Pi

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Most (and perhaps all) mathematicians involved in the teaching of calculus are asking themselves these days about the standard "**techniques of integration**" chapter: How much should be retained now that symbolic antidifferentiation is so readily available on home computers?

Herbert Wilf described a more general question in his "**The disk with the college education**" [this Monthly, 89 (1982) 4-8], and I would like to suggest a general point of view by giving a particular answer to the "**techniques of integration**" question.

The point of view should, it seems to me, be that *the home computer can free the student* from tedious drill and leave the teacher free to discuss the important conceptual aspects of integration. I take the consensus to be that change of variable, integration by parts, and the use of partial fractions must be taught: change of variable because of its theoretical importance and because it gives the student a chance to see how a hard problem becomes easy when looked at in the right way; integration by parts because of its constant use in both pure and applied mathematics as a theoretical tool; and partial fractions because one can show how only two transcendental functions (the logarithm and the arctangent) are necessary for integrating all rational functions.

Having agreed to this, what problems do we discuss to make our points? It seems to me they should be conceptual problems that the student can appreciate. I offer one example that I have found useful.

The students know how to compute the area, πr^2, and the perimeter, $2\pi r$, of a circle of radius r.

The good students may even have wondered why this same mysterious π appears in both formulae.

For those who have not, this question gets their attention.

Assuming the students know how to use the definite integral to find area and arclength, one has

$$A = 4\int_0^r \sqrt{r^2 - x^2}dx,\ \ P = 4\int_0^r \frac{r}{\sqrt{r^2 - x^2}}dx.$$

Now, the change of variable, $x = ru$, gives immediately that

$$A = \left\{4\int_0^1 \sqrt{1 - x^2}dx\right\} r^2$$

and

$$P = 2\left\{2\int_0^1 \frac{1}{\sqrt{1 - x^2}}dx\right\} r.$$

So, it is a question of relating

$$\int_0^1 \sqrt{1 - x^2}dx \text{ to } \int_0^1 \frac{1}{\sqrt{1 - x^2}}dx$$

But the identity $1 = (1 - x^2) + x^2$ and integration by parts yield

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} dx = 2 \int_0^1 \sqrt{1-x^2} dx$$

We've actually proved the existence of a number, π say, with $A = \pi r^2$ and $P = 2\pi r$;
we can calculate its approximate value by numerical integration;
and we've done so using change of variable and integration by parts.
Who could ask for anything more?

References

[1] E.F. Assmus, Jr. Pi, American Mathematical Monthly, vol. 092, 213-214.