数学笔记

May 28, 2019

1 report

1.1 2019-05-04

investigate functional

$$I:\mathcal{A}\to\mathbb{R},\omega\mapsto\int_U F(D\omega)\,\mathrm{d}x.$$
$$\mathcal{A}\equiv\left\{\omega\in W^{1,q}(U;\mathbb{R}^m)\mid\omega=g\ \text{ on }\ \partial U\right\},\quad 1<q<\infty,$$

in which

$$g:\partial U\to\mathbb{R}^m,$$

and

$$F:M^{m\times n}\to\mathbb{R}$$

be given smooth function.

for $P\in M^{m\times n}$, $\alpha>0$, $\beta\geq0$, suppose $F(P)\geq\alpha|P|^q-\beta$.

$\omega\in\mathcal{A}$, so $\omega:U\to\mathbb{R}^m,\ x\mapsto\left(\omega^1(x),\cdots,\omega^m(x)\right)$. and denote

$$D\omega=\begin{pmatrix}\omega^1_{x_1}&\cdots&\omega^1_{x_n}\\ \vdots&\ddots&\vdots\\ \omega^m_{x_1}&\cdots&\omega^m_{x_n}\end{pmatrix}=\frac{\partial\omega}{\partial(x_1,\cdots,x_n)}.$$

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existence problem for a minimizer of $I[\cdot]$ in $\mathcal{A}$ turns the weak lower semicontinuity of $I[\cdot]$.

$I[\cdot]$在class $\mathcal{A}$ 中最小值存在问题转化为$I[\cdot]$的弱下半连续性问题.

F的什么样的非线性结构条件可以推出$I[\cdot]$的弱下半连续性?

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suppose u is a smooth minimizer, $v=\left(v^1,\cdots,v^m\right)$ is Lipschitz function with compact support in U, then

$$i(t)\equiv I[u+tv]=\int_U F(Du+tDv)\,\mathrm{d}x$$

with

$$0\leq i''(0)=\int_U\frac{\partial^2F}{\partial p_i^k\partial p_j^l}(Du)\,v^k_{x_i}v^l_{x_j}\,\mathrm{d}x$$

select

$$v(x)\equiv\varepsilon\zeta(x)\rho\left(\frac{x\cdot\xi}{\varepsilon}\right)\eta$$

send $\varepsilon\to0$

$$\frac{\partial^2F}{\partial p_i^k\partial p_j^l}(Du(x))\,\eta_k\eta_l\xi_i\xi_j\geq0,\ x\in U,\eta\in\mathbb{R}^m,\xi\in\mathbb{R}^n.$$

concern F satisfy **the Hadamard-Legedre inequality**

$$(\eta\otimes\xi)^T D^2F(P)(\eta\otimes\xi)\geq0\quad\left(P\in M^{m\times n},\eta\in\mathbb{R}^m,\xi\in\mathbb{R}^n\right).$$

$\eta\otimes\xi=(\eta_k\xi_i)_{m\times n}$, $P=\left(p^i_j\right)_{m\times n}$, $D^2F(P)=\left(\frac{\partial^2F}{\partial p_i^k\partial p_j^l}\right)_{(m\times n)\times(m\times n)}$, F called **rank-one convex**.

定理 1.1. *If $F : M^{m\times n} \to \mathbb{R}$ is rank-one convex, then for each $P \in M^{m\times n}$, $\eta \in \mathbb{R}^m$, $\xi \in \mathbb{R}^n$, scalar function*

$$f(t) \equiv F(P + t(\eta \otimes \xi)) \quad (t \in \mathbb{R})$$

is convex.

Proof. $f''(t) = (\eta \otimes \xi)^T D^2 F(P + t(\eta \otimes \xi))(\eta \otimes \xi) \geq 0.$ □

the reverse does not imply F is convex.

