Diff Übungen 4

1) $e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \dots$$

$$\frac{d}{dt} e^{At} = 0 + A + A^2 t + \frac{1}{2}A^3 t^2$$

$$= A(I + At + \frac{1}{2}A^2 t^2 + \dots)$$

$$\underline{\underline{= A e^{At}}}$$

2) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 6-\lambda & 2 \\ 2 & 2 & 6-\lambda \end{vmatrix}$$

$(1-\lambda)((6-\lambda)(6-\lambda)-4) - 2(2(6-\lambda)-4) + 2(4-(6-\lambda)(2))$

$= (1-\lambda)(6-\lambda)^2 - 4(1-\lambda) - 4(6-\lambda) + 8 + 8 - 4(6-\lambda)$

$= (1-\lambda)(6-\lambda)^2 - 4(1-\lambda) - 4(6-\lambda) - 4(6-\lambda) + 16$

$= (1-\lambda)(6-\lambda)^2 - 4(1-\lambda+6-\lambda+6-\lambda) + 16$

$= (1-\lambda)(6-\lambda)^2 - 4(13-3\lambda) + 16$

$= (1-\lambda)(6-\lambda)^2 - 52 + 12\lambda + 16$

$= (1-\lambda)(36-12\lambda+\lambda^2) - 36 + 12\lambda$

$= \cancel{36} - \cancel{12\lambda} + \lambda^2 - 36\lambda + 12\lambda^2 - \lambda^3 \cancel{- 36} \cancel{+ 12\lambda}$

$= -\lambda^3 + 13\lambda^2 - 36\lambda = 0$

$\lambda(-\lambda^2 + 13\lambda - 36) = 0$ $\qquad \underline{\lambda_1 = 0}$

$-\lambda^2 + 13\lambda - 36 = 0$

$\lambda^2 - 13\lambda + 36 = 0$

$$\lambda = \frac{13 \pm \sqrt{169-144}}{2} = \frac{13 \pm 5}{2}$$

$\underline{\lambda_2 = 4} \quad \underline{\lambda_3 = 9}$

$$A = \begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 6-\lambda & 2 \\ 2 & 2 & 6-\lambda \end{bmatrix}$$

$\lambda_1 = 0$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 6 & 2 & 0 \\ 2 & 2 & 6 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$x_1 = -4x_2 \qquad x_2 = t$

$v_1 = s\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = \underline{s\begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}}$

$x_2 = x_3$

$\lambda_2 = 4$

$$\left[\begin{array}{ccc|c} -3 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \sim \begin{bmatrix} -5 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$x_1 = 0$

$x_2 = -x_3 \qquad \underline{v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$

$x_3 = s$

$\lambda_1 = 9$

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \sim \begin{matrix} -8 & 2 & 2 \\ 2 & -3 & 2 \\ 0 & 5 & -5 \end{matrix} \sim \begin{matrix} -8 & 2 & 2 \\ 2 & -3 & 2 \\ 0 & 1 & -1 \end{matrix} \sim \begin{matrix} -8 & 0 & 4 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{matrix}$$

$$\sim \begin{matrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{matrix} \sim \begin{matrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$2x_1 = x_2$

$2x_1 = x_3 \qquad \underline{v_3 = s\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$

$x_1 = s$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix} \qquad V = \begin{bmatrix} v_1 \mid v_2 \mid v_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

$$AV = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 1 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-2-2 & 0+2-2 & 1+4+4 \\ 8-6-2 & 0+6-2 & 2+12+4 \\ 8-4-6 & 0+2-6 & 2+4+12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 9 \\ 0 & 4 & 18 \\ 0 & -4 & 18 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \mid \lambda_2 v_2 \mid \lambda_3 v_3 \end{bmatrix}$$

$$AV = \begin{bmatrix} \lambda_1 v_1 \mid \lambda_2 v_2 \mid \lambda_3 v_3 \end{bmatrix} = V \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

$$\underline{\underline{AV = V\Lambda}}$$

3) Diagonaliserbar om en matrise i $\mathbb{R}^{n \times n}$ har n egenverdier

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$(2-\lambda)((2-\lambda)^2 - 1) - 1(2-\lambda+1) + 1(1-2+\lambda)$$

