NUS MA2108 Finals Cheatsheet

Well-ordering principle for $\mathbb{N}$ Every non empty subset S of $\mathbb{N}$ has a least element, i.e. there exists $m \in S$ such that $m \leq k, \forall k \in S$.

The Real Numbers

The Order Property of $\mathbb{R}$

Bernoulli's Inequality If $x > -1$, then $(1+x)^n \geq 1 + nx, \forall n \in \mathbb{N}$.

Absolute Value and the Real Line

Theorem 2.2.2 (Properties of absolute value)
(a) $|ab| = |a||b|, \forall a, b \in \mathbb{R}$
(b) $|a|^2 = a^2, \forall a \in \mathbb{R}$
(c) If $c \geq 0$, then $|a| \leq c \iff -c \leq a \leq c$
(d) $-|a| \leq a \leq |a|, \forall a \in \mathbb{R}$
Theorem 2.2.3 Triangle Inequality If $a, b \in \mathbb{R}$, then $|a+b| \leq |a| + |b|$
Corollary 2.2.4 For $a, b \in \mathbb{R}$, one has
(i) $||a| - |b|| \leq |a-b| \equiv -|a-b| \leq |a| - |b| \leq |a-b|$
(ii) $|a-b| \leq |a| + |b|$
Corollary 2.2.5 If $a_1, a_2, \cdots, a_n$ are any real numbers, then

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

Definition 2.2.7 Let $a \in \mathbb{R}$ and $\epsilon > 0$. Then the $\epsilon-$**neighborhood** of a is the set $V_\epsilon(a) := \{x \in \mathbb{R} : |x-a| < \epsilon\}$.

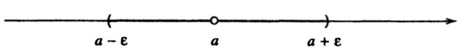


Figure 2.2.4 An ε-neighborhood of *a*

The Completeness Property of $\mathbb{R}$

Definition 2.3.2(a) Let S be a nonempty subset of $\mathbb{R}$. A number u is said to be **supremum** (or least upper bound) of S if it satisfies the conditions
(1) u is an upper bound of S (i.e. $s \leq u, \forall s \in S$). By defn of upper bound, $u \in \mathbb{R}$.
(2) if v is any upper bound of S, then $u \leq v$.
Definition 2.3.2(b) Let S be a nonempty subset of $\mathbb{R}$. A number w is said to be **infimum** (or greatest lower bound) of S if it satisfies the conditions:
(1) w is an lower bound of S (i.e. $w \leq s, \forall s \in S$). By defn of lower bound, $w \in \mathbb{R}$.
(2) if t is any lower bound of S, then $t \leq w$.

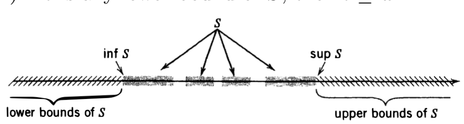


Figure 2.3.1 inf *S* and sup *S*

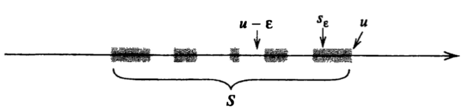


Figure 2.3.2 $u = \sup S$

The Supremum/Infimum Property (Axiom) of $\mathbb{R}$ Every **nonempty** subset of $\mathbb{R}$ that has an upperbound/lowerbound has a supremum/infimum.

Applications of the Supremum Property

2.4.3 Archimedean Property If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N} : x < n_x$.
Corollary 2.4.5 For any $\epsilon > 0, \exists n \in \mathbb{N} : \frac{1}{n} < \epsilon$. Note that the numerator can be any constant $c \in \mathbb{N}$. Let $n' := cn \in \mathbb{N}$.

$$\frac{c}{n'} < \epsilon \iff \frac{1}{n} < \epsilon$$

Theorem 2.4.8 (The Density Theorem of $\mathbb{Q}$) If $x, y \in \mathbb{R}$ with $x < y$, then there exists a rational number $r \in \mathbb{Q} : x < r < y$.
Corollary (The Density Theorem of Irrational Numbers) If $x, y \in \mathbb{R}$ with $x < y$, then there exists an irrational number $z : x < z < y$.

Intervals

Theorem 2.5.1 (Nested Interval Property) If $I_n = [a_n, b_n], n \in \mathbb{N}$ is a nested sequence of closed bounded intervals, then there exists a number $\xi \in \mathbb{R} : \xi \in I_n, \forall n \in \mathbb{N}$.
Theorem 2.5.2 If $I_n = [a_n, b_n], n \in \mathbb{N}$ is a nested sequence of closed bounded intervals such that the lengths $b_n - a_n$ of I_n satisfy

$$\inf\{b_n - a_n : n \in \mathbb{N}\} = 0,$$

then the number ξ contained in all I_n is unique.

