7/25 ME132 Discussion!

Stability of Natural Response

An unstable system can't be designed for a specific transient response or steady-state error requirement. If our system is LTI

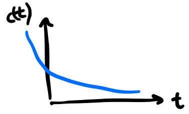
a. **Stable**: natural response approaches 0 as $t \to \infty$
b. **Unstable**: natural response grows unbounded as $t \to \infty$
c. **Marginally Stable**: natural response oscillates (without converging) as $t \to \infty$ (some BI yield unbounded outputs)

the total response is defined as $c(t) = c_{forced}(t) + c_{natural}(t)$

↳ stability ⇒ only $c_{forced}(t)$ remains!

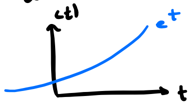
i.e. $G(s) = \dfrac{1}{s+2} \xrightarrow{\mathcal{L}^{-1}} e^{-2t}u(t)$

$\lim_{t\to\infty} \underbrace{e^{-2t}u(t)}_{\text{natural response}} = 0$ ✓ stable!



i.e. $G(s) = \dfrac{1}{s-1} \xrightarrow{\mathcal{L}^{-1}} e^{t}u(t)$

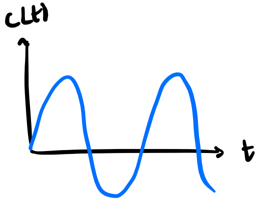
$\lim_{t\to\infty} e^{t}u(t) = \infty$ X unstable



i.e. $G(s) = \dfrac{1}{s^2+4} \xrightarrow{\mathcal{L}^{-1}} \sin(2t)u(t)$

$\lim_{t\to\infty} \sin(2t)u(t)$ DNE!

X marginally stable



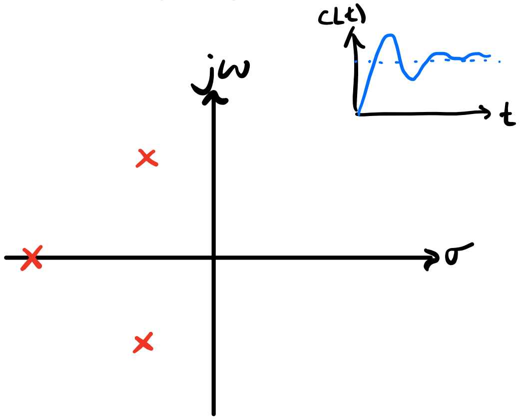
Stability with PZ plot

i.e. $G(s) = \frac{1}{s(s+1)} = \frac{1}{s^2+s}$

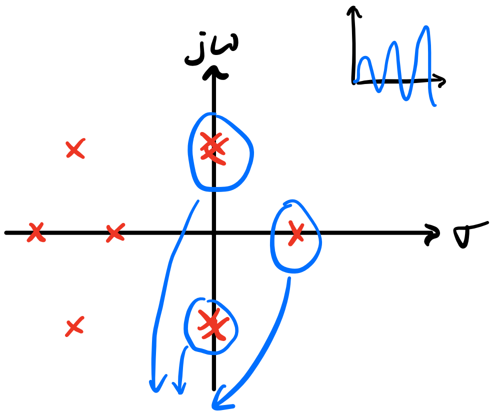
missing constant term

Any missing power of s in the denominator of the TF implies an unstable system! [missing a power generally implies at least one pole at the origin!

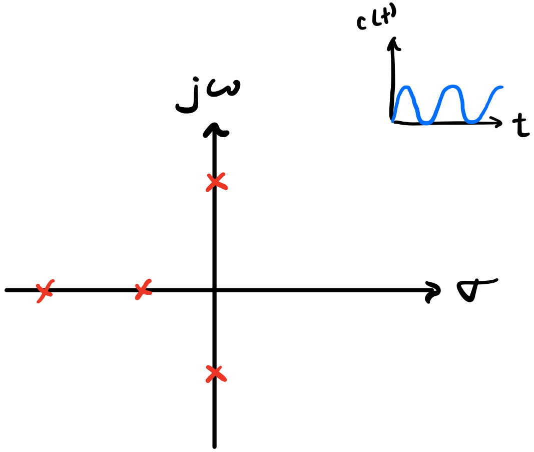
- if poles are only in LHP ⇒ **stable**
- if **ANY** poles are in the RHP ⇒ **unstable**
- if poles are on the $j\omega$ axis (no repeated poles on $j\omega$ allowed!) and in LHP ⇒ **marginally stable**



STABLE



UNSTABLE



MARGINALLY STABLE

BIBO Stability (Formalization of PZ-plot stability)

just FYI. we won't test you on this...

