**问题 1.** (Folland) Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, $m =$ Lebesgue measure, and $\mu =$ counting measure on $\mathcal{M}$.

1. $m \ll \mu$ but $\mathrm{d}m \neq f \,\mathrm{d}\mu$ for any $f$.
2. $\mu$ has no Lebesgue decomposition with respect to $m$.

**问题 2.** (Folland) By the definition:

> If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, define its Hardy-Littlewood maximal function $Mf$ by
> $$Mf(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)| \,\mathrm{d}y.$$

If $f \in L^1(\mathbb{R}^n)$, $f \neq 0$, there exist $C, R > 0$ such that $Mf(x) \geq \frac{C}{|x|^n}$ for $|x| > R$. Hence $m(\{x : Mf(x) > \alpha\}) \geq \frac{C'}{\alpha}$ when $\alpha$ is small, so the maximal theorem is essentially sharp.

**问题 3.** (Folland) If $E$ is a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of $E$ at $x$ is defined as

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))},$$

whenever the limit exists.

1. Show that $D_E(x) = 1$ for *a.e.* $x \in E$ and $D_E(x) = 0$ for *a.e.* $x \in E^c$.
2. Find examples of $E$ and $x$ such that $D_E(x)$ is a given number $\alpha \in (0,1)$, or such that $D_E(x)$ does not exist.

**问题 4.** (Rudin) Let $L^\infty = L^\infty(m)$, where $m$ is Lebesgue measure on $I = [0,1]$. Show that there is a bounded linear functional $\Lambda \neq 0$ on $L^\infty$ that is 0 on $C(I)$, and that therefore there is no $g \in L^1(m)$ that satisfies $\Lambda f = \int_I fg \,\mathrm{d}m$ for every $f \in L^\infty$. Thus $(L^\infty)^* \neq L^1$.

**问题 5.** (Rudin) Suppose $1 < p < \infty$ and prove that $L^q(\mu)$ is the dual space of $L^p(\mu)$ even if $\mu$ is not $\sigma$-finite, $1/p + 1/q = 1$ as usual.

**问题 6.** (GTM 18*) If $\mu$ and $\nu$ are totally $\sigma$-finite signed measures such that $\nu \ll \mu$, then

$$\nu\left(\left\{x : \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = 0\right\}\right) = 0.$$