Sequences and their limits

Definition 3.1.3 (Definition of Limits) $\forall \epsilon > 0, \exists K(\epsilon) \in \mathbb{N} : |x_n - x| < \epsilon, \forall n \geq K(\epsilon)$.
Theorem 3.1.9. Let $X = (x_n : n \in \mathbb{N})$ be a sequence of real numbers and let $m \in \mathbb{N}$. Then the m-tail $X_m = (x_{m+n} : n \in \mathbb{N})$ of X converges iff X converges. In this case, $\lim X_m = \lim X$.
Definition 3.2.1 A sequence $X = (x_n)$ of real numbers is said to be **bounded** if there exists a real number $M > 0 : |x_n| \leq M, \forall n \in \mathbb{N}$.
Theorem 3.2.2 A convergent sequence of real numbers is bounded. However, it is not true that bounded sequence implies convergence.
Theorem 3.2.3. If $\lim_{n \mapsto \infty} x_n = x, \lim_{n \mapsto \infty} y_n = y$ and $c \in \mathbb{R}$, then
(i) $\lim_{n \mapsto \infty}(x_n + y_n) = x + y$;
(ii) $\lim_{n \mapsto \infty}(x_n - y_n) = x - y$;
(iii) $\lim_{n \mapsto \infty}(x_n y_n) = xy$;
(iv) $\lim_{n \mapsto \infty} c(x_n) = cx$;
(v) $\lim_{n \mapsto \infty}(x_n / y_n) = x/y$ provided $y_n \neq 0, \forall n \in \mathbb{N}$ and $y \neq 0$.
Observations
Let $k \in \mathbb{N}$. $\lim_{n \mapsto \infty}(a_n^k) = (\lim_{n \mapsto \infty}(a_n))^k$.
T5Q2 If $x_n > 0$ and $\lim_{n \mapsto \infty} x_n = x \neq 0$, then $\lim_{n \mapsto \infty}(x_n)^r = x^r$ for $r \in \mathbb{Q}$.
Theorem 3.2.7 (Squeeze Theorem) If $x_n \leq y_n \leq z_n, \forall n \in \mathbb{N}$ and $\lim_{n \mapsto \infty} x_n = \lim_{n \mapsto \infty} z_n = a$, then we have

$$\lim_{n \mapsto \infty} y_n = a$$

Theorem 3.2.9 Let the sequence $X = (x_n)$ converge to x. If $x = \lim(x_n)$, then $|x| = \lim(|x_n|)$.
HW2Q1(c) Let (x_n) be a sequence of positive real numbers such that $L := \lim(x_{n+1}/x_n)$ exists. If $|L| < 1$, then (x_n) converges and $\lim(x_n) = 0$.

Theorem 3.3.2 (Monotone Convergence Theorem) Let (x_n) be a **monotone** sequence of real numbers. Then

$$(x_n) \text{ is convergent } \iff (x_n) \text{ is bounded.}$$

Moreover,
(a) If (x_n) is bounded and increasing, then $\lim x_n = \sup\{x_n : n \in \mathbb{N}\}$
(b) If (x_n) is bounded and decreasing, then $\lim x_n = \inf\{x_n : n \in \mathbb{N}\}$
Theorem 3.4.2 If (x_n) converges to x, then any subsequence (x_{n_k}) also converges to x. That is,

$$\lim_{n_k \mapsto \infty} x_{n_k} = \lim_{k \mapsto \infty} x_{n_k} = x.$$

Theorem. If every subsequence of (x_n) converges to the same limit, then the limit of (x_n) is the same.
Theorem 3.4.5 If a sequence (x_n) has either of the following properties, then (x_n) is divergent.
1. (x_n) has two convergent subsequences whose limits are not equal.
2. (x_n) is unbounded.
Theorem 3.4.8 (Bolzano-Weierstrass Theorem) Every bounded sequence has a convergent subsequence.
Definition 3.5.1 (Cauchy Sequence) $\forall \epsilon > 0, \exists H(\epsilon) \in \mathbb{N} : |x_n - x_m| < \epsilon, \forall n, m \geq H(\epsilon)$. For the negation of 3.5.1, note that there can be a relation between n, m.
Theorem 3.5.5 (Cauchy Convergence Criterion) A sequence of real numbers is convergent iff it is a Cauchy sequence.
Definition 3.5.7 A sequence (x_n) is called **contractive** if there exists a constant C, $0 < C < 1$ such that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$$

$\forall n \in \mathbb{N}$. The number C is called the **constant** of the contractive sequence.
Theorem 3.5.8 Every contractive sequence is Cauchy and so is convergent.

Examples for Sequences and their limits.

Example 3.1.6(a) $\lim(\frac{1}{n}) = 0$.
Example 3.1.6(b) $\lim(\frac{1}{n^2+1}) = 0$.
Example 3.1.6(d) $\lim(\sqrt{n+1} - \sqrt{n}) = 0$.
Example 3.1.6(e) If $0 < b < 1$, then $\lim(b^n) = 0$.
Example 3.1.11(c) $c > 0 \implies \lim(c^{1/n}) = 1$.
Example 3.1.11(d) $\lim(n^{1/n}) = 1$.
Example 3.2.8 (f) $\lim(\frac{\sin n}{n}) = 0$.
18/19 Midterm Q3(ii) $\lim(\frac{\sqrt{n^3}}{\sqrt{n^3+n}}) = 1$.

Introduction to Infinite Series

Theorem 3.7.3 (the n-th term test) If the series $\sum x_n$ converges, then $\lim_{n \mapsto \infty} x_n = 0$. Or contrapositively, if $\lim_{n \mapsto \infty} x_n$ does not exist or exists but not 0, then the series diverges.
Theorem 3.7.4 (Cauchy criterion test) The series $\sum x_n$ converges if and only if for every $\epsilon > 0$, there exists $M(\epsilon) \in \mathbb{N}$ such that if $m > n \geq M(\epsilon)$, then

$$|s_m - s_n| = |x_{n+1} + \cdots + x_m| < \epsilon$$

Theorem 3.7.5 (Partial sum bounded test for series with nonnegative terms Suppose $x_n \geq 0, \forall n \in \mathbb{N}$. Then the series $\sum x_n$ converges iff the sequence (s_n) of partial sums is bounded. In this case,

$$\sum x_n = \lim_{n \mapsto \infty} s_n = \sup\{s_n : n \in \mathbb{N}\}$$

Theorem 3.7.7 (Comparison Test) Let $(x_n), (y_n)$ be real sequences and suppose that *for some* $K \in \mathbb{N}$, we have

