nu\|\boldsymbol{u}(t)\|^2 = -2\hat{b}(\boldsymbol{u}(t), \boldsymbol{u}_2(t), \boldsymbol{u}(t)).$$

The inequalities established in the proof of Lemma 8.2 enable us to estimate the right side of this equation by

$$\begin{aligned}c_3|\boldsymbol{u}(t)|\ \|\boldsymbol{u}(t)\|\ \|\boldsymbol{u}_2(t)\| + c_4|\boldsymbol{u}(t)\|^{1/2}\|\boldsymbol{u}(t)\|^{3/2}|\boldsymbol{u}_2(t)|^{1/2}\|\boldsymbol{u}_2(t)\| \\ \leqslant \nu\|\boldsymbol{u}(t)\|^2 + c_5\|\boldsymbol{u}_2(t)\|^2|\boldsymbol{u}(t)|^2 + \nu\|\boldsymbol{u}(t)\|^2 \\ + c_6|\boldsymbol{u}_2(t)|^2\|\boldsymbol{u}_2(t)\|^2|\boldsymbol{u}(t)|^2.\end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\{|\boldsymbol{u}(t)|^2 + \epsilon|p(t)|^2\} \leqslant \sigma(t)|\boldsymbol{u}(t)|^2. \tag{8.63}$$

where σ is a scalar function in $L^1(0, T)$; more precisely,

$$\sigma(t) = (c_5 + c_6 |u_2(t)|^2)\, \|u_2(t)\|^2.$$

By Gronvall's Lemma and since

$$u(0) = 0, \ p(0) = 0,$$

(8.63) implies that

$$|u(t)|^2 + \epsilon |p(t)|^2 = 0, \ 0 \leqslant t \leqslant T,$$

and the uniqueness is proved.

8.2. Convergence of the Perturbed Problems to the Navier-Stokes Equations

Theorem 8.3. *If $n = 2$, as $\epsilon \to 0$, the solutions $\{u_\epsilon, p_\epsilon\}$ of Problem 8.1 converge to the solution u of Problem 3.1 and the associated pressure p in the following sense:*

$$u_\epsilon \to u \text{ in } L^2(0, T; H^1(\Omega)) \text{ strongly,}$$
$$L^\infty(0, T; L^2(\Omega)) \text{ weak-star,} \tag{8.64}$$
$$\operatorname{grad} p_\epsilon \to \operatorname{grad} p \text{ in } H^{-1}(Q). \tag{8.65}$$

Theorem 8.4. *If $n = 3$, there exists a sequence $u_{\epsilon'}, p_{\epsilon'}$, of solutions of Problem 8.1, such that*

$$u_{\epsilon'} \to u \text{ in } L^2(0, T; L^2(\Omega)) \text{ strongly,} \tag{8.66}$$

$L^2(0, T; H_0^1(\Omega))$ weakly, $L^\infty(0, T; L^2(\Omega))$ weak-star,

$$\operatorname{grad} p_{\epsilon'} \to \operatorname{grad} p \text{ in } H^{-1}(Q) \text{ weakly,} \tag{8.67}$$

where u is some solution of Problem 3.1 and p denotes the associated pressure.

For any other sequence $u_{\epsilon'}, p_{\epsilon'}$, such that (8.66) – (8.67) hold, u must be some solution of Problem 3.1 and p the corresponding pressure.

Proof of Theorems 8.3 and 8.4. We pointed out in Remark 8.2 that the solutions u_ϵ, p_ϵ constructed in the proof of Theorem 8.1 satisfy *a priori* estimates independent of ϵ. By virtue of these estimates, there exists a sequence $\epsilon' \to 0$, such that

$$u_{\epsilon'} \to u_* \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak-star,}$$
$$L^2(0, T; H_0^1(\Omega)) \text{ weakly} \tag{8.68}$$

$$\sqrt{\epsilon'}\, p_{\epsilon'} \to \chi \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak-star.} \tag{8.69}$$

Due to (8.68), (8.51), and the compactness theorem (Theorem 2.1), we also have

$$u_{\epsilon'} \to u_* \text{ in } L^2(0, T; L^2(\Omega)) \text{ strongly.} \tag{8.70}$$

We can pass to the limit in (8.14), for the sequence ϵ';

$$\sqrt{\epsilon'}\ \frac{\mathrm{d}}{\mathrm{d}t}(p_{\epsilon'}, q) \to \frac{\mathrm{d}}{\mathrm{d}t}(\chi, q)$$

in the distribution sense; hence

$$\epsilon'\ \frac{\mathrm{d}}{\mathrm{d}t}(p_{\epsilon'}, q) \to 0$$

in the same sense, and (8.14) gives in the limit the equation

$$(\operatorname{div} u_*, q) = 0, \quad \forall q \in L^2(\Omega),$$

which shows that $\operatorname{div} u_* = 0$ and hence

$$u_* \in L^2(0, T; V) \cap L^\infty(0, T; H). \tag{8.71}$$

Let v be an element of $\mathscr{V}$; equation (8.13) then gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_\epsilon, v) + \nu((u_\epsilon, v)) + \hat{b}(u_\epsilon, u_\epsilon, v) = (f, v) \tag{8.72}$$

since

$$(\operatorname{grad} p_\epsilon, v) = (p_\epsilon, \operatorname{div} v) = 0.$$

If ψ is a continuously differentiable scalar function on $[0, T]$ with $\psi(T) = 0$ we can multiply (8.72) by $\psi(t)$ integrate in t, and then integrate by parts, obtaining

$$-\int_0^T (u_\epsilon(t), v\psi'(t))\,\mathrm{d}t + \nu \int_0^T ((u_\epsilon(t), v\psi(t)))\,\mathrm{d}t$$
$$+ \int_0^T \hat{b}(u_\epsilon(t), u_\epsilon(t), v\psi(t))\,\mathrm{d}t \tag{8.73}$$

$$=(\boldsymbol{u}_0, \boldsymbol{v})\psi(0)+\int_0^T (\boldsymbol{f}(t), \boldsymbol{v}\psi(t))\,\mathrm{d}t, \ \forall \boldsymbol{v}\in V.$$

We can pass to the limit in (8.73) using the weak convergence results (8.68) and the strong convergence (8.69). We obtain

$$-\int_0^T (\boldsymbol{u}_*(t), \boldsymbol{v}\psi'(t))\,\mathrm{d}t + \nu\int_0^T ((\boldsymbol{u}(t), \boldsymbol{v}\psi(t)))\,\mathrm{d}t$$

$$+\int_0^T \hat{b}\,(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}\psi(t))\,\mathrm{d}t$$

$$=(\boldsymbol{u}_0, \boldsymbol{v})\psi(0)+\int_0^T (\boldsymbol{f}(t), \boldsymbol{v}\psi(t))\,\mathrm{d}t,$$

$$\forall \boldsymbol{v}\in \mathscr{V}. \tag{8.74}$$

The relation (8.74) is the same as (3.43) and we conclude as for (3.43) that $\boldsymbol{u}_*$ is a solution of Problem 3.1.

If $n = 2$, the solution $\boldsymbol{u}$ of Problem 3.1 is unique and the whole sequence $\boldsymbol{u}_\epsilon$ converges to $\boldsymbol{u}$ in the sense (8.68) – (8.69). The strong convergence in $L^2(0, T; \boldsymbol{H}_0^1(Q))$ is obtained with a technique already used and which we skim over: we first prove that

$$(\boldsymbol{u}_\epsilon(T) - \boldsymbol{u}(T), \boldsymbol{v}) \to 0, \ \ \forall \boldsymbol{v}\in V,$$

and this, along with the previous weak convergence results, suffices to prove that the expression

$$X_\epsilon = |\boldsymbol{u}_\epsilon(T) - \boldsymbol{u}(T)|^2 + \epsilon|p_\epsilon(T)|^2 + 2\nu\int_0^T \|\boldsymbol{u}_\epsilon(t) - \boldsymbol{u}(t)\|^2\,\mathrm{d}t$$

converges to zero, as $\epsilon\to 0$.

It remains to prove (8.65) and (8.67). For this we write (8.24) as

$$\operatorname{grad} p_\epsilon = \boldsymbol{f} - \boldsymbol{u}_\epsilon' + \nu\Delta\boldsymbol{u}_\epsilon - \hat{B}\boldsymbol{u}_\epsilon.$$

The convergence results for $\boldsymbol{u}_\epsilon$ show that the right-hand side converges to

$$\boldsymbol{f} - \boldsymbol{u}' + \nu\Delta\boldsymbol{u} - B\boldsymbol{u} \qquad (\hat{B}\boldsymbol{u} = B\boldsymbol{u})$$

in $\boldsymbol{H}^{-1}(Q)$, and by comparison with (3.129), this is exactly grad p. Hence

$$\operatorname{grad} p_\epsilon \to \operatorname{grad} p \text{ in } \boldsymbol{H}^{-1}(Q);$$

the convergence holding for the whole sequence ϵ, in $\boldsymbol{H}^{-1}(Q)$ strongly if $n = 2$, and for a subsequence ϵ' in $\boldsymbol{H}^{-1}(Q)$ weakly if $n = 3$.

8.3. *Approximation of the Perturbed Problems.*

Our goal is now to approximate the perturbed problems, Problems 8.1 and 8.2. Among the many available methods, we will study the approximation by a fractional step method, with a discretization in the space variables by finite differences. The study will be similar in several respects with the study of the fractional step scheme of section 7.2.

8.3.1. *Description of the scheme.*

The discretization of $H_0^1(\Omega)$ and V is the discretization (APX1) and all the notations of Subsection 7.2.1 are maintained. For the approximation of the pressure we will need an approximation of the space $L^2(\Omega)$; we take simply the space X_h which has already appeared a few times: X_h is the space of step functions of type

$$\pi_h = \sum_{M \in \mathring{\Omega}_h^1} \xi_M \, w_{hM}, \quad \xi_M \in \mathcal{R} \; (\xi_M = \pi_h(M)), \tag{8.78}$$

where w_{hM} is the characteristic function of the block $\sigma_h(M)$ centered at M, whose edges are parallel to the x_i axes and of length h_i, $i = 1, \ldots, n$. The space X_h is equipped with the scalar product induced by $L^2(\Omega)$,

$$(\pi_h, \pi_h') = \int_\Omega \pi_h(x)\, \pi_h'(x)\, dx$$

$$= (h_1 \ldots . h_n) \sum_{M \in \mathring{\Omega}_h^1} \pi_h(M)\, \pi_h'(M). \tag{8.79}$$

Moreover, the data p_0, $\boldsymbol{u}_0$, and f are given satisfying (8.11), (8.4), and (8.5) and we choose some decomposition of f

$$f = \sum_{i=1}^{n} f_i, \quad f_i \in L^2(0, T; H); \tag{8.80}$$

as in (7.73) this decomposition is quite arbitrary and the simplest choice could be $f_1 = f$, $f_i = 0$, $i = 2, \ldots, n$.

The interval $[0, T]$ is divided into N intervals of length k ($T = kN$) and we set

$$f^{m+i/n} = \frac{1}{k}\int_{mk}^{(m+1)k} f_i(t)\,\mathrm{d}t, \quad i = 1, \ldots, n. \tag{8.81}$$

The fractional step scheme to be studied involves n intermediate steps, where n is the dimension of the space ($n = 2$ or 3).

We define a family of pairs $\{u_h^{m+i/n}, \pi_h^{m+i/n}\}$ of $W_h \times X_h$, where $m = 0, \ldots, N-1$ and $i = 1, \ldots, n$. These elements are defined successively in the order of increasing values of the fractional index $m + i/n$.

We start with

$$\begin{aligned} &u_h^0 = \textit{the orthogonal projection of } u_0 \textit{ onto } V_h \textit{ in } L^2(\Omega),\\ &\pi_h^0 = \textit{the orthogonal projection of } p_0 \textit{ on } X_h \textit{ in } L^2(\Omega). \end{aligned} \tag{8.82}$$

These definitions make sense ($V_h \subset L^2(\Omega)$, $X_h \subset L^2(\Omega)$) and it is worth observing that

$$|u_h^0| \leqslant |u_0|, \quad |\pi_h^0| \leqslant |p_0|, \quad \forall h. \tag{8.83}$$

When $u_h^{m+i-1/n}$, $\pi_h^{m+i-1/n}$ are known, we define $u_h^{m+i/n}$, $\pi_h^{m+i/n}$ ($m = 0, \ldots, N-1$, $i = 1, \ldots, n$) by means of the following conditions:

$$\begin{aligned} &u_h^{m+i/n} \in W_h, \quad \pi_h^{m+i/n} \in X_h \text{ and,}\\ &\frac{1}{k}(u_h^{m+i/n} - u_h^{m+i-1/n}, v_h) + \nu((u_h^{m+i/n}, v_h))_{ih}\\ &\qquad + b_{ih}(u_h^{m+i-1/n}, u_h^{m+i/n}, v_h) - (\pi_h^{m+i/n}, \nabla_{ih} v_{ih})\\ &\qquad = (f^{m+i/n}, v_h), \quad \forall v_h \in W_h, \end{aligned} \tag{8.84}$$

$$\begin{aligned} &\frac{\epsilon}{k}(\pi_h^{m+i/n} - \pi_h^{m+i-1/n}, \pi_h') + (\nabla_{ih} u_{ih}^{m+i/n}, \pi_h') = 0,\\ &\forall \pi_h' \in X_h. \end{aligned} \tag{8.85}$$

There are no conditions on the discrete divergence of the computed elements $u_h^{m+i/n}$, since they are allowed to belong to W_h and not only to V_h.

The equations (8.84)–(8.85) form a linear variational equation for the pair $\{u_h^{m+i/n}, \pi_h^{m+i/n}\} \in W_h \times X_h$; the coercivity of the bilinear form

$$\{(u_h, \pi_h), (u'_h, \pi'_h)\} \to \frac{1}{k}(u_h, u'_h) - \nu((u_h, u'_h))_{ih}$$

$$+ \frac{\epsilon}{k}(\pi_h, \pi'_h) + b_{ih}(u_h^{m+i-1/n}, u_h, u'_h)$$

$$- (\pi_h, \nabla_{ih} u'_{ih})(\nabla_{ih} u_{ih}; \pi'_h),$$

is ensured by (7.72). The existence of $u_h^{m+i/n}$, $\pi_h^{m+i/n}$ is a consequence of the Projection Theorem.

Remark 8.3. As in Remark 7.3, we can interpret the equations (8.84)–(8.85) in the following way:

$$\frac{1}{k}(u_h^{m+i/n}(M) - u_h^{m+i-1/n}(M)) - \nu\, \delta_{ih}^2\, u_h^{m+i/n}(M)$$

$$+ \frac{1}{2} u_{ih}^{m+i-1/n}(M)\, \delta_{ih}\, u_h^{m+i/n}(M)$$

$$+ \frac{1}{2} \delta_{ih}(u_{ih}^{m+i-1/n}\, u_h^{m+i/n})(M) + \bar{\nabla}_{ih}\, \pi_h^{m+i/n}(M)$$

$$= f_h^{m+i/n}(M), \quad \forall M \in \mathring{\Omega}_h, \tag{8.86}$$

$$\frac{\epsilon}{k}(\pi_h^{m+i/n}(M) - \pi_h^{m+i-1/n}(M)) + \nabla_{ih}\, u_{ih}^{m+i/n}(M) = 0,$$

$$\forall M \in \mathring{\Omega}_h^1, \tag{8.87}$$

where

$$f_h^{m+i/n}(M) = \frac{1}{h_1, \ldots, h_n} \int_{\sigma_h(M)} f^{m+i/n}(x)\,dx, \quad \forall M \in \mathring{\Omega}_h^1. \tag{8.88}$$

The unknowns when computing $u_h^{m+i/n}$, $\pi_h^{m+i/n}$ are the n components of $u_h^{m+i/n}(M)$ and the numbers $\pi_h^{m+i/n}$. Here again (see Remark 7.3) the main point of the fractional step method, is that the above equations are actually uncoupled into several much smaller subsystems which only involve the unknowns $u_h^{m+i/n}(M)$, $\pi_h^{m+i/n}(M)$, corresponding only to nodes M on the same line parallel to the x_i direction. This makes the resolution of (8.86) and (8.87) very easy.

8.3.2. *Unconditional a priori estimates*

The stability of the Scheme will be established through two types of *a priori* estimates, as in Section 7.2: unconditional *a priori* estimates, leading to unconditional stability results, and conditional *a priori* estimates leading to conditional stability theorems, and also used in the proof of convergence. This subsection deals with unconditional *a priori* estimates.

Lemma 8.5. *The elements $u_h^{m+i/n}$ remain bounded in the following sense:*

$$|u_h^{m+i/n}|^2 \leqslant d_4, \quad \epsilon|\pi_h^{m+i/n}|^2 \leqslant d_4, \quad m = 0, \ldots, N-1,$$
$$i = 1, \ldots, n, \tag{8.89}$$

$$k \sum_{m=0}^{N-1} \|u_h^{m+i/n}\|_{ih}^2 \leqslant \frac{d_4}{\nu}, \quad i = 1, \ldots, n, \tag{8.90}$$

$$\sum_{m=0}^{N-1} |u_h^{m+i/n} - u_h^{m+i-1/n}|^2 \leqslant d_4, \quad i = 1, \ldots, n, \tag{8.91}$$

$$\epsilon \sum_{m=0}^{N-1} |\pi_h^{m+i/n} - \pi_h^{m+i-1/n}|^2 \leqslant d_4, \quad i = 1, \ldots, n \tag{8.92}$$

where

$$d_4 = |u_0|^2 + |p_0|^2 + \frac{d_0^2}{\nu} \sum_{i=1}^{n} \int_0^T |f_i(t)|^2 \, \mathrm{d}t.$$

Proof. We write (8.84) with $v_h = u_h^{m+i/n}$; due to (7.72) we find:

$$\begin{aligned}
|u_h^{m+i/n}|^2 &- |u_h^{m+i-1/n}|^2 + |u_h^{m+i/n} - u_h^{m+i-1/n}|^2 \\
&+ 2k\nu\|u_h^{m+i/n}\|_{ih}^2 - 2k(\pi_h^{m+i/n}, D_h\, u_h^{m+i/n}) \\
&= 2k(f^{m+i/n}, u_h^{m+i/n}) \leqslant 2k|f^{m+i/n}||u_h^{m+i/n}| \\
&\leqslant \text{(by (7.67))} \leqslant 2k\, d_0|f^{m+i/n}|\,\|u_h^{m+i/n}\|_{ih} \\
&\leqslant k\nu\|u_h^{m+i/n}\|_{ih}^2 + \frac{kd_0^2}{\nu}\, |f^{m+i/n}|^2.
\end{aligned} \tag{8.93}$$

After simplification, there remains

$$\begin{aligned}
&|u_h^{m+i/n}|^2 - |u_h^{m+i-1/n}|^2 + |u_h^{m+i/n} - u_h^{m+i-1/n}|^2 \\
&\qquad + k\nu \|u_h^{m+i/n}\|_{ih}^2 - 2k(\pi_h^{m+i/n}, \nabla_{ih}\, u_{ih}^{m+i/n}) \\
&\qquad \leqslant \frac{kd_0^2}{\nu} |f^{m+i/n}|^2 .
\end{aligned} \tag{8.94}$$

Moreover, taking $\pi_h' = \pi_h^{m+i/n}$ in (8.85), we get the relation

$$\begin{aligned}
&\epsilon|\pi_h^{m+i/n}|^2 - \epsilon|\pi_h^{m+i-1/n}|^2 + \epsilon|\pi_h^{m+i/n} - \pi_h^{m+i-1/n}|^2 \\
&\qquad + 2k(\pi_h^{m+i/n}, \nabla_{ih}\, u_{ih}^{m+i/n}) = 0.
\end{aligned} \tag{8.95}$$

Next we add all the relations (8.94) and (8.95) for $i = 1, \ldots, n$, $m = 0, \ldots, N-1$. After some simplification, and after dropping some positive terms, we find:

$$\begin{aligned}
&|u_h^N|^2 + \epsilon|\pi_h^N|^2 + \sum_{i=1}^{n} \sum_{m=0}^{N-1} \{|u_h^{m+i/n} - u_h^{m+i-1/n}|^2 \\
&\qquad + \epsilon|\pi_h^{m+i/n} - \pi_h^{m+i-1/n}|^2\} + k\nu \sum_{i=1}^{n} \sum_{m=0}^{N-1} \|u_h^{m+i/n}\|_{ih}^2 \\
&\qquad \leqslant |u_h^0|^2 + \epsilon|\pi_h^0|^2 + \frac{kd_0^2}{\nu} \sum_{i=1}^{n} \sum_{m=0}^{N-1} |f^{m+i/n}|^2 .
\end{aligned} \tag{8.96}$$

As done in several other places, we bound the right-hand side of this inequality by

$$d_4 = |u_0|^2 + |p_0|^2 + \frac{d_0^2}{\nu} \sum_{i=1}^{m} \int_0^T |f(t)|^2 \, \mathrm{d}t$$

(see (8.83) and (5.29) and recall that $\epsilon \leqslant 1$). With the second member bounded by d_4, relation (8.96) implies (8.90) and (8.91).

