6/27 ME132 Discussion!

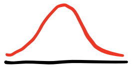
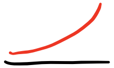
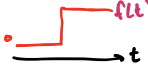
Larry

Basically Generalized Fourier Transform! → weighted one-sided FT for nasty functions!

laplace transform intuition - review

MATLAB

```
syms w t
pretty ( laplace ( sin (w*t)))
```

- We can Fourier transform functions that decay to 0 at $+\infty$ and $-\infty$ i.e. 
  - ↳ but $e^{\lambda t}$  or $u(t)\begin{cases}0, & t<0\\ 1, & t\geq 0\end{cases}$  we cannot!

- So what do we do? We know solutions to diff eq's are made of exponentials and sinusoids
  - ↳ This is true since $\frac{de^{at}}{dt} = ae^{at}$ and $\frac{d^2\sin(at)}{dt^2} = -a^2\sin(at)$ $\Longrightarrow$ so soln must contain $e^{-j\omega t}$ or e^{at}

    original fn appears in soln.

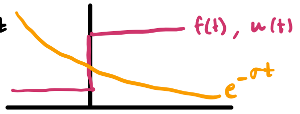
    $= \cos(-\omega t) + j\sin(-\omega t)$

- Recall the Fourier Transform maps a signal in time domain to frequency domain and is given by $\hat{F}(\omega) = \int_{-\infty}^{\infty} F(t)e^{-j\omega t}\,dt$

  basically decomposes any function into sinusoids

  - ↳ How? $x(t)\cdot e^{j\omega t}$ and sum remainder across all time for frequency info.
  - ↳ but this is restrictive ... it only contains complex exponentials (sinusoids) ... where are the exponentials ??? Why not pre-multiply it by $e^{-\sigma t}$?

SOLUTION: let $f(t)$ be any bad behaving function. Then multiply it by an exponential $e^{-\sigma t}$ so that $f(t)e^{-\sigma t} \longrightarrow 0$ as $t\to\infty$. LHS of fn might blow up as $t\to-\infty$ so now what?



↳ multiply by $u(t)$ as well so we have $F(t) = f(t)e^{-\sigma t}u(t) = \begin{cases}0 & t<0\\ f(t)e^{-\sigma t} & t\geq 0\end{cases}$ ← one-sided

← sufficiently stable

Then apply Fourier Transform to pre-multiplied signal to get

$$\hat{F}(\omega) = \int_{-\infty}^{\infty} F(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} \left(f(t)e^{-\sigma t}\right)e^{-j\omega t}\,dt$$

$$= \int_0^{\infty} f(t)e^{-(\sigma+j\omega)t}\,dt \quad , \text{let } s \hat{=} \sigma + j\omega$$

Then $\hat{F}(\omega) = F(s) = \int_0^{\infty} f(t)e^{st}\,dt$

This is the Laplace Transform!

↳ The Fourier transform of $F(t)$ is the Laplace transform of $f(t)$

Beyond scope

why? Dirac-delta $\delta(x)$ has the property $\int_{-\infty}^{\infty} e^{j\omega x}\, d\omega = 2\pi\delta(x)$

so if FT: $\hat{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$

IFT: $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\omega)e^{j\omega t}d\omega$

sub $\hat{F}(\omega)$ into $f(t)$

$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau\right)e^{j\omega t}d\omega$

By Fubini's theorem $\Rightarrow f(t) = \cancel{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(\tau)\Big(\underbrace{\int_{-\infty}^{\infty} e^{j\omega(t-\tau)}d\omega}_{\cancel{2\pi}\delta(t-\tau)}\Big)d\tau$

$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau$

By sifting property

$f(t) = f(t)$ ■

inverse laplace transform (ILT) - intuition

compensates for the use of angular frequency insted of normal freq.

↳ that's why $\frac{1}{2\pi}$

- Takes frequency domain information $F(s)$ and reconstructs the original time domain behavior

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st}F(s)ds,$$

- From above, the inverse laplace transform is the inverse Fourier Transform of $\hat{F}(\omega)$

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(\omega)e^{j\omega t}d\omega$$

← inverse Fourier Transform

recall $F(t) = f(t)\, e^{-\sigma t}u(t)$, multiply $F(t)$ by that as well

$$F(t)e^{\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sigma t}\hat{F}(\omega)e^{j\omega t}d\omega = f(t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{\overbrace{(\sigma+j\omega)}^{s}t}d\omega \quad , \quad s \triangleq \sigma + j\omega, \quad ds = j\,d\omega \Rightarrow d\omega = \frac{1}{j}ds$$

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

MATLAB

```
syms s
sys = 1/(s^2+2*s+2)/(s+3);
pretty(ilaplace(sys))
```

NOTE: $e^{st} = e^{(\sigma+j\omega)t}$

$= e^{\sigma t}e^{j\omega t}$

Real: tells you how fast signal grows/decays

Imaginary: how fast it oscillates

discussion - 1 - ILT table

This looks complicated! So convert it into a sum of simpler terms for which we know the Laplace Transform of each term. $\Rightarrow$ PARTIAL FRACTIONS

$$F(s) = \frac{2s-5}{s^2+5s+6} + \frac{5}{s^2+25}$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$
2. $\mathbf{e}^{a\,t}$	$\frac{1}{s-a}$
3. t^n, $n = 1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
6. $t^{n-\frac{1}{2}}$, $n = 1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$
8. $\cos(at)$	$\frac{s}{s^2+a^2}$

1. Partial Fraction Decomposition (PFD)

↳ If $F_1(s) = N(s)/D(s)$, ONLY able to do PFD if $\mathcal{O}(N(s)) < \mathcal{O}(D(s))$

if $\mathcal{O}(N(s)) \geq \mathcal{O}(D(s)) \Rightarrow$ divide $N(s)$ by $D(s)$ until remainder whose $\mathcal{O}(N(s)) < \mathcal{O}(D(s))$

↳ use synthetic division if $D(s)$ is binomial, o.w. polynomial long div.

