Contents

Chapter 1

抽象代数

定理 1.0.1

同类置换有相同的循环结构.

解. $P, Q, T \in S_n$且$Q = TPT^{-1}$, Q, P属同类, 则Q, P有相同的循环结构.
$P(v) = (1^{v_1}2^{v_2}\cdots m^{v_m})$, $P = C_1C_2\cdots C_r$, $r = \sum_i v_i$, $n = \sum_i iv_i$. $Q = TPT^{-1} = \prod_i TC_iT^{-1} = \prod_i C_i'$.
T是一一映射, $(C_i')_i$两两不交, 因$(C_i)_i$两两不交, C_i'与C_i同阶, 所以$Q(v) = P(v)$. □

问题 1.0.1

在环R中, 对于任意的$x \in R$, 都存在$n \in \mathbb{N}_+$, 使得$x = x^{n+1}$, 证明: 对于任意的$y \in R$, $yx^n = x^n y$.

解. 先证x^n是幂等的, $(x^n)^2 = x^n$.
再证, 若$ab = 0$, 则$ba = (ba)^{\tilde{n}+1} = b(ab)^{\tilde{n}}a = 0$.
再证, $x = x^{n+1}$, 则$yx^n = yx^{2n}$, 所以$(y - yx^n)x^n = 0$, 所以$x^n(y - yx^n) = 0$, 即$x^n y = x^n yx^n$.
最后, 同上面的做法, 由$x^n y = x^{2n}y$, 有$yx^n = x^n yx^n$, 所以$x^n y = yx^n$. □

问题 1.0.2: AMM, E.C.Johnsen, D.L. Outcalt and Adil Yaqub, An Elementary Commutativity Theorem For Rings, Vol. 75, No. 3, 288-289

有幺元的非结合环R中, 若对于任意的$x, y \in R$, 有$(xy)^2 = x^2y^2$, 则R是交换环.

解. 由$(xy)^2 = x^2y^2$, $(x(y+1))^2 = x^2y^2+2x^2y+x^2$, 而$(x(y+1))^2 = (xy+x)^2 = (xy)^2+(xy)x+x(xy)+x^2$, 所以$xyx+xxy+2x^2y$, 将$x+1$代换$x$的位置, 有$xyx + yx + xxy = 2x^2y + xy$, 即得$xy = yx$.

注. 含幺性不可省, $R = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in \mathbb{Z}\right\}$, 或$\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\} \leq \mathrm{GF}(2)$ 有左幺元.

注2. $(xy)^k = x^ky^k$在$k > 2$时有反例, $k \geq 3$固定, p素且满足: k奇时, $p \mid k$, k偶时, $p \mid \frac{k}{2}$.

$$R = \left\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, cd \in \mathrm{GF}(p)\right\} \leq \mathrm{GF}(p),$$

这里的R不可交换. □

问题 1.0.3

R是含幺环, 若对于任意的$x, y \in R$, 存在$m, n \in \mathbb{N}$, 使得$x^{m+1}y^{n+1} = x^m yxy^n$, 则$R$是交换环.

解. $x^m(xy - yx)y^n = 0$, $x^l(xy - yx)(y+1)^k = 0$, 定义$r = \max\{m, l\}$, 则

$$x^r(xy - yx)y^n = 0,\ \ x^r(xy - yx)(y+1)^k = 0,\ \ (y, y+1) = 1 \Longrightarrow (y^n, (y+1)^k) = 1.$$

所以$(y^{2n-1}, (y+1)^ky^{n-1}) = y^{n-1}$, 所以存在$A(y), B(y)$使得$Ay^{2n-1} + B(y+1)^ky^{n-1} = y^{n-1}$, 所以$x^r(xy - yx)y^{n-1} = 0$, 注意到红色部分的$y^n$得到了降次, 所以存在$s$满足$x^s(xy - yx) = 0$.
再设$(x+1)^t(xy - yx) = 0$, 同样辗转相除得到$xy - yx = 0$, 即R可交换. □

1.1 群的定义问题集(MA2008)

问题 1.1.1

Prove that the set

$$G=\{3^k/2^{2k};k\in\mathbb{Z}\}$$

forms a group with respect to multiplication. You may assume that multiplication is associative.