$$\mathbf{0} \leq x_n \leq y_n, \forall n \geq K$$

Then
(a) The convergence of $\sum y_n$ implies the convergence of $\sum x_n$.
(b) The divergence of $\sum x_n$ implies the divergence of $\sum y_n$ (contrapositive of (a)).
Theorem 3.7.8 (Limit comparison Test) Suppose that $(x_n), (y_n)$ are **strictly positive** sequences and suppose that the following limits exists:

$$r := \lim_{n \mapsto \infty}(\frac{x_n}{y_n})$$

(a) If $r > 0$ then $\sum x_n$ is convergent iff $\sum y_n$ is convergent.
(b) If $r = 0$ and if $\sum y_n$ is convergent, then $\sum x_n$ is convergent.
Remark
(i) The comparison tests 3.7.7 and 3.7.8 depend on having a stock of series that one knows to be convergent (or divergent). The reader will find that the p-series is often useful for this purpose.
(ii) Contrapositive argument for divergence test.

Examples for Infinite Series

Example 3.7.6(b) The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
Example 3.7.6(c) The **2-series** $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
Example 3.7.6(d) The **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$.
Example 3.7.6(e) The **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $0 < p \leq 1$.
Example 3.7.6(f) The alternating harmonic series given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

is convergent to $\ln 2$.
T7Q2(b) $\sum_{n=0}^{\infty} \frac{1}{n!} = \underset{n \mapsto \infty}{e_n} = e$. The series converges.
T7Q2(c) $\sum_{n=0}^{\infty}(1 + \frac{1}{n})^n$ converges.

Infinite Series

Absolute Convergence

Definition 9.1.1 The series $\sum x_n$ is **absolutely convergent** if the series $\sum |x_n|$ is convergent. A series is said to be **conditionally (or nonabsolutely) convergent** if it is convergent, but it is **not** absolutely convergent.
Theorem 9.1.2 If a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, then it is convergent.

Tests for Absolute Convergence

Theorem 9.2.1 (Limit Comparison Test II) Suppose that $(x_n), (y_n)$ are non-zero sequences and suppose that the following limits exists:

$$r := \lim_{n\mapsto\infty}\left(\frac{|x_n|}{|y_n|}\right)$$

(a) If $r > 0$, then $\sum x_n$ is *absolutely convergent* iff $\sum y_n$ is *absolutely convergent.*
(b) If $r = 0$ and $\sum y_n$ is *absolutely convergent*, then $\sum x_n$ is absolutely convergent.
Theorem 9.2.2 (Root Test) Let (x_n) be a sequence
(a) If there exist $r \in \mathbb{R}$ with $0 \le r < 1$ and $K \in \mathbb{N}$ such that

$$|x_n|^{\frac{1}{n}} \le r \quad \forall n \ge K$$

then the series $\sum x_n$ is absolutely convergent.
(b) If there exists $K \in \mathbb{N}$ such that

$$|x_n|^{\frac{1}{n}} \ge 1 \quad \forall n \ge K$$

then the series $\sum x_n$ is divergent.
Corollary 9.2.3 Suppose that the limit $r := \lim_{n\mapsto\infty} |x_n|^{\frac{1}{n}}$ exists. Then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.
Remark. (i) If $r = 1$, the root test provides no information.
(ii) Corollary 9.2.3 uses lim while Theorem 9.2.2 uses upper/lower bound.
Theorem 9.2.4 (Ratio Test) Let (x_n) be a sequence of nonzero real numbers.
(a) If there exist r with $0 < r < 1$ and $K \in \mathbb{N}$ such that

$$\left|\frac{x_{n+1}}{x_n}\right| \le r \quad \forall n \ge K$$

then $\sum x_n$ is absolutely convergent.
(b) If there exists $K \in \mathbb{N}$ such that

$$\left|\frac{x_{n+1}}{x_n}\right| \ge 1 \quad \forall n \ge K$$

then $\sum x_n$ is divergent.
Corollary 9.2.5 Let (x_n) be a sequence of nonzero real numbers and suppose that the limit $r := \lim_{n\mapsto\infty} |\frac{x_{n+1}}{x_n}|$ exists. Then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.

Limits

Limits of Functions

Definition 4.1.1 Let A be a subset of $\mathbb{R}$. A point c is called a **cluster point** of A if for every $\delta > 0$, there exists at least one point $x \in A, x \ne c$ such that $|x - c| < \delta$ or $c - \delta < x < c + \delta$. Alternatively,

$$(V_\delta(c)\backslash\{c\}) \cap A \ne \emptyset \text{ for any } \delta > 0$$

Definition 4.1.4 Let $A \subseteq \mathbb{R}$ and c be a cluster point of A. For a function $f : A \mapsto \mathbb{R}$, a real number L is said to be a **limit** of f at c if for any given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$, that is,

$$x \in A \cap (V_\delta(c)\backslash\{c\}) \implies f(x) \in V_\epsilon(L)$$

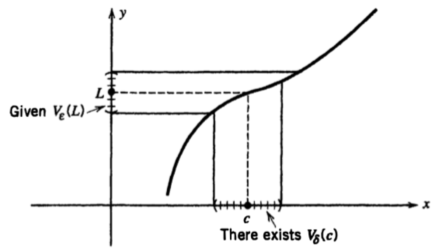
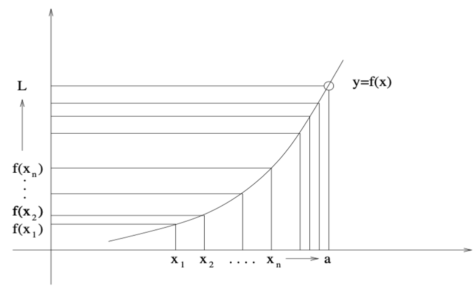


Figure 4.1.1 The limit of f at c is L

Remark
1. The cluster point c does not necessarily belong to A. Thus to discuss the limit of f at a point $x = c$, we do not require f to be defined at $x = c$.
2. Even if $f(c)$ is defined, its value has no bearing on $\lim_{x\mapsto c} f(x)$.

