$$\int E^{a x} \operatorname{Cos}[b x] \, dx = \frac{e^{a x} (a \operatorname{Cos}[b x] + b \operatorname{Sin}[b x])}{a^2 + b^2}$$

$$\prod_{i=1}^{\infty} \left(1 + \frac{1}{i^3}\right) = \frac{\operatorname{Cosh}\left[\frac{\sqrt{3}\,\pi}{2}\right]}{\pi}$$

$$\operatorname{Limit}\left[\frac{E}{2} x + x^2 \left(\left(1 + \frac{1}{x}\right)^x - E\right), x \to +\infty\right] = \frac{11\, e}{24}$$

$$\int_1^{\sqrt{3}} \left(x^{2 x^2+1} + \operatorname{Log}\left[x^{2 x^{2 x^2+1}}\right]\right) dx = 13$$

$$\int \left(x^{2 x^2+1} + \operatorname{Log}\left[x^{2 x^{2 x^2+1}}\right]\right) dx \;//\; \operatorname{Expand} = -2 x^{2 x^2+2} \log(x) + \frac{x^{2 x^2}}{2} + x \log\left(x^{2 x^{2 x^2+1}}\right)$$

N[π, 120] =
3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628\
03482534211706798214808651328230664709384460955094`120.

$$\int \left(x^{2 x^2+1} + \ln\left(x^{2 x^{2 x^2+1}}\right)\right) dx$$

$$\overset{y=2 x^2+1}{=} \int (x^y + 2 x^y \ln(x)) \, dx$$

$$\left(\text{注意：} \frac{d x^{y-1}}{dx} = x^{y-1} \left(y' \ln(x) + \frac{y-1}{x}\right) = x^{y-1} (4 x \ln(x) + 2 x) = 4 x^y \ln(x) + 2 x^y\right)$$

$$= \int \frac{1}{2} d\left(x^{y-1}\right) = \frac{1}{2} x^{2 x^2}$$

$$\int_1^{\sqrt{3}} \left(x^{2 x^2+1} + \operatorname{Log}\left[x^{2 x^{2 x^2+1}}\right]\right) dx = \frac{1}{2} \left(\sqrt{3}\right)^{2 \left(\sqrt{3}\right)^2} - \frac{1}{2} = 13$$

美国月刊 / 外森比克

已知：$a + b + c = 1$，求 / ：$27 (a - b c) (b - c a) (c - a b) \le 8 a b c$

/明：取$a = \frac{1}{4} y^2 z^2,\ b = \frac{1}{4} z^2 x^2,\ c = \frac{1}{4} x^2 y^2$

$$\left(1 - \frac{bc}{a}\right) \left(1 - \frac{ca}{b}\right) \left(1 - \frac{ab}{c}\right) = \left(1 - \frac{x^4}{4}\right) \left(1 - \frac{y^4}{4}\right) \left(1 - \frac{z^4}{4}\right)$$

$$\le \left(\frac{3 - \frac{x^4+y^4+z^4}{4}}{3}\right)^3 \le \left(\frac{3 - \frac{x^2 y^2+y^2 z^2+z^2 x^2}{4}}{3}\right)^3 = \left(\frac{2}{3}\right)^3$$

$$\frac{\sin(x)}{x} = \sum_{i=0}^{+\infty} \frac{\left(-x^2\right)^i}{(2 i + 1)!} = \prod_{i=1}^{+\infty} \left(1 - \frac{x^2}{i^2 \pi^2}\right)$$

$\sum_{i=1}^{+\infty}\frac{1}{i^4}$ 的求解：

$$\frac{\mathrm{Sin}[x]}{x}=1-\frac{1}{3!}x^2+\frac{1}{5!}x^4-\frac{1}{7!}x^6+\ldots$$

$$\frac{\mathrm{Sin}[I\,x]}{I\,x}=1+\frac{1}{3!}x^2+\frac{1}{5!}x^4+\frac{1}{7!}x^6+\ldots$$

$$\left(1-\frac{1}{3!}x^2+\frac{1}{5!}x^4-\frac{1}{7!}x^6\right)\left(1+\frac{1}{3!}x^2+\frac{1}{5!}x^4+\frac{1}{7!}x^6\right)\ //\ \mathrm{Expand}=-\frac{x^{12}}{25\,401\,600}+\frac{x^8}{302\,400}-\frac{x^4}{90}+1$$

$$\sum_{i=1}^{+\infty}\frac{1}{i^4}=\frac{\pi^4}{90}$$

$$\frac{\mathrm{Sin}[\pi x]}{\pi x}=\prod_{i=1}^{+\infty}\left(1-\frac{x^2}{i^2}\right)=1-\frac{\pi^2x^2}{6}+\frac{\pi^4x^4}{120}-\frac{\pi^6x^6}{5040}+\frac{\pi^8x^8}{362\,880}-\frac{\pi^{10}x^{10}}{39\,916\,800}+O\left(x^{11}\right)$$

根据 ˛⁄ · 定理：$\sum_{i=1}^{+\infty}\frac{-1}{i^2}=-\frac{\pi^2}{6}$

$$\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\ldots\left(1-\frac{x^2}{n^2}\right)=$$

$$\frac{(-1)^n x^{2n}}{(n!)^2}+(-1)^{n-1}\frac{x^{2(n-1)}}{(n!)^2\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\right)}+\ldots-(n!)^2\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)x^2+1$$

$$=\ a_{2n}x^{2n}+a_{2(n-1)}x^{2(n-1)}+\ldots+a_2x^2+a_0$$

$$\frac{a_2}{a_0}=-\left(x_1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)$$

