Chapter 1

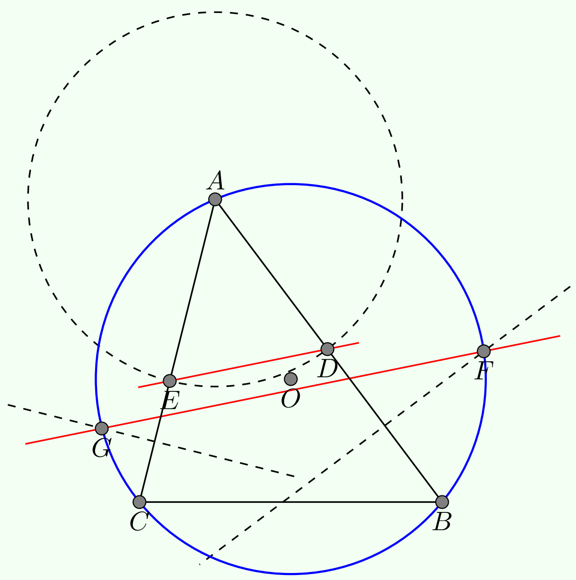
AOPS

1.1 IMO

1.1.1 2018

问题 1.1.1

Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.



问题 1.1.2

Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \dots a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$ and

$$a_i a_{i+1} + 1 = a_{i+2},$$

for $i = 1, 2, \dots, n$.

问题 1.1.3

An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1+2+3+\cdots+2018$?

问题 1.1.4

A site is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.
Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.
Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

问题 1.1.5

Let a_1, a_2, ... be an infinite sequence of positive integers. Suppose that there is an integer $N>1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m=a_{m+1}$ for all $m \geq M$.

问题 1.1.6

A convex quadrilateral $ABCD$ satisfies $AB\cdot CD=BC\cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB=\angle XCD \quad \text{and} \quad \angle XBC=\angle XDA.$$

Prove that $\angle BXA+\angle DXC=180°$.

1.2 2003

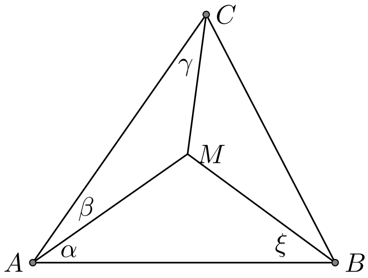
1.2.1 02

问题 1.2.1: USAMO 1996, Problem 5

Let ABC be a triangle, and M an interior point such that $\angle MAB=10°$, $\angle MBA=20°$, $\angle MAC=40°$ and $\angle MCA=30°$. Prove that the triangle is isosceles.

解. 设$\angle MCB=x$, 用Ceva角元公式.

根据AOPS中的问题, 进行推广研究有下面的结论, 现在研究, 当角度α,β,γ,ξ均给定时, 三角形$\triangle ABC$是等腰三角形的条件.



当$AC = AB$时, 由正弦定理有$\sin\gamma\sin(\alpha+\xi) = \sin(\beta+\gamma)\sin\xi$.

当$AC = BC$时, 同样有$\sin(\beta+\gamma)\sin(\alpha+\beta-\xi) = \sin\beta\sin(\alpha+\beta+\gamma+\xi)$; 当$AB = BC$时, 有$\sin(\alpha+\xi)\sin(\alpha+\beta-\gamma) = \sin(\alpha+\beta+\gamma+\xi)\sin\alpha$.

$\triangle ABC$是等腰三角形的充要条件是$(AB-AC)(AB-BC)(AC-BC) = 0$成立. 所以有

$$\begin{gathered}(\sin\gamma\sin(\alpha+\xi) - \sin(\beta+\gamma)\sin\xi)\,(\sin(\beta+\gamma)\sin(\alpha+\beta-\xi) - \sin\beta\sin(\alpha+\beta+\gamma+\xi)) \\ \cdot(\sin(\alpha+\xi)\sin(\alpha+\beta-\gamma) - \sin(\alpha+\beta+\gamma+\xi)\sin\alpha) = 0.\end{gathered}$$

□

问题 1.2.2: Russian olympiad

Let A and B be two sets such that $A\cup B = \{1,2,..,2n\}$ and $|A| = |B| = n$. Let $A = \{a_1,a_2,\dots,a_n\}$ and $B = \{b_1,b_2,\dots,b_n\}$ with $a_1 < a_2 < \cdots < a_n$ and $b_1 > b_2 > \cdots > b_n$. Prove that $|a_1-b_1|+|a_2-b_2|+\cdots+|a_n-b_n| = n^2$.

解. 问题和答案在此

问题可以推广, 当A, B是不交的元素个数相同的数积, $S = A\cup B$, 且仍记$|S| = 2n$, a_i, b_i的增减性条件保持, 则上面的要证的等式等于S中较大的n项和与较小的n项和之差. □

1.2.2 03

问题 1.2.3: Indian olympiad 2003

consider triangle acute angled ABC . let BE and CF be cevians with E and F on AC and AB resp intersecting in P. join EF and AP. denote the intersection of AP and EF by D. draw perpendicular on CB from D and denote the intersction of the perpendicular by K. Prove that KD bisects $\angle EKF$.

问题 1.2.4: IMO Shortlist 1997, Q7

The lengths of the sides of a convex hexagon $ABCDEF$ satisfy $AB = BC$, $CD = DE$, $EF = FA$. Prove that:

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

问题 1.2.5: Iran 1999

Let ABC be a triangle, and let w be a circle passing through A and C. Sides AB and BC meet w again at D and E, respectively. Let q be the incircle of the circular triangle EBD, i. e. the circle which touches the segments BD and BE and internally touches the circle w. Suppose the circle q touches the arc DE at M. Prove that the line MI is the angle bisector of the angle AMC, where I is the incenter of triangle ABC.

1.2.3 05

1.2.4 06

问题 1.2.6: NZ IMO 2003 Team

Show that given a directed graph with n nodes, where n is even, such that: vertex 1 is joined to vertex 2, vertex 2 is joined to vertex 3 and vertex 4, vertex 3 is joined to vertex 5 and vertex 6, ..., vertex $(n/2)$ joined to vertex $n-1$ and vertex n, vertex $(n/2)+1$ joined again to vertex 1 and vertex 2, ..., vertex n joined to vertex $n-1$ [for each k with $1 \leq k \leq n$, the vertex k is joined to the vertices $2k$ and $2k-1 \bmod n$] we can always find an Euler tour of the graph going along the edges in the right direction.

1.2.5 07

问题 1.2.7

We are given n vertices (unjoined). How many trees can we form by joining them? A tree is a graph without cycles.

问题 1.2.8

Let ABC be a triangle. Prove that: $R\sqrt{2pabc} \leq abc$.

问题 1.2.9: IMO ShortList 2003, combinatorics problem 1

Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers t_1, $t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

问题 1.2.10: IMO ShortList 2003, number theory problem 3

Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

问题 1.2.11: IMO ShortList 2003, geometry problem 6

Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\dfrac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

问题 1.2.12: IMO ShortList 2003, geometry problem 1

Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

问题 1.2.13: IMO ShortList 2003, algebra problem 4

Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2 - 1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

问题 1.2.14: IMO ShortList 2003, number theory problem 6

Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

问题 1.2.15

Prove that in every triangle ABC with sides a, b, c we have

$$a^2+b^2+c^2 \geq 4S\sqrt{3}\max\left\{\frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}\right\}$$

where S is the area of $\triangle ABC$ and m_a, m_b, m_c and h_a, h_b, h_c are the medians and altitudes of the triangle corresponding to the sides a, b, c respectively.

