# 7/18 ME132 Discussion!

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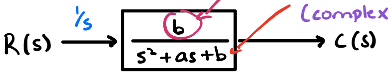
# general 2nd order systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

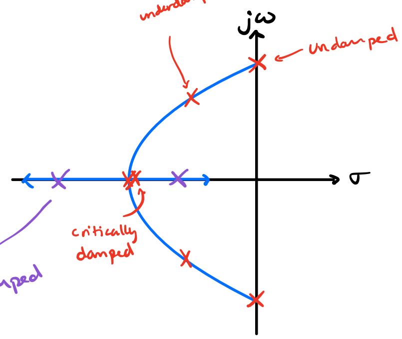
[DC Gain of 1]

no zeros

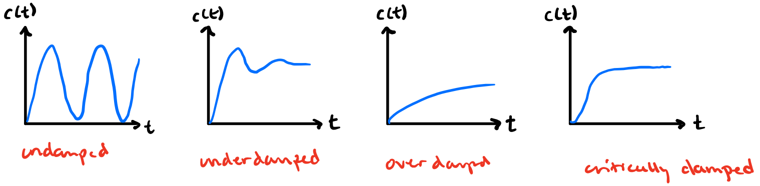
2 finite poles (complex pole pair determined by params a, b)




where $a = 2\zeta\omega_n$, $b = \omega_n^2$, $\zeta = \frac{a}{2\omega_n}$, $\omega_n = \sqrt{b}$




- underdamped $0 < \zeta < 1$
  - ↳ complex-conjugate poles
  - $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
- critically damped $\zeta = 1$
  - ↳ repeated real poles $-\omega_n, -\omega_n$
- overdamped
  - ↳ Two distinct poles
  - $-\zeta\omega_x \pm \omega_n\sqrt{\zeta^2-1}$




## STEP RESPONSE

$C(s) = R(s) \cdot G(s)$

$= \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

partial fractions ...

$= \frac{1}{s} + \frac{(s+\zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$

$\downarrow \mathcal{L}^{-1}$

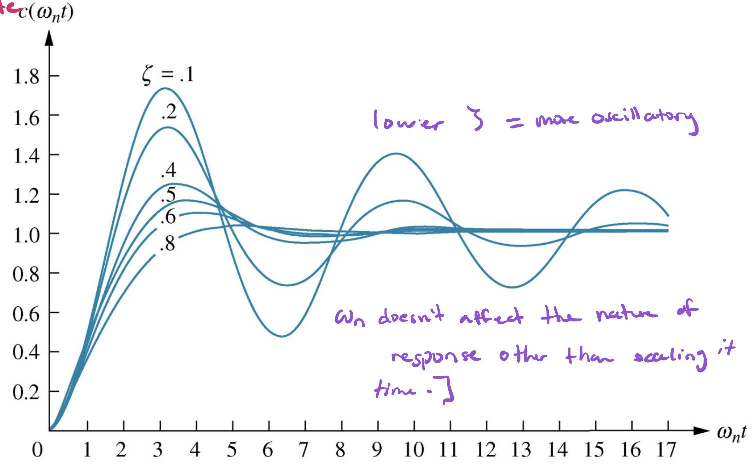
$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos(\omega_n\sqrt{1-\zeta^2})t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2})t\right)$

decay rate

via trigonometry

$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2} - \phi)$

, $\phi \doteq \arctan\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$




lower $\zeta$ = more oscillatory

[$\omega_n$ doesn't affect the nature of response other than scaling it time.]

# underdamped second-order systems

1. Damping Ratio $(\zeta)$

$$\zeta = \frac{\text{exponential decay freq.}}{\text{natural freq (rad/s)}} = \frac{1}{2\pi} \frac{\text{Natural period}}{\text{Exponential time constant}}$$

2. Natural Frequency $(\omega_n)$

↳ frequency of oscillations w/o damping!

