CS2040 Lecture Note #6:
Analysis of Algorithms

Measuring amount of resources to run an algorithm

Lecture Note #6: Analysis of Algorithms

- **Objectives:**
  - To introduce the theoretical basis for measuring the efficiency of algorithms
  - To learn how to use such measure to compare the efficiency of different algorithms

Outline

1. What is an algorithm?
2. What do we mean by analysis of algorithms?
3. Algorithm Growth Rates
4. Big-O notation – Upper Bound
5. How to find the complexity of a program?

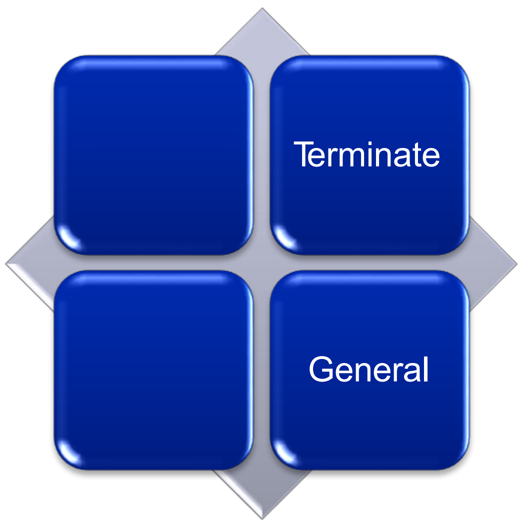
You are expected to know…

- Proof by induction
- Operations on logarithm function
- Arithmetic and geometric progressions
  - Their sums
- Linear, quadratic, cubic, polynomial functions
- ceiling, floor, absolute value

1 What is an algorithm?

1 Algorithm

- A step-by-step procedure for solving a problem.
- Properties of an algorithm:
  - Each step of an algorithm must be exact.
  - An algorithm must terminate.
  - An algorithm must be effective.
  - An algorithm should be general.



2 What do we mean by Analysis of Algorithms?

2.1 What is Analysis of Algorithms?

- Analysis of algorithms
  - Provides tools for contrasting the efficiency of different methods of solution (rather than programs)
  - Complexity of algorithms
- A comparison of algorithms
  - Should focus on significant differences in the efficiency of the algorithms
  - Should not consider reductions in computing costs due to clever coding tricks. Tricks may reduce the readability of an algorithm.

2.2 Determining the Efficiency of Algorithms

- To evaluate rigorously the resources (time and space) needed by an algorithm and represent the result of the analysis with a formula
- Emphasize more on the time requirement
- The time requirement of an algorithm is also called its time complexity

2.3 By measuring the run time?

TimeTest.java

```
class TimeTest {
  public static void main(String[] args) {
    long startTime = System.currentTimeMillis();
    long total = 0;
    for (int i = 0; i < 10000000; i++) {
      total += i;
    }
    long stopTime = System.currentTimeMillis();
    long elapsedTime = stopTime - startTime;
    System.out.println(elapsedTime);
  }
}
```

Note: The run time depends on the compiler and computer used, and the current work load of the computer system.

2.4 Exact run time is not always needed

Using exact run time is not meaningful when we want to compare two algorithms

- coded in different languages,
- using different data sets, or
- running on different computers.

2.5 Determining the Efficiency of Algorithms

- Difficulties with comparing programs instead of algorithms
  - How are the algorithms coded?
  - Which compiler is used?
  - What computer should you use?
  - What data should the programs use?
- Algorithm analysis should be independent of
  - Specific implementations
  - Compilers and their optimizers
  - Computers
  - Data

2.6 Execution Time of Algorithms

- Instead of working out the exact timing, we count the number of some or all of the primitive operations (e.g. +, -, *, /, assignment, …) needed.
- Counting an algorithm's operations is a way to assess its efficiency
  - An algorithm's execution time is related to the number of operations it requires.
  - Examples
    - Traversal of a linked list
    - Towers of Hanoi
    - Nested Loops

3 Algorithm Growth Rates

3.1 Algorithm Growth Rates (1/2)

- An algorithm's time requirements can be measured as a function of the problem size, say n
- An algorithm's growth rate
  - Enables the comparison of one algorithm with another
  - Examples
    - Algorithm A requires time proportional to n^2
    - Algorithm B requires time proportional to n
- Algorithm efficiency is typically a concern for large problems only. Why?