For an LTI system with input u(t) and output y(t), we have the following statement

convolution...

comes from $y(t) = (h * u)(t) = \int_0^t h(\tau) u(t-\tau) d\tau$ ... we bound it by $|y(t)| \leq \left(\sup_\tau |u(\tau)| \right) \int_0^\infty |h(\tau)| d\tau$

BIBO stable ⇔ every bounded input gives a bounded output

Mathematically, we can check this using the impulse response h(t).

$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \iff$ BIBO stable (DT)

$$\int_{-\infty}^{\infty} |h(t)| < \infty \iff \text{BIBO Stable}$$

which implies that $h \in \mathcal{L}^1$ or h(t) is absolutely integrable.

$h \in \ell^1$ (DT)

Internal Stability

SEE: $\dot{x} = Ax + Bu$

$x = Cx + Du$

When a system is LTI and in state-space, we can use the **A** matrix to check stability. The following are the conditions

a. **Internally Stable:** $Re(\lambda_i) < 0$ for all eigenvalues

↳ Recall eigenvalues are found as follows: $\det(A - \lambda I) = 0$ — the eigenvalues are the poles!

b. **Marginally Stable: ALL** eigenvalues satisfy $Re(\lambda_i) \leq 0$ and any $Re(\lambda_i) = 0$ are simple (no repeated poles on $j\omega$ axis **AND** no Jordan blocks larger than 1)

Jordan Canonical Form (JCF)

↳ $J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}_{k \times k}$, a 1×1 Jordan Block is $[\lambda]$

c. **Unstable:** At least one eigenvalue with $Re(\lambda_i) > 0$

Stability Example

$$A = \begin{bmatrix} 2+3j & 0 & 0 \\ 0 & 2-3j & 0 \\ 0 & 0 & -1+4j \end{bmatrix}$$

We need to check the eigenvalues of A!

↳ since it's upper triangular, λ_i are just the diagonals

$\Rightarrow \lambda_1 = 2+3j, \quad \lambda_2 = 2-3j, \quad \lambda_3 = -1+4j$

For stability, $Re(\lambda_i) < 0$, since $Re(\lambda_1), Re(\lambda_2) > 0$

$\Rightarrow$ the system is unstable!

BIBO example: consider a transfer function $G(s) = \frac{1}{s+2}$ ← you can tell its BIBO stable from the poles already...

$G(s) = \frac{1}{s+2} \xrightarrow{\mathcal{L}^{-1}} h(t) = e^{-2t}u(t)$

↳ since

$\int_{-\infty}^{\infty} |h(t)| dt = \int_0^{\infty} e^{-2t} dt$

$= \left[-\frac{1}{2}e^{-2t}\right]_0^{\infty} = \frac{1}{2}$ ✓ FINITE $\Rightarrow$ BIBO stable

e.x. BIBO ≠ internal stability

$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = [1 \; 0]$

$D = [0]$

$\xrightarrow{ss2tf} G(s) = \frac{s-1}{s^2-1} = \frac{\cancel{s-1}}{\cancel{(s-1)}(s+1)} = \frac{1}{s+1}$

this is BIBO stable since

$\int_0^{\infty} e^{-t} dt = 1$ finite!

BUT $Re(\lambda_i)$ for A include a negative $\Rightarrow$ INTERNALLY UNSTABE

↳ This is due to POLE-ZERO CANCELLATION

Routh-Hurwitz Criterion

Tabular Method can be used to determine the stability when the roots of a higher order characteristic polynomial is hard to obtain.