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lower semi-continuous of $I[\cdot]$ implies F is rank-one convex.
$P \in M^{m\times n}$, $U = Q = (0,1)^n \subset \mathbb{R}^n$, $v \in C_c^\infty(Q; \mathbb{R}^m)$, $k \in \mathbb{N}_+$,

$$Q = \bigcup_{l=1}^{2^{kn}} Q_l$$

for each Q_l with side length $\frac{1}{2^k}$, centered at x_l.

$$u_k(x) = \frac{1}{2^k} v\left(2^k(x - x_l)\right) + Px \quad x \in Q_l,$$

$$u(x) = Px$$

then $u_k \rightharpoonup u$ in $W^{1,q}(U; \mathbb{R}^m)$, since $\|u_k - u\|_{L^p} \ll \frac{C}{2^k}$,

$$Du_k - Du = \left(v^i_{x^j}\left(2^k(x - x_l)\right)\right)_{m\times n} \rightharpoonup (0)_{m\times n} = \mathbf{0}.$$

we can use the following theorem

定理 1.2. *f and g are continuous 1-periodic functions. Then*

$$\lim_{n\to\infty} \int_0^1 f(x) g(nx)\, \mathrm{d}x = \int_{[0,1]} f \,\mathrm{d}x \int_{[0,1]} g \,\mathrm{d}x.$$

$u(x) = Px$ is a minimizer on Q subject to its own boundary values. If $I[u] \leq \liminf_{k\to\infty} I[u_k]$, then

$$\int_Q F(p)\, \mathrm{d}x = L^n(Q) F(P) \leq \int_Q F(P + Dv)\, \mathrm{d}x.$$

that is the following theorem:

定理 1.3. *if the functional $I[\cdot]$ is lower semicontinuous with respect to weak convergence in $W^{1,q}(U; \mathbb{R}^m)$, then $F : M^{m\times n} \to \mathbb{R}$ is quasiconvex, moreover, F is automatically rank-one convex.*

定义 1.4. A function $F : M^{m\times n} \to \mathbb{R}$ is called quasiconvex provided

$$L^n(Q) F(P) = \int_Q F(P)\, \mathrm{d}x \leq \int_Q F(P + Dv)\, \mathrm{d}x, \quad \forall P \in M^{m\times n}, v \in C_c^\infty(Q; \mathbb{R}^m).$$

其中Q是$\mathbb{R}^n$中的单位开立方体.

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open question: rank-one convex is quasiconvex function? ref[13] and [93].

定理 1.5. *if F is quadratic, then F is quasiconvex iff F is rank-one convex.*

定理 1.6. *convex function is quasiconvex and rank-one convex, by Jensen's inequality.*

$$F\left(\fint_Q P + Dv \,\mathrm{d}x\right) \leq \fint_Q F(P + Dv)\, \mathrm{d}x.$$

growth condition: $0 \leq F(P) \leq C(1+|P|^q)$, $P \in M^{m\times n}$, C be constant.

$$|P| = \sum_{i=1}^{m}\sum_{j=1}^{n}|p_{ij}|$$

引理 1.7. *F is rank-one convex and verifies growth condition*

$$0 \leq F(P) \leq C(1+|P|^q),$$

$P \in M^{m\times n}$. Then

$$|DF(P)| \leq C\left(1+|P|^{q-1}\right) \qquad P \in M^{m\times n}$$

for C be constant.

Proof. define $f(t) \equiv F(P+t(\eta\otimes\xi)) \quad (t\in\mathbb{R})$, $\eta = e_k$, $\xi = e_i$. then F rank-one convex implies f is a convex function, then

$$f'(0) \leq f'(\delta) = \frac{f(r)-f(0)}{r} \leq \frac{C}{r}\max_{B(r)}|f|, \quad r>0$$

$$f'(0) = DF(P)$$

$$\max_{B(r)}|f| = F(P+te_{ki}) \leq \max_{B(r)} C(1+|P+te_{ki}|^q) \leq C(1+|P|^q+r^q)$$

select $r = 1+|P|$, but not $r = |P|-1$. □

注意, 这一引理是针对rank-one convex而言的, 从而quasiconvex自然也有此引理成立.

1.2 2019-04-05

Concentration and sobolev inequalities

Gagliardo-Nirenberg-Sobolev inequality 可以得出$W^{1,p}(U)$嵌入到$L^{p^*}(U)$ for $1\leq p<n$, $p^* = \frac{pn}{n-p}$.

$W^{1,p}(U)$可以紧嵌入到$L^q(U)$中, for $1\leq q<p^*$.

定理 1.8. *(Estimates for $W^{1,p}$, $1\leq p<n$)*

定理 1.9. *(**Rellich-Kondrachov Compactness Theorem**). Assume U bounded open subset of $\mathbb{R}^n$, and ∂U is C^1. Suppose $1\leq p<n$. Then*

$$W^{1,p}(U) \Subset L^q(U)$$

for each $1\leq q<p^*$.

