if $|z_i| = 1$ prove $|z_1 + 1| + |z_2 + 1| + |z_1z_2 + 1| \geq 2$

[-1] [2] ali

[2016-10-19 06:08:18]

[inequality complex-numbers]

[https://math.stackexchange.com/questions/1975274/if-z-i-1-prove-z-11z-21z-1z-21-ge-2]

problem:if $z_1z_2z_3 = 1$ and $|z_1| = |z_2| = |z_3|$ prove that $|z_1 + 1| + |z_2 + 1| + |z_3 + 1| \geq 2$ I want a clean solution for this problem. I prove it is equivalent to this problem if $|z_1| = |z_2| = 1$ then $|z_1 + 1| + |z_2 + 1| + |z_1z_2 + 1| \geq 2$ but I cant prove it. I want a hint.

(4) I'm not sure your statement is correct, if I take $z_1 = z_2 = z_3 = 1$ I get $3 \leq 2$... - **Tsang**

sorry it is true now - **ali**

Notice $|z_1| = |z_2| = |z_3| = 1$, so we can let $z_1 = e^{i\theta}, z_2 = e^{i\phi}$, then $z_3 = e^{-i(\theta+\phi)}$. Meanwhile $|e^{i\theta} + 1| = 2 + 2cos(\theta)$ so your statement is equivalent to $cos(\theta) + cos(\phi) + cos(-\theta - \phi) \geq -4$. At this point I would use calculus to check this, but perhaps there is a neater way to verify this using inequalities - **Tsang**

[+5] [2016-10-19 08:21:12] user348749 [✔ACCEPTED]

$$\begin{aligned} |z_1 + 1| + |z_2 + 1| + |z_1z_2 + 1| &\geq |z_1 + 1| + |(z_2 + 1) - (z_1z_2 + 1)| \\ &\geq |z_1 + 1| + |z_2 - z_1z_2| \\ &= |z_1 + 1| + |1 - z_1| \\ &\geq |z_1 + 1 + 1 - z_1| \\ &= 2 \end{aligned}$$

We have used $|a \pm b| \leq |a| + |b|$ for any two complex numbers a, b.

1

[+1] [2016-10-19 07:59:11] Jacky Chong

Suppose you have $z_1, z_2, z_3 \in \mathbb{C}$ with the properties $z_1z_2z_3 = 1$ and $|z_1| = |z_2| = |z_3| = 1$, then we could rename them as following

$$z_1 = \zeta_1, \quad z_2 = \bar{\zeta}_2, \quad z_3 = \bar{\zeta}_1\zeta_2$$

which means the original inequality could be written as

$$|\zeta_1 + 1| + |\bar{\zeta}_2 + 1| + |\bar{\zeta}_1\zeta_2 + 1| \geq 2. \quad (*)$$

To prove $(*)$, observe we have

$$\begin{aligned} (|\zeta_1 + 1| + |\bar{\zeta}_2 + 1| + |\bar{\zeta}_1\zeta_2 + 1|)^2 &\geq |\zeta_1 + 1|^2 + |\bar{\zeta}_2 + 1|^2 + |\bar{\zeta}_1\zeta_2 + 1|^2 + 2|\zeta_1 + 1||\bar{\zeta}_2 + 1| \\ &= 6 + \zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2 + \bar{\zeta}_1\zeta_2 + \zeta_1\bar{\zeta}_2 + 2|\zeta_1 + 1||\bar{\zeta}_2 + 1| \\ &= 4 + (\zeta_1 + 1)(\bar{\zeta}_2 + 1) + (\bar{\zeta}_1 + 1)(\zeta_2 + 1) + 2|\zeta_1 + 1||\bar{\zeta}_2 + 1| \\ &= 4 + 2\,\mathrm{Re}[(\zeta_1 + 1)(\bar{\zeta}_2 + 1)] + 2|\zeta_1 + 1||\bar{\zeta}_2 + 1| \\ &\geq 4. \end{aligned}$$

Hence we have our desired inequality

$$|\zeta_1 + 1| + |\bar{\zeta}_2 + 1| + |\bar{\zeta}_1\zeta_2 + 1| \geq 2.$$

2