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & p_{n-2} & p_{n-4} & p_{n-6} & \cdots & \\
s^{n-3} & q_{n-3} & q_{n-5} & q_{n-7} & \cdots & \\
\vdots & \vdots & \vdots & \vdots & & \\
s^1 & y_2 & y_0 & & & \\
s^0 & y_0 & & & &
\end{array}
$$

$$
\begin{array}{c|cccc}
\cdot & a & b & x & \cdot \\
\cdot & c & d & y & \cdot \\
\cdot & \dfrac{bc-ad}{c} & \dfrac{xc-ay}{c} & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Routh Table

closed-loop TF required!

$$G(s) = \frac{s+1}{s^4 + 3s^3 + 6s^2 + 9s + 12}$$

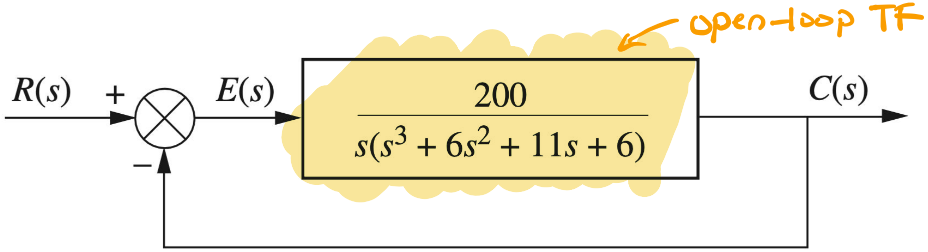
← characteristic polynomial

this is what we use to construct Routh table

# of sign changes in the first column determine the # of RHP poles.

* NOTE: There are special cases (zero only in 1st column & row of zeros) ⟹ look at epsilon procedure; reciprocal roots procedure

Routh-Hurwitz Example



We need closed loop TF ⇒ apply Black's Formula

$$T(s) = \frac{\dfrac{200}{s^4+6s^3+11s^2+6s}}{1+\dfrac{200}{s^4+6s^3+11s^2+6s}}$$

$$= \frac{\dfrac{200}{\cancel{s^4+6s^3+11s^2+6s}}}{\dfrac{s^4+6s^3+11s^2+6s+200}{\cancel{s^4+6s^3+11s^2+6s}}}$$

$$= \frac{200}{s^4+6s^3+11s^2+6s+200}$$

characteristic polynomial

$$\begin{array}{c|ccc}
s^4 & 1 & 11 & 200 \\
s^3 & \cancel{6}\ 1 & \cancel{6}\ 1 & 0 \\
s^2 & \cancel{10}\ 1 & \cancel{200}\ 20 & 0 \\
s^1 & -19 & 0 & 0 \\
s^0 & 20 & 0 & 0
\end{array}$$

$$\frac{-\begin{vmatrix} 1 & 200 \\ 1 & 0 \end{vmatrix}}{1} \qquad \frac{-\begin{vmatrix} 1 & 11 \\ 1 & 1 \end{vmatrix}}{1}$$

$$\frac{-\begin{vmatrix} 1 & 1 \\ 1 & 20 \end{vmatrix}}{1} \qquad \frac{-\begin{vmatrix} 1 & 20 \\ -19 & 0 \end{vmatrix}}{-19}$$

From

$$\begin{array}{cc}
s^2 & 1 \\
s^1 & -19 \\
s^0 & 20
\end{array}$$

1 sgn change

2 sgn change

2 sign change in 1st col ⇒ 2 RHP poles

⇒ no $j\omega$ poles since no row of zeros in Table

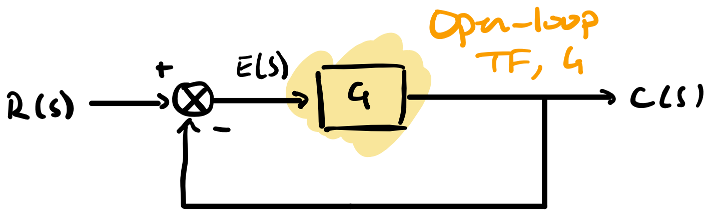
⇒ since degree 4 polynomial, we must have 4 poles.

⇒ 2 poles LHP as well!

Steady-State Errors

→ defined as the difference b/w the input and output for a test input

How does my CL TF behave given certain inputs? Does it eventually converge/follow the form of the input signal?



