MEC ENG 193B/292B: Feedback Control of Legged Robots

Homework 1

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Background: The Three-Link Robot

In this homework, we will derive the kinematics and dynamics of a simple, multi-link legged robot—one with two legs (without knees), a torso and constrained to move in the vertical plane (see Figure 1). Similar to the bouncing ball problem illustrated in class, the robot has a continuous-time dynamics (that we derive in Problem 1) and undergoes a discrete change in its velocities upon a rigid impact (that you will derive in HW 2). It is important to note that the methods used to derive the dynamics in this homework also apply to higher dimensional robots such as *Cassie* and *Atlas*.

Robot Parameters

Table 1 presents the various mechanical parameters of the robot links. Assume the center of mass of each link to be located at the geometric center of the link.

Link	**Mass (kg)**	**Moment of Inertia about CoM** ($kg{-}m^2$)	**Link Length (m)**
Torso	10	1	0.5
Leg	5	0.5	1

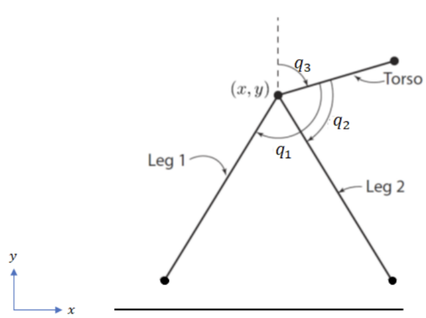
Table 1: Model Parameters for the Three-Link Robot

Configuration Variables

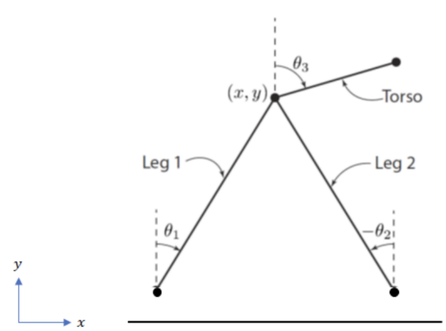
There are several ways to represent the configuration of the robot. *Configuration variables* are the minimum number of variables required to completely define the configuration (i.e. the position and orientation of the various links) of the robot. Figure 1 illustrates two different (but equivalent) representations - Figure 1a illustrates the configuration variables in terms of relative angles, where as, Figure 1b illustrates the configuration variables in terms of absolute angles.

Throughout this homework, unless otherwise, we will assume the order of the configuration variable $q \in \mathbb{R}^5$ as:

$$q = \begin{bmatrix} x & y & q_1 & q_2 & q_3 \end{bmatrix}^\top \tag{1}$$



(a) Relative Coordinates



(b) Absolute Coordinates

Figure 1: Two configuration variable representations of the Three-Link Robot. Here, x, y represent the Cartesian position of the hip joint with respect to the inertial frame, q_3 or θ_3 are the absolute torso angle with respect to the vertical, while q_1, q_2 are the two leg angles relative to the torso and θ_1, θ_2 are the absolute leg angles. Note that each link has distributed mass with the center-of-mass (CoM) at the center of the link. The link length, link mass and link inertia about the link CoM are provided in Table 1.

1 Problems

1.1 Lagrangian Dynamics

We will derive the continuous-time dynamics model for the three-link robot. We will use the configuration variables represented in Figure 1a. We will numerically compute various quantities for the following two configurations and velocities $(q, \dot{q})$:

$$\text{(i)} \quad (q, \dot{q}) = \left(\begin{bmatrix} 0.5\, m \\ \frac{\sqrt{3}}{2}\, m \\ 150^\circ \\ 120^\circ \\ 30^\circ \end{bmatrix}, \begin{bmatrix} -0.8049\, m \cdot s^{-1} \\ -0.4430\, m \cdot s^{-1} \\ 0.0938\, rad \cdot s^{-1} \\ 0.9150\, rad \cdot s^{-1} \\ 0.9298\, rad \cdot s^{-1} \end{bmatrix} \right)$$

$$\text{(ii)} \quad (q, \dot{q}) = \left(\begin{bmatrix} 0.3420\, m \\ 0.9397\, m \\ 170^\circ \\ 20^\circ \\ 30^\circ \end{bmatrix}, \begin{bmatrix} -0.1225\, m \cdot s^{-1} \\ -0.2369\, m \cdot s^{-1} \\ 0.5310\, rad \cdot s^{-1} \\ 0.5904\, rad \cdot s^{-1} \\ 0.6263\, rad \cdot s^{-1} \end{bmatrix} \right).$$

(a) **Position of Link Center-of-Mass**
Symbolically compute the position of the center of mass of each of the links as a function of the configuration variables q.
For the given two numerical configurations, provide the position of the center-of-mass of the three links as a matrix:

$$P := \big[\text{Position of CoM of Link 1},\ \text{Position of CoM of Link 2},\ \text{Position of CoM of Link 3}\big].$$