3.1 Algorithm Growth Rates (2/2)

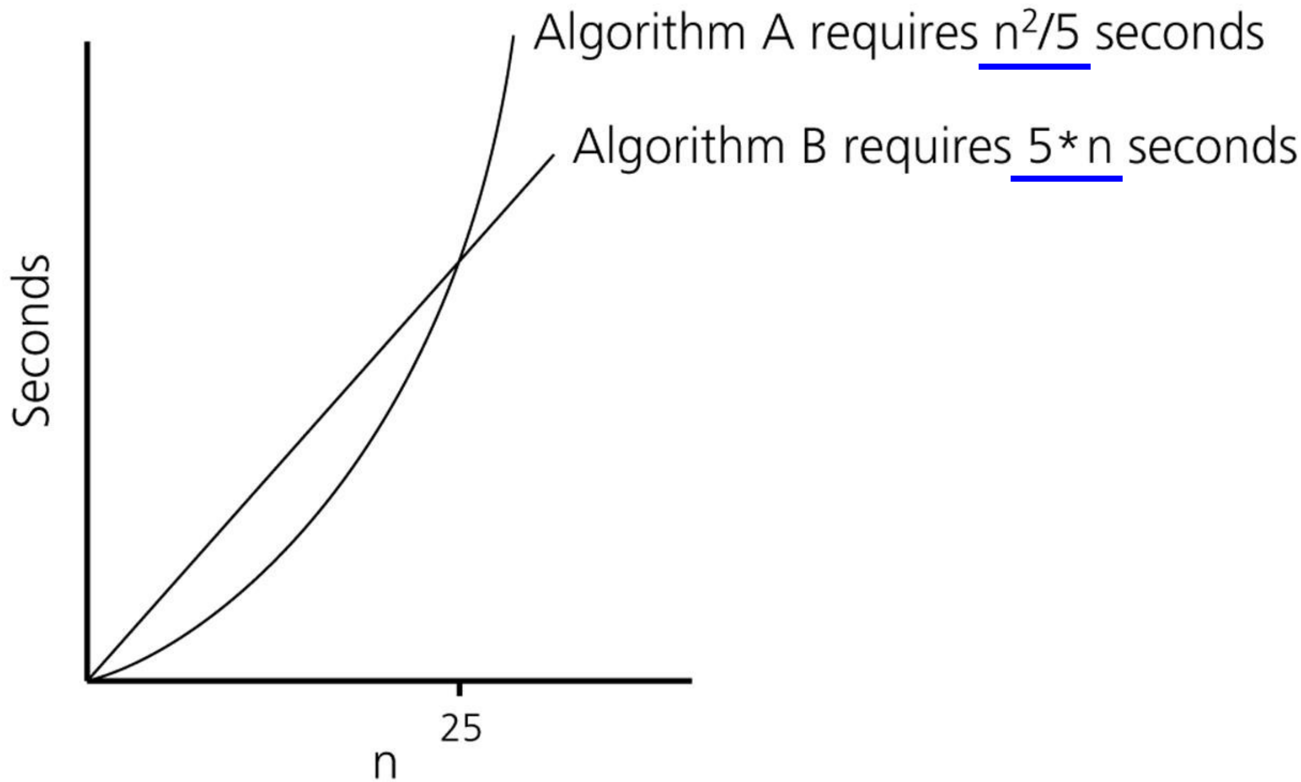


Figure - Time requirements as a function of the problem size n

3.2 Computation cost of an algorithm

- How many operations are required?

```
for (int i=1; i<=n; i++) {
    perform 100 operations;          // A
    for (int j=1; j<=n; j++) {
        perform 2 operations;        // B
    }
}
```

Total Ops = A + B $= \sum_{i=1}^{n} 100 + \sum_{i=1}^{n} (\sum_{j=1}^{n} 2)$

$$= 100n + \sum_{i=1}^{n} 2n \quad = 100n + 2n^2 \quad = 2n^2 + 100n$$

3.3 Counting the number of statements

- To simplify the counting further, we can ignore
  - the different types of operations, and
  - different number of operations in a statement,

  and simply count the number of statement executed.

- So, total number of statements executed in the previous example is $2n^2 + 100n$

3.4 Approximation of analysis results

- Very often, we are interested only in using a simple term to indicate how efficient an algorithm is. The exact formula of an algorithm's performance is not really needed.