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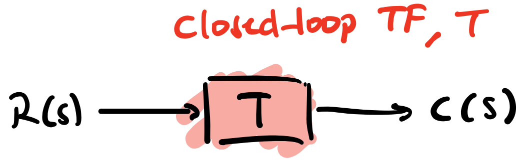


if $E(s) = R(s) - C(s)$, $C(s) = E(s)\,G(s)$

$\Rightarrow E(s) = \dfrac{R(s)}{1+G(s)}$

Apply FVT: $e_{ss} = \lim_{s\to 0} \dfrac{s\,R(s)}{1+G(s)}$

$T \triangleq \dfrac{G}{1+GH}$

SS error in terms of $T(s)$

$E(s) = R(s) - C(s)$, $C(s) = R(s)\,T(s)$

$\Rightarrow E(s) = R(s) - R(s)\,T(s) = R(s)\left[1 - T(s)\right]$

Apply FVT: $e_{ss} = \lim_{s\to 0} s\,R(s)\left[1 - T(s)\right]$

Steady-State Errors System Types

↳ only for unity negative feedback: (For HW, look at 7.5)

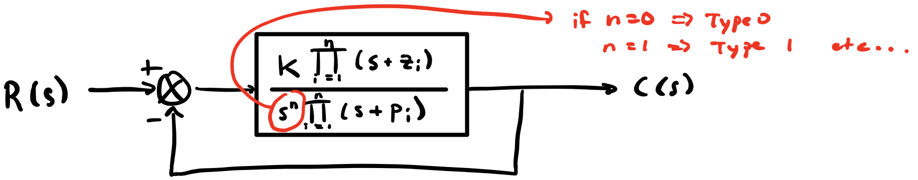
Input	Steady-state error formula	Type 0: Static error constant	Type 0: Error	Type 1: Static error constant	Type 1: Error	Type 2: Static error constant	Type 2: Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Static error constants:

$K_p = \lim_{s\to 0} G(s)$, $\quad e_{step}(\infty) = \dfrac{1}{1 + \lim_{s\to 0} G(s)}$

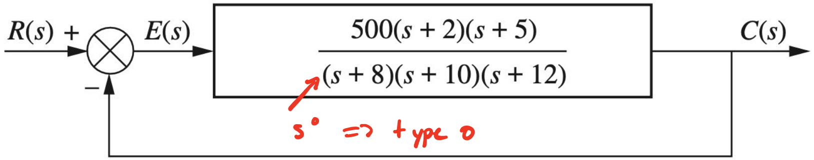
$K_v = \lim_{s\to 0} sG(s)$, $\quad e_{ramp}(\infty) = \dfrac{1}{\lim_{s\to 0} sG(s)}$

$K_a = \lim_{s\to 0} s^2G(s)$, $\quad e_{parabolic}(\infty) = \dfrac{1}{\lim_{s\to 0} s^2G(s)}$



SS Errors Example

PROBLEM: For each system of Figure 7.7, evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs.



$s^0 \Rightarrow$ type 0

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

① Check if system is BIBO stable

all poles in LHP ⇒ stable ✓

② Static error constants

$$K_p = \lim_{s\to 0} G(s) = \frac{500\cdot 2\cdot 5}{8\cdot 10\cdot 12} = 5.208$$

$$K_v = \lim_{s\to 0} s\,G(s) = 0$$

$$K_a = \lim_{s\to 0} s^2\,G(s) = 0$$

③ Expected Errors

$$e_{step} = \frac{1}{1+K_p} = 0.161$$

$$e_{ramp} = \frac{1}{K_v} = \infty$$

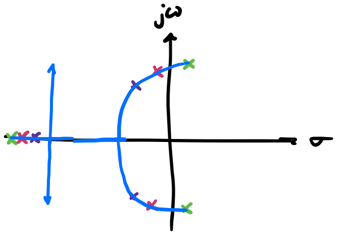
$$e_{parabola} = \frac{1}{K_a} = \infty$$

Root Locus

$$G(s) = \frac{s^2+s+1}{s^3+4s^2+Ks+1}$$

unknown gain or parameter that affects poles!

$$s^3+4s^2+0s+1=0$$
$$s^3+4s^2+1s+1=0$$
$$s^3+4s^2+2s+1=0$$
$$\vdots \quad \leftarrow \text{as } K \to \infty$$



Helps with design and checking the effects of variations.

