问题. (Lieb) Prove the following statements:

1. Suppose Ω is the open subset of $\mathbb{R}^n$, $f, g \in L^1_{\text{loc}}(\Omega)$, if the distributions defined by f and g are equal, **then** $f(x) = g(x)$, *a.e.* $x \in \Omega$.

2. Suppose $T \in \mathscr{S}'(\mathbb{R}^n)$, $\varphi \in \mathscr{S}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, **show that**

$$T(\varphi(\cdot - y)) - T(\varphi(\cdot)) = \int_0^1 \sum_{j=1}^n y_j \langle \partial_j T, \varphi(\cdot - ty)\rangle \, \mathrm{d}t =: \int_0^1 y \cdot \langle \nabla T, \varphi(\cdot - ty)\rangle \, \mathrm{d}t.$$

   We know that $z \mapsto (\partial_j T)(\varphi(\cdot - z))$ (or $z \mapsto \langle \partial_j T, \varphi(\cdot - z)\rangle$) is a C^∞-function, so the integral is well defined. Also this means that the Fundamental theorem of calculus for distributions is still hold, and be aware that ∇T can be measures, i.e. when T is a Heaviside function or functions in $BV(\mathbb{R}^n)$.

3. Convolution and distribution can be interchangable, i.e. if $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\psi \in L^1(\mathbb{R}^n)$, $T \in \mathscr{S}'(\mathbb{R}^n)$, then $\langle \psi * T, \phi\rangle = \langle T, \psi * \phi\rangle$ which means

$$\int \psi(y) \langle T, \phi(\cdot - y)\rangle \, \mathrm{d}y = \left\langle T, \int \psi(y) \phi(\cdot - y) \, \mathrm{d}y \right\rangle.$$

   Let $\varphi \in C_c^\infty(\mathbb{R}^n)$, **show that** there is a function $f \in C^\infty(\mathbb{R}^n)$ s.t.

$$\langle \varphi * T, \phi \rangle = \int_{\mathbb{R}^n} f(y) \phi(y) \, \mathrm{d}y \qquad \forall \phi \in \mathscr{S}(\mathbb{R}^n).$$

   If $\int_{\mathbb{R}^n} \varphi = 1$, $\varphi_\epsilon(x) = \frac{1}{\epsilon^n} \varphi\left(\frac{x}{\epsilon}\right)$, for $\epsilon > 0$, (φ_ϵ is a mollifier), then $\varphi_\epsilon * T \to T$ in $\mathscr{S}'(\mathbb{R}^n)$ as $\epsilon \to 0$.

问题. Suppose $P_t(x) = t^{-n} P\left(\frac{x}{t}\right)$, where $P(x) = \frac{C_n}{1+|x^2|^{\frac{n+1}{2}}}$. Liouville theorem shows that if a harmonic function u is bounded in $\mathbb{R}^n$, it must be a constant.

1. If $f \in L^p(\mathbb{R}^n)$, $p \in (1, \infty)$, we know $u(x,t) := P_t * f(x)$ is harmonic in $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times \mathbb{R}_+$. If $u(x,t)$ is bounded in $\mathbb{R}^{n+1}_+$. **Show that**

$$\int_{\mathbb{R}^n} P_{t_2}(x-y) u(y, t_1) \, \mathrm{d}y = u(x, t_1 + t_2).$$

2. If $u(x,t)$ is harmonic in $\overline{\mathbb{R}^{n+1}_+}$, and there is a $C > 0$, $p \in (1, \infty]$ such that

$$\|u(\cdot, t)\|_p \leq C, \qquad \forall t > 0.$$

   Then there is a f such that $u(x,t) = P_t * f(x)$. (Hint: using Banach Alaoglu theorem)

问题. [1, Sec. 2.4] Suppose $f : \mathbb{C}^+ \to \mathbb{C}$ is analytic, and

$$\sup_{0<y<\infty} \int_{-\infty}^{\infty} |f(x + \mathrm{i}y)|^2 \, \mathrm{d}x = C < \infty.$$

1. Define $G(x) := \int_{x+\mathrm{i}}^{x+\mathrm{i}y} f(z) \mathrm{e}^{-\mathrm{i}tz} \, \mathrm{d}z$, **then** there exists a sequence $x_i \to \infty$ with

$$G(\pm x_i) \to 0.$$

2. If $f_y(x) = f(x + \mathrm{i}y)$, $F(t) = \mathrm{e}^{2\pi t} \widehat{f_1(x)}(t)$, **then**

$$F(t) = \mathrm{e}^{2\pi t} \widehat{f_y(x)}(t) = \int_{-\infty}^{\infty} f(x + \mathrm{i}y) \mathrm{e}^{2\pi \mathrm{i}t(x+\mathrm{i}y)} \, \mathrm{d}x.$$

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3. **Show that** there exists $F \in L^2(0, \infty)$ such that $\forall z \in \mathbb{C}^+$,

$$f(z) = \int_0^\infty F(t) \mathrm{e}^{2\pi \mathrm{i}tz} \, \mathrm{d}t$$

   and $\|F\|_2^2 = C$.

References

[1] 程民德, 邓东皋, 龙瑞麟, 实分析, 高等教育出版社.

[2]