NUS MA2101 Final Cheat Sheet

1: Vector Spaces

Matrices

Matrix Multiplication
(i) (element) Let $(AB) = (c_{ij})$. $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.
(ii) (matrix multiplied by column) $A_{nn} c_{n1} = (\mathbf{a_1}|\cdots|\mathbf{a_n})(c_1 \cdots c_n)^T = c_1\mathbf{a_1} + \cdots + c_n\mathbf{a_n}$.
(iii) (matrix multiplied by matrix) $AB = A(\mathbf{b_1}|\cdots|\mathbf{b_n}) = (A\mathbf{b_1}|\cdots|A\mathbf{b_n})$
Theorem 1.4.1. (Associative Law) If A, B, C are matrices over the field $\mathbb{F}$ st products BC and $A(BC)$ are defined, then $A(BC) = (AB)C$.
Theorem (Distributive Law) If A, B, C are matrices of respective sizes $m \times n, n \times p, n \times p$, then

$$A(B+C) = AB + AC$$

Determinant

Definition A function $D : M_n(\mathbb{F}) \mapsto \mathbb{F}$ is called a **determinant function** if it has the following properties:
(D1) It is *multilinear*. We regard D as a function on $n-tuples$ of column vectors:
$D(X) = D(X_1, \cdots, X_n)$ where $X = (x_{ij})$ and $X_1, \cdots, X_n$ are the columns of X. Then D is a **linear function** of each column when the other columns are held fixed, that is, for each $1 \leq j \leq n, \alpha, \beta \in \mathbb{F}$ and column vectors $X_1, \cdots, X_{j-1}, X_{j+1}, \cdots, X_n, \mathbf{u}, \mathbf{v}$,

$$\begin{aligned} &D(X_1, \cdots, X_{j-1}, \alpha\mathbf{u} + \beta\mathbf{v}, X_{j+1}, \cdots, X_n) \\ &= \alpha D(X_1, \cdots, X_{j-1}, \mathbf{u}, X_{j+1}, \cdots, X_n) \\ &+ \beta D(X_1, \cdots, X_{j-1}, \mathbf{v}, X_{j+1}, \cdots, X_n) \end{aligned}$$

(D2) It is *alternating*. This means that if X has two equal columns, then $D(X) = 0$.
(D3) If I is the $n \times n$ identity matrix, then

$$D(I) = 1$$

Theorem 1.5.2. Let the function $D : M_n(\mathbb{F}) \mapsto \mathbb{F}$ be multilinear and alternating. If $X \in M_n(\mathbb{F})$ and X' is the matrix obtained from X by interchanging two columns then

$$D(X') = -D(X)$$

Theorem 1.5.7. (Cofactor expansion along row i). If $A = (a_{ij}) \in M_n(\mathbb{F})$ and $1 \leq i \leq n$, then

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \tilde{A}_{ij}$$

Note that row i is fixed and i need not necessarily be 1. Differs from theorem 1.5.3 in that it uses theorem 1.5.5 to replace the determinant function with the function's evaluation.
Corollary. Let $A = (a_{ij}) \in M_n(\mathbb{F})$ be an upper or lower triangular matrix. Then

$$\det(A) = \prod_{k=1}^{n} a_{kk}$$

Theorem 1.5.8. If $A \in M_r(\mathbb{F}), B \in M_{rs}(\mathbb{F})$ and $C \in M_s(\mathbb{F})$, then

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = (\det A)(\det C)$$

Theorem 1.5.9. If $A, B \in M_n(\mathbb{F}), then$

$$\det(AB) = \det A \det B$$

Theorem 1.5.10. If $A \in M_n(\mathbb{F})$ and A^t is its transpose, then

$$\det A^t = \det A$$

Theorem 1.5.11. (Cofactor expansion along column j) If $A = (a_{ij}) \in M_n(\mathbb{F})$ and $1 \leq j \leq n$, then

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det \tilde{A}_{ij}$$

Note that column j is fixed and j need not necessarily be 1.
Theorem 1.9.1. Let V be a vector space over $\mathbb{F}$ and S a finite subset of vectors in V. Then:
(i) $Span(S)$ is a subspace of V
(ii) If W is a subspace of V and $S \subseteq W$ then $Span(S) \subseteq W$.

Direct sum of subspaces

Theorem 1.13.2. Let $W_1, \cdots, W_k$ be subspaces of the vector space V and $W = W_1 + \cdots + W_k$. Then the following are equivalent:
(i) $W = W_1 \oplus \cdots \oplus W_k$.
(ii) For each $2 \leq j \leq k$, we have

$$W_j \cap (W_1 + \cdots + W_{j-1}) = \{\mathbf{0}\}$$

(iii) If $\mathcal{B}_i$ is a basis for W_i for $1 \leq i \leq k$ and $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset, \forall i \neq j$, then

$$\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$$

is a basis for W.

Chapter 1 Tutorial Theorems

T2Q5
(a) $S_1 \subseteq S_2 \implies Span(S_1) \subseteq Span(S_2)$
(b) $Span(S_1 \cup S_2) = Span(S_1) + Span(S_2)$
(c) $Span(S_1 \cap S_2) \subseteq Span(S_1) \cap Span(S_2)$
T3Q7. Let V be a finite dimensional vector space and W_1 and W_2 be two subspaces of V.

$$\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$$

T4Q1. Let V be a finite dimensional vector space and let W be a subspace of V.
(i) W has a basis and $\dim W \leq \dim V$.
(ii) If $\dim W = \dim V$, then $W = V$.
Properties of invertible matrix Let A be an $n \times n$ matrix. The following statements are equivalent:
(i) A is invertible
(ii) The linear system $A\mathbf{v} = \mathbf{0}$ has only the trivial solution.
(iii) The RREF of A is an identity matrix.
(iv) A can be expressed as product of elementary matrices.
(v) $\det(A) \neq 0$
(vi) The rows of A form a basis for $\mathbb{R}^n$.
(vii) The columns of A form a basis for $\mathbb{R}^n$.
(viii) $rank(A) = n$
(IX) 0 is not an eigenvalue of A
Properties of determinant.
(i) $\det(A^{-1}) = \frac{1}{det(A)}$
(ii) $\det(cA) = c^n \det(A)$