- *What value of K should I choose to meet my system performance requirements?*
- *What's the effect of a variation of K on my system?*

Accomplishes this by plotting the movement of poles!

↪ The path of closed loop poles as the gain, K, is varied!

$$K = \frac{\prod \text{pole length}}{\prod \text{zero length}}$$

Other info: to approx. higher order system to 2nd order we define DOMINANT POLES!

↳ select 2 dominant poles (CL, $|\sigma_i| \geq 5\sigma_1$, $|\sigma_i| \geq 5\sigma_2$)

any other pole

Root Locus

we need the characteristic eqn in the form $1 + K\underbrace{G(s)}_{\text{OL TF}} = 0$ (ensures that we include feedback!)

i.e. CL TF: $\longrightarrow \boxed{T} \longrightarrow$

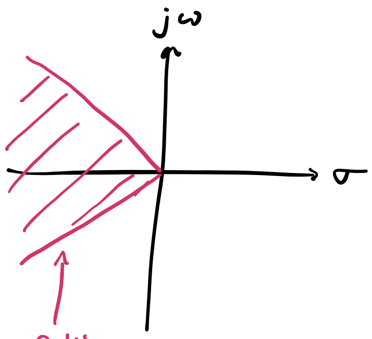
$$T = \frac{s^2+s+1}{s^3+4s^2+Ks+1} \Rightarrow s^3+4s^2+Ks+1=0 \Rightarrow (s^3+4s^2+1)+Ks=0 \Rightarrow 1 + K\underbrace{\frac{s}{s^3+4s^2+1}}_{G(s)} = 0$$

divide by $(\cdot)$

Given damping ratio, time for exponential decay to half, natural frequency requirements:

Damping Ratio

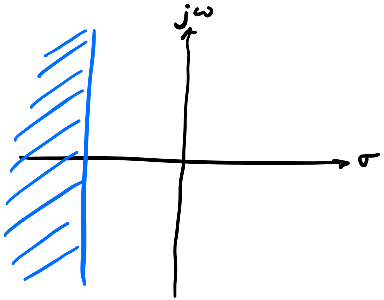
↳ if $\zeta >$ some value ...



poles must lie here!

Time for exponential delay

↳ mode decays to some value in x amt. of time.

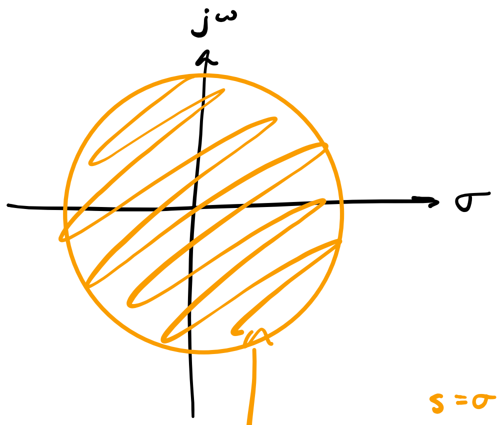


pole lies here

(poles further left ⇒ the faster it decays)

Natural Frequency

↳ mode has to be $< \omega_n$ rad/s



poles lie here

$s = \sigma \pm j\omega_d$

$\omega_n = \sqrt{\sigma^2 + \omega_d^2}$

Root Locus Drawing Rules (BASIC)

→ OL TF

1. # Branches in root locus = # of open-loop poles; there are *n* lines where *n* is the higher degree of {N(s), D(s)}, TF = N(s)/D(s)

i.e. $G(s) = \dfrac{s+2}{(s+3)(s-4)}$

⇒ 2 OL poles so 2 loci/branches!

i.e. $\dfrac{s^2+2s+1}{s^4+7s^3-3s^2+s+1}$ $= N(s)$, $= D(s)$

⇒ $\deg\{D(s)\} > \deg\{N(s)\}$ ⇒ 4 loci

if the degree are the same then use either or for # of branches.