↳ or use MATLAB $\Rightarrow$ syms s t $\Rightarrow$ F1 = N(s) / D(s) $\Rightarrow$ f1_t = ilaplace(F1, s, t)

$$\frac{2s-5}{s^2+5s+6} = \frac{2s-5}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

1. Factor

2. Multiply D(s) both sides

$$2s-5 = A(s+3) + B(s+2)$$

3a. Expand and match coeffs.

$$\left.\begin{aligned} 2s-5 &= As+3A+Bs+2B \\ &= (A+B)s + (3A+2B) \end{aligned}\right\} \Rightarrow \begin{aligned} A+B &= 2 \Rightarrow B = 2-A \\ 3A+2B &= -5 \end{aligned}$$

↳ $3A + 2(2-A) = -5 \Rightarrow A = -9 \Rightarrow B = 11$

OR 3b. make terms 0. (i like this!)

let $s=-3$:

$2(-3)-5 = A\cancel{(-3+3)} + B(-3+2)$

$-11 = -B \Rightarrow B = 11$

let $s=-2$

$\Rightarrow 2(-2)-5 = A(-2+3) + \cancel{B(-2+2)}$

$-9 = A$

4. you're done!

$$\frac{2s-5}{(s+2)(s+3)} = \frac{-9}{s+2} + \frac{11}{s+3}$$

2. Use table for ILT on each term

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{(s+2)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{-9}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{11}{s+3}\right\}$$

$$= -9\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 11\,\mathcal{L}\left\{\frac{1}{s+3}\right\}$$

$$= -9e^{-2t} + 11e^{-3t}$$

2. Second term?

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\} = \sin(5t)$$

3. Now What?

↳ recall LT and ILT is a linear operator so by superposition

the ILT of $F(s)$ is:

$$\boxed{F(s) = -9e^{-2t} + 11e^{-3t} + \sin(5t)}$$

More on PFD

CASE 1: Roots of $D(s)$ are real + distinct

called residues

i.e. $\frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$ [Stop here in discussion]

(multiplied both sides by $D(s)$

$2 = A(s+2) + B(s+1)$

let $s = -2$:

$2 = A\cancel{(-2+2)}^{0} + B(-2+1)$

$2 = B(-1) \Rightarrow B = -2$

let $s = -1$:

$2 = A(-1+2) + B\cancel{(-1+1)}^{0}$

$2 = A(1) \Rightarrow A = 2$

$$\frac{2}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-2}{s+2}$$

CASE 2: Roots of $D(s)$ are real + repeated

i.e. $\frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$ [Stop here in discussion]

CASE 3: Roots of $D(s)$ are complex / imaginary

i.e. $\frac{3}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$ [Stop here in discussion]

Other techniques for ILT:

• use the LT theorems.

i.e. $F_1(s) = \frac{1}{(s+3)^2}$

By frequency shift: $\mathcal{L}[e^{-at}f(t)] = F(s+a)$

and $\mathcal{L}\{tu^t\} = \frac{1}{s^2}$

if $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = tu(t)$, then if

$$\mathcal{L}^{-1}\{F(s+a)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+a)^2}\right\}$$

$$= e^{-at}\,tu(t)$$

$\therefore f_1(t) = e^{-3t}\,tu(t)$

CASE 4: MATLAB

Consider $\frac{B(s)}{A(s)} = \frac{2s^3+5s^2+3s+6}{s^3+6s^2+11s+6}$

num = [2 5 3 6]

den = [1 6 11 6]

[r, p, k] = residue (num, den)

$\Rightarrow$ r = −6, −4, 3   p = −3, −2, −1   k = 2

$$\therefore \frac{B(s)}{A(s)} = \frac{-6}{s+3} + \frac{-4}{s+2} + \frac{3}{s+1} + 2$$

the opposite also works

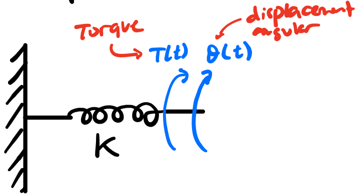
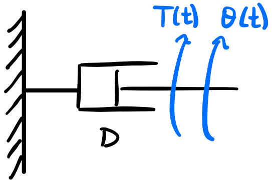
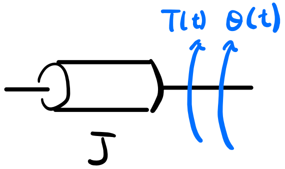
[num, den] = residue (r, p, k)

print(num, den, 's')

$$\text{num/den} = \frac{2s^3+5s^2+3s+6}{s^3+6s^2+11s+6}$$

rotational mechanical systems

- Basically the same as dealing with translational mechanical systems: $F(t)\,[N] \rightarrow T(t)\,[N\cdot m]$, $x(t)\,[m] \rightarrow \theta(t)\,[rad]$
- same components as well: spring; damper; mass but they are rotated instead of stretched.

