■ $\int_{\frac{1}{2}}^{2}\left(1-2x-x^2\right)e^{x+\frac{1}{x}}\,dx=-\frac{15\,e^{5/2}}{4}$

pf1 : $\frac{d}{dx}\left(x^a\,e^{x+\frac{1}{x}}\right)$, when $a=2$ we get $a\,\frac{d}{dx}\left(x^a\,e^{x+\frac{1}{x}}\right)=e^{x+\frac{1}{x}}\left(x^2+2x-1\right)$

pf2 : let $\int_{1/2}^{2}2\,x\,e^{x+1/x}\,dx$ integration by parts $\left(d\,x^2\right)$

■ $\int_0^{\pi/4}\frac{(\text{Cot}[x]-1)^{p-1}}{\text{Sin}^2[x]}\,\text{Log}[\text{Tan}[x]]\,dx=-\frac{\pi}{p}\,\csc[p\,\pi]\,(-1<p<0)$

pf : let $u=\cot x-1$ we get $\int_0^{+\infty}u^{p-1}\ln(1+u)\,du=\frac{\pi}{p}\csc(p\,\pi)$,

then use $d\,u^p=p\,u^{p-1}\,du$ integration by parts and the Beta function

■ $I_n=\text{Assuming}\left[n>-1,\int_0^{\frac{\pi}{2}}\text{Sin}[x]^n\,dx\right];\ I_{2n-2}=a^{2n-1}L_n=a^{2n-1}\int_0^{\infty}\frac{1}{\left(a^2+x^2\right)^n}\,dx(a>0)$

$$\frac{\sqrt{\pi}\;\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{n}{2}+1\right)}$$

$I_n=\frac{n-1}{n}\,I_{n-2}(n\geq 2)$ and $I_0=\frac{\pi}{2},\ I_1=1$

$$\int_0^{\frac{\pi}{2}}\text{Sin}[x]^{2n}\,dx=\frac{(2n-1)!!}{(2n)!!}\,\frac{\pi}{2};$$

$$\int_0^{\frac{\pi}{2}}\text{Sin}[x]^{2n+1}\,dx=\frac{(2n)!!}{(2n+1)!!}.$$

$$L_n=a^{1-2n}\,\frac{\pi}{2}\,\frac{(2n-3)!!}{(2n-2)!!}$$

■ $J_n=\text{Assuming}\left[n>-1,\int_0^{\frac{\pi}{4}}\text{Tan}[x]^n\,dx\right]=K_n=\int_0^{\frac{\pi}{4}}\left(\frac{\text{Cos}[x]-\text{Sin}[x]}{\text{Cos}[x]+\text{Sin}[x]}\right)^n\,dx$

$$\frac{1}{4}\left(\psi^{(0)}\left(\frac{n+3}{4}\right)-\psi^{(0)}\left(\frac{n+1}{4}\right)\right)$$

using $\text{Tan}[x]^2=(\text{Tan}[x])'-1$, we have $J_n=\frac{1}{n-1}-J_{n-2}(n\geq 2)$. so

$$\begin{cases} J_{2n}=\frac{(-1)^n\pi}{4}+\sum_{k=1}^{n}\frac{(-1)^{n-k}}{2k-1}; \\ J_{2n+1}=\frac{(-1)^n\ln 2}{2}+\sum_{k=1}^{n}\frac{(-1)^{n-k}}{2k}. \end{cases}\ (n\geq 1)$$

■ $I_n = \int_0^{\frac{\pi}{2}} \mathrm{Cos}[x]^n \, \mathrm{Cos}[n\,x] \, dx$

ConditionalExpression$\left[\pi\, 2^{-n-1},\ \mathrm{Re}(n) > -1\right]$

$2\, I_n = \int_0^{\frac{\pi}{2}} \mathrm{Cos}[x]^{n-1} \left(\mathrm{Cos}[(n-1)\,x] + \mathrm{Cos}[(n+1)\,x]\right) dx =$