Definition If f has no limit at $x = c$, then we say that f diverges at c.
Theorem 4.1.8 (Sequential Criterion for Limits of Functions) Let $f : A \mapsto \mathbb{R}$ and a be a cluster point of A. The following statements are equivalent:
1. $\lim_{x\mapsto a} f(x) = L$
2. For every sequence (x_n) in A that converges to a such that $x_n \ne a$ for all n, the sequence $(f(x_n))$ converges to L.



Limit Theorems

Definition 4.2.1 Let $f : A \mapsto \mathbb{R}$ and c be a cluster point of A. We say that f is **bounded on a neighbourhood of** c if there exists $V_\delta(c)$ and a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in A \cap V_\delta(c)$.
Theorem 4.2.2 If $f : A \mapsto \mathbb{R}$ has a limit at a cluster point c, then f is bounded on some neighbourhood of c.
Definition 4.2.3 Let $A \subset \mathbb{R}$ and let f and g be functions defined on A. We define the **sum** $f + g$, the **difference** $f - g$, and the **product** fg on A to be the functions given by

$$(f+g)(x) := f(x) + g(x)$$
$$(f-g)(x) := f(x) - g(x)$$
$$(fg)(x) := f(x)g(x)$$

If $b \in \mathbb{R}$, we define the **multiple** bf to be the function given by

$$(bf)(x) := bf(x)$$

If $h(x) \ne 0, \forall x \in A$, we define the **quotient** f/h to be the function given by

$$\left(\frac{f}{h}\right)(x) := \frac{f(x)}{h(x)}$$

Theorem 4.2.4 Suppose that $\lim_{x\mapsto c} f(x) = L$ and $\lim_{x\mapsto c} g(x) = M$. Let $b \in \mathbb{R}$.
(a) $\lim_{x\mapsto c}(f \pm g)(x) = L \pm M$;
(b) $\lim_{x\mapsto c}(fg)(x) = LM$, $\lim_{x\mapsto c}(bf)(x) = bL$;
(c) If $h(x) \ne 0, \forall x \in A$ and $\lim_{x\mapsto c} h(x) = H \ne 0$, then

$$\lim_{x\mapsto c}\left(\frac{f}{h}\right)(x) = \frac{L}{H}.$$

Remark Let $f_1, \cdots, f_n$ be functions on A to $\mathbb{R}$. Assume that $\lim_{x\mapsto c} f_i(x) = L_i, 1 \le i \le n$.
1. $\lim_{x\mapsto c}(f_1 + \cdots + f_n)(x) = L_1 + \cdots + L_n$.
2. $\lim_{x\mapsto c}(f_1 \cdot \cdots \cdot f_n)(x) = L_1 \cdot \cdots \cdot L_n$.
3. If $L = \lim_{x\mapsto c} f$ and $n \in \mathbb{N}$, then $\lim_{x\mapsto c}(f(x)^n) = (\lim_{x\mapsto c} f(x))^n = L^n$.

Theorem 4.2.7 (Squeeze Theorem) Let $A \subseteq \mathbb{R}$, let $f, g, h : A \mapsto \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. Suppose that $f(x) \le g(x) \le h(x)$ for all $x \in A$ and $\lim_{x\mapsto c} f(x) = \lim_{x\mapsto c} h(x) = L$, then

$$\lim_{x\mapsto c} g(x) = L.$$

Theorem 4.2.9 (Lower Bound) If $\lim_{x\mapsto c} f(x) > 0$ (resp. $\lim_{x\mapsto c} f(x) < 0$), then there exists $V_\delta(c)$ of c such that $f(x) > 0$ (resp. $f(x) < 0$) for all $x \in A \cap V_\delta(c), x \ne c$.
Remark. Lower bound theorem can be applied to continuous points as well since $\lim_{n\mapsto\infty} f(x) = f(c)$ iff f is continuous at c.
Definition 4.3.1 (One-sided limit) Let f be a function on A to $\mathbb{R}$.
(i) Let c be a cluster point of $A \cap (c, \infty)$. We say that L is the **right-hand limit** of f at c if for any $\epsilon > 0$, $\exists\delta > 0$:

$$\mathbf{0 < x - c} < \delta(i.e. x \in (c, c+\delta)) \implies |f(x) - L| < \epsilon$$

In this case, we write $\lim_{x\mapsto c^+} f(x) = L$.
(ii) Let c be a cluster point of $A \cap (-\infty, c)$. We say that L is the **left-hand limit** of f at c if for any $\epsilon > 0$, $\exists\delta > 0$:

$$-\delta < \mathbf{x - c < 0}(i.e. x \in (c-\delta, c)) \implies |f(x) - L| < \epsilon$$

