6/20 ME132 Discussion!

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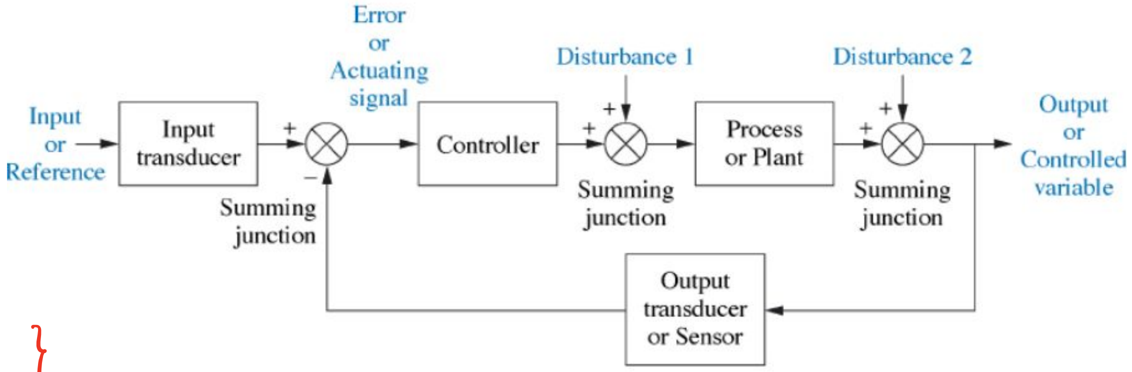
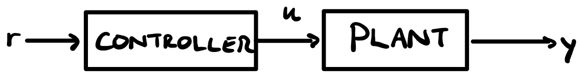
block diagrams



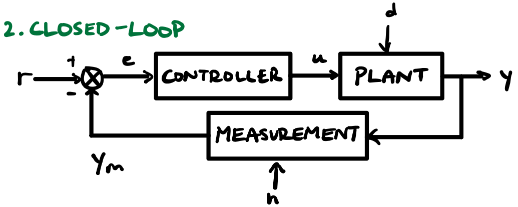
1. OPEN-LOOP



{ NO FEEDBACK
NOT CONTROLLED }

Example. Washer / Dryer

2. CLOSED-LOOP



{ FEEDBACK
CONTROL }

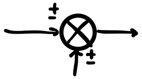
y: output
d: disturbance
u: control input
n: noise

e: error signal
y_m: measured output
r: reference input

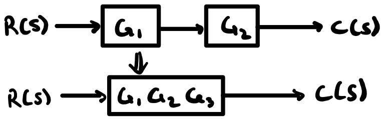
PLANT: Any physical object or process to be controlled

CONTROLLER: A Mechanism that seeks to minimize the difference b/w the desired and actual value.

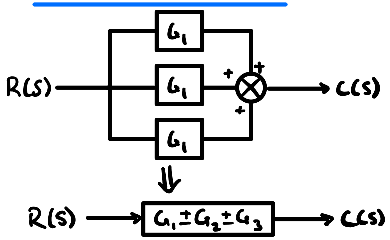
SUMMING JUNCTION: Adds signals based on +/-



CASCADED FORM



PARALLEL FORM



LTI systems

- Linear

ZIZO

INPUT $u(t)$ → system → $y(t)$ output **OR** $y(t) = f(u(t))$

only Linear:

1) Scaling Property: $f(\alpha u(t)) = \alpha f(u(t))$
2) Additivity: $f(u_1(t) + u_2(t)) = f(u_1(t)) + f(u_2(t))$

$$f(\alpha u_1(t) + \beta u_2(t)) = \alpha f(u_1(t)) + \beta f(u_2(t))$$

- Time-Invariant

only Time-invariant if

$$\boxed{f(u(t-\tau)) \overset{\Delta}{=} y(t-\tau)}$$

Ex. $y(t) = u(-t) = f(u(t))$

LHS: $f(u(t-\tau)) = u(-t-\tau)$

RHS: $y(t-\tau) = u(-(t-\tau))$

$= u(-t+\tau)$

Ex. $y(t) = \int_0^t u(\tau)\,d\tau$ for $u(t) = 0$ for $t=0$

LHS: $f(u(t-\tau)) = \int_0^t u(z-\tau)\,dz$

RHS: $y(t-\tau) = \int_0^{t-\tau} u(\tau)\,d\tau$

Same T.I.