Proof. 根据定理 Estimates for $W^{1,p}$, $1\leq p<n$. and since U is Bounded and ∂U is C^1, so

$$W^{1,p}(U) \subset L^q(U), \quad \|u\|_{L^q(U)} \leq C\|u\|_{W^{1,p}(U)}.$$

所以只需证紧嵌入. 设$(u_m)_1^\infty$ 是$W^{1,p}(U)$中的有界序列, 现求它在$L^q(U)$中的收敛子列.

用延拓定理对函数序列(u_m)延拓到V上, V是有界开集 of $\mathbb{R}^n$, and

$$\sup_m \|u_m\|_{W^{1,p}(V)} < \infty.$$

then smooth the functions $u_m^\varepsilon := \eta_\varepsilon * u_m$ $(\varepsilon>0,\ m=1,2,\cdots)$,

$$u_m^\varepsilon \to u_m \text{ in } L^1(V) \text{ as } \varepsilon\to 0, \text{ uniformly in } m.$$

这是计算技巧, 再由插值不等式

$$u_m^\varepsilon \to u_m \text{ in } L^q(V) \text{ as } \varepsilon\to 0, \text{ uniformly in } m.$$

然后, for each fixed $\varepsilon > 0$, the sequence $(u_m^\varepsilon)_1^\infty$ is uniformly bounded and equicontinuous. (计算技术)

对序列$(u_m^\varepsilon)_1^\infty$使用Arzela-Ascoli紧性判别法, 构造出子列$\left(u_{m_j}\right)_{j=1}^\infty$满足

$$\limsup_{j,k\to\infty}\|u_{m_j} - u_{m_k}\|_{L^q(V)} \le \delta.$$

最后取$\delta = 1, \frac{1}{2}, \frac{1}{3}, \cdots$, 并利用对角线原理构造子列$(u_{m_l})_{l=1}^\infty$满足

$$\limsup_{l,k\to\infty}\|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

□

但是一般从$W^{1,q}(U)$到$L^{q^*}(U)$的嵌入不是紧嵌入, 其中$1 \le q < n$.

1.2.1 定义

定义 1.10. Let X and Y be Banach spaces, $X \subset Y$. say X is **compactly embedded** in Y, $X \Subset Y$, provided

1. $\|x\|_Y \le C\|x\|_X$ $(x \in X)$ for some constant C;
2. each bounded sequence in X is precompact in Y.

定义 1.11. Let (X, τ) be a topological space, and let Σ be a σ-algebra containing τ (namely, containing the Borel σ-algebra). A collection of measures M on Σ is called **tight** if for every $\varepsilon > 0$ there is compact set $K_\varepsilon \subset X$ s.t. for each measure $\mu \in M$, we have that $|\mu(X \setminus K_\varepsilon)| < \varepsilon$.

定义 1.12. Let X be a non-empty set, $\mathcal{A}$ is a σ-algebra on X. Given two measure μ and ν on $\mathcal{A}$, say ν has **the Radon-Nikodym property relative to** μ, if there exists a measureable function $f : X \to [0, \infty]$, s.t.

$$\nu(A) = \int_A f \, \mathrm{d}\mu, \quad \forall A \in \mathcal{A}.$$

say f is a **density** for ν relative to μ.

1.2.2 定理

定理 1.13. *设$n \ge 3$, (为了防止$2^* = \infty$, 其实$2^* = \frac{2n}{n-2}$),*

$$\begin{aligned} &f_k \to f \quad \text{strongly in } L^2_{loc}(\mathbb{R}^n), \quad |f_k|^{2^*} \to \nu \quad \text{in } \mathcal{M}(\mathbb{R}^n), \\ &Df_k \rightharpoonup Df \quad \text{in } L^2(\mathbb{R}^n; \mathbb{R}^n), \quad |Df_k|^2 \rightharpoonup \mu \quad \text{in } \mathcal{M}(\mathbb{R}^n). \end{aligned}$$

则

1. There exists an at most countable index set J, and distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^n$, and nonnegative weights $\{\mu_j, \nu_j\}_{j\in J}$ s.t.

$$\nu = |f|^{2^*} + \textstyle\sum_{j\in J}\nu_j\delta_{x_j}, \quad \mu \ge |Df|^2 + \textstyle\sum_{j\in J}\mu_j\delta_{x_j} \ .$$

2. $\nu_j \le C_2^{2^*}\mu_j^{2^*/2}$, for all $j \in J$. here C_2 is the optimal constant for the Gagliardo-Nirenberg-Sobolev inequality.
3. if $f \equiv 0$ and $\nu(\mathbb{R}^n)^{1/2^*} \ge C_2\mu(\mathbb{R}^n)^{1/2}$, then ν is concentrated at a single point.