问题 1.2.16

Please mail names of greatest mathematical geniuses whose performance are sensational at IMOs.

问题 1.2.17: Japan MO 1997, problem #2

Prove that
$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}$
for any positive real numbers a, b, c.

问题 1.2.18

Let N be a point on the longest side AC of a triangle ABC. The perpendicular bisectors of AN and NC intersect AB and BC respectively in K and M. Prove that the circumcenter O of $\triangle ABC$ lies on the circumcircle of triangle KBM.

问题 1.2.19: IMO Shortlist 1997, Q4

An $n\times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a silver matrix if, for each $i = 1, 2, \ldots, n$, the i-th row and the i-th column together contain all elements of S. Show that:
(a) there is no silver matrix for $n = 1997$;
(b) silver matrices exist for infinitely many values of n.

1.2.6 08

问题 1.2.20: IMO Shortlist 2001, C1

What is the largest number of subsequences of the form $n, n+1, n+2$ that a sequence of 2001 positive integers can have? For example, the sequence 1, 2, 2, 3, 3 of 5 terms has 4 such subsequences.

问题 1.2.21

a and b are positive coprime integers. A subset S of the non-negative integers is called admissible if 0 belongs to S and whenever k belongs to S, so do $k+a$ and $k+b$. Find $f(a, b)$, the number of admissible sets.

问题 1.2.22: IMO Shortlist 1999, C3

A chameleon repeatedly rests and then catches a fly. The first rest is for a period of 1 minute. The rest before catching the fly $2n$ is the same as the rest before catching fly n. The rest before catching fly $2n+1$ is 1 minute more than the rest before catching fly $2n$.
How many flies does the chameleon catch before his first rest of 9 minutes? How many minutes (in total) does the chameleon rest before catching fly 98? How many flies has the chameleon caught after 1999 total minutes of rest?

问题 1.2.23: IMO Shortlist 1999, C6

Every integer is colored red, blue, green or yellow. m and n are distinct odd integers such that $m+n$ is not zero. Show that we can find two integers a and b with the same color such that $a-b=m$, n, $m+n$, or $m-n$.

问题 1.2.24: IMO Shortlist 1994, C6

Two players play alternatively on an infinite square grid. The first player puts a X in an empty cell and the second player puts a O in an empty cell. The first player wins if he gets 11 adjacent Xs in a line, horizontally, vertically or diagonally. Show that the second player can always prevent the first player from winning.

问题 1.2.25

Lagrangia posted this problem a day or two ago and asked for some ideas :
A (2k+1) x (2k+1) chessboard in coloured in white and black in the usual way such that the four corners are black. For which k is it possible to cover some squares on the table with trominoes (L-shaped figures made up from 3 squares) such that all the black squares are covered ? For these values of k , what is the minimal number of trominoes ?
I have a solution but I will post it tonight because I don't have time now.Anybody else solved it?

问题 1.2.26: CMO (Canada MO) 1999, problem 5

Let x, y, and z be non-negative real numbers satisfying $x+y+z=1$. Show that

$$x^2y + y^2z + z^2x \leq \frac{4}{27}$$

and find when equality occurs.

问题 1.2.27

Prove that in any choice of $n+1$ numbers from $\{1, 2, .., 2n\}$, there exist 2, a and b, so that $a \mid b$.

问题 1.2.28

Let E be a finite set of point(in the plane), no 3 of them colinear, no 4 of them concyclic. An unordered pair of points A,B is called a good pair iff there exists a disk which contains only A and B but no other point. Let f(E) be the number of good pair in E. -¿Prove that if E has 1003 points, then 2003¡= f(E) ¡=3003

问题 1.2.29: Romanian selection test 2002

Let $n \geq 4$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$a_1^2 + a_2^2 + \cdots + a_n^2 = 1.$$

Prove that the following inequality takes place

$$\frac{a_1}{a_2^2+1} + \cdots + \frac{a_n}{a_1^2+1} \geq \frac{4}{5}\left(a_1\sqrt{a_1} + \cdots + a_n\sqrt{a_n}\right)^2.$$

问题 1.2.30: IMO Shortlist 2000, Problem G1

In the plane we are given two circles intersecting at X and Y. Prove that there exist four points with the following property:
(P) For every circle touching the two given circles at A and B, and meeting the line XY at C and D, each of the lines AC, AD, BC, BD passes through one of these points.

问题 1.2.31

Let $A = \{1, 2, 3, \ldots, 6003\}$. Let B be a subset of A such that $|B| = 4002$. Prove that B has a subset C which satisfies: (i) $|C| = 2001$; (ii) If you arrange the 2001 elements of C in increasing order, then you get 2001 numbers which are even and odd in turn, i.e., even, odd, even, odd, ... or odd, even, odd, even, ...

问题 1.2.32

Consider a circle with a radius of 16 cm. 650 points inside this circle are given. A ring is the part of the plane that's included between two concentric circles of radius 2 cm and 3 cm respectively. Show that a ring can be placed such that at least 10 of the 650 given points are covered by this ring.

问题 1.2.33

Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n-1)) = f(n+1) - f(n)$$

for all $n \geq 2$?

问题 1.2.34: IMO 1994, Problem 5, IMO Shortlist 1994, A3

Let S be the set of all real numbers strictly greater than 1. Find all functions $f : S \to S$ satisfying the two conditions:
(a) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x, y in S;
(b) $\frac{f(x)}{x}$ is strictly increasing on each of the two intervals $-1 < x < 0$ and $0 < x$.

问题 1.2.35

Each vertex of a regular 1997-gon is labeled with an integer, such that the sum of the integers is 1. Starting at some vertex, we write down the labels of the vertices reading counterclockwise around the polygon. Can we always choose the starting vertex so that the sum of the first k integers written down is positive for $k = 1, ..., 1997$?

1.2.7 09

问题 1.2.36

An infinite arithmetic progression whose terms are positive integers contains the square of an integer and the cube of an integer. Show that it contains the sixth power of an integer.

问题 1.2.37

Let S = 1, 2, 3, ..., 1982. Determine the maximum number of elements of a set A such that : (1) A is a subset of S ; (2) There do not exist numbers x, y, z in A such that xy = z.

问题 1.2.38: IMO Shortlist 1997, Q22

Does there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^k$ for all real numbers x
a) if $k = 3$?
b) if $k = 4$?

问题 1.2.39

On a blackboard we have the numbers $1, 2, 3, \ldots, 2001$. An operation is this: we erase a and b and we replace them with one number, $\frac{ab}{a+b+1}$. After 2000 operations, we are left with one number k. What is k?

问题 1.2.40

(by positive i mean zero or more)
consider the equation a*x+b*y=c with a,b,c positive and a and b coprime
it can easily be proven that if c¿a*b-a-b , positive solutions x and y can be found for the equation
but for some reason i can't seem to prove this fact :
exactly half of the integers :1,2,3,4,......a*b-a-b has positive solutions
some experiment suggest that k has positive solutions iff (a*b-a-b)-k has none
but how to prove that problem? anyone? plz help me?

问题 1.2.41

A recurrence relation is defined like that:

(1) $a_1 = 1$;

(2) $a_n + 1 = 1/16 \times (1 + 4a_n + \sqrt{1 + 24a_n})$, and $n \in N$

Determine the explicit formula and prove that it is correct !