Linear example

Ex. $y(t) = f(u(t)) = e^{u(t)}$

$$f(\alpha u_1(t) + \beta u_2(t)) = \alpha f(u_1(t)) + \beta f(u_2(t))$$

LHS: $f(\alpha u_1(t) + \beta u_2(t)) = e^{\alpha u_1(t) + \beta u_2(t)}$

$$= e^{\alpha u_1(t)} e^{\beta u_2(t)}$$

$\times$

RHS: $\alpha f(u_1(t)) + \beta f(u_2(t)) = \alpha e^{u_1(t)} + \beta e^{u_2(t)}$

$$C_0(x) y + C_1(x) \frac{dy}{dx} + \dots + C_k(x) \frac{d^k y}{dx^k} + \alpha(x) -$$

differential equations

- Linear ~ $\frac{dy}{dx} + P(x)\,y = Q(x)$

Ex. $\frac{dy}{dx} + \frac{y}{x} = 3x$

$Q(x) = 3x$

$P(x) = \frac{1}{x}$

$\boxed{I \triangleq \exp \int P(x)\,dx} = \exp \int \frac{1}{x}\,dx = \exp(\ln(x)) = x$

formula.

$I(x)\,y = \int I(x)\,Q(x)\,dx$

$xy = \int x \cdot 3x\,dx$

$xy = \int 3x^2\,dx$

$xy = x^3 + c$

$\boxed{y = x^2 + \frac{c}{x}}$

- Solving Differential Eqns: Homogenous & Particular Solutions

$ay'' + by' + cy + d = 0$ ← homogenous

$Y = Y_h + Y_p$

discussion - 1 - 2nd order diffeq example

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$$\frac{d^2y}{dx^2} = r^2$$

$$\frac{dy}{dx} = r$$

$$y = 1$$

Recall characteristic equation

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$r = -1$, multiplicity 2.

repeated roots: $y(t) = c_1 e^{rt} + c_2 t e^{rt}$

$$\boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t}}$$

$y(0) = 0$

$y'(0) = 1$

discussion - 2- harder diffeq setup

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 5\cos(x)$$

$y(t) = y_h(t) + y_p(t)$

In the form of $y'' + by' + cy = x^m e^{ax} \cos(bx)$ or $\sin(bx)$

try: $y = x^{s}(A_m t^m + \cdots + A_1 t + A_0) e^{at} \cos(bt)$

$\quad + x^s(B_m t^m + \cdots + B_1 t + B_0) e^{at} \sin(bt)$

How to choose s? $\Rightarrow$ check characteristic polynomial

① $a+ib$ not a root : $s=0$, $b\neq 0$

② $a+ib$ is a root : $s=1$, $b\neq 0$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 5\cos(x)$$

$m = 0, \quad a = 0, \quad b = 1$

$$y = x^0 (A_0) e^0 \cos(bt) + x^0 (B_0) e^0 \sin(bt)$$

$b = 1$ $\quad$ $b = 1$

$$y_p = A_0 \cos(bt) + B_0 \sin(bt)$$

$$y_p' = -A_0 \sin(t) + B_0 \cos(t)$$

$$y_p'' = -A_0 \cos(t) - B_0 \sin(t)$$

$\downarrow$ sub. into original eqn.