„ 于 $\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\mathrm{Sin}[x]}}\,dx$

$$=\ -2\,F\left(\frac{1}{2}\left(\frac{\pi}{2}-x\right)\,\middle|\,2\right)=\frac{1}{2}\,B\left(\frac{1}{4},\frac{1}{2}\right)=\sqrt{2}\ \mathrm{EllipticK}\left[\frac{1}{2}\right]$$

求 $\int_{x_0}^{x_1}\sqrt{y[x]\,(1+y'[x])}\,dx$ 的泛函…« 曲ˇ

求非ˇ 性微分方程 $2\,y\,y''=1+(y')^2$

$y=f(x)$ 的曲率 ∶$K=\frac{y''}{\sqrt{\left(1+(y')^2\right)^3}}=\frac{1}{2\,y\sqrt{1+(y')^2}}$

$$\frac{K'}{K}=-\frac{3\,y'(x)}{2\,y(x)}\Leftrightarrow\frac{dK}{dy}=-\frac{3\,K}{2\,y}\Leftrightarrow K=C_1\,y^{\frac{-3}{2}}=\frac{1}{2\,y\sqrt{1+(y')^2}}\Leftrightarrow C\left(1+(y')^2\right)=y$$

$$y'=p\Longrightarrow y=C\left(1+p^2\right);\quad\frac{dx}{dp}=\frac{dx}{dy}\frac{dy}{dp}=\frac{1}{p}\,2\,C\,p=2\,C\Longrightarrow x=2\,C\,p+C_1\Longrightarrow p=\frac{x-C_1}{2\,C}$$

$$y=C\left(1+\left(\frac{x-C_1}{2\,C}\right)^2\right)=C+\frac{(x-C_1)^2}{4\,C^2}$$

$$\frac{\sqrt[3]{\frac{x}{1+y}}+\sqrt[3]{\frac{y}{1+x}}}{2}\le\sqrt[3]{\frac{\frac{x}{1+y}+\frac{y}{1+x}}{2}}=\sqrt[3]{\frac{28}{x\,y+7}-1}$$

$$\mathrm{Limit}\left[\frac{x-\mathrm{Cos}[x]\,x}{2\,x^2\,\mathrm{Sin}[x]\,\mathrm{Cos}[x]},\ x\to 0\right]=\frac{1}{4}$$

$$\text{Limit}\left[\frac{D[x-\text{Cos}[x]\,\text{Sin}[x],\{x,2\}]}{D\left[2x^2\,\text{Sin}[x]\,\text{Cos}[x],\{x,2\}\right]},x\to 0\right]=\frac{1}{3}$$

$$\frac{D[x,\{x,1\}]}{D[x\,\text{Cos}[2x]+2\,\text{Sin}[x],\{x,1\}]}=\frac{1}{-2x\sin(2x)+2\cos(x)+\cos(2x)}$$

$$\text{Limit}\left[\frac{x-\text{Cos}[x]\,x}{x-\text{Cos}[x]\,\text{Sin}[x]},x\to 0\right]=\frac{3}{4}$$

$$F[s]=\int_0^{+\infty}E^{-s\,t}\,\text{Cos}[a\,t]\,dt=\frac{s}{a^2+s^2}$$

$$\sum_{a=1}^{+\infty}F[1]=\sum_{i=0}^{+\infty}\frac{1}{i^2+1}=\sum_{a=1}^{+\infty}\int_0^{+\infty}E^{-t}\,\text{Cos}[a\,t]\,dt$$

$$=\int_0^{+\infty}E^{-t}\sum_{a=1}^{+\infty}\text{Cos}[a\,t]\,dt$$

$x\,y\,z=1\,\&\&\,x,y,z>0$

$$\Longrightarrow\frac{x^3}{(1+y)\,(1+z)}+\frac{y^3}{(1+x)\,(1+z)}+\frac{z^3}{(1+y)\,(1+x)}\geq\frac{3}{4}$$

$$\text{Proof 1}:\sum\left(\frac{1+y}{8}+\frac{1+z}{8}+\frac{x}{4}+\frac{x^3}{(1+y)\,(1+z)}\right)\geq\sum\left(4\sqrt[4]{\frac{1+y}{8}\,\frac{1+z}{8}\,\frac{x}{4}\,\frac{x^3}{(1+y)\,(1+z)}}\right)=$$

$$\sum\left(4\sqrt[4]{\frac{x^4}{8\times 8\times 4}}\right)=\sum x$$

$$\sum\frac{x^3}{(1+y)\,(1+z)}\geq\frac{1}{2}\sum x-\frac{3}{4}\geq\frac{3}{2}\sqrt[3]{x\,y\,z}-\frac{3}{4}=\frac{3}{4}$$

Proof 2 : „„造f (t) = t^4 + t^3 - 1 / 4 (t + 1) ^3

$x\geq 1\geq z>0$

$(x-1)\,g\,(x)\geq(x-1)\,g\,(y)$

$(y-1)\,g\,(y)\geq(y-1)\,g\,(y)$

$(z-1)\,g\,(z)\geq(z-1)\,g\,(y)$

三式相加

$4\,f\,(x)+4\,f\,(y)+4\,f\,(z)\geq[(x-1)+(y-1)+(z-1)]\,g\,(y)\geq 0$

所以$f\,(x)+f\,(y)+f\,(z)\geq 0$

x^3 (1 + x) + y^3 (1 + y) + z^3 (1 + z) ≥ 1 / 4[(x + 1) ^3 + (y + 1) ^3 + (z + 1) ^3]