问题 1.2.42

Let $n \in N$ and $M_n = 1, 2, 3, \ldots, n$. A subset T of M_n is called 'cool' if no element of T is smaller than the number of elements of T. The number of cool subsets of M_n is denoted by $f(n)$.
Determine a formula for $f(n)$. In particular calculate $f(32)$!

问题 1.2.43

The two mathematicians Lagrangia and Galois :D play the following game: They select and take from the set 0,1,2,3,...,1024 512, 256, 128, 64, 32, 16, 8, 4, 2, 1 numbers away alternately. Lagrangia starts and takes 512 numbers away, then Galois 256 numbers etc. Finally there are two remaining numbers a,b (a¡b). Galois pays Lagrangia the following amount of money: abs(b-a). Lagrangia wants to obtain as much money as possible. And vice versa Galois wants to loose at least money as possible.
Exlain how much money Lagrangia can earn at most ! Assume that they all try their best.

问题 1.2.44

Feuerbach's theorem: Prove that in any triangle, the inscribed circle and the 3 exinscribed circles are tangent to Euler's circle.

问题 1.2.45: Bundeswettbewerb Mathematik 1988, stage 2, problem 4

Provided the equation $xyz = p^n(x+y+z)$ where $p \geq 3$ is a prime and $n \in \mathbb{N}$. Prove that the equation has at least $3n+3$ different solutions (x,y,z) with natural numbers x,y,z and $x<y<z$. Prove the same for $p>3$ being an odd integer.

问题 1.2.46

Prove that:

$$\left(\sum_{k=0}^{n} 2^k \binom{2n}{2k}\right)^2 - 2\left(\sum_{k=0}^{n-1} 2^k \binom{2n}{2k+1}\right)^2 = 1$$

$\binom{a}{b}$ denotes the binomial coefficient, $\frac{a!}{b!(a-b)!}$

问题 1.2.47: IMO Shortlist 2000, Problem N2

For a positive integer n, let $d(n)$ be the number of all positive divisors of n. Find all positive integers n such that $d(n)^3 = 4n$.

问题 1.2.48

Let a,b,c be positive real numbers with $abc=1$. Prove that

$$\sum_{cyc}(a+bc) \leq 3 + \frac{a}{c} + \frac{b}{a} + \frac{c}{b}$$

问题 1.2.49

Suppose that each of the n guests at a party acquaited with exactly 8 other guests. Furthermore, suppose that each pair of guests who are acquainted with each other have four acquaintances in common at the party, and each pair of guests who are not acquainted have only two acquaintances in common. What are the possible values of n ?

问题 1.2.50: USAMO 1997/5; also: ineq E2.37 in Book: Inegalitati; Authors:L.Panaitopol,V. Bandila,M.Lascu

Prove that, for all positive real numbers a, b, c, the inequality

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}$$

holds.

问题 1.2.51: IMO 1996 Shortlist

Suppose that $a,b,c>0$ such that $abc=1$. Prove that

$$\frac{ab}{ab+a^5+b^5} + \frac{bc}{bc+b^5+c^5} + \frac{ca}{ca+c^5+a^5} \leq 1.$$

问题 1.2.52

A nice one : Let n be a positive integer. Ann writes down n different positive integers. Then Ivo deletes some numbers (possibly none, but not all). He puts + or - signs in front of each of the remaining numbers and sums them up. If the result is divisible by 2003, Ivo wins. Otherwise Ann wins. For which values of n Ivo has a winning strategy ? For which values of n Ann has a winning strategy ?

问题 1.2.53

The sequence $\{u_n\}$ with n being a positive integer is given by the recurrence
(1) $u_0 = 0$ (2) $U_{2n} = u_n$ (3) $u_{2n+1} = 1 - u_n$
a.) Determine u_{2002} and b.) and u_m with $m = (2^p - 1)^2$ and p a natural number !

问题 1.2.54: JBMO 2002, Problem 4

问题 1.2.55

Show that

$$\frac{1+a^2}{1+b+c^2} + \frac{1+b^2}{1+c+a^2} + \frac{1+c^2}{1+a+b^2} \geq 2$$

for reals $a, b, c \geq -1$.

问题 1.2.56

There are n girls and n boys at the party. Participants who belong to the same sex do not know each other. Moreover, there cannot be found two girls who know the same two boys. At most how many acquaintances can be among the participants of the party ?
a.) n = 5 b.) n = 7

Chapter 2

定理集

定理 2.0.1: 勾股定理

若$(\alpha,\beta)=0$, 则

$$|\alpha+\beta|^2=|\alpha|^2+|\beta|^2.$$

进一步的, 若向量$\alpha_1,\alpha_2,\cdots,\alpha_m$两两正交, 则

$$|\alpha_1+\alpha_2+\cdots+\alpha_m|^2=|\alpha_1|^2+|\alpha_2|^2+\cdots+|\alpha_m|^2.$$

定理 2.0.2

若$\alpha_1,\alpha_2,\cdots,\alpha_m$两两正交, 则$\alpha_1,\alpha_2,\cdots,\alpha_m$线性无关.

定理 2.0.3: 标准正交基的存在性

在任何有限维欧式空间中, 都有标准正交基. 有限维欧式空间V中的任何非零正交向量组都可以扩充为V的一个正交基.

定理 2.0.4

两个有限维欧式空间同构的充要条件是它们的维数相同.
任何n维欧式空间都与欧式空间$\mathbb{R}^n$同构.

定理 2.0.5

若欧式空间V的子空间$V_1,V_2,\cdots,V_s$两两正交, 则它们的和是直和.
反之, 子空间的和为直和时, 子空间之间不一定正交.

定理 2.0.6

设W是欧式空间V的子空间. 则$W^\perp=\{\gamma\mid\gamma\in V,\gamma\perp W\}$是$V$的子空间; 当$W$是有限维时, $V=W\oplus W^\perp$, $(W^\perp)^\perp=W$.
V_1,V_2是欧式空间V(不一定有限维)的两个子空间. 证明:

$$(V_1+V_2)^\perp=V_1^\perp\bigcap V_2^\perp.$$

定理 2.0.7

设T是n维欧式空间V的一个线性变换. 则T是正交变换的充要条件是, T把标准正交基变成标准正交基.

定理 2.0.8

设$s_1, s_2, \cdots, s_n$是n维欧式空间V的一个标准正交基，A是一个n阶实方阵，且$(\eta_1, \eta_2, \cdots, \eta_n) = (s_1, s_2, \cdots, s_n)A$. 则$\eta_1, \eta_2, \cdots, \eta_n$是标准正交基的充要条件是, A为正交方阵.

定理 2.0.9

设T是n维欧式空间V的一个线性变换. 则T是正交变换的充要条件是, T在标准正交基下的矩阵是正交矩阵.
有限维欧式空间V的正交变换有逆变换, 而且是V到V的同构映射.
其中充分性部分, 标准正交性的条件不可省去.

定理 2.0.10

实数域上有限维空间(不要求是欧式空间)的每一个线性变换, 都有一维或二维的不变子空间.

解. 证明分实特征根和复特征根两种情况, 复的情况对特征向量分离实虚部, 得到不变子空间的基. □

定理 2.0.11

设T是有限维欧式空间V的一个正交变换. 若子空间W对T不变, 则$W^\perp$对T也不变.
设T是有限维欧式空间V的一个正交变换, 则V可分解成对T不变的一维或二维子空间的直和.