2. As the gain, *K*, increases from 0 → ∞, the roots move from open-loop poles to open-loop zeros WHY?

ANS: $1 + K\dfrac{N(s)}{D(s)} = 0$

$D(s) + K N(s) = 0$

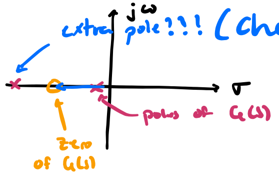
poles of G(s) are when D(s) = 0

zeros of G(s) are when N(s) = 0

when K = 0, what happens?

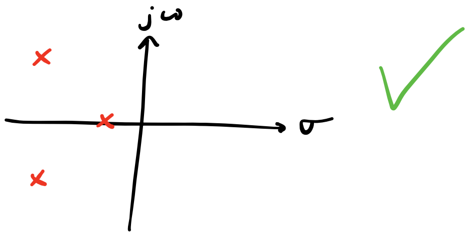
↳ $D(s) + \cancel{0 \cdot N(s)} = 0$ ⇒ $D(s) = 0$

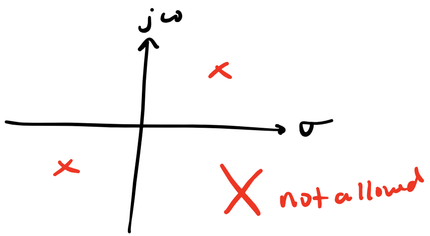
as $K \to \infty$, $D(s) + \infty N(s) = 0$ ⇒ N(s) dominates

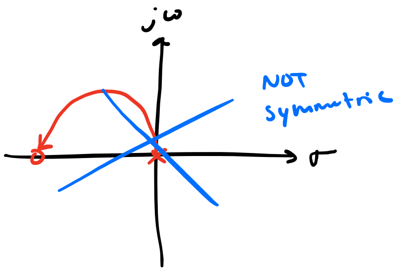


(check rule 7)

3. Root locus is symmetrical about the real axis (when roots are complex, they occur in conjugate pairs)



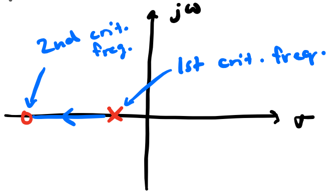


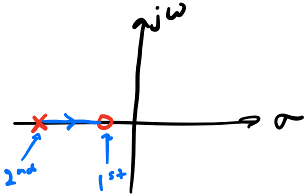


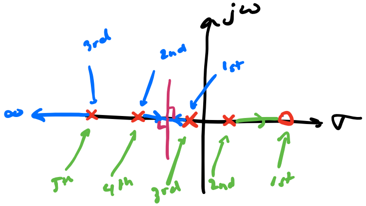
Root Locus Drawing Rules (BASIC)

DOES NOT TELL YOU DIRECTION!

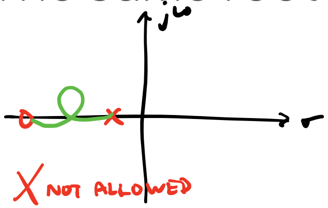
4. The part of the real axis (to the left of the odd number of poles and zeros) are part of the loci. [every other space on the σ-axis b/w critical freq. is a part of the root locus]

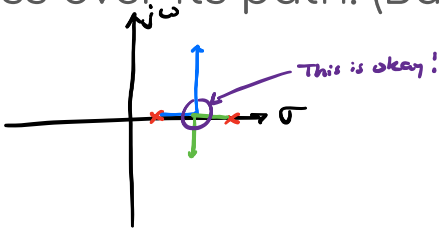




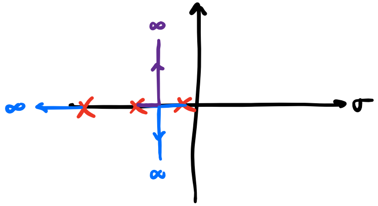


5. The same root cannot cross over its path. (But two roots can).

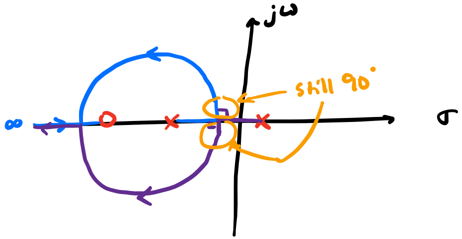




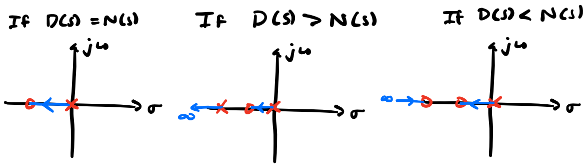
6. Lines leave and enter the real axis at 90°



