ME 132 Dynamic Systems and Feedback
Summer 2025 George Anwar

Final

Instructions

- The exam is open book, open notes, and open web. However, you may not consult or communicate with other people (besides course staff). **Use of ChatGPT or any other LLM is heavily advised against—if you make a mistake on two subparts of a question that matches the mistake made by your LLM of choice, you will receive a 0 on the ENTIRE question.**
- **DISCLAIMER**: If you are ***SUSPECTED*** of using AI, you will receive an oral exam on randomly selected problems. If you cannot produce any coherent or correct answer, you will receive a 0 on the final exam, and receive disciplinary action.
- When you start, **the first thing you should do** is **check that you have all 34 pages and all 8 questions** + the extra credit questions. This exam is worth 80% of the final exam grade.
- You have **35** hours. (If you are in the Disabled Students' Program and have an allowance of 150% or 200% time, that comes to **52.5 hours** or **70 hours**, respectively.)
- The total number of points is 255. There are 3 short answers worth a total of 50 points; 5 long answers worth a total of 204 points; and the extra credit problem worth 30 additional points.
- There are extra credit questions at the end. Please do not attempt them until you have finished the rest of the exam! The points associated with this problem will **NOT** be added to the final exam itself; they are a different category.
- Please write neatly and legibly, because if *we can't read it, we can't evaluate it.* **Box** your final answer.
- In all of the questions, **show your work**, not just the final answer. Unless we explicitly state otherwise, you may expect full credit only if you explain your work succinctly, but clearly and convincingly.
- If you are asked to provide a "sketch," you are allowed a computer plot or *hand-drawn* sketch. Ensure it is well-labeled to indicate all the salient features.

Deliverables

Submit a PDF of your exam to the Gradescope assignment entitled "{Your Name} Final Exam". You must neatly handwrite your solutions (submit PDF format, not .doc/.docx format). Do **not** use a text editing software.

- On the next page, please the following statement and sign your signature next to it for **1 point** (Mac Preview, PDF Expert, and FoxIt PDF Reader, among others, have tools to let you sign a PDF file.) We want to make it *extra* clear so that no one inadvertently cheats.
- You've got this! Good Luck!

First Name	
Last Name	
Student ID	

1 Code of Conduct [1 pt]

Statement of Academic Integrity

- I affirm that this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and I have neither provided to, nor received from, anyone **any** assistance that produces unfair advantage for me or for any of my peers.
- I will adhere to the Berkeley Honor Code: specifically, as a member of the UC Berkeley community, I have acted with honesty, integrity, and respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct.
- I will complete this assignment entirely on my own, and will not discuss its contents or any related concepts with anyone other than to ask clarification questions directly to the course staff. I will not work on this exam in the physical proximity of any student currently or previously enrolled in this course.
- I will not post any part of this assignment, or related questions, to external websites such as Chegg, CourseHero, StackOverflow, or similar platforms. **You may not use ChatGPT/your LLM of your choice**.
- Communication in any other form—oral, written, or electronic; public or private; direct or indirect—with any human being outside the scope permitted above is not allowed.
- I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document.
- More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.
- I hereby acknowledge that I have read and understood the above instructions. I agree that any failure on my part to adhere to these instructions, which may result in a penalty, is solely my responsibility.

Failure to adhere to these guidelines will be considered an academic integrity violation. Please email Professor Anwar ganwar@berkeley.edu or post on Ed privately if you have any questions!

- **Declare and sign the following statement**:

  In the space provided below, **hand-write** the following sentence, verbatim. Then sign and date before uploading your work to Gradescope.

  "I have read the Code of Conduct and certify that all solutions in this document are entirely my own and that I have not looked at anyone else's solution. I have given credit to all external sources I consulted."

  Signature : ______________________ *Date* : ______________________

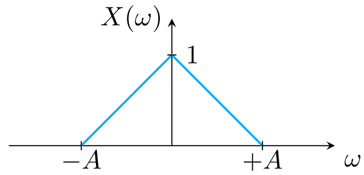
  Everything in your solution must be your (and only your) creation. Furthermore, all external material (i.e., *anything* outside lectures and assigned readings, including figures and pictures) should be cited properly. We wish to remind you that consequences of academic misconduct are *particularly severe*!

- Violation of the Code of Conduct will result in a **zero** on this exam and disciplinary referral. However, we reserve the right to give an instant **F** for even one violation.

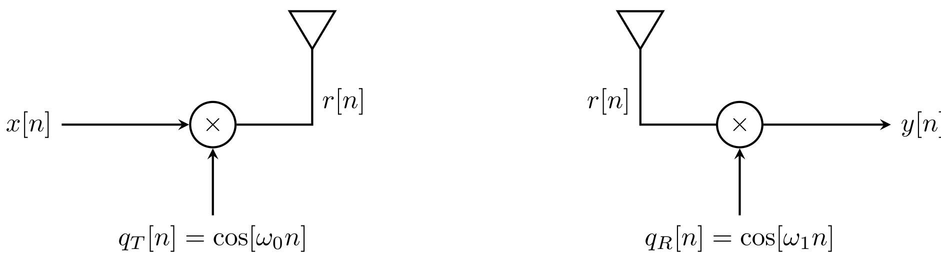
2 Short Answers [50 pts]

1. AM I very LTI? [12 pts]

Suvan, a die-hard RF engineer, is currently working on a top-secret project for the Indian government: designing an HF radio for a fighter jet so advanced that it might just tip the global power scale and make the US reconsider their Air Force budget! Before he revolutionizes aerial warfare, he needs to analyze the transmissions from the fighter jet. The transmission is a bandlimited discrete time signal $x[n]$ with the following triangular spectrum shown below. In discrete time, we define signals at integer-valued time indices n, i.e. $n \in \mathbb{Z}$.



A spectrum is a graph that shows how much of each frequency is present in the signal. The following diagram shows an (AM) amplitude modulation-demodulation scheme to communicate the signal $x[n]$ to a receiver.