Proof. We can write the position of the center of mass of each of the links as a function of the configuration variables q and using relative coordinates (relative to the torso). Since both legs 1 and legs 2 start at (x, y) and the torso angle is in terms of absolute coordinates we have

$$p_{leg1} = \begin{bmatrix} x \\ y \end{bmatrix} + \frac{l_{leg1}}{2} \begin{bmatrix} \sin(q_3 + q_1) \\ \cos(q_3 + q_1) \end{bmatrix}$$

We can apply this to each link and just replace the absolute angles like so

$$p_{leg2} = \begin{bmatrix} x \\ y \end{bmatrix} + \frac{l_{leg2}}{2} \begin{bmatrix} \sin(q_3 + q_2) \\ \cos(q_3 + q_2) \end{bmatrix}$$
$$p_{torso} = \begin{bmatrix} x \\ y \end{bmatrix} + \frac{l_{torso}}{2} \begin{bmatrix} \sin(q_3) \\ \cos(q_3) \end{bmatrix}$$

We can then just assemble the position of link CoM matrix, P, as follows

$$P = \begin{bmatrix} x + \frac{l_{leg}}{2}\sin(q_3 + q_1) & x + \frac{l_{leg}}{2}\sin(q_3 + q_2) & x + \frac{l_{torso}}{2}\sin(q_3) \\ y + \frac{l_{leg}}{2}\cos(q_3 + q_1) & y + \frac{l_{leg}}{2}\cos(q_3 + q_2) & y + \frac{l_{torso}}{2}\cos(q_3) \end{bmatrix}$$

Using MATLAB and the symbolic toolbox, for configuration 1 and 2 we have

$$P_1 = \begin{bmatrix} 0.5 & 0.75 & 0.625 \\ 0.366 & 0.433 & 1.0825 \end{bmatrix} \texttt{m}, \quad P_2 = \begin{bmatrix} 0.171 & 0.725 & 0.467 \\ 0.4699 & 1.2611 & 1.1562 \end{bmatrix} \texttt{m}$$

$\square$

(b) **Velocity of Link Center-of-Mass**
Symbolically compute the velocities of center-of-mass of the three links as a function of the configuration variables q and their velocities $\dot{q}$.

Proof. The velocities are simply the derivatives of the of the position

$$\dot{p}_{leg1,x} = \dot{x} + \frac{l_{leg1}}{2}\cos(q_3 + q_1)(\dot{q}_3 + \dot{q}_1), \quad \dot{p}_{leg1,y} = \dot{y} - \frac{l_{leg1}}{2}\sin(q_3 + q_1)(\dot{q}_3 + \dot{q}_1)$$

Similarly for the second leg

$$\dot{p}_{leg2,x} = \dot{x} + \frac{l_{leg2}}{2}\cos(q_3 + q_2)(\dot{q}_3 + \dot{q}_2), \quad \dot{p}_{leg2,y} = \dot{y} - \frac{l_{leg2}}{2}\sin(q_3 + q_2)(\dot{q}_3 + \dot{q}_2)$$

For the torso, we have

$$\dot{p}_{torso,x} = \dot{x} + \frac{l_{torso}}{2}\cos(q_3)\dot{q}_3, \quad \dot{p}_{torso,y} = \dot{y} - \frac{l_{torso}}{2}\sin(q_3)\dot{q}_3$$

Rewriting in a matrix form, we have the following velocity matrix

$$\dot{p} = \begin{bmatrix} \dot{x} + \frac{1}{2}\dot{q}_1\cos(q_1+q_3) + \frac{1}{2}\dot{q}_3\cos(q_1+q_3) & \dot{x} + \frac{1}{2}\dot{q}_2\cos(q_2+q_3) + \frac{1}{2}\dot{q}_3\cos(q_2+q_3) & \dot{x} + \frac{1}{4}\dot{q}_3\cos(q_3) \\ \dot{y} - \frac{1}{2}\dot{q}_1\sin(q_1+q_3) - \frac{1}{2}\dot{q}_3\sin(q_1+q_3) & \dot{y} - \frac{1}{2}\dot{q}_2\sin(q_2+q_3) - \frac{1}{2}\dot{q}_3\sin(q_2+q_3) & \dot{y} - \frac{1}{4}\dot{q}_3\sin(q_3) \end{bmatrix}$$

Using MATLAB and the symbolic toolbox, for configuration 1 and 2 we have

$$\dot{p}_1 = \begin{bmatrix} -1.3167 & -1.6037 & -0.6036 \\ -0.4430 & -0.9042 & -0.5592 \end{bmatrix}\texttt{m/s}, \quad \dot{p}_2 = \begin{bmatrix} -0.6663 & 0.2685 & 0.0131 \\ -0.0390 & -0.7029 & -0.3152 \end{bmatrix}\texttt{m/s}$$

□

(c) **Total Kinetic Energy**
Symbolically compute the total kinetic energy of the system. For the given two numerical configurations, provide the kinetic energy.