For r and j fixed, $0 \leqslant r \leqslant N-1$, $1 \leqslant j \leqslant n$, we add relations (8.94) and (8.95) for $m = 0, \ldots, r-1$, $i = 1, \ldots, q$, and for $m = r$, $i = 1, \ldots, j$;[1]

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[1] The summation concerns those indices m, i, such that $0 \leqslant m + i/n \leqslant r + j/n$.

dropping several positive terms and simplifying, we obtain

$$|u_h^{r+j/q}|^2 + \epsilon|\pi_h^{r+j/q}|^2 \leqslant |u_h^0|^2 + \epsilon|\pi_h^0|^2$$

$$+ \frac{kd_0^2}{\nu} \sum_{\substack{i,\, m \\ 0 \leqslant m + i/n \leqslant r + j/n}} |f^{m+i/n}|^2 \leqslant |u_h^0|^2 + \epsilon|\pi_h^0|^2$$

$$+ \frac{kd_0^2}{\nu} \sum_{i=1}^{n} \sum_{m=0}^{N-1} |f^{m+i/n}|^2 \leqslant d_4, \quad r = 0, \ldots, N-1,$$

$$j = 1, \ldots, n.$$

Thus (8.89) is established and the proof of Lemma 8.5 is complete.

The approximate functions. We consider now the following approximate functions associated with the $u_h^{m+i/n}$, $\pi_h^{m+i/n}$:

$$u_h^{(i)}: \ [0, T] \to W_h,$$

$$u_h^{(i)}(t) = u_h^{m+i/n}, \text{ for } mk \leqslant t < (m+1)k, \ \ i = 1, \ldots, n. \tag{8.97}$$

$u_h: [0, T] \to W_h$; *u_h is a continuous function linear on each interval* $[mk, (m+1)k]$, $m = 0, \ldots, N-1$; *and*

$$u_h(mk) = u_h^m, \ \ m = 0, \ldots, N. \tag{8.98}$$

The results of Lemma 8.5 can be interpreted as a stability result:

Theorem 8.5. *The functions $u_h^{(i)}$ and u_h, defined by* (8.97) *are unconditionally $L^\infty(0, T; L^2(\Omega))$ stable* $(1 \leqslant i \leqslant n)$. *The functions $\delta_{ih} u_h^{(i)}$ and $\delta_{nh} u_h$, $(1 \leqslant i \leqslant n)$ are unconditionally $L^2(0, T; L^2(\Omega))$ stable.*

Remark 8.4. (i) As a consequence of (8.91), we have

$$|u_h^{(i)} - u_h^{(i-1)}|_{L^2(0, T; L^2(\Omega))} \leqslant \sqrt{kd_4}, \ i = 2, \ldots, n. \tag{8.99}$$

(ii) As in Lemma 7.3 and Remark 7.6,

$$|u_h^{(n)} - u_h|^2_{L^2(0, T; L^2(\Omega))} = \frac{k}{3} \sum_{m=0}^{N-1} |u_h^{m+1} - u_h^m|^2$$

$$\leqslant \frac{k}{3} \sum_{m=0}^{N-1} \left(\sum_{i=1}^{n} |u_h^{m+i/n} - u_h^{m+i-1/n}| \right)^2$$

$$\leqslant \frac{kn}{3} \sum_{m=0}^{N-1} \sum_{i=1}^{n} |u_h^{m+i/n} - u_h^{m+i-1/n}|^2 \leqslant \frac{kn}{3} d_4 .$$

Thus

$$|u_h^{(n)} - u_h|_{L^2(0,T;L^2(\Omega))} \leqslant \sqrt{\frac{knd_4}{3}} . \tag{8.100}$$

Remark 8.5. Let $\pi_h^{(i)}$, π_h denote the function from $[0, T]$ into X_h defined by

$$\pi_h^{(i)}(t) = \pi_h^{m+i/n} \text{ for } mk \leqslant t < (m+1)k,\ i = 1, \ldots, n, \tag{8.101}$$

π_h *is continuous on* $[0, T]$, *linear on each interval* $[mk, (m+1)k]$, *and* $\pi_h(mk) = \pi_h^m$. (8.102)

Then (8.92) amounts to saying that

$$\sqrt{\epsilon}\, |\pi_h^{(i)} - \pi_h^{(i-1)}|_{L^2(0,T;L^2(\Omega))} \leqslant \sqrt{kd_4},\ i = 2, \ldots, n \tag{8.103}$$

On the other hand, we can prove as for (8.100) that

$$\sqrt{\epsilon}\, |\pi_h^{(n)} - \pi_h|_{L^2(0,T;L^2(\Omega))} \leqslant \sqrt{\frac{knd_4}{3}} . \tag{8.104}$$

8.3.3. *Conditional a priori estimates*

The following estimates are obtained by the methods of Lemmas 7.11 and 7.12, using in particular the lemmas in Sections 7.3.1 and 7.3.2. It is convenient to consider also the linear operators, denoted D_{ih}, continuous from X_h into W_h and such that

$$(D_{ih}\, \pi_h, v_h) = (\pi_h, \nabla_{ih}\, v_{ih}),\ \forall \pi_h \in X_h,\ \forall v_h \in W_h . \tag{8.105}$$

Using the operators A_{ih}, D_{ih}, B_{ih}, we reformulate (8.84) as

$$\begin{aligned} &\frac{1}{k}(u_h^{m+i/n} - u_h^{m+i-1/n}) + \nu A_{ih}\, u_h^{m+i/n} \\ &\qquad + B_{ih}(u_h^{m+i-1/n}, u_h^{m+i/n}) + D_{ih}\, \pi_h^{m+i/n} = f_h^{m+i/n}, \\ &\qquad m = 0, \ldots, N-1,\ i = 1, \ldots, n. \end{aligned} \tag{8.106}$$

Lemma 8.6.

$$|D_{ih}\, \pi_h| \leqslant S_i(h)|\pi_h|,\ \forall \pi_h \in X_h,\ S_i(h) = \frac{2}{h_i} . \tag{8.107}$$

---

(1) Essentially $D_{ih}\, \pi_h$ is the vector function whose only non zero component is its i^{th} components which is equal to $\nabla_{ih}\, \pi_h$.

Proof. Due to (8.101),

$$|(D_{ih}\ \pi_h, v_h)| = |(\pi_h, \nabla_{ih}\ v_{ih})| \leqslant |\pi_h||\nabla_{ih}\ v_{ih}|$$
$$= |\pi_h||\delta_{ih}\ v_{ih}| \leqslant |\pi_h|\,\|v_h\|_{ih} \leqslant S_i(h)|\pi_h||v_h|,$$
$$\forall\ v_h \in W_h,$$

and (8.107) is proved.

Lemma 8.7. *We assume that $n = 2$ and that k and h satisfy*

$$k\,S(h)^2 \leqslant M, \tag{8.108}$$

where M is fixed, and arbitrarily large.
Then we have,

$$k \sum_{m=0}^{N-1} \|u_h^{m+i/2}\|_h^2 \leqslant \text{Const.},\ i = 1, 2, \tag{8.109}$$

where the constant depends only on M and the data.

Proof. We write (8.106) with $i = 2$ (and $n = 2$):

$$u_h^{m+1} = u_h^{m+1/2} - k\nu A_{2h}\,u_h^{m+1} - k\,B_{2h}(u_h^{m+1/2}, u_h^{m+1})$$
$$- k\,D_{2h}\,\pi_h^{m+1} + k\,f_h^{m+1}.$$

Taking the norm $\|\cdot\|_{1h}$ of each side, using (7.102), (7.104), (7.107), and (8.107), we estimate the right hand side as follows:

$$\|u_h^{m+1}\|_{1h} \leqslant \|u_h^{m+1/2}\|_{1h} + k\nu\|A_{2h}\,u_h^{m+1}\|_{1h}$$
$$+ k\|B_{2h}(u_h^{m+1/2}, u_h^{m+1})\|_{1h} + k\|D_{2h}\,\pi_h^{m+1}\|_{1h}$$
$$+ k\|f_h^{m+1}\|_{1h} \leqslant \|u_h^{m+1}\|_{2h} + k\nu\,S_1(h)\,S_2(h)\|u_h^{m+1}\|_{2h}$$
$$+ 2\sqrt{3}\,k\,S_1(h)\,S_2(h)|u_h^{m+1/2}|^{1/2}\,\|u_h^{m+1/2}\|_{1h}^{1/2}\,|u_h^{m+1}|^{1/2}\cdot$$
$$\cdot\|u_h^{m+1}\|_{2h}^{1/2} + k\,S_1(h)\,S_2(h)|\pi_h^{m+1}| + k\,S_1(h)|f_h^{m+1}|.$$

By the Schwarz inequality

$$\|u_h^{m+1}\|_{1h}^2 \leqslant 5(1 + k^2\,\nu^2\,S_1(h)^2)\|u_h^{m+1}\|_{2h}^2$$
$$+ 60\,k^2\,S_1(h)\,S_2(h)^2\,|u_h^{m+1/2}|\,\|u_h^{m+1/2}\|_{1h}\,|u_h^{m+1}|\,\|u_h^{m+1}\|_{2h}$$
$$+ 5\,k^2\,S_1(h)^2\,S_2(h)^2|\pi_h^{m+1}|^2 + 5\,k^2\,S_1(h)^2\,|f_h^{m+1}|.$$

Due to the previous estimates of Lemma 8.5 and (8.108) the sum from 0 to $N-1$ of the right-hand side of this inequality is bounded by k^{-1} times a constant and (8.109) is proved for $i = 2$.

In order to prove that

$$k \sum_{m=0}^{N-1} \|u_h^{m+1/2}\|_{2h}^2 \leqslant \text{Const.},$$

we write (8.106) with $i = 1$, and estimate appropriately the norm $\|\cdot\|_{2h}$ of $u_h^{m+1/2}$.

Lemma 8.8. *We assume that $n = 3$ and that k and h satisfy*

$$\frac{k\, S(h)^{11/4}}{\sqrt{\epsilon}} \leqslant M, \tag{8.110}$$

where M is fixed and arbitrarily large.

Then we have

$$k \sum_{m=0}^{N-1} \|u_h^{m+i/n}\|_h^2 \leqslant \text{Const}, \quad i = 1, 2, 3, \tag{8.111}$$

with a constant depending only on M and the data.

Proof. The same as the proof of Lemma 8.7 (and Lemmas 7.11, 7.12), the estimates (7.107) of B_{ih} being replaced by the estimates (7.110).

We infer from these lemmas a new stability result:

Theorem 8.6. *Assuming that k, h and ϵ satisfy the stability condition* (8.108) *(if $n = 2$) or* (8.110) *(if $n = 3$), all the functions*

$$\delta_{jh}\; u^i_{(h)}, \quad 1 \leqslant i,\; j \leqslant n,$$

are $L^2(0, T; L^2(\Omega))$ stable.

8.3.4. *The Convergence Theorems.*

Theorem 8.7. *We assume that the dimension is $n = 2$ and that k, h, ϵ, remain connected by* (8.108).

Then, if k, h and ϵ go to zero and

$$\frac{k}{\epsilon} \to 0, \tag{8.112}$$

the following convergence results hold:

$$u_h^{(i)},\ u_h, \text{ converge to } u \text{ in } L^2(Q) \text{ strongly, } L^\infty(0, T; L^2(\Omega)) \text{ weak-star, } i = 1, 2, \tag{8.113}$$

$$\delta_{jh}\, u_h^{(i)},\ \delta_{jh}\, u_h, \text{ converge to } D_j u \text{ in } L^2(Q) \text{ weakly, } 1 \leqslant i, j \leqslant 2, \tag{8.114}$$

$$\delta_{ih}\, u_h^{(i)} \text{ converge to } D_i u \text{ in } L^2(Q) \text{ strongly, } i = 1, 2, \tag{8.115}$$

where u is the unique solution of Problem 3.1 *corresponding to the data f, u_0, in* (8.4) *and* (8.5).

Theorem 8.8. *We assume that the dimension is $n = 3$. There exists a sequence h', k', ϵ', converging to zero* (1) *such that*

$$u_{h'}^{(i)},\ u_{h'}, \text{ converge to } u \text{ in } L^2(Q) \text{ strongly, } L^\infty(0, T; L^2(\Omega)) \text{ weak-star} \tag{8.116}$$

$$\delta_{jh'},\ u_{h'}^{(i)},\ \delta_{jh'}\, u_{h'}, \text{ converge to } D_j u \text{ in } L^2(Q) \text{ weakly, } 1 \leqslant i, j \leqslant 3, \tag{8.117}$$

where u is some solution of Problem 3.1.

For any other sequence h', k', $\epsilon' \to 0$, with

$$k'/\epsilon' \to 0. \tag{8.118}$$

and such that the convergence results (8.116) *and* (8.117) *hold, u must be a solution of Problem* 3.1.

The main lines of the proof are given in Subsections 8.3.5, and 8.3.6.

8.3.5. *Proof of Convergence*

Lemma 8.9. *Under the conditions* (8.109) *(if $n = 2$) or* (8.110) *(if $n = 3$), and if k/ϵ remain bounded,*

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\, |\hat{u}_h(\tau)|^2 \, d\tau \leqslant \text{Const., for some } 0 < \gamma < \frac{1}{4}, \tag{8.119}$$

where $\hat{u}_h$ is the Fourier transform in t of the function u_h extended by 0 *outside the interval* $[0, T]$, *the constant depending on γ, M, the bound on k/ϵ, and the data.*

---

(1) ϵ', h', k', satisfying (8.106), and $k'/\epsilon' \to 0$.

Proof. We add (separately) the relations (8.84) and (8.85) for $i = 1, \ldots, n$; this gives after an easy calculation

$$\frac{1}{k}(u_h^{m+1} - u_h^m, v_h) + \sum_{i=1}^{n} ((u_h^{m+i/n}, v_h))_{ih}$$

$$+ \sum_{i=1}^{n} b_{ih}(u_h^{m+i-1/n}, u_h^{m+i/n}, v_h) - (\pi_h^{m+1}, D_h v_h)$$

$$= \sum_{i=1}^{n} (f^{m+i/n}, v_h) + \sum_{i=1}^{n-1} (\pi_h^{m+1} - \pi_h^{m+i/n}, \nabla_{ih} v_{ih}),$$

$$\forall v_h \in W_h, \tag{8.120}$$

$$\frac{\epsilon}{k}(\pi_h^{m+1} - \pi_h^m, \pi_h') + (D_h u_h^{m+1}, \pi_h')$$

$$= \sum_{i=1}^{n-1} (\nabla_{ih} u_{ih}^{m+1} - \nabla_{ih} u_{ih}^{m+i/n}, \pi_h')$$

$$\forall \pi_h' \in X_h. \tag{8.121}$$

We reformulate these equations as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_h(t), v_h) + \sum_{i=1}^{n} \nu((u_h^{(i)}(t), v_h))_{ih}$$

$$+ \sum_{i=1}^{n} b_{ih}(u_h^{(i-1)}(t), u_h^{(i)}(t), v_h)$$

$$- (\pi_h^{(n)}(t), D_h v_h) = \sum_{i=1}^{n} (f_{ih}(t), v_h)$$

$$+ (\phi_h(t), v_h), \ \forall v_h \in W_h, \ \forall t \in (0, T), \tag{8.122}$$

$$\epsilon \frac{\mathrm{d}}{\mathrm{d}t}(\pi_h(t), \pi_h') + (D_h u_h^{(n)}(t), \pi_h') = (\theta_h(t), \pi_h'),$$

$$\forall \pi_h' \in X_h, \ \forall t \in (0, T), \tag{8.123}$$

where ϕ_h (and θ_h) are two step functions from $[0, T]$ into W_h (and X_h)

defined by

$$\phi_h(t) = \sum_{i=1}^{n-1} (D_{ih}\,\pi_h^{m+1} - D_{ih}\,\pi_h^{m+i/n}),$$

$$mk \leqslant t < (m+1)k \tag{8.124}$$

$$\theta_h(t) = \sum_{i=1}^{n-1} \nabla_{ih}\,u_{ih}^{m+1} - \nabla_{ih}\,u_{ih}^{m+i/n},\quad mk \leqslant t < (m+1)k,$$

$$m = 0, \ldots, N-1. \tag{8.125}$$

The derivation of (8.119) is based on the same principles as those of similar results (see Lemma 5.6 and the proof of Theorem 8.1, point (iii)) except for the treatment of the terms corresponding to ϕ_h and θ_h, which we must estimate suitably.

