2000年哈尔滨理工大学高等数学竞赛试题简答

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1. (5') 求
$$\lim_{t\to 0}\frac{1+t\mathrm{e}^{\frac{1}{t}}}{t\mathrm{e}^{\frac{1}{t}}-\frac{2}{\pi}\arctan\frac{1}{t}}$$
$t\to 0^+$, 令$\frac{1}{t}=x$. $t\to 0^-$易得. 两者均为1.

2. (6') 求
$$\lim_{x\to 0}\frac{\tan(\sin x)-\sin(\tan x)}{\tan x-\sin x}$$
由
$$\sin x = x-\frac{x^3}{6}+o(x^4),$$
$$\tan x = x+\frac{x^3}{3}+o(x^4).$$
分子$=o(x^3)$, 分母$=O(x^3)$, 故$=0$.

3. (6') 设$f\in C^2$, $f''(0)>0$, $f(0)=f'(0)=0$, t是$y=f(x)$上点$(x,f(x))$处的切线在x轴截距. 求$\lim\limits_{x\to 0}\frac{xf(t)}{tf(x)}$. 用拉格朗日余项的Taylor展开:
$$f(x)=\frac{f''(\eta)}{2!}(x-0)^2, (\eta\text{在}0\text{和}x\text{之间}),$$
$$f'(x)=\frac{f''(\zeta)}{1!}(x-0).$$
以及t满足$0=f'(x)(t-x)+f(x)$代入得$\frac{1}{2}$.

4. (7') 首先$(\mathrm{e}^x f')'>(\mathrm{e}^x f)'$两边$\int_0^x$, 后两边乘$\mathrm{e}^{-2x}$, e^{-2x}为积分因子, 得$(\mathrm{e}^{-x}f(x))'>(f'(0)-f(0))\mathrm{e}^{-2x}$, 两边$\int_0^x$得
$$\mathrm{e}^{-x}f(x)-f(0)>(f'(0)-f(0))\frac{1-\mathrm{e}^{-2x}}{2}>0\Rightarrow f(x)>\mathrm{e}^x.$$

5. (6') 考虑$\int gf'\mathrm{d}y\int\frac{f'}{g}\mathrm{d}y=-1\Rightarrow -f''=\frac{f'}{g}$. 所以$\ln f'=-\int\frac{1}{g}\mathrm{d}y$, $f'=C\mathrm{e}^{-\int\frac{1}{g}\mathrm{d}y}$, 所以$f=\int\mathrm{e}^{-\int\frac{1}{g}\mathrm{d}y}\mathrm{d}y$. 即$f(t)=\int\mathrm{e}^{-\int\frac{1}{1+t+t^2+t^3}\mathrm{d}t}\mathrm{d}t$.

6. (6') $F'(x)$ 是连续的, 首先 $x\neq 0$ 时,
$$F(x)=\frac{\int_0^{\tan x}f(tx^2)\mathrm{d}(tx^2)}{x^2}=\frac{\int_0^{x^2\tan x}f(y)\mathrm{d}y}{x^2}.$$
此时 $F(x)$ 按 $\frac{u}{v}$ 型求导, $F'(0)$ 按定义, $x\neq 0$ 时 $F'(x)$ 连续是显然的. 只需验证 $x=0$ 时 $\lim\limits_{x\to 0}F'(x)=F'(0)$.

7. (6')

$$\int_0^\pi x\ln\sin x\mathrm{d}x=\int_0^\pi(\pi-x)\ln\sin x\mathrm{d}x\Rightarrow\int_0^\pi x\ln\sin x\mathrm{d}x=\frac{\pi}{2}\int_0^\pi\ln\sin x\mathrm{d}x.$$

$$\begin{aligned}\int_0^\pi\ln\sin x\mathrm{d}x&=2\int_0^{\frac{\pi}{2}}\ln\sin x\mathrm{d}x=2\int_0^{\frac{\pi}{2}}\ln\cos x\mathrm{d}x\\&=\int_0^{\frac{\pi}{2}}(\ln\sin x+\ln\cos x)\mathrm{d}x=\int_0^{\frac{\pi}{2}}(\ln\sin 2x-\ln 2)\mathrm{d}x\\&=-\frac{\pi}{2}\ln 2+\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\sin 2x\mathrm{d}2x=-\frac{\pi}{2}\ln 2+\frac{1}{2}\int_0^{\pi}\ln\sin y\mathrm{d}y.\end{aligned}$$

$$\int_0^\pi\ln\sin y\mathrm{d}y=-\pi\ln 2\Rightarrow\int_0^\pi x\ln\sin x\mathrm{d}x=-\frac{\pi^2}{2}\ln 2$$

8. (6')

$$\int_0^1\mathrm{d}x\int_0^x\frac{f'(y)\mathrm{d}y}{\sqrt{(1-x)(x-y)}}=\int_0^1\mathrm{d}y\int_y^1\frac{f'(y)\mathrm{d}x}{\sqrt{(1-x)(x-y)}}$$

计算: $\int_y^1\frac{\mathrm{d}x}{\sqrt{(1-x)(x-y)}}=\pi$, 即得.