问题 1.1.2

Consider the following group of congruence classes of integers modulo 14 with respect to multiplication:

$$G=\{[1]_{14},[3]_{14},[5]_{14},[9]_{14},[11]_{14},[13]_{14}\}$$

Given that $[1]_{14}$ is the identity of the group, find (a) the order of $[13]_{14}$, (b) the order of $[3]_{14}$, (c) the inverse of $[9]_{14}$.

问题 1.1.3

Suppose that G is a group, $x\in G$ is an element of order 3 and $y\in G$ is an element of order N. Prove that

$$\left(x^2yx\right)^N=1$$

问题 1.1.4

Let $G=\{1,a,a^2,b,b^2,ab,ab^2,a^2b,a^2b^2\}$ be a group where $a^3=b^3=1$ and $ab=ba$. (a) Find $\langle b^2\rangle$, the subgroup of G generated by b^2. (b) It is given that $H=\{1,ab,a^2b^2\}$ is a subgroup of G. Find the distinct right cosets of H.

问题 1.1.5

Suppose G is an abelian group with its binary operation written as multiplication. Show that the mapping $\theta:G\to G$ defined by $g\theta=g^{-1}$ is a homomorphism.

问题 1.1.6

Prove that the set of vectors

$$\left\{\begin{pmatrix}a+b\\a-b\\a\end{pmatrix}:a,b\in\mathbb{R}\right\}$$

is a vector subspace of $\mathbb{R}^3$. Calculate a basis and the dimension of this subspace.

问题 1.1.7

Reduce the matrix $A=\begin{pmatrix}2&0&1&1\\1&0&-1&0\\3&0&0&1\\3&0&-3&0\end{pmatrix}$ to row-echelon form. Give a basis for the row-space of A.

问题 1.1.8

Prove that the following vector subspaces V and W of $\mathbb{R}^3$ are equal:

$$V=\operatorname{span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix}\right),\quad W=\operatorname{span}\left(\begin{pmatrix}0\\2\\4\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix},\begin{pmatrix}5\\7\\9\end{pmatrix}\right)$$

问题 1.1.9

Explain carefully which of the following functions define homomorphisms of vector spaces:
(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(a) = a + 1$,
(b) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f\begin{pmatrix} a \\ b \end{pmatrix} = a + b$,
(c) $f : \mathcal{P}_2 \to \mathcal{P}_5$ defined by $f\left(a + bx + cx^2\right) = \left(1 - 3x^2 + 7x^3\right)\left(a + bx + cx^2\right)$.

问题 1.1.10

State the rank-nullity theorem for a homomorphism of finite-dimensional vector spaces. For any linear map $f : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^2$, prove there are at least two linearly independent matrices $A \in M_{2\times 2}(\mathbb{R})$ which satisfy $f(A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

问题 1.1.11

(a) Suppose G is a finite group and g is an element of G. Define the terms i. order of G, ii. order of g.
(b) Use Lagrange's theorem to prove the following. If G is a finite group and $g \in G$ then the order of g divides the order of G.
(c) Now suppose that G is an abelian group and let

$$H_k = \left\{x \in G : x^k = 1_G\right\}$$

where k is a positive integer. Show that H_k is a subgroup of G.
(d) Let G be the group with the following operation table.

	1	a	a^2	a^3	b	ab	a^2b	a^3b	b^2	ab^2	a^2b^2	a^3b^2
1	1	a	a^2	a^3	b	ab	a^2b	a^3b	b^2	ab^2	a^2b^2	a^3b^2
a	a	a^2	a^3	1	ab	a^2b	a^3b	b	ab^2	a^2b^2	a^3b^2	b^2
a^2	a^2	a^3	1	a	a^2b	a^3b	b	ab	a^2b^2	a^3b^2	b^2	ab^2
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b	a^3b^2	b^2	ab^2	a^2b^2
b	b	ab^2	a^2b	a^3b^2	b^2	a	a^2b^2	a^3	1	ab	a^2	a^3b
ab	ab	a^2b^2	a^3b	b^2	ab^2	a^2	a^3b^2	1	a	a^2b	a^3	b
a^2b	a^2b	a^3b^2	b	ab^2	a^2b^2	a^3	b^2	a	a^2	a^3b	1	ab
a^3b	a^3b	b^2	ab	a^2b^2	a^3b^2	1	ab^2	a^2	a^3	b	a	a^2b
b^2	b^2	ab	a^2b^2	a^3b	1	ab^2	a^2	a^3b^2	b	a	a^2b	a^3
ab^2	ab^2	a^2b	a^3b^2	b	a	a^2b^2	a^3	b^2	ab	a^2	a^3b	1
a^2b^2	a^2b^2	a^3b	b^2	ab	a^2	a^3b^2	1	ab^2	a^2b	a^3	b	a
a^3b^2	a^3b^2	b	ab^2	a^2b	a^3	b^2	a	a^2b^2	a^3b	1	ab	a^2