In this case, we write $\lim_{x\mapsto c^-} f(x) = L$.
Theorem 4.3.2 (Sequential Criterion for one-sided limits).
(i) $\lim_{x\mapsto c^+} f(x) = L \iff$ if for every (x_n) converges to c and $x_n > c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.
(ii) $\lim_{x\mapsto c^-} f(x) = L \iff$ for every (x_n) converges to c and $x_n < c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.
Theorem 4.3.3 $\lim_{x\mapsto c} f(x) = L$ exists iff both $\lim_{x\mapsto c^+} f(x)$ and $\lim_{x\mapsto c^-} f(x)$ exist and

$$\lim_{x\mapsto c^+} f(x) = \lim_{x\mapsto c^-} f(x) = L.$$

Continuous Functions

Continuous Functions

Definition 5.1.1 ($\epsilon - \delta$ definition of continuity) Let $A \subset R$, let $f : A \mapsto \mathbb{R}$ and let $c \in A$. We say that f is **continuous at c** if given any number $\epsilon > 0$, there exists $\delta > 0$ such that if x is any point of A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.
Equivalent Definition to 5.1.1. If c is a cluster point in A, $f(x)$ is continuous at c iff

$$f(c) = \lim_{x\mapsto c} f(x)$$

Remark. The equivalent definition is useful as it opens up limit theorems (e.g. squeeze theorem, lower bound etc).

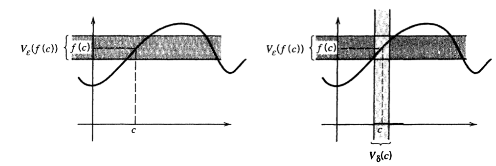
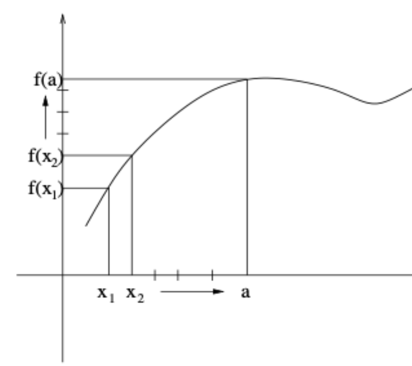


Figure 5.1.1 Given $V_\epsilon(f(c))$, a neighborhood $V_\delta(c)$ is to be determined

Remarks
(i) Limit has to exist (i.e. left and right limit must exist and be the same.)
(ii) Equality must hold
(iii) $f(c)$ must be well defined.
Theorem 5.1.3 (Sequential Criterion for Continuity) A function $f : A \mapsto \mathbb{R}$ is continuous at the point $c \in A$ iff for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.



Theorem 5.1.4 (Discontinuity Criterion) f is discontinuous at $x = a$ iff there **exists** a sequence (x_n) in the domain of f such that $x_n \mapsto a$, but $f(x_n) \nrightarrow f(a)$.
Remark (i) $(f(x_n))$ is divergent or (ii) $\lim_{n\mapsto\infty} f(x_n) \ne f(a)$.

Combinations of Continuous Functions

Theorem 5.2.1 Suppose that f and g are continuous at $x = c$, then
(a) $f \pm g, f \cdot g$, and bf are also continuous at $x = c$, where b is a constant.
(b) If $g(c) \ne 0$, then f/g is also continuous at $x = c$.
Theorem 5.2.2 Suppose that f and g are continuous on A, then
(a) $f \pm g, f \cdot g$, and bf are also continuous on A.
(b) If $g(c) \ne 0$, then f/g is also continuous on A.

Theorem 5.2.6 Let $f : A \mapsto \mathbb{R}, g : B \mapsto \mathbb{R}$ and $f(A) \subseteq B$. If f is continuous at c, and g is continuous at $b = f(c)$, then $g \circ f$ is continuous at c.
Theorem 5.2.7 Let $f : A \mapsto \mathbb{R}, g : B \mapsto \mathbb{R}$ and $f(A) \subseteq B$. If f is continuous at A, and g is continuous at B, then $g \circ f$ is continuous on A.

Continuous Functions on Intervals

Definition 5.3.1 A function $f : A \mapsto \mathbb{R}$ is **bounded** on A if there exists $M > 0$ such that

$$|f(x)| \leq M, \forall x \in A$$

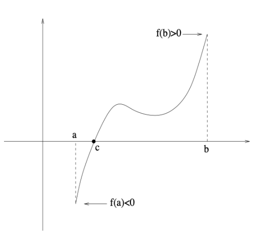
Or, the set $f(A)$ is bounded.
Definition 5.3.3 (i) We say that f has an **absolute maximum** on A if there exists $x^* \in A$ such that

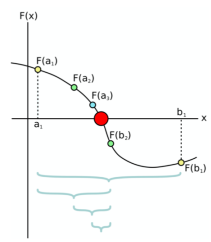
$$f(x^*) \geq f(x), \forall x \in A.$$

So in this case, $f(x^*) = \sup f(A) = \max f(A)$.
(ii) We say that f has an **absolute minimum** on A if there exists $x_* \in A$ such that

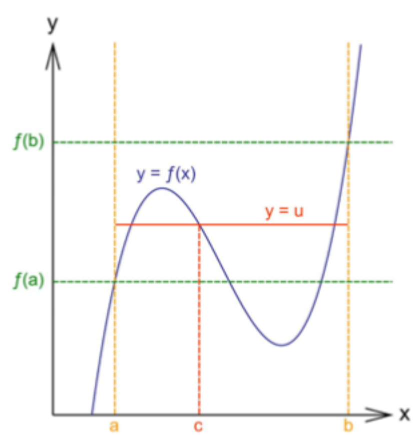
$$f(x_*) \leq f(x), \forall x \in A.$$

So in this case, $f(x_*) = \inf f(A) = \min f(A)$.
Theorem 5.3.4 (Maximum-Minimum Theorem) If f is continuous on $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.
Bisection Method. Essentially Binary Search.
Theorem 5.3.5 (Location of Roots Theorem) If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists a point $c \in (a, b)$ such that $f(c) = 0$.