Component		Torque-angular velocity	Torque-Displacement	Impedences (zero IC's)
Spring		$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T \propto \theta(t)$ so $T(t) = K\theta(t)$ [N·m/rad]	K
damper		$T(t) = D\omega(t)$	$T \propto \omega(t) = \frac{d\theta(t)}{dt}$ [rad/s] so $T(t) = D\frac{d\theta(t)}{dt}$ [N·m·s/rad]	Ds
inertial element		$T(t) = J\frac{d\omega(t)}{dt}$	$T \propto \alpha(t) \Rightarrow T(t) = J\alpha(t)$ $\Rightarrow T(t) = J\frac{d^2\theta(t)}{dt^2}$ [kg·m²]	Js^2

J: moment of inertia
property of rigid body that measures resistance to a change in angular position/velocity.

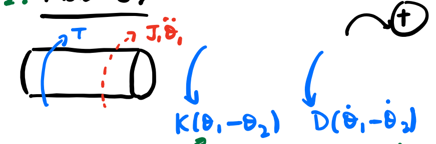
STEPS TO SOLVE:

1. Identify the moments of inertia
2. Draw FBD's
   ↳ only look at forces/torques directly acting on object!
3. Apply $\sum T = J\ddot{\theta}$ for EoMs!
4. Laplace Transform!
   ↳ Assume IC's are 0 i.e. $x(0)=0$, $\dot{x}(0)=0$
5. Solve for Transfer Function
   ↳ apply Cramer's Rule

discussion - 2 - rotational mechanical q

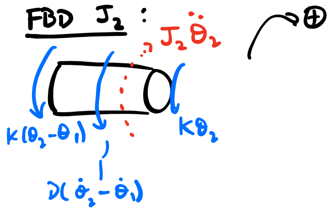
1. J_1 and J_2 are moments of inertia

2. FBD J_1

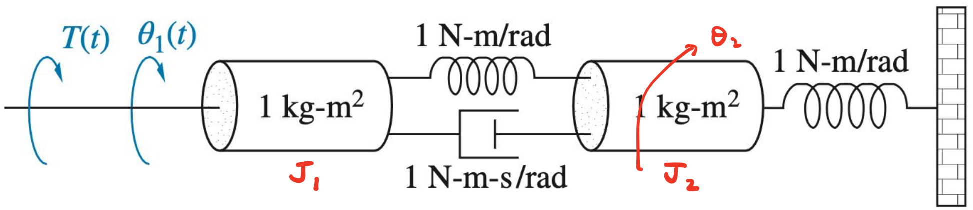


the mass we're looking at
subtract the mass that sandwhiches
the spring

FBD J_2:



34. Find the transfer function, $\dfrac{\theta_1(s)}{T(s)}$, for the system shown in Figure P2.19.



3. FBD J_1:

$\Sigma T = J_1\ddot{\theta}_1$

$T - K(\theta_1-\theta_2) - D(\dot{\theta}_1-\dot{\theta}_2) = J_1\ddot{\theta}_1$

For FBD J_2:

$\Sigma T = J_2\ddot{\theta}_2$

$-K(\theta_2-\theta_1) - K\theta_2 - D(\dot{\theta}_2-\dot{\theta}_1) = J_2\ddot{\theta}_2$

4. Rearrange and Laplace

$T(s) = J_1\theta_1 s^2 + K(\theta_1-\theta_2) + Ds(\theta_1-\theta_2)$

$T(s) = (s^2+s+1)\theta_1 - (s+1)\theta_2$ —— (1)

$0 = J_2\theta_2 s^2 + K_2\theta_2 + K(\theta_2-\theta_1) + Ds(\theta_2-\theta_1)$

$0 = (s^2+s+2)\theta_2 - (s+1)\theta_1$ —— (2)

$$\boxed{\frac{\theta_1}{T(s)} = \frac{s^2+s+2}{s^4+2s^3+3s^2+s+1}}$$

5. Cramer's Rule

$$\begin{bmatrix} s^2+s+1 & -(s+1) \\ -(s+1) & s^2+s+2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} T(s) \\ 0 \end{bmatrix}$$

MATLAB OR/CRAMER

$$\theta_1 = \frac{\det A_1}{\det A} = \frac{\det\begin{bmatrix} T(s) & -(s+1) \\ 0 & s^2+s+2 \end{bmatrix}}{\det\begin{bmatrix} s^2+s+1 & -(s+1) \\ -(s+1) & s^2+s+2 \end{bmatrix}} = \frac{T(s)(s^2+s+2)}{(s^2+s+1)(s^2+s+2)-(s+1)(s+1)}$$

alternative method to representing physical systems

THIS IS MODERN CONTROL THEORY

why? It can deal with MIMO systems, non-linear, time-varying

↳ Examples: LQR, Kalman Filtering, Luenberger Observers

↳ Time domain emphasis

state space (continuous time)

- how the sys. is changing is a function of the current state → "snapshot" of all necessary variables to describe the system at a moment in time.

$$\dot{\vec{x}} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u}$$
$$\vec{y} = \mathbf{C}\vec{x} + \mathbf{D}\vec{u}$$

- For an arbitrary dynamic system, look at how energy changes by analyzing relationship b/w states and derivatives

derivatives → $\dot{x} = f(x)$ ← state

defines stability for system. More to come in the next couple weeks!