Proof. The total kinetic energy is the sum of the kinetic energy of each link. Each link also has a translational and rotational kinetic energy associated with it. We can write it as

$$T_i = \frac{1}{2}\dot{p}_i^\top m_i \dot{p}_i + \frac{1}{2} I_{CoM,i}\omega_i^2 \quad \text{and} \quad T = T_{leg1} + T_{leg2} + T_{torso}$$

For the absolute velocities. Given leg 1's orientation $\theta_1 = q_3 + q_1 \implies \omega_1 = \dot{q}_3 + \dot{q}_1$. For leg 2, we have $\theta_2 = q_3 + q_2 \implies \omega_2 = \dot{q}_3 + \dot{q}_2$. The torso $\theta_3 = q_3 \implies \omega_3 = \dot{q}_3$. Symbolically, we can represent the total kinetic energy as

$$T = \frac{1}{2}\dot{p}_{leg1}^\top m_{leg1}\dot{p}_{leg1} + \frac{1}{2}I_{leg1}(\dot{q}_3+\dot{q}_1)^2 + \frac{1}{2}\dot{p}_{leg2}^\top m_{leg2}\dot{p}_{leg2} + \frac{1}{2}I_{leg2}(\dot{q}_3+\dot{q}_2)^2 + \frac{1}{2}\dot{p}_{torso}^\top m_{torso}\dot{p}_{torso} + \frac{1}{2}I_{torso}(\dot{q}_3)^2$$

Substituting in all our expressions from before, we have

$$\begin{aligned} T = \frac{5}{2}\left(\dot{x} + \tfrac{1}{2}\dot{q}_1\cos(q_1+q_3) + \tfrac{1}{2}\dot{q}_3\cos(q_1+q_3)\right)^2 + \frac{5}{2}\left(\dot{x} + \tfrac{1}{2}\dot{q}_2\cos(q_2+q_3) + \tfrac{1}{2}\dot{q}_3\cos(q_2+q_3)\right)^2 \\ + \tfrac{1}{4}\left(\dot{q}_1+\dot{q}_3\right)^2 + \tfrac{1}{4}\left(\dot{q}_2+\dot{q}_3\right)^2 \\ + \frac{5}{2}\left(\tfrac{1}{2}\dot{q}_1\sin(q_1+q_3) - \dot{y} + \tfrac{1}{2}\dot{q}_3\sin(q_1+q_3)\right)^2 + \frac{5}{2}\left(\tfrac{1}{2}\dot{q}_2\sin(q_2+q_3) - \dot{y} + \tfrac{1}{2}\dot{q}_3\sin(q_2+q_3)\right)^2 \\ + \tfrac{1}{2}\dot{q}_3^2 + 5\left(\dot{x} + \tfrac{1}{4}\dot{q}_3\cos(q_3)\right)^2 + 5\left(\dot{y} - \tfrac{1}{4}\dot{q}_3\sin(q_3)\right)^2 \end{aligned}$$

Using MATLAB, we calculate the numerical values for the kinetic energies for both configurations as follows

$$T_1 = 18.2289\text{J}, \quad T_2 = 3.9277\text{J}$$

□

(d) **Total Potential Energy**
Symbolically compute the total potential energy of the system. For the given two numerical configurations and velocities, provide the potential energy.

Proof. The total potential energy is, again, the sum of the potential energy of each link. We can write the potential energy of each link

$$U_i = m_i g p_i^\top \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, symbolically we have the following

$$\begin{aligned} U &= m_{leg1} g p_{leg1}^\top \begin{bmatrix} 0 \\ 1 \end{bmatrix} + m_{leg2} g p_{leg2}^\top \begin{bmatrix} 0 \\ 1 \end{bmatrix} + m_{torso} g p_{torso}^\top \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{981y}{5} + \frac{981\cos(q_1+q_3)}{40} + \frac{981\cos(q_2+q_3)}{40} + \frac{981\cos(q_3)}{40} \end{aligned}$$

Using MATLAB and symbolic toolbox for each configuration, we have $U_1 = 145.3892\texttt{J}$ and $U_2 = 198.3268\texttt{J}$. □

(e) **Dynamical Model**
The dynamics of the three-link robot can be computed in the form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)u,$$

where $q \in \mathbb{R}^5$ is the vector of configuration variables, $u \in \mathbb{R}^2$ is a vector of motor torques representing the two actuators that actuate the relative leg angles for Leg 1 and Leg 2 respectively (i.e., the two motors actuate q_1 and q_2), $D \in \mathbb{R}^{5\times5}$ is the *Inertia Matrix*, $C \in \mathbb{R}^{5\times5}$ is the *Coriolis Matrix*, $G \in \mathbb{R}^{5\times1}$ is the *Gravity Vector*, and $B \in \mathbb{R}^{5\times2}$ is the *Input Mapping Matrix*.

Use the Matlab function `LagrangianDynamics.m` to symbolically compute the dynamics by computing the terms $D(q)$, $C(q,\dot{q})$, $G(q)$, $B(q)$. For the two numerical configurations, provide the numerical values of the matrix $D(q)$, the vectors $C(q,\dot{q})$, $G(q)$, and the matrix $B(q)$.