- Example:

  Given the formula: $3n^2+2n+\log n + 1/(4n)$

  - the dominating term $3n^2$ can tell us approximately how the algorithm performs.

- What kind of approximation of the analysis of algorithms do we need?

3.5 Asymptotic analysis

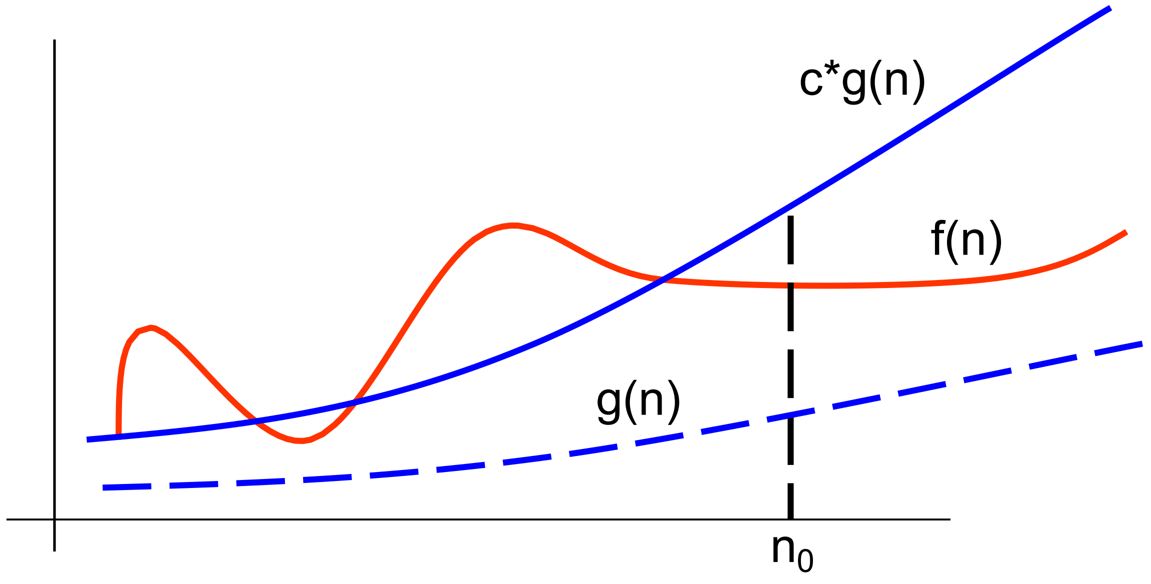
- Asymptotic analysis is an analysis of algorithms that focuses on
  - analyzing the problems of large input size,
  - considering only the leading term of the formula, and
  - ignoring the coefficient of the leading term
- Some notations are needed in asymptotic analysis

4 Big O notation

4.1 Definition

- Given a function f(n), g(n) is an (asymptotic) upper bound of f(n), denoted as f(n) = O(g(n)), if there exist a constant c>0, and a positive integer n_0 such that f(n) ≤ c*g(n) for <u>all</u> n ≥ n_0.

  - f(n) is said to be bounded from above by g(n).
  - O() is called the "big O" notation.



4.2 Ignore the coefficients of all terms

- Based on the definition, $2n^2$ and $30n^2$ have the same upper bound n^2, i.e.,
  - $2n^2 = O(n^2)$
  - $30n^2 = O(n^2)$

  They differ only in the choice of c.
- Therefore, in big O notation, we can omit the coefficients of all terms in a formula:
  - Example: $f(n) = 2n^2 + 100n = O(n^2) + O(n)$

4.3 Finding the constants c and n_0

- Given $f(n) = 2n^2 + 100n$, prove that $f(n) = O(n^2)$.

$2n^2 + 100n < 2n^2 + n^2 = 3n^2$ whenever $n > 100$.

→Set the constants to be $c=3$ and $n_0 = 100$.

By definition, we have $f(n) = O(n^2)$.

Notes:

1. $n^2 < 2n^2 + 100n$ for all n, i.e., $g(n) < f(n)$, and yet $g(n)$ is an asymptotic upper bound of $f(n)$
2. c and n_0 are not unique.
   For example, we can choose $c = 2 + 100 = 102$, and $n_0 = 1$

Q: Can we write $f(n) = O(n^3)$?

