问题. (Rudin or Folland) 设$0<\mu(X)<\infty$, 记

$$A_p\left(f\right)=\left(\frac{1}{\mu\left(X\right)}\int_X\left|f\right|^p\,\mathrm{d}\mu\right)^{1/p}=:\left(⨍_X\left|f\right|^p\,\mathrm{d}\mu\right)^{1/p},\quad 0<p\leq\infty.$$

证明:

1. $\forall f\in L^p\left(X,\mu\right)$, $A_p\left(f\right)$是p的增函数.
2. 若有$0<p<q\leq\infty$, 使$A_p\left(f\right)=A_q\left(f\right)$, 证明$f$除了一个0测度集外是一个常数.
3. 证明:
$$\lim_{p\to 0^+}A_p\left(f\right)=\exp\left\{⨍_X\ln\left|f\right|\,\mathrm{d}\mu\right\}.$$
4. 设$f\in L^\infty\left(X,\mu\right)$, 证明:
$$\lim_{n\to\infty}A_n\left(f\right)=\left\|f\right\|_\infty.$$

问题. 设$1\leq p<\infty$, 且$f\in L^p_{\mathrm{loc}}\left(\mathbb{R}\right)$, 则对于$\mathbb{R}$中的任何有限测度集$M$, 有

$$\lim_{h\to 0}\int_M\left|f\left(x+h\right)-f\left(x\right)\right|^p\,\mathrm{d}x=0.$$

问题. (Folland) If $1\leq p<r\leq\infty$, show that L^p+L^r is a Banach space with norm

$$\left\|f\right\|=\inf\left\{\left\|g\right\|_p+\left\|h\right\|_r:f=g+h\in L^p+L^r\right\},$$

and if $p<q<r$, the inclusion map $L^q\to L^p+L^r$ is continuous, ~~i.e. $L^q\hookrightarrow L^p+L^r$, or $\forall f\in L^p+L^r$, $\|f\|\leq c\|f\|_q$, where $c>0$ independent of p,q,r and f.~~