CS2040 – Data Structures and Algorithms

Lecture 13 – Maze Exploration ~ Graph Traversal

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Annoucement

- PE on Saturday 6th April
  - Will be **harder** then SIL 1&2
  - There will be **4 problems**
  - Topics tested will be as follows

  Q1. Linear Data Structure Applications
  - Applications of LinkedList, Stack and Queue

  Q2. Non-Linear Data Structure Applications
  - Applications of HashSet/HashMap, TreeSet/TreeMap and PriorityQueue

  Q3. Graph Representation and Traversals Applications
  - Applications of DFS and BFS

  Q4. Mix and Match Applications
  - Applications of any topics covered up to Week 11 Lab Session and Week 10 Lectures (Thursday/Friday). Multiple topics can be combined.

Outline

Two algorithms to traverse a graph

- Depth First Search (DFS) and Breadth First Search (BFS)
- Plus some of their interesting applications

https://visualgo.net/en/dfsbfs

Reference: Mostly from CP3 Section 4.2

- Not all sections in CP3 chapter 4 are used in CS2040!
  - Some are quite advanced :O

GRAPH TRAVERSAL ALGORITHMS

Review – **Binary Tree** Traversal

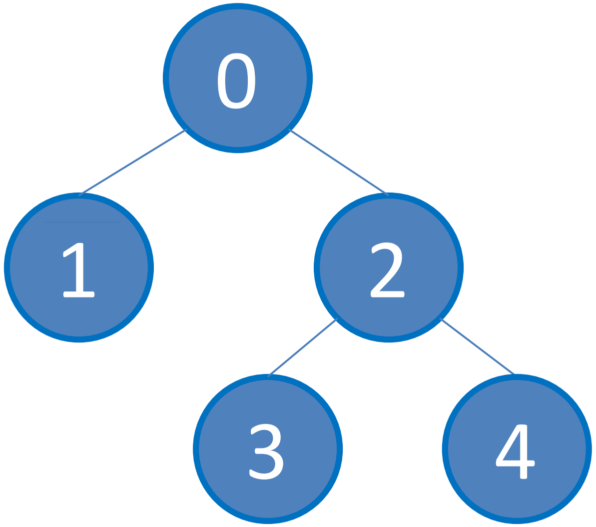
In a binary tree, there are three standard traversal:

- Preorder
- **Inorder**
- Postorder

`pre(u)`
`visit(u);`
`pre(u->left);`
`pre(u->right);`
	`in(u)`
	`in(u->left);`
	`visit(u);`
	`in(u->right);`
		`post(u)`
		`post(u->left);`
		`post(u->right);`
		`visit(u);`

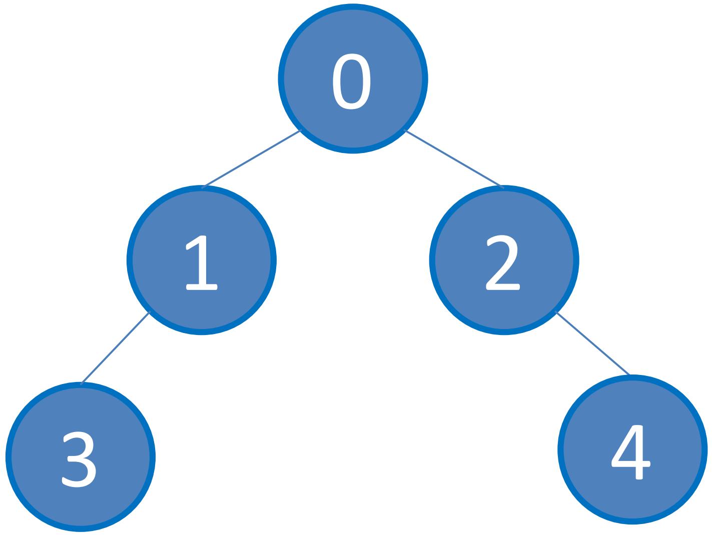
We start binary tree traversal from root:

- pre(root)/in(root)/post(root)
  - pre = 0, 1, 2, 3, 4
  - in = 1, 0, 3, 2, 4
  - post = 1, 3, 4, 2, 0



What is the **Post**Order Traversal of this Binary Tree?

1. 0 1 2 3 4
2. 0 1 3 2 4
3. 3 4 1 2 0
4. 3 1 4 2 0



Traversing a Graph (1)

Two ingredients are needed for a **traversal:**

1. The start
2. The movement

Defining the start ("source")

- In tree, we *normally* start from root
  - Note: Not all tree are rooted though!
    - In that case, we have to select one vertex as the "source", see below
- In general graph, we do not have the notion of root
  - Instead, we start from a distinguished vertex
    - We call this vertex as the **"source" s**

Traversing a Graph (2)

Defining the movement:

- In (binary) tree, we only have (at most) two choices:
  - Go to the **left subtree** or to the **right subtree**
- In general graph, we can have more choices:
  - If **vertex u** and **vertex v** are adjacent/connected with edge (**u**, **v**); and we are now in **vertex u**; then we can also go to **vertex v** by traversing that edge (**u**, **v**)
- In (binary) tree, there is **no cycle**
- In general graph, we **may have (trivial/non trivial) cycles**
  - We need a way to avoid revisiting **u** → **v** → **w** → **u** → **v** … indefinitely