$\dot{x} = f(x, u)$ ← inputs

defines how a system changes due to external Inputs

- state-space: repackaging High order diff. eq. into set of 1st order diff. eq. that focus on this relationship.

how the state vector changes

Lin. combo.

State vector: vector of all states

- SEE: $\dot{x}(t) = \boxed{A}\, x(t) + \boxed{B}\, u(t)$

lin comb of inputs

how internal states connect to each other

how inputs enter system, which states do they affect?

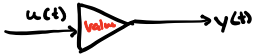
- OE: $y(t) = \boxed{C}\, x(t) + \boxed{D}\, u(t)$

output of system ≠ State vars.

how states combine to get outputs

↳ allow input to bypass the system to feed forward to output

i.e.



$y(t) = \text{value} \cdot u(t)$

$\Rightarrow A = B = C = 0$, $D = \text{value}$

- Difference b/w TF's?

↳ TF: dynamics of system end-to-end

↳ SS: dynamics of each state variable and how they relate.

partitioned matrix

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \Leftarrow \overbrace{\left[\begin{array}{c|c} \underset{\text{STATE DYNAMICS}}{\mathbf{A}_{n\times n}} & \underset{\text{INPUT MATRIX}}{\mathbf{B}_{n\times m}} \\ \hline \underset{\text{OUTPUT MATRIX}}{\mathbf{C}_{k\times n}} & \underset{\text{DIRECT FEEDTHROUGH}}{\mathbf{D}_{k\times m}} \end{array}\right]}^{\text{partitioned matrix}}$$

$A \in \mathbb{R}^{n\times n}$ $\quad C \in \mathbb{R}^{k\times n}$

$B \in \mathbb{R}^{n\times m}$ $\quad D \in \mathbb{R}^{k\times m}$

PROCEDURE

1. system vars: select a subset of all possible sys. vars. as states and determine I/Os
2. State Diff. Eq.: Write n- simultaneous, 1st order diff. eq. of the states in terms of the states and inputs for an n^{th} order system.
3. Initial condition: If we know the IC's for all the states at t_0 and inputs for $t \geq t_0$, we can solve the simultaneous set of diff eq's. for states $t \geq t_0$
4. output-state relation equations: write linear relations of outputs in terms of the states and inputs for $t \geq t_0$
5. State-Space representation: state and output eqn's represent a viable representation of the system

What are state-variables?

any var. that responds to an input or initial condition in a system.

- Formally: "non-unique set of L.I. system variables s.t. the values of the members of the set at time t_0 along with known inputs completely determine the value of all system variables $\forall t > t_0$.

↳ BASICALLY: min. set. of variables that $\underbrace{\text{fully describe}}_{\text{enough info to predict future behaviour}}$ the system.

L.C.: a lin combo of n vars, x_i for $i \in [1, n]$ is given by the following sum S

$$S = \alpha_n x_n + \alpha_{n-1} x_{n-1} + \dots + \alpha_1 x_1$$

where $\alpha \in \mathbb{R}$ i.e. constant

L.I.: none of the vars can be written as a L.C. of the others.

Vars $x_1, \dots, x_n$ are L.I. if $S = 0 \iff \alpha_i = 0$ and no $x_i = 0 \ \forall t \geq 0$

NOTE: • min number of states required to describe a system equals the order of the system diff. eq.

↳ you can define more than the min set; however the minimal set of states must be L.I.

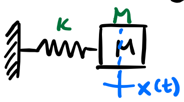
• IO: SISO systems are a unique case of MIMO systems

↳ IO for SISO ⇒ scalar

↳ IO for MIMO ⇒ vector

state space for mechanical systems

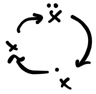
- For a spring-mass system



$M\ddot{x}(t) = -Kx(t)$

$\ddot{x}(t) = -\frac{K}{M}x(t)$ ← acceleration is a function of position (the current state)

[consider initializing the sys w/ energy i.e. stretching spring]

↳ 

1. What are the states?

- where will the system be in 1 second? What do we need to know?

IC's:

① is mass moving currently? we know this from $\boxed{\text{velocity}}$

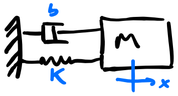
② how hard is spring pulling? $F_s = K \cdot \Delta x$ ↑ $\boxed{\text{position}}$

2 state vars ↓ 2nd order system. ↓ typically min states is the order of the diff. eq.

Why not acceleration?

$a = \frac{F}{m} = \frac{K \cdot \Delta x}{m}$ ← we already have this.

If you added a damper?



$F_{damper} = b \cdot \boxed{\text{velocity}}$

$\therefore$ 2nd order system

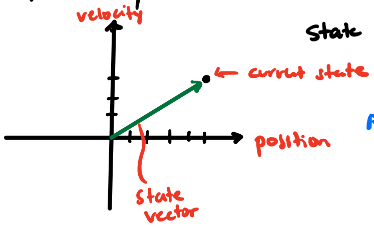
2. STATE SELECTION

- choose position and velocity of each point of L.I. motion

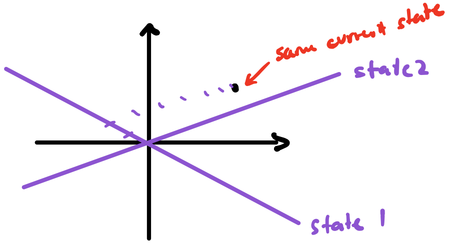
3. Why not 2 diff. states?

↳ well they can be! SS representation are NOT UNIQUE!