这个定理是为了展示$W^{1,2}(U)$不能紧嵌入到$L^{2^*}(U)$中, 这个定理指明了这种非紧性的构造方法.

Proof. first assume $f \equiv 0$, then $Df \equiv 0$.

μ, ν都是非负有限测度, 这是条件中的弱收敛保证的.

consider

$$D = \{x \in \mathbb{R}^n \mid \mu(\{x\}) > 0\}$$

then D is at most countable for μ is positive finite measure. then suppose $(x_j)_{j \in J} = D$, $\mu_j = \mu(\{x_j\})$, for subadditivity of μ

$$\mu \geq \sum_{j \in J} \mu_j \delta_{x_j}.$$

by $|f_k|^{2^*} \to \nu$ in $\mathcal{M}(\mathbb{R}^n)$ and $|Df_k|^2 \rightharpoonup \mu$ in $\mathcal{M}(\mathbb{R}^n)$, for all $\phi(x) \in C_c^\infty(\mathbb{R}^n)$, $\phi f_k \in W^{1,2}(\mathbb{R}^n)$ with compact support. use the Gagliardo-Nirenberg-Sobolev inequality.

$$\|\phi f_k\|_{2^*} \leq C_2 \|D(\phi f_k)\|_2$$

by $f_k \to 0$ in L^2_{loc}, so

$$\left(\int_{\mathbb{R}^n} |\phi|^{2^*} \, \mathrm{d}\nu\right)^{1/2^*} \leq C_2 \left(\int_{\mathbb{R}^n} |\phi|^2 \, \mathrm{d}\mu\right)^{1/2}$$

by regularity of μ and ν

$$\nu(E) \leq C_2^{2^*} \mu(E)^{2^*/2}$$

for all E to be Borel set of $\mathbb{R}^n$. then when $\mu(E) = 0$ implies $\nu(E) = 0$, i.e. $\nu \ll \mu$. then

$$\nu(E) = \int_E D_\mu \nu(x) \, \mathrm{d}\mu$$

for all E to be Borel set, and

$$D_\mu \nu(x) \equiv \lim_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))}, \qquad \mu - a.e. \ \ x \in \mathbb{R}^n$$

因为$\nu(E) \leq C_2^{2^*} \mu(E)^{2^*/2}$, 所以在$\mathbb{R}^n \setminus D$上$\mu - a.e.$有$D_\mu \nu = 0$. 从而可令$\nu_j = D_\mu \nu(x_j) \mu_j$. 所以

$$\nu(E) = \int_E D_\mu \nu(x) \, \mathrm{d}\mu = \int_{E \cap D} D_\mu \nu(x) \, \mathrm{d}\mu = \sum_{x \in E \cap D} D_\mu \nu(x_j) \mu_j = \sum_{x \in E} \nu_j \delta_{x_j}(x)$$

从而$\nu_j \leq C_2^{2^*} \mu_j^{2^*/2}$. 最后证(3). 首先有$\nu(\mathbb{R}^n) = C_2^{2^*} \mu(\mathbb{R}^n)^{2^*/2}$. 用Holder不等式有

$$\left(\int_{\mathbb{R}^n} |\phi|^{2^*} \, \mathrm{d}\nu\right)^{1/2^*} \leq C_2 \left(\int_{\mathbb{R}^n} |\phi|^2 \, \mathrm{d}\mu\right)^{1/2} \Longrightarrow \left(\int_{\mathbb{R}^n} |\phi|^{2^*} \, \mathrm{d}\nu\right)^{1/2^*} \leq C_2 \mu(\mathbb{R}^n)^{1/n} \left(\int_{\mathbb{R}^n} |\phi|^{2^*} \, \mathrm{d}\mu\right)^{1/2^*}$$

所以$\boxed{\nu = C_2^{2^*} \mu(\mathbb{R}^n)^{2/(n-2)} \mu}$. 因此

$$\left(\int_{\mathbb{R}^n} |\phi|^{2^*} \, \mathrm{d}\nu\right)^{1/2^*} \leq C_2 \left(\int_{\mathbb{R}^n} |\phi|^2 \, \mathrm{d}\mu\right)^{1/2}$$

