Data ^{X}

About Me:

Markov and State Space

Data X: A Course on Data, Signals, and Systems

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Data Applications and State Space

Input X → Data Application → Output Y

Typically inferred from
training from past X,Y

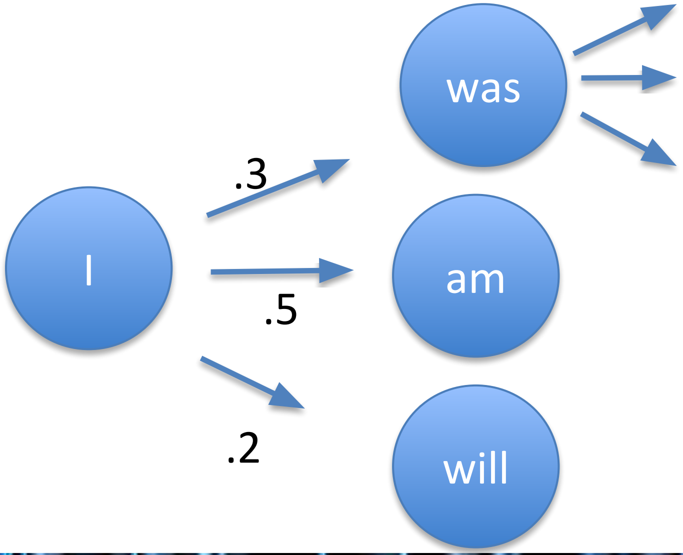
Input X → Data Application With State → Output Y

Depends also on the last
previous value of X

Next Word Predictor

A Word in Text: Input X → Data Application With State → Predict next word In Text: Y

For state X, there will be a different likelihood for the next word



Transition Probability

$$P = \begin{array}{cc} & \text{to} \\ \text{from} & \begin{pmatrix} .1 & .4. & .5 \\ .4 & .2 & .4 \\ .9 & .1 & 0 \end{pmatrix} \end{array}$$

More Formally, what is a "State"

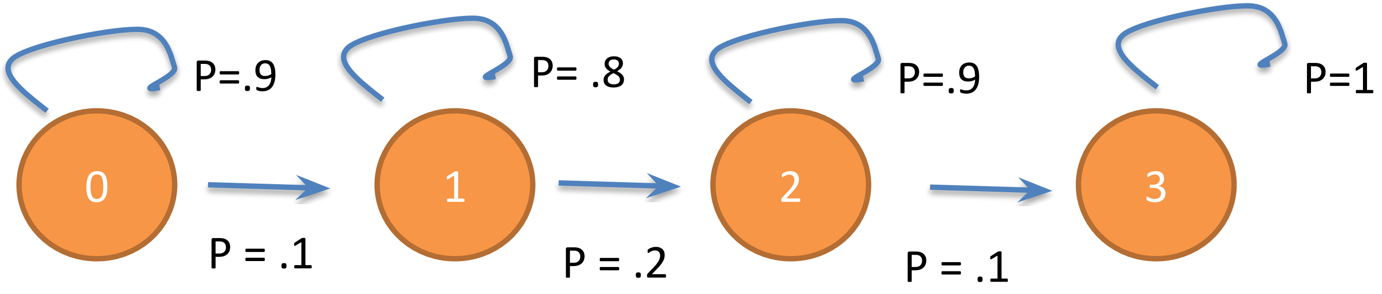
- State space is a set
- For example: $\boldsymbol{S} = \{ 0, 1, 2, 3 \}$

- Typically States have a meaning, for example:
- State 0 = Customer signed up for free (freemium) service
- State 1= Customer upgrades to paid service ($10/mo)
- State 2 = Customer upgrades to premium service ($20/mo)
- State 3 = Customer stops paying for service