We observe that for $t \in (mk, (m+1)k)$,

$$|(\phi_h(t), v_h)| = \left| \sum_{i=1}^{n-1} (\pi_h^{m+1} - \pi_h^{m+i/n}, \nabla_{ih}\,v_{ih}) \right|$$

$$\leqslant \left(\sum_{i=1}^{n-1} |\pi_h^{m+1} - \pi_h^{m+i/n}|^2 \right)^{1/2}$$

$$\left(\sum_{i=1}^{n} |\nabla_{ih}\,v_{ih}|^2 \right)^{1/2}$$

$$\leqslant c_1 \|v_h\|_h \left(\sum_{i=2}^{n} |\pi_h^{m+i/n} - \pi_h^{m+(i-1)/n}|^2 \right)^{1/2}$$

Thus, denoting by $\|\cdot\|_{*h}$ the dual norm of $\|\cdot\|_h$ on W_h,

$$\|\phi_h(t)\|_h^2 \leqslant c_1^2 \sum_{i=2}^{n} |\pi_h^{m+i/n} - \pi_h^{m+(i-1)/n}|^2,$$

$$mk \leqslant t < (m+1)k$$

so that

$$\int_0^T \|\phi_h(t)\|_{*h}^2\,\mathrm{d}t \leqslant (\text{by } (8.92)) \leqslant c_1^2\, d_4 \frac{k}{\epsilon} \leqslant \text{Const.} \tag{8.126}$$

Similarly, if $t \in (mk, (m+1)k)$,

$$|\theta_h(t)| \leqslant \sum_{i=1}^{n-1} |\nabla_{ih}\, u_{ih}^{m+1} - \nabla_{ih}\, u_{ih}^{m+i/n}|,$$

$$|\theta_h(t)|^2 \leqslant C_2\, S(h)^2 \sum_{i=2}^{n} |u_h^{m+i/n} - u_h^{m+i-1/n}|^2,$$

and thus (using (8.91), (8.108) and (8.110))

$$\int_0^T |\theta_h(t)|^2\, \mathrm{d}t \leqslant \text{Const.} \tag{8.127}$$

The estimates (8.126) and (8.127) suffice in order to bound the terms corresponding to ϕ_h and θ_h in (8.122) and (8.123).

Proof of Theorems 8.7 and 8.8. (ii) In virtue of Theorem 8.5, there exists a sequence $h', k' \to 0$, such that

$$u_{h'}^{(i)} \to u^{(i)} \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak-star, } 1, \dots, n \tag{8.128}$$

$$u_{h'} \to u_* \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak-star.} \tag{8.129}$$

Due to Remark 8.4, $u_h^{(i)} - u_h^{(i-1)}$ converges to 0 in $L^2(Q)$ strongly $(2 \leqslant i \leqslant n)$ and $u_{h'} - u_{h'}^{(n)}$ does too. Therefore, all the limits are equal:

$$u^{(1)} = \dots = u^{(n)} = u_*, \tag{8.130}$$

and we would like to prove that u_* is a solution of Problem 3.1.

According to Theorem 8.6, the sequence $h', k' \to 0$, can be chosen in such a way that we also have

$$\delta_{jh'} u_{h'}^{(i)},\ \delta_{jh'} u_{h'} \to D_j u_* \text{ in } L^2(Q) \text{ weakly.} \tag{8.131}$$

We can also choose the sequence so that $k'/\epsilon' \to 0$. Then, by virtue of Lemma 8.9 and the property (5.92).[1]

$$u_{h'} \to u_* \text{ in } L^2(Q) \text{ strongly.} \tag{8.132}$$

Again using Remark 8.4, we get

$$u_{h'}^{(i)} \to u_* \text{ in } L^2(Q) \text{ strongly, } 1 \leqslant i \leqslant n. \tag{8.133}$$

---

[1] See the proof in Subsection 6.1.3 for finite differences

(ii) Let us show now that div $\boldsymbol{u}_* = 0$ by passing to the limit in (8.85). Let σ be a function in $\mathscr{D}(\Omega)$ and let π'_h be defined by

$$\pi'_h(M) = \sigma(M), \ \ \forall M \in \mathring{\Omega}^1_h.$$

It is clear that

$$\pi'_h \to \sigma \text{ in } L^\infty(\Omega), \ \text{ as } h \to 0. \tag{8.134}$$

Let ψ denote a continuously differentiable scalar function on $[0, T]$ with $\psi(T) = 0$. If we multiply (8.85) by $\psi^m = \psi(mk)$, and then add these relations for $m = 0, \dots, N-1$, $i = 1, \dots, m$, we get

$$\begin{aligned}\epsilon \sum_{m=1}^{N} (\pi_h^m, (\psi^m - \psi^{m-1})\pi'_h) \\ + k \sum_{i=1}^{n} \sum_{m=0}^{N-1} (\nabla_{ih}\, u_{ih}^{m+i/n}, \psi^m \pi'_h) \\ = \epsilon(\pi_h^0, \pi'_h \psi(0)).\end{aligned} \tag{8.135}$$

It is easy to pass to the limit in (8.135) using the estimates (8.89) on the π_h^m, and the above weak convergence. The limit relation is

$$\int_0^T (\operatorname{div} \boldsymbol{u}_*(t), \ \sigma\, \psi(t))\, \mathrm{d}t = 0, \tag{8.136}$$

and since $\sigma \in \mathscr{D}(\Omega)$ and ψ are arbitrary, (8.136) is equivalent to

$$\operatorname{div} \boldsymbol{u}_* = 0$$

so that

$$\boldsymbol{u}_* \in L^2(0, T; V) \cap L^\infty(0, T; H). \tag{8.137}$$

(iii) It remains to pass to the limit in (8.122). Let $\boldsymbol{v}$ be an element of $\mathscr{V}$ and let us write (8.122) with $\boldsymbol{v}_h = r_h\, \boldsymbol{v}$ where r_h = the restriction operator of the approximation (APX1) ($D_h\, r_h \boldsymbol{v} = 0$). Let ψ be as before and multiply (8.122) by $\psi(t)$, integrate in t, and then integrate by parts; we find

$$-\int_0^T (\boldsymbol{u}_h(t), \ \psi'(t)\, \boldsymbol{v}_h\, \mathrm{d}t + \sum_{i=1}^{n} \nu \int_0^T ((\boldsymbol{u}_h^{(i)}(t), \boldsymbol{v}_h\, \psi(t)))_{ih}\, \mathrm{d}t$$

$$+ \sum_{i=1}^{n} \int_0^T b_{ih}(u_h^{(i-1)}(t), u_h^{(i)}(t), \psi(t)\, v_h)\, dt$$

$$= \sum_{i=1}^{n} \int_0^T (f_{ih}(t), v_h\, \psi(t))\, dt + (u_h^0, v_h)\psi(0)$$

$$+ \int_0^T (\phi_h(t), v_h\, \psi(t))\, dt. \tag{8.138}$$

Passing to the limit in (8.138) for all the terms except the last (and for the sequence h', k', ϵ') is standard. The last term in the right-hand member converges to zero because of estimate (8.126) on ϕ_h, and since we assumed that $k'/\epsilon' \to 0$. This is the only point of the proof where we need this hypothesis on the ratio k/ϵ.

In the limit we get an equation similar to (3.43) (with u replaced by u_*), from which we infer that u_* is a solution of Problem 3.1 (as done in Theorem 3.1).

If $n = 2$, the solution of Problem 3.1 is unique: thus $u_* = u$ and the preceding weak convergence results hold for the whole sequence $k, h, \epsilon \to 0$ (with (8.108) and $k/\epsilon \to 0$).

The strong convergence results (8.115) are proved by showing that the expressions

$$X_h = |u_h^N - u(T)|^2 + \epsilon|\pi_h^N|^2$$

$$+ 2\nu \sum_{i=1}^{2} \int_0^T |D_i u(t) - \delta_{ih}\, u_h^{(i)}(t)|\, dt$$

converge to 0, as $h, k, \epsilon \to 0$.

APPENDIX I

Properties of the curl operator and application to the steady-state Navier–Stokes equations

We give in this appendix some functional properties of the curl operator on a bounded set of $\mathcal{R}^n$, $n = 2$ or 3, and we derive from these properties and improved form of the existence result contained in Theorem 1.5, Chap. II. Essentially, we follow Section 1 of C. Foias – R. Temam [3].

1. Functional Properties of the curl operator

Let Ω be an open bounded set of $\mathcal{R}^n$, $n = 2$ or 3. We assume that:

$$\begin{gathered}\Omega \text{ is connected and of class } \mathcal{C}^2 \text{ (cf. (1.3) Chap. I) and its}\\ \text{boundary } \Gamma \text{ has a finite number of connected components}\\ \text{denoted } \Gamma_1, \ldots, \Gamma_k \ (k \geqslant 1).\end{gathered} \tag{1.1}$$

The open set Ω may be simply or multi connected. In the latter case it is clear that we can make it simply connected with a finite number of smooth cuts. More precisely:

$$\begin{gathered}\text{We denote by } \Sigma_1, \ldots, \Sigma_N, N \text{ manifolds of dimension } n-1\\ \text{and of class } \mathcal{C}^2 \ (N \geqslant 0) \text{ such that } \Sigma_i \cap \Sigma_j = \emptyset \text{ for } i \neq j, \text{ and}\\ \text{the open set } \dot{\Omega} = \Omega \setminus \Sigma,\ \Sigma = \bigcup_{i=1}^{N} \Sigma_i \text{ is simply connected and}\\ \text{lipshitzian (i.e. the } \Sigma_i\text{'s are not tangent to } \Gamma)\end{gathered} \tag{1.2}$$

The notations are otherwise the same as elsewhere, in particular for the function spaces the notations introduced in § 1 and 2 of Chapter I, § 1 of Chapter II ($\mathcal{V}$, V, H, $H^m(\Omega)$, $\boldsymbol{H}^m(\Omega)$, $H^{1/2}(\Gamma)$, $\boldsymbol{H}^{1/2}(\Gamma)$ ...). We will

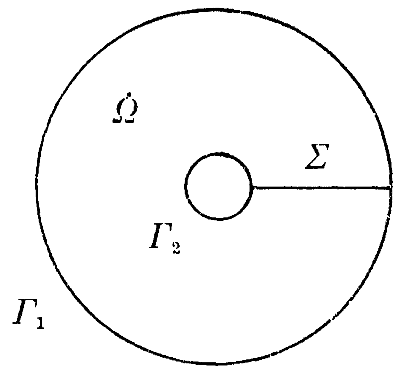

Fig. 1.

sometime mention the space $H^{m-1/2}(\Gamma)$, m integer $\geqslant 1$, which can be defined in a simple way as in the footnote of page 33: this is the space $\gamma_0 H^m(\Omega)$, i.e. the space of traces of functions in $H^m(\Omega)$, and the space is equipped with the quotient norm

$$\|\psi\|_{H^{m-1/2}(\Gamma)} = \operatorname*{Inf}_{\gamma_0 u = \psi} \|u\|_{H^m(\Omega)} \tag{1.3}$$

A systematic study of these spaces is otherwise available in J.L. Lions–E. Magenes [1].

We recall also the orthogonal decomposition of $\boldsymbol{L}^2(\Omega)$ which is given in Theorem 1.5, Chapter I,

$$\boldsymbol{L}^2(\Omega) = H \oplus H_1 \oplus H_2; \tag{1.4}$$

H_1, H_2, defined by I.(1.41), I.(1.42).

An orthogonal decomposition of H will be found below.

1.1. Kernel of the curl operator

There is no need to recall the definition of the curl operator in the three dimensional case. If the dimension of space is $n = 2$, we set

$$\text{curl}\, u = \{-D_2 u, D_1 u\}, \text{ if } u : \Omega \to \mathcal{R} \text{ is a scalar function,}$$

$$\text{curl}\, \boldsymbol{u} = D_1 u_2 - D_2 u_1, \text{ if } \boldsymbol{u} = \{u_1, u_2\} \text{ is a vector function : } \Omega \to \mathcal{R}^2.$$

The curl operator maps $\boldsymbol{L}^2(\Omega)$ into $\boldsymbol{H}^{-1}(\Omega)$ if $n = 3$ (or $H^{-1}(\Omega)$ if $n = 2$). Our aim is now to describe its kernel in $\boldsymbol{L}^2(\Omega)$, Ker(curl).

It is clear that Ker(curl) $\supset H_1 \oplus H_2$; let us find the elements of Ker(curl) $\cap H$. If $\boldsymbol{u} \in$Ker(curl) $\cap H$, then u is locally a gradient and more precisely $\boldsymbol{u} = \operatorname{grad} q$ in $\dot{\Omega}$, with div $\boldsymbol{u} = \Delta q = 0$, so that q is $\mathcal{C}^\infty$ and one to one on $\dot{\Omega}$, q is $\mathcal{C}^\infty$ in $\dot{\bar{\Omega}}$ except in the neighborhood of $\Gamma \cap \Sigma$. We have also with I.(1.34),

$$\gamma_\nu u = \frac{\partial q}{\partial \nu} = 0 \text{ on } \Gamma.$$

We denote Σ_i^+ and Σ_i^- the two sides of Σ_i and ν_i the unit normal on Σ_i oriented from Σ_i^+ toward Σ_i^-; if a function θ takes different values on Σ_i^+ and Σ_i^-, then we set

$$[\theta]_i = \theta|_{\Sigma_i^+} - \theta|_{\Sigma_i^-}.$$

For the function q above, we have $[q]_i$ = constant since grad q is $\mathcal{C}^\infty$ and single valued; we write $[q]_i = a_i \in \mathcal{R}$. In order to caracterise completely $\boldsymbol{u}$ and q we have to write that div $\boldsymbol{u} = 0$ in Ω. If $\phi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned}\langle \operatorname{div} \boldsymbol{u}, \phi\rangle = -\langle \operatorname{grad} q, \operatorname{grad} \phi\rangle &= -\int_{\Omega \text{ or } \dot{\Omega}} \operatorname{grad} q . \operatorname{grad} \phi \, dx \\ &= (\text{since } \Delta q = 0) = -\int_{\partial\dot{\Omega}} \frac{\partial q}{\partial \nu} \phi \, d\Gamma \\ &= -\sum_{i=1}^{N} \int_{\Sigma_i} \left[\frac{\partial q}{\partial \nu_i}\right]_i \phi \, d\Sigma = 0.\end{aligned}$$

It follows that

$$\left[\frac{\partial q}{\partial \nu_i}\right]_i = 0, \quad i = 1, \ldots, N.$$

Finally

Lemma 1.1. *The space* H_c = Ker(curl) $\cap H$ *is composed of the gradients of the harmonic functions q, multivalued in* Ω, *single valued in* $\dot{\Omega}$, *and such that*

$$\frac{\partial q}{\partial \nu} = 0 \text{ on } \Gamma, \tag{1.5}$$

$$[q]_i = \text{constant}, \quad i = 1, \ldots, N, \tag{1.6}$$

$$\left[\frac{\partial q}{\partial \nu_i}\right]_i = 0, \quad i = 1, \ldots, N. \tag{1.7}$$

We are going to exhibit a basis of H_c. Before that we prove the

Lemma 1.2. *For $i = 1, \ldots, N$ there exists a function q_i, unique up to an additive constant, such that*

$$\begin{aligned} &\Delta q_i = 0 \text{ in } \dot{\Omega}, \\ &\frac{\partial q_i}{\partial \nu} = 0 \text{ on } \Gamma, \\ &\left[\frac{\partial q_i}{\partial \nu_j}\right]_j = 0, \quad j = 1, \ldots, N, \\ &[q_j]_j = 0 \quad j \neq i, \\ &[q_i]_i = 1. \end{aligned} \tag{1.8}$$

Proof. We consider the following problem which will be shown to be equivalent to problem (1.8):

$$\begin{aligned} &\Delta q_i' = 0 \text{ in } \dot{\Omega}, \\ &\frac{\partial q_i'}{\partial \nu} = 0 \text{ on } \Gamma, \\ &\left[\frac{\partial q_i'}{\partial \nu_j}\right]_j = 0 \quad j = 1, \ldots, N, \\ &[q_i']_j = 0, \quad j \neq i, \\ &[q_i']_i = \text{(undetermined) constant}, \\ &\int_{\Sigma_i} \frac{\partial q_i'}{\partial \nu_i}\, \mathrm{d}\Sigma = 1. \end{aligned} \tag{1.9}$$

We first observe that (1.9) is a variational problem. Let

$$\mathrm{X}_i = \{p \in H^1(\dot{\Omega}),\ [p]_i = \text{constant},\ [p]_j = 0, \quad j \neq i\}.$$

It is elementary to check that (1.9) amount of looking for $q_i' \in X_i$ such that

$$\int_\Omega \text{grad } q_i' . \text{ grad } p \text{ d}x = [p]_i, \forall p \in X_i. \tag{1.10}$$

The left-hand side of (1.10) defines on $X_i/\mathcal{R}$ is a bilinear continuous coercive form and the right-hand side is a linear form on X_i which vanishes on the constants and which therefore induces a linear continuous form on $X_i/\mathcal{R}$. We obtain from Theorem 2.2, Chapter I, the existence and uniqueness of q_i' in $X_i/\mathcal{R}$.

If q_i' is solution of (1.10), then $\alpha_i = [q_i']_i \neq 0$ and $q_i = (1/\alpha_i)\, q_i'$ is solution of (1.8): $\alpha_i \neq 0$ since otherwise q_i' can be extended as a function of $H^1(\Omega)$ which is solution of the homogeneous Neumann problem in Ω, so that q_i' is a constant; this contradicts (1.10).

Conversely if q_i is solution of (1.8), then

$$\beta_i = \int_{\Sigma_i} \frac{\partial q_i}{\partial \nu_i} \text{ d}\Sigma$$

is not zero and $q_i' = (1/\beta_i)\, q_i$ is solution of (1.9): $\beta_i \neq 0$ since otherwise q_i would be a solution of the equation (1.10) with the left-hand side replaced by 0; we would have q_i = constant (=0 in $X_i/\mathcal{R}$), in contradiction with $[q_i]_i = 1$.

Lemma 1.3. $H_c = \text{Ker(curl)} \cap H$ *is the space spanned by grad* $q_1, \ldots,$ *grad* q_N [(1)]. *Its dimension is N.*

Proof. If $u \in H_c$, then $u = \text{grad } q$, where u satisfies the conditions given in Lemma 1.1. Let $a_i = [q]_i$ and $r = q - \sum_{i=1}^{N} a_i q_i$. It is clear that r is analytic in $\dot{\Omega}$ and

$$\Delta r = 0 \quad \text{in } \dot{\Omega}, \qquad \frac{\partial r}{\partial \nu} = 0 \quad \text{on } \Gamma,$$

$$\left[\frac{\partial r}{\partial \nu}\right]_i = 0, \quad [r]_i = 0, \quad i = 1, \ldots, N.$$

---

(1) grad q_i is understood in the classical sense. The distribution gradients of the q_i's are the sum of these functions and of Dirac distributions located on Σ_i: $-[q]_i \nu_i \delta_{\Sigma_i}$

Whence r is a constant and $u = \sum_{i=1}^{N} a_i \operatorname{grad} q_i$.

Finally, the functions grad q_i are obviously linearly independant.

Remark 1.1. (i) If Ω is simply connected, $N = 0$ and $H_c = \{0\}$,

(ii) The space H_c is isomorphic to the first cohomology space of Ω, i.e. the quotient of the space of closed differential forms on Ω by the space of exact differential forms on Ω (cf. G. de Rham [1]).

Lemma 1.4. Let H_0 be the orthogonal of H_c into H. We have

$$H_0 = \{\boldsymbol{v} \in H, \int_{\Sigma_i} \boldsymbol{v}.\, \nu \mathrm{d}\Sigma = 0, \quad i = 1, \ldots, N\}. \tag{1.11}$$

Proof. If $\boldsymbol{v} \in H$, then $\boldsymbol{v} \in H_0$ if and only if $(\boldsymbol{v}, \operatorname{grad} q_i) = 0$, $i = 1, \ldots, N$. But, by the generalized Stokes formula I.(1.19),

$$(\boldsymbol{v}, \operatorname{grad} q_i) = \int_{\dot{\Omega}} \boldsymbol{v}.\, \operatorname{grad} q_i \, \mathrm{d}x$$

$$= \int_{\partial\dot{\Omega}} \boldsymbol{v}.\, \nu q_i \mathrm{d}\Gamma = \int_{\Sigma_i} \boldsymbol{v}.\, \nu \mathrm{d}\Sigma.$$

Whence (1.11).

