The math behind the math behind the PMM of dodo

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November 2023

1 Relation between curves and prices

In an automated market maker (AMM) there is a relation between the supply of two coins. This relation allows to fix a price in a unique way. We call the assets X and Y from now on and their quantities are denoted x and y, respectively. There is a set of requirements on such curves:

- Supplies cannot be negative, meaning we have $x, y \leq 0$
- Prices should be unique. This implies that as a function of x, the price has to be either continuously falling or increasing, and the opposite as a function of y.
- The function that relates x and y has to be continuous and the first derivative has to be defined and continuous everywhere. Second derivatives are allowed to be discontinuous.
- The so-called spot price corresponds to the first derivative of the curve relating x to y.

I will demonstrate this with the simplest AMM of the all (apart from constant), which is the constant function market maker. The formula that is usually used to describe it is

$$x \cdot y = L^2 \tag{1}$$

For the purpose of the discussion and more in line with discussing general functions, it is more instructive to write y as a function of x according to

$$y(x) = \frac{L^2}{x} \tag{2}$$

In a next step I will argue why the derivative of said curve is the price. For simplicity I assume that Y is a stablecoin worth 1$ although this is not a requirement (it serves as the numeraire for X which could be ETH). In a swap, we now pay δx tokens into the pool and want to extract δy. The relation between

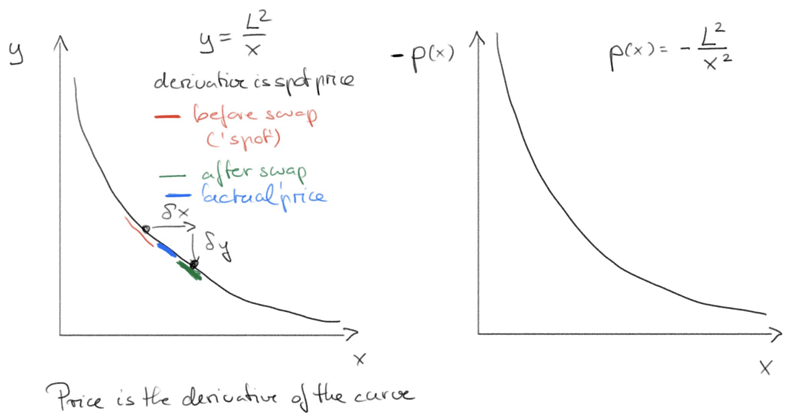

Figure 1: LHS: A standard CFMM and how it relates tokens X to Y. The spot price is the derivative of the curve at the given concentrations. The actual swap price suffers price impact from non-linear effects. RHS: A plot of the price as a function of x.

the two is what we call the price. There are two ways to approach this. We can go back to Eq. (1) and use that we are on a constant curve. This means

$$(x + \delta x) \cdot (y + \delta y) = L^2 \tag{3}$$

Using $(x + \delta x) \cdot (y + \delta y) = x \cdot y$, we can solve this for

$$\delta y = -\frac{y}{x + \delta x}\delta x \; . \tag{4}$$

We would now call $\frac{y}{x+\delta x}$ the average price. It includes a 'dilution effect' from adding δx. This is different from the spot price which is $\frac{y}{x}$. To make this more clear let us consider a different way to determine δy starting from Eq. (2). The amount δy can be related to the curve itself via

$$\delta y = y(x + \delta x) - y(x) = L^2 \left(\frac{1}{x + \delta x} - \frac{1}{x} \right) = -\frac{L^2}{x}\frac{1}{x + \delta x}\delta x = -\frac{y}{x + \delta x}\delta x \; . \tag{5}$$

The spot price is the price that we get for a swap assuming infinite liquidity or no price impact. This means

$$\delta y \approx -\frac{y}{x}\delta x = \frac{dy(x)}{dx}\delta x \tag{6}$$

The spot market price is consequently the derivative of the curve meaning

$$p_{\text{spot}}(x) = -\frac{dy(x)}{dx} = \frac{L^2}{x^2} \; . \tag{7}$$

There are three prices in total: price before swap, actual price of the swap, and price after the swap:

$$\begin{aligned} p_{\text{before}}(x) &= -\frac{dy(x)}{dx} \\ p_{\text{actual}}(x) &= -\frac{y(x+\delta x)-y(x)}{\delta x} \\ p_{\text{after}}(x+\delta x) &= -\frac{dy(x+\delta x)}{dx} \\ . \end{aligned} \tag{8}$$

Whenever $\delta x \ll x$, all three prices are the same. The whole discussion is summarized in Fig. 1.