成为

$$\left(\int_{\mathbb{R}^n} |\phi|^{2^*} \, \mathrm{d}\nu\right)^{1/2^*} (\nu(\mathbb{R}^n))^{1/n} \leq \left(\int_{\mathbb{R}^n} |\phi|^2 \, \mathrm{d}\nu\right)^{1/2}$$

when $f \not\equiv 0$, then $|Df_k - Df|^2 = |Df_k|^2 - 2Df_k Df + |Df|^2 \rightharpoonup |Df_k|^2 - |Df|^2 \rightharpoonup \mu - |Df|^2$ to be a positive measure, also we can get

$$\mu - |Df|^2 \geq \sum_{j \in J} \mu_j \delta_{x_j}.$$

如何证明,

$$\int \phi \left(|f_k|^{2^*} - |f_k - f|^{2^*}\right) \mathrm{d}x \to \int \phi |f|^{2^*} \, \mathrm{d}x$$

□

1.2.3 问题

1. (solved)为什么Sobolev函数的弱导数满足牛顿莱布尼兹公式:

$$\int_U \varphi \left(u\left(x+1\right)-u\left(x\right)\right)\,\mathrm{d}x=\int_U \varphi \int_0^1 Du\left(x+t\right)\,\mathrm{d}t\,\mathrm{d}x.$$

1.2.4 反例

2 notes for Toro, Geometric Measure Theory - Recent Applications

Let $\Omega\subset\mathbb{R}^{n+1}$ be a bounded domain, $f\in C\left(\partial\Omega\right)$. The classical Dirichlet problems asks whether there is a function $u\in C\left(\overline{\Omega}\right)\cap W^{1,2}\left(\Omega\right)$ such that

$$\begin{cases}\Delta u=0 & in\ \Omega\\ u=f & on\ \partial\Omega.\end{cases}$$

Here $u\in W^{1,2}\left(\Omega\right)$ means: u and its weak derivatives are in $L^2\left(\Omega\right)$ and $\Delta u=0$ is interpreted in the weak sense; that is, for any $\zeta\in C_c^1\left(\Omega\right)$,

$$\int\left\langle \nabla u,\nabla\zeta\right\rangle =0.$$

The questions here are whether a sol. u exists, how regular it is, whether there is a formula in terms of f.

say Ω is regular if for all $f\in C\left(\partial\Omega\right)$, any sol. u is in $C\left(\overline{\Omega}\right)\cap W^{1,2}\left(\Omega\right)$.

a characterization of regular domains using capacity.

the Maximum Principle $\left|u\left(x\right)\right|\le\max_{\partial\Omega}\left|f\right|$.

for $x\in\Omega$, $T_x:C\left(\partial\Omega\right)\to\mathbb{R}$, $f\mapsto u\left(x\right)$.

$$u\left(x\right)=\int_{\partial\Omega}f\left(q\right)\,\mathrm{d}\omega^x\left(q\right).$$

3 BV space

3.1 Kreuzer - Bounded variation and Helly's selection theorem.pdf

全体左连续的有界变差函数不是完备的, 有界变差函数的定义不能推广到大于1维.

4 Sobolev space

弱导数（Weak Derivative）是一个函数的微分（强微分）概念的推广，它可以作用于那些勒贝格可积（Lebesgue Integrable）的函数，而不必预设函数的可导性（事实上大部分可以弱微分的函数并不可微）。

弱导数作用于那些勒贝格可积的函数，而不必预设函数的可微性。一个典型的勒贝格可积函数的空间是$L^1\left(\left[a,b\right]\right)$。在分布中，可以定义一个更一般的微分概念。

令u是一个在$L^1\left(\left[q,p\right]\right)$中的勒贝格可积的函数，称$v\in L^1\left(\left[q,p\right]\right)$是$u$的一个弱导数，如果

$$\int_q^p u\left(t\right)\varphi'\left(t\right)\,\mathrm{d}t=-\int_q^p v\left(t\right)\varphi\left(t\right)\,\mathrm{d}t,$$

其中φ是任意一个连续可微的函数，并且满足$\varphi\left(p\right)=\varphi\left(q\right)=0$。

推广到n维的情形，如果u和v是$L_{loc}^1\left(U\right)$中的函数（在某个开集$U\subset\mathbb{R}^n$中局部可积），并且α是一个多重指标，那么v称为u的α次弱微分，如果

$$\int_U uD^\alpha\varphi=\left(-1\right)^\alpha\int_U v\varphi,$$

其中$\varphi \in C_c^\infty(U)$是一个任意给定的函数，即给定的支撑集含于$U$的无穷可微的函数。

如果u的弱导数存在，一般被记为$D^\alpha u$。可以证明，一个函数的弱微分在测度意义是唯一的，即如果有两个不同的弱导数，其仅可能在一个零测集上存在差异.