Theorem 5.3.7 (Intermediate Value Theorem) Let I be an interval, f be continuous on I, and $a, b \in I$ with $f(a) \leq f(b)$. For any $k \in [f(a), f(b)]$, then there exists a point c in I such that $f(c) = k$.
Remark. $a, b \in I$. (i) Case $a \leq b$: $[a, b] \subseteq I$ (ii) Case $b > a$: $[b, a] \subseteq I$.



Theorem 5.3.10 (Preservation of Closed Intervals Theorem) If f is continuous on $[a, b]$, then

$$f([a, b]) := \{f(x) : x \in [a, b]\} = [m, M]$$

where $m = \inf f([a, b]) = f(x_*)$ and $M = \sup f([a, b]) = f(x^*)$. That is, for any $m \leq k \leq M$, there exists $c \in [a, b]$ such that $f(c) = k$.
Remark. Let f be a continuous function on $[a, b]$.
1. $f([a, b]) = [m, M]$ does not imply $f([a, b]) = [f(a), f(b)]$. Image of the closed interval might not equal the closed interval given by $f(a), f(b)$.
2. If we replace the closed bounded interval $[a, b]$ by an arbitrary interval I (e.g. open or half-open), then $f(I)$ could be of any type such as half-open or unbounded.

Uniform Continuity

Uniform Continuity Let $A \subset \mathbb{R}$, $f : A \mapsto \mathbb{R}$. We say that f is **uniformly continuous** on A if for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ ($\delta(\epsilon)$ only depends on ϵ) such that

$$\forall x, y \in A, |x - y| < \delta(\epsilon) \implies |f(x) - f(y)| < \epsilon$$

Negative Definition. f is **not uniformly continuous** on A if there exists an $\epsilon_0 > 0$ such that for all $\delta > 0$, there are points x_δ, y_δ such that $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon_0$.
(Sequential Criterion for Uniform Continuity) The function $f : A \mapsto \mathbb{R}$ is uniformly continuous on A iff for any two sequences $(x_n), (y_n)$ in A such that $\lim_{n\mapsto\infty} x_n - y_n = 0$, we have $\lim_{n\mapsto\infty} f(x_n) - f(y_n) = 0$.
Negative Definition. There exists an $\epsilon_0 > 0$ and two sequences $(x_n), (y_n)$ in A such that $\lim_{n\mapsto\infty} x_n - y_n = 0$ and we have $\lim_{n\mapsto\infty} |f(x_n) - f(y_n)| \geq \epsilon_0$.
Theorem 5.4.3 (Uniform Continuity Theorem) If f is continuous on a closed bounded interval $[a, b]$, then it is uniformly continuous on $[a, b]$.
Definition 5.4.4 A function $f : A \mapsto \mathbb{R}$ is said to be a **Lipschitz function** (or satisfy a **Lipschitz condition**) on A if there exists a $K > 0$ such that

$$\frac{|f(x) - f(y)|}{|x - y|} \leq K, \text{ for } x \neq y \in A$$

which implies that its *derivative* $f'(x)$ (if exists) is bounded on A.
Theorem 5.4.5 If $f : A \mapsto \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.
Theorem 5.4.7 (Uniformly continuous functions preserve Cauchy sequence) If $f : A \mapsto \mathbb{R}$ is uniformly continuous on A and (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence.
Theorem 5.4.8 (Continuous Extension Theorem) A function f is uniformly continuous on the interval (a, b) iff $\lim_{x\mapsto a^+} f(x)$ and $\lim_{x\mapsto b^-} f(x)$ can be defined at the endpoints a and b such that the extended function is continuous on $[a, b]$.

Monotone and Inverse Functions

Theorem 5.6.1 (One-sided Limits for Monotone Functions Exist Theorem) Let $I \subset \mathbb{R}$ be an interval and let $f : I \mapsto \mathbb{R}$ be **increasing** on I. Suppose that $c \in I$ is not an endpoint of I. Then
(i) $\lim_{x\mapsto c^-} f(x) = \sup\{f(x) : x \in I, x < c\}$
(ii) $\lim_{x\mapsto c^+} f(x) = \inf\{f(x) : x \in I, x > c\}$
Remark. By showing limit exists, we can also use theorems related to limits such as the sequential criterion for limits theorem.
Definition If $f : I \mapsto \mathbb{R}$ is **increasing** on I and if c is not an endpoint of I, we define the **jump of f at c** to be

$$j_f(c) := \lim_{x\mapsto c^+} f(x) - \lim_{x\mapsto c^-} f(x)$$
$$= \inf\{f(x) : x \in I, x > c\} - \sup\{f(x) : x \in I, x < c\}$$

At the endpoints a or b, define

$$j_f(c) := \begin{cases} \lim_{x\to a^+} f(x) - f(a) & \text{if } c = a \\ f(b) - \lim_{x\to b^-} f(x) & \text{if } c = b \end{cases}$$

Theorem 5.6.3 Let $f : I \mapsto \mathbb{R}$ be **increasing** on I. Then f is continuous at c iff $j_f(c) = 0$.
Remark. Theorem 5.6.1, Corollary 5.6.2, defn of Jump, and theorem 5.6.3 can be reformulated from **increasing** to **decreasing** functions.
Theorem 5.6.4 (Discontinuous points of monotone functions) Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \mapsto \mathbb{R}$ be monotone on I. Then the set of points $D \subseteq I$ at which f is discontinuous is a countable set.
Theorem 5.6.5 (Continuous Inverse Theorem) Let $I \subset \mathbb{R}$ be an interval and $f : I \mapsto \mathbb{R}$ be **strictly** monotone and continuous. Then the inverse function f^{-1} is also strictly monotone and continuous on $J := f(I) = \mathcal{R}(f)$.
Remark. $f^{-1} : J \mapsto \mathbb{R}$, where J might not equal the codomain of f.
Rational Power Function. $x^r, r \in \mathbb{Q}$ is defined.