What is a State

- State space is a set
- For example: $\boldsymbol{S} = \{ 0, 1, 2, 3 \}$

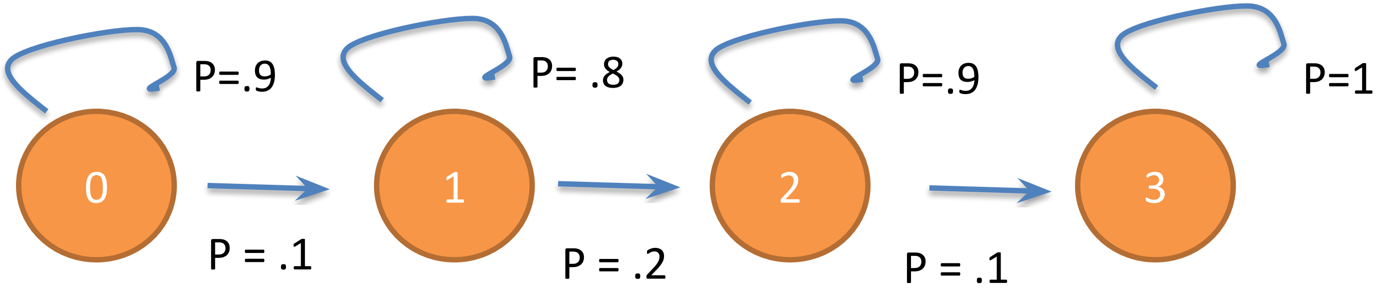
- State 0, 1, 3, 3 = Free, paid, premium, discontinues
- Time is changing: n = 1, 2, 3, .. In discrete steps
- There are a sequence of random variables that take on the State values



Markov basically means that the next State Depends only on the last one

- For example: ***S*** = { 0, 1, 2, 3 }
- n = number of months = 1, 2, 3 ..
- State 0, 1, 3, 3 = Free, paid, premium, discontinues

- X[n] is a random sequence, takes on x1, x2, x3…
- So a sequence of X might be 0, 0, 0, 1, 1, 1, 2, 3, 3, 3, 0



Key Concept: It's a Markov state only if the next state depend on the current state, also known as a memoryless property

Same thing but in proper notation
Discrete Time Markov Chain Definition

A stochastic process $\{ X_n \}$ is called a Markov chain if

$$\Pr\{X_{n+1} = j \mid X_0 = k_0, \ldots, X_{n-1} = k_{n-1}, X_n = i\}$$

$$= \Pr\{X_{n+1} = j \mid X_n = i\} \quad \leftarrow \text{transition probabilities}$$

for every $i, j, k_0, \ldots, k_{n-1}$ and for every n.

Discrete time means $n \in N = \{0, 1, 2, \ldots\}$.

The future behavior of the system depends only on the current state i and not on any of the previous states.

Same thing but in proper notation
Discrete Time Markov Chain Definition

A stochastic process $\{X_n\}$ is called a Markov chain if

$$\Pr\{X_{n+1} = j \mid X_0 = k_0, \ldots, X_{n-1} = k_{n-1}, X_n = i\}$$

Note: Academics love Markov because it has very nice properties.

$$= \Pr\{X_{n+1} = j \mid X_n = i\} \quad \leftarrow \text{transition probabilities}$$

It does not mean most things have Markov nature

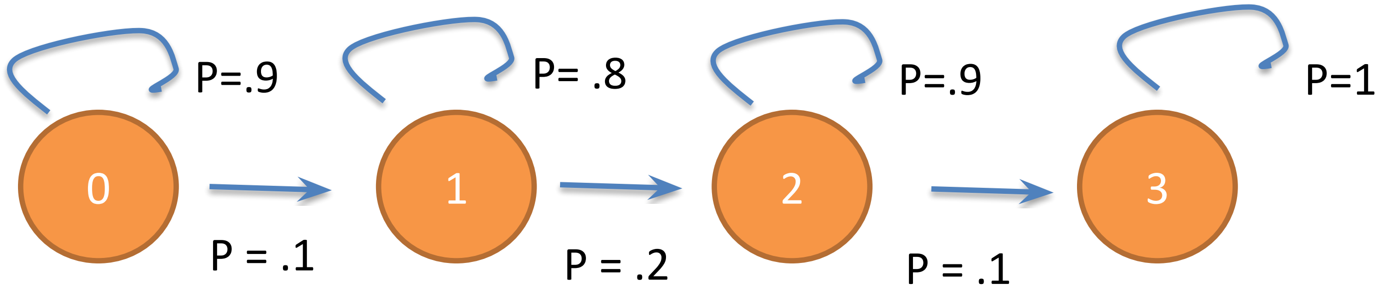
for every $i, j, k_0, \ldots, k_{n-1}$ and for every n.

With Data applications, some of those properties don't really apply – but some also do.

Discrete time means $n \in N = \{0, 1, 2, \ldots\}$.

The future behavior of the system depends only on the current state i and not on any of the previous states.