除一下得 ⁄

根据Laplace- »»$L\left[E^{at}\right]=\frac{1}{s-a}\,(s>a)$，有

$$\int_0^{+\infty}\frac{E^{n\,t}-1}{E^{(n+1)\,t}-E^{n\,t}}\,dt=\text{Sum}\left[\frac{1}{i},\{i,1,n\}\right]//N$$

$$\int_0^{+\infty}\frac{E^{n\,t}-1}{E^{(n+1)\,t}-E^{n\,t}}\,dt=\text{If}\left[\text{Re}\,(n)>0,H_n,\text{Integrate}\left[\frac{e^{n\,t}}{e^{(n+1)\,t}-e^{n\,t}}-\frac{1}{e^{(n+1)\,t}-e^{n\,t}},\right.\right.$$
$$\left.\left.\{t,0,\infty\},\text{Assumptions}\to\text{Re}\,(n)\leq 0\right]\right]\qquad(*\text{HarmonicNumber}*)$$

$v+u+w=2$, 求$v^\wedge 2\, u^\wedge 2+w^\wedge 2\, u^\wedge 2+w^\wedge 2\, v^\wedge 2$的取 范¸§

$(u v+v w+w u)^2-4 u v w=v^\wedge 2\, u^\wedge 2+w^\wedge 2\, u^\wedge 2+w^\wedge 2\, v^\wedge 2=t$

$z^3-2 z^2+a z-b==0$

$u v w=b=$

$u=a+b-c;$

$v=a-b+c;$

$w=-a+b+c;$

$u^2 v^2+v^2 w^2+w^2 u^2+4\left(a^3+b^3+c^3\right)(a+b+c)-4 a b c\,(a+b+c)$ // Expand =

$7 a^4+2 a^2 b^2+2 a^2 c^2+7 b^4+2 b^2 c^2+7 c^4$

公式：$\sum_{i=1}^{+\infty} \frac{x^2}{x^2+i^2}=\frac{\pi x \coth \pi x}{2}-\frac{1}{2} \quad (x \in R)$

Proof：考´˙¶数 $\frac{d}{dx}\left[\ln\left(\frac{\operatorname{sh} \pi x}{\pi x}\right)\right]=\frac{\pi x}{\operatorname{sh}(\pi x)} \frac{\pi \operatorname{ch}(\pi x)\, \pi x-\operatorname{sh}(\pi x)\, \pi}{\pi^2 x^2}$

$=\frac{\pi \operatorname{ch}(\pi x)\, x-\operatorname{sh}(\pi x)}{x \operatorname{sh}(\pi x)}$

` 一方面：$\frac{\operatorname{sh}(\pi x)}{\pi x}=\prod_{i=1}^{+\infty}\left(1+\frac{x^2}{i^2}\right)$

$\Longrightarrow \ln\left(\frac{\operatorname{sh} \pi x}{\pi x}\right)=\ln\left(\prod_{i=1}^{+\infty}\left(1+\frac{x^2}{i^2}\right)\right)=\sum_{i=1}^{+\infty} \ln\left(1+\frac{x^2}{i^2}\right)$

$\Longrightarrow \frac{d}{dx}\left[\ln\left(\frac{\operatorname{sh} \pi x}{\pi x}\right)\right]=\frac{d}{dx}\left[\sum_{i=1}^{+\infty} \ln\left(1+\frac{x^2}{i^2}\right)\right]=\sum_{i=1}^{+\infty} \frac{d}{dx} \ln\left(1+\frac{x^2}{i^2}\right)$

$=\sum_{i=1}^{+\infty} \frac{\frac{2x}{i^2}}{1+\frac{x^2}{i^2}}=\sum_{i=1}^{+\infty} \frac{2x}{i^2+x^2}$

于是 $\sum_{i=1}^{+\infty} \frac{2x}{i^2+x^2}=\frac{d}{dx}\left[\ln\left(\frac{\operatorname{sh} \pi x}{\pi x}\right)\right]=\frac{\pi \operatorname{ch}(\pi x)\, x-\operatorname{sh}(\pi x)}{x \operatorname{sh}(\pi x)}=\pi \coth(\pi x)-\frac{1}{x} \quad (x \neq 0)$

整理化… 得：$\sum_{i=1}^{+\infty} \frac{x^2}{i^2+x^2}=\frac{\pi x \coth(\pi x)}{2}-\frac{1}{2} \quad (x \in R)$

($\because x=0$ ˚-左- ¸零，右- 的…«限¸˝零，广 意 下"ˇ¸在R上是相等的)

由上式我ˆ˙可以得到：

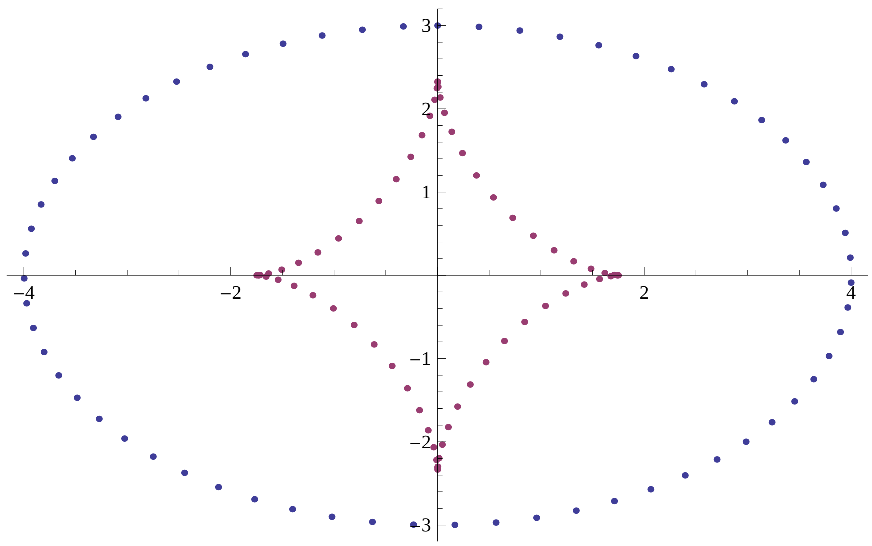
$$\frac{1}{1^2+1}+\frac{1}{2^2+1}+\frac{1}{3^2+1}+\ldots+\frac{1}{n^2+1}+\ldots=\frac{\pi \coth(\pi)}{2}-\frac{1}{2}=\frac{\pi\left(e^\pi+e^{-\pi}\right)}{2\left(e^\pi-e^{-\pi}\right)}-\frac{1}{2}$$