State space

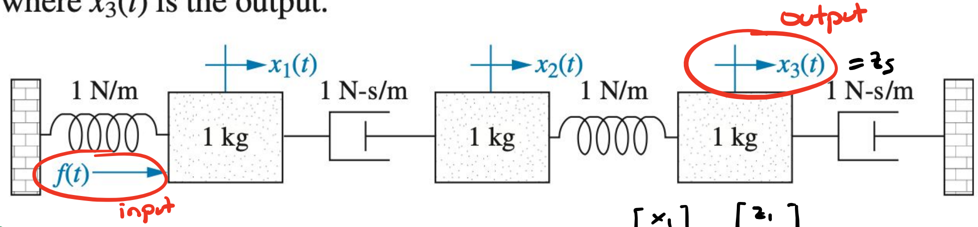


same state
same total energy
diff. coordinates



discussion - 3 - mechanical system state space

PROBLEM: Represent the translational mechanical system shown in Figure 3.9 in state space, where $x_3(t)$ is the output.



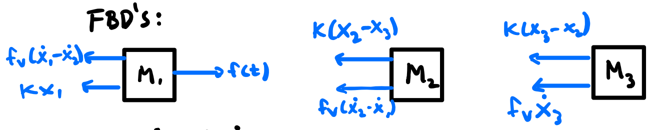
1. STATE VARIABLES

- Each mass has pos. + vel. and 3 masses so 6 state vars

$$\underline{z} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix}$$

2. EoM

FBD's:



$$\Sigma F_{x_1} = m\ddot{x}_1 = f(t) - f_v\dot{x}_1 + f_v\dot{x}_2 - Kx_1$$

$$\ddot{x}_1 = f(t) - \dot{x}_1 + \dot{x}_2 - x_1 \quad \text{—— (1)}$$

$$\Sigma F_{x_2} = m\ddot{x}_2 = Kx_3 - Kx_2 + f_v\dot{x}_1 - f_v\dot{x}_2$$

$$\ddot{x}_2 = x_3 - x_2 + \dot{x}_1 - \dot{x}_2 \quad \text{—— (2)}$$

$$\Sigma F_{x_3} = m\ddot{x}_3 = Kx_2 - Kx_3 - f_v\dot{x}_3$$

$$\ddot{x}_3 = x_2 - x_3 - \dot{x}_3 \quad \text{—— (3)}$$

$z_1 = x_1$
$z_2 = \dot{x}_1$
$z_3 = x_2$
$z_4 = \dot{x}_2$
$z_5 = x_3$
$z_6 = \dot{x}_3$

State vars

3. Take Derivatives of state vars to rewrite 2nd order EoMs into 1st order

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = (1) = f(t) - \dot{x}_1 + \dot{x}_2 - x_1 = -z_1 - z_2 + z_4 + f(t)$$

$$\dot{z}_3 = z_4$$

$$\dot{z}_4 = (2) = x_3 - x_2 + \dot{x}_1 - \dot{x}_2 = z_2 - z_3 - z_4 + z_5$$

$$\dot{z}_5 = z_6$$

$$\dot{z}_6 = (3) = x_2 - x_3 - \dot{x}_3 = z_3 - z_5 - z_6$$

4. Recall SEE and OE

$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \\ \dot{z}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} f(t)$$

$$y = Cx + Du$$

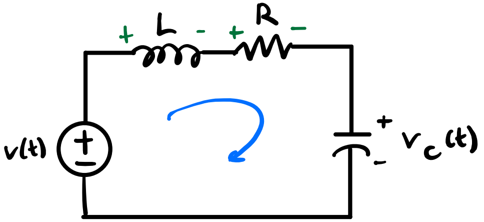
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix}$$

circuits rundown (KVL+KCL)

→ Read 2.4 for help. ME100 should be required.

1. Mesh Analysis

• Use KVL + Ohm's Law : KVL: In a closed loop $\sum V = 0$. +ve voltage contribution $- \rightarrow +$
-ve voltage contribution $+ \rightarrow -$
(if loop flows with current)



KVL: $\sum V = 0$

$$v(t) - v_L - v_R - v_c = 0$$

$$\mathcal{L}\left\{v(t) - L\frac{di(t)}{dt} - Ri(t) - \frac{1}{C}\int_0^t i(\tau)\,d\tau\right\} = 0$$

$$V(s) - LsI(s) - RI(s) - \frac{1}{Cs}I(s) = 0 \Rightarrow V(s) = \left(Ls + R + \frac{1}{Cs}\right)I(s)$$

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}$$

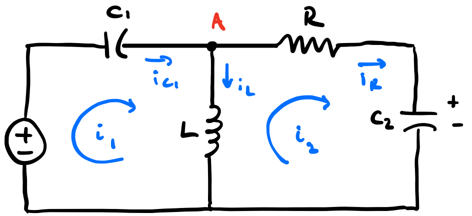
but $V_c(s) = I(s)\frac{1}{Cs}$

$\Rightarrow I(s) = CsV_c(s)$

$$\frac{CsV_c(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}} \quad \ldots$$

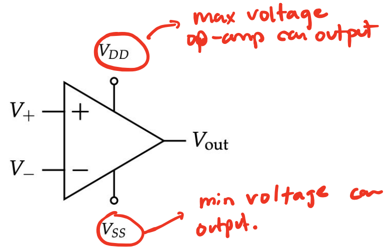
2. Nodal Analysis (Usually Easier)