Traversing a Graph (2)

Solution: BFS and DFS ☺

Idea: If a vertex **v** is reachable from **s**, then all neighbors of **v** will also be reachable from **s** (recursive definition)

Breadth First Search (BFS) – Ideas

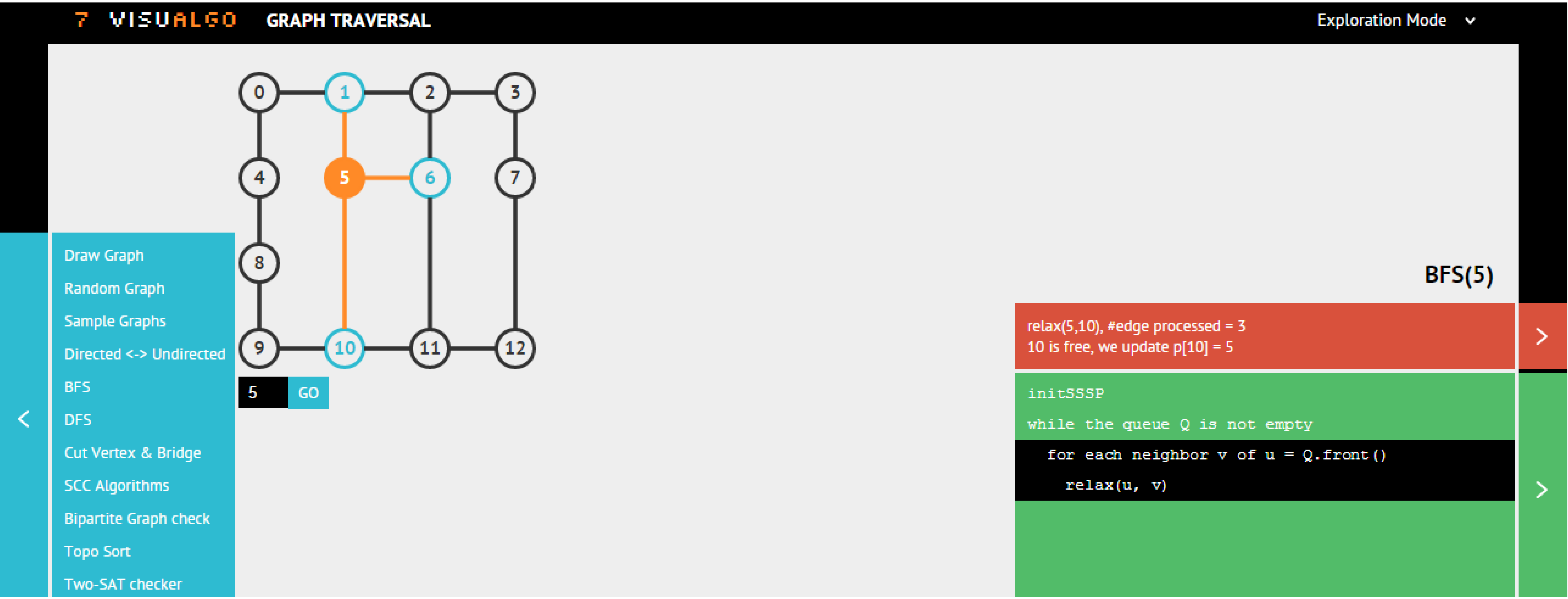
- Start from **s**
- BFS visits vertices of G in *breadth-first* manner (when viewed from source vertex s)
  - Q: How to maintain such order?
    - A: Use queue **Q**, initially, it contains only **s**
  - Q: How to differentiate visited vs unvisited vertices (to avoid cycle)?
    - A: 1D array/Vector **visited** of size V,
      visited[v] = 0 initially, and **visited[v] = 1** when **v** is visited
  - Q: How to memorize the path?
    - A: 1D array/Vector **p** of size V,
      p[v] denotes the **p**redecessor (or **p**arent) of **v**

- Edges used by BFS in the traversal will form a BFS "spanning" tree of G (tree that includes all vertices of G) stored in **p**

Graph Traversal: BFS(s)

Ask VisuAlgo to perform various Breadth-First Search operations on the sample Graph (CP3 4.3, Undirected)

In the screen shot below, we show the start of **BFS(5)**



BFS Pseudo Code

```
for all v in V
  visited[v] ← 0
  p[v] ← -1
Q ← {s} // start from s
visited[s] ← 1
```

Initialization phase

```
while Q is not empty
  u ← Q.dequeue()
  for all v adjacent to u // order of neighbor
    if visited[v] = 0 // influences BFS
      visited[v] ← true // visitation sequence
      p[v] ← u
      Q.enqueue(v)
```