Remark 1.2. (i) It is easy to see that for $\boldsymbol{v} \in H$, $\int_{\Sigma_i} \boldsymbol{v}.\, \nu \mathrm{d}\Sigma$ is independent of the cut Σ_i, which means that its value does not change by a continuous deformation of Σ_i:

$$\int_{\Sigma_i} \boldsymbol{v}.\, \nu \mathrm{d}\Sigma = \int_{\Sigma_i'} \boldsymbol{v}.\, \nu \mathrm{d}\Sigma.$$

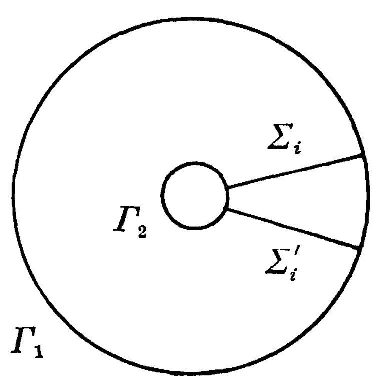

Fig. 2.

(ii) One can check directly that if $\boldsymbol{v} \in H_c$, $\boldsymbol{v} = \operatorname{grad} q$ and

$$\int_{\Sigma_i} \boldsymbol{v}.\,\nu \mathrm{d}\Sigma = \int_{\Sigma_i} \frac{\partial q}{\partial \nu} \mathrm{d}\Sigma = 0, \qquad i = 1, \ldots, N,$$

then $\boldsymbol{v} = 0$. $\square$

We summarize the previous results in the
Proposition 1.1. *Under the assumptions (1.1) and (1.2),*

$$\boldsymbol{L}^2(\Omega) = H_c \oplus H_0 \oplus H_1 \oplus H_2, \tag{1.12}$$

$$\operatorname{Kerl}(\operatorname{curl}) = H_c \oplus H_1 \oplus H_2, \tag{1.13}$$

the spaces H_c, H_0, H_1, H_2, being described above.

Remark 1.3. (i) We denote by P_{H_c}, P_{H_i}, the orthogonal projectors in $\boldsymbol{L}^2(\Omega)$ onto H_c, H_i, $i = 0, 1, 2$; P_{H_1} and P_{H_2} were implicitly introduced and defined in the proof of Theorem 1.5 and in Remark 1.6, Chapter I. If $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$,

$$\boldsymbol{u}_2 = P_{H_2}\boldsymbol{u} = \operatorname{grad} p \ , \tag{1.14}$$

where p is the solution of the Dirichlet problem

$$\Delta p = \operatorname{div} \boldsymbol{u} \in H^{-1}(\Omega), \quad p \in H_0^1(\Omega), \tag{1.15}$$

$$\boldsymbol{u}_1 = P_{H_1}\boldsymbol{u} = \operatorname{grad} q, \tag{1.16}$$

q being the solution of the Neumann problem

$$\Delta q = 0 \quad \text{in } \Omega, \quad \frac{\partial q}{\partial \nu} = \gamma_\nu(\boldsymbol{u} - \operatorname{grad} p) \quad \text{on } \Gamma \tag{1.17}$$

which is well defined (cf I.(1.44) and the remarks following that relation).

Now, the definition of P_{H_c} is as follows: $P_{H_c}\boldsymbol{u} = \sum_{i=1}^{N} a_i \operatorname{grad} q_i$, where the a_i's are solutions of the linear system

$$\sum_{i=1}^{N} \alpha_{ij} q_i = \int_{\Sigma_j} \boldsymbol{u}.\,\nu \mathrm{d}\Sigma, \quad 1 \leqslant j \leqslant N, \tag{1.18}$$

with

$$\alpha_{ij} = \int_{\Sigma_j} \frac{\partial q_i}{\partial \nu_j} \, d\Sigma = \int_{\Sigma_i} \frac{\partial q_j}{\partial \nu_i} \, d\Sigma = (\operatorname{grad} q_i, \operatorname{grad} q_j),$$

so that the matrix of elements α_{ij} is *regular.*

(ii) We know (cf Remark 1.6, Chapter I), that P_{H_1} and P_{H_2} map $\boldsymbol{H}^m(\Omega)$ into $H_i \cap \boldsymbol{H}^m(\Omega)$ if Ω is of class $\mathcal{C}^r$, $r \geqslant m+2$. Since $H_c \subset \mathcal{C}^\infty(\Omega) \cap \mathcal{C}^r(\bar{\Omega})$, the same is true for P_{H_c} and P_{H_0}.

1.2. The space curl $(\boldsymbol{H}^1(\Omega))$.

First of all:

Lemma 1.5. curl $\boldsymbol{H}^1(\Omega) = \operatorname{curl}(\boldsymbol{H}^1(\Omega) \cap H_0)$.

Proof. We observe that if $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{v} = P_{H_0}\boldsymbol{u} = \boldsymbol{u} - \operatorname{grad} q$, where grad $q \in \operatorname{Ker}(\operatorname{curl})$, and thus curl $\boldsymbol{v}$ = curl $\boldsymbol{u}$.

Lemma 1.6. *There exists a constant* $c_0 = c_0(\Omega)$ *such that,*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leqslant c_0 |\operatorname{curl} \boldsymbol{u}|, \tag{1.19}$$

$$|\boldsymbol{u}| \leqslant c_0 |\operatorname{curl} \boldsymbol{u}| \tag{1.19}$$

for every $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \cap H_0$.

Proof. It is proved in G. Duvaut–J.L. Lions [1] (Theorem 6.1, Chapter 7) that

$$\{ u \in \boldsymbol{H}^1(\Omega), \gamma_\nu u = u \,.\, \nu \,|_\Gamma = 0\} =$$
$$= \{ u \in \boldsymbol{L}^2(\Omega), \operatorname{div} u \in \boldsymbol{L}^2(\Omega), \operatorname{curl} u \in \boldsymbol{L}^2(\Omega), \gamma_\nu u = 0 \text{ on } \Gamma\} \tag{1.21}$$

and that there exists a constant $c_1 = c_1(\Omega)$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leqslant c_1 \{ |\boldsymbol{u}| + |\operatorname{div} \boldsymbol{u}| + |\operatorname{curl} \boldsymbol{u}|\} \tag{1.22}$$

for every $\boldsymbol{u}$ in the space (1.21).

Therefore (1.19) is an obvious consequence of (1.20) and (1.22).

In order to prove (1.20) we proceed by contradiction: if (1.20) is not true, there exists a sequence $\boldsymbol{u}_m \in \boldsymbol{H}^1(\Omega) \cap H_0$, such that

$$|\boldsymbol{u}_m| > m|\mathrm{curl}\, \boldsymbol{u}_m|, \quad \forall m. \tag{1.23}$$

We may assume that $|\boldsymbol{u}_m| = 1$. Then by (1.22), $\boldsymbol{u}_m$ is bounded in $\boldsymbol{H}^1(\Omega)$. We can extract a subsequence still denoted $\boldsymbol{u}_m$, which converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \cap H_0$; the convergence holds in $\boldsymbol{L}^2(\Omega)$ strongly and then $|u| = 1$. On the other hand by (1.23), curl $\boldsymbol{u} \in H_0 \cap \mathrm{Ker}(\mathrm{curl})$ and therefore $\boldsymbol{u} = 0$ by contradiction with $|\boldsymbol{u}| = 1$.

Lemma 1.7. curl $\boldsymbol{H}^1(\Omega)$ *is closed in* $\boldsymbol{L}^2(\Omega)$ *if* $n = 3$ *(in* $L^2(\Omega)$ *if* $n = 2$*).*

Proof. By (1.19), curl is an isomorphism from $\boldsymbol{H}^1(\Omega) \cap H_0$ into $\boldsymbol{L}^2(\Omega)$ (for $L^2(\Omega)$ if $n = 2$).

Characterisation of (curl $\boldsymbol{H}^1(\Omega))^\perp$: We denote by (curl $\boldsymbol{H}^1(\Omega))^\perp$ the orthogonal of curl $\boldsymbol{H}^1(\Omega)$ in $\boldsymbol{L}^2(\Omega)$ ($n = 3$).

Proposition 1.2. *If* $n = 3$, (curl $\boldsymbol{H}^1(\Omega))^\perp = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega), \boldsymbol{u} = \mathrm{grad}\, p$, $p \in H^1(\Omega)$, $p =$ *constant on each* $\Gamma_i\}$.

Proof. If $\boldsymbol{u} \in (\mathrm{curl}\, \boldsymbol{H}^1(\Omega))^\perp$, then

$$(\boldsymbol{u}, \mathrm{curl}\, \phi) = 0, \quad \forall \phi \in \mathcal{D}(\Omega)$$

and thus curl $\boldsymbol{u} = 0$.

Then since $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ and curl $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ we can, by a trace theorem of G. Duvaut–J.L. Lions [1] which is similar to Theorem 1.2, Chapter I, define the trace of $\boldsymbol{u} \wedge \nu$ on Γ $\boldsymbol{u} \wedge \nu|_\Gamma \in \boldsymbol{H}^{-1/2}(\Gamma)$, and we have the generalized Stokes formula

$$(\boldsymbol{u}, \mathrm{curl}\, \phi) = (\mathrm{curl}\, \boldsymbol{u}, \phi) + \langle \boldsymbol{u} \wedge \nu|_\Gamma, \gamma_0 \phi\rangle \quad \forall \phi \in \boldsymbol{H}^1(\Omega). \tag{1.24}$$

Then

$$(\boldsymbol{u}, \mathrm{curl}\, \phi) = \langle \boldsymbol{u} \wedge \nu|_\Gamma, \gamma_0\phi\rangle = 0, \quad \forall \phi \in \boldsymbol{H}^1(\Omega)$$

so that $\boldsymbol{u} \wedge \nu|_\Gamma = 0$ and

$$(\mathrm{curl}\, \boldsymbol{H}^1(\Omega))^\perp = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega), \mathrm{curl}\, \boldsymbol{u} = 0, \boldsymbol{u} \wedge \nu|_\Gamma = 0\}.$$

We can be more precise: for such a $\boldsymbol{u}$, curl $\boldsymbol{u} = 0$ and $\boldsymbol{u} = \operatorname{grad} p$, $\boldsymbol{u} \in H_c \oplus H_1 \oplus H_2$. Since $\operatorname{grad} p = (\partial p/\partial \nu)\, \nu + \nabla_\tau p$ where $\nabla_\tau p$ is the tangential component of grad p on Γ, $\boldsymbol{u} \wedge \nu|_\Gamma = 0$ amounts to

$$\nabla_\tau p = 0 \quad \text{on } \Gamma,$$

and p is constant on each Γ_i. We see that $P_{H_c}(\operatorname{grad} p)$ is necessarily 0 and therefore $p \in H^1(\Omega)$.

The reciprocal is easy and the result is proved.

Proposition 1.3. *Under the assumptions* (1.1) *and* (1.2) *and if* $n = 3$,

$$\operatorname{curl} \boldsymbol{H}^1(\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \operatorname{div} \boldsymbol{u} = 0,\ \int_{\Gamma_i} \boldsymbol{u}.\ \nu \mathrm{d}\Gamma = 0,\ \forall i\}.$$

Proof. Let Y be the space in the right-hand side of the above inequality. Since $\operatorname{curl}(\boldsymbol{H}^1(\Omega))$ is closed, it suffices to show that $Y = (\operatorname{curl} \boldsymbol{H}^1(\Omega))^{\perp\perp}$. But if $\boldsymbol{v} = \operatorname{grad} p \in (\operatorname{curl} \boldsymbol{H}^1(\Omega))^\perp$, and if $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$, $\operatorname{div} \boldsymbol{u} = 0$, then,

$$(\boldsymbol{u}, \boldsymbol{v}) \int_\Omega \boldsymbol{u}.\ \operatorname{grad} p \,\mathrm{d}x = \int_\Gamma \gamma_\nu \boldsymbol{u} p \mathrm{d}\Gamma = \sum_{i=1}^{k} p(\Gamma_i) \int_{\Gamma_i} \gamma_\nu \boldsymbol{u} \mathrm{d}\Gamma,$$

$p(\Gamma_i)$ = the value of p on Γ_i. Since the $p(\Gamma_i)$'s are arbitrary numbers, the result follows.

Remark 1.4. If Γ is connected, $k = 1$, there remains

$$\operatorname{curl} \boldsymbol{H}^1(\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \operatorname{div} \boldsymbol{u} = 0\}.$$

Remark 1.5. For $n = 2$, we have analogous results with the same proofs for curl $H^1(\Omega)$ and $(\operatorname{curl} H^1(\Omega))^\perp$.

1.3. Remark on the regularity

We can somehow complete the result of G. Duvaut–J.L. Lions in (1.21), (1.22);

Proposition 1.4. *Let Ω be an open bounded set of $\mathcal{R}^n$, $n = 2, 3$, and let m be an integer $\geqslant 1$. We assume that* (1.1) *and* (1.2) *hold and that Ω is of class $\mathcal{C}^r$, $r \geqslant m + 1$. Then*

$$\boldsymbol{H}^m(\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \operatorname{curl} \boldsymbol{u} \in \boldsymbol{H}^{m-1}(\Omega),\ \operatorname{div} \boldsymbol{u} \in H^{m-1}(\Omega),$$

$$\gamma_\nu \boldsymbol{u} \in H^{m-1/2}(\Gamma)\} \tag{1.25}$$

and there exists $c_2 = c_2(m, \Omega)$ *such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^m(\Omega)} \leqslant c_2\{|\boldsymbol{u}| + \|\operatorname{curl}\boldsymbol{u}\|_{\boldsymbol{H}^{m-1}(\Omega)} + |\operatorname{div}\boldsymbol{u}|_{H^{m-1}(\Omega)} + |\gamma_\nu \boldsymbol{u}|_{H^{m-1/2}(\Gamma)}\}$$

for every $\boldsymbol{u} \in \boldsymbol{H}^m(\Omega)$.

Proof. (i) We start with the case $m = 1$. The space in the right-hand side of (1.25) contains $\boldsymbol{H}^1(\Omega)$ and we have to prove the other inclusion. Let $\boldsymbol{u}$ be an element of this space and grad p the projection of $\boldsymbol{u}$ on $H_1 \oplus H_2$. We know that p is solution of the Neumann problem

$$\Delta p = \operatorname{div}\boldsymbol{u} \quad \text{in } \Omega, \qquad \frac{\partial p}{\partial \nu} = \boldsymbol{u}.\nu \quad \text{on } \Gamma. \tag{1.27}$$

Then $\boldsymbol{v} = \boldsymbol{u} - \operatorname{grad} p$ satisfies $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, curl $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, div $\boldsymbol{v} \in L^2(\Omega)$ and $\gamma_\nu \nu = 0$ on Γ, so that $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ by (1.21) and $\boldsymbol{u}$ too since $p \in H^2(\Omega)$ by the standard regularity results for the Neumann problem (cf. Agmon–Douglis–Nirenberg [1] or Lions–Magenes [1]). The inequality (1.26) then follows from the inequality (1.22) and from

$$\|p\|_{H^2(\Omega)/\mathcal{R}} \leqslant c_3\{|\Delta p|_{L^2(\Omega)} + |\frac{\partial p}{\partial \nu}|_{H^{1/2}(\Gamma)}\}.$$

(cf. Agmon–Douglis–Nirenberg [1].)

(ii) We proceed by induction when $m > 1$.

We assume that (1.25) and (1.26) are proved at the order $m - 1$. We have first to show that if $\boldsymbol{u}$ belongs to the space in the right-hand side of (1.25), then $\boldsymbol{u} \in \boldsymbol{H}^m(\Omega)$. By the induction assumption, we have already $\boldsymbol{u} \in \boldsymbol{H}^{m-1}(\Omega)$. If D^{m-1} is a differential operator of order $m - 1$, we see that $\boldsymbol{v} = D^{m-1}\boldsymbol{u}$ satisfies

$$\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \qquad \operatorname{curl}\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \qquad \operatorname{div}\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),$$

$$\boldsymbol{v} \cdot \nu = D^{m-1}(\boldsymbol{u} \cdot \nu) - \sum_{i=1}^{m-1} \binom{i}{m-i} D^i \nu D^{m-1-i}\boldsymbol{u} \in H^{1/2}(\Gamma),$$

and by the first part, $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$. It follows that $\boldsymbol{u} \in \boldsymbol{H}^m(\Omega)$ and (1.26) is easy.

Remark 1.6. (i) We can deduce this proposition directly from Agmon–Douglis–Nirenberg [1].

(ii) We can replace (1.26) by

$$\|\boldsymbol{u} - P_{H_c}\boldsymbol{u}\|_{\boldsymbol{H}^m(\Omega)} \leqslant c_3\{\|\operatorname{curl}\boldsymbol{u}\|_{\boldsymbol{H}^{m-1}(\Omega)} + |\operatorname{div}\boldsymbol{u}|_{H^{m-1}(\Omega)} + |\gamma_\nu\boldsymbol{u}|_{H^{m-1/2}(\Gamma)}\}. \tag{1.28}$$

2. Application to the non-homogeneous steady-state Navier–Stokes equations

The following result completes Theorem 1.5, of Chapter II, when $n = 2$ or 3.

Theorem 2.1. *We assume that* $n = 2$ *or* 3 *and that* (1.1) *and* (1.2) *are satisfied. Let* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ *and* $\phi \in \boldsymbol{H}^{1/2}(\Gamma)$ *be given such that*

$$\int_{\Gamma_i} \phi \cdot \nu \mathrm{d}\Gamma = 0, \quad i = 1, \ldots, k. \tag{2.1}$$

Then the problem II(1.62)–II(1.64) *possesses at least one solution* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ *and* $p \in L^2(\Omega)$.

Proof. Because of Theorem 1.5, Chapter II, we just have to show that the conditions II(1.65), II(1.66) are satisfied by ϕ, which means (cf. Remark 1.5 (i), Chapter II) that

$$\phi \in \gamma_0\{\operatorname{curl}\boldsymbol{H}^2(\Omega)\}. \tag{2.2}$$

Let ϕ be given satisfying (2.1). By solving the non-homogeneous Stokes problem

$$\begin{aligned} -\Delta\Phi + \operatorname{grad}\pi &= 0 && \text{in } \Omega,\\ \operatorname{div}\Phi &= 0 && \text{in } \Omega,\\ \Phi &= \phi && \text{on } \Gamma, \end{aligned}$$

we find by Theorem 2.4, Chapter I, a function $\Phi \in \boldsymbol{H}^1(\Omega)$ such that $\operatorname{div}\Phi = 0$ and $\phi = \gamma_0\Phi$. By (2.1), Lemma 1.5 and Proposition 1.3, there exists $\zeta \in \boldsymbol{H}^1(\Omega) \cap H_0$ such that $\operatorname{curl}\zeta = \Phi$. Observing that $\operatorname{curl}\zeta \in \boldsymbol{H}^1(\Omega)$

and using Proposition 1.4, we see that $\zeta \in H^2(\Omega)$ and (2.2) or(II(1.65), II(1.66)) are verified.