Properties of Trace
(i) $Tr(A+B) = Tr(A) + Tr(B)$
(ii) $Tr(cA) = cTr(A)$
(iii) $Tr(A) = Tr(A^T)$
(iv) $Tr(A^TB) = Tr(AB^T)$
(v) $Tr(AB) = Tr(BA)$
(vi) (cyclic property) $Tr(ABCD) = Tr(BCDA) = Tr(CDAB) = Tr(DABC)$

2: Linear Transformations

Operations on Functions

(i) $(f \pm g)(x) = f(x) \pm g(x)$
(ii) $(f \cdot g)(x) = f(x) \cdot g(x)$
(iii) $(f/g)(x) = f(x)/g(x)$

Definition and Examples

Definition.

$$\begin{aligned} &\mathbf{v}_1, \mathbf{v}_2 \in V \text{ and } c_1, c_2 \in \mathbb{F} \implies \\ &T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) \end{aligned}$$

Observation 3. You can recover the full linear transformation if you know the image of a basis.
Theorem 2.1.1. Let $\mathbb{F}$ be a field and let $T : \mathbb{F}^n \mapsto \mathbb{F}^m$ be a linear transformation. Then there exists a unique matrix $A \in M_{mn}(\mathbb{F})$:

$$T(\mathbf{x}) = A\mathbf{x}$$

$\forall \mathbf{x} \in \mathbb{F}^n$. That is, $T = L_A$ is a *left-multiplication transformation.*
Theorem 2.1.2. Let V and W be vector spaces over the same field $\mathbb{F}$. Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis for V and $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ be **any set** of n vectors in W. Then there is a unique linear transformation $T : V \mapsto W$ with the property that

$$T(\mathbf{v}_i) = \mathbf{w}_i, \;\; i = 1, 2, \cdots, n.$$

Remark. $\mathbf{w}_i$ can be repeated vectors.

The range and kernel of a linear transformation

Kernel. $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$.
Range. $\mathcal{R}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$
Remarks
(1) Since $T(\mathbf{0}) = \mathbf{0}, \mathbf{0} \in \ker(T)$.
(3) T is *surjective* iff $\mathcal{R}(T) = W$.
Lemma 2.2.2. A linear transformation $T : V \mapsto W$ is injective iff

$$\ker(T) = \{\mathbf{0}\}$$

Definition Let $T : V \mapsto W$ be a linear transformation.
Rank. $rank(T) = \dim \mathcal{R}(T)$
Nullity. $nullity(T) = \dim \ker(T)$
Theorem 2.2.3. (Dimension Theorem) Let $T : V \mapsto W$ be a linear transformation. Suppose that V is finite dimensional. Then

$$rank(T) + nullity(T) = \dim V$$

Proof of Dimension Theorem Let $T : V \mapsto W$ be a linear transformation. Let $\mathbf{v_1}, \cdots, \mathbf{v_r}$ be the basis for $\ker(T)$. We extend this basis to

$$\{\mathbf{v_1}, \cdots, \mathbf{v_r}, \mathbf{v_{r+1}}, \cdots, \mathbf{v_n}\}$$

There are two useful properties of this new basis:
(i) $\{T(\mathbf{v_{r+1}}) \cdots T(\mathbf{v_n})\}$ is a basis for $\mathcal{R}(T)$
(ii) $T(\mathbf{v}_i) = 0, 1 \leq i \leq r$
Theorem 2.2.4. Let V and W be a finite dimension vector spaces with $\dim V = \dim W$ and $T : V \mapsto W$ a linear transformation. Then the following statements are equivalent:
(i) T is injective (i.e. one-to-one)
(ii) T is surjective (i.e. onto)
(iii) T is bijective (i.e. one-to-one and onto)

The vector space of LT

Addition of LTs. If $T_1, T_2 : V \mapsto W$ are linear transformations, then we define the map $T_1 + T_2 : V \mapsto W$ by

$$(T_1 + T_2)(\mathbf{v}) = T_1(\mathbf{v}) + T_2(\mathbf{v}), \quad \forall \mathbf{v} \in V$$

Scalar multiplication of LTs. If $T : V \mapsto W$ is a linear transformation and $a \in \mathbb{F}$, then we define the map $\alpha T : V \mapsto W$ by

$$(\alpha T)(\mathbf{v}) = \alpha T(\mathbf{v}), \quad \forall \mathbf{v} \in V.$$

The maps $T_1 + T_2$ and αT are LTs.
Theorem 2.3.3. Let V and W be vector spaces over the same field $\mathbb{F}$, and let

$$\mathcal{L}(V, W) = \{\textbf{all linear transformations } T : V \mapsto W\}$$

Then $\mathcal{L}(V, W)$ forms a vector space over addition and scalar multiplications defined in equations above. In addition, if V and W are finite dimensional, then

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

Isomorphism

Invertible Linear Transformation.
$ST = I_V \quad and \quad TS = I_W$. Denote $S := T^{-1}$.
Lemma 2.5.1. If $T : V \mapsto W$ is an invertible linear transformation, then T^{-1} is also a linear transformation.
Definition. An invertible linear transformation $T : V \mapsto W$ is called an *isomorphism* of V onto W. So for a linear transformation $T : V \mapsto W$,

$$T \text{ isomorphism} \iff \text{invertible} \iff \text{bijective}$$

T5Q6. An isomorphism sends a basis for U to a basis for V, and finite dimensional vector spaces which are isomorphic to each other have the same dimension.
Definition. We say that the vector space V is isomorphic to the vector space W if there is an isomorphism $T : V \mapsto W$. In this case, we write $V \cong W$.
Remarks. Roughly speaking, isomorphic vector spaces may look different but they have the same structure.
Transitive Property of isomorphism. Let V, W, and U be vector spaces over the same field. If $V \cong W$ and $W \cong U$, then $V \cong U$.
Theorem 2.5.2. If V and W are finite dimensional vector spaces over a field $\mathbb{F}$, then V is isomorphic to W iff $\dim V = \dim W$.