$\leftarrow$ method of undetermined coeffs.

$$\boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t} + A_0 \cos(t) + B_0 \sin(t)}$$

laplace transforms - intuition

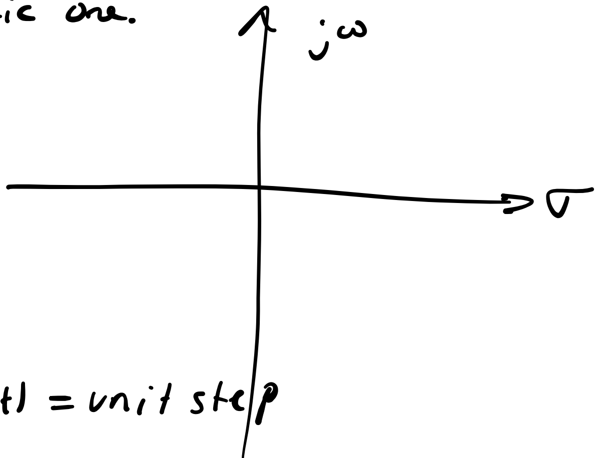
- We use it to represent systems in time and frequency. Laplace transform allows us to go from time domain $\rightarrow$ freq.
  $f(t)$ $F(s)$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$s = \sigma + j\omega$$

Real   Im

- For ODEs, they turn a calculus problem into an algebraic one.

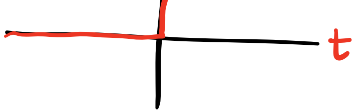


$$\text{ODE} \xrightarrow[\mathcal{L}]{} \text{Algebraic fn.} \xrightarrow[\mathcal{L}^{-1}]{} \text{solution.}$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s)\, e^{-st}\, ds = f(t)\, u(t), \quad u(t) = \text{unit step}$$

$$= \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$u(t-\tau)$



discussion - 3 - laplace derivation

$$f(t) = \sin(5t)$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(5t)\}$$

$$\mathcal{L}\{\sin(5t)\} = \int_0^\infty e^{-st}\sin(5t)\,dt$$

$u = \sin(5t)$ $\qquad dv = e^{-st}$

u-sub

$du = 5\cos(5t)$ $\qquad v = -\frac{1}{s}e^{-st}$

$$= -\frac{1}{s}e^{-st}\sin(5t) + \frac{5}{s}\int e^{-st}\cos(5t)$$

$u = \cos(5t)$ $\qquad dv = e^{-st}$

$du = -5\sin(5t)$ $\qquad v = -\frac{1}{s}e^{-st}$

$$\int_0^\infty e^{-st}\sin(5t)\,dt = -\frac{1}{s}e^{-st}\sin(5t) - \frac{5}{s^2}e^{-st}\cos(5t) - \frac{25}{s^2}\int e^{-st}\sin(5t)\,dt$$

$$\frac{s^2+25}{s^2}\int_0^\infty e^{-st}\sin(5t)\,dt = \left[-\frac{1}{s}e^{-st}\sin(5t) - \frac{5}{s^2}e^{-st}\cos(5t)\right]_0^\infty$$

$$\frac{s^2+25}{s^2}\int_0^\infty e^{-st}\sin(5t)\,dt = 0 - \left(0 - \frac{5}{s^2}\right)$$

$$\boxed{F(s) = \frac{5}{s^2+25}}$$

$$\mathcal{L}\{\sin(5t)\} = \int_0^\infty e^{-st}\sin(5t)\,dt$$

$\downarrow$ Euler's Formula!

$$e^{i(\omega t+\theta)} = \underbrace{\cos(\omega t+\theta)}_{Re} + i\underbrace{\sin(\omega t+\theta)}_{Im}$$

so! $$\int_0^\infty e^{-st}\sin(5t) = \int_0^\infty e^{-st}\,Im\{e^{i5t}\}\,dt$$

$$= \cancel{Im}\left\{\int_0^\infty e^{-(s-5i)t}\,dt\right.$$

R

$$= Im\left\{\frac{-1}{s-5i}\left(e^{-\infty}-e^{0}\right)\right\}$$

$$= Im\left\{\frac{1}{s-5i}\right\}$$

$$= Im\left\{\frac{s+5i}{s^2+25}\right\}$$

$$\boxed{F(s) = \frac{5}{s^2+25}}$$

discussion - 4 - laplace table

$$f(t) = 3\left(\sin(6t) + 2\sqrt{t}\right)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3\sin(6t) + 2\sqrt{t}\}$$

