MEC ENG 132: Dynamical Systems and Feedback Controls

Discussion 3 Supplement

Summer 2025: Professor George Anwar
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Summary and Notes

This set of notes will cover all the topics in homework 3 since we do not have discussion this week as it is the Fourth of July. Namely, we will look into state-space representation in phase-variable form and example MATLAB code; state-space to transfer functions; a review of translational motion; output responses and step responses; poles and zeros as well as nature of each response.

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1. State-Space Representation in Phase-Variable Form

So far what you have done in the previous homework has been to represent mechanical, rotational, and electrical systems in state-space. If you need a refresher on state-space check out Athul's Discussion 2 notes, as they are extremely comprehensive and will be useful especially CCF and the formula to change from state-space to transfer functions. This section will cover how we convert an arbitrary transfer function into state space!

A convenient way to represent transfer functions in state-space is using the Controllable Canonical Form (CCF) and pattern match the coefficients of the transfer function into the A, B, C, and D matrices. You might be wondering: *What's the difference? Why do we do this?* Well actually, it turns out that phase-variable form is a specific type of CCF where the state variables are defined as successive derivatives of the output. This is particularly useful when dealing with systems where the output and its derivatives are easily measurable or controllable (more to come on this later!!).

Like before, we need to choose a set of state variables, called **phase variables**, to define our system, where each subsequent state variable is defined to be the derivative of the previous state variable. In general, we start with an n-th-order linear constant coefficient differential equation (LCCDE)

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u \Rightarrow \sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = b_0 u \quad \text{where } a_n \triangleq 1$$

Then, we can define a series of state-variables (phase variables) and then differentiate both sides

$$x_1 = y x_2 = \dot{y} x_3 = \ddot{y} \quad \vdots x_n = \frac{d^{n-1}y}{dt^{n-1}} \Rightarrow \dot{x}_1 = \dot{y} = x_2 \dot{x}_2 = \ddot{y} = x_3 \dot{x}_3 = \dddot{y} = x_4 \quad \vdots \dot{x}_{n-1} = x_n \dot{x}_n = \frac{d^n y}{dt^n} = -a_0 x_1 - a_1 x_2 \cdots -$$

We can now make a **sparse matrix representation** of this relationship, A, to describe the first part of the State Evolution Equation (SEE) $\dot{x}(t) = Ax(t) + Bu(t)$. Here $u(t)$ is the constant term that is left over. $B \triangleq \begin{bmatrix} 0 & \cdots & 0 & b_0 \end{bmatrix}^\top$. By definition, also, $C \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$, and $D \triangleq [0]$. Vector-matrix form is below:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u$$

We can write the output equation (OE) by recognizing that $y(t)$ is the output of the LCCDE

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

1.1. Procedure and Examples

The following steps will help you solve these types of problems. To convert a transfer function into state equations in **phase-variable** form, we do the following:

1. Convert the transfer function to a differential equation by cross-multiplying
2. Take the inverse Laplace Transform, assuming zero initial conditions $\dot{x}(0) = 0, x(0) = 0$.
3. Represent the differential equation in state-space in phase variable form.

For an example, consider the following case where the numerator of the transfer function is constant:

Example 1.1 Find the state-space representation in phase-variable form for the transfer unction shown below.

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

Solution: We cross multiply the transfer function to get

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

The corresponding differential equation is found by taking the ILT assuming zero initial conditions:

$$\mathcal{L}^{-1}\{(s^3 + 9s^2 + 26s + 24)C(s)\} = \mathcal{L}^{-1}\{24R(s)\}$$
$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

Next, we select our state variables as successive derivatives of c:

$$x_1 = c$$
$$x_2 = \dot{c}$$
$$x_3 = \ddot{c}$$

Then, since the SEE requires the state-derivative vector, we need to differentiate the states to get $\dot{x}_1$ and $\dot{x}_2$ and we just substitute in our state variables into the ODE and rearrange to isolate for $\dddot{c} = \dot{x}_3$.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

In vector-matrix form, we have the following SEE

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

Recall that our transfer function is defined as the output over the input so $C(s)/R(s)$ so our output is c. Look at the state variables and you will see that $x_1 = c$. Then, write our OE as

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

1.2. What about a polynomial in the numerator?

If the transfer function has a polynomial in terms of s that is lower order than the polynomial of the denominator, we handle the numerator and denominator separately—we separate the transfer function by cascading it into 2 separate ones.