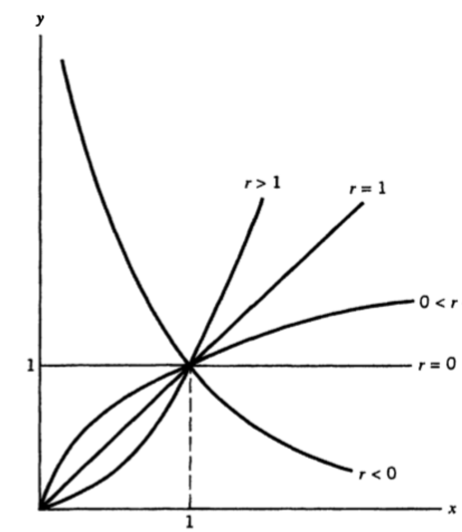


Figure 5.6.8 Graphs of $x \to x^r$ $(x \geq 0)$

Theorem 5.6.7 If $m \in \mathbb{Z}, n \in \mathbb{N}$ and $x > 0$, then $(x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}$

A Glimpse into Topology

Open and Closed Sets in $\mathbb{R}$

Definition 11.1.1 A set V is called a **neighbourhood** of a point $x \in \mathbb{R}$ if $\exists \epsilon > 0 : V_\epsilon(x) \subseteq V$.

Definition 11.1.2

(i) A subset G of $\mathbb{R}$ is **open** in $\mathbb{R}$ if G is a neighbourhood of any point in G, that is, $\forall x \in G, \exists \epsilon_x > 0 : V_{\epsilon_x}(x) \subseteq G$.

(ii) A subset F of $\mathbb{R}$ is **closed** in $\mathbb{R}$ if the complement $C(F) := \mathbb{R} \backslash F$ is open in $\mathbb{R}$. Equivalently, for any $y \notin F, \exists \epsilon_0 > 0 : V_{\epsilon_0}(y) \cap F = \emptyset$.

Theorem 11.1.4 (Open Set Properties)

(a) The union of an arbitrary collection of open subsets in $\mathbb{R}$ is open.

(b) The intersection of any **finite** collection of open sets in $\mathbb{R}$ is open.

Theorem 11.1.5 (Closed Set Properties)

(a) The intersection of an arbitrary collection of closed subsets in $\mathbb{R}$ is closed

(b) The union of any finite collection of closed sets in $\mathbb{R}$ is closed.

Theorem 11.1.7 (Characterisation of Closed Sets) A subset F of $\mathbb{R}$ is closed iff for any convergent sequence (x_n) in F, $\lim_{n\mapsto\infty} x_n$ belongs to F.

Theorem 11.1.8 A subset F of $\mathbb{R}$ is closed iff it contains all of its cluster points.

Theorem 11.1.9 (Characterisation of Open Sets) A subset of $\mathbb{R}$ is open iff it is the union of countably many disjoint open intervals in $\mathbb{R}$.

Continuous Functions

Lemma 11.3.1 A function $f : A \mapsto \mathbb{R}$ is continuous at the point c in A iff for every (open) neighborhood U of $f(c)$, there exists a (open) neighborhood V of c such that if $x \in V \cap A$, then $f(x) \in U$. That is, $\forall$ neighborhood U of $f(c)$, $\exists$ a neighborhood V of c st. $f(V \cap A) \subseteq U$.

Theorem 11.3.2 (Global Continuity Theorem) Let $f : A \mapsto \mathbb{R}$ be a function with domain A. Then the following are equivalent:

(a) f is continuous at every point of A.

(b) For every open set G in $\mathbb{R}$, there exists an open set H in $\mathbb{R}$ such that $H \cap A = f^{-1}(G)$.

Note that for a set G, $f^{-1}(G) := \{x \in A : f(x) \in G\}$ (preimage, not inverse!).

Corollary 11.3.3 A function $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous iff $f^{-1}(G)$ is open in $\mathbb{R}$, whenever G is open. Or, the preimage of an open set is always open.

Remark. In general, for an open set G, the direct image $f(G)$ is not necessarily open.

Metric Spaces

Definition 11.4.1 A **metric** on a set S is a function $d : S \times S \mapsto \mathbb{R}$ that satisfies the following properties:

(a) $d(x, y) \geq 0, \forall x, y \in S$ (positivity);

(b) $d(x, y) = 0 \iff x = y$ (definiteness);

(c) $d(x, y) = d(y, x), \forall x, y \in S$ (symmetry);

(d) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in S$ (triangle inequality);

Definition 11.2.1 An **open cover** of a subset A of a metric space S is a collection of $\mathcal{G} := \{G_\lambda : \lambda \in \Lambda\}$ of open subsets of S whose union contains A; that is,

$$A \subseteq \cup_{\lambda \in \Lambda} G_\lambda$$

If $\mathcal{G}'$ is a subcollection of sets from $\mathcal{G}$, such that the union of the sets in $\mathcal{G}'$ also contains A, then $\mathcal{G}'$ is called a **subcover of** $\mathcal{G}$.