linear operator

$$= 3\mathcal{L}\{\sin(6t)\} + 6\mathcal{L}\{\sqrt{t}\}$$

$$= 3\left(\frac{6}{s^2+6^2}\right) + 6\left(\frac{\sqrt{\pi}}{2s^{3/2}}\right)$$

$$F(s) = \frac{18}{s^2+36} + \frac{3\sqrt{\pi}}{s^{3/2}}$$

$$f(t) = \frac{P(x)}{Q(x)}$$

partial

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$
2. $\mathbf{e}^{at}$	$\frac{1}{s-a}$
3. $t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
6. $t^{n-\frac{1}{2}}, \quad n = 1, 2, 3, \ldots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$
8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
10. $t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$

discussion - 5 - transfer function representation

$$\mathcal{L}\left\{\frac{d^3z}{dx^3}+6\frac{d^2z}{dx^2}+9\frac{dz}{dx}+12z\right\} = \mathcal{L}\left\{\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+5\frac{dy}{dx}+7y\right\}$$

$$\boxed{\text{TF: } \frac{\text{output}}{\text{input}} = G(s)}$$

$\mathcal{L}$

$$\boxed{\mathcal{L}\{f^n(t)\} \triangleq s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}$$

Assume 0 IC's : $f(0) = 0$

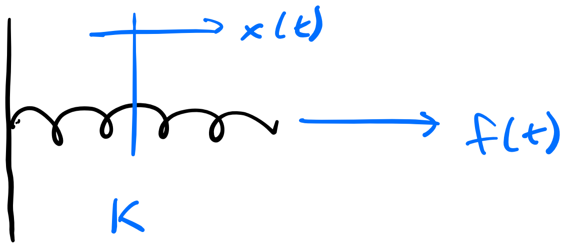
$$s^3 F(s) - \cancel{s^2 f(0)} - \cancel{sf'(0)} - \cancel{f''(0)}$$

$$\boxed{\frac{Y(s)}{Z(s)} = \frac{s^3+6s^2+9s+12}{s^3+3s^2+5s+7}}$$

$$s^3 Z(s) + 6s^2 Z(s) + 9s\, Z(s) + 12 Z(s) = s^3 Y(s) + 3s^2 Y(s) + 5s Y(s) + 7Y(s)$$

$$Z(s)\left[s^3+6s^2+9s+12\right] = Y(s)\left[s^3+3s^2+5s+7\right]$$

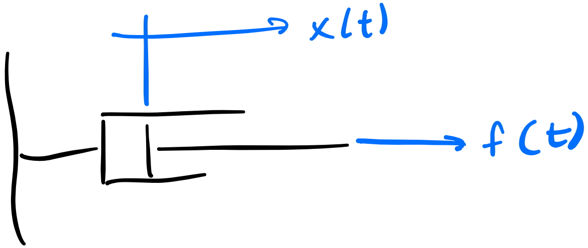
transfer functions for physical systems



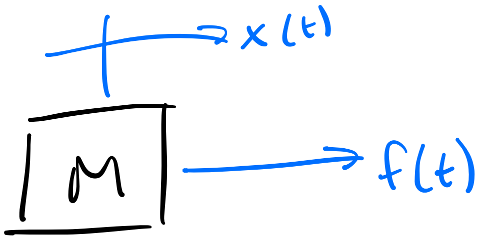
F-displacement

Impedance (Laplace space)