Main loop

```
// after BFS stops, we can use info stored in visited/p
```

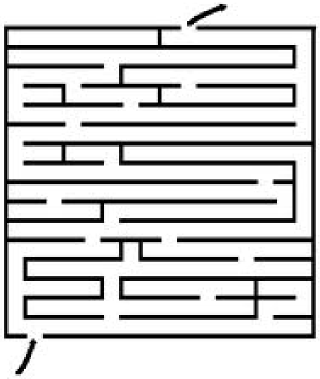
BFS Analysis

```
for all v in V
  visited[v] ← 0
  p[v] ← -1
Q ← {s} // start from s
visited[s] ← 1

while Q is not empty
  u ← Q.dequeue()
  for all v adjacent to u // order of neighbor
    if visited[v] = 0 //  influences BFS
      visited[v] ← true // visitation sequence
      p[v] ← u
      Q.enqueue(v)

// we can then use information stored in visited/p
```

Time Complexity: O(**V**+**E**)

- Each vertex is only in the queue once ~ O(**V**)
- Every time a vertex is dequeued, all its **k** neighbors are scanned; After all vertices are dequeued, all **E** edges are examined ~ O(**E**) → assuming that we use **Adjacency List**!
- Overall: O(**V**+**E**)

Depth First Search (DFS) – Ideas

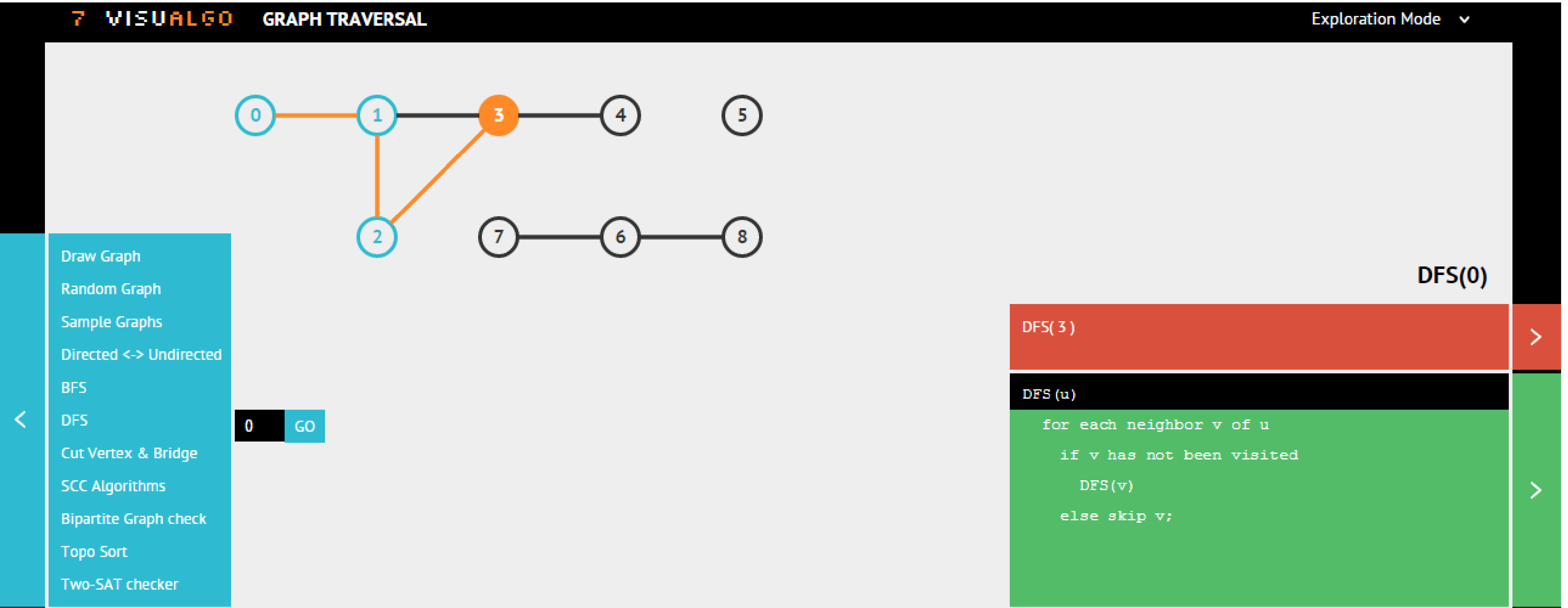


- Start from **s**
- DFS visits vertices of G in *depth-first* manner (when viewed from source vertex s)
  - Q: How to maintain such order?
    - A: Stack **S**, but we will simply use recursion (an implicit stack)
  - Q: How to differentiate visited vs unvisited vertices (to avoid cycle)?
    - A: 1D array/Vector **visited** of size V,
      visited[v] = 0 initially, and **visited[v] = 1** when **v** is visited
  - Q: How to memorize the path?
    - A: 1D array/Vector **p** of size V,
      p[v] denotes the **p**redecessor (or **p**arent) of **v**
- Edges used by DFS in the traversal will form a DFS "spanning" tree of G (tree that includes all vertices of G) stored in **p**

Graph Traversal: DFS(s)

Ask VisuAlgo to perform various Depth-First Search operations on the sample Graph (CP3 4.1, Undirected)