$I_{n-1} + I_n - \int_0^{\frac{\pi}{2}} \mathrm{Cos}[x]^{n-1} \,\mathrm{Sin}[x]\, \mathrm{Sin}[n\,x]\, dx = I_{n-1}$(Integral by parts)

so $I_n = \dfrac{\pi}{2^{n+1}}$

■ $J_n = \int_0^{\frac{\pi}{2}} \mathrm{Cos}[x]^n \, \mathrm{Sin}[n\,x] \, dx$

ConditionalExpression$\left[\dfrac{2^{-n-1}\left(\left((-1)^n - e^{3\,i\,\pi\,n}\right) \mathrm{B}_{-1}(-n,\, n+1) - i\,\pi\left(1 + e^{2\,i\,\pi\,n}\right)\right)}{-1 + e^{2\,i\,\pi\,n}},\ \mathrm{Re}(n) > -1\right]$

$J_n = \dfrac{1}{2\,n} + \dfrac{1}{2}\, J_{n-1} = \dfrac{1}{2^{n+1}} \sum_{k=1}^{n} \dfrac{2^k}{k}$ the same method as foregoing

■ $I_n = \int_0^{\frac{\pi}{2}} \dfrac{\mathrm{Sin}[(2\,n-1)\,x]}{\mathrm{Sin}[x]} \, dx$

$$\left(\Gamma(n+1)\left((n+1)\left(-2\,{}_2F_1\left(\frac{1}{2}, 1;\, n+\frac{3}{2};\, 1\right)\sin(\pi\,n) + 2\,\pi\,n + \pi\right) - \sin(\pi\,n)\,{}_3F_2\left(1, \frac{3}{2}, n+1;\, n+\frac{3}{2}, n+2;\, 1\right)\right) - \sqrt{\pi}\,\left(2\,n^2 + 3\,n + 1\right)\Gamma\left(n+\frac{1}{2}\right){}_1F_0(-n;\, ;\, 1)\right) \Big/ \left(2\,(n+1)\,(2\,n+1)\,\Gamma(n+1)\right)$$

$I_n = \int_0^{\frac{\pi}{2}} \dfrac{\mathrm{Sin}[(2\,n-3)\,x]\,\mathrm{Cos}[2\,x] + \mathrm{Cos}[(2\,n-3)\,x]\,\mathrm{Sin}[2\,x]}{\mathrm{Sin}[x]} \, dx$

$= I_{n-1} + 2 \int_0^{\frac{\pi}{2}} [\mathrm{Cos}[(2\,n-3)\,x]\,\mathrm{Cos}[x] - \mathrm{Sin}[(2\,n-3)\,x]\,\mathrm{Sin}[x]]\, dx$

$= I_{n-1} (n \geq 1);$

so $I_n = I_1 = \dfrac{\pi}{2}$, the same, we have :

$J_n = \int_0^{\frac{\pi}{2}} \dfrac{\mathrm{Cos}[(2\,n-1)\,x]}{\mathrm{Cos}[x]} \, dx = -J_{n-1} = (-1)^{n-1} J_1 = (-1)^{n-1} \dfrac{\pi}{2}\ (n \geq 1);$

and $K_n = \int_0^{\frac{\pi}{2}} \left(\dfrac{\mathrm{Sin}[n\,x]}{\mathrm{Sin}[x]}\right)^2 dx = K_{n-1} + \dfrac{\pi}{2} = K_1 + \dfrac{(n-1)\,\pi}{2} = \dfrac{n\,\pi}{2}\ (n \geq 1);$