4.4 Is the bound tight?

- The complexity of an algorithm can be bounded by many functions.
- Example:
  - $2n^2 + 100n$ is bounded by n^2, n^3, n^4 and many others according to the definition of big O notation.
- We are more interested in the tightest bound which is n^2 for this case.

4.5 Growth Terms: Order-of-Magnitude

- In asymptotic analysis, a formula can be simplified to a single term with coefficient 1
- Such as term is called a growth term (rate of growth, order of growth, order-of-magnitude)
- The most common growth terms can be ordered as follows:

  $O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n)$

Note:

- "log" = log base 2, or $\log_2$; "$\log_{10}$" = log base 10; "ln" = log base e. In big O, all these log functions are the same. (Why?)

4.6 Examples on big O notation

- $f1(n) = \frac{1}{2}n + 4$
  $= O(n)$
- $f2(n) = 240n + 0.001n^2$
  $= O(n^2)$
- $f3(n) = n \log n + \log n + n \log (\log n)$
  $= O(n \log n)$

Why?

4.7 Exponential Time Algorithms

- Suppose we have a problem that, for an input consisting of n items, can be solved by going through 2^n cases
  - We say the complexity is exponential time
  - Q: What sort of problems?
- We use a supercomputer that analyses 200 million cases per second
  - Input with 15 items, 164 microseconds
  - Input with 30 items, 5.36 seconds
  - Input with 50 items, more than two months
  - Input with 80 items, 191 million years!

4.8 Quadratic Time Algorithms

- Suppose solving the same problem with another algorithm will use $300n^2$ clock cycles on a 80386, running at 33MHz (very slow old PC)
  - We say the complexity is quadratic time
  - Input with 15 items, 2 milliseconds
  - Input with 30 items, 8 milliseconds
  - Input with 50 items, 22 milliseconds
  - Input with 80 items, 58 milliseconds
- What observations do you have from the results of these two algorithms? What if the supercomputer speed is increased by 1000 times?
- It is very important to use an efficient algorithm to solve a problem

4.9 Order-of-Magnitude Analysis and Big O Notation (1/2)

(a)