定理 2.0.12

欧式空间中正交变换的特征值为± 1.
正交方阵的特征根的模为1.

定理 2.0.13

设T是二维欧式空间V的一个正交变换, 且无特征值, 则T在标准正交基下的矩阵具有形状

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

定理 2.0.14

设T是n维欧式空间V的一个正交变换, 则存在标准正交基, 使T在此基下的矩阵成下面形状:

$$\begin{pmatrix} 1 &&&&&&&& \\ & \ddots &&&&&&& \\ && 1 &&&&&& \\ &&& -1 &&&&& \\ &&&& \ddots &&&& \\ &&&&& -1 &&& \\ &&&&&& S_1 && \\ &&&&&&& \ddots & \\ &&&&&&&& S_r \end{pmatrix}$$

其中

$$S_i = \begin{pmatrix} \cos\varphi_i & -\sin\varphi_i \\ \sin\varphi_i & \cos\varphi_i \end{pmatrix}, \qquad i = 1, 2, \cdots, r.$$

对于任何n阶正交方阵A, 都存在正交方阵U, 使$U^{-1}AU$为上面方阵的形式.

定理 2.0.15

设T是n维欧式空间V的一个线性变换. 则T是对称变换的充要条件是, T在标准正交基下的矩阵为对称方阵.

定理 2.0.16

实对称方阵的特征根全是实数.

定理 2.0.17

设T是n维欧式空间V的对称变换, 则T的属于不同特征值的特征向量相互正交.

定理 2.0.18

设T是n维欧式空间V的一个对称变换, W是对T不变的非零子空间, 则W中有关于T的特征向量.
如果α是它的一个特征向量, 则与α正交的全体向量是T的$n-1$维不变子空间.
对V的每个对称变换T, 都存在标准正交基, 使T在此基下的矩阵为对角矩阵.
对每个实对称方阵A, 都存在正交方阵U, 使$U^{-1}AU$为对角矩阵.
任何实二次型

$$f(x_1, x_2, \cdots, x_n) = X'AX$$

都可经过正交线性代换$X = UY$化成

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

实对称方阵A是正定的充要条件是, A的特征根全是正的.

2.1 数学分析

定理 2.1.1: 关于有界, 无界的充分条件

(1) $\lim\limits_{x\to x_0^-} f(x)$存在, 则$\exists\delta > 0$, 当$-\delta < x - x_0 < 0$时, $f(x)$有界; 对$x \to x_0^+$, $x \to x_0$有类似结论.
(2) $\lim\limits_{x\to\infty} f(x)$存在, 则存在$X > 0$, 当$|x| > X$时, $f(x)$有界. $x \to \pm\infty$有类似结论.
(3) $f(x) \in C[a, b]$, 则$f(x)$在$[a, b]$上有界.
(4) $f(x)$在集U上有最大(小)值, 则$f(x)$在U上有上(下)界.
(5) 有界函数间的和, 积运算封闭.
(6) $\lim\limits_{x\to\square} f(x) = \infty$, 则$f(x)$在$\square$的空心邻域内无界. $\square$可为$x_0, x_0^-, x_0^+, \infty, \pm\infty$.

定理 2.1.2: Stolz定理

证明: 若

(a) $y_{n+1} > y_n (n \in \mathbb{N}_+)$;

(b) $\lim_{n\to\infty} y_n = +\infty$;

(c) $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$存在.

则

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}.$$

定理 2.1.3: Cauchy定理

若函数$f(x)$定义于区间$(a,+\infty)$, 并且在每一个有限区间(a,b)内是有界的, 则

(a) $\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}[f(x+1)-f(x)]$;

(b) $\lim_{x\to+\infty}[f(x)]^{1/x}=\lim_{x\to+\infty}\frac{f(x+1)}{f(x)}$, $(f(x)\geq C>0)$,

假定等是右端的极限都存在且可为$\pm\infty$.

问题: 对于上下极限是否仍有类似结论?.

定理 2.1.4

假设$f_n:[a,b]\to\mathbb{R}$, 每个在$[a,b]$上均可积, 且$f_n(x)\rightrightarrows f(x)$, $n\to\infty$. 则$f(x)$可积, 且

$$\int_a^b f\,\mathrm{d}x=\lim_{n\to\infty}\int_a^b f_n\,\mathrm{d}x.$$

解. 类似?? □

定理 2.1.5: (一致收敛级数)逐项积分

$u_k:[a,b]\to\mathbb{R}$, 对每个$k\in\mathbb{N}_+$均可积, $\sum_{k=1}^\infty u_k(x)$在$[a,b]$上一致收敛. 则$f(x)=\sum_{k=1}^\infty u_k(x)$可积, 且

$$\int_a^b f(x)\,\mathrm{d}x=\sum_{k=1}^\infty\int_a^b u_k(x)\,\mathrm{d}x.$$

解. 让$S_n(x)=\sum_{k=1}^n u_k(x)$并用2.1.4 □

定理 2.1.6: 逐项微分

设$u_k:[a,b]\to\mathbb{R}$, $k\in\mathbb{N}_+$, 每项均有连续导数(端点处单边可微), 若有:

(i) $\sum_{k=1}^\infty u_k(x_0)$在某些点$x_0\in[a,b]$收敛.

(ii) $\sum_{k=1}^\infty u_k'(x)$在$[a,b]$上一致收敛到$f(x)$.

则

(1) $\sum_{k=1}^\infty u_k(x)$在$[a,b]$上收敛且和函数$F(x)=\sum_{k=1}^\infty u_k(x)$在$[a,b]$上可微且$F'(x)=f(x)$.

(2) $\sum_{k=1}^\infty u_k(x)\rightrightarrows F(x)$.

解. (1). u_k'连续$(k\in\mathbb{N}_+)$, $\sum_{k=1}^\infty u_k'\rightrightarrows f$, 则$f\in C[a,b]$, 所以$f$在$[a,b]$上可积. 让$x\in[a,b]$, 对$u_k'$和$f$在区间$[x_0,x]$上使用2.1.5, (或$[x,x_0]$, 如果$x<x_0$), 则有

$$\sum_{k=1}^\infty\int_{x_0}^x u_k'(x)\,\mathrm{d}x=\int_{x_0}^x f(x)\,\mathrm{d}x=\lim_{n\to\infty}\sum_{k=1}^n(u_k(x)-u_k(x_0)).$$

由假设(i), $\sum_{k=1}^\infty u_k(x_0)$收敛, 所以级数$F(x)=\lim_{n\to\infty}\sum_{k=1}^n u_k(x)$对任意$x\in[a,b]$均收敛, 所以$F:[a,b]\to\mathbb{R}$是良定义的, 于是$F(x)-F(x_0)=\int_{x_0}^x f(x)\,\mathrm{d}x$, 由$f$连续, 两边求导, 便有$F'(x)=f(x)$.

