

Properties of the Classical Fourier Transform; Some Examples

Definition of the Classical Fourier Transform Let $f(x)$ be a (possibly complex-valued) function defined for $-\infty \leq x \leq \infty$ and square integrable, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty.$$

There are certain *regularity requirements* inherent in this definition which we do not want to get into right now; we comment further on this in the section on **proof of Fourier transform properties**. The transform produces from $f(x)$, another function, $\hat{f}(\xi)$, by the formula

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx.$$

It turns out that the transform is invertible, with the inverse transform being given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \hat{f}(\xi) d\xi.$$

Minor variations common in the literature involve replacement of the factor $\frac{1}{\sqrt{2\pi}}$ common to both of our formulas with $\frac{1}{2\pi}$ in one of the formulas and just 1 in the other formula. The factor $\frac{1}{\sqrt{2\pi}}$ plays much the same role for the classical, or *continuous* Fourier transform as the factor $\frac{1}{\sqrt{N}}$ does in the case of the **discrete Fourier transform**. With the transform as we have defined it, the function $\hat{f}(\xi)$ is square integrable just in case $f(x)$ is square integrable and we have the *Plancherel identity*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi.$$

It is also true that if $f(x)$ and $g(x)$ are both square integrable functions and $\hat{f}(\xi)$ and $\hat{g}(\xi)$ are their respective transforms, then

$$\langle f, g \rangle \equiv \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx = \int_{-\infty}^{\infty} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi = \langle \hat{f}, \hat{g} \rangle.$$

The second property reduces to the first when we set $g(x) = f(x)$, of course. We express both of these properties by saying that the Fourier transform is *unitary*.

Fourier Transform Properties The properties of the Fourier transform are much the same as, but not identical to, the properties of the **Laplace transform**. This is to be expected because, as shown in the section on the **relationship between the two**, the Laplace transform is just a special case of the Fourier transform together with a change of variable in the complex plane. In many cases the proofs of the Fourier transform properties are essentially the same as those given for the **Laplace transform**; we do not repeat those here. The property of *linearity* is more or less obvious; we refer the reader to the Laplace transform discussion. In the subsequent discussion of the Fourier transform and its properties we will use the symbols $\hat{f}(\xi)$ or $\text{calF}(f)(\xi)$, whichever is more convenient under the circumstances.

Property I: Fourier Transform of $e^{iax} f(x)$. We have

$$\begin{aligned} \text{calF} \left(e^{iax} f(x) \right) (\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} e^{iax} f(x) dx \\ &= \int_{-\infty}^{\infty} e^{-i(\xi-a)x} f(x) dx = \text{calF}(f)(\xi - a). \end{aligned}$$

Property II: Fourier Transform of $(ix)^n f(x)$

$$\text{calF} (((ix)^n f(x))) (\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} (ix)^n f(x) dx$$

$$= (-1)^n \frac{d^n}{d\xi^n} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx \right) = (-1)^n \frac{d^n}{d\xi^n} \text{calF}(f)(\xi).$$

Property III: Fourier Transform of $\mathbf{F}(\mathbf{x}) = \int_0^{\mathbf{x}} \mathbf{f}(\mathbf{y}) d\mathbf{y}$ If $f(x)$ and $F(x)$, as indicated, lie in $L^2(-\infty, \infty)$ then $\frac{\text{calF}(f)(\xi)}{\xi}$ lies in $L^2(-\infty, \infty)$ and, using integration by parts, we have

$$\begin{aligned} \text{calF}(F)(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \int_0^x f(y) dy dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{e^{-i\xi x}}{i\xi} \right) f(x) dx = \frac{1}{i\xi} \text{calF}(f)(\xi). \end{aligned}$$

Property IV: Fourier Transform of $\mathbf{f}'(\mathbf{x}), \mathbf{f}^{(\mathbf{k})}(\mathbf{x})$ If $f(x)$ and $f'(x)$ lie in $L^2(-\infty, \infty)$ then $\xi \text{calF}(f)(\xi)$ lies in $L^2(-\infty, \infty)$ and, again using integration by parts,

$$\begin{aligned} \text{calF}(f')(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f'(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\xi e^{-i\xi x} f(x) dx = i\xi \text{calF}(f)(\xi). \end{aligned}$$

This process can be repeated to see that if $f(x), f'(x), \dots, f^{(k)}(x)$ all lie in $L^2(-\infty, \infty)$ then $\xi^j \text{calF}(f)(\xi)$ lies in $L^2(-\infty, \infty)$, $j = 1, 2, \dots, k$ and

$$\text{calF}(f^{(j)})(\xi) = (i\xi)^j \text{calF}(f)(\xi), \quad j = 1, 2, \dots, k.$$

Property V: Fourier Transform of $\mathbf{f}(\mathbf{x} - \mathbf{a})$ The behavior of the Fourier transform with respect to shifted functions is simpler than the corresponding behavior of the Laplace transform. No distinction needs to be made between the definitions of right and left hand shifts; we simply observe that for any real number a

$$\text{calF}(f(x - a))(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x - a) dx$$

$$= e^{-ia\xi} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi(x-a)} f(x-a) d(x-a) = e^{-ia\xi} \text{calF}(f)(\xi).$$

Property VI: Fourier Transform of the Convolution $(f * g)(x)$ The convolution product of two functions $f(x)$, $g(x)$ in $L^2(-\infty, \infty)$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y) dy.$$

