Abstract

题目较难的, 大家量力而为.

问题 1. (Rudin) Let $(X, \mathfrak{M}, \mu)$ be a positive measure space. Call a set $\Phi \subset L^1(\mu)$ uniformly integrable if $\forall \epsilon > 0$ corresponds a $\delta > 0$ such that

$$\left| \int_E f \, \mathrm{d}\mu \right| < \epsilon$$

whenever $f \in \Phi$ and $\mu(E) < \delta$.

1. Prove the following convergence theorem of Vitali: If
   (a) $\mu(X) < \infty$,
   (b) $\{f_n\}$ is uniformly integrable,
   (c) $f_n(x) \to f(x)$ *a.e.* as $n \to \infty$,
   (d) $|f(x)| < \infty$ *a.e.*, then $f \in L^1(\mu)$ and

$$\lim_{n \to \infty} \int_X |f_n - f| \, \mathrm{d}\mu = 0.$$

2. Show that (1) fails if μ is Lebesgue measure on $(-\infty, \infty)$, even if $\{\|f_n\|_1\}$ is assumed to be bounded. Hypothesis (a) can therefore not be omitted in (1).
3. Show that hypothesis (d) is redundant in (1) for some μ (for instance, for Lebesgue measure on a bounded interval), but that there are finite measures for which the omission of (d) would make (1) false.
4. Show that Vitali's theorem implies Lebesgue's dominated convergence theorem (DCT), for finite measure spaces. Construct an example in which Vitali's theorem applies although the hypotheses of Lebesgue's theorem do not hold.
5. Construct a sequence $\{f_n\}$, say on $[0, 1]$, so that $f_n(x) \to 0$ for every x, $\int f_n \to 0$, but $\{f_n\}$ is not uniformly integrable (with respect to Lebesgue measure).
6. The following converse of Vitali's theorem is true: If $\mu(X) < \infty$, $f_n \in L^1(\mu)$, and

$$\lim_{n \to \infty} \int_E f_n \, \mathrm{d}\mu$$

   exists for every $E \in \mathfrak{M}$, then $\{f_n\}$ is uniformly integrable.

prove this by completing the following outline.

Define $\rho(A, B) = \int |\chi_A - \chi_B| \, \mathrm{d}\mu$. Then $(\mathfrak{M}, \rho)$ is a complete metric space (modulo sets of measure 0), and $E \to \int_E f_n \, \mathrm{d}\mu$ is continuous for each n. If $\epsilon > 0$, there exist E_0,δ, N so that

$$\left| \int_E (f_n - f_N) \, \mathrm{d}\mu \right| < \epsilon \quad \text{if } \rho(E, E_0) < \delta, \quad n > N. \tag{1}$$

If $\mu(A) < \delta$, (1) holds with $B = E_0 - A$ and $C = E_0 \cup A$ in place of E. Thus (1) holds with A in place of E and 2ϵ in place of ϵ. Now apply (a) to $\{f_1, \cdots, f_N\}$: There exists $\delta' > 0$ such that

$$\left| \int_A f_n \, \mathrm{d}\mu \right| < 3\epsilon \quad \text{if } \mu(A) < \delta', \quad n = 1, 2, 3, \cdots.$$

问题 2. (GTM18) Does the set of all totally finite signed measures on a σ-algebra form a Banach space with respect to the norm defined by $\|\mu\| \equiv |\mu|(X)$?

问题 3. Prove the following statement:

1. Suppose $1 \le p < \infty$, Ω be bounded domain and $u_n \rightharpoonup u \quad \text{in } L^p(\Omega)$, Then u_n is bounded in $L^p(\Omega)$ and
$$\|u\|_{L^p(\Omega)} \le \liminf_{n\to\infty} \|u_n\|_{L^p(\Omega)}.$$

2. Let $1 \le p < \infty$, u be a 2π periodic function in $L^p(\mathbb{R})$. for $n = 1, 2, \cdots$, set
$$u_n(x) := u(nx).$$
Then
$$u_n \rightharpoonup \frac{1}{2\pi}\int_0^{2\pi} u(y)\,\mathrm{d}y \quad \text{in } L^p(U) \text{ for any bounded open set } U \subset \mathbb{R}.$$

问题 **4.** (Lecture) Find the closure of $C([0,1])$ under weakly sequential convergence.

问题 **5.** (Lecture) Discuss the compactness criterion of $L^p(\mathbb{R}^n, \mathrm{d}\mu)$, Here μ is a Borel measure.

问题 **6.** (Schur's Theorem*) In l^1, weak and norm convergences of sequences coincide.

问题 **7.** (GTM92, 证明以下任何一部分结论(除指明不必证的结论)).

If (f_n) is a **weakly null sequence** (a sequence weakly converge to 0) in $L^2[0,1]$, then (prove this if you like) there is a subsequence (f_{k_n}) such that
$$\lim_n \sup_{j_1<\cdots<j_n} \left\| \frac{1}{n}\sum_{i=1}^n f_{k_{j_i}} \right\|_2 = 0.$$

1. (Banach-Saks*) Suppose $1 < p \le 2$. It is easy to prove (donot prove this statement) that there is a constant $A > 0$ for which
$$|a+b|^p \le |a|^p + p|a|^{p-1} b\,\mathrm{sign}(a) + A|b|^p \quad a, b \in \mathbb{R}.$$
If (f_n) is a weakly null sequence in $L^p[0,1]$, **then** (f_n) admits a subsequence (f_{k_n}) for which $\|\sum_{i=1}^n f_{k_i}\| = O\left(n^{1/p}\right)$. **Hence**, any bounded sequence in $L^p[0,1]$ admits a subsequence whose arithmetic means (or Cesaro means) converge in norm $(1 < p \le 2)$.

2. (Szlenk*) In $L^1[0,1]$, every weakly convergent sequence has a subsequence whose arithmetic means are norm convergent.

3. (Banach-Saks*) Suppose $p > 2$. Let $p > 2$, $[p]$ is the greatest positive integer $\le p$. Using the conclusion below

There are $A, B > 0$ such that (donot prove this) for any real numbers a, b
$$|a+b|^p \le |a|^p + p|a|^{p-1} b\,\mathrm{sign}(a) + A|b|^p + B\sum_{j=2}^{[p]} |a|^{p-j}|b|^j.$$

If (f_n) is a weakly null sequence in $L^p[0,1]$, **then** (f_n) has a subsequence (g_n) such that
$$\begin{aligned}\int_0^1 \left|\sum_{i=1}^n g_i(t)\right|^p \mathrm{d}t \le &\int_0^1 \left|\sum_{i=1}^{n-1} g_i(t)\right|^p \mathrm{d}t + p\int_0^1 \left|\sum_{i=1}^{n-1} g_i(t)\right|^{p-1} \mathrm{sign}\left(\sum_{i=1}^{n-1} g_i(t)\right) g_n(t)\,\mathrm{d}t \\ &+ A\int_0^1 |g_n|^p\,\mathrm{d}t + B\sum_{j=2}^{[p]}\int_0^1 \left|\sum_{i=1}^{n-1} g_i(t)\right|^{p-j} |g_n(t)|^j\,\mathrm{d}t.\end{aligned}$$

And **hence** we have $\|\sum_{i=1}^n g_i\|_p \le M\sqrt{n}$, for some $M > 0$.