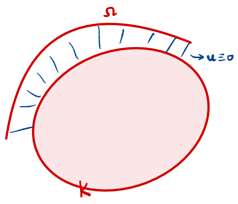
Thm 1. 设 $u(x)\in L^1(\Omega)$，且 $\operatorname{supp} u\subseteq K\subset\subset\Omega$，且当 $\delta>0$ 充分小时，

$$u_\delta(x)=\int_\Omega u(y)j_\delta(x-y)dy=(u*j_\delta)(x)\in C_0^\infty(\mathbb{R}^n).$$



Pf. 记 K_δ 为 K 的 δ 闭邻域，　$K_\delta=\{x\in\mathbb{R}^n: dist(x,K)\le\delta\}$

① $x\notin K_\delta$，$\forall y\in K \Rightarrow \|x-y\|>\delta \Rightarrow j_\delta(x-y)=0$.

$$u_\delta(x)=\int_\Omega u(y)j_\delta(x-y)dy=\int_K u(y)j_\delta(x-y)dy=0$$

②. ① $\Rightarrow \operatorname{supp}u_\delta\subseteq K_\delta$.

$$\frac{\partial u_\delta}{\partial x_1}=\lim_{h\to0}\frac{1}{h}\int_\Omega[j_\delta(x+he_1-y)-j_\delta(x-y)]u(y)dy$$

$$=\lim_{h\to0}\int_\Omega\frac{\partial}{\partial x_1}j_\delta(x+\theta he_1-y)\,u(y)\qquad \theta\in(0,1).$$

$$=\int_\Omega\frac{\partial}{\partial x_1}j_\delta(x-y)\,u(y)dy\qquad (DCT)$$

③. 同理，$\forall p\in\mathbb{N}^n$. $D^pu_\delta=D^pj_\delta*u \Rightarrow u_\delta\in C_0^\infty(\Omega)$.

例. 设 $\Omega\subseteq\mathbb{R}^n$ 为非空开集，$f(x)\in L^1_{loc}(\Omega)$，即 $\forall K\subset\subset\Omega$，有 $f(x)\in L^1(K)$. 定义 $\mathcal{D}$ 上的泛函

$$f^*(\varphi)=\int_\Omega f(x)\varphi(x)dx$$

则 f^* 是广义函数.

Pf. ① $f^*(\varphi)$ 是良定的.

$f(x)\in L^1_{loc}(\Omega)\Rightarrow\forall\varphi\in\mathcal{D}$，$f(x)\varphi(x)\in L^1(\operatorname{supp}\varphi)$

② f^* 是线性的.

③ f^* 连续.

$$\left.\begin{array}{l}\{\varphi_i\}\subseteq\mathcal{D}\\ \varphi_i\to\varphi\in\mathcal{D}\ in\ \mathcal{D}\end{array}\right\}\Rightarrow\exists K\subset\subset\Omega,\ s.t.\ \begin{array}{l}supp(\varphi_i)\subseteq K\\ supp(\varphi)\subseteq K\end{array}\bigwedge\ \varphi_j\rightrightarrows\varphi\ in\ K$$

$\Rightarrow\exists M>0$, s.t. $\|\varphi_j(x)\|_{L^\infty}\le M$

$$\Rightarrow f^*(\varphi_j)=\int_\Omega f(x)\varphi_j(x)dx=\int_K f(x)\varphi_j(x)dx$$

$$\to\int_K f(x)\varphi(x)dx,\ (j\to\infty).\qquad (DCT)$$

$$=\int_\Omega f(x)\varphi(x)dx.$$

$$\Rightarrow f^*(\varphi_j)\to f^*(\varphi).\ (j\to\infty)$$

例. δ_a 不是函数型广义函数. $(a\in\Omega)$.　$\underline{\delta_a:\mathcal{D}(\Omega)\to\mathbb{R},\ \varphi\mapsto\varphi(a)}$

Pf. (反证).