Continue Example



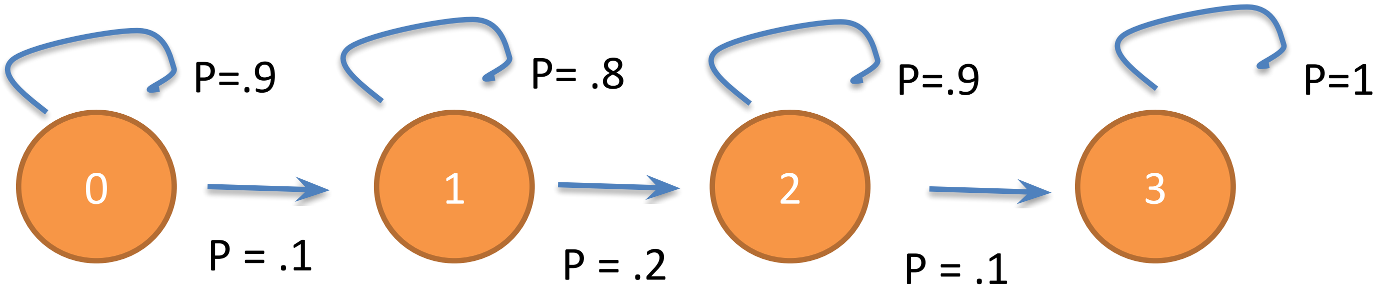
$$P\{X_{n+1} = 0 \mid X_n = 0\} = .9$$

$$P\{X_{n+1} = 1 \mid X_n = 0\} = .1$$

$$P\{X_{n+1} = 2 \mid X_n = 0\} = 0$$

$$P\{X_{n+1} = 0 \mid X_n = 0\} = 0$$

Continue to Example



$P\{X_{n+1} = 0 \mid X_n = 0\} = .9$

$P\{X_{n+1} = 1 \mid X_n = 0\} = .1$

$P\{X_{n+1} = 2 \mid X_n = 0\} = 0$

$P\{X_{n+1} = 0 \mid X_n = 0\} = 0$

Or in 1-step Transition matrix form

$$P = \begin{pmatrix} .9 & .1 & 0 & 0 \\ 0 & .8 & .2 & 0 \\ 0 & 0 & .9 & .1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$P_{i,j}$ = prob to go from state i to state j

Transition Probabilities

- For a customer starting at state 0, what is the probability that they will be in the other states after 2, 5, or 10 months?

Define Distribution of $P^{(i)}$ (Xi) at step i at = $[p1\ p2\ p3\ p4]^{(i)}$, ie p1 is prob of being in state 1

$$P(X1) \quad = \quad P(X0) \quad \text{X} \quad \begin{pmatrix} .9 & .1 & 0 & 0 \\ 0 & .8 & .2 & 0 \\ 0 & 0 & .9 & .1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transition Probabilities

- For a customer starting at state 0, what is the probability that they will be in the other states after 2, 5, or 10 months?

Define Distribution of P(Xi) at step I as [p1 p2 p3 p4], ie p1 is prob of being in state 1

$$P(X1) = P(X0) \times \begin{pmatrix} .9 & .1 & 0 & 0 \\ 0 & .8 & .2 & 0 \\ 0 & 0 & .9 & .1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P(X1) = [.9\ .1\ 0\ 0] = [1\ 0\ 0\ 0] \times \begin{pmatrix} .9 & .1 & 0 & 0 \\ 0 & .8 & .2 & 0 \\ 0 & 0 & .9 & .1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transition Probabilities

- For a customer starting at state 0, what is the probability that they will be in the other states after 2, 5, or 10 months?

Define Distribution of P(Xi) at step I as [p1 p2 p3 p4], ie p1 is prob of being in state 1

$$[p1\ p2\ p3\ p4]^{(1)} = [p1\ p2\ p3\ p4]^{(0)} \times \begin{pmatrix} .9 & .1 & 0 & 0 \\ 0 & .8 & .2 & 0 \\ 0 & 0 & .9 & .1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^{(1)}(X1) = [.9\ .1\ 0\ 0] = [1\ 0\ 0\ 0\ ]^{(0)} \times \begin{pmatrix} .9 & .1 & 0 & 0 \\ 0 & .8 & .2 & 0 \\ 0 & 0 & .9 & .1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

n-Step Transition Probabilities

This idea generalizes to an arbitrary number of steps.