Note: that the circle with $\times$ means multiplication and the triangle is the antenna that transmits and receives. There is also a frequency mismatch (ϵ) between the transmitter (LHS) and receiver (RHS) carrier signals q_T and q_R, respectively. In particular, assume that

$$0 < \epsilon \ll A \quad \text{and} \quad A < \omega_0 = \omega_1 + \epsilon$$

Suvan needs to answer the following question in order to proceed to help topple the US' air supremacy!

(a) [4 pts] Determine reasonably simple expressions for the signals r and y in terms of ω_0. You may or may not find the following trigonometric identity useful:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

(b) Consider the transmitter from the discrete-time AM system above, represented by F.

$$x[n] \longrightarrow \boxed{\mathsf{F}} \longrightarrow r[n]$$

where ω_0 is the constant frequency of the carrier signal.

(i) [4 pts] Select the strongest assertion from the list below **and** provide a clear and detailed explanation.

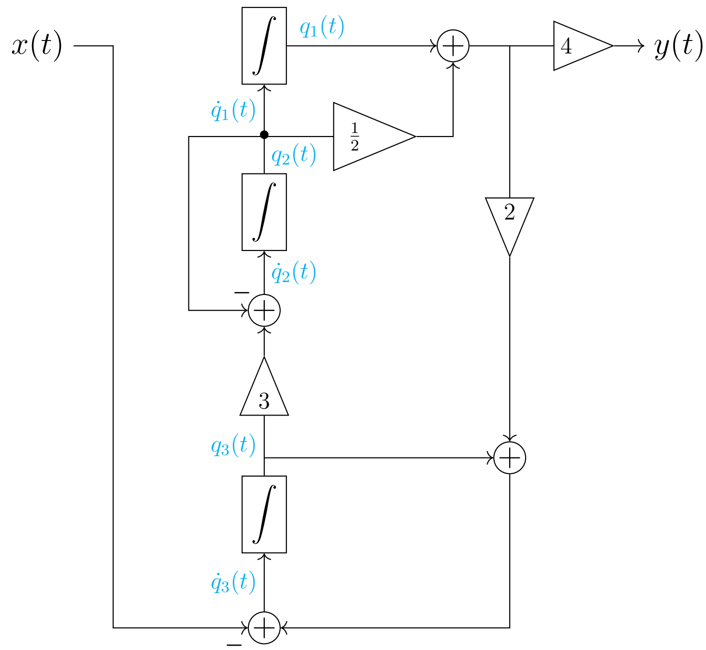
- ◯ The system must be *linear*. (If you choose this, give a short proof.)
- ◯ This system could be *linear*, but does not have to be. (If you choose this, explain why and specify additional conditions needed to determine linearity.)
- ◯ The system cannot be *linear*. (If you choose this, give a counterexample.)

(ii) [4 pts] Select the strongest true assertion from the list below **and** provide a clear and detailed explanation.

- ◯ The system must be *time invariant*. (If you choose this, give a short proof.)
- ◯ This system could be *time invariant*, but does not have to be. (If you choose this, explain why and specify additional conditions needed to determine time invariance.)
- ◯ The system cannot be *time invariant*. (If you choose this, give a counterexample.)

2. State-(Space) Your Business! [22 pts]

You are given the following IAG-block diagram for a system G with the internal states $q_1(t)$, $q_2(t)$, $q_3(t)$ and and output $y(t)$. *Note*: The junction with a dot indicates all paths are connected. You may check your answers with MATLAB, but **do not use it to solve ANY part of the question unless otherwise directed**.



(a) [8 pts] Find the state-space representation for G, where the state vector $\mathbf{q}(t) \triangleq \begin{bmatrix} q_1(t) & q_2(t) & q_3(t) \end{bmatrix}^\top$. Explicitly write out the SEE and OE, identifying the values of the A, B, C, and D where

$$\begin{aligned} \text{SEE}: \quad & \dot{\mathbf{q}}(t) = A\mathbf{q}(t) + Bx(t) \\ \text{OE}: \quad & y(t) = C\mathbf{q}(t) + Dx(t) \end{aligned}$$

(b) [3 pts] You are given a different continuous-time LTI system H with the following state-space representation:

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} -1/3 & 0 & 0 & 0 \\ 0 & -3/2 & 0 & 0 \\ 0 & 0 & -0.6+0.8i & 0 \\ 0 & 0 & 0 & -0.6-0.8i \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x(t)$$

$$y(n) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{q}(t) - 3x(t)$$

where $i \in \mathbb{C}$ is the imaginary unit. Is H internally stable? Show your work. Explain in a sentence when BIBO stability **does not** implies internal stability?

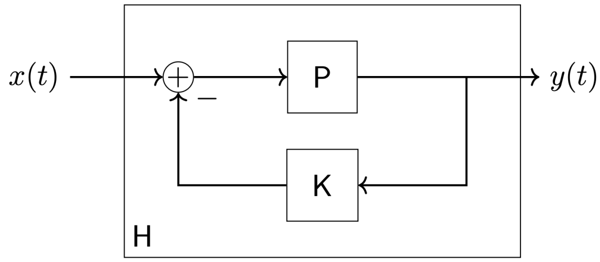
(c) Professor Anwar has random gadgets in Hesse 50B! Out of spite, you want to create a system out of the components you find lying around to get revenge on Larry and Athul for making an appalling final! You find components, $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$, and F, and know the following:

- Component X has impulse response $h_X(t) = 10e^{-\frac{1}{3}t}u(t)$.
- Component Y has the LCCDE $y(t) - 2\dot{y}(t) = 100x(t) + 35\dot{x}(t) - 10000\ddot{x}(t)$.
- Component Z has frequency response $Z(j\omega) = \frac{0.0001}{5+j\omega-\omega^2}$.
- Component F has frequency response $F(j\omega) = \omega^2$.