i. Write down the elements of $H_4 = \left\{x \in G : x^4 = 1_G\right\}$.
ii. Is H_k always a subgroup of G, even if G is not abelian? Explain your answer.

解. $H_4 = \{1, a, a^2, a^3, b, ab, a^3b, ab^2, a^3b^2\}$. 其中$(ab)^4 = (a^2)^2 = 1$, $(a^3b)^4 = (a^2)^2 = 1$, $(ab^2)^4 = (a^2)^2 = 1$, $(a^3b^2)^4 = a^2 = 1$. □

问题 1.1.12

(a) Suppose that

$$G=\left\{1, a, a^2, a^3, b, ab, a^2 b, a^3 b\right\}$$

where

$$a^4=1, \quad a^2=b^2, \quad ba=a^3 b$$

and

$$G^{\prime}=\left\{1, c, c^2, c^3, d, cd, c^2 d, c^3 d\right\}$$

where

$$c^4=d^2=1 \text{ and } dc=cd.$$

Construct a non-trivial homomorphism $\theta: G \rightarrow G^{\prime}$. Find the image and kernel of your homomorphism.
(b) Consider the linear map $f: \mathcal{P}_2 \rightarrow \mathcal{P}_2$ defined by

$$f\left(a+bx+cx^2\right)=(a+b)+(2b+c)x+3cx^2.$$

Find three eigenvectors of f which form a basis $\mathcal{B}$ of $\mathcal{P}_2$. Write down the following matrices and the relation between them:
- The change of basis matrix P, from $\mathcal{B}$ to the standard basis $\left\{1, x, x^2\right\}$.
- The matrix C which represents f with respect to the standard basis.
- The matrix D which represents f with respect to the basis $\mathcal{B}$.

问题 1.1.13

Consider the function

$$f: \mathcal{P}_2 \longrightarrow M_{2 \times 2}(\mathbb{R})$$

defined by

$$f\left(a+bx+cx^2\right)=\left(\begin{array}{cc} a+b & 0 \\ a-c & b+c \end{array}\right)$$

(a) Prove that the nullspace of f has dimension one, and give a basis $\{v\}$ for it.
(b) Prove that the matrices $f(x)$ and $f\left(x^2\right)$ form a basis for the image of f.
(c) Find the matrix A which represents the linear map f with respect to the standard bases for $\mathcal{P}_2$ and $M_{2 \times 2}(\mathbb{R})$. Using part (b), write down a basis for the column-space of this matrix.
(d) Extend the set $\left\{f(x), f\left(x^2\right)\right\}$ to a basis $\mathcal{B}$ for $M_{2 \times 2}(\mathbb{R})$. Write down:
i. the change of basis matrix P from the basis $\mathcal{B}$ to the standard basis for $M_{2 \times 2}(\mathbb{R})$,
ii. The change of basis matrix Q from the basis $\left\{v, x, x^2\right\}$, where v is the polynomial you gave in part (a), to the standard basis for $\mathcal{P}_2$. Show that

$$AQ=P\left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Bibliography

[PH] The Man Who Loved Only Numbers, The Story of Paul Erdos and The Search for Mathematical Truth; Paul Hoffman; 1999.

[YN] 数域的上同调; 尤尔根·诺伊基希, 亚历山大[德], 哈尔滨工业大学.

[TH] Holder不等式及其应用; 田景峰, 哈明虎, 清华大学.

[HKZ2013on] Hu, W., Kukavica, I., Ziane, M.: On the regularity for the Boussinesq equations in a bounded domain, J. Math. Phys. 54(8), 081507, 10 (2013)

[T1997Inf] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences Vol. 68 (Springer, 1997).