(2). Cauchy判别法, 取$\varepsilon>0$, 则$\exists N_1$使任意$n\geq m\geq N_1$有

$$\left|\sum_{k=m}^n u_k(x_0)\right|<\frac{\varepsilon}{2}$$

存在N_2使任意$n \geq m \geq N_2$有

$$\left|\sum_{k=m}^{n} u_k'(x)\right| < \frac{\varepsilon}{2(b-a)}, \quad x \in [a,b]$$

故可取$N = \max\{N_1, N_2\}$, $g(x) = \sum_{k=m}^{n} u_k(x)$, 则$g(x) - g(x_0) = g'(\xi)(x - x_0)$. 于是

$$|g(x)| \leq |g(x_0)| + |g(x) - g(x_0)| \leq \frac{\varepsilon}{2} + |g'(\xi)| \cdot |x - x_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)} \cdot (b-a) = \varepsilon.$$

故由Cauchy判别法, $\sum_{k=1}^{\infty} u_k(x)$一致收敛. □

定理 2.1.7: 求导与极限的交换

函数列$f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}_+$, 在$[a,b]$上连续可微, $f_n(x) \to f(x)$, $x \in [a,b]$. $f_n'(x) \rightrightarrows \varphi(x)$, $x \in [a,b]$, 则f可微且$f'(x) = \varphi(x)$, 从而$f_n \rightrightarrows f$.

解. 取$u_1 = f_1$, $u_n = f_n - f_{n-1}$, $n > 1$, 并用2.1.6. □

2.2 微分方程

定理 2.2.1: 伯努力方程

$\frac{\mathrm{d}y}{\mathrm{d}x} = p(x)y + q(x)y^n$, 其中$p(x)$, $q(x)$是所考虑区域上的连续函数, $n(\neq 0, 1)$是常数.

解.

(1) 当$n > 0$时, $y = 0$是方程的解.

(2) 当$y \neq 0$时, 两边同除以y^n, 令$z = y^{1-n}$, 即得一阶线性方程.

□

定理 2.2.2

设$y_1(x)$和$y_2(x)$是方程

$$y'' + P(x)y' + Q(x)y = 0$$

的两个线性无关解, 齐次边值问题

$$\begin{cases} y'' + P(x)y' + Q(x)y = 0; \\ \alpha_1 y(a) + \beta_1 y'(a) = 0; \\ \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

存在非平凡解(即不恒等于零的解)当且仅当

$$\begin{vmatrix} \alpha_1 y_1(a) + \beta_1 y_1'(a) & \alpha_1 y_2(a) + \beta_1 y_2'(a) \\ \alpha_2 y_1(b) + \beta_2 y_1'(b) & \alpha_2 y_2'(b) + \beta_2 y_2'(b) \end{vmatrix} = 0$$

2.3 泛函分析

定理 2.3.1: Arzela-Ascoli定理

设$\{f_n\}$是$[0,1]$上一致有界, 等度连续函数族, 则存在某一子序列$\{f_{n(i)}\}$在$[0,1]$上一致收敛.

定理 2.3.2: Hahn-Banach, $\mathbb{R}-version$

设$\mathcal{X}$是定义在$\mathbb{R}$上的向量空间, $q:\mathcal{X}\to\mathbb{R}$是拟半范数. 若给定线性子空间$\mathcal{Y}\subset\mathcal{X}$和其上的线性映射$\phi:\mathcal{Y}\to\mathbb{R}$, 使得

$$\phi(y)\le q(y),\forall y\in\mathcal{Y}.$$

则存在线性映射$\varphi:\mathcal{X}\to\mathbb{R}$满足

(i) $\varphi\mid_{\mathcal{Y}}=\phi$;

(ii) $\varphi(x)\le q(x)$, $\forall x\in\mathcal{X}$.

解. 先证$\mathcal{X}/\mathcal{Y}=1$的情况. 即有$x_0\in\mathcal{X}$使得

$$\mathcal{X}=\{y+sx_0:y\in\mathcal{Y},s\in\mathbb{R}\}.$$

于是只需找到$\alpha\in\mathbb{R}$使得映射$\varphi(y+sx_0)=\phi(y)+s\alpha$, $\forall y\in\mathcal{Y}$, $s\in\mathbb{R}$满足条件(ii), 于是$s>0$时有

$$\alpha\le q(z+x_0)-\phi(z),\forall z\in\mathcal{Y},z=s^{-1}y,s>0$$

对于$s<0$时有

$$\alpha\ge\phi(w)-q(w-x_0),\forall w\in\mathcal{Y},w=t^{-1}y,s<0$$

然而$\phi(w)-q(w-x_0)\le q(z+x_0)-\phi(z)$, $\forall w,z\in\mathcal{Y}$恒成立. 然后用Zorn引理. □

未知 2.3.1: Hahn-Banach定理, $\mathbb{C}$-version

设$\mathcal{X}$是复数域$\mathbb{C}$上的向量空间, $q:\mathcal{X}\to\mathbb{R}$是$\mathcal{X}$上的拟半范数, 给定线性子空间$\mathcal{Y}\subset\mathcal{X}$和$\mathcal{Y}$上的线性映射$\phi:\mathcal{Y}\to\mathbb{C}$使得

$$\mathrm{Re}\phi(y)\le q(y),\quad\forall y\in\mathcal{Y}.$$

则存在线性映射$\psi:\mathcal{X}\to\mathbb{C}$满足:

(i) $\psi\mid_{\mathcal{Y}}=\phi$;

(ii) $\mathrm{Re}\psi(x)\le q(x)$, $\forall x\in\mathcal{X}$.

解. 设$\phi_1=\mathrm{Re}\phi$, 因ϕ_1是$(\mathcal{Y},\mathbb{R})$上的线性映射且被拟半范数q控制, 则由$\mathbb{R}$-Hahn Banach定理, ϕ_1可延拓到$(\mathcal{X},\mathbb{R})$上的实线性映射ψ_1且满足

(i') $\psi_1\mid_{\mathcal{Y}}=\phi_1$;

(ii') $\psi_1(x)\le q(x)$, $\forall x\in\mathcal{X}$.

但注意这里用的是实的Hahn Banach定理, 所延拓的ψ_1是针对实向量空间$(\mathcal{X},\mathbb{R})$的, 要得到复向量空间的ψ_1, 则在$(\mathcal{Y},\mathbb{C})$上考虑$\psi_1(y)=\phi_1(y)$, 但新定义的ψ_1是实域上的线性映射, 而不是复域上的线性映射, 显然所求线性映射ψ的实部$\mathrm{Re}\psi$在实线性空间中也满足以上两条件. 若取$\mathrm{Re}\psi=\psi_1$, 则ψ在实的情况已满足条件(ii). 而$\mathrm{Im}\psi(y)=\mathrm{Re}(-\mathrm{i}\psi(y))=\mathrm{Re}\psi(-\mathrm{i}y)=\psi_1(-\mathrm{i}y)$, 于是$\psi(y)=\psi_1(y)+\mathrm{i}\psi_1(-\mathrm{i}y)$, 要证$\psi\mid_{\mathcal{Y}}=\phi$, 只需证$\mathrm{Im}\psi(y)=\mathrm{Im}\phi(y)$, $\forall y\in\mathcal{Y}$, 然而

$$\mathrm{Im}\phi(y)=\mathrm{Re}(-\mathrm{i}\phi(y))=\mathrm{Re}(\phi(-\mathrm{i}y))=\phi_1(-\mathrm{i}y)=\psi_1(-\mathrm{i}y)=\mathrm{Im}\psi(y).$$

最后证明线性映射$\psi(x)=\psi_1(x)+\mathrm{i}\psi_1(-\mathrm{i}x)$在复域上满足(ii), 注意这里的$\psi_1(-\mathrm{i}x)$是怎么定义的? □

2.4 拓扑

定理 2.4.1: 杨忠道定理

证明: 拓扑空间中的每一子集的导集为闭集的充分必要条件是此空间中的每一个单点集的导集为闭集.