(i) [2 pts] Find the frequency response $X(j\omega)$ of component X. State any assumptions made.

(ii) [2 pts] Find the frequency response $Y(j\omega)$ of component Y. State any assumptions made.

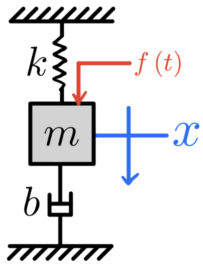
(iii) [7 pts] Professor Anwar, although reticent, hints that by creating a system that inverts F, you will be able to torment them! You find the gadget H with the block diagram shown below.

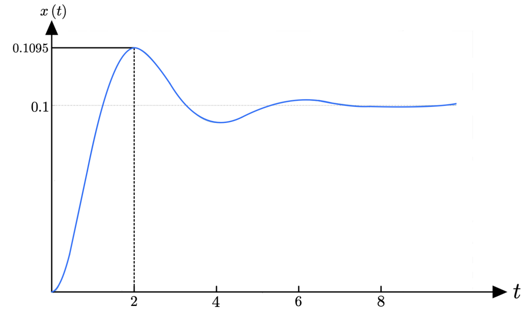


You want to pick P and K from the set of $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$, and F in order to make H invert F when placed in series for $\omega > 1$. Select the best substitution strategy you should employ. Justify your answer.

3. Steady as She Goes! [16 pts]

You and Athul are analyzing a scaled down version of a rudimentary suspension system, that is appalling, for Jaguar, specifically the 57' Jaguar XKSS that Larry wants to buy! You two have modeled it as the following spring-damper system to the left. You can assume there is no gravity.

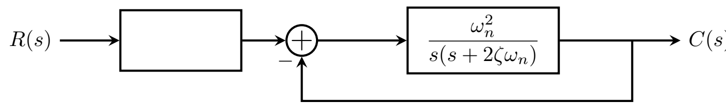




Athul, with a modicum of mischief, decides to leave Larry to drive in his messed up suspension so he doesn't mention what the mass, spring constant, and damping coefficient m, k, b are. All he leaves you with is a response plot when $f(t) = 2$ `N` is applied as a step-input to the system on the right.

(a) [6 pts] Given the response curve, what are the values of the mass, damping coefficient, and the spring constant, $m, b,$ and k respectively?

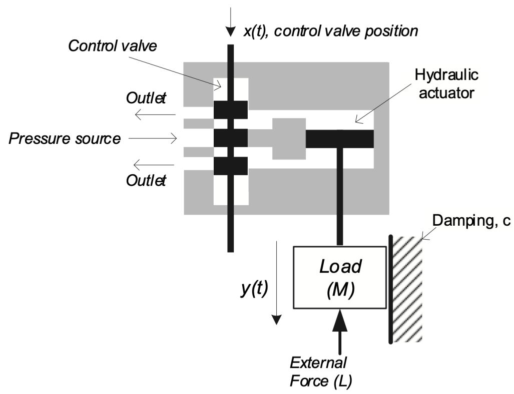
(b) [10 pts] You now need to design a filter to eliminate the steady-state error from the ramp for the arm dampener in a Maneki-neko (招き猫)! What is the the steady-state error from an unfiltered system with the same open-loop transfer function? Since, the EE team also needs to control the frequency of the arm, you must use the same values of ω_n and ζ from the original suspension design to save time. Fill in the block diagram with a **selected filter on the reference**. What gain, k, is required to eliminate the steady-state error from the ramp with the lowest system type?



3 Long Answer [214 pts]

1. F1? [34 pts]

Larry has signed as an F1 driver for Scuderia Ferarri! Unfortunately, due to strategy issues in the beginning of the season (of course!), he has sacrificed many points! Athul, the chief engineer, has decided to lighten the DRS (Drag-Reduction System) hydraulic actuator on the SF-25 for their upgrade package to help Larry's pace. A schematic of the system is shown below.



The DRS control valve position $x(t)$ modulates the inflow of hydraulic fluid from the high-pressure pump to the actuator cylinder. This, in turn, adjusts the rear-wing flap position $y(t)$. Let M and c be the combined mass and viscous friction coefficient of the flap assembly, and A the piston area. The volumetric flow rate $Q(t)$ into the cylinder can be expressed in terms of the piston area A.

$$Q(t) = A\frac{dy}{dt}$$

The pressure across the piston can be expressed by the following relation

$$P(t) = bx(t) - Q(t), \quad b = \text{constant}$$

If $L(t)$ is an external aerodynamic disturbance force, the net force $F(t)$ acting on the flap is:

$$F(t) = AP(t) - L(t)$$

(a) [5 pts] Write the equation of motion and transfer function of the rear-wing flap assembly (piston and load) in response to the net DRS hydraulic force $F(t)$. You may leave your answer in terms of $F(s)$.

(b) [11 pts] The flap position $y(t)$ is controlled in closed-loop to follow a driver's demand $y_d(t)$, and the DRS valve opening $x(t)$ is given by

$$x(t) = K_p(y_d(t) - y(t)), \quad K_p = \text{constant}$$

Draw the overall block diagram in the s-domain of the closed-loop system showing the variables $L(t), y(t), y_d(t)$ and the transfer functions corresponding to various blocks. Label all intermediate signals.

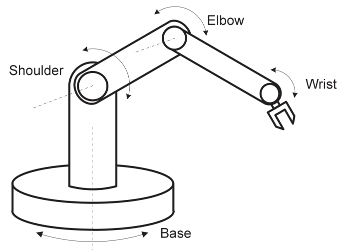
(c) [6 pts] For this DRS actuator, $A = 0.01\texttt{m}^2$, $b = 2\ \texttt{m}^3/\texttt{s}$, $c = 10\texttt{N} \cdot \texttt{s/m}$, and $M = 1\ \texttt{kg}$. Write the open-loop transfer function assuming unity negative feedback. Assume no external aerodynamic disturbance.