Proof. Symbolically, the Inertia matrix is defined as

$$D = \begin{bmatrix} 20 & 0 & \frac{5}{2}\cos(q_1+q_3) & \frac{5}{2}\cos(q_2+q_3) & \frac{5}{2}\cos(q_1+q_3)+\frac{5}{2}\cos(q_2+q_3)+\frac{5}{2}\cos(q_3) \\ 0 & 20 & -\frac{5}{2}\sin(q_1+q_3) & -\frac{5}{2}\sin(q_2+q_3) & -\frac{5}{2}\sin(q_1+q_3)-\frac{5}{2}\sin(q_2+q_3)-\frac{5}{2}\sin(q_3) \\ \frac{5}{2}\cos(q_1+q_3) & -\frac{5}{2}\sin(q_1+q_3) & 7/4 & 0 & 7/4 \\ \frac{5}{2}\cos(q_2+q_3) & -\frac{5}{2}\sin(q_2+q_3) & 0 & 7/4 & 7/4 \\ \frac{5}{2}\cos(q_1+q_3)+\frac{5}{2}\cos(q_2+q_3)+\frac{5}{2}\cos(q_3) & -\frac{5}{2}\sin(q_1+q_3)-\frac{5}{2}\sin(q_2+q_3)-\frac{5}{2}\sin(q_3) & 7/4 & 7/4 & 41/8 \end{bmatrix}$$

The Coriolis Matrix is defined as

$$C = \begin{bmatrix} 0 & 0 & -\frac{5}{2}\sin(q_1+q_3)(\dot{q}_1+\dot{q}_3) & -\frac{5}{2}\sin(q_2+q_3)(\dot{q}_2+\dot{q}_3) & -\frac{5}{2}\dot{q}_1\sin(q_1+q_3)-\frac{5}{2}\dot{q}_2\sin(q_2+q_3)-\dot{q}_3\left(\frac{5}{2}\sin(q_1+q_3)+\frac{5}{2}\sin(q_2+q_3)+\frac{5}{2}\sin(q_3)\right) \\ 0 & 0 & -\frac{5}{2}\cos(q_1+q_3)(\dot{q}_1+\dot{q}_3) & -\frac{5}{2}\cos(q_2+q_3)(\dot{q}_2+\dot{q}_3) & -\dot{q}_3\left(\frac{5}{2}\cos(q_1+q_3)+\frac{5}{2}\cos(q_2+q_3)+\frac{5}{2}\cos(q_3)\right)-\frac{5}{2}\dot{q}_1\cos(q_1+q_3)-\frac{5}{2}\dot{q}_2\cos(q_2+q_3) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The gravity vector G and the input mapping matrix B

$$G = \begin{bmatrix} 0 \\ 981/5 \\ -\frac{981}{40}\sin(q_1+q_3) \\ -\frac{981}{40}\sin(q_2+q_3) \\ -\frac{981}{40}\sin(q_1+q_3)-\frac{981}{40}\sin(q_2+q_3)-\frac{981}{40}\sin(q_3) \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The numerical values of the matrices using configurations 1 are

$$D_1(q) = \begin{bmatrix} 20 & 0 & -2.5 & -2.1651 & -2.5 \\ 0 & 20 & 0 & -1.25 & -2.5 \\ -2.5 & 0 & 1.75 & 0 & 1.75 \\ -2.1651 & -1.25 & 0 & 1.75 & 1.75 \\ -2.5 & -2.5 & 1.75 & 1.75 & 5.125 \end{bmatrix}, \quad C_1(q,\dot{q}) = \begin{bmatrix} 0 & 0 & 0 & -2.3060 & -3.4682 \\ 0 & 0 & 2.559 & 3.9941 & 4.54 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_1(q) = \begin{bmatrix} 0 \\ 196.2 \\ 0 \\ -12.2625 \\ -24.525 \end{bmatrix}, \quad B_1(q) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The numerical values of the matrices using configurations 2 are

$$D_2(q) = \begin{bmatrix} 20 & 0 & -2.3492 & 1.607 & 1.4228 \\ 0 & 20 & 0.8551 & -1.9151 & -2.3101 \\ -2.3492 & 0.8551 & 1.75 & 0 & 1.75 \\ 1.607 & -1.9151 & 0 & 1.75 & 1.75 \\ 1.4228 & -2.3101 & 1.75 & 1.75 & 5.125 \end{bmatrix}, \quad C_2(q,\dot{q}) = \begin{bmatrix} 0 & 0 & 0.9895 & -2.3301 & -2.1234 \\ 0 & 0 & 2.7188 & -1.9552 & -0.5924 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_2(q) = \begin{bmatrix} 0 \\ 196.2 \\ 8.388 \\ -18.7872 \\ -22.6617 \end{bmatrix}, \quad B_2(q) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\square$