5 Holder space

6 Lorentz space

7 Lipschitz space

8 Functional Analysis

8.1 稠密性

定理 8.1. ***[Evans, Appendix, C.4. Theorem 6]*** *If $1 \le p < \infty$ and $f \in L^p_{loc}(U)$, U is open set of $\mathbb{R}^n$, then there is smooth functions $f^\epsilon \to f$ in $L^p_{loc}(U)$.*

If $f \in C(U)$, then there is smooth functions such that $f^\epsilon \rightrightarrows f$ on compact subsets of U.

for any function f, there is smooth functions such that $f^\epsilon \to f$ a.e. as $\epsilon \to 0$.

事实 8.2. *continuous functions not dense in L^∞, i.e. the Heaviside function.*

8.2 空间关系

命题 8.3. *for any bounded open set U, there is*

$$C\left(\overline{U}\right) \subset Lipchitz \subset Holder \subset W^{k,p}.$$

9 论文

9.1 古典导数与弱导数

有弱导数不一定有古典导数. 几乎处处有古典导数, 不一定有弱导数: 除一点有跳跃间断点的C^1函数均无弱导数.

没有跳跃间断点, 几乎处处可导函数不一定有弱导数. $I = (-1, 1)$.

引理 9.1. 设函数f, 对任意的ϕ, 有

$$\int_I f(x)\,\phi(x)\ \mathrm{d}x = \phi(0),$$

则f在I是非局部*Lebesgue*可积函数.

引理 9.2. 设函数$f, g \in L^1_{loc}(I)$且在分布意义下相等, 即对于任意的ϕ,

$$\int_I f\phi\,\mathrm{d}x = \int_I g\phi\,\mathrm{d}x,$$

则f与g几乎处处相等. 反之, 若f与g几乎处处相等且其中之一属于$L^1_{loc}(I)$, 则另一个也属于$L^1_{loc}(I)$且分布意义下必相等.

引理 9.3. 设连续函数u几乎处处可导且导函数$\frac{\mathrm{d}u}{\mathrm{d}x} \in L^1_{loc}(I)$, 则$u$弱可导, 弱导数$u'$几乎处处等于$\frac{\mathrm{d}u}{\mathrm{d}x}$.

引理 9.4. 设u存在弱导数, 那么存在函数$\widetilde{u} \in C(I)$, 使得在I上几乎处处有$u = \widetilde{u}$且对于任意的$x, y \in I$, 有

$$\widetilde{u}(x) - \widetilde{u}(y) = \int_y^x u'(t)\ \mathrm{d}t.$$

引理 9.5. 引理*9.4*中连续表示$\widetilde{u}$存在弱导数, 且弱导数$\widetilde{u}' = u'$ *a.e.*

引理 9.6. u存在弱导数, 则存在零测集I_0, 使得u在$I \setminus I_0$上是相对连续的且当$x \in I \setminus I_0$, $u(x) = \widetilde{u}(x)$.

引理 9.7. u存在弱导数, 则连续表示$\widetilde{u}$的不可导点是零测集且$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}x}(x) = u'(x)$ $a.e.$

引理 9.8. 设u在点x_0古典意义下可导, 则$\widetilde{u}$在点x_0也古典意义下可导且$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}x}(x_0) = \frac{\mathrm{d}u}{\mathrm{d}x}(x_0)$.

定理 9.9. 设函数u几乎处处可导且u弱可导, 则$u' = \frac{\mathrm{d}u}{\mathrm{d}x}$ $a.e.$

定理 9.10. 设函数u定义在I上, 则u存在弱导数的充要条件是存在连续表示$\widetilde{u} \in C(I)$, 且$\widetilde{u}$古典意义下几乎处处可导, 且古典意义下的导数$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}x} \in L^1_{loc}(I)$.

定理 9.11. $u \in W^{1,p}(I)$的充要条件是存在连续表示$\widetilde{u} \in C(I)$, $\widetilde{u}$古典意义下几乎处处可导, 且古典意义下的导数$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}x} \in L^p(I)$.

在高维空间中上述定理不成立.