$$\frac{1}{1^2+2}+\frac{1}{2^2+2}+\frac{1}{3^2+2}+\ldots+\frac{1}{n^2+2}+\ldots=\frac{\sqrt{2}\, \pi \coth\left(\sqrt{2}\, \pi\right)}{4}-\frac{1}{4}$$

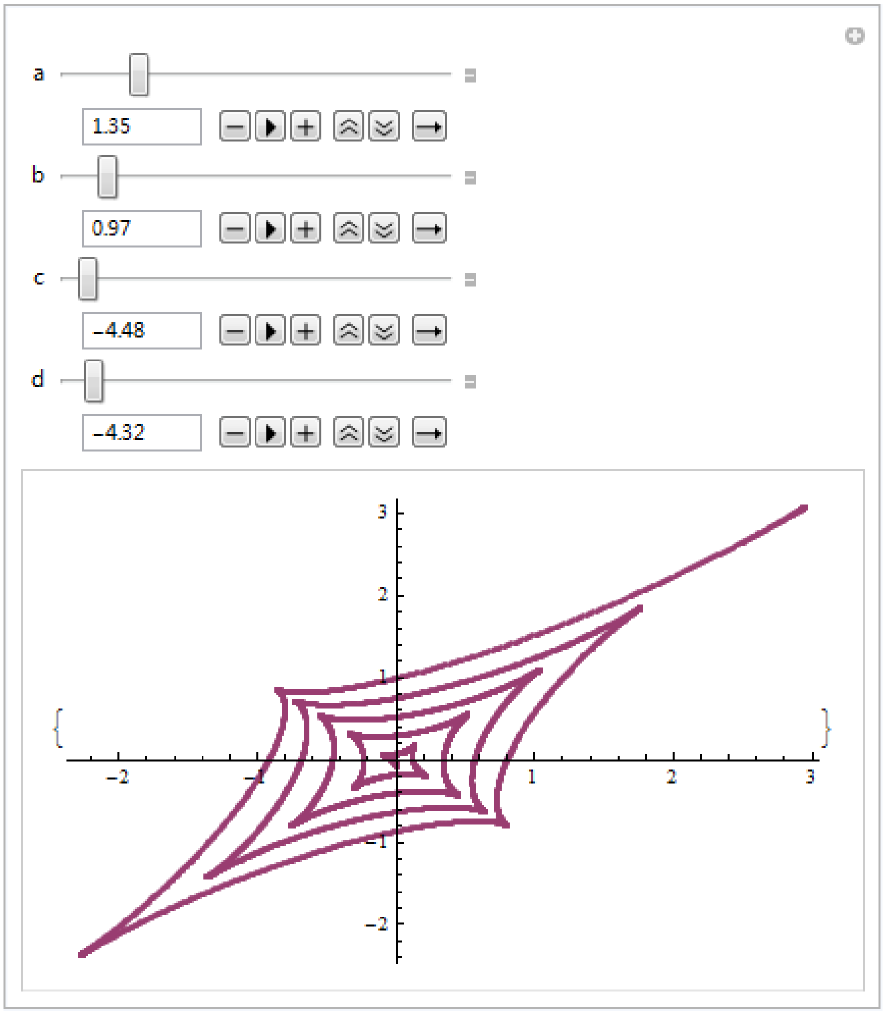
$$\frac{1}{1^2+3}+\frac{1}{2^2+3}+\frac{1}{3^2+3}+\ldots+\frac{1}{n^2+3}+\ldots=\frac{\sqrt{3}\, \pi \coth\left(\sqrt{3}\, \pi\right)}{6}-\frac{1}{6}$$

... ...

```
r = 4 Sin[t];
f = 3 Cos[t];
data = Table[Table[n, {t, 0, 2 Pi, 0.1}],
    {n, {{x, f}, {r - (D[f, t] (D[r, t]^2 + D[f, t]^2))/(D[r, t] D[f, {t, 2}] - D[f, t] D[r, {t, 2}]),
       f + (D[r, t] (D[r, t]^2 + D[f, t]^2))/(D[r, t] D[f, {t, 2}] - D[f, t] D[r, {t, 2}])}}}];
ListPlot[data]
```



```
Manipulate[{r = Sin[c t];
  f = Cos[d t];
  data = Table[Table[n, {t, 0, 2 Pi, 0.001}],
    {n, {{x, f}, {r - (D[f, t] (D[r, t]^2 + D[f, t]^2))/(D[r, t] D[f, {t, 2}] - D[f, t] D[r, {t, 2}]),
        f + (D[r, t] (D[r, t]^2 + D[f, t]^2))/(D[r, t] D[f, {t, 2}] - D[f, t] D[r, {t, 2}])}}}];
  ListPlot[data]}, {a, 0.5, 5}, {b, 0.5, 5}, {c, -5, 5}, {d, -5, 5}]
```



$$\sum_{i=0}^{n} (-1)^{i} \left(\frac{n!}{i!\,(n-i)!}\right)^{2} = \frac{\sqrt{\pi}\,2^{n}}{\Gamma\left(\frac{1-n}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}$$

$$\prod_{i=1}^{n} \left(1+\frac{1}{i^{3}}\right) \text{ // Factor } = \frac{(n+1)\cosh\left(\frac{\sqrt{3}\,\pi}{2}\right)\Gamma\left(n-\frac{i\sqrt{3}}{2}+\frac{1}{2}\right)\Gamma\left(n+\frac{i\sqrt{3}}{2}+\frac{1}{2}\right)}{\pi\,\Gamma(n+1)^{2}}$$