In the screen shot below, we show the start of **DFS(0)**



DFS Pseudo Code

```
DFSrec(u)
  visited[u] ← 1 // to avoid cycle
  for all v adjacent to u // order of neighbor
    if visited[v] = 0 //  influences DFS
      p[v] ← u // visitation sequence
      DFSrec(v) // recursive (implicit stack)
```

Recursive phase

```
// in the main method
for all v in V
  visited[v] ← 0
  p[v] ← -1
DFSrec(s) // start the
recursive call from s
```

Initialization phase, same as with BFS

DFS Analysis

```
DFSrec(u)
  visited[u] ← 1 // to avoid cycle
  for all v adjacent to u // order of neighbor
    if visited[v] = 0 //  influences DFS
      p[v] ← u // visitation sequence
      DFSrec(v) // recursive (implicit stack)

// in the main method
for all v in V
  visited[v] ← 0
  p[v] ← -1
DFSrec(s) // start the
recursive call from s
```

Time Complexity: O(**V**+**E**)

- Each vertex is only visited once O(**V**), then it is flagged to avoid cycle
- Every time a vertex is visited, all its **k** neighbors are scanned; Thus after all vertices are visited, we have examined all **E** edges ~ O(**E**) → assuming that we use **Adjacency List**!
- Overall: O(**V**+**E**)

Path Reconstruction Algorithm (1)

```
// iterative version (will produce reversed output)
Output "(Reversed) Path:"
i ← t // start from end of path: suppose vertex t
while i != s
  Output i
  i ← p[i] // go back to predecessor of i
Output s



// try it on this array p, t = 4
// p = {-1, 0, 1, 2, 3, -1, -1, -1}
```

Path Reconstruction Algorithm (2)

```
void backtrack(u)
  if (u == -1) // recall: predecessor of s is -1
    stop
  backtrack(p[u]) // go back to predecessor of u
  Output u // recursion like this reverses the order

// in main method
// recursive version (normal path)
Output "Path:"
backtrack(t); // start from end of path (vertex t)
// try it on this array p, t = 4
// p = {-1, 0, 1, 2, 3, -1, -1, -1}
```

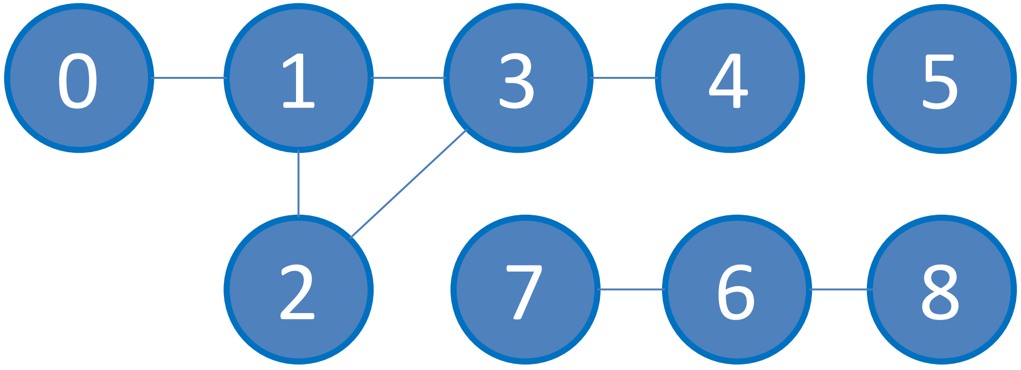
SOME GRAPH TRAVERSAL APPLICATIONS

What can we do with BFS/DFS? (1)

Several stuffs, let's see ***some of them***:

- Reachability test
  - Test whether vertex **v** is reachable from vertex **u**?
  - Start BFS/DFS from **s = u**
  - If **visited[v] = 1** after BFS/DFS terminates,
    then **v** is *reachable* from **u**; otherwise, **v** is *not reachable* from **u**