Remark 2.1. As in I(2.49), a necessary condition for ϕ is

$$\int_\Gamma \phi \cdot \nu \mathrm{d}\Gamma = 0 \tag{2.3}$$

which is weaker than (2.1).

Remark 2.2. For the regularity of u and p, the reader is refered to Remark 1.6 and Proposition 1.1, Chapter II.

Remark 2.3. For f and ϕ given, several times continuously differentiable, ϕ satisfying (2.1), J. Leray proved in [1] the existence of solutions of II(1.62)–II(1.64) which are several times continuously differentiable.

Proposition 2.1. *Under the assumptions of Theorem* 2.1, *for any* $\delta > 0$, *there exists a linear continuous mapping* Λ_δ *from*

$$\dot{H}^{3/2}(\Gamma) = \{\phi \in H^{3/2}(\Gamma), \int_{\Gamma_i} \phi \cdot \nu \mathrm{d}\Gamma = 0, i = 1, \ldots, k\} \tag{2.4}$$

into $H^2(\Omega)$, *such that*

$$\operatorname{div} \Lambda_\delta\phi = 0 \quad \text{in } \Omega, \tag{2.5}$$

$$\gamma_0\Lambda_\delta\phi = \phi \quad \text{on } \Gamma, \tag{2.6}$$

$$|b(v, \Lambda_\delta\phi, v)| \leqslant \delta\|\phi\|_{H^{3/2}(\Gamma)}\|v\|^2, \qquad \forall \phi \in \dot{H}^{3/2}(\Gamma), \quad \forall v \in V. \tag{2.7}$$

Proof. We just have to analyze the proofs of Theorem 2.1 above and Theorem 1.5 in Chapter II. In the proof of Theorem 2.1 above the mapping $\phi \to \Phi$ is linear continuous from $\dot{H}^{3/2}(\Gamma)$ into $H^2(\Omega)$ (Theorem 2.4, Chapter I). By Proposition 1.3 and 1.4,

$$\operatorname{curl} H^2(\Omega) = \{u \in H^1(\Omega), \operatorname{div} u = 0, \int_{\Gamma_i} u \cdot \nu \mathrm{d}\Gamma = 0, \forall i\},$$

and curl is an isomorphism from $H^2(\Omega) \cap H_0$ onto curl $H^2(\Omega)$. Therefore $\Phi \in \operatorname{curl} \boldsymbol{H}^2(\Omega)$, and $\Phi = \operatorname{curl} \zeta$, $\zeta \in \boldsymbol{H}^2(\Omega) \cap H_0$, the mapping $\Phi \to \zeta$ being linear continuous from curl $\boldsymbol{H}^2(\Omega)$ (equipped with the norm of $\boldsymbol{H}^1(\Omega)$) into $\boldsymbol{H}^2(\Omega) \cap H_0$.

We then have to analyze the proof of Theorem 1.5, i.e. the correspondence $\phi = \operatorname{curl} \zeta \to \psi = \operatorname{curl}(\theta_\epsilon \zeta)$. The mapping $\zeta \to \theta_\epsilon \zeta$ is obviously linear continuous in $\boldsymbol{H}^2(\Omega)$, and for ϵ sufficient by small, $\epsilon \leqslant \epsilon_0 = \epsilon_0(\delta)$, we can take $\Lambda_\delta \phi = \psi = \operatorname{curl}(\theta_\epsilon \zeta)$, and (2.7) is satisfied, (2.5) and (2.6) being obvious (for (2.7), cf. II(1.85)).

Remark 2.4. It is easy, using Proposition 2.1. to establish existence and uniqueness results, analog to those in Chapter III §§ 3, 4, but with a nonhomogeneous boundary condition for u:

$$u(x, t) = \phi(x, t), \quad x \in \Gamma, \quad 0 < t < T.$$

This is left as an exercise.

APPENDIX II

by F. Thomasset

Implementation of non-conforming linear finite elements
(Approximation APX5 – Two dimensional case)

0. Test Problems

Two test problems are considered.

Problem 1: *Flow in a cavity*

Ω is the square $(0, 1) \times (0, 1)$ of the plane.

$\boldsymbol{u}$ is given on $\partial\Omega$, $\boldsymbol{u} = 0$ except on the side $y = 1$ where $\boldsymbol{u} = \{U, 0\}$

(U to be specified).

Problem 2: *Flow between nonconcentric rotating cylindars*

Ω is the annular domain of the plane, Oxy, which is limited by the two circles

$$C_1 \quad : \quad x^2 + y^2 - 25 = 0$$

$$C_2 \quad : \quad (x - 1)^2 + y^2 - 4 = 0$$

C_1 and C_2 are rotating with algebraic angular velocities, σ_1 and σ_2 to be specified.

In both cases, no body forces are applied.

1. The triangulation

The triangulation $\mathcal{T}_h$ is defined by numbering the vertices and the triangles. An algorithm of automatic triangulation provide us the following entries

	X(.)	Y(.)	
i	X(i)	Y(i)	coordinates of the i^{th} vertice.

Nu(.,.)

Nu(1,j)	Nu(2,j)	Nu(3,j)	number of the vertices of the j^{th} triangle.

It is necessary to be able to recognize in a simple way the boundary vertices: an economical way of doing it is to provide them with numbers $> N$, where N is the total number of interior nodes.[1] There are two different types [2] of automatic triangulation algorithms

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[1] In the case of a domain with several boundaries (i.e. multi-connected domain), we are led to introduce another set of integers in order to distinguish the points belonging to different boundaries.

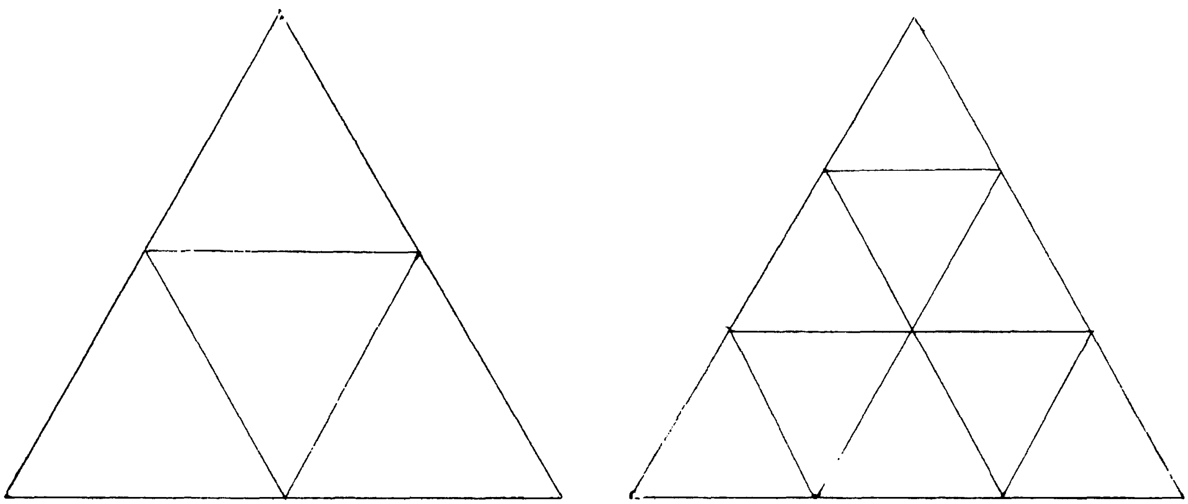
[2] We are concerned with the treatment of general domain shapes. Of course, for a specific geometry which must be considered many times, one may find a triangulation more economical than that given by a general program.

– the method of "successive subdivision" leading to regular nets

– the method of "contraction of the domain" by Alan George [1], which is convenient for a domain with a curved boundary.

The method of successive subdivision

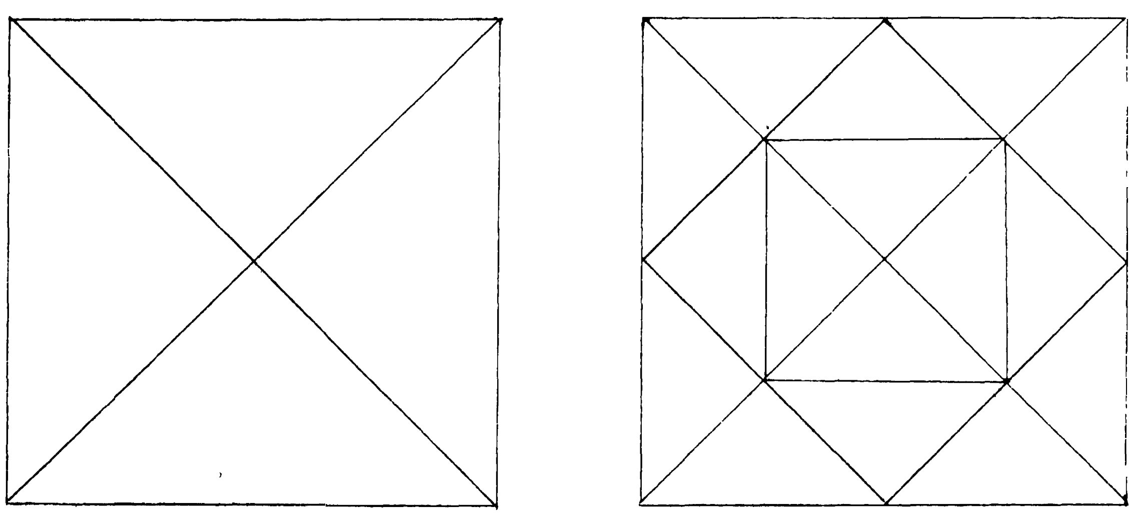
The domain is initially covered with a coarse triangulation; then at each step, every triangle is divided into k^2 smaller triangles, similar to the initial triangle.



k = 2 k = 3

The procedure can be reiterated until we reach the appropriate number of triangles.

Example ($k = 2$).



Initial triangulation 1st subdivision

N = 1 , N1 = 5 , M = 4 N = 5 , N1 = 13, M = 16

For a given triangulation, let N be the number of interior vertices, N_1 the total number of vertices, M the number of triangles. Let N', N'_1, M', denote the corresponding parameters after the subdivision into k^2 triangles has been performed. Then observing that

$$2C + C_B = 3M, \quad C_B = N_1 - N,$$

(C and C_B denoting the number of interior and boundary edges), we find

$$M^1 = k^2 \times M$$

$$N^1 = \frac{(k-1)(k-2)}{2} M + \frac{(k-1)(3M + N - N_1)}{2} + N$$

$$N_1^1 = N' + k(N_1 - N).$$

After n subdivisions, we note that

$$M^n = k^{2n} M^0, \quad C_B^n = k^n \, C_B^0$$

$$M^n = 2N^n + C_B^n + 2(T-1)$$

$$M^0 = 2N^0 + C_B^0 + 2(T-1)$$

where T is the number of wholes in the domain.

The parameters of the triangulation are then

$$N^n = \frac{1}{2}((k^{2n} - 1)M^0 - (k^n - 1)C_B^0) + N^0$$

$$N_1^n = \frac{1}{2}((k^{2n} - 1)M^0 + (k^n + 1)C_B^0) + N^0$$

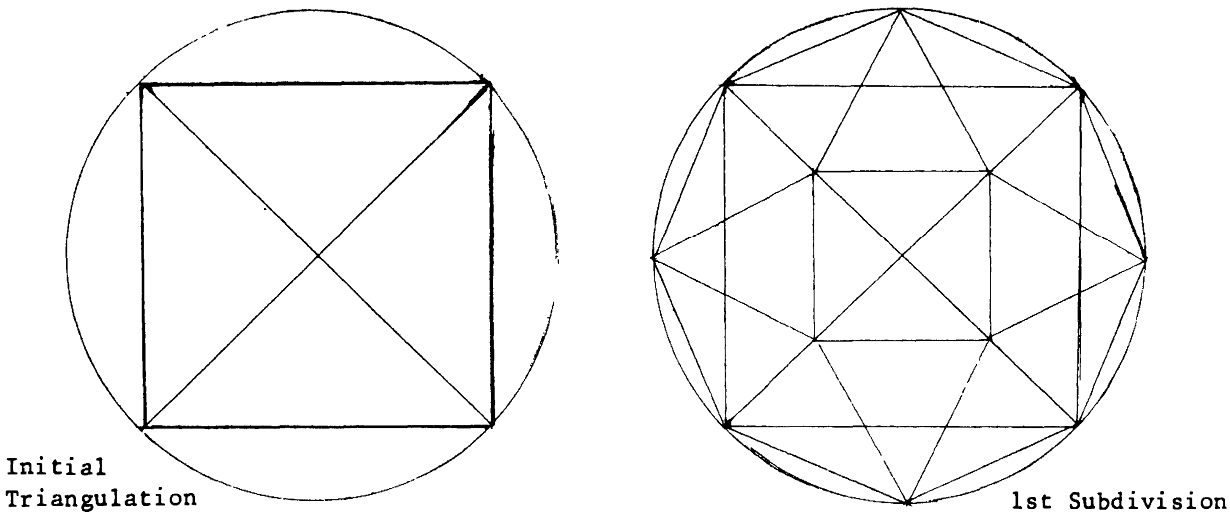
$$M^n = k^{2n} M^0, \quad C_B^0 = N_1^0 - N^0$$

$(M^0 = M, \ldots)$.

We emphasize that the triangles obtained by subdivision are similar to the triangles of the initial triangulation: therefore the angles do not become smaller than those of the initial triangulation.

The method of successive subdivision can be adapted to domains with a simple curved boundary: we just replace the new vertices appearing on the boundary edges by their projection on the boundary of Ω.

Example ($k = 2$)



Method of Contraction of the domain

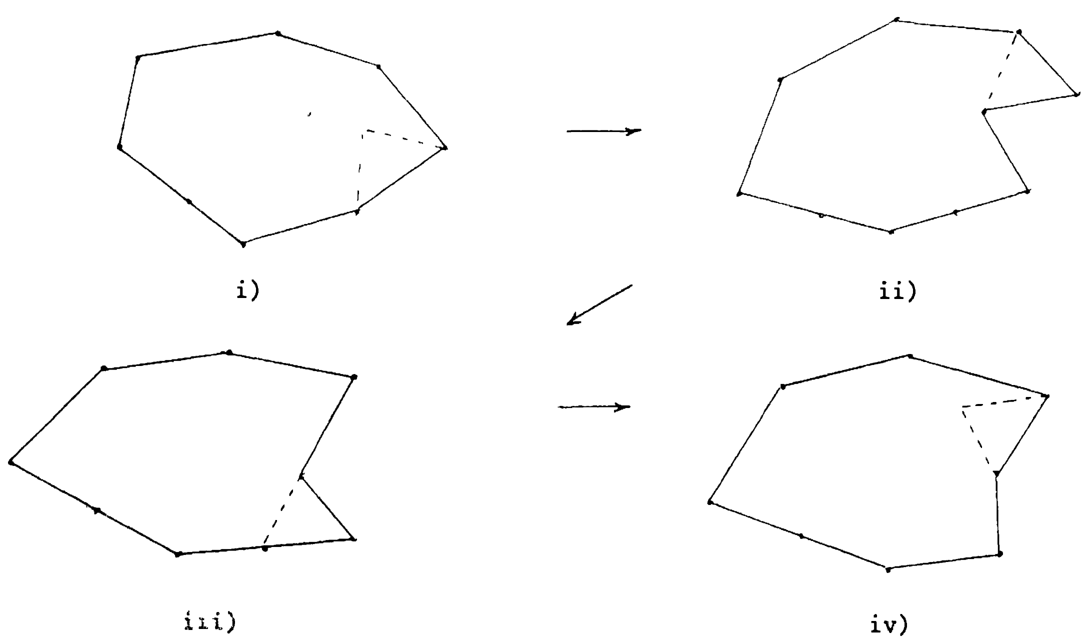
The principle of the method is to cover Ω with triangles as similar as possible to equilateral triangles (for which the numbers σ_T are minimal and thus also, the bound of the error). An initial discretisation of $\partial\Omega$ is given, constituted by a closed connected sequence of segments. If Ω is multi-connected, we must introduce artificial boundaries in Ω, and these boundaries will be removed after the triangulation is done.

Given the boundary polygon, the algorithm can construct a triangle in two different ways:

– by introducing a new node in the interior (notching)

– by joining two consecutive boundary vertices (trimming).

Then we start again with a new domain constituted with the initial domain less the last triangle, and so on until the domain is reduced to one triangle.



Examples. Figures 1–4 show two triangulations of the two considered geometries.

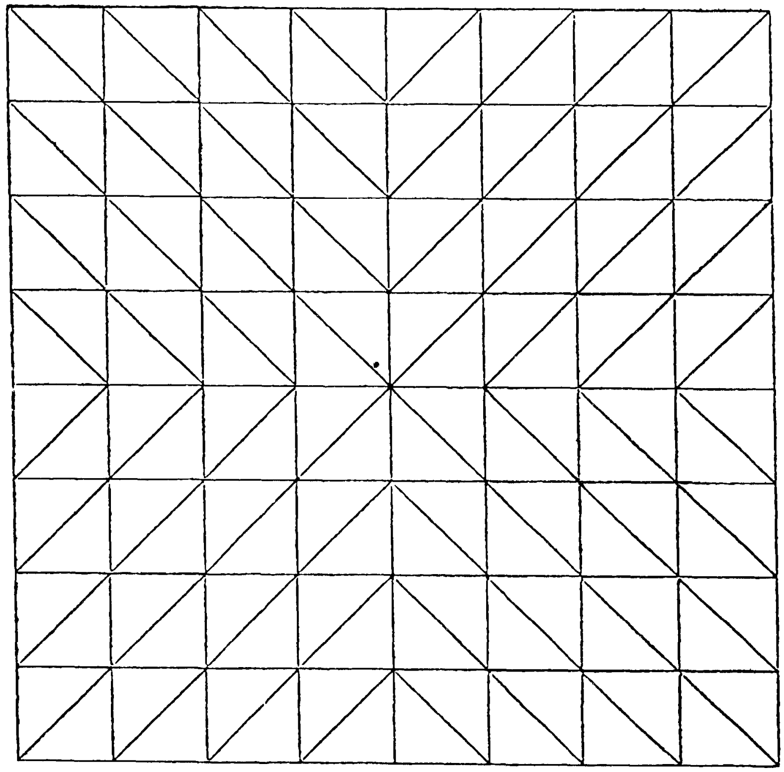


Fig. 1.