$$f(t) = K\,x(t) \xrightarrow{\mathcal{L}} K$$

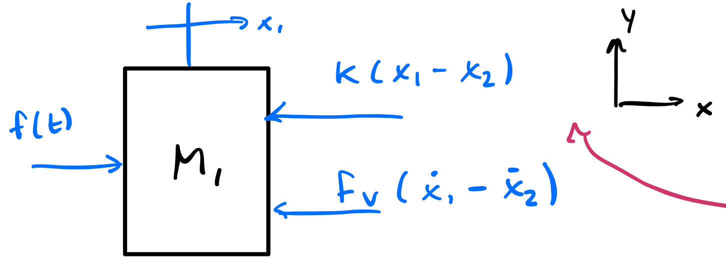


$$f(t) = f_v \frac{dx(t)}{dt} \xrightarrow{\mathcal{L}} f_v\, s$$

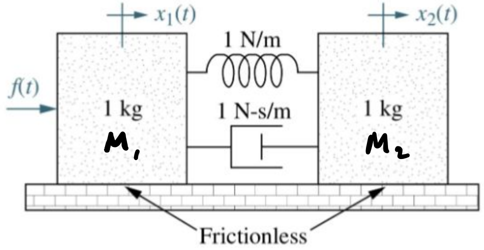


$$f(t) = M\frac{d^2x(t)}{dt^2} \xrightarrow{\mathcal{L}} Ms^2$$

discussion - 6 - mechanical system TF

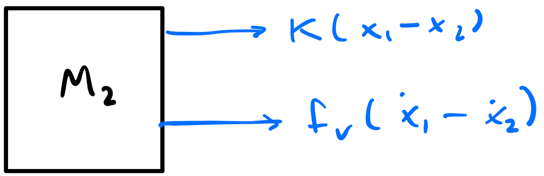


Find the transfer function, $G(s) = X_2(s)/F(s)$, for the translational mechanical network below.



Newton 3rd law

force pair



$$\Sigma F = m_1 \ddot{x}_1$$

$$f(t) - \cancel{K}^{1}(x_1 - x_2) - \cancel{f_v}^{1}(\dot{x}_1 - \dot{x}_2) = \cancel{m_1}^{1} \ddot{x}_1$$

$$\ddot{x}_1 + \dot{x}_1 + x_1 - \dot{x}_2 - x_2 = f(t) \quad (1)$$

$$\downarrow \mathcal{L}$$

$$(s^2 + s + 1)X_1 - (s+1)X_2 = F(s)$$

$$(s^2 + s + 1)\frac{s^2+s+1}{s+1} X_2 - (s+1)X_2 = F(s)$$

$$\left(\frac{(s^2+s+1)^2 - (s+1)^2}{s+1} \right) X_2 = F(s)$$

$$\downarrow$$

$$\Sigma F = m_2 \ddot{x}_2$$

$$\cancel{K}^{1}(x_1 - x_2) + \cancel{f_v}^{1}(\dot{x}_1 - \dot{x}_2) = \cancel{m_2}^{1} \ddot{x}_2$$

$$\ddot{x}_2 + \dot{x}_2 + x_2 = \dot{x}_1 + x_1 \quad (2)$$

$$(s^2 + s + 1)X_2 = (s+1)X_1$$

↳ rearrange for $X_1 \Rightarrow \frac{s^2+s+1}{s+1} X_2$

Cramer's Rule

$$\underbrace{\begin{bmatrix} s^2+s+1 & -(s+1) \\ -(s+1) & s^2+s+1 \end{bmatrix}}_{A} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