Coordinates

Definition. The $n \times 1$ matrix $[\mathbf{v}]_{\mathcal{B}}$ is called the coordinate matrix of $\mathbf{v}$ relative to the ordered basis $\mathcal{B}$.
Theorem 2.7.1. Let V be an $n-$dimensional vector space over a field $\mathbb{F}$ and let $\mathcal{B}$ be an ordered basis for V. Then the map $T : V \mapsto \mathbb{F}^n$ defined by

$$T(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}, \quad (\mathbf{v} \in V)$$

is an isomorphism.

Representation of LT by matrices

Definition. The matrix of T relative to the ordered bases $\mathcal{B}$ and $\mathcal{B}'$ and is denoted by

$$[T]_{\mathcal{B}',\mathcal{B}} = ([T(\mathbf{v}_1)]_{\mathcal{B}'} \mid \cdots \mid [T(\mathbf{v}_n)]_{\mathcal{B}'})$$

Note that T is a LT, $[T]_{\mathcal{B}',\mathcal{B}}$ is a matrix. So, equation can be written as

$$[T(\mathbf{v})]_{\mathcal{B}'} = [T]_{\mathcal{B}',\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$$

Note that $T(\mathbf{v})$ is a vector in W, $[T(\mathbf{v})]_{\mathcal{B}'}$ is a $m \times 1$ matrix, $[T]_{\mathcal{B}',\mathcal{B}}$ is a $m \times n$ matrix.
Properties of Coordinates
(i) $[\mathbf{v}]_{\mathcal{B}_s} = \mathbf{v}$.
(ii) Let $\mathcal{B}_s$ denote the standard basis for $\mathbb{F}^n$. Consider $L_M : \mathbb{F}^n \mapsto \mathbb{F}^n, L_M(\mathbf{v}) = M\mathbf{v}$. Then $[L_M]_{\mathcal{B}_s} = M$
(iii) $[cT]_{\mathcal{B}} = c[T]_{\mathcal{B}}$
(iv) $[T^n]_{\mathcal{B}} = ([T]_{\mathcal{B}})^n$

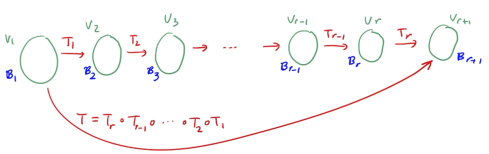
Matrices for composition of LTs

Theorem 2.9.1. Let $\mathbb{F}$ be a field. For $1 \leq i \leq r+1$, let V_i be a finite dimensional vector space over $\mathbb{F}$ and let B_i be an ordered basis for V_i. For $1 \leq j \leq r$, let $T_j : V_j \mapsto V_{j+1}$ be a linear transformation. Consider the composition

$$T = T_r \circ \cdots \circ T_1 : V_1 \mapsto V_{r+1}$$

Then

$$[T]_{\mathcal{B}_{r+1},\mathcal{B}_1} = [T_r]_{\mathcal{B}_{r+1},\mathcal{B}_r} \cdots [T_1]_{\mathcal{B}_2,\mathcal{B}_1}$$



Matrices for invertible LTs

Theorem 2.10.1. Let $T : V \mapsto W$ be a linear transformation, $\mathcal{B}$ an ordered basis for V and $\mathcal{B}'$ be an ordered basis for W. Then T is an isomorphism iff the matrix $[T]_{\mathcal{B}',\mathcal{B}}$ is invertible. In this case,

$$[T^{-1}]_{\mathcal{B},\mathcal{B}'} = ([T]_{\mathcal{B}',\mathcal{B}})^{-1}$$

Corollary 2.10.2. (Inverse of matrix of identity operator) Consider the identity operator $I_V : V \mapsto V$ on V and let $\mathcal{B}$ and $\mathcal{B}'$ be two ordered bases for V. Then

$$[I_V]_{\mathcal{B},\mathcal{B}'} = ([I_V]_{\mathcal{B}',\mathcal{B}})^{-1}$$

Remarks. Forward direction of Theorem 2.10.1. If $\mathcal{B} = \mathcal{B}'$ and $\dim V = n$, then $[I_V]_{\mathcal{B}} = [I_V]_{\mathcal{B},\mathcal{B}} = I_n$ is the identity matrix.

Change of basis

Theorem 2.11.1. Let V be a finite dimensional vector space and let $\mathcal{B}$ and $\mathcal{B}'$ be two ordered bases of V. Then for any $\mathbf{v} \in V$,

$$[\mathbf{v}]_{\mathcal{B}} = [I_V]_{\mathcal{B},\mathcal{B}'}[\mathbf{v}]_{\mathcal{B}'}$$

Note that $I_V : V \mapsto V$ is the identity linear transformation.
Definition. The matrix $P = [I_V]_{\mathcal{B},\mathcal{B}'}$ is called the *transition* matrix from $\mathcal{B}'$ to $\mathcal{B}$.
Theorem 2.11.2. Let V be a finite dimensional vector space and $T : V \mapsto V$ a linear operator on V. If $\mathcal{B}$ and $\mathcal{B}'$ are two ordered bases of V and $P = [I_V]_{\mathcal{B},\mathcal{B}'}$ is the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$, then

$$[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P = [I_V]_{\mathcal{B}',\mathcal{B}}[T]_{\mathcal{B}}[I_V]_{\mathcal{B},\mathcal{B}'}$$

Definition (similar matrices) Let A and B be two matrices over the field $\mathbb{F}$. We say B is *similar* to A over $\mathbb{F}$ if there is an invertible $n \times n$ matrix P such that $B = P^{-1}AP$.
Definition (determinant of a linear operator). If $T : V \mapsto V$ is a linear operator, then the *determinant* of T is the determinant of *any* matrix which represents T. More precisely, if $\mathcal{B}$ is an ordered basis of V and $A = [T]_{\mathcal{B}}$ the matrix which represents T wrt $\mathcal{B}$, then