10 术语表

1. 集合函数理论
2. 全变差测度
3. 一致p可积性
4. 弱完备
5. Gelfand理论
6. 有界正规复值Borel测度
7. 弱可分离
8. Lipschitz空间Λ_α, $\Lambda_{p,\alpha}$
9. Harday空间H_p
10. 光滑化子
11. BMO
12. Besov $B^s_{p,q}$
13. geophysical models
14. Navier Stokes equations
15. rotating fluids
16. Navier Stokes Coriolis equations (NSC$_\varepsilon$)
17. coM: smallest convex set containing M
18. $\overline{co}M$: smallest closed convex set containing M
19. $ext\ M$: $int\,(M^c)$
20. Frechet组合
21. Lebesgue积分的性质有哪些
22. Volterra型积分方程
23. Fredholm积分方程
24. Schwartz space

25. 古典分析的内容

26. Levy processes

27. direct limit of topological spaces

28. real-valued Radon measure and signed measures

29. upper integral

11 method in mathematics

11.1 approximation method

11.2 increase or decrease dimensions

11.3 sliding hump method

11.4 solve in more large space

11.5 biconditional

11.6 construct a counterexample

11.7 construct from nothing

12 reading

12.1 weakly compact and weakly sequentially compact

12.1.1 weakly compact from wikipedia

weakly compact cardinal, an infinite cardinal number on which every binary relation has an equally large homogeneous subset.

weakly compact set, a compact set in a space with the weak topology.

weakly compact set, a set that has some but not all of the properties of compact sets, for example:

1. **sequentially compact space**, a set in which every infinite sequence has a convergent subsequence.

2. **limit point compact**, a set in which every infinite subset of X has a limit point.

12.1.2 compactness in weak* topology from MSE/27423

X be a Banach space, X^* be dual space.

Under the weak* topology, do compactness and sequential compactness coincide?

that is: If a subset of X^* weakly* compact iff it is weakly* sequentially compact?

Is the weak* topology on X^* Hausdorff? Is the weak topology on X Hausdorff?

If a subset of X^* is weakly* compact, then it is weakly* closed. If a subset of X is weakly compact, then it is weakly closed. Let $\mathcal{F} \subset \mathcal{F}'$

12.2 Fourier transforms

Fourier transform of functions defined on some Abelian group.

denote $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $1 \leq p \leq \infty$, function f defined on $\mathbb{T}$ can be thought of a function defined on $\mathbb{R}$ with $f(t+1) = f(t)$ for all $t \in \mathbb{R}$. The space $L^p(\mathbb{T};\mathbb{C})$ can be identified with the spaces $L^p([0,1];\mathbb{C})$, but $\mathcal{C}(\mathbb{T};\mathbb{C})$ is not the same space as $\mathcal{C}([0,1];\mathbb{C})$.

定义 12.1. If $f \in L^1(\mathbb{T};\mathbb{C})$, (or $f \in L^1([0,1];\mathbb{C})$) then its Fourier transform is the sequence $\hat{f}$ defined by

$$\hat{f}(k) = \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi kt} f(t)\,\mathrm{d}t, \qquad k \in \mathbb{Z}.$$

If $F \in L^1(\mathbb{R};\mathbb{C})$, then its Fourier transform is the function $\hat{F}$ defined by

$$\hat{F}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi\omega t} F(t)\,\mathrm{d}t, \qquad \omega \in \mathbb{R}.$$

If $\phi \in l^1(\mathbb{Z};\mathbb{C})$, then its Fourier transform is the function $\hat{\phi}$ defined by

$$\hat{\phi}(\omega) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-2\pi\mathrm{i}\omega k}\phi_k, \qquad \omega \in \mathbb{R}.$$

定理 12.2. *(Riemann-Lebesgure lemma)*

1. If $f \in L^1(\mathbb{T};\mathbb{C})$, then $\hat{f} \in c_0(\mathbb{Z};\mathbb{C})$.
2. If $F \in L^1(\mathbb{R};\mathbb{C})$, then $\hat{F} \in C_0(\mathbb{R};\mathbb{C})$.
3. If $\phi \in l^1(\mathbb{Z};\mathbb{C})$, then $\hat{\phi} \in C(\mathbb{T};\mathbb{C})$.

定义 12.3.

12.3 questions to be solved

12.3.1 question from MSE/3170661:

More generally, the sum of $p-1$ consecutive Fibonacci numbers is divisible by the prime p as soon as the polynomial $x^2 - x - 1$ is reducible in $F_p[x]$ (and 1 is not a root, which can never occur).