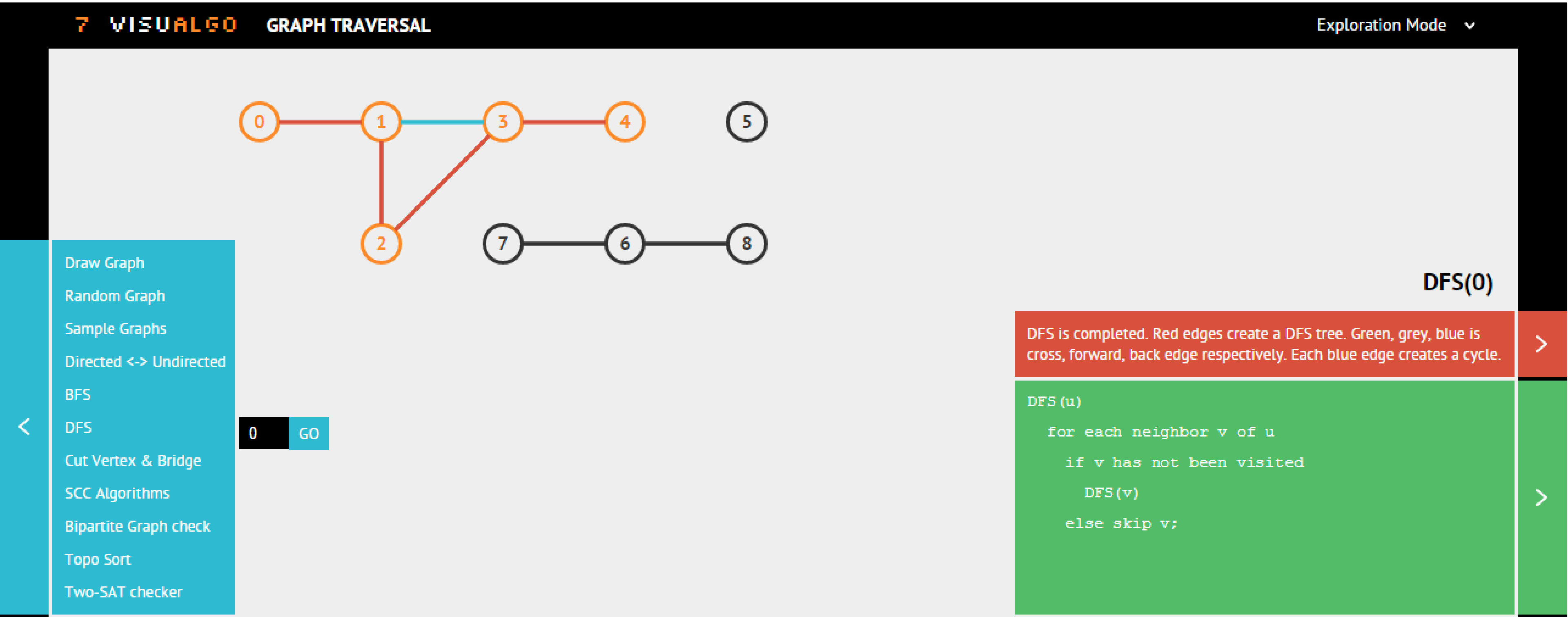
```
BFS(u) // DFSrec(u)
if visited[v] == 1
  Output "Yes"
else
  Output "No"
```



Reachability Test

Ask VisuAlgo to perform various DFS (or BFS) operations on the sample Graph (CP3 4.1, Undirected)

Below, we show vertices that are reachable from vertex 0



What can we do with BFS/DFS? (2)

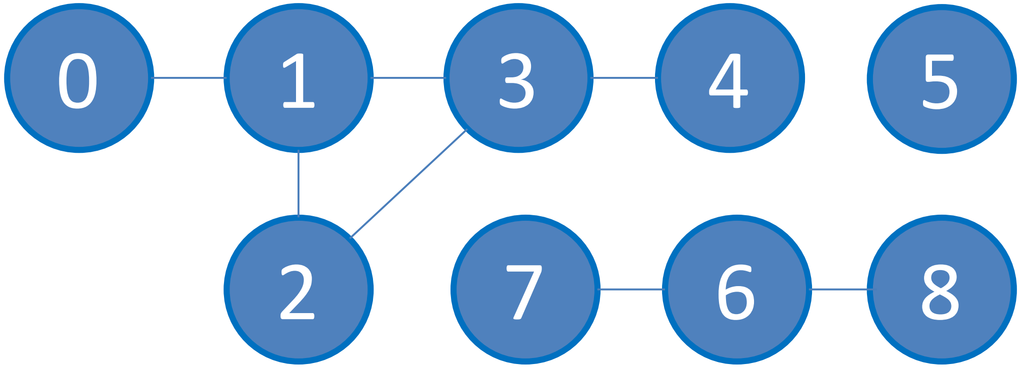
- Find Shortest Path between 2 vertices in an unweighted graph/graph where edges have same weight
  - When the graph is **unweighted*/edges have same weight**, shortest path between any 2 vertices **u,v** is finding the **least number of edges** traversed from u to v
  - The O(**V**+**E**) Breadth First Search (BFS) traversal algorithm precisely measures this
    - Run BFS from u as source
    - Construct shortest path from u to v from **p** after BFS finishes
    - Cost of shortest path from u to v is (number of edges in the path)×(edge weight for weighted edges)

* Can treat the edge weight as 1

What can we do with BFS/DFS? (3)

- Identifying component(s)
  - Component is sub graph in which any 2 vertices are connected to each other by at least one path, and is connected to no additional vertices
  - With BFS/DFS, we can identify components by labeling/counting them in graph G
  - Solution:

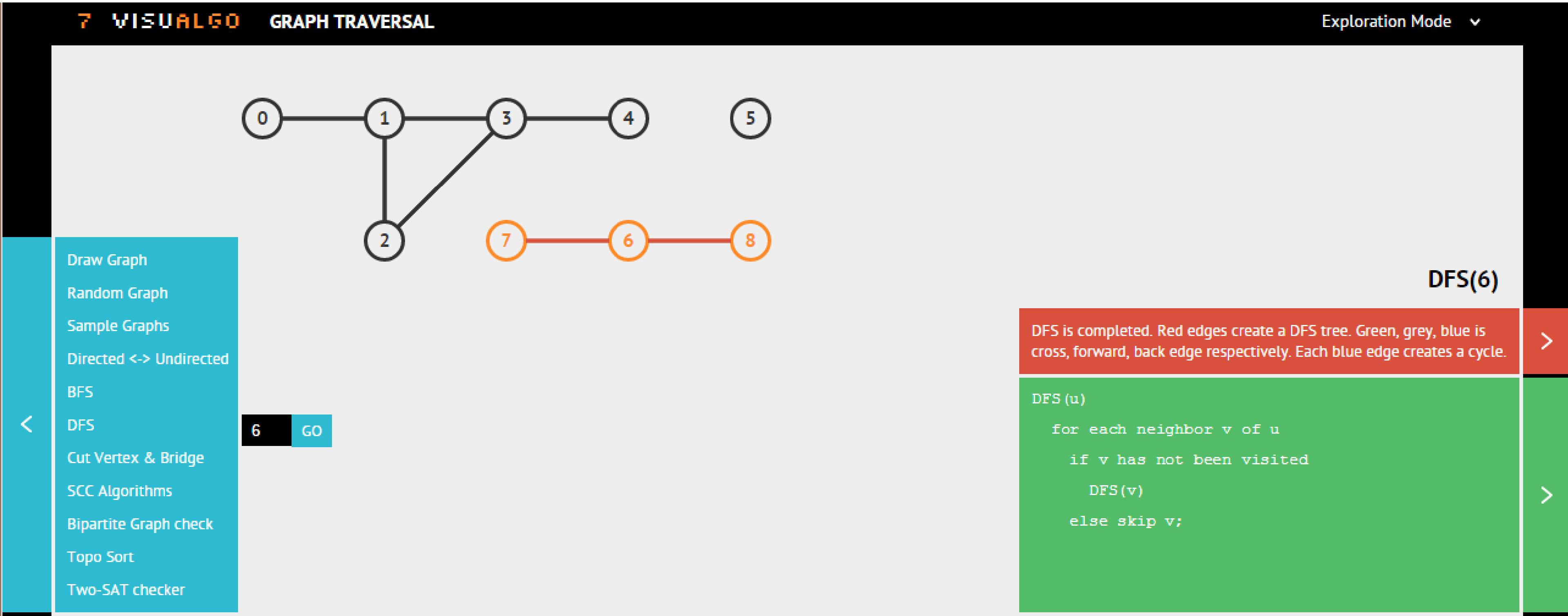
```
CC ← 0
for all v in V
  visited[v] ← 0
for all v in V // O(V)?
  if visited[v] == 0
    CC ← CC + 1
    DFSrec(v)//O(V+E)?
    // BFS from v
    // is also OK
```



Identifying Components

Ask VisuAlgo to perform various DFS (or BFS) operations on the sample Graph (CP3 4.1, Undirected)

Call **DFS(0)/BFS(0)**, **DFS(5)/BFS(5)**, then **DFS(6)/BFS(6)**

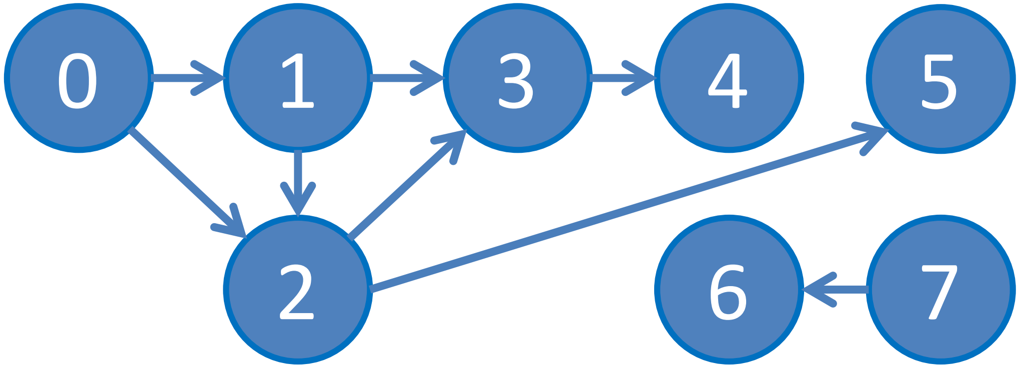


What is the time complexity for "counting connected component"?

1. Hm… you can call O(**V**+**E**) DFS/BFS up to **V** times…
   I think it is O(**V***(**V**+**E**)) =
   O(**V^2** + **VE**)
2. It is O(**V**+**E**)...
3. Maybe some other time complexity, it is O(______)

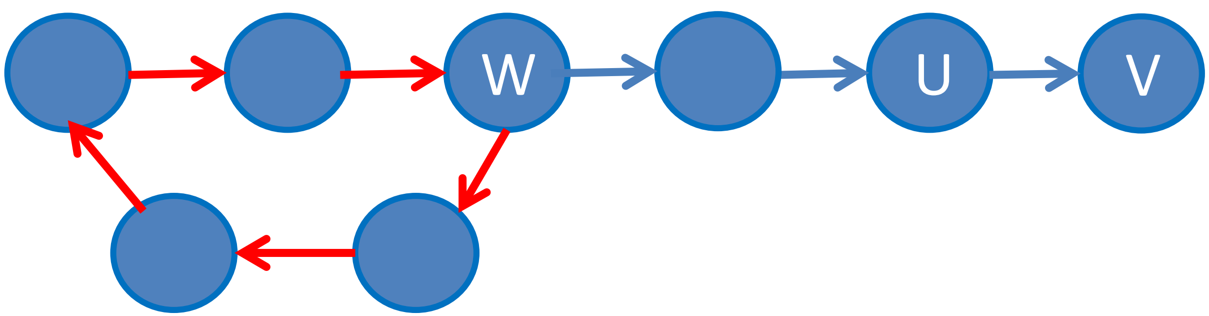
What can we do with BFS/DFS? (4)