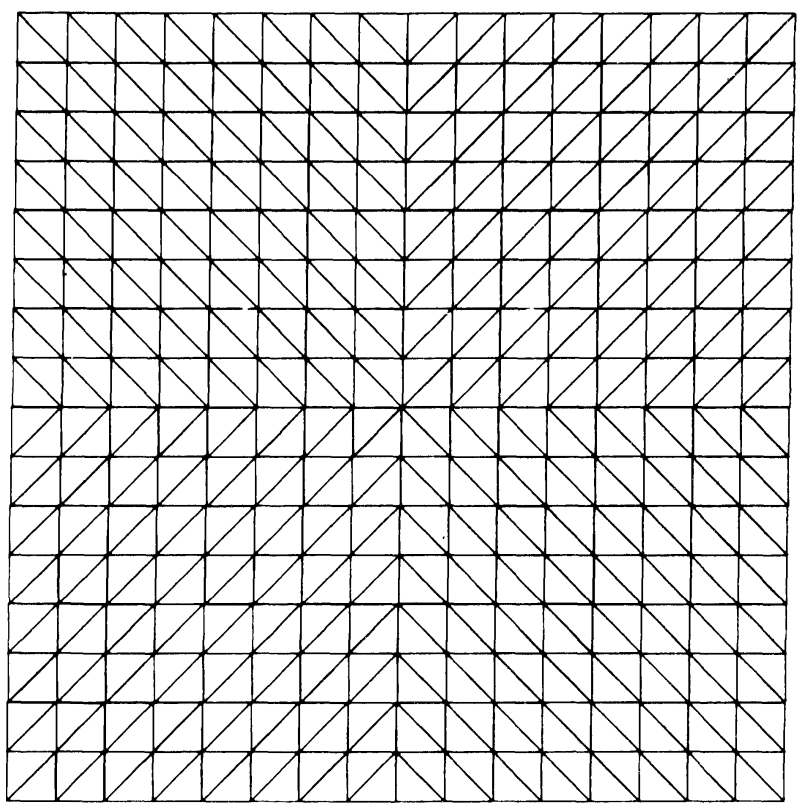


Fig. 2.

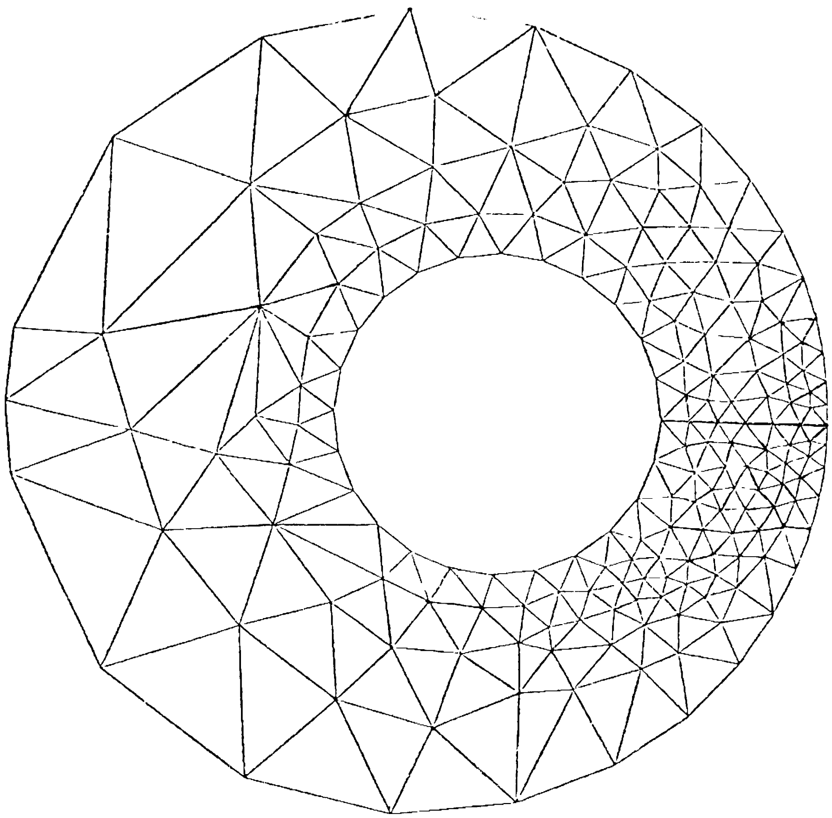


Fig. 3.

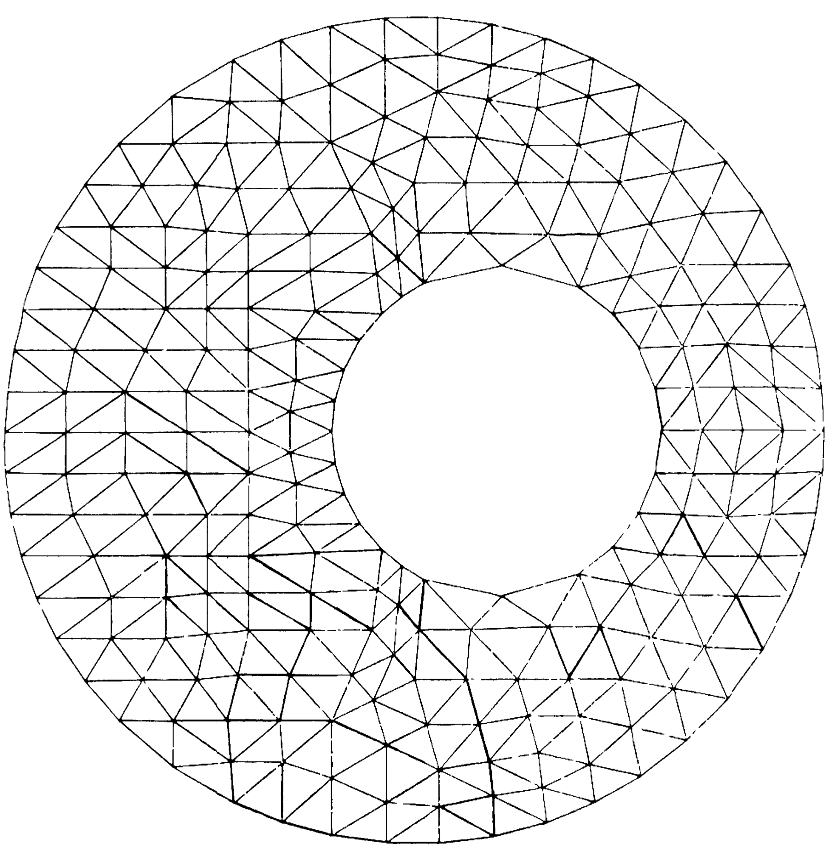


Fig. 4.

2. The nodes

The nodal variables of a function ϕ are the values of ϕ at the mid-edges which will be called the nodes.

Given the entries X, Y, Nu, we form the corresponding entries for the nodes

	$X_m(.)$	$Y_m(.)$	
i	$X_m(i)$	$Y_m(i)$	coordinates of the i^{th} node

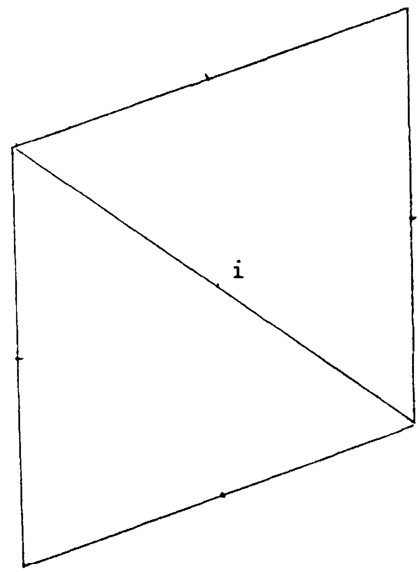
(the boundary nodes are numbered with numbers $> N_m$, N_m = the number of interior nodes).

$Nu_m(.,.)$

$Nu_m(1,j)$	$Nu_m(2,j)$	$Nu_m(3,j)$	numbers of the nodes of the j^{th} triangle

(this amount to numbering the edges).

It will appear later that we need to know rapidly the numbers of the nodes which are neighbors of a given nodes (i.e. those belonging to a same triangle: there are at most 4 neighbors).



For this reason we will compute from Nu_m the entry of the neighbors:

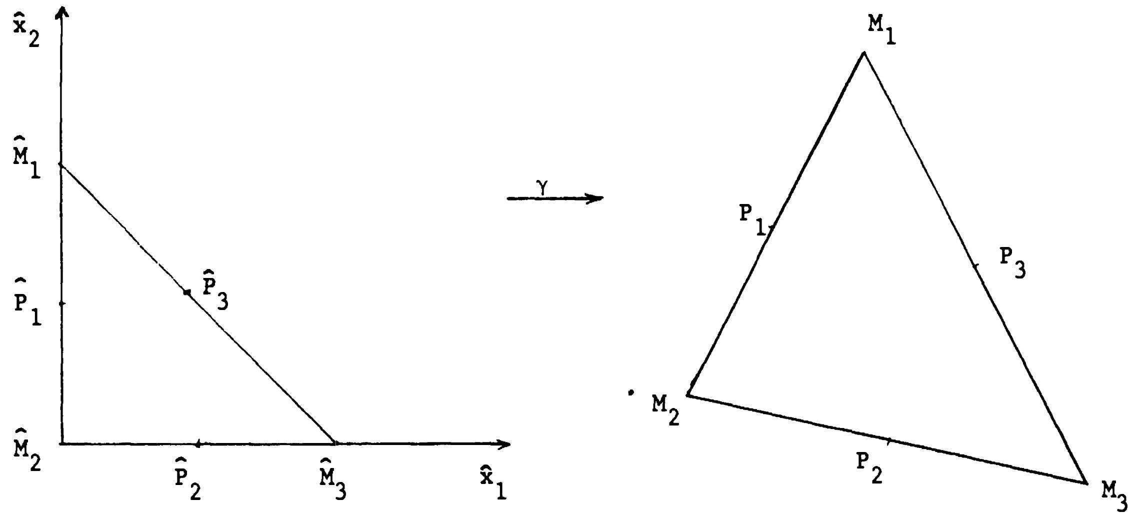
TABV(.,.)

TABV(1,i)	TABV(2,i)	TABV(3,i)	TABV(4,i)	Numbers of the nodes which are neighbors of the i^{th} node

3. Computation of the basis function on a given triangle

Let there be given a triangle T with vertices M_1, M_2, M_3, mid-edges P_1, P_2, P_3, and let v_1, v_2, v_3 denote the corresponding basis functions ($v_i(P_j) = \delta_{ij}$).

We introduce a reference triangle $\hat{T}$ and the linear mapping transforming $\hat{T}$ into T



$$M = \gamma(\hat{M}) = \begin{pmatrix} G(1,1) & G(2,1) \\ G(1,2) & G(2,2) \end{pmatrix} \hat{M} + \begin{pmatrix} G(3,1) \\ G(3,2) \end{pmatrix}$$

We compute the $G(\alpha, \beta)$ by using the equations $M_i = \gamma(\hat{M}_i)$:

$$\begin{pmatrix} G(1,1) & G(2,1) \\ G(1,2) & G(2,2) \end{pmatrix} = \begin{pmatrix} (M_1 - M_3)_1 & (M_2 - M_3)_1 \\ (M_1 - M_3)_2 & (M_2 - M_3)_2 \end{pmatrix} \begin{pmatrix} (\hat{M}_1 - \hat{M}_3)_1 & (\hat{M}_2 - \hat{M}_3)_1 \\ (\hat{M}_1 - \hat{M}_3)_2 & (\hat{M}_2 - \hat{M}_3)_2 \end{pmatrix}^{-1}$$

$$= \frac{1}{(\hat{M}_1 - \hat{M}_3)_1 (\hat{M}_2 - \hat{M}_3)_2 - (\hat{M}_1 - \hat{M}_3)_2 (\hat{M}_2 - \hat{M}_3)_1}$$

$$\times \begin{pmatrix} (M_1 - M_3)_1 & -(M_2 - M_3)_2 \\ (M_1 - M_3)_2 & (M_2 - M_3)_2 \end{pmatrix} \begin{pmatrix} (\hat{M}_2 - \hat{M}_3)_2 & (\hat{M}_2 - \hat{M}_3)_1 \\ -(\hat{M}_1 - \hat{M}_3)_2 & (\hat{M}_1 - \hat{M}_3)_1 \end{pmatrix}$$

$$= \begin{pmatrix} (M_1 - M_3)_1 & (M_2 - M_3)_1 \\ (M_1 - M_3)_2 & (M_2 - M_3)_2 \end{pmatrix} \cdot \hat{C}$$

where $\hat{C}$ is a 2×2 matrix independent of T.

$$\begin{pmatrix} G(3,1) \\ G(3,2) \end{pmatrix} = M_1 - \begin{pmatrix} G(1,1) & G(2,1) \\ G(1,2) & G(2,2) \end{pmatrix} \cdot \hat{M}_1.$$

We easily deduce γ^{-1} from γ and:

$$\hat{M} = \gamma^{-1}(M) = \begin{pmatrix} G1(1,1) & G1(2,1) \\ G1(1,2) & G1(2,2) \end{pmatrix} \cdot M + \begin{pmatrix} G1(3,1) \\ G1(3,2) \end{pmatrix}$$

We then possess all what we need to construct the basis functions. Indeed, let λ_i denote the barycentric coordinates in T with respect to the M_j ($\lambda_i(M_j) = \delta_{ij}$). We have

$$\begin{aligned}
\nu_1 &= \lambda_1 + \lambda_2 - \lambda_3 \\
\nu_2 &= \lambda_2 + \lambda_3 - \lambda_1 \\
\nu_3 &= \lambda_3 + \lambda_1 - \lambda_2 .
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\lambda_1(M) = \hat{x}_2(\hat{M}) \\
&\lambda_2(M) = 1 - \hat{x}_1(\hat{M}) - \hat{x}_2(\hat{M}) \\
&\lambda_3(M) = \hat{x}_2(\hat{M}) \\
&\int_T \hat{x}_1^{i_1} \hat{x}_2^{i_2} \, \mathrm{d}\hat{x}_1 \, \mathrm{d}\hat{x}_2 = \frac{i_1 ! \, i_2 !}{(i_1 + i_2 + 2)!}
\end{aligned}$$

from which we infer that

$$\text{area } T = \frac{1}{2} \det G$$

$$\int_T \nu_i \, \mathrm{d}M = \frac{1}{3} \text{ area } (T)$$

$$\int_T \nu_i \nu_j \, \mathrm{d}M = \frac{1}{3} \delta_{ij} \text{ area } (T).$$

It is interesting to notice that the basis functions are orthogonal in

$L^2(T)$. The gradient of the v_i's can be computed by using the matrix G_1.

$$\frac{\partial\lambda_i}{\partial u_j} = \sum_{k=1,2} \frac{\partial\hat{x}_k}{\partial x_j}\frac{\partial(\lambda_i\circ\gamma)}{\partial\hat{x}_k} = \sum_{k=1,2} G1(j,\,k)\frac{\partial(\lambda_i\circ\gamma)}{\partial\hat{x}_k}$$

$$\frac{\partial\lambda_2}{\partial x_j} = G1(j,\,2),\ \frac{\partial\lambda_2}{\partial x_j} = -\,G1(j,\,1) - G1(j,\,2),\ \frac{\partial\lambda_3}{\partial x_j} = G1(j,\,1)$$

$$\frac{\partial v_1}{\partial x_j} = -\,2\,G1(j,\,1)$$

$$\frac{\partial v_2}{\partial x_j} = -\,2\,G1(j,\,2)$$

$$\frac{\partial v_3}{\partial x_j} = 2\,G1(j,\,1) + 2\,G1(j,\,2)\,.$$

Numerical integration of the right hand side

$$\int_T f(M)\,v_i(M)\,\mathrm{d}M \approx \frac{\text{area }(T)}{3}\,f(P_i),\quad i = 1,\,2,\,3.$$

4. Solution of Stokes' Problem

4.1. *The basis functions*

A basis of W_h (the approximation of $H_0^1(\Omega)$) is constituted with the w_i ($i = 1, \ldots, N_m$):

- w_i is linear on each triangle
- $w_i = 0$ on $\partial\Omega_h$
- $w_i = 0$ at each node (= at each mid-edge), except i^{th} node, where $w_i(i) = 1$.

The restriction of w_i to a triangle T is one of the three functions v_j defined in Section 3. The support of w_i is the union of the two triangles containing i.

A basis of W_h (approximation of $\boldsymbol{H}_0^1(\Omega)$) is provided by the

$$\{(w_i, 0), (0, w_i), i = 1, \ldots, N_m\}.$$

With N_m = number of interior nodes.

The discrete homogeneous Stokes problem leads to the solutions of

$$u_h = \begin{cases} \sum_{i=1}^{N_m} X_i w_i \\ \sum_{i=1}^{N_m} Y_i w_i \end{cases}$$

$$\nu \sum_{j=1}^{N_m} X_j \alpha_{jk} = \int_\Omega f_x w_k \, dM + \sum_T \pi_h(T) \text{ area } (T) \frac{\partial w_k}{\partial x} (T) \; (1 \leqslant k \leqslant N_m)$$

$$\nu \sum_{j=1}^{N_m} Y_j \alpha_{jk} = \int_\Omega f_y w_k \, dM + \sum_T \pi_h(T) \text{ area } (T) \frac{\partial w_k}{\partial y} (T)$$

with $\operatorname{div}(u_h) = 0$ on T, $\forall T$.

The α_{jk} are computed by the following expressions

$$\alpha_{jk} = \sum_{T(j,k)} \text{area } (T) \,.\, \nabla w_j(T) \,.\, \nabla w_k(T) \,,$$

where the summation is over all the triangles T adjacent to the nodes j and k.

For the nonhomogeneous problem, we also introduce the w_i associated to the boundary nodes ($i > N_m$), and the linear system has the same form as (4.1), provided we respectively add to the right-hand sides the expressions

$$- \sum_{j > N_m} X_j \, \alpha_{jk}, \quad - \sum_{j > N} Y_j \, \alpha_{jk}.$$

(The X_j and Y_j for $j > N$ are known and given by the boundary data).

4.2. *Uzawa algorithm*

Let us now write the discrete Uzawa Algorithm:

– we start with an arbitrary $\pi_h^0 = \{\pi_h^0(T), T \in \mathscr{T}_h\}$

(for example $\pi_h^0(T) = 0, \forall T$,

– when π_h^n is known, we compute u_h^{n+1} by

$$u_h^{n+1} \begin{cases} \sum X_i^{n+1} \, w_i \\ \sum Y_i^{n+1} \, w_i \end{cases}$$

$$\nu \sum_{j=1}^{N_m} X_j^{n+1} \, \alpha_{jk} = \int_\Omega f_x \, w_k \, \mathrm{d}M - \nu \sum_{j > N_m} X_j \, \alpha_{jk}$$

$$+ \sum_T \pi_h^n(T) \text{ area } (T) \frac{\partial w_k}{\partial x}(T) \; k = 1, . . ,$$

$$\nu \sum_{j=1}^{N_m} Y_j^{n+1} \, \alpha_{jk} = \int_\Omega f_y \, w_k \, \mathrm{d}M - \nu \sum_{j > N_m} Y_j \, \alpha_{jk}$$

$$+ \sum_T \pi_h^n(T) \text{ area } (T) \frac{\partial w_k}{\partial y}(T) \; k = 1, . . . , \tag{4.2}$$

and then we compute π_h^{n+1} by

$$\pi_h^{n+1}(T) = \pi_h^n(T) - \rho \, \mathrm{div}(u_h^{n+1})(T), \; \forall T \, . \tag{4.3}$$

The two components of the velocity in (4.2) are actually uncoupled; we just have to solve a linear system of the type

$$\sum_{j=1}^{N_m} Z_j \, \alpha_{jk} = f_k, \quad k = 1, . . ., N_m \, . \tag{4.4}$$

We proceed by overrelaxation (S.O.R.) (cf. Varga [1]).