¶ 于 $\sum_{i=2}^{+\infty} \frac{1}{n\,\mathrm{Ln}(n)}$ 考´˙ $\int_{2}^{+\infty} \frac{1}{x\,\mathrm{Ln}(x)}\,dx = \int_{2}^{+\infty} \frac{1}{\mathrm{Ln}(x)}\,d\,\mathrm{Ln}(x) = \mathrm{Ln}(\mathrm{Ln}(x))\,\Big|_{2}^{+\infty} \to +\infty$

故…¶敛 $\sum_{i=2}^{+\infty} \frac{1}{n\,\mathrm{Ln}(n)}$ •散

$$\sum_{i=2}^{+\infty} (\mathrm{Ln}\,(i))^{\mathrm{Ln}\,(i)} \leq \sum_{i=2}^{+\infty} \left(e^{\mathrm{Ln}[\mathrm{Ln}[i]]}\right)^{\mathrm{Ln}\,(i)} = \sum_{i=2}^{+\infty} e^{\mathrm{Ln}[\mathrm{Ln}[i]]\,\mathrm{Ln}[i]} \leq \sum_{i=2}^{+\infty} e^{-2\,\mathrm{Ln}[i]} = \sum_{i=2}^{+\infty} i^{-2}$$

…¶数右端收`†故 $\sum_{i=2}^{+\infty} (\mathrm{Ln}\,(i))^{\mathrm{Ln}\,(i)}$ 收`†

$$\lim_{x\to 0} \frac{x^2 \sin\left(x^{-4}\right)}{1-e^x} = \lim_{x\to 0}\left\{\frac{x^2 \sin\left(x^{-4}\right)}{x} \times \frac{x}{1-e^x}\right\} = \lim_{x\to 0} \frac{x^2 \sin\left(x^{-4}\right)}{x} = \lim_{x\to 0} x\sin\left(x^{-4}\right) = 0$$

即尽管´ 必塔法 第一†‰ ：$\lim\limits_{x\to 0} \dfrac{x^2 \sin\left(x^{-4}\right)}{1-e^x} = \lim\limits_{x\to 0} \dfrac{2\,x \sin\left(\frac{1}{x^4}\right) - \frac{4\cos\left(\frac{1}{x^4}\right)}{x^3}}{-e^x}$ ‥致右端…«限不存在，

但不意味着原…«限不存在， 是´ 必塔法 需要´œ足的` 一个条件

$$A^{-1} = \frac{A^*}{|A|} \Longrightarrow A^* = |A| \times A^{-1}$$

于是 $\left(A^{-1}\right)^* = \left|A^{-1}\right| \times \left(A^{-1}\right)^{-1} = \dfrac{A}{|A|}$

$$(A^*)^{-1} = \left(|A| \times A^{-1}\right)^{-1} = |A|^{-1} \times A = \frac{A}{|A|}$$

由此 $\left(A^{-1}\right)^* = (A^*)^{-1}$

$$B \times B^T = E = B \times B^{-1}$$

$$|B - \lambda E| = 0 = \left|B^T - \lambda E\right|$$

于是 ：$|B - \lambda E| \times \left|B^T - \lambda E\right| = 0 = \left|E - \lambda^2 E - \lambda\left(B + B^T\right)\right|$

$$\iint\limits_{\substack{y=\sin(x)\ x\in(0,\pi)\\ y=0}} \sin(y)\, E^x\, dx\, dy + \int_0^{\pi} E^x\, dx =$$

$$\int_0^{\pi}\int_0^{\sin(x)} y E^x\, dy\, dx + E^{\pi} - 1 =$$

$$E^{\pi} - 1 - \int_0^{\pi} E^x \frac{1}{2} (\sin(x))^2\, dx =$$

$$E^{\pi} - 1 - \frac{1}{5}\left(-1 + e^{\pi}\right) = \frac{4}{5}\left(-1 + e^{\pi}\right)$$

$$-\frac{1}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}}+\frac{1}{\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}}$$

$$=\frac{\frac{1}{x_1\,(x_1+1)}+\frac{1}{x_2\,(x_2+1)}+\frac{1}{x_3\,(x_3+1)}}{\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}\right)}$$

$$=\frac{\frac{1}{x_1\,(x_1+1)}+\frac{1}{x_2\,(x_2+1)}+\frac{1}{x_3\,(x_3+1)}}{\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}\right)}$$