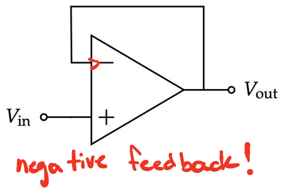
• Use KCL at nodes. KCL: $\sum i_{in} = \sum i_{out}$ at node, $I = \frac{V}{Z}$



$i_{C_1} = i_R + i_L$ at node A

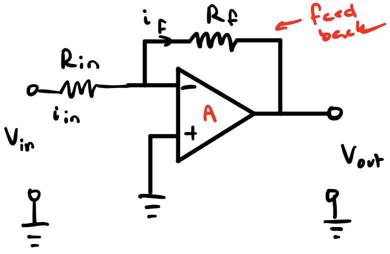
quick note on op-amps





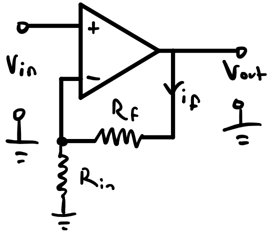
An op-amp (operational amplifier) in its basic form, amplifies small input voltage signal into a larger output voltage. ACTIVE component i.e. requires power

1. Since **some** students are not familiar with op-amps, or circuits, you will **NOT** have a circuit question on the **Midterm or Final**, only on the homeworks.
2. We will always assume IDEAL op-amps which means we follow the “golden rules”
   a. Zero input current into signal terminals: $I_+ = I_- = 0$.
   b. For negative feedback, $V_+ = V_-$
3. The following gain formulas will be helpful in Q1 of HW2



$A = \frac{V_{out}}{V_{in}} = -\frac{R_f}{R_{in}}$

INVERTING



$A = \frac{V_{out}}{V_{in}} = 1 + \frac{R_f}{R_{in}}$

NON-INVERTING

electrical network components

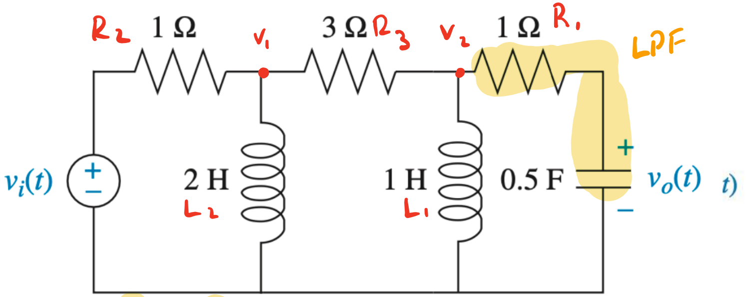
(passive) component	Voltage-current	current-voltage	Voltage-charge (probably never use)	Impedance $Z(s)=V(s)/I(s)$	Admittance $(Y(s)=I(s)/V(s))$
capacitor (stores energy in electric field) ↳ opposes change in voltage	$\underbrace{v(t)}_{[V]}=\frac{1}{\underbrace{C}_{[F]}}\int_0^t i(\tau)d\tau$	$i(t)=C\frac{dv(t)}{dt}$	$v(t)=\frac{1}{C}q(t)$	$1/Cs$	Cs
resistor (resists flow of e^-)	Ohm's Law $v(t)=\underbrace{R}_{[\Omega]}\underbrace{i(t)}_{[A]}$	$i(t)=\frac{1}{R}v(t)$	$v(t)=R\frac{dq(t)}{dt}$	R	$\frac{1}{R}=\underbrace{G}_{[\Omega\cdot10^{-3}]}$
inductor (stores energy in magnetic field) ↳ opposes change in current	$v(t)=\underbrace{L}_{[H]}\frac{di(t)}{dt}$	$i(t)=\frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t)=L\frac{d^2q(t)}{dt^2}$	Ls	$1/Ls$

$$i(t)=\frac{dq(t)}{dt}$$

discussion - 4a - circuit TF

$\| \Rightarrow$ parallel i.e. $\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2}$

$\Rightarrow Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2}$



$$V_o = \frac{Z_c}{Z_{R_1} + Z_c} V_2 = \frac{1/Cs}{R_1 + 1/Cs} = \frac{1}{0.5s + 1} V_2 =$$

$$V_2 = \left[1 - \frac{Z_{R_3}}{Z_{R_3} + (Z_{R_1} + Z_c) \| Z_{L_1}} \right] V_1$$ ← voltage divider

$$\frac{(Z_{R_1} + Z_c) \| Z_{L_1}}{Z_{R_3} + (Z_{R_1} + Z_c) \| Z_{L_1}}$$

$$= \left[1 - \frac{3}{3 + \frac{(1 + \frac{2}{s})s}{\cancel{4}s + 1 + \frac{2}{s}}} \right] V_1 = \left[1 - \frac{3}{3 + \frac{(s+2)s}{s^2 + s + 2}} \right] V_1$$

$$= \left[1 - \frac{3s^2 + 3s + 6}{4s^2 + 5s + 6} \right] V_1$$

$$= \left[\frac{s^2 + 2s}{4s^2 + 5s + 6} \right] V_1$$

Now, deal with inside loop

$$V_1 = \left[1 - \frac{R_2}{R_2 + Z_{L_2} \| R_3 + (Z_{R_1} + Z_c) \| Z_{L_1}} \right] V_i$$

$$= \left[1 - \frac{1}{1 + \frac{2s\left(3 + \frac{(s+2)s}{s^2+s+2}\right)}{\underbrace{2s}_{Z_{L_2}} + 3 + \frac{(s+2)s}{s^2+s+2}}} \right]$$