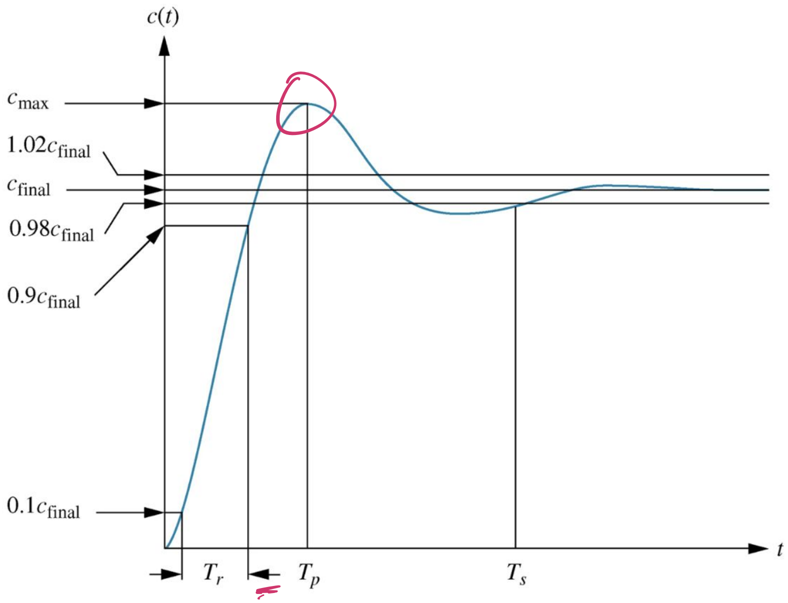
↳ multiple modes of vibration are possible...

↳ $M\ddot{x} + Kx = 0 \Rightarrow x = \underbrace{\phi e^{i\omega t}}_{\text{impulse}} \Rightarrow \underbrace{(K - \omega^2 M)\phi = 0}_{\text{eigenvalues = modes}}$

3. Settling Time $(T_s)$

↳ time req. for transient damped oscillations to reach and stay within 2% of the final steady-state value.

$$T_s = \frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n} \doteq \frac{4}{\zeta\omega_n}$$



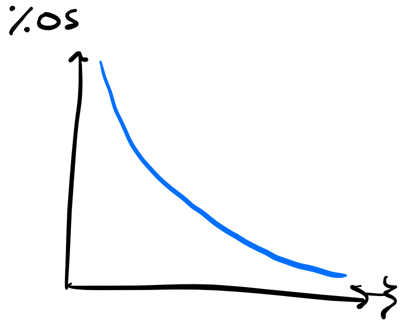
# underdamped second-order systems cont.

3. Peak Time

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

4. Rise Time

$$\omega_n T_r = (1.768\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$$




5. % Overshoot

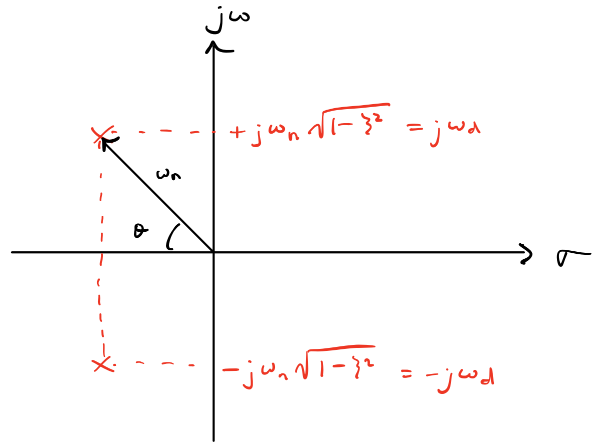
$$\%PO = 100\% \times e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}}$$

$$\%OS = \frac{C_{max} - C_{final}}{C_{final}} \cdot 100$$

## Natural Frequency

↳ radial distance from the origin to the pole




## Damping Ratio

↳ ratio of the magnitude of the Re part of poles over the $\omega_n$

$$\cos\theta = \frac{-\zeta\omega_n}{\omega_n} = \zeta$$

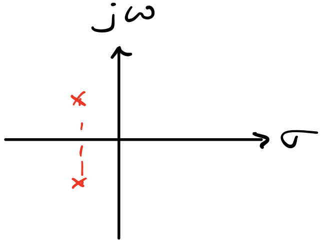
# underdamped second-order systems example

$$H(s) = \frac{1}{s^2 + s + 1}$$

① PZ plot

P: $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} j$

Z: $0$




② step-response

$$C(s) = \frac{1}{s} \cdot \frac{1}{s^2+s+1}$$

↓ PFD

$$= \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

↓ $\mathcal{L}^{-1}$

$$c(t) = A + Be^{-t}\cos t + Ce^{-t}\sin t$$

④ system responses ($\zeta$, $\omega_n$, $T_s$, $T_p$, $T_r$, %OS)