A concentrated liquidity position is just a twist on this so there is nothing special about it.

2 From prices to curves

The most common AMMs fix the curve and determine the price from it. But one can also determine the curve $y(x)$ if the price $p(x)$ is known. The reason is that the price $p(x)$ is the derivative of the curve $y(x)$ and consequently, one can invert the operation. The corresponding object is calle the integral. In order to fix this integral, one needs to know a price p_0 at one specific point on the curve, meaning a given x_0 and the corresponding y_0. The function $y(x)$ can then be written as

$$y(x) = -\int_{x_0}^{x} dx' p(x') + y_0 \ . \tag{9}$$

Checking this for the CFMM is straightforward and I am not showing this here.

3 DODO math

So how does dodo approach the AMM? Instead of the costumary approach that starts with a curve $y(x)$, it starts with a price p(x). It comes up with the following rule. There is a curve $y(x)$ that is continuous and differentiable. The derivative of it, however, is not necessarily differentiable, but still continuous. It assumes there are two asset, X and Y. It furthermore defines an equilibrium position x_0 and y_0, where the price is p_0 (note that this corresponds to the starting position of the integral discussed above). The price is defined piecewise via

$$p(x) = \begin{cases} p_0\left(1-k+k\left(\frac{x_0}{x}\right)^2\right) & \text{I} \quad x < x_0; \text{ and y(x)} > \text{y}_0 \\ p_0 & \quad\ \ x = x_0; \text{ and y(x)} = \text{y}_0 \\ p_0\frac{1}{1-k+k\left(\frac{y_0}{y(x)}\right)^2} & \text{II} \quad x > x_0; \text{ and y(x)} < \text{y}_0 \end{cases} \tag{10}$$

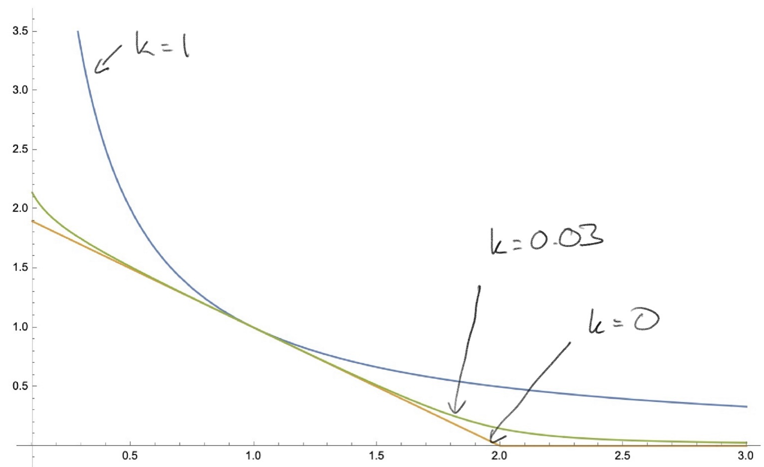

Figure 2: Comparison of different curves as a function of the concentration parameter k. We find that one can interpolate between a linear stable line and a standard CFMM profile.

where I introduced the notation I for the region $x < x_0$ and $y(x) > y_0$ and II for the region $x > x_0$ and $y(x) < y_0$ It is important that this function is continuous and therefore can be integrated in an elementary fashion (although region II is cumbersome and one has to solve a proper differential equation since it is implicit). Consequently, there are two functions $y_{\mathrm{I}}(x)$ and $y_{\mathrm{II}}(x)$ and they are continuously connected at y_0 and x_0. The full function reads

$$y(x) = \begin{cases} p_0 \frac{(x-x_0)((-1+k)x-kx_0)}{x} + y_0 & \mathrm{I} \\ \frac{-p_0x+p_0x_0+y_0-2ky_0+\sqrt{-4(-1+k)ky_0^2+(p_0(x-x_0)+(-1+2k)y_0)^2}}{2(1-k)} & \mathrm{II} \end{cases} \tag{11}$$