$$\det T = \det A$$

Chapter 2 Tutorial Theorems

T5Q3. Let U and V be vector spaces over a field $\mathbb{F}$ and let $T : U \mapsto V$ be a linear transformation. Suppose that $\{\mathbf{u_1}, \cdots, \mathbf{u_n}\}$ spans U.
(a) $\{T(\mathbf{u_1}), \cdots, T(\mathbf{u_n})\}$ spans $\mathcal{R}(T)$.
(c) If T is injective and $\{\mathbf{u_1}, \cdots, \mathbf{u_n}\}$ is a basis for U, then $\{T(\mathbf{u_1}), \cdots, T(\mathbf{u_n})\}$ is a basis for $\mathcal{R}(T)$.
(e) If T is an isomorphism, then $\{T(\mathbf{u_1}, \cdots \mathbf{u_n})\}$ is a basis for V and $\dim U = \dim V$. So an isomorphism sends a basis for U to a basis for V, and finite dimensional vector spaces which are isomorphic to each other have the same dimension.
T5Q6. Let $\mathbb{F}$ be a field. For each $A \in M_{mn}(\mathbb{F})$, let $T_A : \mathbb{F}^n \mapsto \mathbb{F}^m$ be defined by

$$T_A(\mathbf{u}) = A\mathbf{u}, \quad \mathbf{u} \in \mathbb{F}^n$$

(i) $T : \mathbb{F}^n \mapsto \mathbb{F}^M$ is a linear transformation iff there exists $A \in M_{mn}(\mathbb{F})$ such that $T = T_A$.
(ii) If $A, B \in M_n(\mathbb{F})$ and $AB = I$, then $BA = I$. Recall that the inverse of A is the matrix B such that $AB = BA = I$. The above says that to prove that B is the inverse of A, we only need to check $AB = I$.
T6Q3 Let V and W be finite dimensional vector spaces and $T : V \mapsto W$ a linear transformation.
(a) If $\dim V < \dim W$, then T is not surjective.
(b) If $\dim V > \dim W$, then T is not injective.
T6Q5. Let $T : V \mapsto V$ be a linear operator such that $T^2 = T$. Such an operator is called a **projection** of V. The direct sum decomposition of $V = \mathcal{R}(T) \oplus \ker(T)$ is equivalent to projections of V.
Theorem. The composition of injective/surjective/bijective functions is injective/surjective/bijective.

3: Diagonalization and Jordan Canonical Forms

Problem: Find an ordered basis $\mathcal{B}$ of V such that the matrix $[T]_{\mathcal{B}}$ of T wrt $\mathcal{B}$ is in an especially simple form.

Diagonalizable linear operators.

Theorem 3.1.1. A linear operator $T : V \mapsto V$ is diagonalizable iff V has a basis $\mathcal{B}$ such that each of its element is an eigenvector of T. In this case, $[T]_{\mathcal{B}}$ is diagonal, and the diagonal entries of $[T]_{\mathcal{B}}$ are the eigenvalues of T.
Definition
(i) The polynomial $c_T(x)$ defined by

$$c_T(x) = \det(xI - T) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n$$

is called the *characteristic polynomial* of T.
(ii) The equation

$$c_T(x) = 0$$

is called the *characteristic equation* of T.
(iii) If λ is an eigenvalue of the linear operator $T : V \mapsto V$, then the subspace of V

$$E_\lambda(T) = \ker(T - \lambda I_V) = \{\mathbf{v} \in V : T(\mathbf{v}) = \lambda\mathbf{v}\}$$

is called the *eigenspace* of T corresponding to the eigenvalue λ.
Fact (T8Q2(b).) If $c_A(x) = a_0 + a_1x + \cdots a_nx^n$, then $\det A = (-1)^n a_0 = (-1)^n c_A(0)$.
Definition If the leading coefficient of a polynomial $p(x)$ is 1, that is $p(x)$ is of the form

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

then $p(x)$ is called a *monic* polynomial.
Theorem 3.1.2. If $A \in M_n(\mathbb{F})$, then the characteristic polynomial $c_A(x)$ of A is a monic polynomial of degree n.

Conditions for diagonalizability

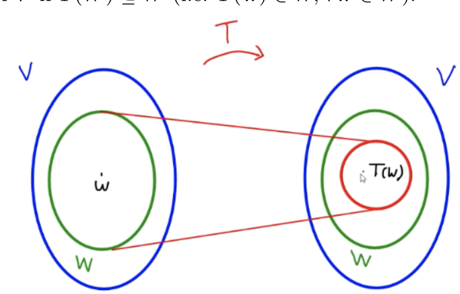
Theorem 3.2.3. Let $T : V \mapsto V$ be a linear operator and let $\lambda_1, \cdots, \lambda_r$ be the distinct eigenvalues of T. For each $1 \leq i \leq r$, let E_{λ_i} be the eigenspace of T corresponding to λ_i. The following are equivalent:
(i) T is diagonalizable
(ii) The characteristic polynomial for T is

$$c_T(x) = (x - \lambda_1)^{d_1} \cdots (x - \lambda_r)^{d_r}$$

where $\dim E_{\lambda_i}(T) = d_i$ for $i = 1, 2, \cdots, r$.
(iii) $\dim E_{\lambda_1}(T) + \cdots + \dim E_{\lambda_r}(T) = \dim V$.
(iv) $V = E_{\lambda_1}(T) \oplus \cdots \oplus E_{\lambda_r}(T)$

Invariant subspaces

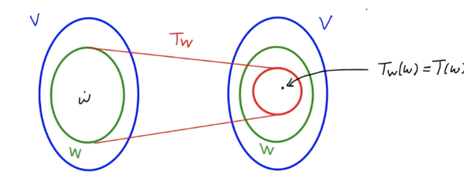
Definition Let $T : V \mapsto V$ be a linear operator. A subspace W of V is called a **T-invariant subspace** of V if $T(W) \subseteq W$ (i.e. $T(\mathbf{w}) \in W, \forall \mathbf{w} \in W$).



Definition Let $T : V \mapsto V$ be a linear operator and let $W \subseteq V$ be a $T-$invariant subspace of V. Define $T_W : W \mapsto W$ by

$$T_W(\mathbf{w}) = T(\mathbf{w}) \quad \forall \mathbf{w} \in W$$

Then T_W is a linear operator on W, called the *restriction of T to W*.



Definition Let $T : V \mapsto V$ be a linear operator and let $\mathbf{u}$ be a nonzero vector in V. The subspace

$$W = Span\{\mathbf{u}, T(\mathbf{u}), T^2(\mathbf{u}) \cdots\}$$

is called the *T-cyclic subspace of V generated by* $\mathbf{u}$. It is a T-invariant subspace of V.