Function	10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

Figure - Comparison of growth-rate functions in tabular form

4.9 Order-of-Magnitude Analysis and Big O Notation (2/2)

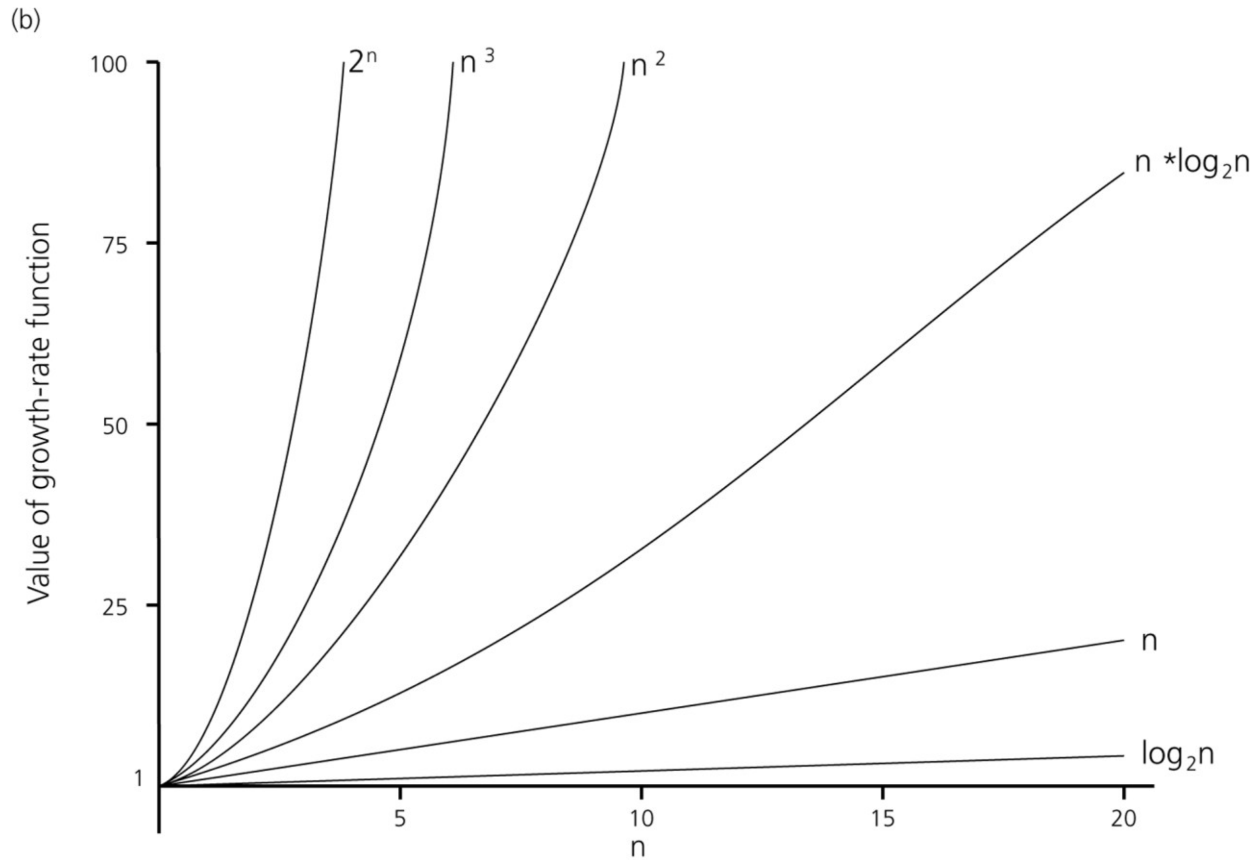


Figure - Comparison of growth-rate functions in graphical form

4.10 Summary: Order-of-Magnitude Analysis and Big O Notation

- Order of growth of some common functions:

  $O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n)$

- Properties of growth-rate functions
  - You can ignore low-order terms
  - You can ignore a multiplicative constant in the high-order term
  - $O(f(n)) + O(g(n)) = O(f(n) + g(n))$

5 How to find the complexity of a program?

5.1 Some rules of thumb and examples

- Basically just count the number of statements executed.
- If there are only a small number of simple statements in a program – O(1)
- If there is a 'for' loop dictated by a loop index that goes up to n – O(n)
- If there is a nested 'for' loop with outer one controlled by n and the inner one controlled by m – O(m*n)
- For a loop with a range of values n, and each iteration reduces the range by a fixed fraction (usually it is 0.5, i.e., half) – O(log n)
- For a recursive method, each call is usually O(1). So
  - if n calls are made – O(n)
  - if n log n calls are made – O(n log n)

5.2 Examples on finding complexity (1/2)

- What is the complexity of the following code fragment?

```
sum = 0;
for (i=1; i<n; i=i*2)
   sum++;
```

- It is clear that sum is incremented only when

  $i = 1, 2, 4, 8, \ldots, 2^k = n$ where $k = \log n$.

  So the complexity is $O(\log_2 n)$

Note:

- When 2 is replaced by 10 in the 'for' loop, the complexity is $O(\log_{10} n)$ which is the same as $O(\log_2 n)$. (Why?)

5.2 Examples on finding complexity (2/2)

- What is the complexity of the following code fragment?

```
sum = 0;
for (i=1; i<n; i=i*3)
  for (j=1; j<=i; j++)
    sum++;
```

- $f(n) = 1 + 3 + 9 + 27 + \ldots + 3^{(\log_3 n)}$

  $= n + n/3 + n/9 + \ldots + 1$

  $= n(1 + 1/3 + 1/9 + \ldots)$

  $\leq 3n/2$

  $= O(n)$

5.3 Eg: Analysis of Tower of Hanoi

- Number of moves made by the algorithm is 2^n-1. Prove it!
  - Hints: $f(1)=1$, $f(n)=f(n-1)+1+f(n-1)$, and proof by induction
- Assume each move takes t time, then:

  $f(n) = t * (2^n-1) = O(2^n)$.
- The Tower of Hanoi algorithm is an exponential time algorithm.