(d) [6 pts] Find the gain K_p that would result in a damping ratio of $\zeta = 0.7$ for the closed-loop system analytically. Assume no external aerodynamic disturbance. Plot the root locus in MATLAB to verify your solution.

(e) [6 pts] The system experiences a constant external load $L(t) = 10$ `N`. If $y_d(t) = 0$, find the steady-state error for the gain K_p found in (d).

2. Sensible Robots [35 pts]

The shoulder joint of a remote-control robotic arm in the figure below is powered by a permanent magnet DC servomotor through a 60 : 1 speed reduction gearbox. The moment of inertia J_r of the robot arm and the external disturbance torque T_d both vary with the robot pose. The position of the motor shaft is measured using a sensor producing $v_0 = 5$ `V/rad`. The demanded joint position is set by a remote-control pendant as $v_i = 5$ `V/rad`. The motor is powered by an amplifier with a gain K_a. The weight of the arm results in a load torque T_d which varies with the arm position. The table below provides the relevant system parameters.



Mechanical / Electrical			Control / Sensor		
Parameter	Value	Description	Parameter	Value	Description
J_R	6–12 kg m^2	Moment of inertia (robot arm)	K_b	0.5 V s rad^{-1}	Motor back emf constant
J_m	0.002 kg m^2	Moment of inertia (motor)	K_t	0.5 N m A^{-1}	Motor torque constant
R_a	5 Ω	Motor armature resistance	K_{pos}	5 V rad^{-1}	Position sensor sensitivity
L	0 H	Motor armature inductance	K_a	10	Power amplifier gain
f	0	Friction	N	60 : 1	Gearbox speed ratio

Electromechanical Systems We recognize the electromechanical systems were out of scope for the midterm, so here is a brief overview of the concepts relevant to this question. They are covered in further depth in the Nise section 2.8, as well as Homework 2 question 8.

- **Controller** - outputs a voltage as the input to the motor
- **Motor** - converts a voltage to a torque or vice versa ($T = \frac{K_t}{R_a} V_{in}$)
- **Amplifier** - multiplies the input voltage by a constant K_a, ($V_{out} = KV_{in}$)
- **Back EMF** - given a motor shaft speed ω, generates a negative voltage in the motor ($V_{back} = K_b \omega$)

(a) [15 pts] Draw the overall block diagram of the closed-loop **positional** control system employing a proportional controller with a gain K, include the external load T_d and write the expressions for relevant transfer functions. Label all intermediate signals and blocks. You may leave the inertia as J_{eff}.

(b) [8 pts] Find the range of the total effective inertia for the system, as it varies with the arm position. **Show all your work including the derivation of the total effective inertia**. Additionally, the mass of the robot arm moved by the shoulder is 35`kg` and its center of gravity at full extension is 0.6`m` from the shoulder axis. Find T_d, the maximum disturbance torque on the motor shaft due to gravitational loads, and the equivalent disturbance voltage v_d at the motor input.

(c) [8 pts] The closed-loop system is required to maintain a damping ratio $\zeta \geq 1/\sqrt{2}$ throughout the robot envelope. Show whether the controller should be designed for the largest or the smallest effective inertia J and find the required proportional control gain K analytically. You may assume no external torque disturbance and check your gain in MATLAB.

(d) [4 pts] The robot designer understands the the gravitational load will cause a deflection of the end-effector from a target position and he is considering using PI or PD controllers as a means of reducing the positioning error. Explain one advantage and one disadvantage of each of these schemes in relation to the desired control performance.

3. MY EARS!!! [45 pts]

After coming back from Tomorrowland, Larry has been vibing to Tiësto and Armin van Buuren on his new headphones, and is excited to test out their lightning-fast noise cancellation while using his CDJ3000! Being the engineer he is, he tries to figure out how exactly the noise cancellation of his headphones work!

A naive overview of active noise cancellation is that, for a given noise signal $g(t)$, the headphones output a corresponding *anti-noise* signal $h(t)$, where $h(t) = -g(t)$. Within the headphones, we will simplify the signal generation and assume the headphones have a step signal generator! The headphone will then modify the step signal through various transformations to match the desired output signal.

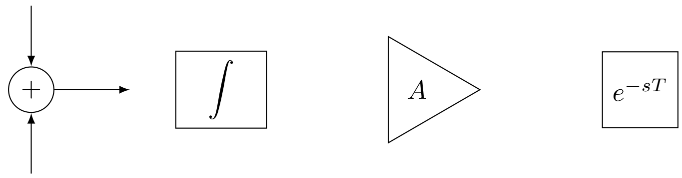
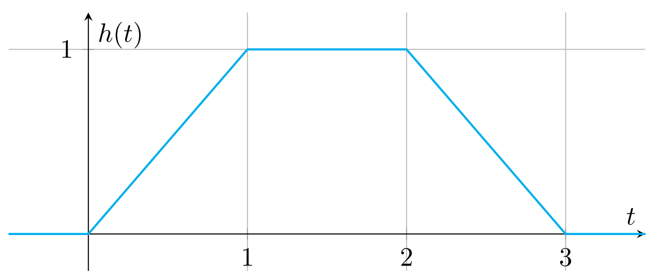


Figure 1: Left to Right: Summing Junction, Gain Block with label A, Delay Operator with delay T