Cayley-Hamilton Theorem

Remark If $\mathcal{B}$ is an ordered basis for V and $[T]_{\mathcal{B}} = A$, then

$$[f(T)]_{\mathcal{B}} = f(A)$$

where $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial over $\mathbb{F}$.
Cayley Hamilton Theorem. Let T be a linear operator on a finite-dimensional vector space V, and let $c_T(x)$ be the characteristic polynomial of T. Then

$$c_T(T) = T_0$$

where T_0 is the zero transformation.

Minimal Polynomial

Definition Let V be a finite dimensional vector space over a field $\mathbb{F}$, and let $T : V \mapsto V$ be a linear operator. A polynomial $m_T(x) \in P(\mathbb{F})$ is called a *minimal polynomial* of T if it has the following properties:
(i) $m_T(x)$ is *monic.*
(ii) $m_T(T) = T_0$.
(iii) If $p(x)$ is a nonzero polynomial and $p(T) = T_0$, then

$$\deg(m_T(x)) \leq \deg(p(x))$$

that is, $m_T(x)$ has the smallest degree among all the nonzero polynomials $p(x)$ with the property $p(T) = T_0$.
Theorem 3.5.1. Let V be a finite dimensional vector space, $T : V \mapsto V$ a linear operator and $m_T(x)$ a minimal polynomial of T.
(i) If $p(x)$ is a polynomial such that $p(T) = T_0$, then $m_T(x)$ divides $p(x)$, that is,

$$p(x) = q(x)m_T(x)$$

for some polynomial $q(x) \in P(\mathbb{F})$. In particular, $m_T(x)$ divides $c_T(x)$.
(ii) The minimal polynomial of T is unique.
Theorem 3.5.2 Let V be a finite dimensional vector space and let $T : V \mapsto V$ be a linear operator. Then the minimal polynomial $m_T(x)$ of T and the characteristic polynomial $c_T(x)$ of T have the same roots, that is, for any scalar λ,

$$m_T(\lambda) = 0 \iff c_T(\lambda) = 0$$

Hence, λ is an eigenvalue of T iff $m_T(\lambda) = 0$.

Theorem 3.5.3. Let $T : V \mapsto V$ be a linear operator and let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues of T. Then T is diagonalizable iff the minimal polynomial of T is the form

$$m_T(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_k)$$

Generalized Eigenspace

Throughout this section, assume that (i) $\mathbb{F} = \mathbb{C}$, (ii) V is a finite dimensional **complex** vector space and (iii) $T : V \mapsto V$ is a linear operator.

Fundamental Theorem of Algebra. The characteristic polynomial of T can be factorized as

$$c_T(x) = (x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \cdots (x - \lambda_k)^{m_k}$$

For each $1 \leq i \leq k$, we call the positive integer m_i the **multiplicity** of the eigenvalue λ_i.

Definition Let $\lambda \in \mathbb{C}$. A nonzero vector $\mathbf{v} \in V$ is called a **generalized eigenvector** of T corresponding to λ if there exists a positive integer m such that

$$(T - \lambda I_V)^m(\mathbf{v}) = \mathbf{0}$$

Every eigenvector of T is a generalized eigenvector of T . However, a generalized eigenvector of T may not be an eigenvector of T .

Theorem 3.6.1 If $\mathbf{v}$ is a generalized eigenvector of T corresponding to λ, then λ is an eigenvalue of T.

Definition Let λ be an eigenvalue of $T : V \mapsto V$. THe subset $K_\lambda(T)$ of V defined by

$$K_\lambda(T) = \{\mathbf{v} \in V : (T - \lambda I_V)^m(\mathbf{v}) = \mathbf{0} \\ \text{for some positive integer } m\ \}$$

is called **the generalized eigenspace of** T **corresponding to** λ.

Theorem 3.6.2. If λ be an eigenvalue of T, then the generalized eigenspace $K_\lambda(T)$ is a $T-$invariant subspace of V.

Theorem 3.6.3. Let $T : V \mapsto V$ be a linear operator and let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues of T. Then

$$V = K_{\lambda_1}(T) \oplus K_{\lambda_2}(T) \cdots \oplus K_{\lambda_k}(T)$$

Theorem 3.6.4. Let λ be an eigenvalue of T with multiplicity m.

(i) $E_\lambda(T) \subseteq K_\lambda(T)$.

(ii) If $\mu \in \mathbb{C}$ and $\mu \neq \lambda$, then $E_\mu(T) \cap K_\lambda(T) = \{\mathbf{0}\}$.

(iii) If $\mu \in \mathbb{C}$ and $\mu \neq \lambda$, then the restriction of $T - \mu I_V$ to $K_\lambda(T)$ is injective and

$$(T - \mu I_V)(K_\lambda(T)) = K_\lambda(T)$$

(iv) $\dim K_\lambda(T) \leq m$. Remark: Proved later that $\dim K_\lambda(T) = m$.

(v) $K_\lambda(T) = Ker(T - \lambda I_V)^m$ (m is fixed)

Jordan Canonical Form

Throughout this section, assume that (i) $\mathbb{F} = \mathbb{C}$, (ii) V is a finite dimensional **complex** vector space and (iii) $T : V \mapsto V$ is a linear operator.

Definition Let $\mathbf{v} \in V$ be a generalized eigenvector of T corresponding to the eigenvalue λ, and let p be the smallest positive integer such that $(T - \lambda I_V)^p(\mathbf{v}) = \mathbf{0}$. Then the ordered set of vectors

$$C = \left\{(T - \lambda I_V)^{p-1}(\mathbf{v}), \ldots, (T - \lambda I_V)(\mathbf{v}), \mathbf{v}\right\}$$

is called **a cycle of generalized eigenvectors of** T **corresponding to** λ.

(i) The vector $(T - \lambda I_V)^{p-1}(\mathbf{v})$ is called the **initial vector** of the cycle. It is the only eigenvector in the cycle.

(ii) The vector $\mathbf{v}$ is called the **end vector** of the cycle.

(iii) The number of vectors p in the cycle is called the **length of the cycle.**

Observation. If $C = \left\{(T - \lambda I_V)^{p-1}(\mathbf{v}), (T - \lambda I_V)^{p-2}(\mathbf{v}), \ldots, (T - \lambda I_V)(\mathbf{v}), \mathbf{v}\right\}$ is a cycle of generalized eigenvectors of T corresponding to λ and $\mathbf{u} \in C$, then for any positive integer m,
$(T - \lambda I_V)^m(\mathbf{u}) = \mathbf{0}$ or $\in C$.