1.2 Change of Coordinates

Consider the configuration variable $q \in \mathbb{R}^5$ specified in (1) for the configuration variables illustrated in Figure 1a. Suppose the configuration variable $\tilde{q} \in \mathbb{R}^5$ specified in (3) represents the configuration variables illustrated in Figure 1b.

$$\tilde{q} = \begin{bmatrix} x \\ y \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}.$$

(a) Show that the two sets of configuration variables are equivalent, i.e. show that there exists an invertible function that maps the configuration variables q to the configuration variables $\tilde{q}$.
Hint: In this simple case, the transformation relating the two sets of configuration variables is affine, i.e. there exists a constant matrix $T \in \mathbb{R}^{5\times5}$ and constant vector $d \in \mathbb{R}^5$ such that $\tilde{q} = Tq + d$, with T invertible and $q = T^{-1}\tilde{q} - T^{-1}d$.

Proof. Given the absolute coordinates $\tilde{q} = \begin{bmatrix} x & y & \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^\top$ and relative coordinates $q = \begin{bmatrix} x & y & q_1 & q_2 & q_3 \end{bmatrix}^\top$, we can write the absolute angles in terms of the relative coordinates as

$$\theta_1 = q_3 + q_1 - \pi, \quad \theta_2 = \pi - (q_3 + q_2), \quad \theta_3 = q_3$$

Then, there exists a constant matrix $T \in \mathbb{R}^{5\times5}$ and constant vector $d \in \mathbb{R}^5$ such that $\tilde{q} = Tq + d$, with T invertible and $q = T^{-1}\tilde{q} - T^{-1}d$. By observation, we find that T can be written as

$$\tilde{q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\pi \\ \pi \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ q_3 + q_1 - \pi \\ \pi - (q_3 + q_2) \\ q_3 \end{bmatrix}$$

We also verify that the inverse mapping. T is invertible since its determinant is 1 i.e. full rank as it is an upper triangular matrix. We can do Gauss-Jordan elimination on the T matrix augmented with the identity to get the inverse

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can also rewrite the configuration variable inverses as follows

$$\theta_1 = q_3 + q_1 - \pi \implies q_1 = \theta_1 - q_3 + \pi, \quad \theta_2 = \pi - (q_3 + q_2) \implies q_2 = \pi - \theta_2 - q_3, \quad \theta_3 = q_3 \implies q_3 = \theta_3$$

Substituting in $q_3 = \theta_3$, we have the following

$$q_1 = \theta_1 - \theta_3 - \pi, \quad q_2 = \pi - \theta_2 - \theta_3, \quad q_3 = \theta_3$$

By observation, we can then construct T^{-1} with $d = [0 \quad 0 \quad -\pi \quad \pi \quad 0]^\top$ as

$$q = T^{-1}\tilde{q} - T^{-1}d$$

$$\begin{bmatrix} x \\ y \\ \theta_1 - \theta_3 - \pi \\ \pi - \theta_2 - \theta_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\pi \\ \pi \\ 0 \end{bmatrix}$$

This inverse T matrix is the same as the Gauss-Jordan elimination method to find the inverse above, therefore, q and $\tilde{q}$ must be equivalent sets via an affine transformation with T being invertible. □

(b) What is the mapping between the velocities $\dot{q}$ and $\dot{\tilde{q}}$? What is the mapping between the accelerations $\ddot{q}$ and $\ddot{\tilde{q}}$?

Proof. If we took the derivatives of the above expressions, we would get the following mapping for velocities

$$\dot{\tilde{q}} = T\dot{q} \implies \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{q}_1 + \dot{q}_3 \\ -\dot{q}_2 - \dot{q}_3 \\ \dot{q}_3 \end{bmatrix} = T\dot{q}, \quad \dot{q} = T^{-1}\dot{\tilde{q}}$$

for acceleration, we have the following

$$\ddot{\tilde{q}} = T\ddot{q} \implies \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{q}_1 + \ddot{q}_3 \\ -\ddot{q}_2 - \ddot{q}_3 \\ \ddot{q}_3 \end{bmatrix} = T\ddot{q}, \quad \ddot{q} = T^{-1}\ddot{\tilde{q}}$$

Note that all the entries of T are constant, so the mapping is actually the same T matrix. □

(c) For the two numerical configurations and velocities for $(q, \dot{q})$ given in Problem 1, compute the corresponding configuration and velocities for $(\tilde{q}, \dot{\tilde{q}})$.