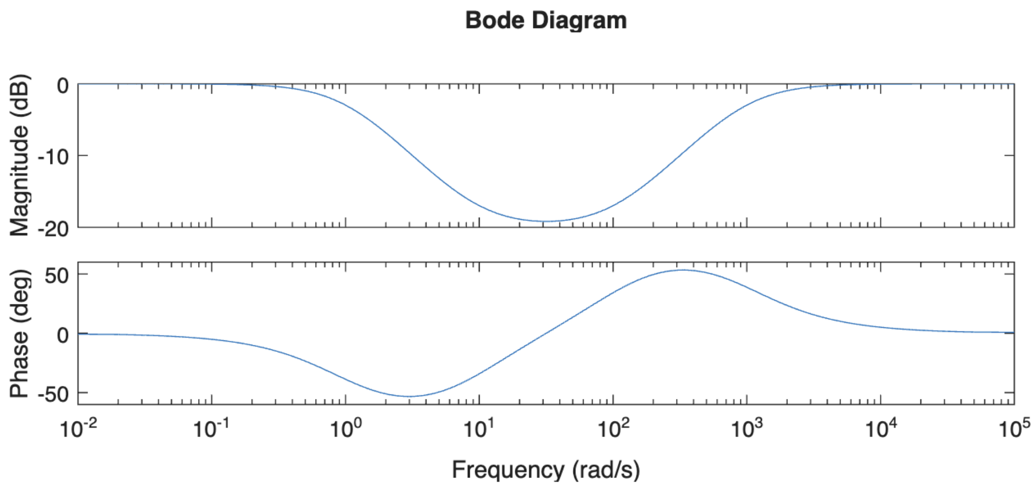
In terms of transfer functions: The integrator is equivalent to multiplying the input by $\frac{1}{s}$. The gain block with label A is equivalent to multiplying the input by a scalar A. The delay block of T seconds is equivalent to multiplying the input by e^{-sT}. *Hint: For all subparts, you will be required to split connections in the block diagram.*

(a) [8 pts] Fill in the block diagram below that creates the given signal $h(t)$ from a step function. **You are allowed to use at most 1 summing junction, 5 delay blocks, 1 integrator, and 5 gain blocks. You must provide a value for each parameter used (e.g. a block with gain 3 has $A = 3$).**



(b) [3 pts] What is the equivalent transfer function $\frac{H(s)}{R(s)}$ for the block diagram in part (ii)?

(c) Initially, Larry believes that the headphones have a fixed internal circuit with a specific transfer function that attenuates specific frequency ranges. He checks a few pure frequencies and records his data below:



i. [5 pts] Based on the Bode plots above, what is the measured transfer function of the headphones? *Hint: all poles and zeroes are powers of 10.*

ii. [2 pts] Using MATLAB, plot the magnitude and phase Bode plots for your transfer function of the headphones. Make sure it matches the one above.

(d) Larry then tries to induce a manual step signal into the headphones, using a power supply tapping directly into the microphone terminals. He puts the headphones on to hear the resulting anti-noise signal, but the headphones explode with a loud POP!

i. [5 pts] Concussed and slowly reorienting himself from the sharp pain in his ears, Larry acquires a Savant-level knowledge of computer architecture! He immediately realizes that the transfer function is actually implemented in some sort of code, and the headphones exploded due to something known as a **floating point error**.

As an oversimplification, a floating point error in this context means the **computer is occasionally unable to compute numbers with full precision**. For example, the number 0.0000067 could be computed as 0!

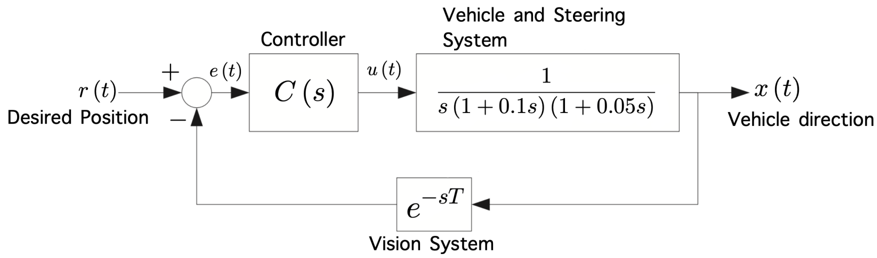
Provide an example as to how this could result in a transfer function whose gain spikes for a step response.

ii. [2 pts] Augment your original transfer function to account for this behavior.

(e) [4 pts] Using the Bode plot transfer function **from part (c)**, Write the output signal $y(t)$ given the following input signal $x(t)$:

$$x(t) = 10\cos(0.1t) + 10\cos(t) + 0.01\cos(1000t) + \cos(0.001t)$$

Suddenly dizzy from the piercing noise from the headphones, Larry relies on the self-driving capabilities of his car to get him to the hospital. The figure below shows the automatic control system for his car, designed to steer the car along a highway lane. The road markings are sensed by a vision system which computes the actual vehicle direction in relation to the direction of the road. In normal driving, $r = 0$ would command the vehicle to follow the road direction. The vision system potentially introduces a time delay T because it requires a finite time to acquire and process the images.



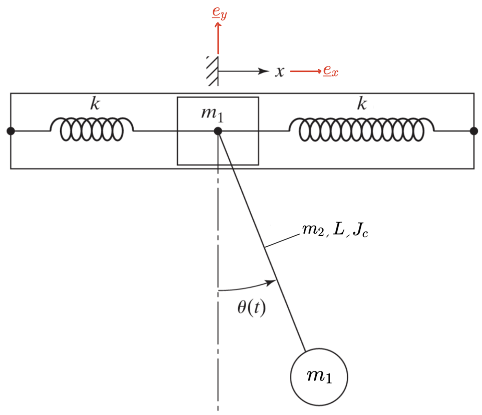
(a) [6 pts] During an initial phase of controller development it was assumed that $T = 0$ and a proportional controller $C(s) = K$ was employed. Plot the corresponding Bode plot and find the value of K that would result in a closed-loop system phase margin $PM = 40°$. **Do not iterate or guess and check using MATLAB**. Draw on the Bode plot and calculate the corresponding gain margin and determine the closed-loop system bandwidth (you can use MATLAB)?

(b) [2 pts] Assuming $T = 0$ and we are still implementing a proportional controller $C(s) = K$, determine the value of gain, K, to yield a step response with 20% overshoot and a closed-loop system phase margin $PM = 40°$.

