$X \neq \phi$：复数域$\mathbb{C}$上的赋范线性空间.　　$\lambda \in \mathbb{C}$

$T: D(T) \to X$：线性算子　　I：X上恒等算子

$D(T) \subseteq X$：T的定义域　　$T_\lambda := T - \lambda I : D(T) \to X$ —— 线性算子.

若T_λ有逆. 则称$R_\lambda(T) = T_\lambda^{-1} = (T-\lambda I)^{-1}$：$T$的预解算子.

T的正则值：$\lambda \in \mathbb{C}$. 若

① $R_\lambda(T)$存在

② $R_\lambda(T)$有界

③ $R_\lambda(T)$的定义域在X中稠.

$\rho(T) = \{\lambda \in \mathbb{C} : \lambda$是$T$的正则值$\}$：$T$的预解集

$\sigma(T) = \mathbb{C} \setminus \rho(T)$：$T$的谱集(谱)

$\lambda \in \sigma(T)$：T的一个谱值

用于 研究谱的线性空间与线性算子的例

①. $A: \mathbb{C}^n \to \mathbb{C}^n$. 只有纯点谱. 没有连续谱/剩余谱

②: $T: l^2 \to l^2$. $x = (x_1, x_2, \cdots) \mapsto (0, x_1, x_2, \cdots)$

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③ $R_f \subseteq \overline{A_f}$

Pf: $\forall w \in R_f$. 则 $\forall \varepsilon > 0$. 取 $E = \{x : |f(x) - w| < \varepsilon\}$. 则 $\mu(E) > 0$.

$$\left| \frac{1}{\mu(E)} \int_E f d\mu - w \right| \leq \frac{1}{\mu(E)} \int_E |f(x) - w| d\mu \leq \varepsilon, \quad \forall \varepsilon > 0.$$

则 $w \in \overline{A_f}$

④ A_f不一定是闭的.

Pf: 令 $f(x) = x \chi_{[0,1]}(x)$, 则

$$\frac{1}{b-a} \int_a^b x dx = \frac{a+b}{2}, \quad \forall a \neq b \in [0,1].$$

$\Rightarrow A_f = (0, 1)$.

⑤. 证明存在测度μ使$\forall f \in L^\infty(\mu)$. A_f是凸集

Pf: 固定x_0,　$\forall E \in \mathcal{M}$, 定义

$$\mu(E) = \begin{cases} 1, & x_0 \in E \\ 0, & x_0 \notin E \end{cases}$$

则 $A_f = \{f(x_0)\}$是凸集.

⑥是否存在μ使A_f对某个$f \in L^\infty(\mu)$不是凸集?

Pf: 固定 x_0, x_1,　$x_0 \neq x_1$.

$$\mu(E) = \begin{cases} 1, & x_0, x_1 \text{仅一个属于} E \\ 2, & x_0, x_1 \in E \\ 0, & x_0, x_1 \notin E \end{cases}$$

则 $A_f = \left\{ f(x_0), f(x_1), \frac{f(x_0) + f(x_1)}{2} \right\}$

⑦ 若L^∞换为L^1如何?

Pf. (1) R_f是闭集但未必有界.

$X = (0,1)$. $f(x) = \frac{1}{x^\alpha}$, $0 < \alpha < 1$.

(2) 若$f \notin L^\infty$, $\sup\limits_{w \in R_f} |w| = \infty = \|f\|_\infty$

③, ④, ⑤, ⑥相同.

μ：X上正测度.　$f(x) = u + iv$.　u, v：X上实可测.　$f, g \in L^1(\mu)$.　$\alpha, \beta \in \mathbb{C}$

$L^1(\mu) := \{f : X \to \mathbb{C} : \int_X |f| d\mu < \infty\}$.　$f \in L^1(\mu)$.

Thm: $f \in L^1(\mu) \Rightarrow \left| \int_X f d\mu \right| \leq \int_X |f| d\mu$

$z = \int_X f d\mu = a + bi$,　$a = \int_X u d\mu$,　$b = \int_X v d\mu$

$$\sqrt{a^2 + b^2} \leq \int_X \sqrt{u^2 + v^2} d\mu$$

$$\Leftrightarrow \left(\int_X u d\mu\right)^2 + \left(\int_X v d\mu\right)^2 \leq \left(\int_X \sqrt{u^2 + v^2} d\mu\right)^2$$

$$\Leftrightarrow \iint u(x)u(y) + v(x)v(y) \, d\mu(x) d\mu(y) \leq \iint \sqrt{u^2(x) + v^2(x)} \cdot \sqrt{u^2(x) + v^2(x)} \, d\mu(x) d\mu(y).$$