Some curves are shown in Fig. 2. It is straightforward to see that one can invert this due to the symmetry $x \to y$ and $p_0 \to 1/p_0$ yielding

$$x(y) = \begin{cases} \frac{-p_0^{-1}y+p_0^{-1}y_0+x_0-2kx_0+\sqrt{-4(-1+k)kx_0^2+\left(p_0^{-1}(y-y_0)+(-1+2k)x_0\right)^2}}{2(1-k)} & \mathrm{I} \\ p_0^{-1} \frac{(y-y_0)((-1+k)y-ky_0)}{y} + x_0 & \mathrm{II} \end{cases} \tag{12}$$

The nice thing about this is that one can interpolate between a completely flat price profile ($k = 0$), a conventional CFMM ($k \to 1$), thereby mimicking the effect of Uniswap v3. The big advantage is that one has a price range from $0 \to \infty$ which makes management easier, takes away the need for tick maths, and is more flexible in terms of producing liquidity profiles. To show how the price behaves as a function of x, please see Fig. 3

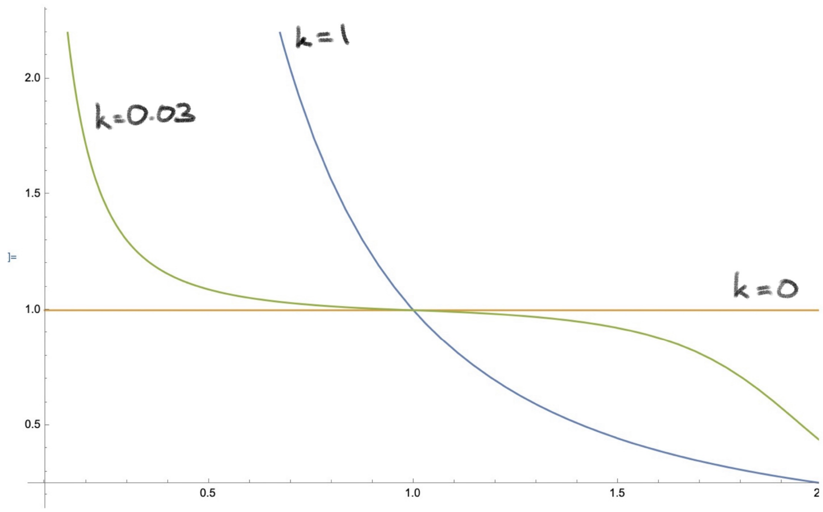

Figure 3: Comparison of different price curves as a function of the concentration parameter k. We find that one can interpolate between a completely flat and a standard CFMM profile.

4 Dynamics of a liquidity pool

We now study how a Dodo pool performs elementary operations.

4.1 How can we perform a swap?

Knowing the curves, swaps are carried out in a standard way. Using the concrete expressions, we find in region I

$$\begin{aligned}
\Delta y &= y(x+\Delta x) - y(x) = -p_0(1-k)\Delta x - p_0 \frac{k x_0^2}{x(x+\Delta x)}\Delta x \\
\Delta x &= \frac{p_0\sqrt{-4(-1+k)k x_0^2 + \left((-1+2k)x_0 + \frac{y-y_0}{p_0}\right)^2}}{2(-1+k)p_0} \\
&- \frac{p_0\sqrt{-4(-1+k)k x_0^2 + \left((-1+2k)x_0 + \frac{y-y_0+\Delta y}{p_0}\right)^2}}{2(-1+k)p_0} \\
&+ \frac{\Delta y}{2(-1+k)p_0}
\end{aligned} \tag{13}$$

