问题 1. Given $(\Omega, \mathcal{F})$ be measuable space, let $\{f_n\}$ be a sequence of probability measures. If for every $A \in \mathcal{F}$, $\lim_{n\to\infty} f_n(A) = f(A)$, then f is a probability measure.

问题 2. Given (X, Σ, μ) be measure space, let $\{f_n\} \subset L^1(X, \mu)$ have the property that for any $E \in \Sigma$,

$$\lim_{n\to\infty} \int_E f_n \, \mathrm{d}\mu$$

exists and is finite. Then for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\mu(E) < \delta \Longrightarrow \int_E |f_n| \, \mathrm{d}\mu < \varepsilon.$$

问题 3. (Banach Alaoglu thm, both versions)

1. (version I) Let X be a reflexive Banach space. Let

   $$B = \{x \in X : \|x\| \leq 1\}$$

   be its closed unit ball. Then B is weakly sequentially compact(w.s.c).

2. (version II, Helly's thm) Let X be a separable Banach space, let $U = X^*$. Then the closed unit ball B of U is weak* sequentially compact.

3. (original version) Let X be a Banach space. Then the unit ball B of X^* is compact in the weak* topology.

问题 4. The weak topology on a normed space X is metrizable iff X is finite dimensional.

问题 5. (Kakutani*) A Banach space X is reflexive iff $\overline{B_X} = \{x \in X : \|x\| \leq 1\}$ is weakly compact.

问题 6. Let X be a Banach space such that X^* is separable. Then X is separable.

References

[1] Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2011.

[2] Lecomte D., Weak topologies, 2006.