$\geq$

$$\frac{1}{x_1\,(x_1+1)}+\frac{1}{x_2\,(x_2+1)}+\frac{1}{x_3\,(x_3+1)}\geq\frac{1}{x_1\,(x_1+1)}+\frac{1}{x_2\,(x_2+1)}+\frac{1}{x_3\,(x_3+1)}$$

$$\frac{1}{x_1\,(x_1+1)}+\frac{1}{x_2\,(x_2+1)}+\frac{1}{x_3\,(x_3+1)}\geq\frac{1}{x_1\,(x_2+1)}+\frac{1}{x_2\,(x_3+1)}+\frac{1}{x_3\,(x_1+1)}$$

$$\frac{1}{x_1\,(x_1+1)}+\frac{1}{x_2\,(x_2+1)}+\frac{1}{x_3\,(x_3+1)}\geq\frac{1}{x_1\,(x_3+1)}+\frac{1}{x_2\,(x_1+1)}+\frac{1}{x_3\,(x_2+1)}$$

$$3\left(\frac{1}{x_1\,(x_1+1)}+\frac{1}{x_2\,(x_2+1)}+\frac{1}{x_3\,(x_3+1)}\right)\geq\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}\right)$$

$$\frac{d^n}{dx^n}\left[e^{a\,t}\,\mathrm{Sin}[b\,t]\right]=\left[a^2+b^2\right]^{\frac{n}{2}}e^{a\,t}\,\mathrm{Sin}\left[b\,t+n\,\mathrm{ArcTan}\left[\frac{b}{a}\right]\right]$$

$$\prod_{\text{p是素数}}^{+\infty}\frac{1}{1-\frac{1}{p^s}}=\sum_{n=1}^{+\infty}\frac{1}{n^s}$$

$$\sum_{i=1}^{+\infty}\frac{1}{i^8}=\frac{\pi^8}{9450}\qquad\sum_{i=1}^{+\infty}\frac{1}{(2\,i-1)^2}=\frac{\pi^2}{8}$$

$$a_1 a_2 a_3 \ldots a_n = 1 \Rightarrow \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} + \frac{4n}{n + a_1 + a_2 + \ldots + a_n} \geq n + 2$$

$$n = 1\ \text{°–:}\ \frac{1}{a_1} + \frac{4}{1 + a_1} = 1 + 2 = 3 \geq 1 + 2$$

$$\text{Łn°–成立,}\ a_1 a_2 \ldots a_n = 1 \Rightarrow \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} + \frac{4n}{n + a_1 + a_2 + \ldots + a_n} \geq n + 2$$

$$n + 1\ \text{°¬,}\ a_1 a_2 \ldots \left(a_n a_{n+1}\right) = 1 \Rightarrow \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} + \frac{1}{a_{n+1}} + \frac{4(n+1)}{n + 1 + a_1 + a_2 + \ldots + a_n + a_{n+1}} \geq$$

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n a_{n+1}} + \frac{4n}{n + a_1 + a_2 + \ldots + a_n a_{n+1}} -$$

$$\left(\frac{1}{a_n a_{n+1}} + \frac{4n}{n + a_1 + a_2 + \ldots + a_n a_{n+1}}\right) + \frac{1}{a_n} + \frac{1}{a_{n+1}} + \frac{4(n+1)}{n + 1 + a_1 + a_2 + \ldots + a_n + a_{n+1}} \geq$$

$$n + 2 - \left(\frac{1}{a_n a_{n+1}} + \frac{4n}{n + a_1 + a_2 + \ldots + a_n a_{n+1}}\right) + \frac{1}{a_n} + \frac{1}{a_{n+1}} + \frac{4(n+1)}{n + 1 + a_1 + a_2 + \ldots + a_n + a_{n+1}} =$$

$$n + 2 + \frac{a_n + a_{n+1} - 1}{a_n a_{n+1}} + \frac{4(n+1)\left(n + a_1 + \ldots + a_n a_{n+1}\right) - 4n\left(n + 1 + a_1 + .. + a_n + a_{n+1}\right)}{\left(n + a_1 + a_2 + \ldots + a_n a_{n+1}\right)\left(n + 1 + a_1 + a_2 + \ldots + a_n + a_{n+1}\right)} =$$

$$n + 3 - \frac{(a_n - 1)\left(a_{n+1} - 1\right)}{a_n a_{n+1}} + \frac{\left(a_1 + \ldots + a_{n-1} + a_n + a_{n+1}\right) + 4(n+1)\left(a_n a_{n+1} - a_n - a_{n+1}\right)}{\left(n + a_1 + a_2 + \ldots + a_n a_{n+1}\right)\left(n + 1 + a_1 + a_2 + \ldots + a_n + a_{n+1}\right)} \geq$$

$$n + 3 - \frac{(a_n - 1)\left(a_{n+1} - 1\right)}{a_n a_{n+1}} + \frac{(n+1) + 4(n+1)\left(a_n a_{n+1} - a_n - a_{n+1}\right)}{\left(n + a_1 + a_2 + \ldots + a_n a_{n+1}\right)\left(n + 1 + a_1 + a_2 + \ldots + a_n + a_{n+1}\right)}$$

$$\frac{(n+1)(2a_n - 1)\left(2a_{n+1} - 1\right)}{\left(n + a_1 + a_2 + \ldots + a_n a_{n+1}\right)\left(n + 1 + a_1 + a_2 + \ldots + a_n + a_{n+1}\right)} \geq \frac{(a_n - 1)\left(a_{n+1} - 1\right)}{a_n a_{n+1}}$$

$$\int_0^{+\infty}\int_0^1 \frac{x}{\left(x^2+1\right)\left(x^2 z^2+1\right)}\,dz\,dx = \int_0^{+\infty}\left[\frac{\tan^{-1} x z}{x^2+1}\right]\Big|_0^1\,dx = \int_0^{+\infty}\frac{\tan^{-1} x}{x^2+1}\,dx = \frac{\pi^2}{8}$$