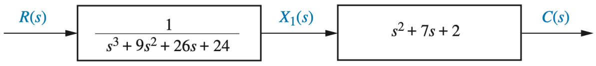

Figure 1: Cascaded System Example 1.2

An illustrative example is shown below and is modeled in Figure 1:

Example 1.2 Find the state-space representation in phase-variable form for the transfer unction shown below.

$$\frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{(s^3 + 9s^2 + 26s + 24)}$$

Solution: First, we write the system into two cascaded blocks. The first block will contain the denominator and the second block contains the numerator as shown above in Figure 1.

We will use the 1st block and do the exact same as the previous example—finding the state equations. From **Example 1.1**, the SEE was found to be:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 24\end{bmatrix} r$$

The only change that needs to be made here is that we had 24 in the numerator of the transfer function in the previous example so that's why our B vector contains 24. In this example, we have 1 in the numerator so our corresponding SEE would be

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} r$$

Now, we use. The second block will be used strictly for our output equation. Using the same state variables as **Example 1.1**:

$$\begin{aligned}x_1 &= c\\ x_2 &= \dot{c}\\ x_3 &= \ddot{c}\end{aligned}$$

We take the Laplace transform of our numerator (the second block) and then replace the c's with x_1, x_2, x_3. Thus, we get y to be

$$y = \begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

2. Time Responses and System Properties

2.1. Poles and Zeros

The poles and zeros of a system can help us gain a qualitative understanding of stability, oscillatory dynamics, etc. of a system without having to explicitly solve for the time-domain system output. They require analyzing the system in frequency by its transfer function, which we consider to be rational. Consider a system given by

$$G(s) = \frac{N(s)}{D(s)}$$

Then, the poles of G are the values of s that make $G(s) \to \infty$, or those that make $D(s) = 0$. In case $N(s)$and $D(s)$ have a common root, meaning that G(s) takes on a 0/0 form, that value of s is still considered a pole. If zeros cancel out poles i.e. a root from the numerator cancels a root from the denominator, it is called *pole-zero cancellation*. This will become important later.

The quick way for us to find **poles** of a system are just to factor the denominator and find the zeros of the denominator. The way to find the **zeros** are to factor the numerator and find the zeros of the numerator. Consider the following example:

Example 1.3 Consider the following transfer function.

$$G(s) = \frac{(s+2)(s+5)}{(s+5)(s+3)}$$

Here $s = -5$ and $s = -3$ are the poles; $s = -2$ and $s = -5$ are the zeros of the system. We graph these in the s-domain. Zeros are represented with ◯ and poles with X on the graph.

Note, in this example the transfer function had real roots. In this case complex roots exist, you would graph the $\sigma \pm j\omega$ where σ is the x or real axis, and $j\omega/y$ is the imaginary axis.

2.2. First-Order Systems

The generic way we define a first-order system with no zeros is

$$G(s) = \frac{a}{s+a}$$

If we give this system a step input, we get the following

$$C(s) = R(s)G(s) = \frac{s}{s(s+a)} \Rightarrow c(t) = c_F(t) + c_N(t) = 1 - e^{-at}$$

We can analyze the time-domain behaviour of this system by a few parameters: rise time, settling time, peak time, and overshoot are the main ones. Below, we will look more into these definitions.

2.2.1. Time Constant

The time constant is defined in the above equation as $\frac{1}{a}$. It is the time for e^{-at} to decay to 37% of its initial value. Alternatively, the time it takes for the step response to reach 63% of its final value. Graphically, the time constant is interpreted as $1/$initial slope.