The integral is defined for all real x because $f(y)$ and $g(x-y)$ lie in $L^2(-\infty, \infty)$ if $f(x)$ and $g(x)$ lie in that space and the **Schwarz inequality** then shows the product $f(y) g(x-y)$ to be (absolutely) integrable. However, $(f * g)(x)$ does not necessarily lie in $L^2(-\infty, \infty)$. In fact we have the result: $(f * g)(x) \in L^2(-\infty, \infty)$ if and only if the product $\text{calF}(f)(\xi) \text{calF}(g)(\xi) \in L^2(-\infty, \infty)$ and, with the change of variable $r = x - y$,

$$\begin{aligned} \text{calF}((f * g)(x))(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \int_{-\infty}^{\infty} f(y) g(x-y) dy dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\xi(r+y)} f(y) g(r) dy dr \\ &= \sqrt{2\pi} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi y} f(y) dy \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi r} g(r) dr \right) \\ &= \sqrt{2\pi} \text{calF}(f)(\xi) \text{calF}(g)(\xi). \end{aligned}$$

Computing the Fourier Transform A major distinction between the Fourier transform and its Laplace counterpart lies in the fact that many, in some sense most, of the familiar elementary functions do not have Fourier transforms in the standard sense of that term. That is true because, in most cases, they do not decay rapidly enough at $\pm\infty$ for the Fourier integral to be defined. Thus we do not have standard Fourier transforms, e.g., for $f(x) = x^n$, $f(x) = e^{ax}$, $f(x) = \sin ax$, etc. (it is possible to interpret the

transforms of the first and second of these as **distributions**, however). Even the transforms of such functions such as $f(x) = \frac{1}{1+x^2}$, which are defined in the standard sense, turn out to involve combinations of different elementary functions. As a consequence, while the Laplace and Fourier transforms are, mathematically, very closely related they are, in fact, used in rather different contexts. Nevertheless, we begin with some examples where the Fourier transform can be computed without too much difficulty.

Example 1 We compute the Fourier transform of $f(x) = e^{-ax^2}$, $a > 0$. In this case

$$\begin{aligned}\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} e^{-ax^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax^2+i\xi x)} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(x^2+i\frac{\xi}{a}x-\frac{\xi^2}{4a^2}\right)} e^{-\frac{\xi^2}{4a}} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(x+i\frac{\xi}{2a}\right)^2} dx.\end{aligned}$$

Using the methods of *contour integration* in the complex plane one can show that

$$\int_{-\infty}^{\infty} e^{-a\left(x+i\frac{\xi}{2a}\right)^2} dx = \int_{-\infty}^{\infty} e^{-ax^2} dx$$

for any real ξ . To compute the last integral we note that

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-ax^2} dx\right)^2 &= \int_{-\infty}^{\infty} e^{-ax^2} dx \cdot \int_{-\infty}^{\infty} e^{-ay^2} dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dx dy = \int_0^{2\pi} \int_0^{\infty} e^{-ar^2} r dr d\theta,\end{aligned}$$

the last identity following from conversion to polar coordinates in the plane.

Then

$$\begin{aligned}\int_0^{2\pi} \int_0^{\infty} e^{-ar^2} r dr d\theta &= \frac{1}{2a} \int_0^{2\pi} \int_0^{\infty} e^{-ar^2} 2ar dr d\theta \\ &= \frac{\pi}{a} \int_0^{\infty} -\frac{d}{dr} (e^{-ar^2}) dr = \frac{\pi}{a} (-e^{-ar^2}) \Big|_0^{\infty} = \frac{\pi}{a}.\end{aligned}$$

Thus we obtain

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{a}} e^{-\frac{\xi^2}{4a}} = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}.$$

When we take $a = \frac{1}{2}$ corresponding to $f(x) = e^{-\frac{x^2}{2}}$ we find, significantly, that $\hat{f}(\xi) = e^{-\frac{\xi^2}{2}}$. This shows that $f(x) = e^{-\frac{x^2}{2}}$ is an *eigenfunction* of the Fourier transform operator corresponding to the eigenvalue $\lambda = 1$.

Example 2 Let us define

$$f(x) = \begin{cases} e^{-ax}, & x > 0, \\ e^{ax}, & x < 0. \end{cases}$$

Then

$$\begin{aligned} \hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-i\xi x} e^{ax} dx + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\xi x} e^{-ax} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{(a-i\xi)x} dx + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(a+i\xi)x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a-i\xi} + \frac{1}{a+i\xi} \right) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \xi^2}. \end{aligned}$$

Example 3 Observing that the result of the previous example is unchanged if we replace $-\xi$ by $|xi|$, we have, with $f(x)$ as given in that example,

$$\sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \xi^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{i\xi x} f(x) dx.$$

Interchanging the roles of ξ and x we have

$$g(x) \equiv \frac{a}{a^2 + x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \sqrt{\frac{\pi}{2}} f(\xi) d\xi.$$

Applying the Fourier transform to both sides and using property i) stated in the **introductory Fourier transform section**, we obtain

$$\sqrt{\frac{\pi}{2}} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-i\xi x} g(x) dx.$$

Thus, with

$$\hat{g}(\xi) \equiv \sqrt{\frac{\pi}{2}} f(\xi) = \begin{cases} \sqrt{\frac{\pi}{2}} e^{-ax}, & x \geq 0, \\ \sqrt{\frac{\pi}{2}} e^{ax}, & x < 0 \end{cases},$$

we have

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \frac{a}{a^2 + x^2} dx.$$

This example illustrates the point, made earlier, that the Fourier transform of a function having a simple algebraic expression may turn out not to be expressible in terms of a single elementary function. In other cases the Fourier transform may have no expression in terms of any number of elementary functions.