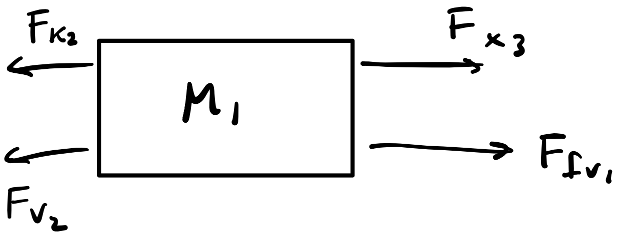
solve

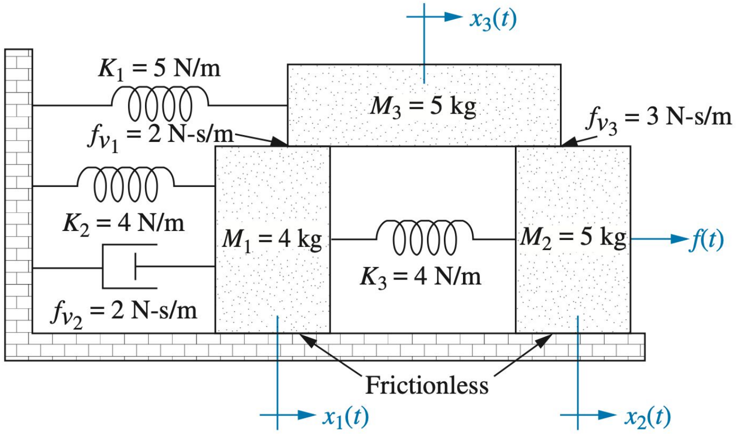
$$\frac{X_2(s)}{F(s)} \Rightarrow X_2(s) \overset{\text{Cramer}}{=} \frac{\det A_2}{\det A}$$

$$A_2 = \begin{bmatrix} s^2+s+1 & F(s) \\ -(s+1) & 0 \end{bmatrix}$$

$$\boxed{\frac{X_2}{F(s)} = \frac{s+1}{(s^2+s+1)^2-(s+1)^2}} = \frac{s+1}{s^4+2s\ \cdots}$$

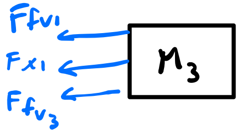
discussion - 7 - mechanical system TF setup





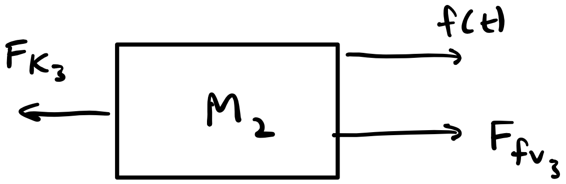
$$\Sigma F = m_1 \ddot{x}_1 = K_3(x_2 - x_1) - K_2 x_1 - f_{v_2}\dot{x}_1 + f_{v_1}(\dot{x}_3 - \dot{x}_1)$$

$$\overset{\mathcal{L}}{\Rightarrow}\quad 4s^2X_1 = K_3(X_2 - X_1) - K_2X_1 - f_{v_2}sX_1 - f_{v_1}(sX_3 - sX_1) \quad \text{— (1)}$$



$$\Sigma F = m_3 \ddot{x}_3$$

$$5s^2X_3 = -K_1X_3 - f_{v_1}(sX_3 - sX_1) - f_{v_3}(sX_3 - sX_2)$$



$$\Sigma F = m_2 \ddot{x}_2 = f(t) + f_{v_3}(\dot{x}_3 - \dot{x}_2) - K_3(x_2 - x_1)$$

$$\overset{\mathcal{L}}{=}\quad 5s^2X_2 = F(s) + f_{v_3}(sX_3 - sX_2) - K_3(X_2 - X_1)$$

transfer functions - MATLAB symbolic toolbox

```
syms x y

x^2+3*x+4

A = [3*x 0 8*y; 0 4*x 4*y; 0 4*x*y 7]

detA = det(A)

B = [y; 3*x*y; 3]
x = A \ B
```

tf (num, den) =

A =

$$\begin{pmatrix} 3\,x & 0 & 8\,y \\ 0 & 4\,x & 4\,y \\ 0 & 4\,x\,y & 7 \end{pmatrix}$$

detA = $84\,x^2 - 48\,x^2\,y^2$

B =

$$\begin{pmatrix} y \\ 3\,x\,y \\ 3 \end{pmatrix}$$

x =

$$\begin{pmatrix} \dfrac{0.3333\,y\,(4\,y^2 - 24\,x\,y^2 + 17)}{x\,(4\,y^2 - 7)} \\ -\dfrac{0.7500\,y\,(7\,x - 4)}{x\,(4\,y^2 - 7)} \\ \dfrac{3\,(x\,y^2 - 1)}{4\,y^2 - 7} \end{pmatrix}$$

office hours!