- Topological Sort
  - Topological sort of a DAG is a linear ordering of its vertices in which each vertex comes before all vertices to which it has outbound edges
  - Every DAG has one *or more* topological sorts



Proof that every DAG has a Topological ordering (1)

- Lemma: If G is a DAG, it has a node with no incoming edges
- Proof by contradiction:
  - Assume every node in G has an incoming edge
  - Pick a node **V** and follow one of it's incoming edge backwards e.g (**U**,**V**) which will visit **U**
  - Do the same thing with **U**, and keep repeating this process
  - Since every node has an incoming edge, at some point you will visit a node **W** 2 times. Stop at this point
  - Every vertex encountered between successive visits to **W** will form a cycle (contradiction that G is a DAG)



Proof that every DAG has a Topological ordering (2)

- Lemma: If G is a DAG, then it has a topological ordering
- Constructive proof:
  - Pick node V with no incoming edge (must exist according to previous lemma)
  - remove V from G and number it 1
  - G-{V} must still be a DAG since removing V cannot create a cycle
  - Pick the next node with no incoming edge W and number it 2
  - Repeat the above with increasing numbering until G is empty
  - For any node it cannot have incoming edges from nodes with a higher numbering
  - Thus ordering the nodes from lowest to highest number will result in a topological ordering

- This constructive proof is the basis for the BFS based algorithm (Khan's algorithm) to compute topological ordering of a DAG

What can we do with BFS/DFS? (5)

- Topological Sort – Khan's algorithm
  - If graph is a DAG, then running a modified version of BFS (Khan's algorithm) on it will give us a valid topological order
    - Replace **visited** array with an integer array **indeg** that keeps track of the in-degree of each vertex in the DAG
    - Use an ArrayList **toposort** to record the vertices visited
  - See pseudo code in the next slide

Khan’s Algorithm Pseudo Code

modifications from BFS in red

```
for all v in V
   indeg[v] ← 0
   p[v] ← -1
for each edge (u,v) // get in-degree of vertices
   indeg[v] ← indeg[v] + 1
for all v’ where indeg[v’] = 0
   Q ← {v’} // enqueue v’

while Q is not empty
   u ← Q.dequeue()
   append u to back of toposort
   for all v adjacent to u // order of neighbor
      if indeg[v] > 0
         indeg[v] ← indeg[v] - 1
      if indeg[v] = 0 // add to queue
         p[v] ← u
         Q.enqueue(v)

Output toposort as the topological order
```

Initialization phase

Main loop

What can we do with BFS/DFS? (5)

- Topological Sort – DFS based algorithm
  - Running a slightly modified **DFS** on the DAG (and at the same time record the vertices in "post-order" manner) will also give us one valid topological order
    - "Post-order" = process vertex **u** after all **neighbors** of **u** have been visited
    - Use an ArrayList **toposort** to record the vertices
  - See pseudo code in the next slide

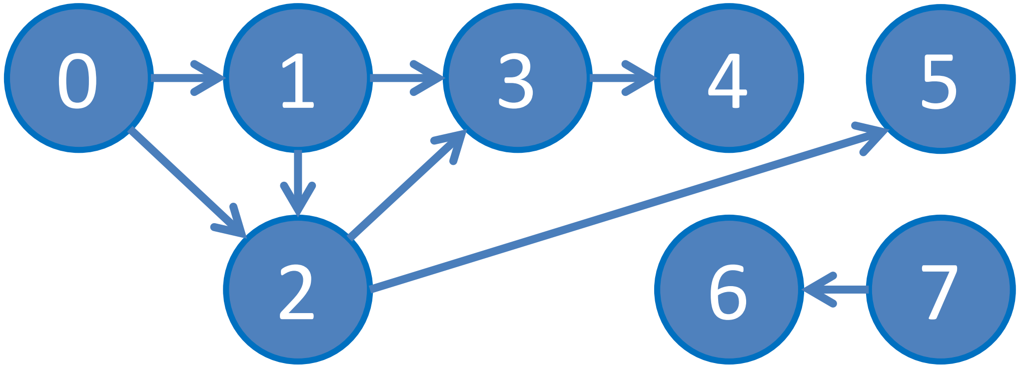
DFS for TopoSort – Pseudo Code

Simply look at the codes in red/underlined

```
DFSrec(u)
  visited[u] ← 1 // to avoid cycle
  for all v adjacent to u // order of neighbor
    if visited[v] = 0 //  influences DFS
      p[v] ← u // visitation sequence
      DFSrec(v) // recursive (implicit stack)
  append u to the back of toposort // "post-order"

// in the main method
for all v in V
  visited[v] ← 0
  p[v] ← -1
clear toposort
for all v in V
  if visited[v] == 0
    DFSrec(s)  // start the recursive call from s
reverse toposort and output it
```