BUT



Root Locus Drawing Rules (BASIC)



7. Not enough poles or zeros to make a pair? the extra lines go to/come from ∞.

8. Lines go to infinity along asymptotes:

(where asymptotes meet)

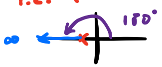
a. real axis intercept (centroid) σ_a : $\sigma_a = \dfrac{\sum \text{value of the finite poles} - \sum \text{value of the finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$

b. angle of asymptote θ_a: $\theta_a = \dfrac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}$

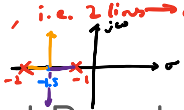
$k = 0, 1, 2, \ldots, (\#\text{finite poles} - \#\text{finite zeros} - 1)$

(basically 1 less than the # of unmatched pz-pairs)

i.e. 1 line → ∞ ⇒ π



, i.e. 2 lines → ∞ ⇒ $\frac{\pi}{2}$, $\frac{3\pi}{2}$

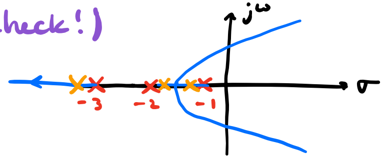


$\sigma_a = \frac{(-2-1)-0}{2} = -1.5$

9. Break-in and Break-out points: roots of D'N-DN'=0. (only on real axis)

↳ where the loci splits off the real axis. This is different than σ_a

10. If there are **at least 2 lines** to infinity, then the sum of all roots is constant for all *K*. (just a sanity check!)



Σ roots = −6

X are the moving CL poles

Σ CL poles = −6 ← still must be −6!

Root Locus Drawing Rules (DETAILED)

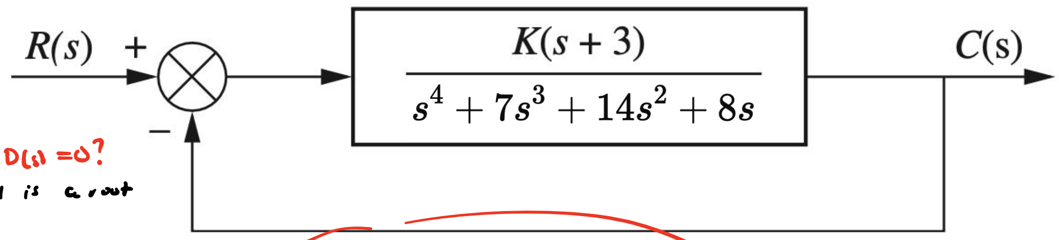
11. For $j\omega$-crossings, use Routh-Hurwitz Criterion
    - Forcing a row of zeros in the Routh Table yields the gain
    - Going one row back to the even polynomial equation and solving for the roots yields the frequency at the $j\omega$ crossing.

OR

- at the $j\omega$ crossing, the sum of angles from the finite open-loop poles and zeros must add to $(2k+1)180^\circ$. Search the $j\omega$-axis for a point that meets this angle condition.

12. angle of departure and arrivals (only for complex poles) (you will not use this really...)

Root Locus Example



① Factor denominator

↳ MATLAB: roots([1, 7, 14, 8 , 0])

↳ Synthetic division + remainder theorem!

What value of s will make D(s) = 0?

−1: 1 − 7 + 14 − 8 = 0 ⇒ −1 is a root

−1	1	7	14	8
		−1	−6	−8
	1	6	8	0

stop when 0.