Theorem 3.7.3. Let $T : V \mapsto V$ be a linear operator and let λ be an eigenvalue of T. Then $K_\lambda(T)$ has basis consisting of a union of disjoint cycles of generalized eigenvectors of T corresponding to λ.

Theorem 3.7.4. Let $T : V \mapsto V$ be a linear operator. Then V has a basis $\mathcal{B}$ such that $\mathcal{B}$ is a disjoint union of cycles of generalized eigenvectors of T.

Jordan Block The matrix $J_p(\lambda)$ is called a **Jordan block** of order p corresponding to λ.

$$J_1(\lambda) = (\lambda), \quad J_2(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Theorem 3.7.5. Let $T : V \mapsto V$ be a linear operator. Then V has a Jordan canonical basis $\mathcal{B}$, that is, the matrix $[T]_\mathcal{B}$ of T relative to $\mathcal{B}$ is of the form

$$[T]_\mathcal{B} = \begin{pmatrix} J_{p_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{p_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & J_{p_r}(\lambda_r) \end{pmatrix}$$

Moreover, the Jordan blocks $J_{p1}(\lambda_1), \cdots, J_{pr}(\lambda_r)$ are unique up to re-ordering.

Remark

1. The theorem says that the Jordan canonical form of a linear operator T is essentially unique. If we have found one JCF of T , then all other JCFs of T are obtained simply by rearranging the Jordan blocks. They are the matrices of T relative to bases obtained by reordering the elements of B.

2. Since the JCF of T is unique up to the reordering of its Jordan blocks, we may call any of the JFCs of T the Jordan canonical form of T .

3. We have proved that a Jordan canonical basis exists. For a proof for the uniqueness of the Jordan blocks, see Section 7.2 of [FIS].

Remark A linear operator T is diagonlizable iff all the jordan blocks in its Jordan canonical form are of order 1, that is,

$$\mathcal{B} = \begin{pmatrix} J_1(\lambda_1) & 0 & \cdots & 0 & 0 \\ 0 & J_1(\lambda_2) & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & J_1(\lambda_{r-1}) & 0 \\ 0 & 0 & \cdots & 0 & J_1(\lambda_r) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_r \end{pmatrix}$$

Definition. Let $A \in M_n(\mathbb{C})$. Then the Jordan canonical form of A is defined to be the Jordan canonical form of the linear operator $L_A : \mathbb{C}^n \mapsto \mathbb{C}^n$.

Characteristic polynomial. The characteristic polynomial of T is given by $c_T(x) = c_J(x) = (x - \lambda_1)^{k_1}(x - \lambda_2)^{k_2} \cdots (x - \lambda_r)^{k_r}$.

The minimal polynomial of a matrix in Jordan canonical form.

Theorem 3.7.6. Let $T : V \mapsto V$ be a linear operator and let J be a Jordan canonical form of T. Suppose that $\lambda_1, \cdots, \lambda_k$ are the distinct eigenvalues of T, and for each $1 \leq i \leq k$, p_i **is the order of the largest Jordan block corresponding to** λ_i **in** J. Then the minimal polynomial of T is given by

$$m_T(x) = (x - \lambda_1)^{p_1}(x - \lambda_2)^{p_2} \cdots (x - \lambda_k)^{p_k}$$

Fact. If the eigenspace $E_\lambda(A)$ of A corresponding to the eigenvalue λ has dimension k, then the number of Jordan blocks corresponding to λ in the JCF of A is also k.

4: Inner Product Spaces

Inner Product

Conditions.
(IP1) $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}, \forall \mathbf{u}, \mathbf{v} \in V$
(IP2+IP3) (linear in the first variable)
$\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$
(IP4) $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ for all nonzero $\mathbf{v} \in V$, and $\langle \mathbf{0}, \mathbf{0} \rangle = 0$.

Remarks
1. (conjugate linear in the second variable)
$\langle \mathbf{w}, \alpha\mathbf{u} + \beta\mathbf{v} \rangle = \bar{\alpha}\langle \mathbf{w}, \mathbf{u} \rangle + \bar{\beta}\langle \mathbf{w}, \mathbf{v} \rangle$
2. $\langle v, x \rangle = \langle v, y \rangle, \forall v \in V \implies x = y$.

Definition Let $A \in M_n(\mathbb{C})$. The *conjugate transpose* of A is the matrix $A^* = \bar{A^T}$.

Properties of Conjugates

(i) $|z| = \sqrt{a^2 + b^2} = |\bar{z}|$
(ii) $\bar{z}z = |z|^2$
(iii) $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$
(iv) $\overline{z_1 \times z_2} = \bar{z_1} \times \bar{z_2}$

Norm and Distance

(a) (*norm*) $||\mathbf{u}|| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$
(b) (*distance*) $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$.

Cauchy-Schwarz Inequality. If $\mathbf{u}, \mathbf{v}$ are two vectors in an inner product space V, then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}||||\mathbf{v}||$$

Equality holds iff $\mathbf{u} = c\mathbf{v}$ for some scalar c.

Triangle Inequality. If $\mathbf{u}, \mathbf{v}$ are two vectors in an inner product space V, then

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

Orthogonal Sets

The Pythagoras Theorem If $\mathbf{u}, \mathbf{v}$ are orthogonal vectors in a inner product space V, then

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

Theorem 4.4.1 An orthogonal set of nonzero vectors is linearly independent.