For n = 3: $P^{(3)} = P^{(2)}\,P = P^2\,P = P^3$

or more generally, $P^{(n)} = P^{(m)}\,P^{(n\text{-}m)}$

The ij th entry of this reduces to

$$p_{ij}^{(n)} = \sum_{k=0}^{m} p_{ik}^{(m)}\, p_{kj}^{(n\text{-}m)} \quad 1 \le m \le n-1$$

Chapman - Kolmogorov Equations

Interpretation:

RHS is the probability of going from i to k in m steps & then going from k to j in the remaining $n - m$ steps, summed over all possible intermediate states k.

Brand Switching Example □

We approximate $q_i(0)$ by dividing total customers using brand i in week 27 by total sample size of 500:

$$\mathbf{q}(0) = (125/500, 230/500, 145/500) = (0.25, 0.46, 0.29)$$

To predict market shares for, say, week 29 (that is, 2 weeks into the future), we simply apply equation with $n = 2$:

$$\mathbf{q}(2) = \mathbf{q}(0)\mathbf{P}^{(2)}$$

$$\mathbf{q}(2) = (0.25, 0.46, 0.29)\begin{bmatrix} 0.90 & 0.07 & 0.03 \\ 0.02 & 0.82 & 0.16 \\ 0.20 & 0.12 & 0.68 \end{bmatrix}^2$$

$$= (0.327, 0.406, 0.267)$$

= expected market share from brands 1, 2, 3

Example from Internet, U of Texas

Steady-State Probabilities

Property 2: Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$ is the m-dimensional row vector of steady-state (unconditional) probabilities for the state space $\boldsymbol{S} = \{1,\ldots,m\}$. To find steady-state probabilities, solve linear system:

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P},\ \ \Sigma_{j=1,m}\, \pi_j = 1,\ \ \pi_j \geq 0,\ \ j = 1,\ldots,m$$

Brand switching example:

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3)\begin{bmatrix} 0.90 & 0.07 & 0.03 \\ 0.02 & 0.82 & 0.16 \\ 0.20 & 0.12 & 0.68 \end{bmatrix}$$

$$\pi_1 + \pi_2 + \pi_2 = 1,\ \ \pi_1 \geq 0,\ \ \pi_2 \geq 0,\ \ \pi_3 \geq 0$$

Yes, steady state probably is easy to calculate

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3)\begin{bmatrix} 0.90 & 0.07 & 0.03 \\ 0.02 & 0.82 & 0.16 \\ 0.20 & 0.12 & 0.68 \end{bmatrix}$$

$$\pi_j = \lim_{n \to \infty} p_{ij}^{(n)}$$

Small matter: Ergodic: Aperiodic and and allows the attainment of any future state

$$\pi_1 = 0.90\pi_1 + 0.02\pi_2 + 0.20\pi_3$$
$$\pi_2 = 0.07\pi_1 + 0.82\pi_2 + 0.12\pi_3$$
$$\pi_3 = 0.03\pi_1 + 0.16\pi_2 + 0.68\pi_3$$
$$\pi_1 + \pi_2 + \pi_3 = 1$$
$$\pi_1 \geq 0,\ \pi_2 \geq 0,\ \pi_3 \geq 0$$

Total of 4 equations, 3 unknowns

Discard 3rd equation and solve the remaining system to get :

$$\pi_1 = 0.474,\ \pi_2 = 0.321,\ \pi_3 = 0.205$$

Steady-State Probabilities

Again, this is one of the coolest things you can do with Markov Chains:
Ie calculate the steady state probability