$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = G(s)$$

$$\omega_n^2 = 1 \Rightarrow \boxed{\omega_n = 1 \text{ rad/s}}$$

$$2\zeta\omega_n = 1 \Rightarrow \boxed{\zeta = 0.5}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.5} = \boxed{8s}$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{3}{\sqrt{0.75}} = \boxed{3.6s}$$

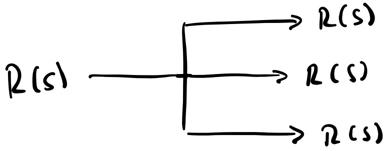
$$\omega_n T_r = ( \cdots$$

$$\%OS = 100\% \cdot \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

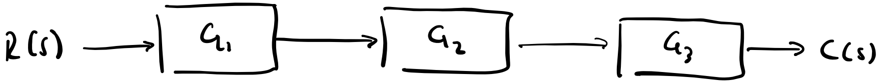
# block diagram reduction

Some terminology (most of this should be review)

1. Pickoff Point

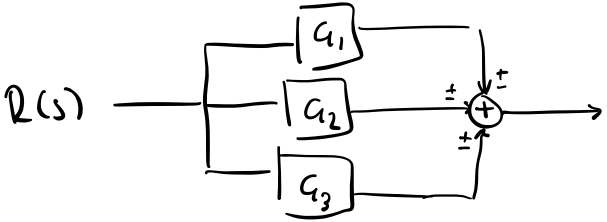



2. Cascade Form




$\equiv \quad R(s) \cdot G_1 G_2 G_3 = C(s)$

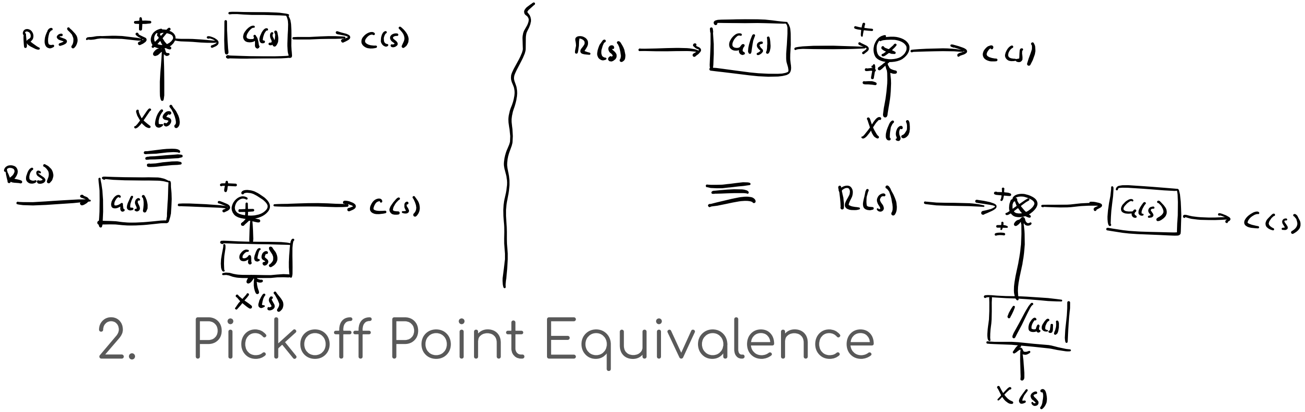
3. Parallel Form



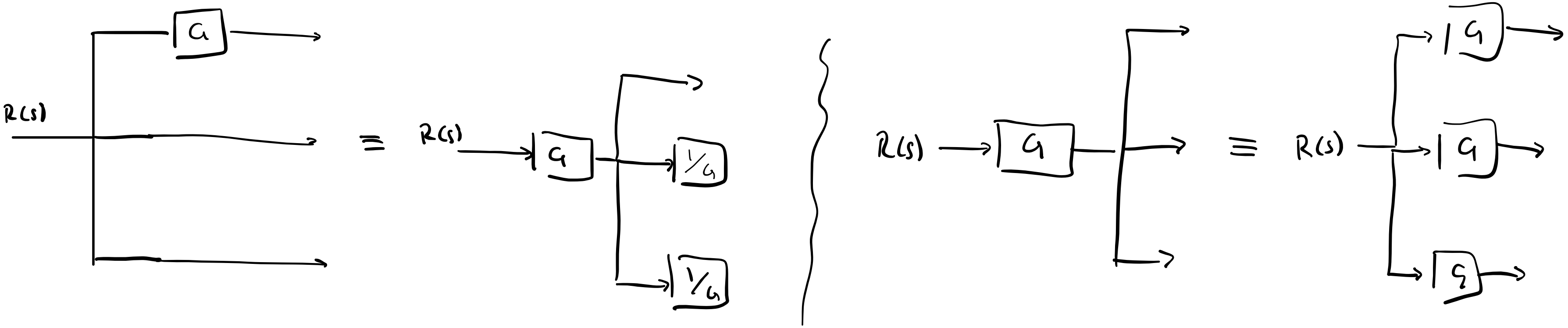

$C(s) = (\pm G_1 \pm G_2 \pm G_3) R(s)$