Lemma 4.5.1. Let $\mathcal{B} = \{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ be an orthonormal basis of V. Then for any $\mathbf{v} \in V$,

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{v}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{v}, \mathbf{v}_n \rangle \mathbf{v}_n$$

Corollary 4.5.2. (Matrix of a linear operator relative to an orthonormal basis) Let $\mathcal{B} = \{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ be an orthonormal basis of V, $T : V \mapsto V$ a linear operator and $[T]_\mathcal{B} = (a_{ij})$. Then for all i, j,

$$a_{ij} = \langle T(\mathbf{v}_j), \mathbf{v}_i \rangle$$

Gram-Schmidt Orthogonalization Process.
Step 1: Choose $\mathbf{v_1} = \frac{1}{||\mathbf{u_1}||}\mathbf{u_1}$
Step 2: Let $\mathbf{v}_2' = \mathbf{u}_2 - \langle \mathbf{u}_2, \mathbf{v}_1 \rangle \mathbf{v}_1$. Then normalise $\mathbf{v}'$.
Step 3: Let $\mathbf{v}_{k+1}' = \mathbf{u}_{k+1} - \sum_{j=1}^{n} \langle \mathbf{u}_{k+1}, \mathbf{v}_j \rangle \mathbf{v}_j$.
Then normalise $\mathbf{v}_{k+1}$
Step 4: Repeat until we obtain an orthonormal basis $\{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ of V.

Theorem 4.5.3. Every finite dimensional inner product space has an orthonormal basis

Orthogonal Complement

Definition Let V be an inner product space and W a subspace of V. The *orthogonal complement* of W is the set

$$W^\perp = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall \mathbf{w} \in W\}.$$

Lemma 4.6.1. If W is a subspace of an inner product space V, then its orthogonal complement $W^\perp$ is a subspace of V. In addition, we have

$$W \cap W^\perp = \{\mathbf{0}\}$$

Theorem 4.6.2. If W is a finite dimensional subspace of an inner product space V, then

$$V = W \oplus W^\perp$$

Remark. This theorem is false without the assumption that W is finite dimensional.

Orthogonal Projections

Definition The linear operator $proj_W$ is called the orthogonal projection of V on W.

$$proj_W(\mathbf{v}) = \mathbf{w} = \mathbf{v} - \mathbf{w}'$$

Remarks
(b) If $\{\mathbf{w_1}, \cdots, \mathbf{w_k}\}$ is an orthonormal basis of W, then

$$proj_W(\mathbf{v}) = \sum_{j=1}^{k} \langle \mathbf{v}, \mathbf{w_j} \rangle \mathbf{w_j}$$

(c) If V is finite dimensional, then so is $W^\perp$. In this case,

$$proj_{W^\perp}(\mathbf{v}) = \mathbf{w}' = \mathbf{v} - \mathbf{w} = (I_V - proj_W)(\mathbf{v}), \forall \mathbf{v} \in V$$

Theorem 4.8.1 (Best Approximation) If W is a finite dimensional subspace of an inner product space V and $\mathbf{v} \in V$, then

$$||proj_{W^\perp}(\mathbf{v})|| = ||\mathbf{v} - proj_W(\mathbf{v})|| < ||\mathbf{v} - \mathbf{w}||$$

for every vector $\mathbf{w} \in W$ different from $proj_W(\mathbf{v})$.
Theorem 4.9.1 (Least Square Solution) For any real linear system $A\mathbf{x} = \mathbf{b}$, the associated normal system

$$(A^t A)\mathbf{x} = A^t \mathbf{b}$$

is consistent, and all its solutions are least square solutions of $A\mathbf{x} = \mathbf{b}$.

The Adjoint of a Linear Operator

Theorem 4.10.2. (adjoint of a linear operator) Let $T : V \mapsto V$ be a linear operator on a finite dimensional inner product space V. Then there exists a unique linear operator $T^* : V \mapsto V$ such that

$$\langle T(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{u}, T^*(\mathbf{v}) \rangle, \forall \mathbf{u}, \mathbf{v} \in V$$

The operator T^* is called the adjoint of T. Generally, adjoint takes the form $(Ax, y) = (x, By)$.
Theorem 4.10.3. (Matrix of the adjoint is the conjugate transpose of the matrix) Let $T : V \mapsto V$ be a linear operator on a finite dimensional inner product space V and $\mathcal{B}$ be an orthonormal basis of V. Then

$$[T^*]_\mathcal{B} = ([T]_\mathcal{B})^*$$

Lemma 4.10.4. Let T, T_1, T_2 be linear operators on a finite dimensional inner product space V. Then:
(i) $(T_1 + T_2)^* = T_1^* + T_2^*$.
(ii) $(cT)^* = \bar{c}T^*$.
(iii) $(T_1T_2)^* = T_2^*T_1^*$
(iv) $(T^*)^* = T$.

Normal and Self-adjoint operators

A linear operator $T : V \mapsto V$ is **orthogonally diagonalizable** if V has an 1) orthonormal basis that 2) consists of eigenvectors of T.
Definition (self-adjoint) A linear operator $T : V \mapsto V$ on a finite dimensional inner product space V is called *self-adjoint* if

$$T = T^*$$

Lemma 4.11.1 All eigenvalues of a self-adjoint operator on a finite dimensional *complex* vector space are real.
Corollary 4.11.2 A self-adjoint operator on a finite dimensional real inner product space has at least one eigenvalue.
Theorem 4.11.3 A linear operator on a finite dimensional *real* inner product space is orthogonally diagonalizable iff it is self-adjoint.
Definition (normal) A linear operator T on a finite dimensional inner product space is called *normal* if

$$TT^* = T^*T$$

Remark. All self-adjoint operators are normal. Converse is not true.
Theorem 4.11.4 Let $T : V \mapsto V$ be a linear operator on a finite dimensional complex inner product space. T is a normal operator iff T is orthogonally diagonalizable
T11Q6 $T(\mathbf{v}) = \lambda\mathbf{v} \implies T^*(\mathbf{v}) = \bar{\lambda}\mathbf{v}, \forall \mathbf{v} \in V$ (T11Q6)