$\vdots$

$$V_1 = \frac{8s^3 + 10s^2 + 12s}{10s^3 + 16s^2 + 21s + 6} V_i$$

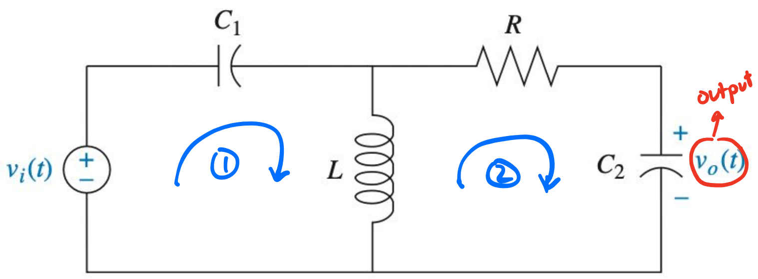
To find TF: $\frac{V_o}{V_i}$, multiply

$$\frac{V_o}{V_i} = \frac{V_o}{V_2} \cdot \frac{V_2}{V_1} \cdot \frac{V_1}{V_i}$$

$$= \left(\frac{1}{\cancel{0.5s+1}} \right) \left(\frac{\cancelto{2s}{s^2 + 2s}}{\cancel{4s^2 + 5s + 6}} \right) \left(\frac{\cancelto{2s}{8s^3 + 10s^2 + 12s}}{10s^3 + 16s^2 + 21s + 6} \right)$$

$$\boxed{\frac{V_o}{V_i} = \frac{4s^2}{10s^3 + 16s^2 + 21s + 6}}$$

discussion - 4b - circuit state space



1. Choose state variables based on energy-storage elements

$$x_1 = v_{c_1},\quad x_2 = i_L,\quad x_3 = v_{c_2} \Rightarrow \underline{x} = \begin{bmatrix} v_{c_1} \\ i_L \\ v_{c_2} \end{bmatrix}$$

Also

$$\dot{v}_{c_1} = \frac{i_{c_1}}{C_1},\quad \dot{i}_L = \frac{v_L}{L},\quad \dot{v}_{c_2} = \frac{i_{c_2}}{C_2} \qquad \dot{\underline{x}} = \begin{bmatrix} \frac{dv_{c_1}}{dt} \\ \frac{di_L}{dt} \\ \frac{dv_{c_2}}{dt} \end{bmatrix}$$

2. Develop state equations

KCL @ Node A: $\sum i_{in} = \sum i_{out}$

$$i_{c_1} = i_R + i_L$$

$$C_1 \dot{v}_{c_1} = i_R + i_L$$

KVL loop 2

$$v_L = v_R + v_{c_2}$$

$$v_L = i_R R + v_{c_2}$$

$$i_R = \frac{1}{R}(v_L - v_{c_2})$$

$$C_1 \dot{v}_{c_1} = i_L + \frac{1}{R}(v_L - v_{c_2})$$

$$\dot{v}_{c_1} = \frac{1}{C_1}\left[i_L + \frac{1}{R}(v_i - v_{c_1} - v_{c_2})\right]$$

$$= \frac{i_L}{C_1} + \frac{1}{RC_1}v_i - \frac{1}{RC_1}v_{c_1} - \frac{1}{RC_1}v_{c_2} \quad \text{— (1)}$$

$v_L = v_i - v_{c_1}$ from KVL loop 1

$$\dot{i}_L = \frac{1}{L}(v_i - v_{c_1}) \Rightarrow \frac{di_L}{dt} = \frac{1}{L}v_i - \frac{1}{L}v_{c_1} \quad \text{— (2)}$$

Now, find $\dot{v}_{c_2}$:

$$i_R = \frac{1}{R}(v_L - v_{c_2})\ ;\ v_L = v_i - v_{c_1},\ i_R = i_{c_2}$$

$$i_{c_2} = \frac{1}{R}(v_i - v_{c_1} - v_{c_2})$$

$$C_2\frac{dv_{c_2}}{dt} = \frac{1}{R}(v_i - v_{c_1} - v_{c_2}) \Rightarrow \dot{v}_{c_2} = \frac{1}{RC_2}(v_i - v_{c_1} - v_{c_2}) \Rightarrow \frac{dv_{c_2}}{dt} = \frac{1}{RC_2}v_i - \frac{1}{RC_2}v_{c_1} - \frac{1}{RC_2}v_{c_2} \quad \text{— (3)}$$

3. Plug into state-space

SEE:

$$\begin{bmatrix} \frac{dv_{c_1}}{dt} \\ \frac{di_L}{dt} \\ \frac{dv_{c_2}}{dt} \end{bmatrix} = \begin{bmatrix} -1/RC_1 & 1/C_1 & -1/RC_1 \\ -1/L & 0 & 0 \\ -1/RC_2 & 0 & -1/RC_2 \end{bmatrix} \begin{bmatrix} v_{c_1} \\ i_L \\ v_{c_2} \end{bmatrix} + \begin{bmatrix} 1/RC_1 \\ 1/L \\ 1/RC_2 \end{bmatrix} v_i$$

OE:

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{c_1} \\ i_L \\ v_{c_2} \end{bmatrix} + [0]\, v_i$$

state space & transfer functions

$$\dot{\mathbf{x}} = \overbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{bmatrix}}^{A} \underline{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}^{B} \underline{u}$$

$$y = \underbrace{\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}}_{C} \underline{\mathbf{x}} \,.\, + D\underline{u}$$

$$H(s) = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1} s^{n-1}}{b_0 + b_1 s + b_2 s^2 + \ldots b_{n-1} s^{n-1} + s^n}$$

Controllable Canonical Form

$$H(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Given state space evolution eqn (1) and output eqn (2)
(SEE) (OE)

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \quad (1)$$

inputs

$$\underline{y} = C\underline{x} + D\underline{u} \quad (2)$$

Take Laplace Transform of both (1) and (2) with zero initial conditions

component wise operation in the vector

$$\mathcal{L}\{\dot{\underline{x}}\} = \mathcal{L}\{\underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}\}$$

$$s\underline{X}(s) = \underline{\underline{A}}\,\underline{X}(s) + \underline{\underline{B}}\,\underline{U}(s)$$

$$\Downarrow$$

$$s\underline{X}(s) - \underline{\underline{A}}\,\underline{X}(s) = \underline{\underline{B}}\,\underline{U}(s)$$

$$(s\underline{\underline{I}} - \underline{\underline{A}})X(s) = \underline{\underline{B}}\,\underline{U}(s)$$

$$X(s) = (s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}}\,\underline{U}(s)$$

and

$$\mathcal{L}\{\underline{y}\} = \mathcal{L}\{\underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u}\}$$

$$\underline{Y}(s) = \underline{\underline{C}}\,\underline{X}(s) + \underline{\underline{D}}\,\underline{U}(s)$$

substitute

$$\underline{Y}(s) = \underline{\underline{C}}[(s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}}\,\underline{U}(s)] + \underline{\underline{D}}\,\underline{U}(s)$$

$$= [\underline{\underline{C}}(s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}} + \underline{\underline{D}}]\,\underline{U}(s)$$

same but expanded / compact

$$\therefore \boxed{H(s) = \frac{Y(s)}{U(s)} = \underline{\underline{C}}(s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}} + \underline{\underline{D}}}$$

Why CCF?

↳ ensures controllability matrix is full rank

discussion - 5 - state space to TF

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$y = [1\ 0\ 0]\ \underline{x}$$

1. By hand

$$(s\underline{\underline{I}} - \underline{\underline{A}})^{-1} = \left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \right)^{-1}$$

$$= \left(\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix} \right)^{-1}$$

$$\left[\begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & s & -1 & 0 & 1 & 0 \\ 1 & 2 & s+3 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_1 : R_1/s$$

$$\left[\begin{array}{ccc|ccc} 1 & -1/s & 0 & 1/s & 0 & 0 \\ 0 & s & -1 & 0 & 1 & 0 \\ 1 & 2 & s+3 & 0 & 0 & 1 \end{array}\right]$$

...

$$\frac{\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1} = (sI-A)^{-1}$$

$$C(sI-A)^{-1}B + \cancel{D}^{\,0} = H(s)$$

$$[1\ 0\ 0]\ (sI-A)^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} = H(s)$$

$$\boxed{H(s) = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix} \mathbf{x}\ .$$

$$H(s) = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1} s^{n-1}}{b_0 + b_1 s + b_2 s^2 + \ldots b_{n-1} s^{n-1} + s^n}$$

discussion - 6 - TF to state space

$$G(s) = \frac{Y(s)}{U(s)} \qquad \frac{s+3}{s^3 + 9s^2 + 24s + 20} \cdot \frac{X(s)}{X(s)}$$

$$Y(s) = sX(s) + 3X(s)$$

$$U(s) = s^3X(s) + 9s^2X(s) + 24sX(s) + 20X(s)$$

$\downarrow \mathcal{L}^{-1}$

$$y(t) = \dot{x}(t) + 3x(t)$$

$$u(t) = \dddot{x}(t) + 9\ddot{x}(t) + 24\dot{x}(t) + 20x(t)$$

let $x_1 = x$ $\qquad y(t) = x_2 + 3x_1$

$x_2 = \dot{x}_1 = \dot{x}$ $\Rightarrow$ $u(t) = \dot{x}_3 + 9x_3 + 24x_2 + 20x_1$

$x_3 = \ddot{x}$ $\qquad \dot{x}_3 = -20x_1 - 24x_2 - 9x_3 + u(t)$

$$\left.\begin{aligned} \Rightarrow \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \underline{x} \end{aligned}\right\} \text{CCF}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix} \mathbf{x} \ .$$

$$H(s) = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1} s^{n-1}}{b_0 + b_1 s + b_2 s^2 + \ldots b_{n-1} s^{n-1} + s^n}$$

ss2tf - MATLAB symbolic toolbox

```
A = [0 1; -1 -2]
B = [0; 1]
C = [1 0]
D = 0
```

```
syms s

transfunc = 1/(s^2+2*s+1)
[num, den] = ss2tf(A, B, C, D)

tf(num, den)
```

```
C*inv(s*eye(2)-A)*B+D
```

```
A = 2x2
     0     1
    -1    -2

B = 2x1
     0
     1

C = 1x2
     1     0

D = 0
transfunc =
```

$$\frac{1}{s^2+2s+1}$$

```
num = 1x3
     0     0     1

den = 1x3
     1     2     1

ans =

       1
  -------------
  s^2 + 2 s + 1

Continuous-time transfer function.
Model Properties
ans =
```

$$\frac{1}{s^2+2s+1}$$

office hours!