■ let $a, b > 0$. if $f \in C[0, \infty)$, $f(\infty)$ exist and finite, then $\int_0^\infty \frac{f(a x) - f(b x)}{x} \, dx = [f(\infty) - f(0)] \ln \frac{a}{b}$;

if $f \in C[0, \infty)$, the integration $\int_1^\infty \frac{f(x)}{x} \, dx$ convergenced, then : $\int_0^\infty \frac{f(a x) - f(b x)}{x} \, dx = f(0) \ln \frac{b}{a}$;

if $f \in C[0, \infty)$, the integration $\int_0^1 \frac{f(x)}{x} \, dx$ and $f(\infty)$ exist and finite,

then : $\int_0^\infty \frac{f(a x) - f(b x)}{x} \, dx = f(\infty) \ln \frac{a}{b}$.

pf : $\lim_{p \to 0^+, q \to \infty} \int_p^q \frac{f(a x) - f(b x)}{x} \, dx$

$= \lim_{p \to 0^+, q \to \infty} \left[\int_{a p}^{a q} \frac{f(z)}{z} \, dz - \int_{b p}^{b q} \frac{f(z)}{z} \, dz \right] = \lim_{p \to 0^+, q \to \infty} \left[\int_{a p}^{b p} \frac{f(z)}{z} \, dz - \int_{a q}^{b q} \frac{f(z)}{z} \, dz \right]$

$= \lim_{p \to 0^+} f(\xi) \ln \frac{b}{a} - \lim_{q \to \infty} f(\eta) \ln \frac{b}{a}$ $(\xi \in [a p, b p], \eta \in [a q, b q]) = [f(\infty) - f(0)] \ln \frac{a}{b}$.

same to the others.

■ $a, b > 0$.

$\int_0^\infty \frac{\left(e^{-a x} - e^{-b x}\right)}{x} \, dx = \ln \frac{b}{a}$;

$\int_0^\infty \frac{\arctan(a x) - \arctan(b x)}{x} \, dx = \frac{\pi}{2} \ln \frac{a}{b}$;

$\int_0^\infty \ln \frac{p + q e^{-a x}}{p + q e^{-b x}} \, dx = \ln \frac{p}{p + q} \ln \frac{a}{b}$ $(p, q > 0)$;

$\int_0^\infty \frac{\left(e^{-a x^2} - e^{-b x^2}\right)}{x} \, dx = \frac{1}{2} \ln \frac{b}{a}$;

$\int_0^\infty \frac{b \sin a x - a \sin b x}{x^2} \, dx = a b \ln \frac{b}{a}$;

$\int_0^1 \frac{x^{a-1} - x^{b-1}}{\ln x} \, dx = \ln \frac{a}{b}$;

$\int_0^\infty \frac{\cos a x - \cos b x}{x} \, dx = \ln \frac{b}{a}$;

$\int_0^\infty \frac{\sin a x \sin b x}{x} \, dx = \frac{1}{2} \ln \left| \frac{a + b}{a - b} \right|$ $(a \neq b)$;

$\int_0^\infty (1 - \cos a x) \frac{\cos b x}{x} \, dx = \ln \frac{\sqrt{|a^2 - b^2|}}{b}$ $(a \neq b)$;

$\int_0^\infty \frac{\sin^4 a x - \sin^4 b x}{x} \, dx = \frac{3}{8} \ln \frac{a}{b}$;

■ $\int_0^1 \frac{\mathrm{ArcSin}\left[\sqrt{x}\right]}{\sqrt{x(1 - x)}} \, dx = \frac{\pi^2}{4}$

pf : let $t = \mathrm{ArcSin}\left[\sqrt{x}\right]$

- $$\int_0^{\pi} \frac{1}{1 + a\,\mathrm{Cos}[x]}\, dx = \frac{\pi}{\sqrt{1 - a^2}} \; (|a| < 1)$$

$$\text{pf}: = \int_0^{\frac{\pi}{2}} \frac{2}{1 - a + 2a\,\mathrm{Cos}[x]}\, dx \left(\text{using } \cos 2x = 1 - 2\cos^2 x\right)$$