2.2.2. Exponential Frequency

The exponential frequency is defined as a or the reciprocal of the time constant. The initial rate of change of the exponential at $t = 0$, since the derivative of e^{-at} is $-a$ when $t = 0$. Since the pole of the TF is at $-a$, the farther the pole is from the imaginary axis, the faster the transient response.

2.2.3. Rise Time

The rise time is defined as T_r and is the time for the waveform to go from 0.1 to 0.9 of its final value. The different in time between $c(t) = 0.9$ and $c(t) = 0.1$. Mathematically, it is defined as

$$T_r \triangleq \frac{2.2}{a}$$

2.2.4. 2% Settling Time

The 2% settling time is defined as T_s and is the time for the response to reach, and stay within, 2% (arbitrary) of its final value. The time when $c(t) = 0.98$. Mathematically, it is defined as

$$T_s \triangleq \frac{4}{a}$$

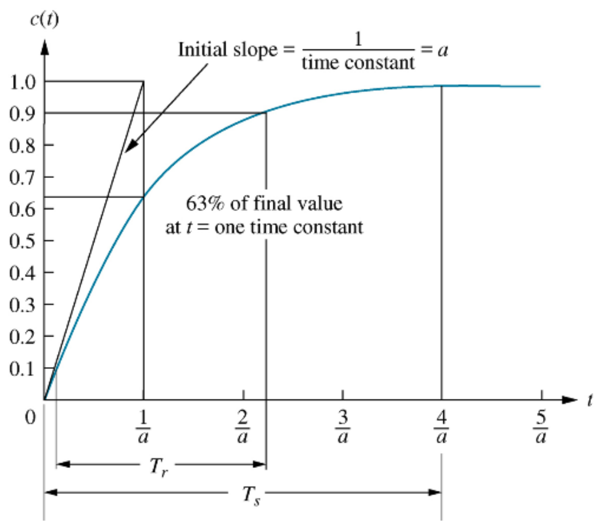

Figure 2: 1st Order System Response to a Unit Step

3. Second-Order Systems

In a second-order system, of the general form

$$G(s) = \frac{b}{s^2 + as + b}$$

There are a few different types of possible responses due to the presence of two poles. Similarly to first-order systems, we are interested in how the system will behave in response to a unit step input. Thus, we multiply the transfer function by a unit step $R(s) = \frac{1}{s}$ and analyze that system.

$$C(s) = R(s) \cdot G(s) = \frac{b}{s(s^2 + as + b)}$$

Let's now look at the different types of responses for second-order systems.

3.1. Overdamped

If both system poles are real (the poles that are NOT from the unit step), negative, and distinct, the system is called **overdamped**. Let the poles be defined by $s - (\sigma + j\omega)$. In this case $\sigma > 0$ (negative real part with the negation up front) and $\omega = 0$ so that the poles are of the form $s + \sigma$. Then the system frequency response can be decomposed into partial fractions as

$$C(s) = R(s)G(s) = \frac{1}{s}G(s) = \frac{1}{s} + \frac{k_1}{s + \sigma_1} + \frac{k_2}{s + \sigma_2}$$

where the last two terms together are called the system's **natural response**, $C_N(s)$ and the pole $s = 0$ is called the system's **forced response**. We can apply the Inverse Laplace Transform to get

$$c_N(t) = k_1 + k_2 e^{-\sigma_1 t} + k_3 e^{-\sigma_2 t}$$

where as $t \to \infty$, we see that $c(t) \to 1$. In summary, there are 2 real system poles at σ_1, σ_2. The time constants are $-\sigma_1, -\sigma_2$.