Overrelation Optimal parameter

It is easy to verify that the matrix (α_{jk}) is symmetrical and positive definite. Let ω denote the relaxation parameter. We start with an arbitrary vector Z_j^0; when Z_j^n is known, we compute Z_j^{n+1} by

$$Z_k^{n+1} = (1-\omega)\,Z_k^n - \frac{\omega}{\alpha_{k,k}}\left(\sum_{j=1}^{k-1} \alpha_{jk}\,Z_j^{n+1} + \sum_{j=k+1}^{N_m} \alpha_{jk}\,Z_j^n - \frac{1}{\nu} f_k\right).$$

The stoping test is, as usual, of the type,

$$\operatorname*{Max}_k |Z_k^{n+1} - Z_k^n| < \epsilon_{\text{rel}}$$

The optimal value of the relaxation parameter ω can be determined by the following algorithm:

We apply the Gauss-Seidel method to the solution of the homogeneous system

$$\sum_j Z_j\,\alpha_{jk} = 0\,,$$

starting with a vector Z^0 of components $Z_j^0 > 0$. Then

$$Z_k^{n+1} = -\frac{1}{\alpha_{kk}}\left(\sum_{j=1}^{k-1} \alpha_{jk}\,Z_j^{n+1} + \sum_{j=k+1}^{N_m} \alpha_{jk}\,Z_j^n\right). \tag{4.6}$$

We set, for each n:

$$\rho_p^{n+1} = \operatorname*{Min}_k \frac{Z_k^{n+1}}{Z_k^n}\,,\quad \rho_g^{n+1} = \operatorname*{Max}_k \frac{Z_k^{n+1}}{Z_k^n} \tag{4.7}$$

$$\omega_p^{n+1} = \frac{2}{1+\sqrt{1-\rho_p^{n+1}}}\,,\quad \omega_g^{n+1} = \frac{2}{1+\sqrt{1-\rho_g^{n+1}}}\,. \tag{4.8}$$

The numerical tests performed show that both ω_p^{n+1} and ω_g^{n+1} converge to ω_{opt}.(1)

Storage of the matrix.

A					TABV			
$A_o(i)$	A(1,.)	A(2,.)	A(3,.)	A(4,.)	TABV(1,.)	TABV(2,.)	TABV(3,.)	TABV(4,.)
α_i^o	α_{i,i_1}	α_{i,i_2}	α_{i,i_3}	α_{i,i_4}	i_1	i_2	i_3	i_4

i_1, i_2, i_3, i_4 = the neighbourhood nodes of i

Direct methods (Cholesky, Frontal method, . . .) may be used to solve (4.2) provided a renumbering of the nodes is done. However it will be uneasy to use these methods in the non-linear case.

4.3. *Numerical results*

The choice of the parameter ρ in the Uzawa algorithm is done by experiment. We perform several tests with different ρ, and we note the results for a given number of Uzawa iterations (say, 10, 20, 30, . . .). A good test for the convergence is the quantity

$$C_h^{(n)} = \operatorname*{Max}_T |\operatorname{div}(u_h^n)_T|.\ ^{(2)}$$

Hereafter we give the results for the geometry 2, with $\nu = 1$, $\sigma_1 = 0$, $\sigma_2 \doteq 1$ (C_1 is at rest and C_2 is rotating with a unit angular velocity, no volumic forces, and $\nu = 1$).

The results given in Fig.5, 6, clearly show the existence of an optimal ρ,

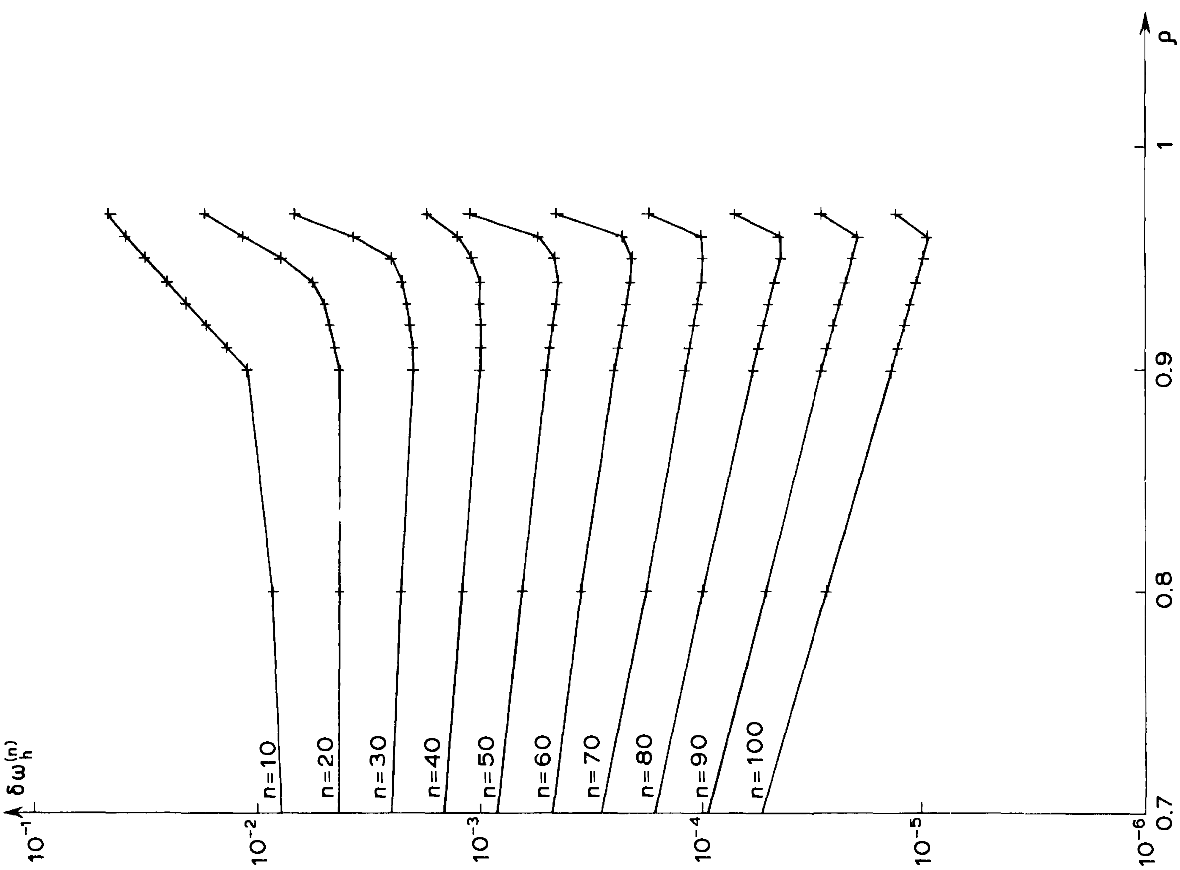
$$\rho_{opt} = 0.96 \text{ in this case.}$$

(1) ρ_p^{n+1} and ρ_g^{n+1} converge to $\rho(\mathcal{L}_1)$ = the spectral radius of the Gauss-Seidel matrix, and

$$\rho(B)^2 = \rho(\mathcal{L}_1),\quad \omega_{opt} = \frac{2}{1+\sqrt{1-\rho(B)^2}}\,.$$

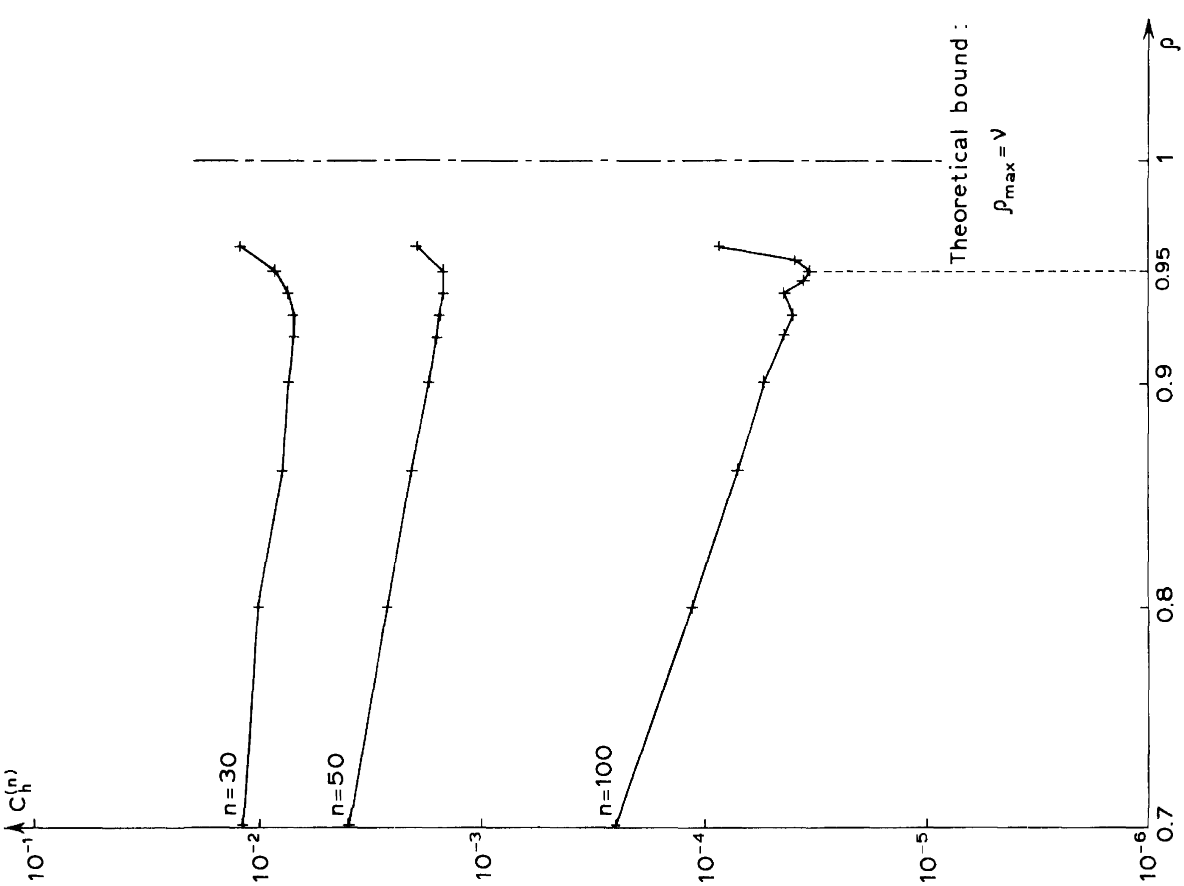
(2) This number, equal to $(1/\rho)\operatorname*{Max}_T |\pi_h^{(n+1)}(T) - \pi_h^n(T)|$ measures the convergence of discrete pressures.

We also consider $\delta w_h^{(n)} = \operatorname*{Max}_i\,[|X_i^{n+1} - X_i^n|, |Y_i^{n+1} - Y_i^n|]$ which measures the convergence of discrete velocities.



$\delta\omega_h^{(n)}$
ρ
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0.7
0.8
0.9
1
n=10
n=20
n=30
n=40
n=50
n=60
n=70
n=80
n=90
n=100

Fig. 6.



$C_h^{(n)}$
ρ
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0.7
0.8
0.9
0.95
1
n=30
n=50
n=100
Theoretical bound :
$\rho_{max} = \nu$

Fig. 5.

4.4. *The augmented Lagrangian technique*

This is a combination of penalty and Uzawa techniques.

We simply add to I (4.20.g) a penalisation term $(1/\epsilon)|\mathrm{div}_h \boldsymbol{v}_h|^2$, and this leads to the following set of equations (Fortin–Glowinski):

$$\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h + \frac{1}{\epsilon}(\mathrm{div}_h\, \boldsymbol{u}_h, \mathrm{div}_h\, \boldsymbol{v}_h) - (\pi_h, \mathrm{div}_h\, \boldsymbol{v}_h)$$

$$= (f, \boldsymbol{v}_h),\ \forall \boldsymbol{v}_h \in W_h\,. \tag{4.9}$$

We write the Uzawa algorithm for (4.9). Setting as before

$$u_h^{n+1} = \left\{ \begin{array}{l} \sum X_i^{(n+1)}\, w_i\,, \\ \\ \sum Y_i^{(n+1)}\, w_i\,, \end{array} \right.$$

we just have to replace the left-hand side of (4.2) by

$$\nu \sum_{j=1}^{N_m} X_j^{n+1}\, \alpha_{jk} + \frac{1}{\epsilon} \sum_{j=1}^{N_{1m}} \{X_j^{n+1}(D_{hx}\, w_j, D_{hx}\, w_k)$$

$$+ Y_j^{n+1}(D_{hy}\, w_j, D_{hx}\, w_k)\}$$

$$\nu \sum_{j=1}^{N_m} Y_j^{n+1}\, \alpha_{jk} + \frac{1}{\epsilon} \sum_{j=1}^{N_{1m}} \{X_j^{n+1}(D_{hx}\, w_j, D_{hy}\, w_k)$$

$$+ Y_j^{n+1}(D_{hy}\, w_j, D_{hy}\, w_k)\}\,. \tag{4.10}$$

We see on (4.10) that the equations for the X and Y components are no longer decoupled and then we must solve a linear system twice bigger than in the case $1/\epsilon = 0$ previously studied. Moreover, although the S.O.R. method is applicable for the solution of (4.10), the parameter ω cannot be found any more with the technique shown in Section 4.2; hence, the choice of ω is made empirically.

For the tests made on Problem 2, the value $\omega = 1, 6$ has been selected. The number n_ϵ of iterations which is necessary to achieve convergence ($C_h^{(n)} \leqslant 10^{-5}$ and $\delta w_h^n \leqslant 10^{-5}$) *decreases* as $1/\epsilon$ increases (starting from 0); it attains a minimum for $1/\epsilon = 8$, with the best $\rho = \rho(\epsilon) = 9.2$ in this case. The convergence was achieved after only 15 iterations ($n_\epsilon = 15$).

5. Solution of Navier–Stokes Equations

We will denote

$\widetilde{\boldsymbol{W}}_h =$ the approximation of $\boldsymbol{H}^1(\Omega)$

$\widetilde{W}_h =$ the approximation of $H^1(\Omega)$

$\boldsymbol{W}_h =$ the approximation of $\boldsymbol{H}_0^1(\Omega)$

$W_h =$ the approximation of $H_0^1(\Omega)$.

X_h is the space of step functions π_h which are constant on each $\mathscr{S} \in \mathscr{T}_h$ and vanish outside $\Omega(h)$.

The basis functions v_k which span W_h correspond to $k \leqslant N_m$. We set

$$S_k = \int_{\Omega_h} v_k^2 \, \mathrm{d}M$$

and we recall that

$$\int_{\Omega_h} v_k v_\ell \, \mathrm{d}M = 0 \text{ if } k \neq \ell.$$

As previously we introduce the discrete differentiation operators $D_{hx} v$, $D_{hy} v$ and the discrete divergence

$$D_h v = D_{hx} v_x + D_{hy} v_y \,.$$

The notations b_h, a_h are the same as in Chap. II, (3.78) and (3.80).

The discrete problem is stated in Chap. II (3.93). We have tried two of the algorithms presented in Chapter II.

Algorithm I. (Uzawa, cf. II.(3.107), II.(3.108)).

We start with some $p_h^{(0)} \in X_h$ (for instance $p_h^{(0)} = 0$). When $p_h^{(m)}$ is known ($m \geqslant 0$), we compute $\boldsymbol{u}_h^{m+1}$ such that

$$\nu((u_h^{m+1}, v_h))_h + b_h(u_h^{m+1}, u_h^{m+1}, v_h) + \frac{1}{\epsilon}(D_h u_h^{m+1}, D_h v_h)$$
$$= (p_h^m, D_h v_h) + (f, v_h) \;\; \forall v_h \in W_h. \qquad (5.1)$$

((5.1) may be solved by an under-relaxation technique).

Then we compute p_h^{m+1}

$$p_h^{m+1} = p_h^m - \rho D_h \, u_h^{m+1} \, . \tag{5.2}$$

Algorithm II. (Implicit Arrow-Hurwicz. Cf. II.(3.122), II.(3.123)).

u_h^{m+1} is computed according to

$$\begin{aligned} ((u_h^{m+1} - u_h^m, v_h))_h + \rho \Big[& \nu((u_h^m, v_h))_h \\ & + \frac{1}{\epsilon} (D_h \, u_h^{m+1}, D_h \, v_h) + b_h(u_h^m, u_h^{m+1}, v_h) \\ & - (p_h^m, D_h \, v_h) - (f, v_h) \Big] = 0 \ \forall \, v_h \in W_h \, . \end{aligned} \tag{5.3}$$

Then p_h^{m+1} is defined by

$$\alpha(p_h^{m+1} - p_h^m) + \rho \, D_h \, u_h^{m+1} = 0 \, .$$

Examples.

Fig. 7 shows the streamlines of the flow for Problem 1 for the best experimental choice of ρ, α, ϵ (Algorithm II, $\nu = 10^{-2}$, $U = 1$, 512 triangles; $1/\epsilon = 0, 5, 7, 10$ has been tested).

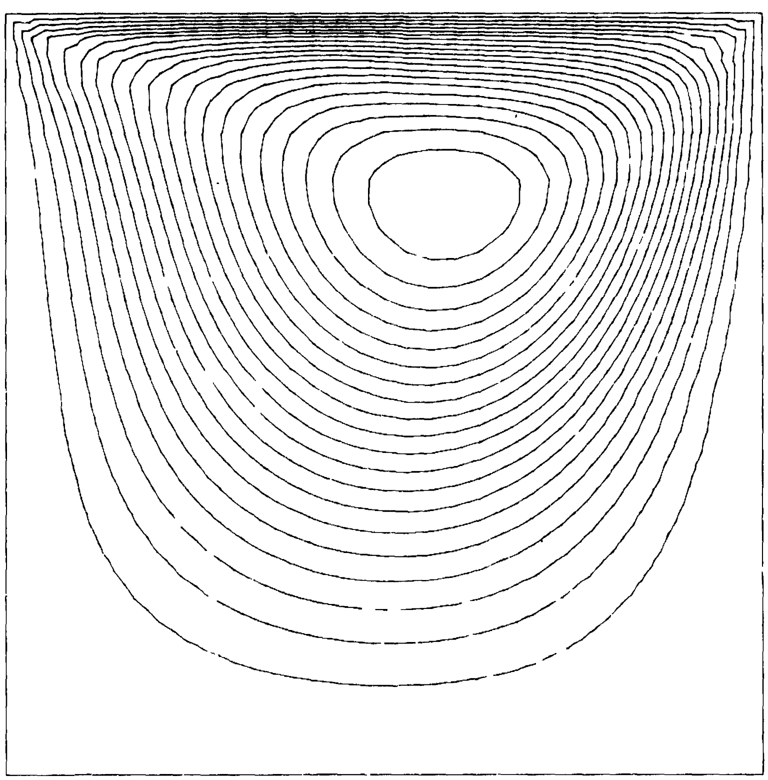


Fig. 7.

Fig. 8 shows the streamlines of the flow for Problem 2 ($\nu = 4.10^{-2}$, $\sigma_1 = 1$, $\sigma_2 = 0$, 412 triangles, 655 nodes (Fig. 4); the selected values are $\rho = 0.1$, $1/\epsilon = 10$, $\alpha = 0.004$).