Proof. Substituting in the given numerical configurations and velocities for $(q, \dot{q})$ into our mappings above, we have the following for configuration 1

$$\tilde{q}_1 = \begin{bmatrix} 0.5000 \\ 0.8660 \\ 0 \\ 0.5236 \\ 0.5236 \end{bmatrix}, \quad \dot{\tilde{q}}_1 = \begin{bmatrix} -0.8049 \\ -0.4430 \\ 1.0236 \\ -1.8448 \\ 0.9298 \end{bmatrix}$$

For configuration 2, we have the following

$$\tilde{q}_2 = \begin{bmatrix} 0.3420 \\ 0.9397 \\ 0.3491 \\ 2.2689 \\ 0.5236 \end{bmatrix}, \quad \dot{\tilde{q}}_2 = \begin{bmatrix} -0.1225 \\ -0.2369 \\ 1.1573 \\ -1.2167 \\ 0.6263 \end{bmatrix}$$

$\square$

(d) For the absolute coordinates $(\tilde{q}, \dot{\tilde{q}})$, recompute the dynamics of the system resulting in:

$$\tilde{D}(\tilde{q})\ddot{\tilde{q}} + \tilde{C}(\tilde{q}, \dot{\tilde{q}})\dot{\tilde{q}} + \tilde{G}(\tilde{q}) = \tilde{B}(\tilde{q})u.$$

For the two numerical configurations and velocities $(\tilde{q}, \dot{\tilde{q}})$ you computed in part (c) above, numerically compute the values of the matrix $\tilde{D}(\tilde{q})$, the vectors $\tilde{C}(\tilde{q}, \dot{\tilde{q}})\dot{\tilde{q}}$, $\tilde{G}(\tilde{q})$ and the matrix $\tilde{B}(\tilde{q})$.

Proof. For configuration 1, we have

$$\tilde{D}(\tilde{q}) = \begin{bmatrix} 20.0000 & 0 & -2.5000 & 2.1651 & 2.1651 \\ 0 & 20.0000 & 0 & 1.2500 & -1.2500 \\ -2.5000 & 0 & 1.7500 & 0 & 0 \\ 2.1651 & 1.2500 & 0 & 1.7500 & 0 \\ 2.1651 & -1.2500 & 0 & 0 & 1.6250 \end{bmatrix}, \tilde{C}(\tilde{q}, \dot{\tilde{q}}) = \begin{bmatrix} 0 & 0 & 0 & 2.3060 & -1.1623 \\ 0 & 0 & 2.5590 & -3.9941 & -2.0131 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{G}(\tilde{q}) = \begin{bmatrix} 0 \\ 196.2000 \\ 0 \\ 12.2625 \\ -12.2625 \end{bmatrix}, \quad \tilde{B}(\tilde{q}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}, \quad \tilde{C}(\tilde{q}, \dot{\tilde{q}})\dot{\tilde{q}} = \begin{bmatrix} -5.3348 \\ 8.1160 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For configuration 2, we have

$$\tilde{D}(\tilde{q}) = \begin{bmatrix} 20.0000 & 0 & -2.3492 & -1.6070 & 2.1651 \\ 0 & 20.0000 & 0.8551 & 1.9151 & -1.2500 \\ -2.3492 & 0.8551 & 1.7500 & 0 & 0 \\ -1.6070 & 1.9151 & 0 & 1.7500 & 0 \\ 2.1651 & -1.2500 & 0 & 0 & 1.6250 \end{bmatrix}, \tilde{C}(\tilde{q}, \dot{\tilde{q}}) = \begin{bmatrix} 0 & 0 & 0.9895 & 2.3301 & -0.7829 \\ 0 & 0 & 2.7188 & 1.9552 & -1.3560 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{G}(\tilde{q}) = \begin{bmatrix} 0 \\ 196.2000 \\ 8.3880 \\ 18.7872 \\ -12.2625 \end{bmatrix}, \quad \tilde{B}(\tilde{q}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}, \quad \tilde{C}(\tilde{q}, \dot{\tilde{q}})\dot{\tilde{q}} = \begin{bmatrix} -2.1802 \\ -0.0817 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

□

(e) Compute the inertial matrix $\tilde{D}(\tilde{q})$ in terms of the inertial matrix $D(q)$ without recomputing the dynamics (as we did in part (d)).
Hint: Note that the kinetic energy computed for either coordinates are equivalent, i.e.

$$\tfrac{1}{2}\dot{q}^T D(q)\dot{q} = \tfrac{1}{2}\dot{\tilde{q}}^T \tilde{D}(\tilde{q})\dot{\tilde{q}}.$$

Proof. Starting with the energy balance, we have

$$\frac{1}{2}\dot{q}^\top D(q)\dot{q} = \frac{1}{2}\dot{\tilde{q}}^\top \tilde{D}(\tilde{q})\dot{\tilde{q}} \implies \dot{q}^\top D(q)\dot{q} = \dot{\tilde{q}}^\top \tilde{D}(\tilde{q})\dot{\tilde{q}}$$

We can then substitute our mapping for velocity $\dot{q} = T^{-1}\dot{\tilde{q}}$

$$(T^{-1}\dot{\tilde{q}})^\top D(q)(T^{-1}\dot{\tilde{q}}) = \dot{\tilde{q}}^\top \tilde{D}(\tilde{q})\dot{\tilde{q}} \implies \dot{\tilde{q}}^\top (T^{-1})^\top D(q)T^{-1}\dot{\tilde{q}} = \dot{\tilde{q}}^\top \tilde{D}(\tilde{q})\dot{\tilde{q}}$$