3.2. Underdamped

Like above, a pole at the origin comes from the unit step input and contributes to the **forced response** of the system. If the system poles are in the left half-plane with non-zero imaginary parts (complex poles), the system is called **underdamped**. For example, say $G(s) = \frac{9}{s^2+2s+9}$. The system poles are at $-1 \pm 2\sqrt{2}j$. Then, we find the time-domain output by taking the Inverse Laplace Transform:

$$C(t) = \mathcal{L}^{-1}(R(s)G(s)) = 1 - e^{-t}\left(\cos(2\sqrt{2}t) + \frac{1}{2\sqrt{2}}\sin 2\sqrt{2}t\right)$$

Here, the real part gives the decay factor of e^{-t} and the imaginary part shows the oscillation through the sine and cosine terms. In summary, there are 2 complex system poles at $\sigma_d \pm j\omega_d$. The natural response is a damped sinusoid with an exponential envelope $c(t) = k_1 e^{-\sigma_d t}\cos(\omega_d t - \phi)$ with a time constant of σ_d and frequency ω_d in rad/s.

3.2.1. Underdamped Response Characteristics

We define the system's **transient response** as the exponentially decaying amplitude generated by the real part of the system pole times a sinusoidal waveform generated by the imaginary part of the system pole.

The **damped frequency of oscillation,** ω_d is the imaginary part of the system poles. The **steady state response** is generated by the input pole located at the origin. And in general, an **underdamped response** approaches a steady-state value via a transient response that is a damped oscillation.

3.3. Undamped

This case is characterized by two distinct poles on the imaginary axis. For example, let $G(s) = \frac{9}{s^2+9}$. Then, we find the time-domain output by taking the Inverse Laplace Transform:

$$c(t) = \mathcal{L}^{-1}(R(s)G(s)) = 1 - \cos(3t)$$

Here, there is no growth or decay term, and the system just oscillates. In summary, there are 2 imaginary system poles at $\pm j\omega_1$ with a natural response of an undamped sinusoid taking the form $c(t) = A\cos(\omega_1 t - \phi)$ with frequency ω_1 in rad/s.

3.4. Critically Damped

In this case, there are 2 poles that are real and repeated, in the left half-plane (i.e. both are negative). Consider the case $G(s) = \frac{9}{s^2+6s+9}$. Both the poles are at -3, so the solution comes out to

$$c(t) = 1 - 3te^{-3t} - e^{-3t}$$

This is a special case of the **overdamped system**. In summary, there are 2 of the same real system poles with a natural response taking the form of a summation of an exponential and a product of a time and an exponential: $c(t) = k_1 e^{-\sigma_1 t} + k_2 t e^{-\sigma_1 t}$ with a time constant of σ_1. Note that this is the fastest response without an overshoot.

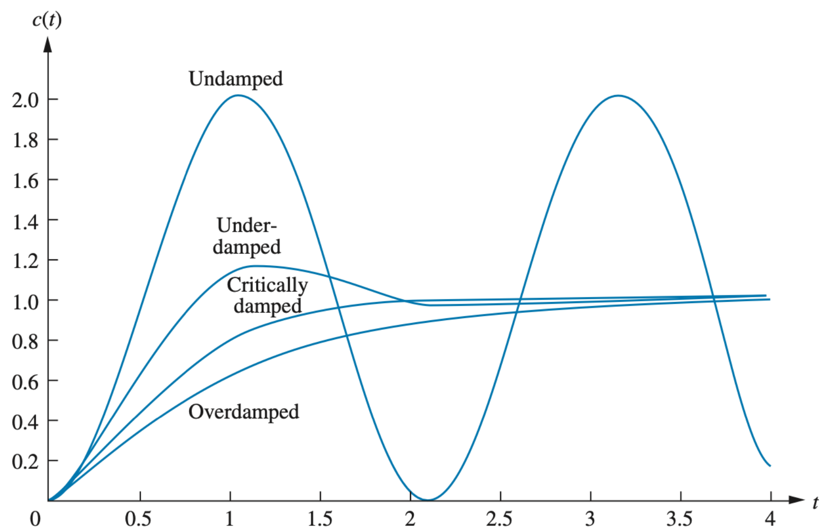

Figure 3: Step Responses for 2nd Order System Damping Cases

3.5. Some Coding Tips

The following functions in MATLAB will be of use. `tf(Num, Den)`, `step(TF)`, `info = stepinfo(TF)`.