` 一方面 ：

$$\int_0^{+\infty}\int_0^1 \frac{x}{\left(x^2+1\right)\left(x^2 z^2+1\right)}\,dz\,dx = \int_0^1\int_0^{+\infty}\frac{x}{\left(x^2+1\right)\left(x^2 z^2+1\right)}\,dx\,dz =$$

$$\int_0^1\int_0^{+\infty}\frac{1}{2\left(z^2-1\right)}\left[\frac{2xz^2}{x^2z^2+1} - \frac{2x}{x^2+1}\right]dx\,dz = \int_0^1 \frac{1}{2\left(z^2-1\right)}\left[\ln\left(\frac{x^2z^2+1}{x^2+1}\right)\right]\Big|_0^{+\infty}\,dz =$$

$$\int_0^1 \frac{\ln\left(z^2\right)}{2\left(z^2-1\right)}\,dz = \int_0^1 \frac{\ln(z)}{z^2-1}\,dz \underset{dv=\frac{dz}{z^2-1}}{\overset{u=\ln(z)}{=}} \text{(分部» 分)}\left[-\ln(z)\tanh^{-1}(z)\right]\Big|_0^1 - \int_0^1 \frac{-\tanh^{-1}(z)}{z}\,dz =$$

$$\int_0^1 \frac{\tanh^{-1}(z)}{z}\,dz \underset{\text{…¶数展¿"}}{\overset{\text{Maclaurin}}{=}} \sum_{n=0}^{+\infty}\int_0^1 \frac{z^{2n}}{2n+1}\,dz = \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2}$$

$$\int_0^1 \frac{\text{Log}[1-x]}{x}\, dx = -\frac{\pi^2}{6}$$

$$\text{Series}\left[\frac{\text{Log}[1-x]}{x}, \{x, 0, 20\}\right] = -1 - \frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4} - \frac{x^4}{5} - \frac{x^5}{6} - \frac{x^6}{7} - \frac{x^7}{8} - \frac{x^8}{9} - \frac{x^9}{10} - \frac{x^{10}}{11} - \frac{x^{11}}{12} - \frac{x^{12}}{13} - \frac{x^{13}}{14} - \frac{x^{14}}{15} - \frac{x^{15}}{16} - \frac{x^{16}}{17} - \frac{x^{17}}{18} - \frac{x^{18}}{19} - \frac{x^{19}}{20} - \frac{x^{20}}{21} + O\left(x^{21}\right)$$

$$\int_0^1 \frac{\text{Log}[1-x]}{x}\, dx = \lim_{a\to 1}\lim_{b\to 0}\int_b^a \frac{\text{Log}[1-x]}{x}\, dx$$

$$= \lim_{a\to 1}\lim_{b\to 0}\int_b^a \left[-1 - \frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4} - \frac{x^4}{5} - \ldots\right] dx$$

$$= \lim_{a\to 1}\lim_{b\to 0}\left[\frac{-x}{1} - \frac{x^2}{2^2} - \frac{x^3}{3^2} - \frac{x^4}{4^2} - \frac{x^5}{5^2} - \ldots\right]\Big|_b^a$$

$$\text{Limit}\left[\left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}, x \to 0\right] = \frac{1}{\sqrt{e}}$$

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\phi}\; d\phi$$

$$F(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} \quad (0 < k < 1)$$

$$E'(k) = -\int_0^{\frac{\pi}{2}} \frac{k \sin^2\phi}{\sqrt{1 - k^2 \sin^2\phi}}\, d\phi$$

$$= \frac{1}{k}\int_0^{\frac{\pi}{2}} \frac{1 - k^2 \sin^2\phi - 1}{\sqrt{1 - k^2 \sin^2\phi}}\, d\phi$$

$$= \frac{1}{k}\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\phi}\; d\phi - \frac{1}{k}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}$$

$$= \frac{E(k) - F(k)}{k}$$

$$ds = \sqrt{(dr\cos[t] - r\sin[t])^2 + (dr\sin[t] + r\cos[t]\,dt)^2}$$

$$= \sqrt{dr^2 + r^2\, dt} = \sqrt{(r'')^2 + r^2}\; dt$$

$$\int_0^1 x^m (\ln x)^n\, dx$$

$$\underset{x=e^t}{=} \int_{-\infty}^0 e^{(m+1)\,t}\, t^n\, dt$$

$$\underset{s=-(m+1)\,t}{=} \frac{(-1)^n}{(m+1)^{n+1}}\int_0^{+\infty} e^s s^n\, ds \quad \left(m \neq -1 \text{ ˚–}\right)$$

$$= \int_{-\infty}^0 t^n\, dt = \ldots \quad \left(m = -1 \text{ ˚–,} \quad \text{´ n与} -1 \text{ 的„ 系}\right)$$

$$\int_0^\pi \frac{\sin nx}{\sin x}\,dx \left(\sin x = \frac{e^{ix}-e^{-ix}}{2\,i},\ \sin nx = \frac{e^{inx}-e^{-inx}}{2\,i}\right)$$

$$= \int_0^\pi \frac{e^{inx}-e^{-inx}}{e^{ix}-e^{-ix}}\,dx$$

$$= \int_0^\pi \left(e^{i\,(n-1)\,x} + e^{i\,(n-2)\,x}\,e^{-ix} + \ldots + e^{ix}\,e^{-i\,(n-2)\,x} + e^{-i\,(n-1)\,x}\right) dx$$

$$= \int_0^\pi \left(e^{i\,(n-1)\,x} + e^{i\,(n-3)\,x} + \ldots + e^{-i\,(n-3)} + e^{-i\,(n-1)\,x}\right) dx$$