$$= 2\int_0^{+\infty} \frac{1}{1 + a + (1 - a)\,t^2}\, dt (t = \mathrm{tgx})$$

$$= \frac{\pi}{\sqrt{1 - a^2}}$$

$$\int_0^{\pi} \frac{1}{1 \pm \mathrm{Cos}[x]}\, dx = +\infty$$

- $\int_0^{\pi} \mathrm{Log}[1 + a\,\mathrm{Cos}[x]]\, dx = \pi\, \mathrm{Log}\left[\frac{1+\sqrt{1-a^2}}{2}\right] (|a| < 1)$

pf : $= I(a)$

$$\frac{d}{da} I(a) = \frac{\pi}{a} - \frac{\pi}{a\sqrt{1-a^2}}$$

$$I(a) = I(0) + \int_0^a I'(a)\, da = \pi \ln \frac{1+\sqrt{1-a^2}}{2}$$

going on calculating :

$$\int_0^{\pi} \mathrm{Log}[1 + a\,\mathrm{Cos}[x]]\, dx = \int_0^{\pi} \sum_{i=1}^{+\infty} \frac{\left(-a^{2i}\right) \mathrm{Cos}[x]^{2i}}{2i}\, dx = -\sum_{i=1}^{+\infty} \frac{a^{2i}\,(2i)!}{((2i)!!)^2\,(2i)} \pi$$

so wo get $-\sum_{i=1}^{+\infty} \frac{a^{2i}\,(2i)!}{((2i)!!)^2\,(2i)} = \mathrm{Log}\left[\frac{1+\sqrt{1-a^2}}{2}\right]$,

the diameter of the left hand series is $R = \sqrt{\frac{1}{\limsup \sqrt[i]{\left|\frac{-(2i)!}{((2i)!!)^2\,(2i)}\right|}}} = 1$,

and $\lim i\, a_i = 0$ and $\lim_{r\to\pm 1} \mathrm{Log}\left[\frac{1+\sqrt{1-a^2}}{2}\right]$ exist,

so $-\sum_{i=1}^{+\infty} \frac{(2i)!}{((2i)!!)^2\,(2i)} = -\ln 2$ and $\int_0^{\pi} \mathrm{Log}[1 \pm \mathrm{Cos}[x]]\, dx = -\pi \ln 2 \Longleftrightarrow \int_0^{\pi} \mathrm{Log}[\mathrm{Sin}[x]]\, dx = -\pi \ln 2$

$\Longleftrightarrow \int_0^{2\pi} \mathrm{Log}\left[\left|1 - e^{ix}\right|\right] dx = 0$

here is another way to proof $\int_0^{\pi} \mathrm{Log}[\mathrm{Sin}[x]]\, dx = -\pi \ln 2$

$\int_0^{\pi} \mathrm{Log}[\mathrm{Sin}[x]]\, dx =$

$$2\int_0^{\pi/2} \mathrm{Log}[\mathrm{Sin}[x]]\, dx = \int_0^{\pi} \mathrm{Log}\left[2\,\mathrm{Sin}\left[\frac{x}{2}\right] \mathrm{Cos}\left[\frac{x}{2}\right]\right] dx = \pi \ln 2 + 4\int_0^{\pi/2} \mathrm{Log}[\mathrm{Sin}[x]]\, dx$$

so $2\int_0^{\pi/2} \mathrm{Log}[\mathrm{Sin}[x]]\, dx = -\pi \ln 2$

$$\int_0^{+\infty} \left(\frac{1}{\mathrm{Sinh}[x]} - \frac{1}{x}\right) \frac{1}{x}\, dx \sim \int_0^{+\infty} \left(\frac{x}{e^x - e^{-x}} - \frac{1}{2}\right) \frac{1}{x^2}\, dx$$

$$\int_0^{\pi} \frac{x\,\mathrm{Sin}[x]}{2 + \mathrm{Cos}[x]}\, dx$$

$$\pi\, \mathrm{Log}\left[\frac{1}{2}\left(2 + \sqrt{3}\right)\right]$$