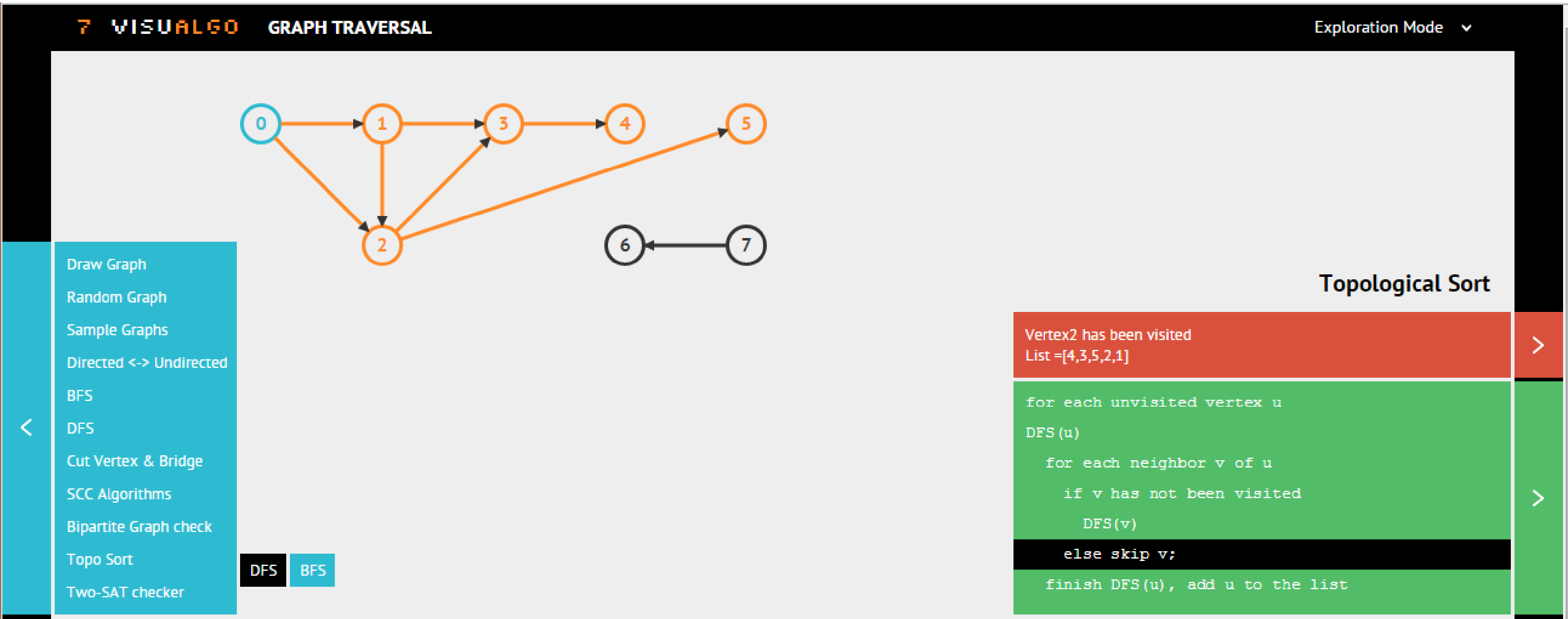
What can we do with BFS/DFS? (6)

- Topological Sort with DFS
  - Suppose we have visited all neighbors of 0 recursively with DFS
  - toposort list = [[list of vertices reachable from 0], vertex 0]
    - Suppose we have visited all neighbors of 1 recursively with DFS
    - toposort list = [[[list of vertices reachable from 1], vertex 1], vertex 0]
    - and so on…
  - We will eventually have = [4, 3, 5, 2, 1, 0, 6, 7]
  - Reversing it, we will have = [7, 6, 0, 1, 2, 5, 3, 4]



Topological Sort

Ask VisuAlgo to perform Topo Sort (BFS/DFS) operation on the sample Graph (CP3 4.4, Directed)



Trade-Off

O(V+E) DFS

- Pros:
  - Slightly easier? to code (this one depends)
  - Use less memory
- Cons:
  - Cannot solve SSSP on unweighted graphs

O(V+E) BFS

- Pros:
  - Can solve SSSP on unweighted graphs (revisited in latter lectures)
- Cons:
  - Slightly longer? to code (this one depends)
  - Use more memory (especially for the queue)

Summary

In this lecture, we have looked at:

- Graph Traversal Algorithms: Start+Movement
  - Breadth-First Search: uses queue, breadth-first
  - Depth-First Search: uses stack/recursion, depth-first
  - Both BFS/DFS uses “flag” technique to avoid cycling
  - Both BFS/DFS generates BFS/DFS “Spanning Tree”
  - Some applications: Reachability, SP in unweighted/same weight graph, CC, Toposort