解. 只证充分性. 设拓扑空间X的每一个单点集的导集为闭集, 任意$A \subset X$, 设$x \in d(d(A))$, 对x的任意开邻域U, 有$U \cap (d(A) \setminus \{x\}) \neq \varnothing$, 因$d(\{x\})$是闭集, 且$x \notin d(\{x\})$, 令$V = U \setminus d(\{x\})$, V是x的开邻域, 从而有

$$y \in V \cap (d(A) \setminus \{x\}).$$

由$y \in V$, $y \notin d(\{x\})$, 且$y \neq x$, 于是存在$W \in \mathscr{U}_y$, 使得$x \notin W$, 因$V \in \mathscr{U}_y$, 令$K = W \cap V$, $K \in \mathscr{U}_y$, 由$y \in d(A)$, 存在$z \in K \cap (A \setminus \{y\}) \neq \varnothing$. 由$z \in K \subset W$, $z \neq x$, 因此$z \in U \cap (A \setminus \{x\})$, 故$U \cap (A \setminus \{x\}) \neq \varnothing$, 即$x \in d(A)$, 所以$d(d(A)) \subset d(A)$, $d(A)$为闭集. □

2.5 数论

定理 2.5.1: 恒等定理

设$f(x), g(x) \in D[x]$, 若有无穷多个$\alpha \in D$使$f(\alpha) = g(\alpha)$, 则$f(x) = g(x)$.

定理 2.5.2: 拉格朗日定理

设$f(x)$是整系数多项式, 模p的次数为n, 则同余方程

$$f(x) \equiv 0 \pmod p \tag{2.1}$$

至多有n个互不相同的解.

解. $n = 1$时结论显然成立, 对n归纳. 假设$n-1$时已正确, 当f的次数是n时, 若同余方程无解, 则无需证明. 若$x = a$是一个解, 用$(x-a)$除$f(x)$得$f(x) = g(x)(x-a) + A$, $(A \in \mathbb{Z})$, 若同余方程(2.1)除$x \equiv a \pmod p$外无解, 则证毕, 否则设$x = b$是(2.1)的另一个解, 且$a \not\equiv b \pmod p$, 则

$$0 \equiv f(b) = g(b)(b-a) + A \pmod p, \text{又由} 0 \equiv f(a) = g(a)(a-a) + A = A \pmod p.$$

所以$g(b) \equiv 0 \pmod p$, 这表明(2.1)的解除$x \equiv a \pmod p$之外, 其余的解均是$g(x) \equiv 0 \pmod p$的解, 但$g(x)$模p的次数显然是$n-1$, 由归纳假设, $g(x) \equiv 0 \pmod p$, 至多有$n-1$个互不同余的解, 从而同余方程(2.1)至多有n个解. □

定理 2.5.3: 整系数多项式的有理根

$f(x) = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$, $n \geq 1$, $a_n a_0 \neq 0$且$(a_n, \cdots, a_0) = 1$, 若$\frac{b}{c}$是$f(x)$的一个有理根$(b, c) = 1$, 则$c \mid a_n$, $b \mid a_0$. 特别地, 首项系数为± 1的整系数多项式的有理根必是整数.

解. 由$f\left(\frac{b}{c}\right) = 0$, 得$a_n b^n + \cdots + a_1 b c^{n-1} + a_0 c^n = 0$, 所以$a_0 \mid b$, $a_n \mid c$.
一种证明有理数是整数的证明途径: 证复数是整数, 先证其是有理数, 且找到作为零点的首一多项式. □

定理 2.5.4: Gauss引理

$\mathbb{Z}[x]$中两个本原多项式的乘积仍是一个本原多项式.

解. 反证法, $f(x) = a_n x^n + \cdots + a_0$, $g(x) = b_m x^m + \cdots + b_0$, 若$f(x)g(x)$不是本原多项式, 则有素数$p$整除$f(x)g(x)$的所有系数. 设$r$是$a_i$不被$p$整除的最小角标, s是b_i不被p整除的最小角标, 则$f(x)g(x)$的x^{r+s}项系数不能被p整除. □

定理 2.5.5: 艾森斯坦判别法

设$f(x) = a_n x^n + \cdots + a_0$是一个整系数多项式, 其中$n \geq 1$. 若存在一个素数$p$, 使得$p \nmid a_n$, $p \mid a_i (i = 0, 1, \cdots, n-1)$, 但$p^2 \nmid a_0$, 则$f(x)$在$\mathbb{Z}$上不可约.

定理 2.5.6: 科恩定理

设$p = \overline{a_n \cdots a_1 a_0}$是一个十进制素数, $0 \leq a_i \leq 9 (i = 0, 1, \cdots, n)$, $a_n \neq 0$. 则多项式

$$f(x) = a_n x^n + \cdots + a_1 x + a_0.$$

在$\mathbb{Z}$上不可约.

解. 先用**??**, 再用**??**. □

定理 2.5.7

Every nonzero integer can be written as a product of primes.

解. Assume that there is an integer that cannot be written as a product of primes. Let N be the smallest positive integer with this property. Since N cannot itself be prime we must have $N = mn$, where $1 < m, n < N$. However, since m and n are positive and smaller than N they must each be a product of primes. But then so is $N = mn$. This is a contradiction.

The proof can be given in a more positive way by using mathematical induction. It is enough to prove the result for all positive integers. 2 is a prime. Suppose that $2 < N$ and that we have proved the result for all numbers m such that $2 \leq m < N$. We wish to show that N is a product of primes. If N is a prime, there is nothing to do. If N is not a prime, then $N = mn$, where $2 \leq m, n < N$. By induction both m and n are products of primes and thus so is N. □

定理 2.5.8: m进(m-adic)表示

正整数$m \geq 2$, $\forall a \in \mathbb{N}_+$, 有表示$a = a_0 + a_1 m + \cdots + a_s m^s$.

定理 2.5.9: Bézout's identity(贝祖等式)

任意两整数$a, b (b \neq 0)$的正最大公因子$d = (a, b)$唯一存在, 而且存在整数u, v使得$ua + vb = d$, u, v称为Bézout系数, Bézout系数不唯一, 若设$a' = \frac{a}{d}$, $b' = \frac{b}{d}$, 则恰有两系数对满足$|u| < |b'|$, $|v| < |a'|$.

解. 若$a > b$, $a = bq + r$, 则$(a, b) = (r, b)$, 于是由碾转相除法的逆过程可得u, v. □

解. 不用碾转相除法. $M = \{ax + by \mid x, y \in \mathbb{Z}\}$, d是M中的最小正整数(自然数良序性). 则若$d = ax_0 + by_0$知$(a, b) \mid d$, 所以只需证$d \mid (a, b)$. 若$d \nmid a$, 取$a = dq + r$, 则$r = a\hat{x_0} + b\hat{y_0} < d$与$d$的选取矛盾. □

推论 2.5.1

a, b互素等价于: 存在整数u, v使$ua + vb = 1$.
$a, b \in \mathbb{Z}$, 则:

$$\begin{aligned}(a, b) = d &\Rightarrow \{am + bn \mid m, n \in \mathbb{Z}\} = \{dk \mid k \in \mathbb{Z}\} \\ &\Rightarrow a\mathbb{Z} + b\mathbb{Z} = d\mathbb{Z} \\ &\Rightarrow (a, b) = (d)\end{aligned}$$

推论 2.5.2: Bézout等式

任s个非零整数$a_1, \cdots, a_s$的最大公因子$d = (a_1, \cdots, a_s)$存在唯一, 且$(a_1, \cdots, a_{s-1}, a_s) = ((a_1, \cdots, a_{s-1}), a_s)$, 且存在整数$u_1, \cdots, u_s$使, $u_1 a_1 + \cdots + u_s a_s = d$.