(c) [8 pts] Propose a phase lag controller that would reduce the settling and peak time of the response while maintaining a phase margin $PM = 40°$. Explain your selection of control parameters. Plot the Bode plot.

4. Bowser's Optimal Trap! [40 pts]

Bowser with the help of Kamek, the Magikoopa, has rigged his castle with a spring-loaded trap to stop Mario's advance to save Princess Peach! A heavy stone block of mass m_1 forms a platform on which Mario must stand. The platform is held in place by two identical springs (each of stiffness k) attached to the castle walls as shown below.



From the underside of the platform, there hangs a long pendulum with a bar mass m_2, length L, and moment of inertia from the center of mass of $J_c = \frac{1}{12}m_2L^2$. The spiked ball at the bottom can be considered a point mass m_1. We need to find the equations of motion before Mario jumps on the platform shifting it horizontally by x and some force pendulum swings by some degree θ. The equations of motion are

$$(m_1 + m_2)\ddot{x} + \frac{1}{2}m_2L(-\dot{\theta}^2\sin(\theta)) + 2kx = 0$$
$$m_2\left(\frac{L^2}{12}\right)\ddot{\theta} + \frac{1}{2}m_2\dot{x}L\dot{\theta}\sin(\theta) + m_2g\frac{L}{2}\sin(\theta) = 0$$

(a) [5 pts] Rewrite the homogeneous EoMs as a set of first-order differential equations using state-variables.

(b) [3 pts] When Mario lands, his impulse delivers a horizontal force u at the platform and—through the pivot acceleration—a matching generalized torque u into the pendulum equation. Rewrite the set of differential equations in part (a) to account for this.

(c) [12 pts] Suppose now that Bowser flips a secret switch that turns on a small motor that simultaneously drives the platform horizontally at a constant velocity u and twists the pendulum at a constant rotational velocity u. Using this and part (b), linearize the system about the equilibrium point, x_e, where all states are zero except $\dot{x}(0) = t$ where $t \in \mathbb{R}$ is constant in order to find the matrix A, B, C, and D. Explicitly write out the SEE and OE. Assume that the outputs are x and θ.

For the following subparts, assume the following values: $m_1 = 3\texttt{kg}$, $m_2 = 1\texttt{kg}$, $L = 1.5\texttt{m}$, $k = 3\texttt{N/m}$. **For the purposes of the following subparts, assume your output is just x, and adjust your C and D matrices accordingly.**

(d) [6 pts] Is it possible to place all the poles of the system anywhere in the left-hand plane? If so, use full-state feedback to obtain the gain vector K to place the closed-loop poles at $-1, -2, -3, -4$.

(e) [10 pts] If you were able to correctly place all poles as desired, **extract the new open loop transfer function $G(s)$ from the closed loop transfer function after pole placement**, and plot the root locus for the system before and after pole placement. If you were not able to correctly place all poles as desired, explain why using the root locus of the system.

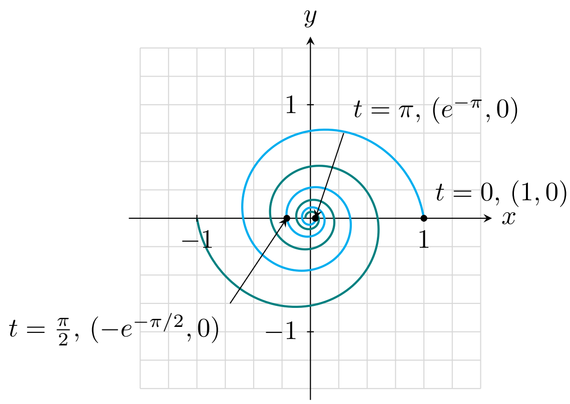
Hint: The root locus for the system after pole placement would be the root locus of $G(s)$.

(f) [4 pts] **Using the root locus of the system after pole placement**, show whether the pole placement was successful or not.

5. Hypnotism [50 pts]

Note: This question is more mathematically involved than the previous questions on the exam. If you're not as comfortable with linear algebra and differential equations, try the other questions before this one!

Larry and Athul have been experimenting with robotic hypnotism! To test whether autonomous systems can "fall under a trance", they fed a robot's visual processor a flashing spiral on a screen. They want to try to hypnotize Professor Anwar into making this exam harder! The hypnotic stimulus is mathematically shown below in the graph. The robot's eye movement $(x(t), y(t))$ is governed by an unknown linear system. Two robots are tested at once—both follow the same system of differential equation, but they have from different initial conditions as illustrated below.



For all subparts, show **all of your work, including algebra**. You may check your answers with MATLAB, but **you may not use it to solve the question for parts (a)-(c)**. **There may be multiple answers, but only solve for one**.

Hint: Expressing points in polar form may be helpful.

(a) [12 pts] **For one of the curves**, you are given some points, labeled on the graph below. Your task is to construct a system of coupled differential equations relating x and y, and give both initial conditions. The system must be of the following form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}, \mathbf{A} \in \mathbb{R}^{2\times 2}$$

(b) [18 pts] Let the system of differential equations be expressed in matrix vector form below:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},\ a, b > 0$$

Directly solve the system of differential equations and obtain $x(t)$ and $y(t)$.

(c) [8 pts] Using the solution to the previous subpart, assume the plot is on the interval $0 \leq t \leq T$. In terms of a, b and T, show each of the following:

- The conditions for which the graph is a converging spiral, a circle, or a diverging spiral.
- In the case of a spiral, the angle that the spiral traverses. (e.g. 6 full loops would be an angle of 12π)

(d) [12 pts] Now, we are going to utilize **polar LTI systems**, to form graphs that are otherwise impossible with standard Cartesian LTI systems. Let our system be defined by the following:

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ \theta \end{bmatrix}$$

First, solve the differential equation above for $r(t)$ **and** $\theta(t)$**, assuming** $(r_0, \theta_0) = (1, 0)$. Show that this forms a lemniscate (∞ symbol), or explain why it is impossible to form a lemniscate using the given state space formulation.