# block diagram algebra

1. Associative Property of Summing Junctions




2. Pickoff Point Equivalence



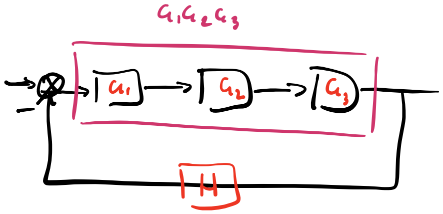
# black's formula

For negative feedback, we use **Black's Formula**

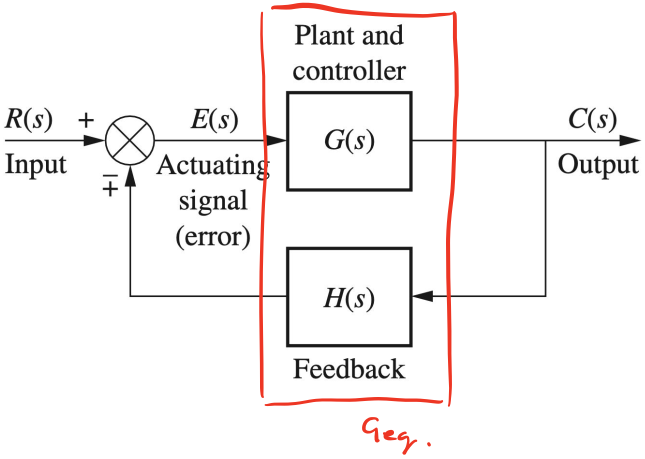
$$G_{eq} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\prod_{i=0}^{n} G_i(s)}{1 + \prod_{i=0}^{n} G_i H_i}$$

For positive feedback, we use the same, but switch the signs

$$G_{eq} = \frac{\prod_{i=0}^{n} G_i(s)}{1 - \prod_{i=0}^{n} G_i H_i}$$

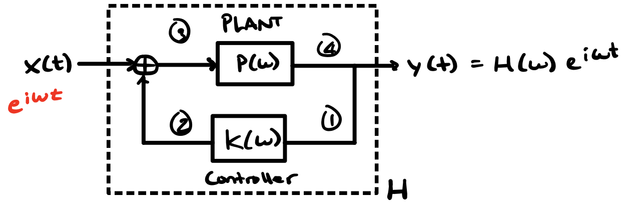



$$\frac{G_1 G_2 G_3}{H G_1 G_2 G_3}$$




*fun fact:* **Black's Formula** is named after Ralph Beebe Blackman, and engineer who worked in information theory and controls!

His colleagues included Claude Shannon (Whittaker-Nyquist-Kotelnikov-Shannon Theorem) and Hendrik Bode (Bode Plots)!