Unitary Operators

Definition (Unitary Operator). A linear operator $T : V \mapsto V$ on a finite dimensional inner product space V is called *unitary* if

$$TT^* = T^*T = I_V$$

that is, T is invertible and $T^{-1} = T^*$.
Lemma 4.12.1. Let $T : V \mapsto V$ be a linear operator on a finite dimensional inner product space V. Then the following are equivalent:
(i) T is unitary;
(ii) (Unitary operator preserves inner product) $\langle T(\mathbf{u}), T(\mathbf{v}) \rangle = \langle \mathbf{u}, \mathbf{v} \rangle, \forall \mathbf{u}, \mathbf{v} \in V$
(iii) T sends an orthonormal basis of V to an orthonormal basis
(iv) (Linear isometry) $||T(\mathbf{v})|| = ||\mathbf{v}||, \forall v \in V$
Remarks.
(i) A unitary operator on a complex inner product space is orthogonally diagonalizable because it is normal.
(ii) A unitary operator on a real inner product space need not be orthogonally diagonalizable (as a normal operator need not be self-adjoint.)
Definition. (orthogonal and unitary matrix)
(Orthogonal) $AA^t = A^tA = I_n$
(Unitary) $AA^* = A^*A = I_n$
Lemma 4.12.2. Let $T : V \mapsto V$ be a unitary operator on a finite dimensional inner product space V over $\mathbb{F}$. Let $\mathcal{B}$ be an orthonormal basis of V and let

$$A = [T]_\mathcal{B}$$

(i) If $\mathbb{F} = \mathbb{R}$, then A is a real orthogonal matrix.
(ii) If $\mathbb{F} = \mathbb{C}$ then A is a unitary matrix.
Columns/Rows of an orthogonal/unitary matrix. A is an orthogonal/unitary matrix $\iff$ the rows/columns of A form an orthonormal basis of $\mathbb{R}^n/\mathbb{C}^n$.
Orthogonally/Unitarily diagonalizable matrices.
(Real) $A \in M_n(\mathbb{R})$ is orthogonally diagonalizable iff A is symmetric.
(Complex) $A \in M_n(\mathbb{C})$ is unitarily diagonalizable iff $\exists$ unitary matrix P st P^*AP is diagonal iff A is normal.

Appendix

Properties of Norms

(i) (Relation between sum of squared norm and squared norm of sum)
$||\sum_{i=1}^n x_i||^2 = \sum_{i=1}^n ||x_i||^2 + \sum_{i\neq j} \langle x_i, x_j \rangle$.

List of Questions

Chapter 1
State matrix multiplication properties
State direct sum of subspaces theorem
State equivalent statements of A is invertible
How to show that a subspace is subset of another subspace with only a set of vectors?
State condition for a subspace to be equal its superspace.
State relationship between summation of dimension of vector spaces and dimension of summation
State multilinear, alternating properties.
$D(X') =?$
Determinant of lower or upper triangular matrix?
$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} =?$
If $c_A(x) = a_0 + a_1x + \cdots + x^n, \det(A) =?$
$\det(AB) =?$
$\det(A^T) =?$
$\det(A^{-1}) =?$
$\det(cA) =?$
Chapter 2
State definition of linear transformation
State unique linear transformation theorem
State definition of kernel, range, rank, nullity.
State conditions of surjective (1) and injective (1) and bijective (1)
State dimension theorem and it's useful construction of basis.
State relation between $\mathcal{L}(V, W)$ and V, W.
State isomorphism definition and properties (2).
State implication of vector spaces which are isomorphic
State $[\mathbf{v}]_\mathcal{B}$ naming.
(i) $[\mathbf{v}]_{\mathcal{B}_s} =?$.
(ii) $[L_M]_{\mathcal{B}_s} =?$
(iii) $[cT]_\mathcal{B} =?$
(iv) $[T^n]_\mathcal{B} =?$
(v) $[ST]_\mathcal{B} =?$
State $[T]_{B',B}$ naming and decomposition and $[T(v)]_{B'}$ decomposition
State relation between isomorphism and ordered basis
State matrices for composition of LTs
How to check if a linear transformation is an isomorphism from its matrix representation?
How do I easily obtain the inverse of matrix of identity operator?
State change of basis for coordinate matrix and linear transformation.
State composition of injective and surjective functions.
State relation between projections and direct sum decomposition.
Let $T : V \mapsto W$.
(i) If $\dim V < \dim W$, what can you say about T?
(ii) If $\dim V > \dim W$, what can you say about T?
Chapter 3
State diagonalizable implication with basis
State characteristic equation, polynomial and eigenspace definition.
State conditions for diagonalizability
State definition of invariant subspaces and restriction.
How is the linear operator $f(T)$ related to the matrix $f(A)$?
State Cayley Hamilton theorem
State minimal polynomial definition
State theorems which relate minimal polynomial with characteristic polynomial (2)
State theorem which relates minimal polynomial and diagonalizable
State definition of generalized eigenvector and eigenspace
State fundamental theorem of algebra
State how to decompose V to generalised eigenspaces.
State general properties of generalised eigenspace
Define cycle of generalised eigenvectors of T and properties.
Explain how JCF is constructed.
State general fact relating $\dim E_\lambda(A)$ and JCF of A.
Chapter 4
State conditions for inner product
State properties of Conjugates
Define normal and distance
State Cauchy-Schwarz Inequality
State triangle inequality
State pythagoras theorem
State property of an orthogonal set of nonzero vectors
How to express a vector $\mathbf{v}$ as the linear combination of orthogonal basis vectors?
State matrix of a linear operator relative to an orthonormal basis.
State purpose of Gram-Schmidt Orthogonalization Process
State orthogonal complement definition
State relation between $W^\perp$ and W. (2)
State definition of projection and how to compute it.
State best approximation and least square solution definition and problem statement.
State adjoint of linear operator and property.
$[T^*]_\mathcal{B} =?$
State general fact of inner product.
State properties of adjoint operators
State definition of orthogonally diagonalizable
State self-adjoint definition and property (1).
State normal definitions definition and properties.
State relation between self-adjoint and normal.
State definition of unitary operator
State properties of unitary operators and conditions of the operator being orthogonally diagonalizable.
State definitions of orthogonal and unitary matrix.
State properties of columns/rows or orthogonal/unitary matrix
$||\sum_{i=1}^n x_i||^2 =?$.
$(T - S)T =?$
How to show that 2 vector spaces are equal to one another? (3)

Exercises

P29 eg, P30 eg, 1.13.2 proof, T1Q1, T1Q2(a, b), T1Q4 (optional), T1Q6(i, ii), T1Q7, T2H2, T2Q1, T2Q3(c, d, f, g), T2Q4(i (opt), ii, iii), T2Q5, T3Q1(b), T3Q3, T3Q4, T3Q7, T4H1(iii), T4H3(i), T4Q1, T4Q2(ii), T4Q3(c), T4Q5,

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