4 Extra Credit [30 pts]

This exam contains **2 problems** that are provided as an extra credit opportunity. You can choose **one** of the following to do. If you attempt both, the higher score will be selected. These questions are designed to challenge you and require you to utilize all the controls knowledge you have acquired over the summer and are not required to complete the exam. Please do not begin this section until you have finished all required questions and are confident in your responses.

Note: the points received on these problems will not be added to this exam! They are a separate category!

For your convenience, since we know you guys are burnt out, we have decided to compile a list of the extra credit questions ranked by difficulty so you can maximize the amount of points you can earn!

Easier

1. **"Terrain, Terrain, Pull Up"** [15 pts] will focus on block diagram analysis, steady-state errors, root locus, and disturbances. Nothing crazy, just a simple question.

Difficult

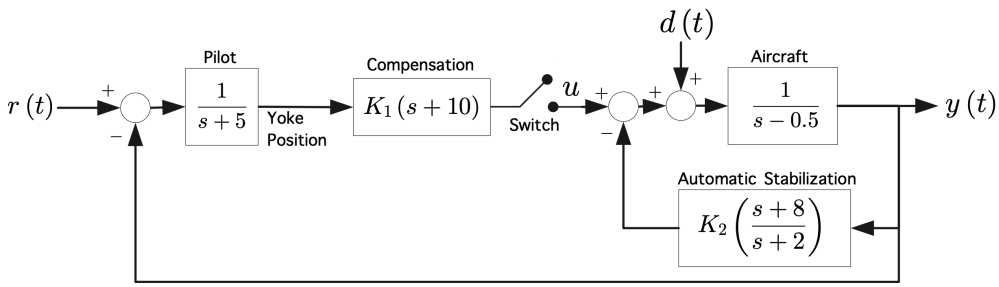
1. **I'm Fast AF (Kalman Filter)** [30 pts] This is the conceptually hardest question in this exam. It requires some statistics intuition including the ideas behind covariance propagation, probabilistic state estimation, and matrix algebra. It also involves a MATLAB component implementing a Kalman Filter! For any material that is out of scope (including the statistics knowledge), the necessary information is still given in the question; we do not expect you to know all of these ideas beforehand. **This is far less tedious but requires a good intuitive understanding of the material!**

Below is a box. Use it to make us laugh or put a funny drawing and *perhaps* we will award you some extra points.

1. “Terrain, Terrain, Pull Up” [15 pts]

The figure below shows the control scheme for the pitch angle $y(t)$ of a Boeing 737 MAX 8. It has been redesigned in the wake of recent disasters including: the maneuvering characteristics augmentation system (MCAS) malfunctioning; doors flying off; and general safety concerns! In particular, a new automatic stabilization system (linearized from flight-test data) has been added to the aircraft, intending to aid the pilot in maintaining safe flight attitudes despite sensor faults and aerodynamic uncertainties.

When the switch in the block diagram below is **open**, the automatic stabilization is **intended to maintain zero pitch angle**. When the switch is **closed**, the pilot acts as a part of the control loop and the corresponding transfer function represents the pilot's reactions. The aircraft is subjected to an external aerodynamic disturbance, i.e. turbulence $d(t)$.



(a) For the following, assume the switch is open making $u(t) = 0$ and there is no disturbance $d(t)$.

(i) [1 pts] Write the characteristic equation of the system.

(ii) [3 pts] Plot the root locus diagram for $0 < K_2 < \infty$. What K is required so that the system is stable? You may use MATLAB.

(iii) [2 pts] Find the gain K_2 analytically (± 0.5 is fine) that would result in a damping ratio $\zeta = 0.6$. Verify your result with MATLAB (use the same plot as above).

(b) For the following questions, the automatic stabilization is set such that $K_2 = 9$ and the switch is closed, putting the pilot in control of the aircraft.

(i) [1 pts] Write the characteristic equation of the system. Assume no disturbance $d(t)$.

(ii) [4 pts] Plot the **new** root locus diagram. Assume no disturbance $d(t)$. Is the system now stable?

(iii) [3 pts] Find the gain K_1 analytically (± 0.5 is fine) that would result in the system being critically damped. Verify your result with MATLAB (use the same plot as above).

(iv) [4 pts] With this gain setting, find the pitch angle corresponding to level flight with $r(t) = 0$ following a sudden disturbance $d(t) = 10$.

2. I'm Fast AF ᕕ(⌐■_■)ᕗ ᕕ(⌐■_■)ᕗ [30 pts]

Note: This question is conceptually more difficult, but less tedious than the rest of the extra credit questions. Depending on your comfort level with the topics in the class, it may be better to attempt all other questions before this one!

You're a delivery robot, and your precious food has just been snatched out of your little robot arms!

To catch the thief, you need to track him down and ensure he doesn't gain any distance from you! Unfortunately for you, the thief is skilled at changing directions and making random turns, making this harder than expected.

You don't have time to figure out where to rewire your internal closed loop observer poles, so let's create a **Kalman Filter** observer to help you **optimally** track the thief's movement and stay hot on his tail, without explicit pole placement!

Notation:

Let's introduce a concept known as a **Multivariate Gaussian Distribution**. For the terms below, assume we have a set of data defined by the probability distribution x.

- μ: The average (expected) value of x. As an example, for a fair coin flip, if you represented heads as 1 and tails as 0, the average value of a toss as the number of tosses, n approaches ∞ is $\mu = 0.5$.
- Σ: the Covariance matrix, represents how the data is spread along different directions. For simplicity, in the diagonal case, the values on the diagonal correspond to how much the data is spread out along the corresponding axis. As an example, in 2D, if $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, the fact that $3 > 2$ would mean that the data is more spread out along the y axis. For a visual explanation, see this link.
- $x \sim \mathcal{N}(\mu, \Sigma)$: x is distributed according to a Gaussian distribution with mean μ and covariance matrix Σ. This means that the data is centered at μ and spread out according to Σ.

