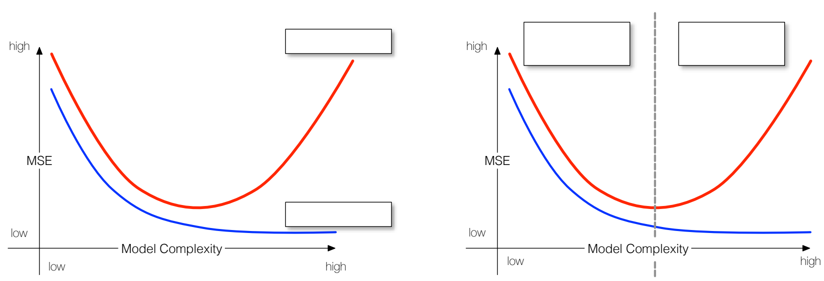
2413, Machine Learning, Tutorial 6
Universität Bern

Simon Jenni (jenni@inf.unibe.ch)

Bias/Variance

1. The two graphs below show train and test-error of some model as a function of the model complexity. In the left graph, indicate which of the two curves shows the 'training error' or 'test error'. In the right graph, indicate which regions show 'high variance' or 'high bias'.



Solution: 1.) Upper curve is test-error, lower training error. 2.) Left part is high bias, right is high variance

VC Dimension

2. Give the VC-dimension for the following 2D-hypothesis spaces for binary classification:

   - The set of origin-centered circles in the plane (+1 inside and -1 outside the circle).
   - The set of all circles in the plane (+1 inside and -1 outside the circle).

Solution: 1.) 1 and 2.) 3 (see also https://en.wikipedia.org/wiki/Shattered_set)

3. Consider the class $\mathcal{H}^d$ of linear classifiers in $\mathcal{R}^d$. Each classifier in this class is parametrized by a vector $w \in \mathcal{R}^d$ and has the form:

$$h_w(x) = \begin{cases} 1 & \text{if } w^T x > 0 \\ -1 & \text{otherwise} \end{cases} \tag{1}$$

Note that we do not allow a bias term w_0, and thus the separating hyperplanes must pass through the origin.

Show that there is a set of d points in $\mathcal{R}^d$ that can be shattered by $\mathcal{H}^d$.

Solution

We want to show the existence of a set of d points $x_1, \ldots, x_d \in \mathcal{R}^d$ such that for any arbitrary choice of labels $y_1, \ldots, y_d \in \{-1, +1\}$ there exists a $w \in \mathcal{R}^d$ such that $h_w(x_k) = y_k$ for $k = 1, \ldots, d$. Then, by definition, $\mathcal{H}^d$ shatters the set $X = \{x_1, \ldots, x_d\}$.

Let's define the k-th point x_k to have all zero entries except for a one in entry k.

$$x_k^i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Given an arbitrary set of labels $Y = (y_1, \ldots, y_d)$ let's define the corresponding w by $w_k(Y) = y_k$. Then,

$$h_{w(Y)}(x_k) = sign(w(Y)^T x_k) = sign(x_k^k y_k) = y_k \tag{3}$$

which is precisely what we wished to show. Hence, we have shown a set X with d points which is shattered by $\mathcal{H}^d$.