$$\left.\begin{array}{l}\exists f\in L^1_{loc}(\Omega),\ s.t.\ \forall\varphi\in\mathcal{D}.\ \delta_a(\varphi)=\int_\Omega f(x)\varphi(x)dx=\varphi(a)\\ \varphi_{a,r}(x)=\begin{cases}e^{\frac{r^2}{\|x-a\|^2-r^2}}, & x\in B_r(a)\\ 0, & x\in\Omega\setminus B_r(a)\end{cases}\Rightarrow\varphi_{a,r}(x)\in\mathcal{D}(x)\end{array}\right\}\Rightarrow\left.\begin{array}{l}\int_\Omega f(x)\varphi_{a,r}(x)dx=\varphi_{a,r}(a)=\frac{1}{e}\\ DCT\Rightarrow\lim_{r\to0^+}\int_\Omega f(x)\varphi_{a,r}(x)dx=0\end{array}\right\}\Rightarrow 矛盾$$

例. 在 $\mathbb{R}$ 上，函数列

$$f_j(x)=\frac{1}{\pi}\frac{\sin jx}{x}\in L^1_{loc}(\mathbb{R})\quad,(j=1,2,\cdots)$$

则

$$f_j\longrightarrow\delta,\ in\ \mathcal{D}'$$

Pf: $\forall\varphi\in\mathcal{D}(\mathbb{R})$，$\Rightarrow\exists T_0>0$, s.t. $\operatorname{supp}\varphi\subseteq[-T_0,T_0]$

$$f_j(\varphi)=\int_{\mathbb{R}}f_j(x)\varphi(x)dx=\int_{-T}^Tf_j(x)\varphi(x)dx.\ (T>T_0)$$

$$=\int_{-T}^Tf_j(x)(\varphi(x)-\varphi(0))dx+\int_{-T}^Tf_j(x)\varphi(0)dx$$

$$=\int_0^Tf_j(x)(\varphi(x)-\varphi(0))dx+\varphi(0)+o(1),\quad j\to+\infty.$$
$$+\int_0^Tf_j(x)(\varphi(-x)-\varphi(0))dx$$

$$=\frac{1}{\pi}\int_0^T\sin jx\,\frac{\varphi(x)+\varphi(-x)-2\varphi(0)}{x}dx+\varphi(0)+o(1).\quad j\to+\infty.$$

$$=\varphi(0)+o(1),\ j\to+\infty.\qquad (\text{Riemann Lebesgue 引理})$$

性质1. 设 $f\in\mathcal{D}'$，定义 $g(\varphi)=-f(\frac{\partial\varphi}{\partial x_1})$，$(\varphi\in\mathcal{D})$，则 $g\in\mathcal{D}'$.

Pf. ① g 是 $\mathcal{D}$ 上线性泛函.

② 设 $\{\varphi_j\}\subseteq\mathcal{D}$，$\varphi\in\mathcal{D}$，$\varphi_j\to\varphi$ in $\mathcal{D}$

$\Rightarrow\partial_1\varphi_j\to\partial_1\varphi$ in $\mathcal{D}$

$\Rightarrow f(\partial_1\varphi_j)\to f(\partial_1\varphi)$

$\Rightarrow g(\varphi_j)\to g(\varphi)\Rightarrow g\in\mathcal{D}'$.

性质3. 设 $f_j\in\mathcal{D}'$，$f\in\mathcal{D}'$，若 $f_j\to f$ in $\mathcal{D}'$，则

$$\partial_if_j\to\partial_if\quad in\ \mathcal{D}',\ (i=1,2,\cdots,n).$$

Pf. $\forall\varphi\in\mathcal{D}$.

$$\partial_if(\varphi)=-f(\partial_i\varphi)=-\lim_{j\to\infty}f_j(\partial_i\varphi)=\lim_{j\to\infty}\partial_if_j(\varphi)$$

$\Rightarrow\partial_if_j\to\partial_if$ in $\mathcal{D}'$. $(i=1,2,\cdots,n)$.

性质4. 若 $\sum_{j=1}^\infty f_j=f\in\mathcal{D}'$，则 $\sum_{j=1}^\infty\partial_if_j=\partial_if$. $(i=1,2,\cdots,n)$.

Pf. $\sum_{j=1}^m f_j\to f$ in $\mathcal{D}'$，$m\to+\infty$.

$\Rightarrow\sum_{j=1}^m\partial_if_j\to\partial_if$ in $\mathcal{D}'$，$m\to+\infty$. $(i=1,2,\cdots,n)$　(性质3)