$$= \int_0^\pi 2\,(\cos\,(n-1)\,x + \cos\,(n-3)\,x + \ldots + \cos\,(1)\,x)\,dx\ \left(n\text{、"偶数˚–}\right) = 0$$

$$= \int_0^\pi (2\cos\,(n-1)\,x + 2\cos\,(n-3)\,x + \ldots + \cos\,(0)\,x)\,dx\ \left(n\text{、"奇数˚–}\right) = \pi$$

$$\int_0^\pi \cos^n x\cos nx\,dx \left(\cos nx = \frac{e^{inx}+e^{-inx}}{2}\quad \cos^n x = \frac{\left(e^{ix}+e^{-ix}\right)^n}{2^n}\right)$$

$$= \frac{1}{2^{n+1}}\int_0^\pi \left(e^{inx}+e^{-inx}\right)\left(e^{ix}+e^{-ix}\right)^n dx$$

$$= \frac{1}{2^n}\int_0^\pi \left(\cos 2\,nx + \binom{n}{1}\cos 2\,(n-1)\,x + \ldots + \binom{n}{n-1}\cos 2\,x + 1\right) dx$$

$$= \frac{\pi}{2^n}$$

$$\frac{1\,!}{x+1} + \frac{2\,!}{(x+1)\,(x+2)} + \frac{3\,!}{(x+1)\,(x+2)\,(x+3)} + \ldots$$

$$\frac{1}{x-1} = \frac{x+1}{x-1}\,\frac{1}{x+1} = \frac{1}{x+1}\left(1+\frac{2}{x-1}\right)$$

$$= \frac{1}{x+1} + \frac{1}{x+1}\,\frac{2}{x-1} = \frac{1}{x+1} + \frac{1}{x+1}\,\frac{2}{x+2}\left(\frac{x+2}{x-1}\right)$$

$$= \frac{1}{x+1} + \frac{1}{x+1}\,\frac{2}{x+2} + \frac{1}{x+1}\,\frac{2}{x+2}\,\frac{3}{x-1}$$

$$= \ \ldots$$

$$= \frac{1\,!}{x+1} + \frac{2\,!}{(x+1)\,(x+2)} + \frac{3\,!}{(x+1)\,(x+2)\,(x+3)} + \ldots + \frac{n\,!}{(x+1)\,(x+2)\,\ldots\,(x+n)}\,\frac{n+1}{x-1}$$

$$= \frac{1\,!}{x+1} + \frac{2\,!}{(x+1)\,(x+2)} + \frac{3\,!}{(x+1)\,(x+2)\,(x+3)} + \ldots + R_n$$

其中 $R_n = \frac{1}{x-1}\prod_{k=1}^{n}\frac{k+1}{k+x} = \frac{1}{x-1}\prod_{k=1}^{n}(1+a_k)$

$$a_k = \frac{1-x}{k} + O\left(\frac{1}{k^2}\right)$$

$\prod_{k=1}^{n}(1+a_k)$ 与 $\sum_{k=1}^{n} a_k$ `†散性相同，

当$x > 1$ ˚–，$\sum_{k=1}^{n} a_k$ 向 $-\infty$•散，此˚– $\prod_{k=1}^{n}(1+a_k) = R_n \to 0$，

‰і而 $\frac{1\,!}{x+1} + \frac{2\,!}{(x+1)\,(x+2)} + \frac{3\,!}{(x+1)\,(x+2)\,(x+3)} + \ldots = \frac{1}{x-1}$

当$x = 1$ ˚–...

当$x < 1$ 且不取‚"整数，$\sum_{k=1}^{n} a_k$ 向 $+\infty$•散，此˚– $\prod_{k=1}^{n}(1+a_k) = R_n \to +\infty$

$(a+b)^x = a^y + b^y$ (下面只考´˙$1 < x < y,\ a,\ b > 1$ 且 $(a,\ b) = 1$ 的情况，
其他情况‰ˇ…　)

考´˙方程$z^x = (z-b)^y + b^y$ 有根$z = a+b$

y ¸"偶数˚–, $a+b \mid 2b^y \Longrightarrow a+b \mid 2$ 不可能

y ¸"奇数˚–, $a+b \mid C_y^1 b^{y-1} \Longrightarrow a+b \mid y$

所以y位奇数，$a+b$是奇数，并 Ɫb ¸"奇数

因 $(a,\ b) = 1 \Longrightarrow la + mb = 1$

$$\begin{cases} (a+b)^x (la+mb)^{y-x} = a^y + b^y \Longrightarrow 1^{y-x} a^y \equiv a^y \pmod b \Longrightarrow 1^{y-x} \equiv 1 \pmod b \Longrightarrow \phi(b) \mid y-x \\ (a+b)^x (la+mb)^y = a^y + b^y \Longrightarrow 1^y a^{y+x} \equiv a^y \pmod b \Longrightarrow (la)^x \equiv 1 \pmod b \Longrightarrow \phi(b) \mid x \end{cases}$$

所以$\phi(b) \mid y$，因$b > 1$ ¸"奇数，$\phi(b)$ ¸"偶数$\Longrightarrow y$ ¸"偶数 矛盾

映射 $f : X \to R$ 且 $\sum_{n \in X} f(n)$ Ɫ¶收ˋ†，　$\{x \mid f(x) \neq 0\}$ 是可数集

$$4 \int_0^{\frac{\pi}{2}} (\text{Log}[2 \, \text{Sin}[x]] \, \text{Log}[2 \, \text{Cos}[x]]) \, dx = -\frac{\pi^3}{12}$$

$$\int_0^{\frac{\pi}{2}} \frac{x}{n^2} \left(\frac{\text{Sin}[n x]}{\text{Sin}[x]} \right)^4 dx$$