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现代分析 I 期末考试试题

20/0 10' **问题 1.** suppose $f : X \to [0, \infty]$ is measurable, and

$$\nu(E) = \int_E f \,\mathrm{d}\mu \quad (E \in \mathfrak{M}).$$

Then ν is a measure on $\mathfrak{M}$, and

$$\int_X g \,\mathrm{d}\nu = \int_X g f \,\mathrm{d}\mu$$

$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} f(x) e^{ix^5}\,dx = \int_{\mathbb{R}} f(x) \cos(x^5+\theta)\,dx$

for every measurable g on X with range in $[0, \infty]$.

40/1 15' 3问 **问题 2.** Define the *essential range* of a function $f \in L^\infty(\mu)$ to be the set R_f consisting of all complex numbers w such that

$$\mu(\{x : |f(x) - w| < \epsilon\}) > 0$$

for every $\epsilon > 0$. Prove that R_f is compact. What relation exists between the set R_f and the number $\|f\|_{L^\infty}$?

Let A_f be the set of all averages

$$\frac{1}{\mu(E)} \int_E f \,\mathrm{d}\mu$$

where $E \in \mathfrak{M}$ and $\mu(E) > 0$. What relations exist between A_f and R_f? Is A_f always closed? Are there measure μ such that A_f is convex for every $f \in L^\infty(\mu)$? Are there measure μ such that A_f fails to be convex for some $f \in L^\infty(\mu)$?

How are these results affected if $L^\infty(\mu)$ is replaced by $L^1(\mu)$, for instance?

40/0 **问题 3.** (Folland p200) If f is a measurable function on X, define the essential range R_f of f to be the set of all $z \in \mathbb{C}$ such that $\{x : |f(x) - z| < \epsilon\}$ has positive measure for all $\epsilon > 0$.

1. R_f is closed.
2. If $f \in L^\infty$, then R_f is compact and $\|f\|_\infty = \max\{|z| : z \in R_f\}$.

21' 3问 **问题 4.** The mapping $f \to \widehat{f}$ is a one-to-one bounded linear transformation of $L^1(T)$ into (but not onto) c_0. 5.15

6.8. 20'. 4问 **问题 5.** Suppose, μ, λ, λ_1, and λ_2 are measures on a σ-algebra $\mathfrak{M}$, and μ is positive.

1. If λ is concentrated on A, so is $|\lambda|$.
2. If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.
3. If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.
4. If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$.
5. If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.
6. If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.
7. If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.

5 **问题 6.** Suppose μ and λ are measures on a σ-algebra $\mathfrak{M}$, μ is positive, and λ is complex. Then the following two condition are equivalent:

1. $\lambda \ll \mu$.

2. To every $\epsilon > 0$ corresponds a $\delta > 0$ such that $|\lambda(E)| < \epsilon$ for all $E \in \mathfrak{M}$ with $\mu(E) < \delta$.

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2 i(2).

问题 7. Let $\{f_n\}$ be a sequence of continuous complex functions on a (nonempty) complete metric space X, such that $f(x) = \lim f_n(x)$ exists (as a complex number) for every $x \in X$.

1. Prove that there is an open set $V \neq \emptyset$ and a number $M < \infty$ such that $|f_n(x)| < M$ for all $x \in V$ and for $n = 1, 2, 3, \ldots$.

2. If $\epsilon > 0$, prove that there is an open set $V \neq \emptyset$ and an integer N such that $|f(x) - f_n(x)| \leq \epsilon$ if $x \in V$ and $n \geq N$.

Hint for (2): For $N = 1, 2, 3, \ldots$, put

$$A_N = \{x : |f_m(x) - f_n(x)| \leq \epsilon \text{ if } m \geq N \text{ and } n \geq N\}.$$

Since $X = \bigcup A_N$, some A_N has a nonempty interior.

问题 8. (Folland) Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

1. If $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.

2. Say that $E \sim F$ if $\mu(E \Delta F) = 0$; then $\sim$ is an equivalence relation on $\mathcal{M}$.

3. For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence clases.

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问题 9. (Folland P77) Suppose that $\mu(X) < \infty$ and $f : X \times [0, 1] \to \mathbb{C}$ is a function such that $f(\cdot, y)$ is measurable for each $y \in [0, 1]$ and $f(x, \cdot)$ is continuous for each $x \in X$.

1. If $0 < \epsilon, \delta < 1$ then $E_{\epsilon,\delta} = \{x : |f(x, y) - f(x, 0)| < \epsilon \text{ for all } y < \delta\}$ is measurable.

2. For any $\epsilon > 0$ there is a set $E \subset X$ such that $\mu(E) < \epsilon$ and $f(\cdot, y) \to f(\cdot, 0)$ uniformly on E^c as $y \to 0$.

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问题 10. (Folland p102) Suppose that ν and μ are finite measure on $(X, \mathcal{M})$. Either $\nu \perp \mu$, or there exist $\epsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\nu \geq \epsilon\mu$ on E (that is, E is a positive set for $\nu - \epsilon\mu$).

问题 11. (Folland p211) If $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$ and ϕ is a nonnegative Borel measurable function on $(0, \infty)$, then

$$\int_X \phi \circ |f| \, \mathrm{d}\mu = -\int_0^\infty \phi(\alpha) \, \mathrm{d}\lambda_f(\alpha).$$

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问题 12. (Folland p265) If $f \in L^1$ and $\widehat{f} \in L^1$, then f agrees almost everywhere with a continuous function f_0, and $\left(\widehat{f}\right)^{\vee} = \widehat{(f^{\vee})} = f_0$.