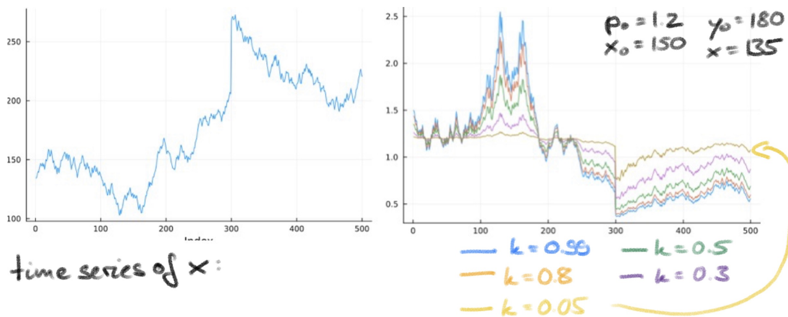

Figure 4: LHS: Time series of x. RHS: Price for different values of k.

and in region II

$$\begin{aligned}
\Delta x &= x(y+\Delta y) - x(y) = -p_0^{-1}(1-k)\Delta y - p_0^{-1}\frac{ky_0^2}{y(y+\Delta y)}\Delta y \\
\Delta y &= \frac{\sqrt{-4(-1+k)ky_0^2 + (p_0(x-x_0)+(-1+2k)y_0)^2}}{2(-1+k)} \\
&- \frac{\sqrt{y_0^2 + 2(-1+2k)p_0y_0(x-x_0+\Delta x) + p_0^2(x-x_0+\Delta x)^2}}{2(-1+k)} \\
&+ \frac{p_0\Delta x}{2(-1+k)} \qquad (14)
\end{aligned}$$

Both can be inverted in straightforward manner to account for the other swap (although it leads to a quadratic equation). On a computer, I have simulated a time series of token x as a random walk with a drift towards higher concentration. I made sure there was a singular even in it (corresponds to someone dumping token y). The parameters were chosen as $p_0 = 1.2$, $y_0 = 180$, $x_0 = 150$, and initial $x = 135$. Now I simulated a time series for x (left-hand side of Fig.~4) and calculated the corresponding curve for y and the resulting price series (right-hand-side) for values of $k = 0.99$ (blue), $k = 0.8$ (orange), $k = 0.5$ (green), $k = 0.3$ (violet), and $k = 0.05$ (light green). The value $k = 0.99$ corresponds to a conventional constant product market maker function. We see that the price gets increasingly stiff and stable with decreasing k.

4.2 Deposits and withdrawals

We now want to study how to make deposits and withdrawals. One appealing aspect of this implementation of a liquidity curve is that it allows for single-side deposits and withdrawals at constant price. This is quite different from Uniswap v2 where deposits and withdrawals are only possible at the current price and at an eact ratio. Uniswap v3 allows to make deposits and withdrawals on and off price at different ratios. Single-sided deposits, however, are only possible off price. Dodo allows a twist on this which is single-sided deposits at the price. The reason for that flexibility lies in the multitude of parameters that are available to accomodate everything. I will divide the following discussion into two cases. Single-side deposits/withdrawals in region I and single-side deposits/withdrawals in region II. Eventually, I will check whether deposits/withdrawals can move the function between regions which is not, a priori, excluded.

4.2.1 Region I

There are three important functions in this region

$$\begin{aligned}
p(x) &= p_0\left(1-k+k\left(\frac{x_0}{x}\right)^2\right) \\
x(y) &= \frac{-p_0^{-1}y+p_0^{-1}y_0+x_0-2kx_0+\sqrt{-4(-1+k)kx_0^2+\left(p_0^{-1}(y-y_0)+(-1+2k)x_0\right)^2}}{2(1-k)} \\
y(x) &= p_0\frac{(x-x_0)\left((-1+k)x-kx_0\right)}{x}+y_0\ . \qquad (15)
\end{aligned}$$

We want to make single side deposits/withdrawals at the current price. I first check the simpler case which is that we make a deposit/withdrawal of Δy (deposit meaning $\Delta y > 0$ and withdrawal $\Delta y < 0$), meaning we go from y to $y' = y + \Delta y$ and $x' = x$. We want to ensure that $p(x, y) = p'(x', y')$. In a deposit we find that the parameters will be adjusted to p_0', k', x_0', and y_0'. The easiest way to do that is to keep $p_0' = p_0$, $k' = k$, and $x_0' = x_0$. Looking at the remaining two equations, we find the following condition

$$y' - y = y_0' - y_0 \qquad (16)$$

meaning

$$y_0' = y_0 + \Delta y \qquad (17)$$

is the correct update. **To summarize, we have** $y' = y + \Delta y$**,** $x' = x$**,** $p_0' = p_0$**,** $k' = k$**,** $x_0' = x_0$**, and** $y_0' = y_0 + \Delta y$.