12.3.2 note from MSE/103208

Radon测度的一些不同定义之间的关系, Radon测度似乎是定义在不同拓扑空间上的Borel sigma代数上的, 比如Hausdorff空间, 局部紧空间, 或者局部紧Hausdorff空间. Can these definitions or most of them be unified? if the definitions are related in some way?

From "Measure Theory, Volumes 1-2" by Valdimir I. Bogachev:

定义 12.4. Let X be a topological space. A Borel measure μ on X is called a Radon measure if for every B in $B(X)$ and $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset B$ s.t. $|\mu|(B \setminus K_\varepsilon) < \varepsilon$.

设X是一个拓扑空间, X上的Borel测度μ称为是Radon测度, 如果对于任意的$B \in B(X)$和$\varepsilon > 0$, 存在紧集$K_\varepsilon \subset B$使得$|\mu|(B \setminus K_\varepsilon) < \varepsilon$.

这定义了一个有限符号测度.

定义 12.5. From Wikipedia

On the Borel σ-algebra of a Hausdorff topological space X, a measure is called a Radon measure if it is locally finite, and inner regular.

在Hausdorff拓扑空间X上的Borel σ-代数, 一个测度称为是Radon测度如果它是局部有限的(紧集上有限测度), 并且是内正则的.

定义 12.6. From ncatlab

If X is a locally compact Hausdorff topological space, a Radon measure on X is a Borel measure on X that is finite on all compact set, outer regular on all Borel sets, and inner on open sets.

X是局部紧Hausdorff空间, X上的Radon测度是一个X上的所有紧集上是有限Borel测度, 在所有Borel集上是外正则的, 且在开集上是内正则的.

定义 12.7. From planetmath

Let X be a Hausdorff space. A Borel measure μ on X is said to be a Radon measure if it is finite on compact sets and inner regular (tight).

设X是Hausdorff空间. X上的一个Borel测度μ称为是Radon测度, 如果它在紧集上有限并且内正则.

定义 12.8. From Wikipedia's Radon measures on locally compact spaces

When the underlying measure space is a locally compact topological space, the definition of a Radon measure can be expressed in terms of continuous linear functionals on the space of continuous functions with compact support.

当测度空间的底空间是局部紧拓扑空间, 则Radon测度可以表示为具有紧支集的连续函数空间上的连续线性泛函.

在正测度情况下, 定义1和定义4是等价的. 在局部紧Hausdorff空间中定义2和定义4等价.

在第二可数的局部紧Hausdorff空间中, 每一个局部有限测度都满足定义3和4

在sigma紧的局部紧Hausdorff空间中, 定义3和4等价. 这一等价性的证明可以在 "**Arveson - NOTES ON MEASURE AND INTEGRATION IN LOCALLY COMPAC.pdf**" 和 "**Integral representation theory: applications to convexity, Banach spaces and potential theory**"中找到, 而避免使用Riesz定理.

一般定义3和4是不等价的, 甚至是在局部紧度量空间中.

在局部紧Hausdorff空间中, 以下存在双射

1. 满足定义3的测度
2. 满足定义4的测度
3. 有紧支集的连续函数空间上的正线性泛函

Riesz表示定理给出1等价于3或者2等价于3, 1 and 2 equivalent is in the Schwarz book mentioned by Joe Lucke; see also Ex 7.14 of "**G.B. Folland, Real Analysis: Modern Techniques and Their Applications**". 在这个文献中, Radon测度是按照定义3给出的.

Do you have an example of a locally compact σ-compact (non-second countable) space which admits a locally finite measure which is not Radon?

For a finite measure on a compact space which is not Radon, I think you can take the measure μ on $\{0,1\}^{\mathbb{R}}$ s.t. $\mu(A)=1$ if A contains $\{0\}^{S}\times\{0,1\}^{\mathbb{R}\setminus S}$ for some countable set S, and $\mu(A)=0$ otherwise.

cylindrical measures?

Schwartz (Radon measures on arbitrary topological spaces and cylindrical measures, 1973) defines Radon measures as comprising two measures.

The first is the measure given in version 3 above and the second is the essential measure defined as locally finite, tight measure. He then shows that each can generate the other. On LCH spaces, version 3 equivalent to version 5. Prinz (Regularity of Riesz measures, 1986, Proc Amer Math Soc) calls version 3 a "Riesz" measure and the locally finite, tight version a "Radon" measure and refers to Schwartz to give their duality.

12.3.3 Vitali Hahn Saks笔记

References

[Evans] Evans, Partial differential equations.