ODE Summary Sheet

1. Differential Equations

↳ Equations involving functions and their derivatives where the **order** is the highest appearing derivative

↳ The **general solution** is <u>all</u> functions that satisfy the diff. eq.

" **particular solution** is a single choice of solution

↳ **linear**: $a_n(x)\dfrac{d^n y}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = F(x)$

where $a_n(x), \dots, a_0(x)$, $F(x)$ depend ONLY on the independent variable x.

Sorry, I accidentally said it is homogenous if there's a constant term in Discussion.

↳ **homogenous**: a **linear** diff. eq. is homogenous if the RHS = 0 i.e. NO constant term

a **non-linear** diff. eq. is homogenous if all terms are the same degree

↳ **constant-coeff**: all coefficients are constant

Definition: An n-th order linear ODE is an equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y^{(1)} + a_0 y = f$$

for an unknown function $y = g(t) : \mathbb{R} \to \mathbb{R}$

Notation $y^{(k)}$ = k^{th} derivative of y

Ex. $y^{(1)} = y'$, $y^{(2)} = y''$, ...

a_k = real number **constant coefficient** or

more generally a function $a_k = a_k(t) : \mathbb{R} \to \mathbb{R}$

f = function $f = f(t) : \mathbb{R} \to \mathbb{R}$

2. Linear 1^{st} order Equations

If $I(x) = e^{\int P(x)dx}$, then a general solution to

$y' + P(x)y = Q(x)$ is $y = \dfrac{1}{I(x)}\left(\int I(x)\, Q(x)\, dx\right)$

integrating factor

3. Linear 2^{nd} order homogenous equations

given $ay'' + by' + cy = 0$ where $a \neq 0$ and a, b, c constant

↓ transform to characteristic poly.

$ar^2 + br + c = 0$ $\quad r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$b^2 - 4ac > 0$: $r_1 \neq r_2$ <u>real</u> roots

↓ general soln.

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$b^2 - 4ac = 0$: r_1 repeated <u>real</u> root

↓ general soln.

$y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

$b^2 - 4ac < 0$: $r_1 = \alpha + \beta i$, $r_2 = \alpha - \beta i$, $\beta \neq 0$ non-real roots

↓ general soln.

$y = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$

from Euler's formula

4. Linear 2^{nd} order Non-homogenous equations

$ay'' + by' + cy = g(t)$ ← linear, 2^{nd} order, constant coeff., non-homogenous diff. eq.

$\underbrace{\text{non-homogenous}}_{g(t) \neq \text{zero function}}$

general solution: $y = y_h + y_p$

homogenous ↗ ↖ particular

To find a soln to $y'' + by' + cy = t^m e^{rt}$ we will

try $y_p = t^s (A_m t^m + \cdots + A_1 t + A_0) e^{rt}$ → $y_p = t^s \left(\sum_{i=0}^{m} A_i t^i \right) e^{rt}$

(i) for a wise choice of s

(ii) then solve for $A_m, \ldots, A_0$

Note special cases $r=0$: $y'' + by' + cy = t^m$

$m=0$: $y'' + by' + cy = e^{rt}$

CHOOSING S:

① $r \neq r_1, r_2$: $s=0$ and try $y_p = (A_m t^m + \cdots + A_1 t + A_0) e^{rt}$

② $r = r_1 \neq r_2$: $s=1$ and try $y_p = t(A_m t^m + \cdots + A_1 t + A_0) e^{rt}$

③ $r = r_1 = r_2$: $s=2$ and try $y_p = t^2 (A_m t^m + \cdots + A_1 t + A_0) e^{rt}$

To find a soln to $y'' + by' + cy = t^m e^{at} \cos(bt)$ OR $t^m e^{at} \sin(bt)$

try $y_p = t^s (A_m t^m + \cdots + A_1 t + A_0) e^{at} \cos(bt) + t^s (B_m t^m + \cdots + B_1 t + B_0) e^{at} \sin(bt)$

$\Rightarrow y_p = t^s \left(\sum_{i=0}^{m} A_i t^i \right) e^{at} \cos(bt) + t^s \left(\sum_{i=0}^{m} B_i t^i \right) e^{at} \sin(bt)$

(i) for a wise choice of s

(ii) then solve for $A_m, \ldots, A_0$ and $B_m, \ldots, B_0$

CHOOSING S:

① $a+ib$ not a root : $s=0$, $b \neq 0$

try $y_p = \left(\sum_{i=0}^{m} A_i t^i \right) e^{at} \cos(bt) + \left(\sum_{i=0}^{m} B_i t^i \right) e^{at} \sin(bt)$

② $a+ib$ a root : $s=1$, $b \neq 0$

try $y_p = t \left(\sum_{i=0}^{m} A_i t^i \right) e^{at} \cos(bt) + t \left(\sum_{i=0}^{m} B_i t^i \right) e^{at} \sin(bt)$

- then find y_p', y_p'' and substitute into ODE. Match coefficients to solve for A's and B's.
- add $y_p + y_h$ and that's it!