If $\mathcal{G}'$ consists of finitely many sets, then we call $\mathcal{G}'$ a **finite subcover of** $\mathcal{G}$.

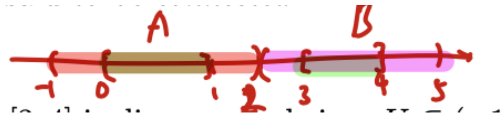
Definition 11.2.2 A subset K of a metric space S is said to be **compact** if every open cover of K has a finite subcover.

Definition. Let (S, d) be a metric space. A subset A of S is called **bounded** if there exist a positive real number M and a point $x \in S$ such that $A \subseteq V_M(x)$ (i.e. $d(a, x) < M, \forall a \in A$).

Theorem 11.2.5. Let (S, d) be a metric space. A subset K of S is compact iff it is closed and bounded.

Theorem 11.4.13 (Preservation of Compactness) If (S, d) is a compact metric space and if the function $f : S \mapsto \mathbb{R}$ is continuous, then $f(S)$ is compact in $\mathbb{R}$.

Definition (disconnected) A subset U of S is called **disconnected** if U has an open cover $\{A, b\}$ such that $A \cap B \cap U = \emptyset$, $A \cap U \neq \emptyset$, $B \cap U \neq \emptyset$. Otherwise, U is said to be *connected.*



Theorem A subset E of $\mathbb{R}$ is connected iff E is an interval.

Theorem (Preservation of connectedness) Let $f : S \mapsto \mathbb{R}$ be a continuous function. If E is a connected subset of S, then $f(E)$ is connected.

Corollary (Generalised form of preservation of closed intervals). Let $f : S \mapsto \mathbb{R}$ be a continuous function. If K is a connected compact subset of S, then $f(K) = [\inf f(K), \sup f(K)]$, a closed bounded interval.

Calculus

L'Hospital's Rule. Suppose that we have one of the following cases,

$$\lim_{x \mapsto \alpha} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ OR } \frac{\pm\infty}{\infty}$$

where α can be any real number, $\pm\infty$. In these cases, we have,

$$\lim_{x \mapsto \alpha} \frac{f(x)}{g(x)} = \lim_{x to \alpha} \frac{f'(x)}{g'(x)}$$

Partial Fraction Decomposition. (1) Factor the bottom. $\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x-2)(x+1)}$.
(2) Write one partial fraction for each of those factors. $\frac{5x-4}{(x-2)(x+1)} = \frac{A_1}{x-2} + \frac{A_2}{x+1}$.
(3) Multiply through by the bottom so we no longer have fractions. $5x - 4 = A_1(x + 1) + A_2(x - 2)$.
(4) Now find constants A_1 and A_2 by substituting the roots (i.e. $x = -1, 2$).

Inequalities
(i) **Multiplicative Inverse** If $a < b$, then $\frac{1}{a} < \frac{1}{b}$
(ii) **Square Root Property** If $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$
(iii) **Exponential** $f(x) = a^x$ is monotonely increasing/decreasing for $a > 1$ and $0 < a < 1$ respectively.

sin **properties**
(i) $\sin x \leq x$
(ii) $\sin x - \sin y = \sin \frac{x-y}{2} \cos \frac{x-y}{2}$

List of Questions

The Real Numbers
Define Bernoulli's Inequality.
Define Triangle Inequality and it's two corrollaries.

The Completeness Property of $\mathbb{R}$
Define $\epsilon-$neighborhood of a.
Define sup and inf and sup/inf property.
Define Archimedean Property and corollary.
Define Density theorem (both rational and irrational).
Describe nested interval property.
How to prove convergence and calculate limits: five ways to identify limits (for cauchy criterion, state negation too).
Define divergent properties.
State relation between limits of subsequences and sequences (2).
State and define five ways to test if infinite series converge/diverge
State and define three ways to test for absolute convergence.
State relation between limit of modulus and modulus of limit

Limits
Define cluster point.
State ways to prove limits of functions (3, excluding one-sided limit)
State lower bound theorem.

Continuous Functions
State ways to show that a function is continuous at a point (4)
Define continuous functions and sequential criterion for continuity. State usefulness of relation to limit.
State discontinuity criterion.
State combinations/composition of functions.
State max-min theorem.
State location of roots and intermediate value theorem.
Define preservation of closed intervals theorem.
How to prove uniform continuity (4)?
How to prove non-uniform continuity (2)?
Define theorems of uniform continuity (2x)
Define one-sided limits for monotone function exist theorem and it's usefulness.
Define jump and it's usefulness.
Define discontinuous points of monotone functions theorem.
Define continuous inverse theorem.
State usefulness of rational power functions.

A glimpse into topology
Define neighbourhood, open set, **closed set.**
Define open set properties, closed set properties.
Define characterisation of **closed sets** and open sets.
Define Global Continuity Theorem corollary
Define metric conditions
Define open cover
State intuitive understanding of disconnected
State theorems related to compactness (x2) and connectedness (x2)
State Generalised form of preservation of closed intervals

Calculus
What to do when you see $\frac{2n+1}{n^2(n+1)^2}$?
If $x < y$ and $0 < a < 1$, what's the inequality relating a^x and a^y?

Examples to relook

2.1.13 (1, 2), 2.2.6(b), 2.4.1(b), 3.1.6(d), 3.2.8 (e), 3.4.3(a), 3.5.6(c), 3.7.6(c), T3Q2 (FD), T3Q5, T5Q2(a), T6Q3, 18/19 (Q2), 13/14(Q1(a), Q5(ii))

Prepared by Larry, AY2020/2021 Semester 1