Multiplying both sides by $(\dot{\tilde{q}}^\top)^{-1}$ and $(\dot{\tilde{q}})^{-1}$ we have

$$(T^{-1})^\top D(q)T^{-1} = \tilde{D}(\tilde{q}) \implies \tilde{D}(\tilde{q}) = T^{-\top}D(q)T^{-1}$$

□

(f) Compute the Coriolis matrix $\tilde{C}(\tilde{q}, \dot{\tilde{q}})$, Gravity vector $\tilde{G}(\tilde{q})$, and input mapping matrix $\tilde{B}(\tilde{q})$ in terms of $C(q, \dot{q})$, $G(q)$, $B(q)$ without recomputing the dynamics.
Hint: Write (4) in the form of (2) and use results from part (e). You can evaluate your result by numerically evaluating what you obtain and comparing with the dynamics you derived in part (d).

Proof. Starting with the original dynamics and substituting our change of coordinate mappings for q, $\dot{q}$, and $\ddot{q}$, we have

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = B(q)u$$
$$D(q)T^{-1}\ddot{\tilde{q}} + C(q, \dot{q})T^{-1}\dot{\tilde{q}} + G(q) = B(q)u$$

To get it into the same form as part e), notice that we can left-multiply both sides by $T^{-\top}$

$$T^{-\top}D(q)T^{-1}\ddot{\tilde{q}} + T^{-\top}C(q, \dot{q})T^{-1}\dot{\tilde{q}} + T^{-\top}G(q) = T^{-\top}B(q)u$$

By pattern matching the coefficients with the original dynamics, we have the following relations

$$\tilde{D}(\tilde{q}) = T^{-\top}D(q)T^{-1}, \quad \tilde{C}(\tilde{q}, \dot{\tilde{q}}) = T^{-1}C(q, \dot{q})T^{-1}, \quad \tilde{G}(\tilde{q}) = T^{-\top}G(q), \quad \tilde{B}(\tilde{q}) = T^{-\top}B(q)$$

After transforming the $D(q), C(q, \dot{q}), G(q)$, and $B(q)$ matrices, we get the exact same values as the matrices built in absolute coordinates in part d). □