For this question, you are not responsible for knowning any of the probability math utilizing these concepts, just an intuitive understanding of what each term means.

The dynamics of the system are defined by the following set of equations:

$$x[t+1] = Ax[t] + Bu[t],\ x[t] \sim \mathcal{N}(\mu_{x[t]}, \Sigma_{x[t]})$$
$$y[t+1] = Cx[t] + \psi_t,\ \psi_t \sim \mathcal{N}(\mu_\psi, \Sigma_\psi)$$

Here, $u[t]$ and $y[t]$ are known values at each time step. This is because $u[t]$ is the input to the system that we provide ourselves, and $y[t]$ is our observation of the system state. What we *want* to know is the true value of $x[t]$ at any point.

(a) Let's begin by defining how uncertain we are of our true state $x[t]$ at any point in time:

   i. Assume that $u[t]$ is known. Given the above information, find an expression for the estimated uncertainty (covariance) of $x[t+1]$, in terms of $\Sigma_{x[t]}, A$, and/or B.
   You will need to use the formula that for a variable $x \sim \mathcal{N}(\mu, \Sigma)$ and matrix K, the covariance of Kx is given by $K\Sigma K^T$.

ii. Now, assume that between each timestep, we need to account for some unknown disturbances (such as our robber tracking sensors being slightly off), that could be added to our measurement of $x[t+1]$. This disturbance has a covariance of Σ_ϵ. **For the purposes of this question, you should assume that covariances can be added up!**
Write an updated expression for the uncertainty of $x[t+1]$, based on your answer from the previous part and this additional requirement.

(b) Now, we can look at our standard Luenberger Observer Equation through the lens of Kalman Filtering, and gain an intuition of what exactly is going on! We can define our estimate for $x[t+1]$ as follows:

$$\hat{x}[t+1] = \bar{x}[t+1] + K(y[t+1] - C\bar{x}[t+1])$$

Here, $\hat{x}[t+1]$ corresponds to the *final* estimate for $x[t+1]$, and $\bar{x}[t+1]$ is our estimate for x from the previous part. In other words, we use our uncertainty estimate $\bar{x}[t+1]$ as *part* of the equation for our final estimate, $\hat{x}[t+1]$.

Let's assume our observation $y[t+1]$ comes from a set of sensors! As a result, $y[t+1] \sim \mathcal{N}(\mu_y, \Sigma_y)$.

i. In two sentences or less, explain when K should be chosen to be small.

ii. In two sentences or less, explain when K should be chosen to be large.

(c) Let's organize the flow of knowledge here, with a few new given equations! Fill in the blanks below given what you have solved for so far.

First, we find our initial guess $\bar{x}[t+1]$, and our initial estimate for its variance/uncertainty $\bar{\Sigma}_{x[t+1]}$

$\bar{x}[t+1] =$ ____

$\bar{\Sigma}_{x[t+1]} =$ ____

Then, using our limited knowledge of the system, we define a parameter K known as the *Kalman Gain*:

$$K = A\bar{\Sigma}_{x[t]}C^\top(C\bar{\Sigma}_{x[t]}C^\top + \Sigma_y)$$

Using the Kalman Gain, we can finalize our estimates for the value and uncertainty of $x[t+1]$, which are $\hat{x}[t+1]$ and $\hat{\Sigma}_{x[t+1]}$!

$\hat{x}[t+1] =$ ____

$$\hat{\Sigma}_{x[t+1]} = (A - KC)\bar{\Sigma}_{x[t+1]}A^\top + \Sigma_\epsilon$$

We will use our final guess and the uncertainty update in our initial estimates for the following timestep, and the process repeats!

Hints: Use information from previous subparts. Based off the last line, the term $\hat{\Sigma}_{x[t]}$ appears as part of the second blank. The answer to the third blank is also given in the question.

The question continues on the following page!

(d) Analyze the code and fill in the blanks below to successfully implement a Kalman Filter and catch the robber!

```
clear; clc; close all;
dt = 0.1;
A = [1 dt; 0  1];             % state matrix
C = [1 0];                    % measure position only
Q = [1e-4  0;  0  1e-4];      % process-noise covariance, sigma_epsilon
R = 1e-2;                     % measurement-noise variance, sigma_y
N = 100;                      % number of time steps
t = (0:N-1)*dt;               % time vector
x_true = [0;  1];
x_hat = [0.5; 0.0];           %initial guess
P = eye(2);                   %guess for sigma_x
u = 0.5*sin(10*t) + cos(2*t); %input function
x_store = zeros(2,N);
xhat_store = zeros(2,N);
y_store = zeros(1,N);

for k = 1:N
    w_k = transpose(mvnrnd([0 0],Q));
    v_k = sqrt(R)*randn;
    y_k = C*x_true + v_k;   %sensor reading

    % prediction
    x_bar = ___________________________________;
    P_bar = ___________________________________;

    % Kalman gain
    K = ___________________________________;

    % guess update
    x_hat = ___________________________________;
    P = ___________________________________;

    x_true = A*x_true + u(k) + w_k;
    x_store(:,k) = x_true;
    xhat_store(:,k) = x_hat;
    y_store(k) = y_k;
end
```

(e) Using the plotting code below and the code from the previous section, show the graphs of your working Kalman Filter!

```
figure;
subplot(2,1,1);
plot(t,x_store(1,:),'k-',  t,xhat_store(1,:),'b--',  t,y_store,'r.');
legend('true position','KF estimate','measurement','Location','best');
xlabel('time [s]'); ylabel('position [m]');
title('Kalman filter tracking a pesky food robber');
subplot(2,1,2);
plot(t,x_store(2,:),'k-',  t,xhat_store(2,:),'b--');
legend('true velocity','KF estimate','Location','best');
xlabel('time [s]'); ylabel('velocity [m/s]');
```

END OF EXAM