① $H(\omega)e^{i\omega t}$

② $H(\omega)K(\omega)e^{i\omega t}$

③ $(1+H(\omega)K(\omega))\,e^{i\omega t}$

④ $P(\omega)\left[1+H(\omega)K(\omega)\right]e^{i\omega t}$

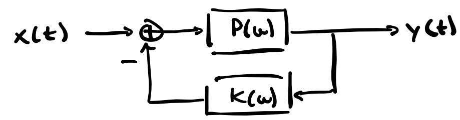
but ④ = ①

$$H(\omega)\,\cancel{e^{i\omega t}} = P(\omega)\left[1+H(\omega)K(\omega)\right]\cancel{e^{i\omega t}}$$

$$\boxed{H(\omega) = \frac{P(\omega)}{1-K(\omega)P(\omega)} \qquad H(\omega) = \frac{\text{Forward loop gain}}{1-\text{feedback gain}}}$$

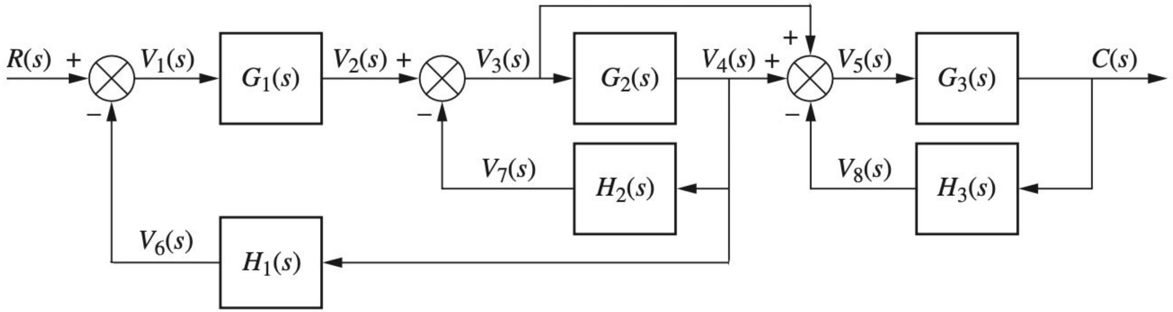
for negative feedback: Black's Formula

$$\boxed{H(\omega) = \frac{P(\omega)}{1+K(\omega)P(\omega)}}$$

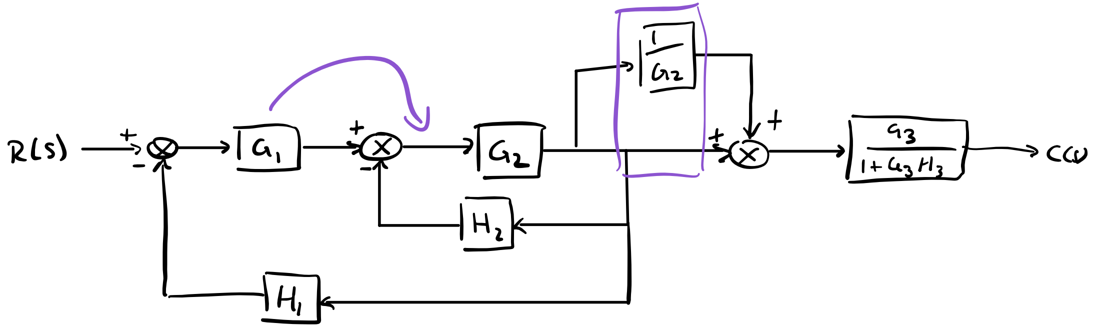


# block diagram reduction example

**PROBLEM:** Reduce the system shown in Figure 5.11 to a single transfer function.

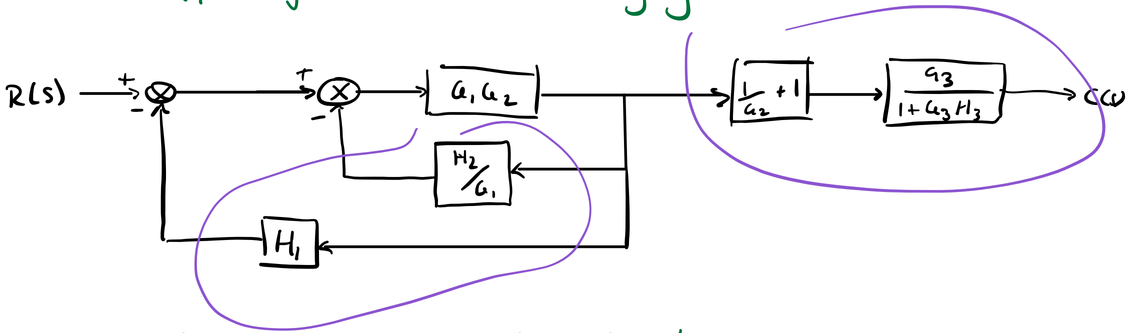



① move $G_2$ left of pickoff + Black's formula $H_3 G_3$

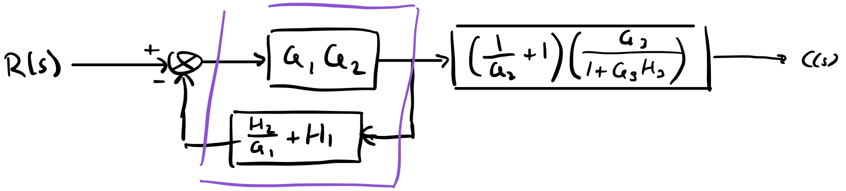