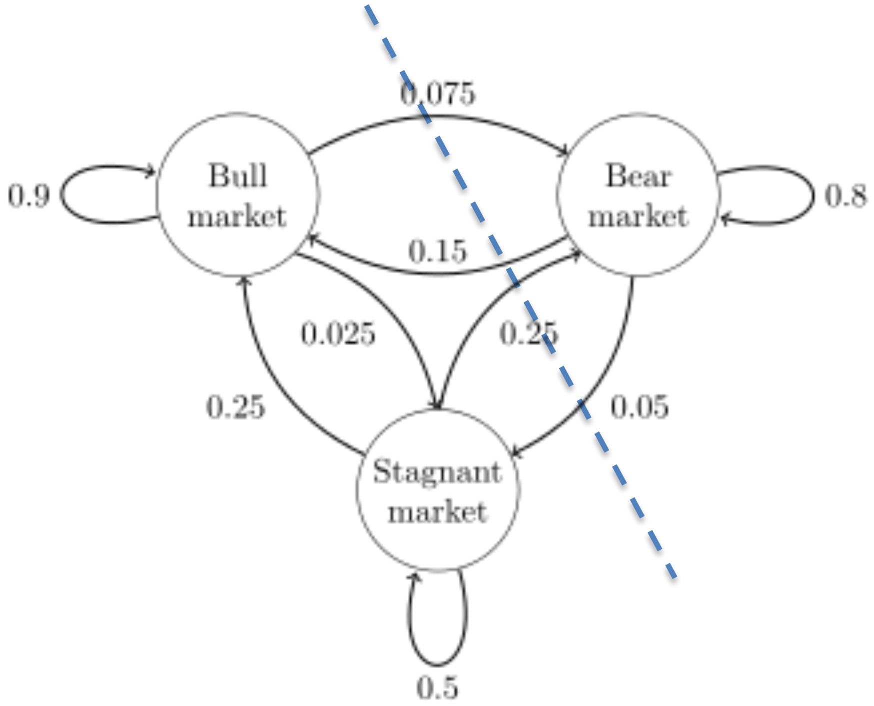
$$\pi = \pi P$$

Unfortunately, this is not as commonly used from a data (empirical) view
More often the transition matrix is collected in training
as in the Next Word predictor case

$$\begin{bmatrix} 0.20 & 0.12 & 0.68 \end{bmatrix}$$

$$\pi_1 + \pi_2 + \pi_2 = 1,\ \ \pi_1 \geq 0,\ \ \pi_2 \geq 0,\ \ \pi_3 \geq 0$$

Probability of Being in Particular State and Balancing Flow



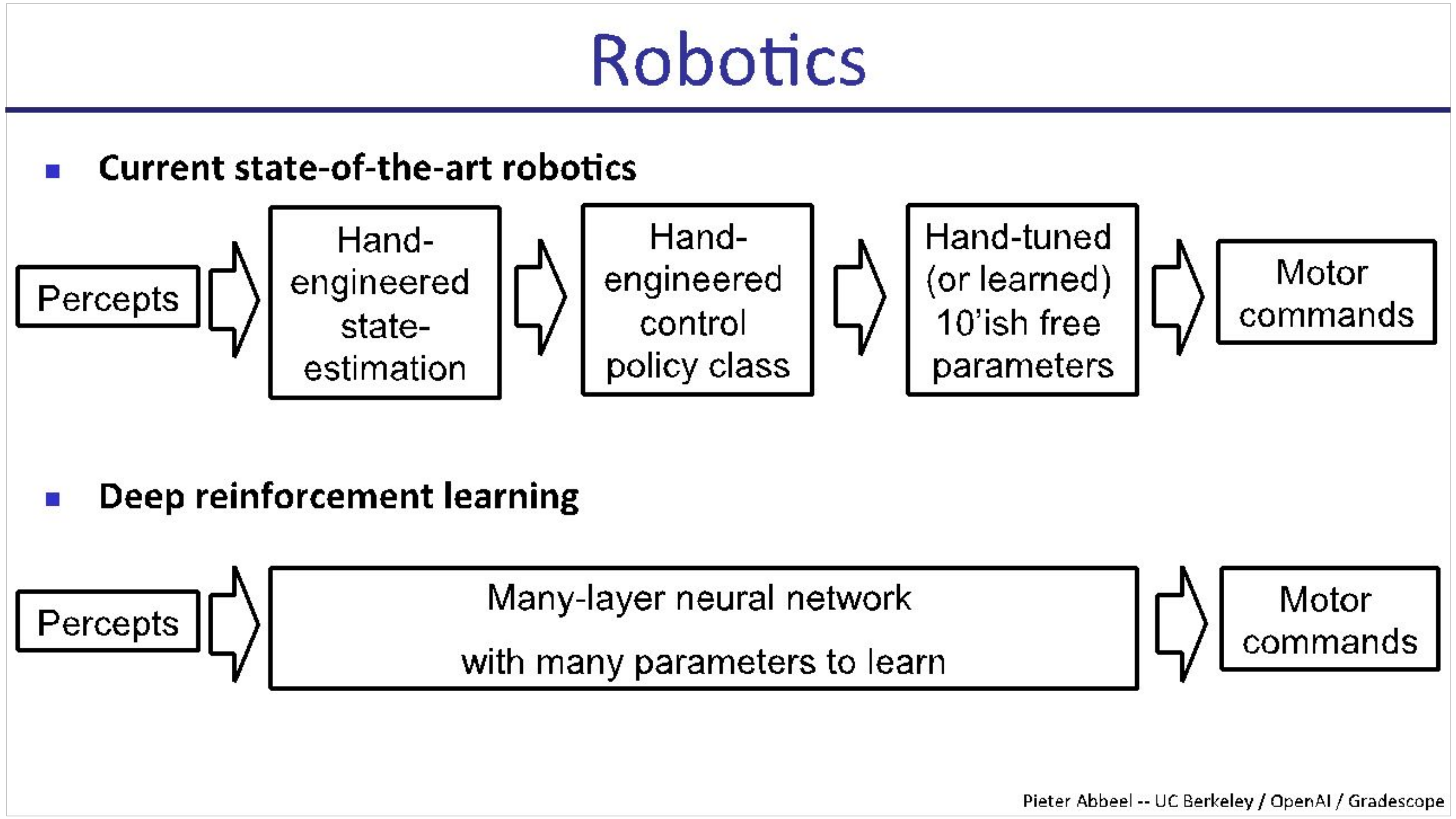
1. You can start with initial condition: p0