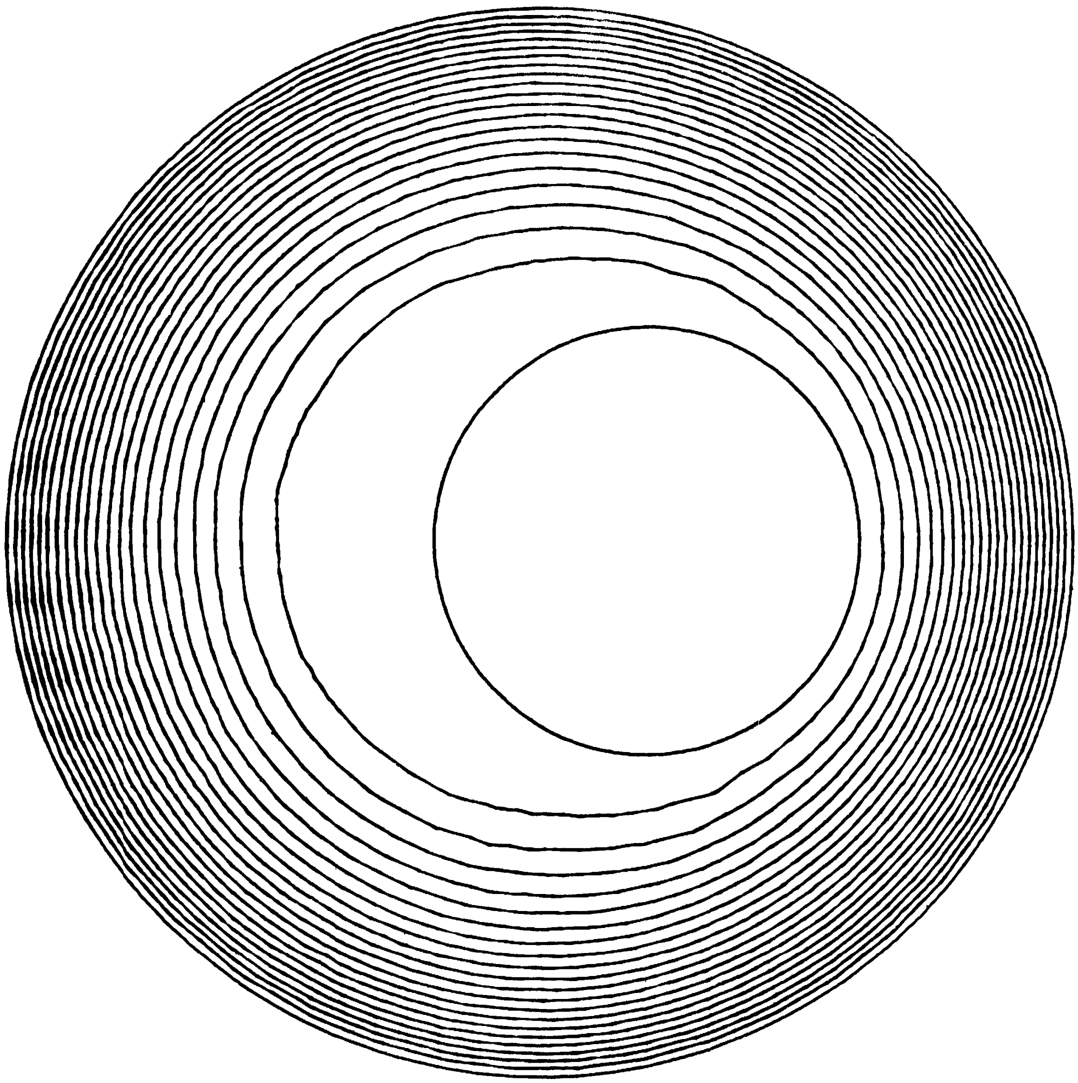


Fig. 8.

Chapter I

Section 1 contains a preliminary study of the basic spaces V and H; the trace theorem is proved by the methods of J. L. Lions–E. Magenes, see ref. [1]. The characterization of $H^{\perp}$ given here is based on a theorem of De Rham of the current theory. A more elementary proof is given in O. A. Ladyzhenskaya [1] for $n = 3$. A simplified version of O. A. Ladyzhenskaya's proof valid for all dimensions, was given in R. Temam [9]. Remark 1.9 gives another way for avoiding De Rham's theorem.

We have not given any systematic study nor review concerning the Sobolev spaces. We restricted ourselves to recalling properties of theses spaces when needed (Sections 1.1 of Chapters I and II in particular). As mentioned in the text, the reader is referred for proofs and further material to J. L. Lions [1], J. L. Lions and E. Magenes [1], J. Nečas [1], L. Sobolev [1], and others.

The variational formulation of Stokes equations was first introduced (in the general frame of the non-linear case) by J. Leray [1], [2], [3], for the study of weak or turbulent solutions of the Navier–Stokes equations. The existence of a solution of the Stokes variational problem is easily obtained by the classical Projection Theorem, whose proof is recalled for the sake of completeness. The study of the non-variational Stokes problem, and the regularity of solutions is based on the paper of Cattabriga [1] (if $n = 3$) and on the paper of Agmon, Douglis and Nirenberg [1] on elliptic systems (any dimension); these results are recalled without proofs. For another approach to the regularity cf. Solonnikov–Scadilov [1]. See also V.A. Solonnikov [4], Vorovich–Yudovich [1].

The concept of approximation of a normed space and of a variational problem was studied in particular by J. P. Aubin [1] and J. Cea [1]; the presentation followed here is that of R. Temam [8]. The discrete Poincare Inequality (Section 3.3) and the approximation of V by finite differences are in J. Cea [1]. The approximation of V by conforming finite elements was first studied and used by M. Fortin [2]; our description of the approximations (APX 2), (APX 3) (conforming finite elements), follows essentially M. Fortin [2]. In this reference one can also find many results of computations using this type of discretization. The idea of using the bulb function is due to P. A. Raviart; the presentation of the approximation (APX 2′) given here, is new. The approximation (APX 4) has been studied and used by J. P. Thomasset [1]. The material related to the nonconforming finite elements for the approximation of divergence free vector functions is due to Crouzeix, R. Glowinski, P.A. Raviart, and the author. Other aspects of the subject (non-conforming finite elements of higher degree and more refined error estimates) can be found in Crouzeix and P. A. Raviart [1]; for numerical experiments, see F. Thomasset [2] and also P. Lailly [1] in the case of the axisymmetric three dimensional flow.

For other applications of finite elements in fluid mechanics, see Oden, Zienkiewicz, Gallagher, Taylor, [1], and the proceedings of a conference held in Italy, June 1976 (to appear). Concerning the general theory of finite elements, let us mention the synthesis works of I. Babushka and A. K. Aziz [1], P. G. Ciarlet [1], P. A. Raviart [2], G. Strang and G. Fix [1], and the proceedings edited by A. K. Aziz [1]. For more references on finite elements (in general situations) the reader is referred to the bibliography of these works. The description of finite element methods given here is almost completely self-contained; we only assume a few specific results whose proofs would necessitate an introduction of tools quite remote from our scope.

After discretization of the Stokes problem, we have to solve a finite dimensional linear problem where the unknown is an element u_h of a finite dimensional space V_h. There are then two possibilities:

(a) either this space V_h possesses a natural and simple basis, such that the problem is reduced to a linear system with a sparse matrix for the components of u_h in this basis; in this case we solve the problem by resolution of this linear system;

(b) or, if not, the finite-dimensional problem is not so simple to solve (ill-conditioned or non-sparse matrix), even if it possesses a unique solution. In this case, appropriate algorithms must be introduced in order to solve these problems; this is the purpose of Section 5.

The algorithms described in Section 5 were introduced in the frame of optimization theory and economics in Arrow, Hurwicz and Uzawa [1]; the application of these procedures to problems of hydrodynamics is studied in J. Céa, R. Glowinski and J. C. Nedelec [1], M. Fortin [2], M. Fortin, R. Peyret, and R. Temam [1]. See in D. Bégis [1], M. Fortin [2], an experimental investigation of the optimal choice of the parameter ρ (or ρ and α); a theoretical resolution of this problem in a very particular case is given in Crouzeix [2].

The approximation of incompressible fluids by slightly compressible fluids, as in Section 6, has been studied by J. L. Lions [4] and

R. Temam [2]. The full asymptotic development of u_ϵ given here is new and due to M. C. Pelissier; for further investigations of this point the reader is referred to M. C. Pelissier [1]. Cf. also new related developments in R. S. Falk [1] and M. Bercovier [1].

Chapter II

Section 1 develops a few standard results concerning the existence and uniqueness of solutions of the nonlinear stationary Navier–Stokes equations. We follow essentially O. A. Ladyzhenskaya [1] and J. L. Lions [2]. A more complete discussion of the regularity of solutions and of the theory of hydrodynamical potentials can be found in O. A. Ladyzhenskaya [1]; for regularity, see also H. Fujita [1]. The stationary Navier–Stokes equations in an unbounded domain have been studied by R. Finn [1]–[5], R. Finn and D. R. Smith [1], [2], and J.G. Heywood [1], [3].

Some recent theoretical results concerning the stationary Navier–Stokes equations will be given in C. Foias and R. Temam [2], R. Temam [11].

Section 2 gives discrete Sobolev inequalities and compactness theorems, whose proofs are very technical. The principle of the proofs in the case of finite-differences parallels the corresponding proofs in the continuous case (see, for instance, J. L. Lions [1], J. L. Lions–E. Magenes [1]). The proof of discrete Sobolev inequalities has not been published before, the proof of the discrete compactness theorem can be found in P. A. Raviart [1]. For conforming finite elements the proofs are much simpler: in particular, for the discrete compactness theorem, the problem is reduced by a simple device to the continuous case. For non-conforming finite elements the proof of the Sobolev inequality is based on specific techniques of non-conforming finite element theory. The discrete compactness theorem is proved by comparison between conforming and non-conforming elements; these results are new.

The discussion of the discretization of the stationary Navier–Stokes equations follows the principles developed in Chapter I. The general convergence theorem is similar to that of Chapter I and the same types of discretization of V are considered; the differences lie in the lack of uniqueness of solutions of the exact problem. The numerical algorithms of Section 3.3 have been introduced and tested in Fortin, Peyret and Temam [1].

The non-uniqueness of stationary solutions of the Navier–Stokes and related equations has been investigated in recent years. The main results in this direction are due to P. H. Rabinowitz [2] and W. Velte [1], [2].

In [2] Rabinowitz establishes the non-uniqueness of solutions of the convection problem by explicitly constructing two different solutions (the first is the trivial one when the fluid is at rest, the second is constructed by an iterative procedure). The works of W. Velte are based on topological methods, the bifurcation theory and the topological degree theory; the problem considered in [1] is the convection problem as in Rabinowitz [2]. In [2], W. Velte proves the non–uniqueness of solutions of the Taylor problem and the situation is very similar to the problem for which existence is proved in Section 1, although not identical. Section 4 follows closely this presentation. For other applications of bifurcation theory see in particular, J. B. Keller and S. Antman [1], L. Nirenberg [1], P. H. Rabinowitz [4] and volume 3, number 2 of the Rocky Mountain J. of Math. (1973).

Chapter III

The existence and uniqueness results for the linearized Navier–Stokes equations (Section 1) are a special case of general results of existence and uniqueness of solutions of linear variational equations (see for instance, J. L. Lions–E. Magenes [1], vol. 2). For completeness, we have given an elementary proof of some technical results which are usually established as easy consequences of deeper results {i.e. Lemma 1.1 which is more natural in the frame of vector valued distribution theory (L. Schwartz [2]) or Lemma 1.2 which can be proved by interpolation methods (J. L. Lions–E. Magenes [1])}.

Theorem 2.1 is one of the standard compactness theorems used in the theory of nonlinear evolution equations. Other compactness theorems are proved and used in J. L. Lions [2].

The existence and uniqueness results related to the non-linear Navier–Stokes equations and given in Sections 3 and 4 are now classical and prolong the early works of J. Leray [1], [2], [3]; see E. Höpf [1], [2], O. A. Ladyzhenskaya [1], J. L. Lions [2], [3], J. L. Lions and G. Prodi [1], and J. Serrin [3]. Further results on the regularity of solutions and the study of the existence of classically differentiable solutions of the Navier–Stokes equations can be found in the second edition of O. A. Ladyzhenskaya [1]. For the analyticity of the solutions see C. Foias and G. Prodi [1], H. Fujita and K. Masuda [1], C. Kahane [1], K. Masuda [1], and J. Serrin [3].

Let us mention also two recent and completely different approaches to the existence and uniqueness theory that we did not treat here. The first one is that of E. B. Fabes, B. F. Jones, and N. M. Riviere [1] based on singular integral operator methods and giving existence and uniqueness results in L^p spaces. The other one is the method of D. G. Ebin and J. Marsden [1] connecting the Navier–Stokes initial value problem with the geodesics of a Riemann manifold and thus using the methods of global analysis.

The material of Section 5 containing a discussion of the stability and convergence of simple discretization schemes for the Navier–Stokes equations is essentially new; a similar study for different equations or different schemes was presented in R. Temam [2], [3], [4]. Stability and convergence of some unconditionally stable one step schemes are given in O. A. Ladyzhenskaya [5]; for fractional step schemes see A. J. Chorin [2], O. A. Ladyzhenskaya and V. I. Rivkind [1]. In all these references except in A. J. Chorin [2] the convergence is proved, as here, by obtaining appropriate *a priori* estimates of the approximated solutions and the utilization of a compactness theorem; in [2], A. J. Chorin assumes the existence of a very smooth solution and compares the approximated and exact solutions.

Section 7.1 is essentially an introduction to Section 7.2. The fractional step scheme described in Section 7.2 (the Projection Method) was independently introduced by A. J. Chorin [1], [2], [3] and the author R. Temam [3]; A. J. Chorin considers a slightly different form of the scheme, without the stabilizing term ½ (div $\boldsymbol{u}$) $\boldsymbol{u}$ (i.e. without replacing b by $\tilde{b}$). Applications and other aspects of this scheme are developed in particular in C. K. Chu and G. Johansson [1], C. K. Chu, K. W. Morton and K. V. Roberts [1], M. Fortin, R. Peyret and R. Temam [1], M. Fortin [1], M. Fortin and R. Temam [1], G. Marshall [1], [2] and C. S. Peskin [1], [2]. This scheme is an interpretation of the fractional step method introduced and studied by G. I. Marchuk [1] and N. N. Yanenko [1] (see Section 8).

The approximation of the Navier–Stokes equations by the equations of slightly compressible fluids (subsection 8.1) was introduced by N. N. Yanenko [1] who considers slightly more complicated perturbed equations. The introduction of these perturbations enabled N. N. Yanenko to use the fractional step method which is studied in Subsection 8.2. Let us point out that the schemes of Section 7 are fractional step schemes not needing the consideration of perturbed equations.

The proof of convergence of the fractional step scheme which is given here is due to R. Temam [3], [4] and follows the method introduced in R. Temam [1]. For other aspects of the Fractional Step Method, see G. I. Marchuk [1], N. N. Yanenko [1], [2] and their bibliographies; see also R. Temam [1], [6], [7].

Other types of perturbed problems, whose purpose is to overcome the difficulties of the constraint "div $\boldsymbol{u} = 0$" (but not to apply fractional step methods) are studied in J. L. Lions [4] and R. Temam [2]. For the alternating direction methods and further results on fractional steps methods, see O.A. Ladyzhenskaya and V.I. Rivkind [1], V.I. Rivkind and B.S. Epstein [1], and B.S. Epstein [1].

The material of Sections 5 to 8 is only a very small part of a considerable amount of work on the approximation of fluid mechanic equations; up-to-date results and very useful references can be found in the proceedings edited by O. M. Belotserkovskij [1], M. Holt [2], H. Cabannes and R. Temam [1], and Richtmyer [1]. See also K. Bryan [1], C. K. Chu and H. O. Kreiss [1], and the list of references compiled by the Los Alamos Scientific Laboratory.

Many other problems can be handled by the methods used here. For the Navier–Stokes equations properly speaking one can consider different boundary conditions (see Iooss [1]), or periodic solutions (G. Prouse [1], [2]), variational inequalities (Lions [2]); Stochastic Navier–Stokes equations are studied in Bensoussan–Temam [1], Foias [1] and Vishik [1]. Optimal control problems for systems governed by the Navier–Stokes equations appear in Cuvelier [1].

The difficulties encountered in the mathematical theory of the Navier–Stokes equations lead several authors to reconsider the fluid mechanic hypotheses leading to these equations and to propose new models with a better mathematical behaviour; see Kaniel [1], Ladyzhenskaya [1].

Similar models involving other equations (most often the Navier–Stokes equations coupled with other equations) are: the convection equations whose treatment is almost identical to the treatment of the Navier–Stokes equations, several fluid models, pollution (Marshall [1]) or blood models (Peskin [1]), and oceanography models (having the appearance of a concentration equation). More elaborated are the magnetohydrodynamic equations and the Bingham equations (see Duvaut–Lions [1], [2]) which are an example of non-Newtonian fluids.

The mathematical theory of the Euler equations has not been developed here. For a treatment based on analytical methods, cf. C. Bardos [1], T. Kato [1], [2], J L. Lions [2], R. Temam [10], [12], Yudowitch [1].

Some results related to the behaviour of the Navier–Stokes equations as $\nu \to 0$ are given in Lions [2], Yudowitch [1]. A similar problem for a model equation related to the Burgers equation is completely studied in C. M. Brauner, P. Penel and R. Temam [1], P. Penel [1]; cf. also C. Bardos, U. Frish, P. Penel and P. L. Sulem in R. Temam [12].

ADDITIONAL COMMENTS FOR THE REVISED EDITION

We mention here the most recent results concerning the theory and numerical analysis of Navier Stokes equations.

For the theory, slight improvements of the existence theorems for the steady-state nonlinear equations has been obtained and are presented in Appendix I. Various results concerning the structure of the set of solutions of stationary (and time periodic) Navier–Stokes equations are presented in the References: C. Foias–R. Temam [2], [3], R. Temam [11] which appeared since the first edition; cf. also R. Temam [a], and J.C. Saut–R. Temam [a], [b] below, where topological (transversality) methods are applied leading to the study of a Cauchy problem for the stationary linear Stokes equations with lower order perturbation terms. The singularities which may present the solutions of the time-dependent Navier–Stokes equations in dimension 3 are investigated by V. Schaffer [a], [b] where the Hausdorff dimension of the singularities is estimated. The Navier–Stokes equations in domains with "many" wholes are considered by J.L. Lions [a] in relation with the theory of homogenization, the flow in porous media and the Darcy law. For the progress in bifurcation theory cf. P.H. Rabinowitz [a]. For the time dependent Navier–Stokes equations cf. also C. Foias–R. Temam [a], [b].

In the numerical analysis of Navier–Stokes equations among many publications dealing in particular with the application of finite element and alternating direction methods which appeared, cf. R.H. Gallagher, O.C. Zienckiewicz, J.T. Oden, M. Morandi-Cecchi, C. Taylor [a], M. Fortin and F. Thomasset [a], R. Temam [b], R. Temam and F. Thomasset [a], F. Thomasett [a], H.J. Wirz [a]; for mixed and hybrid finite elements cf. M. Fortin [a], P.A. Raviast [a]. See also the bibliography of these references.

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