Consequently, the whole curve is simply shift up or down which makes intuitive sense.

The deposit of Δx is more complicated. The shift goes from x to $x' = x+\Delta x$ First, we want to ensure that

$$p(x, y) = p'(x', y')\ . \qquad (18)$$

Since we do single-side, we have $y' = y$. There is no unique way of doing a deposit but there is a simplest version according to the following recipe. We first shift

$$x_0' = x_0 + \frac{x_0}{x}\Delta x\ , \qquad (19)$$

while keeping $p_0' = p_0$, and $k' = k$. This ensures that $p(x, y) = p'(x', y)$. Using the above expression, we find that this leads to

$$y_0' = y_0 + \left(p_0 \frac{(x - x_0)\left((-1+k)x - kx_0\right)}{x} - p_0 \frac{(x' - x_0')\left((-1+k)x' - kx_0'\right)}{x'} \right) \quad (20)$$

which completes the operation.

At $x = x_0$ the updates are simple and just a special case of this update.

4.2.2 Region II

Region II is in principle the perfect analogue of region I. The important equations are

$$\begin{aligned} p(y) &= \frac{p_0}{1 - k + k\left(\frac{x_0}{x}\right)^2} \\ y(x) &= \frac{-p_0 x + p_0 x_0 + y_0 - 2ky_0 + \sqrt{-4(-1+k)ky_0^2 + \left(p_0(x - x_0) + (-1+2k)y_0\right)^2}}{2(1-k)} \\ x(y) &= p_0^{-1} \frac{(y - y_0)\left((-1+k)y - ky_0\right)}{y} + x_0 \; . \end{aligned} \quad (21)$$

We want to make single side deposits/withdrawals at the current price. I first check the simpler case which is that we make a deposit/withdrawal of Δx (deposit meaning $\Delta x > 0$ and withdrawal $\Delta x < 0$), meaning we go from x to $x' = x + \Delta x$ and $y' = y$. We want to ensure that $p(x, y) = p'(x', y')$. In a deposit we find that the parameters will be adjusted to p_0', k', x_0', and y_0'. The easiest way to do that is to keep $p_0' = p_0$, $k' = k$, and $y_0' = y_0$. Looking at the remaining two equations, we find the following condition

$$x' - x = x_0' - x_0 \quad (22)$$

meaning

$$x_0' = x_0 + \Delta x \quad (23)$$

is the correct update. **To summarize, we have $x' = x + \Delta x$, $y' = y$, $p_0' = p_0$, $k' = k$, $y_0' = y_0$, and $x_0' = x_0 + \Delta$.**

The deposit of Δy is more complicated. The shift goes from y to $y' = y + \Delta y$ First, we want to ensure that

$$p(x, y) = p'(x', y') \; . \quad (24)$$

Since we do single-side, we have $x' = x$. There is no unique way of doing a deposit but there is a simplest version according to the following recipe. We first shift

$$y_0' = y_0 + \frac{y_0}{y} \Delta y \; , \quad (25)$$

while keeping $p_0' = p_0$, and $k' = k$. This ensures that $p(x, y) = p'(x, y')$. Using the above expression, we find that this leads to

$$x_0' = x_0 + \left(p_0^{-1} \frac{(y - y_0)\,((-1 + k)y - ky_0)}{y} - p_0^{-1} \frac{(y' - y_0')\,((-1 + k)y' - ky_0')}{y'} \right) \tag{26}$$

which completes the operation.

4.3 Rebalancing

The last operation that needs to be possible is rebalancing. A rebalancing operation has the following properties. It needs to keep x, y, and p, constant. This implies that a generic rebalancing operation leaves a lot of options, since we have p_0, k, x_0, and y_0 as adjustable parameters. Simple counting reveals that it has three conditions but 4 parameters.