2. N step distribution = pn = p0 P^n

3. Steady State: $\pi = \pi P$

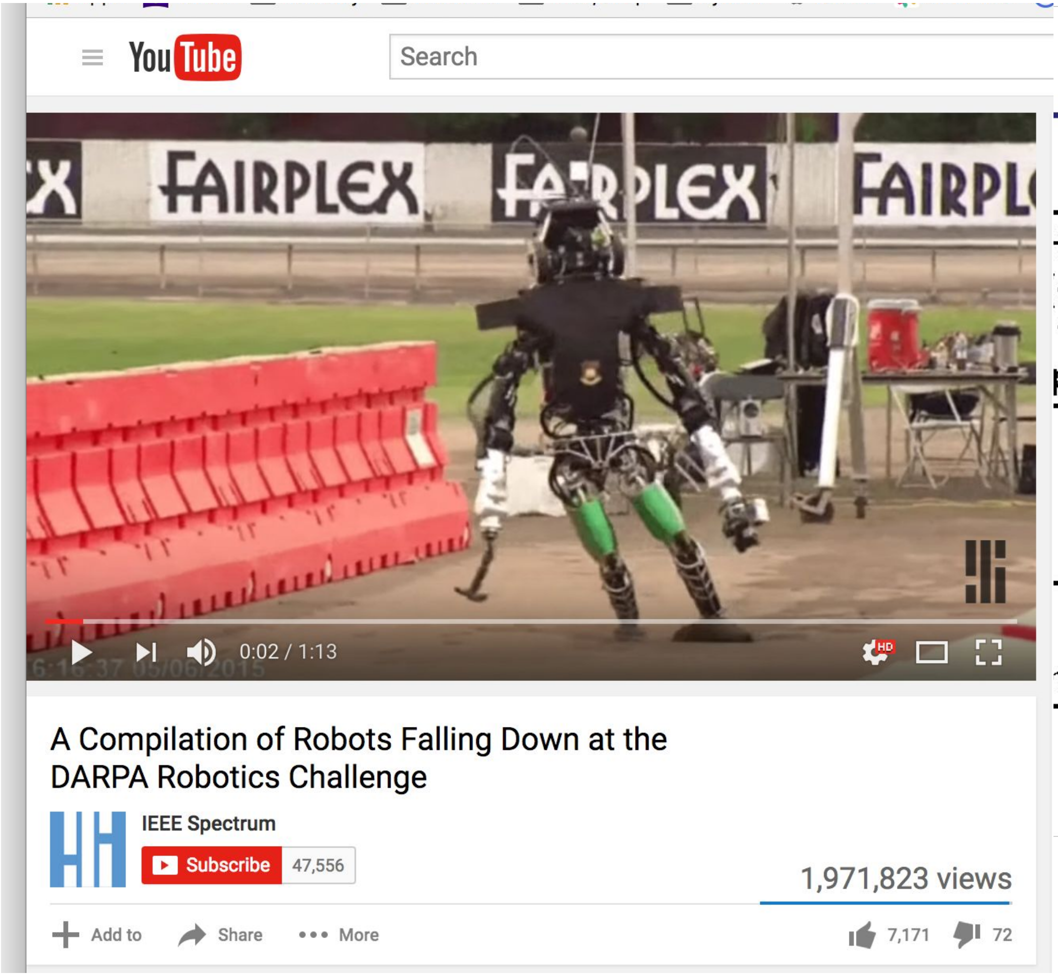
4. And also: use balance equations:

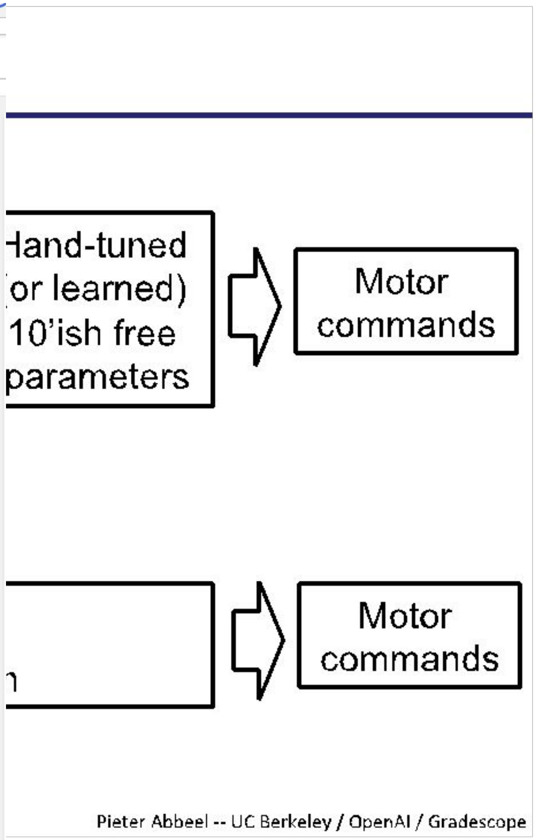
e.g. cut anywhere into two chains:
p(bear) x (.05 + 0.15) =
p(stagnant) x .25 +
P(bull) x 0.75



These models all have problems because of lack of feedback

https://www.youtube.com/watch?v=g0TaYhjpOfo





These models all have problems because of lack of feedback

https://www.youtube.com/watch?v=g0TaYhjpOfo

Markov State is being Combined with ML
to take different actions depending on the state/situation

Reinforcement Learning (RL)

probability of taking action a in state s

$$\pi_\theta(a|s)$$

Robot +
Environment

$$\max_\theta \; E\left[\sum_{t=0}^{H} R(s_t) \,\middle|\, \pi_\theta\right]$$

- Robotics
- Marketing / Advertising
- Dialogue
- Optimizing operations / logistics
- Queue management
- ...

Pieter Abbeel -- UC Berkeley / OpenAI / Gradescope

Markov State is being Combined with ML
to take different actions depending on the state/situation

Reinforcement Learning (RL)

probability of taking action a in state s

$\pi_\theta(a|s)$

Robot +
Environment

- Robotics
- Marketing / Advertising
- Dialogue
- Optimizing operations /

The goal here is not to introduce reinforcement leaning, but instead to introduce the idea that your project could track a “state”
and use a different ML approach to depending on the state.

End of Section

Markov State is being Combined with ML
to take different actions depending on the state/situation

Definition

- Markov Decision Process (MDP) defined by $(\mathcal{S}, \mathcal{A}, P)$, where
  - $\mathcal{S}$: **state space**
  - $\mathcal{A}$: **action space**
  - $P(r, s' \mid s, a)$: a transition probability distribution
- Extra objects defined depending on problem setting
  - μ: Initial state distribution
  - γ: discount factor

Markov Decision Process (S, A, T, R, H)

Given

- S: set of states
- A: set of actions
- T: S x A x S x {0,1,…,H} [0,1], $T_t(s,a,s') = P(s_{t+1} = s' \mid s_t = s, a_t = a)$
- R: S x A x S x {0, 1, …, H} < $R_t(s,a,s')$ = reward for $(s_{t+1} = s', s_t = s, a_t = a)$
- H: horizon over which the agent will act

Goal:

- Find ¼ : S x {0, 1, …, H} A that maximizes expected sum of rewards, i.e.,

Markov Decision Process

Assumption: agent gets to observe the state

[Drawing from Sutton and Barto, Reinforcement Learning: An Introduction, 1998]

Training Models from Real Life

Problem Case

Balancing Flow

Famous Processes

Making Models

A Text Processing Example