9. (7') (1). $S_1(0)=0$, $S_1(0)>0$, $S_2(1)>0$, $S_2(0)=0$. $S=S_1-S_2\searrow$, $S(0)>0$, $S(1)<0$. 所以$\exists|t\in[0,1]\ni S(t)=0$. $(S_1,S_2,S$连续$)$.

(2). $S_1(t)=\int_t^1(f(1)-f(x))\mathrm{d}x$, $S_2(t)=\int_0^t(f(x)-f(0))\mathrm{d}x$, 令$S=S_1+S_2$, 求$S'=0$得$t=f^{-1}\left(\frac{f(0)+f(1)}{2}\right)$时取得最小值.

10. (6') $\overrightarrow{l}=(1,1,-1)$, $\overrightarrow{n}=(1,-1,2)$, $\cos\theta=\frac{\overrightarrow{l}\cdot\overrightarrow{n}}{|\overrightarrow{l}|\cdot|\overrightarrow{n}|}=-\frac{\sqrt{2}}{3}$.

$\overrightarrow{h}=\overrightarrow{l}+|\overrightarrow{l}|\cdot|\cos\theta|\cdot\frac{\overrightarrow{n}}{|\overrightarrow{n}|}$. $\frac{\mathrm{d}f}{\mathrm{d}\overrightarrow{h}}=\overrightarrow{\nabla}f\cdot\frac{\overrightarrow{h}}{|\overrightarrow{h}|}=\pm\frac{1}{\sqrt{21}}(2\sqrt{2}\mathrm{d}x-2\mathrm{d}y+\frac{\sqrt{2}}{2}\mathrm{d}z)$.

11. (6') 利用恒等式$\sum\limits_{i=1}^{\infty}\frac{1}{x^2+i^2}=\frac{\pi\coth\pi x}{2x}-\frac{1}{2x^2}$, 所以

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n}{m^2+n^2}=\sum_{n=1}^{\infty}a_n\left(\frac{\pi\coth\pi n}{2n}-\frac{1}{2n^2}\right)$$

注: $\sum\limits_{n=1}^{\infty}\frac{a_n}{2n^2}<\sum\frac{1}{2n^2}=\frac{1}{2}\frac{\pi^2}{6}$.

从而$\sum\frac{a_n}{2n^2}$收敛, 又$\exists N$, 当$n>N$时$\coth\pi n<2$, 所以$\sum\limits_{N}\frac{\pi\coth\pi n}{2n}a_n<\sum\limits_{N}\frac{\pi}{n\sqrt{n}}$右边是收敛的.

12. (6') 主要利用Green公式

$$\int_{\widehat{ACB}}=\int_{\overline{AB}}+\int_{\widehat{ACBA}}\qquad l_{AB}:y=\frac{x}{\pi}+1.$$

$$\int_{\overline{AB}}=\int_\pi^{3\pi}\left[f\left(\frac{x}{\pi}+1\right)\cos x-(x+\pi)\right]\mathrm{d}x+\left[f'\left(\frac{x}{\pi}+1\right)\sin x-\pi\right]\frac{1}{\pi}\mathrm{d}x=-6\pi^2-2\pi.$$

$$\int_{\widehat{ACBA}=\partial D}=\iint_D\pi\mathrm{d}x\mathrm{d}y=\pi\Rightarrow\int_{\widehat{ACB}}=-6\pi^2-\pi.$$

13. (7') (1) $\mathrm{d}u = z = yf(xy)\mathrm{d}x + xg(xy)\mathrm{d}y$, 所以$\frac{\partial yf(xy)}{\partial y} = \frac{\partial xg(xy)}{\partial x}$. 令$F = f - g \Rightarrow F + xF' = 0 \Rightarrow F = \frac{C}{x}$, 即$f - g = \frac{C}{x}$, C为任意常数.