定理 2.5.10

$v_p(n)$使使得$p^k \| n$的整数k. 则

$$v_p(n!) = \sum_k \left[\frac{n}{p^k}\right].$$

定理 2.5.11: 威尔逊定理

p是素数, 则有$(p-1)! \equiv -1 \pmod p$.

解. 当$p=2$时, 命题显然. 若$p \geq 3$, 由于对每个与p互素的a在模p下均有逆a^{-1}. 故可得$1, 2, \cdots, p-1$的每个与其逆配对, 而特别的当$a=a^{-1}$时是例外. 此时对应$a^2 \equiv 1 \pmod p$有解$a=1$或$a=p-1$, 而$2, \cdots, p-2$可两两配对使积为1. 所以$(p-1)! \equiv 1 \cdot (p-1) \equiv -1 \pmod p$. □

解. 用 Euler 恒等式

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} i^n = \begin{cases} 0, n < m \\ (-1)^n n!, n = m \end{cases}$$

取$m=n=p-1$, 当 $p>2$ 时及 Fermat 小定理有

$$(-1)^{p-1} \cdot (p-1)! \equiv \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} i^{p-1} \equiv \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} \equiv -1 \pmod p.$$

□

解. 当 $p \geq 3$ 时, 由 Fermat 小定理 $p-2$ 次同余方程

$$f(x) = (x-1)(x-2)\cdots(x-p+1) - x^{p-1} + 1 \equiv 0 \pmod p$$

有 $p-1$ 个不同得解, 所以 $f(x)$ 的系数模 p 余零, 所以常数项 $(p-1)!+1 \equiv 0 \pmod p$. □

2.6 不等式

定理 2.6.1: Generalized Schur Inequality

设六个非负实数a, b, c, x, y, z满足(a, b, c)和(x, y, z)均单调, 则

$$\sum_{cyc} x(a-b)(b-c) \geq 0.$$

解. 不妨设$a \geq b \geq c$, 分$x \geq y \geq z$与$x \leq y \leq z$两种情况分别讨论. □

推论 2.6.1

记$S = \sum_{cyc} x(a-b)(a-c)$. 下面几条条件的任何一个均可证明$S \geq 0$.

(1) 当$a \geq b \geq c \geq 0$, $x \geq y \geq 0$且$z \geq 0$时.

(2) 当$a \geq b \geq c \geq 0$, $z \geq y \geq 0$且$x \geq 0$时.

(3) 当$a \geq b \geq c \geq 0$, 且$ax \geq by \geq 0$或者$by \geq cz \geq 0$时.

解. (1)和(2)显然, 对于(3)有

$$\begin{aligned}&\frac{1}{abc}(x(a-b)(a-c)+y(b-a)(b-c)+z(c-a)(c-b))\\&=ax\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{c}\right)+by\left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{c}\right)+cz\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}-\frac{1}{b}\right).\end{aligned}$$

便转化为前面的两种情况了. □

定理 2.6.2: Cauchy不等式

对于欧式空间中任意向量α,β都有

$$|(\alpha,\beta)|\le|\alpha||\beta|.$$

而且即当α与β线性相关时等号成立.

定理 2.6.3: 三角形不等式

对欧式空间中任意向量α,β有

$$|\alpha+\beta|\le|\alpha|+|\beta|.$$

对欧式空间中任意向量$\alpha_1,\alpha_2,\cdots,\alpha_m$都有

$$|\alpha_1+\alpha_2+\cdots+\alpha_m|\le|\alpha_1|+|\alpha_2|+\cdots+|\alpha_m|.$$

Chapter 3

定义集

3.1 初等数论

定义 3.1.1: 模p同余

若两多项式$f(x)$与$g(x)$同次幂系数均关于模p同余, 则称$f(x)$和$g(x)$对模p同余或模p恒等.

$$f(x) \equiv g(x) \pmod p.$$

定义 3.1.2: 多项式模p的次数

若$f(x)$的系数不全被p整除, 其中系数不被p整除的最高幂次称为$f(x)$模p的次数.

定义 3.1.3: 容度, 本原多项式

设$f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$, 且$f(x) \neq 0$, 将$a_0, a_1, \cdots, a_n$的最大公约数$(a_0, a_1, \cdots, a_n)$, 称为$f(x)$的容度. 容度为1的多项式称为本原多项式.

3.2 高等代数

定义 3.2.1: 欧式空间

设V是实数域$\mathbb{R}$上的一个线性空间. 如果V中存在一个二元运算$(\cdot,\cdot) : V^2 \to \mathbb{R}$, 且满足

1. $(\alpha, \beta) = (\beta, \alpha)$,
2. $(k\alpha, \beta) = k(\alpha, \beta)$, $(k \in \mathbb{R})$,
3. $(\alpha_1 + \alpha_2, \beta) = (\alpha_1, \beta) + (\alpha_2, \beta)$,
4. 当$\alpha \neq \theta$时, $(\alpha, \alpha) > 0$,

则称在V上定义了一个内积, 并把V叫做一个欧式空间. 在欧式空间中, 常把实数(α, β)叫做向量α与β的内积.

定义 3.2.2

称非负实数$\sqrt{(\alpha, \alpha)}$为向量α的长, 并用$|\alpha|$表示, 即

$$|\alpha| = \sqrt{(\alpha, \alpha)}.$$

设α,β为两个非零向量, 称实数$\varphi=\arccos\frac{(\alpha,\beta)}{|\alpha||\beta|}$为向量$\alpha$与$\beta$的夹角, 亦即

$$\cos\varphi=\frac{(\alpha,\beta)}{|\alpha||\beta|}.$$

定义 3.2.3: 正交

如果欧式空间中两个向量α与β的内积等于零, 即$(\alpha,\beta)=0$, 则称α与β正交.

定义 3.2.4

设V是n维欧式空间. 如果V的基$\alpha_1,\alpha_2,\cdots,\alpha_n$中每两个向量都正交, 则称此基为正交基. 如果正交基中每个向量的长都是1, 则称该基为标准正交基.

定义 3.2.5: 欧式空间的同构映射

设V和V'是两个欧式空间, 如果φ是线性空间V到V'的一个同构映射, 而且对V中任意向量α,β都有

$$(\alpha,\beta)=(\varphi(\alpha),\varphi(\beta)),$$

则称φ是欧式空间V到V'的一个同构映射.
如果欧式空间V到V'存在同构映射, 则称欧式空间V与V'同构.

定义 3.2.6: Gram矩阵

设$\alpha_1,\alpha_2,\cdots,\alpha_n$为欧式空间的一组向量, 则称实对称方阵

$$G=\begin{pmatrix}(\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & \cdots & (\alpha_1,\alpha_n)\\ (\alpha_2,\alpha_1) & (\alpha_2,\alpha_2) & \cdots & (\alpha_2,\alpha_n)\\ \cdots & \cdots & \cdots & \cdots\\ (\alpha_n,\alpha_1) & (\alpha_n,\alpha_2) & \cdots & (\alpha_n,\alpha_n)\end{pmatrix}$$

为这组向量的Gram矩阵. G满秩当且仅当$\alpha_1,\alpha_2,\cdots,\alpha_n$线性无关.