5.4 Eg: Analysis of Sequential Search (1/2)

- Check whether an item x is in an unsorted array a[]
  - If found, it returns position of x in array
  - If not found, it returns -1

```
public int seqSearch(int[] a, int len, int x) {
    for (int i = 0; i < len; i++) {
        if (a[i] == x)
            return i;
    }
    return -1;
}
```

5.4 Eg: Analysis of Sequential Search (2/2)

- Time spent in each iteration through the loop is at most some constant t_1
- Time spent outside the loop is at most some constant t_2
- Maximum number of iterations is n, the length of the array
- Hence, the asymptotic upper bound is:

  $t_1 n + t_2 = O(n)$

- **Rule of Thumb:**

  In general, a loop of n iterations will lead to O(n) growth rate (complexity is linear).

```
public int seqSearch(int[] a, int len, int x) {
    for (int i = 0; i < len; i++) {
        if (a[i] == x)
            return i;
    }
    return -1;
}
```

5.5 Eg: Binary Search Algorithm

- Requires array to be sorted in ascending order
- Maintain subarray where x (the search key) might be located
- Repeatedly compare x with m, the middle element of current subarray
  - If x = m, found it!
  - If x > m, continue search in subarray after m
  - If x < m, continue search in subarray before m

5.6 Eg: Non-recursive Binary Search (1/2)

- Data in the array a[] are sorted in ascending order

```
public static int binSearch(int[] a, int len, int x)  {
      int mid, low = 0;
      int high = len - 1;
      while (low <= high) {
            mid = (low + high) / 2;
            if (x == a[mid]) { return mid; }
            else if (x > a[mid]) low = mid + 1;
            else high = mid - 1;
      }
      return -1;
}
```

5.6 Eg: Non-recursive Binary Search (2/2)

- Time spent outside the loop is at most t_1
- Time spent in each iteration of the loop is at most t_2
- For inputs of size n, if we go through at most f(n) iterations, then the complexity is
  $t_1 + t_2 f(n)$
  or O(f(n))

```
public static int binSearch(int[] a, int len, int x) {
	int mid, low = 0;
	int high = len - 1;
	while (low <= high) {
		mid = (low + high) / 2;
		if (x == a[mid]) { return mid; }
		else if (x > a[mid]) low = mid + 1;
		else high = mid - 1;
	}
	return -1;
}
```

5.6 Bounding f(n), the number of iterations (1/2)

- At any point during binary search, part of array is "*alive*" (might contain the point x)
- Each iteration of loop eliminates at least half of previously "*alive*" elements
- At the beginning, all n elements are "*alive*", and after
  - One iteration, at most $n/2$ are left, or alive
  - Two iterations, at most $(n/2)/2=n/4=n/2^2$ are left
  - Three iterations, at most $(n/4)/2=n/8=n/2^3$ are left

    :
  - i iterations, at most $n/2^i$ are left
  - At the final iteration, at most 1 element is left

5.6 Bounding f(n), the number of iterations (2/2)

In the worst case, we have to search all the way up to the last iteration k with only one element left.

We have:

$$n/2^k = 1$$
$$2^k = n$$
$$k = \log n$$

Hence, the binary search algorithm takes O(f(n)) , or O(log n) times

Rule of Thumb:

- In general, when the domain of interest is reduced by a fraction (eg. by 1/2, 1/3, or 1/10, etc.) for each iteration of a loop, then it will lead to O(log n) growth rate.
- The complexity is $\log_2 n$.

5.6 Analysis of Different Cases

Worst-Case Analysis

- Interested in the worst-case behaviour.
- A determination of the maximum amount of time that an algorithm requires to solve problems of size n

Best-Case Analysis

- Interested in the best-case behaviour
- Not useful

Average-Case Analysis

- A determination of the average amount of time that an algorithm requires to solve problems of size n
- Have to know the probability distribution
- The hardest

5.7 The Efficiency of Searching Algorithms

- Example: Efficiency of Sequential Search (data not sorted)
  - Worst case: O(n)

    Which case?
  - Average case: O(n)
  - Best case: O(1)

    Why? Which case?
  - Unsuccessful search?
- Q: What is the best case complexity of Binary Search (data sorted)?
  - Best case complexity is not interesting. Why?

5.8 Keeping Your Perspective

- If the problem size is always small, you can probably ignore an algorithm's efficiency
- Weigh the trade-offs between an algorithm's time requirements and its memory requirements
- Compare algorithms for both style and efficiency
- Order-of-magnitude analysis focuses on large problems
- There are other measures, such as big Omega (Ω), big theta (Θ), little oh (o), and little omega (ω). These may be covered in more advanced module.