(2) 因为$\mathrm{d}u = yf(xy)\mathrm{d}x + xg(xy)\mathrm{d}y$, 即$\frac{\partial u}{\partial x} = yf(xy)$, $F'(x) = f(x)$. 所以$u = F(xy) + h(y)$, (此时h为y的任意函数, 下面确定h). 则$\frac{\partial u}{\partial y} = xf(xy) + h'(y) = xg(xy)$, 由$f - g = \frac{C}{x}$, 所以$x(f(xy) - g(xy)) + h'(y) = 0 = x\frac{C}{xy} + h'(y) \Rightarrow h'(y) = -\frac{C}{y}$. 所以$h(y) = -C\ln y + C_2$, 于是$u = F(xy) + C_1 \ln y + C_2$, ($C_1, C_2$为任意常数.)

14. (7') 令$u = x^2 + y^2 + z^2$. 由$\Delta w = 0 \Rightarrow 2u^2 f''(u) + 7uf'(u) + 3f(u) = 0$, 这是Euler方程, 所以$f(u) = C_1 u^{-1} + C_2 u^{-3}$由$f(1) = -1$, $f'(1) = \frac{3}{2} \Rightarrow f(u) = -\frac{3}{4u} - \frac{1}{4u^3}$, $f'(u) > 0$. 所以$\min\limits_{u\in[1,+\infty)} f(u) = f(1) = -1$.

15. (7') 此微分方程是可解的, 积分因为$\mu = \frac{\mathrm{e}^{\arctan t}}{\sqrt{1+t^2}}$, 所以$\sqrt{1+x^2}\mathrm{e}^{\arctan x} f(x) = \int_0^x \frac{\mathrm{e}^{\arctan t}}{\sqrt{1+t^2}}\mathrm{d}t$, 可证$f$单调(在$[0,1]$), $f'(0) = 1$, $f'(x)$在$[0,1]$上恒正但单调减小. $\sqrt{1+x^2}\mathrm{e}^{\arctan t} f(x) \geq \int_0^x \frac{\mathrm{e}^{\arctan t}}{1+t^2}\mathrm{d}t \Rightarrow f(x) \geq \frac{1}{\sqrt{1+x^2}} > \frac{1}{4}$的证明. 用中值定理:

$$\begin{aligned}
\sum_{n=1}^{m} g\left(\frac{1}{n}\right) &= \sum_{n=1}^{m-1} n \cdot \left(g\left(\frac{1}{n}\right) - g\left(\frac{1}{n+1}\right)\right) + m\left(\frac{1}{m}\right) \\
&= \sum_{n=1}^{m-1} \frac{1}{n+1} f(\xi_n) + \frac{g\left(\frac{1}{m}\right)}{\frac{1}{m}} (m \to 0\text{时}, \frac{g\left(\frac{1}{m}\right)}{\frac{1}{m}} \to f(0) = 0.) \\
&\geq \sum_{n=1}^{m-1} \frac{1}{n+1} \cdot \frac{1}{4n} + o\left(\frac{1}{m}\right) \\
&\left(f(\xi_n) \geq \frac{1}{\sqrt{1+\xi_n^2}} \geq \frac{1}{\sqrt{1+\frac{1}{n^2}}} \geq \frac{1}{4n} \Leftrightarrow 4n \geq \sqrt{1+\frac{1}{n^2}}\text{让} n \to \infty.\right)
\end{aligned}$$

< 1的证明, 只需证$f(\xi_n) \leq \frac{1}{n}$即可, 强化不等式证$f(\xi_n) \leq \xi_n$. 因为

$$\frac{f(\xi_n)}{\xi_n} = f'(\eta_n)(\text{中值定理}) = \frac{1 - (1+\eta_n)f(\eta_n)}{1+\eta_n^2} \leq 1.\text{即证.}$$

注: 这里只需要保证f是非负的结果, 其他结果是不必要的副产品.

16. (6')

$$\begin{aligned}
I &= \iint_{\Sigma} (2f + x)\mathrm{d}y\mathrm{d}z + f\mathrm{d}z\mathrm{d}x + (3f + z)\mathrm{d}x\mathrm{d}y \\
&= \iint_{\Sigma} (2f + x, f, 3f + z) \cdot \frac{1}{\sqrt{3}} \cdot (-1, -1, 1)\mathrm{d}S \\
&= \iint_{\Sigma} (x, 0, z)(-1, -1, 1)\mathrm{d}S_{xy} \\
&= \iint_{\Sigma_{xy}} (z - x)\mathrm{d}x\mathrm{d}y \\
&= \int_{-5}^{0} \mathrm{d}x \int_{-5-x}^{0} (y + 5)\mathrm{d}y \\
&= \frac{250}{3}.
\end{aligned}$$