2 Code Appendix

```
%% Question 1, part a)

% rel. coords
syms x y q1 q2 q3 xdot ydot q1dot q2dot q3dot real

% generalized coords
q = [x y q1 q2 q3];

% generalized velocity
dq = [xdot ydot q1dot q2dot q3dot];

% constants
m_torso = 10;  % kg
m_leg = 5;     % kg
I_torso = 1;   % kg-m^2
I_leg = 0.5;   % kg-m^2
l_torso = 0.5; % m
l_leg = 1;     % m
g = 9.81;      % m/s^2

% Configuration 1
conf1 = [0.5, sqrt(3)/2, deg2rad(150), deg2rad(120), deg2rad(30)];

% Configuration 2
conf2 = [0.3420, 0.9397, deg2rad(170), deg2rad(20), deg2rad(30)];

p_leg1 = [x; y] + (l_leg/2) * [sin(q1 + q3); cos(q1 + q3)];
p_leg2 = [x; y] + (l_leg/2) * [sin(q2 + q3); cos(q2 + q3)];
p_torso = [x; y] + (l_torso/2) * [sin(q3); cos(q3)];

P = [p_leg1 p_leg2 p_torso];

P1 = double(subs(P, q, conf1));
P2 = double(subs(P, q, conf2));

%% Question 1, part b)

qdot1 = [-0.8049, -0.4430, 0.0938, 0.9150, 0.9298];
qdot2 = [-0.1225, -0.2369, 0.5310, 0.5904, 0.6263];

dP_leg1 = simplify(jacobian(p_leg1, q) * dq');
dP_leg2 = simplify(jacobian(p_leg2, q) * dq');
dP_torso = simplify(jacobian(p_torso, q) * dq');

dP = [dP_leg1 dP_leg2 dP_torso];

dP1 = double(subs(dP, [q, dq], [conf1, qdot1]));
dP2 = double(subs(dP, [q, dq], [conf2, qdot2]));

%% Question 1, part c)

T_leg1 = 0.5 * m_leg * (dP_leg1' * dP_leg1) + 0.5 * I_leg * (q3dot + q1dot)^2;
T_leg2 = 0.5 * m_leg * (dP_leg2' * dP_leg2) + 0.5 * I_leg * (q3dot + q2dot)^2;
T_torso =  0.5 * m_torso * (dP_torso' * dP_torso) + 0.5 * I_torso * (q3dot)^2;

T = T_leg1 + T_leg2 + T_torso;

T1 = double(subs(T, [q, dq], [conf1, qdot1]));
T2 = double(subs(T, [q, dq], [conf2, qdot2]));

%% Question 1, part d)

e2 = [0; 1];

U_leg1 = m_leg * g * (p_leg1' * e2);
```

```
U_leg2 = m_leg * g * (p_leg2' * e2);
U_torso = m_torso * g * (p_torso' * e2);

U = simplify(U_leg1 + U_leg2 + U_torso);

U1 = double(subs(U, [q, dq], [conf1, qdot1]));
U2 = double(subs(U, [q, dq], [conf2, qdot2]));

%% Question 1, part e)

q_act = [q1; q2];

[D, C, G, B] = LagrangianDynamics(T, U, q', dq', q_act);

D1 = double(subs(D, q', conf1'));
D2 = double(subs(D, q', conf2'));

C1 = double(subs(C, [q'; dq'], [conf1'; qdot1']));
C2 = double(subs(C, [q'; dq'], [conf2'; qdot2']));

G1 = double(subs(G, q', conf1.'));
G2 = double(subs(G, q', conf2.'));

B1 = double(subs(B, q', conf1.'));
B2 = double(subs(B, q', conf2.'));

%% Question 2, part c)
T = [1 0 0 0 0;
     0 1 0 0 0;
     0 0 1 0 1;
     0 0 0 -1 -1;
     0 0 0 0 1];

d = [0; 0; -pi; pi; 0];

qt1 = T * conf1' + d;
qtdot1 = T * qdot1';
qt2 = T * conf2' + d;
qtdot2 = T * qdot2';

%% Question 2, part d)

% reinitialize in abs. coords
syms x y th1 th2 th3 xdot ydot th1dot th2dot th3dot real

qt = [x; y; th1; th2; th3];
dqt = [xdot; ydot; th1dot; th2dot; th3dot];

q_invmap = T\qt - T\d;
dq_invmap = T\dqt;

% positions
p_leg1 = [x; y] + (l_leg/2) * [sin(th1+pi); cos(th1+pi)];
p_leg2 = [x; y] + (l_leg/2) * [sin(pi-th2); cos(pi-th2)];
p_torso = [x; y] + (l_torso/2) * [sin(th3); cos(th3)];

Pt = [p_leg1 p_leg2 p_torso];

% velocities
dpt_leg1  = jacobian(p_leg1, qt) * dqt;
dpt_leg2  = jacobian(p_leg2, qt) * dqt;
dpt_torso = jacobian(p_torso, qt) * dqt;

% kinetic energy
T_leg1 = 0.5 * m_leg * (dpt_leg1' * dpt_leg1) + 0.5 * I_leg * th1dot^2;
T_leg2 = 0.5 * m_leg * (dpt_leg2' * dpt_leg2) + 0.5 * I_leg * th2dot^2;
T_torso =  0.5 * m_torso * (dpt_torso' * dpt_torso) + 0.5 * I_torso * th3dot^2;
```

```
Tt = T_leg1 + T_leg2 + T_torso;

% potential energy

e2 = [0; 1];

U_leg1 = m_leg * g * (p_leg1' * e2);
U_leg2 = m_leg * g * (p_leg2' * e2);
U_torso = m_torso * g * (p_torso' * e2);

Ut = simplify(U_leg1 + U_leg2 + U_torso);

q_act_t = [th1 - th3 + pi; pi - th2 - th3];

% lagrangian dynamics
[Dt, Ct, Gt, Bt] = LagrangianDynamics(Tt, Ut, qt, dqt, q_act_t);

Dt1 = double(subs(Dt, qt, qt1));
Dt2 = double(subs(Dt, qt, qt2));

Ct1 = double(subs(Ct, [qt; dqt], [qt1; qtdot1]));
Ct2 = double(subs(Ct, [qt; dqt], [qt2; qtdot2]));
Cqdot_t1 = Ct1 * qtdot1;
Cqdot_t2 = Ct2 * qtdot2;

Gt1 = double(subs(Gt, qt, qt1));
Gt2 = double(subs(Gt, qt, qt2));

Bt1 = double(subs(Bt, qt, qt1));
Bt2 = double(subs(Bt, qt, qt2));

%% Lagrangian Dynamics
% Function to output the dynamics matrices. Uses the lagrangian method.
% Inputs:
%   T: Kinetic Energy scalar
%   U: Potential Energy scalar
%   q: Generalized coordinates
%   dq: Time-derivative of the generalized coordinates
%   q_act: Actuated angles of the system
% Outputs:
%   D: D(q) Inertia matrix
%   C: C(q,dq) Coriollis matrix
%   G: G(q) Gravity matrix
%   B: B(q) Input Matrix?

function [D, C, G, B] = LagrangianDynamics(T, U, q, dq, q_act)

D = simplify( jacobian(jacobian(T,dq), dq) ) ;
for k=1:length(q)
    for j=1:length(q)
        C(k,j) = sym(0) ;
        for i=1:length(q)
            C(k,j) = C(k,j) + 1/2 * ( diff(D(k,j),q(i)) + diff(D(k,i),q(j)) - diff(D(i,j),q(k)) )
                 * dq(i) ;
        end
    end
end
C = simplify(C) ;
G = simplify( jacobian(U,q) )' ;
B = jacobian(q_act, q)' ;

end
```