$\Rightarrow (s+1)(s^3+6s^2+8s)$

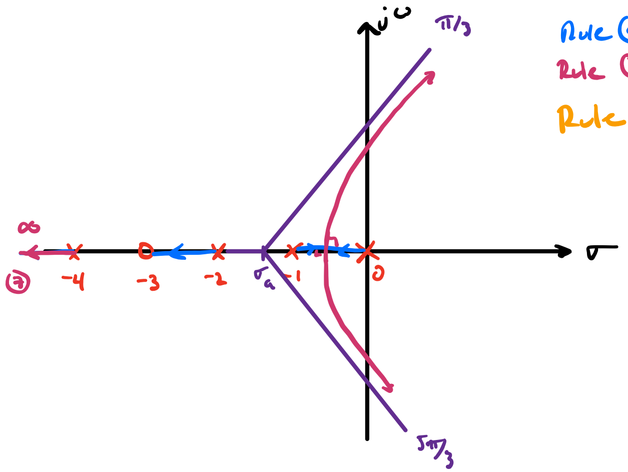
$s(s+1)(s^2+6s+8)$

$s(s+1)(s+2)(s+4)$ ✓

POLES: s = 0, s = −1, s = −2, s = −4 ⟹ ① 4 branches

ZEROS: s + 3 ⇒ s = −3

② PLOT POLES + ZEROS



Rule ④

Rule ⑦

Rule ⑨

Rule ⑧ ⟹ $\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$

$\theta_a = \frac{(2k+1)\pi}{\#fp - \#fz}$, k = #fp − #fz − 1 = 4 − 1 − 1 = 2

k=0: $\theta_a = \frac{\pi}{3} = 60°$

k=1: $\theta_a = \pi = 180°$

k=2: $\theta_a = \frac{5\pi}{3} = 300°$

Rule ⑨ : intuitively b/w 0 and 1

Use $\frac{dD}{ds} N(s) - \frac{dN}{ds} D(s) = 0$

MATLAB tools

`rlocus(sys)`: plots the root locus of the TF

`roots(polynomial)`: returns roots of polynomial

`G = zpk(zeros, poles, gain)`: create zero-pole gain, can do `rlocus(G)` after.

For question 10, some helpful functions:

`readtable(filename)`: creates table from a spreadsheet

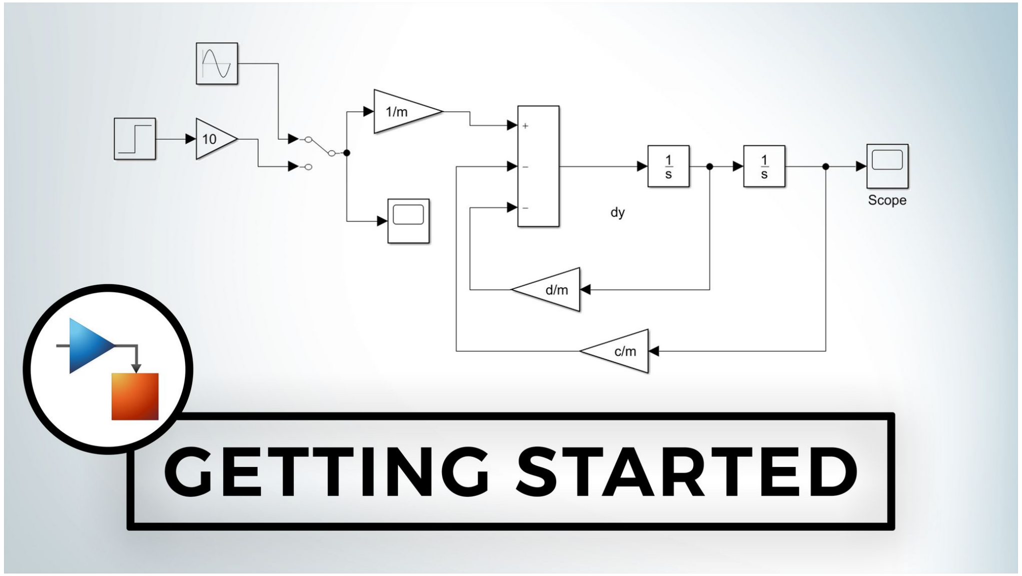
`findpeaks(y)`: returns vector of local maxima of a input signal `y`

`impulse(TF)`: plots the impulse response

Simulink tools

You can look at Larry's Simulink Supplement for more information:

On bCourses: Files > Discussion Summer 2025 > Larry's Discussion > Simulink Supplement. Also found by clicking here!



office hours!