定义 3.2.7: 正交

设W是欧式空间V的子空间. 如果V中向量α与W中每个向量都正交, 则称α与W正交, 记为$(\alpha,W)=0$或$\alpha\perp W$.
如果子空间V_1中每个向量与子空间V_2中每个向量都正交, 则称子空间V_1与V_2正交, 记为$(V_1,V_2)=0$或$V_1\perp V_2$.

定义 3.2.8

设W和W'是欧式空间V(不一定是有限维)的两个子空间. 如果

$$V=W+W' \quad 且 \quad W\perp W',$$

则称W'为子空间W的正交补.

定义 3.2.9: 正交变换

设T是欧式空间V的一个线性变换, 如果T保持V中任何向量的长都不变, 亦即对V中任意的α都有

$$(T\alpha,T\alpha)=(\alpha,\alpha),$$

则称T是V的一个正交变换.

设T是欧式空间V的线性变换. 则T为正交变换的充要条件是, T保持向量的内积不变, 即对V中任意向量α, β都有

$$(T\alpha, T\beta) = (\alpha, \beta).$$

定义 3.2.10

设A是一个实n阶方阵. 如果$AA' = E$, 则称A为正交方阵. 正交方阵A中的行向量是欧式空间$\mathbb{R}^n$中的一个标准正交基.

定义 3.2.11

设T是n维欧式空间V的一个正交变换, 且在某标准正交基下的方阵为A. 若$|A| = 1$, 则称为旋转或第一类的; 若$|A| = -1$, 则称T为第二类的.

定义 3.2.12

设T是欧式空间V的一个线性变换, 如果对V中任意向量α, β都有

$$(T\alpha, \beta) = (\alpha, T\beta),$$

则称T是V的一个对称变换.

3.3 数学分析

定义 3.3.1: 数学分析习题集

- 分析引论
  1. 实数
  2. 数列理论
  3. 函数的概念
  4. 函数图像表示法
  5. 函数的极限
  6. 符号$\mathcal{O}$
  7. 函数连续性
  8. 反函数, 用参数形式表示的函数
  9. 函数的一致连续性
  10. 函数方程

定义 3.3.2: 分析引论

- 实数
  - 数学归纳法
  - 分割$\rightarrow$实数
  - 绝对值(模)$\rightarrow$三角不等式, 开区间, 半开区间, 闭区间
  - 上, 下确界的定义
  - 绝对误差, 相对误差$\rightarrow$精确数字
- 数列理论
  - 数列极限的概念

- 收敛, 发散, 无穷小量, 无穷极限
- 极限存在的判别法
  - 夹逼定理
  - 单调有界
  - Cauchy准则
- 数列极限的基本定理
  - 保序性
  - 唯一性
  - 保加, 减, 乘, 除(分母不为0)运算
  - Stolz公式
- 极限点, 上下极限, 运算和不等式
- 重要极限
  - $\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
  - $\gamma = \lim_{n\to\infty}(H_n - \log n)$

• 函数的概念
- (单值)函数的定义, 定义域(存在域), 值域
- 反函数
- (严格)单调
- 复合函数

• 函数的图像表示, 函数的零点

• 函数的极限
- 函数的有界性, 上确界, 下确界, 振幅
- 函数在某一点的极限, (与数列极限的关系)
- 重要极限
  - $\lim_{x\to 0}\frac{\sin x}{x} = 1$
  - $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \mathrm{e}$
- Cauchy准则(函数极限存在的充要条件)
- 单侧极限, 左右极限
- 无穷极限, $\lim_{x\to a} f(x) = \infty \iff \forall E > 0, \exists \delta = \delta(E) > 0, \ni \forall 0 < |x-a| < \delta(E)$, 均有$|f(x)| > E$.
- 子列极限, 下极限, 上极限

• 函数的连续性
- (点)连续
- 间断点
  - 第一类间断点
    - 可去间断点
    - 跳跃间断点
  - 第二类间断点(无穷型间断点)
- 左右连续
- (点)连续, 保加, 减, 乘, 除(分母不为0)
- 复合函数的连续
- 初等函数的连续
- 基本定理
  - 闭区间上连续函数有界

    * 闭区间上连续函数达到上下确界(Weierstrass定理)
    * 闭区间上连续函数定义在$(\alpha,\beta)\subset[a,b]$, f取到$f(\alpha)$, $f(\beta)$之间的所有值(Cauchy定理)
    * 闭区间上连续函数的零点定理

- 反函数
  - 反函数的存在性和连续性
  - 单值连续分支
  - 参数形式表示的函数的连续性
- 函数的一致连续性
  - 一致连续性的定义
  - Cantor定理

3.4 微分方程

定义 3.4.1: 标准形式下的边值问题

二阶线性微分方程: $y''+P(x)y'+Q(x)y=\phi(x)$, $P(x),Q(x),\phi(x)\in C[a,b]$在满足边界条件:

$$\begin{cases}\alpha_1y(a)+\beta_1y'(a)=\gamma_1, & \alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2\in\mathbb{R},\\ \alpha_2y(b)+\beta_2y'(b)=\gamma_2, & \alpha_1^2+\beta_1^2\neq0,\alpha_2^2+\beta_2^2\neq0.\end{cases}$$

的问题称为标准形式下的边值问题. 边值问题是齐次的, 若$\phi(x)\equiv0$, $\gamma_1=\gamma_2=0$. 否则称为非齐次的.

定义 3.4.2: 更一般的齐次边值问题

更一般的齐次边值问题是有如下形式的问题

$$\begin{cases}y''+P(x,\lambda)y'+Q(x,\lambda)y=0;\\ \alpha_1y(a)+\beta_1y'(a)=0;\\ \alpha_2y(b)+\beta_2y'(b)=0\end{cases}$$

3.5 泛函分析

定义 3.5.1: 紧算子

设 X 是 Banach 空间, 若线性算子 T 把每一有界集映成列紧集, 则称线性算子 T 为紧算子.

定义 3.5.2: Banach空间中的凸集

设 X 是 Banach 空间, 集合 $K\subset X$称为是凸的, 若 $(1-t)K+tK\subset K$, $(0\leq t\leq1)$.

定义 3.5.3: 拟半范数, 半范数

设$\mathbb{K}$是$\mathbb{R}$或$\mathbb{C}$, $\mathcal{X}$是域$\mathbb{K}$上的向量空间.

A. 映射$q:\mathcal{X}\to\mathbb{R}$称为拟半范数, 如果

  (i) $q(x+y)\leq q(x)+q(y)$, 对于任意$x,y\in\mathcal{X}$.
  (ii) $q(tx)=tq(x)$, 对任意的$x\in\mathcal{X}$和$t\in\mathbb{R}$, $t\geq0$.

B. 映射$q:\mathcal{X}\to\mathbb{R}$称为是半范数, 如果上面的两个条件中(ii)改为

(ii') $q(\lambda x) = |\lambda| q(x)$, 对任意$x \in \mathcal{X}$和$\lambda \in \mathbb{K}$.

注: 若$q : \mathcal{X} \to \mathbb{R}$是半范数, 则对于任意的$x \in \mathcal{X}$, $q(x \geq 0)$. (因$2q(x) = q(x) + q(-x) \geq q(0) = 0$).