# block diagram reduction example cont.

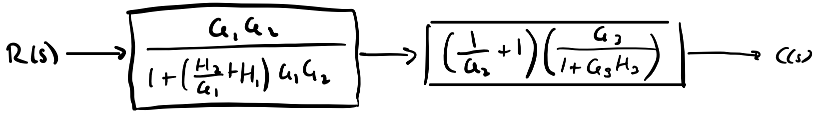
② move $G_1$ right of summing junction +



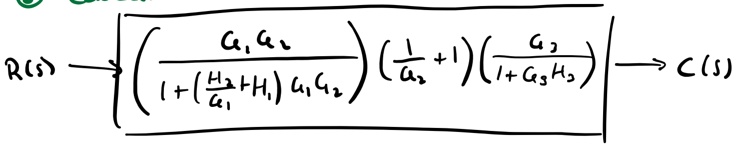

③ Cascade last 2 blocks, feedback parallel

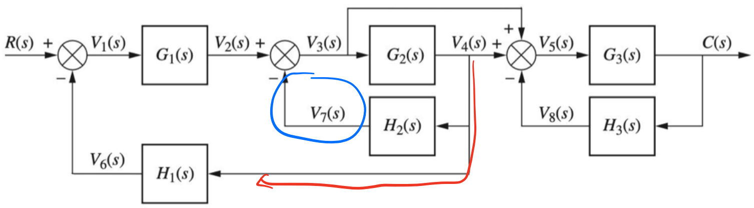


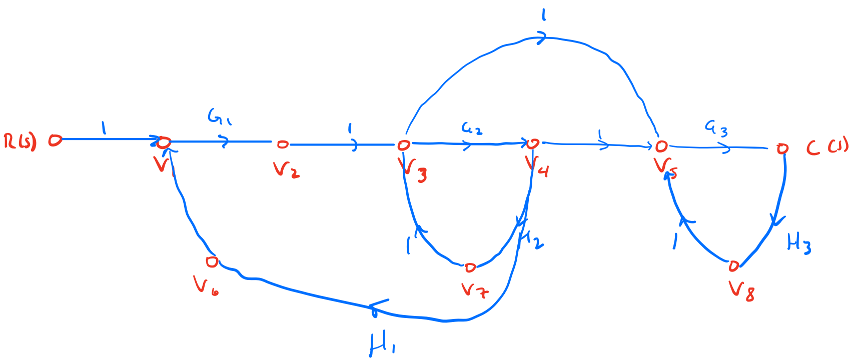

④ Black's Formula




⑤ Cascade






Signal flow graph

Look up Mason's Rule for a more formulaic way to redue block diagrams

# MATLAB tools

- **`stepinfo(sys)`**
  - retrieves step-response characteristics
- **`ord2(sys)`**
  - Gives state-space representation of a 2nd order system.
  - `[A,B,C,D]=ord2(w_n, zeta)`
- **`damp(sys)`**
  - displays poles, damping, frequency, and time constants
- **`series(sys1,sys2)`** and **`parallel(sys1, sys2)`**
  - cascaded blocks, parallel blocks
- **`feedback(sys1, sys2, sign)`**
  - Blacks Formula where sys1 = feedforward, sys2=feedback, sign = +ve/-ve feedback

Block reduction

# office hours!