$$= (2-\lambda)^3 - 2 + \lambda - 2 + \lambda - 1 + 1 - 2 + \lambda$$

$$= (2-\lambda)^3 - 6 + 3\lambda$$

$$= (2-\lambda)^3 - 3(2-\lambda)$$

$$= (2-\lambda)((2-\lambda)^2 - 3)$$

$$= (2-\lambda)(4 - 4\lambda + \lambda^2 - 3)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 1)$$

$$\lambda = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

<u>Er diagonaliserbar</u>

7) $\dot{x} = Ax \quad x = Vz \quad \dot{x} = V\dot{z}$

$A = V\Lambda V^{-1}$

$V\dot{z} = Ax = AVz$

$V\dot{z} = V\Lambda V^{-1} V z$

$V\dot{z} = V\Lambda z$

$V^{-1}V\dot{z} = V^{-1}V\Lambda z$

$\underline{\underline{\dot{z} = \Lambda z}}$

8)

$$e^{A} = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots$$

$$e^{\Lambda t} = \sum_{n=0}^{\infty} \frac{(\Lambda t)^n}{n!} = I + \Lambda t + \frac{1}{2}\Lambda^2 t^2 + \frac{1}{3!}\Lambda^3 t^3 + \dots$$

$$= \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & 0 \\ & \lambda_2 t & \\ & & \ddots \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\lambda_1^2 t^2 & & \\ & \frac{1}{2}\lambda_2^2 t^2 & \\ & & \ddots \end{bmatrix} + \dots$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & e^{\lambda_3 t} & \\ & & & \ddots \end{bmatrix}$$

9) $A = V\Lambda V^{-1} \Rightarrow e^{A} = Ve^{\Lambda}V^{-1}$

$A^m = V\Lambda^m V^{-1}$

$$e^{A} = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots$$

$$= I + V\Lambda V^{-1} + \frac{1}{2}V\Lambda^2 V^{-1} + \frac{1}{3!}V\Lambda^3 V^{-1}$$

$$= V(I + \Lambda + \frac{1}{2}\Lambda^2 + \frac{1}{3!}\Lambda^3)V^{-1}$$

$$= Ve^{\Lambda}V^{-1}$$

1 0)

$\dot{x}_1 = -x_1$

$\dot{x}_2 = x_1 - x_2$

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\begin{vmatrix} -1-\lambda & 0 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$(-1-\lambda)^2 = 0$

$\lambda = -1$

$$\begin{bmatrix} -1+1 & 0 \\ 1 & -1+1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = 0 \\ x_2 = s \end{matrix} \quad a_1 = s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Ikke diagonaliserbar, bare én egenvektor

1 3)

$$e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = I + z + \frac{1}{2} z^2 + \frac{1}{3!} z^3 + \dots$$

$$e^{z}e^{w} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{z^n}{n!} \frac{w^m}{m!} \qquad ?$$

$$e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{l=0}^{n} \frac{n!}{l!(n-l)!} z^n w^{n-l} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{1}{n!} \frac{n!}{l!(n-l)!} z^n w^{n-l}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} z^n w^{n-l} \qquad ?$$

1 4)

$$\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \lambda I + N$$

$IN = NI$?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+1 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 1+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$IN = NI$

1 5)

$$e^{(\lambda I + N)t} = e^{\lambda I t}\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \quad ?$$

$$e^{\lambda I t + Nt} = e^{\lambda I t} e^{Nt}$$

$$e^{Nt} = I + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \dots$$

$$N^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^{Nt} = I + Nt + \underbrace{\frac{1}{2}(Nt)^2 + \dots}_{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

$$e^{Nt} = I + Nt = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

$$e^{\lambda I t + Nt} = e^{\lambda I t} e^{Nt} = e^{\lambda I t}\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

16)

$\lambda = -1$

$$e^{At}x_0 = e^{(\lambda I + N)t}\begin{bmatrix}1\\1\end{bmatrix} = e^{-t}\begin{bmatrix}1 & 0\\ t & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = e^{-t}\begin{bmatrix}1\\ t+1\end{bmatrix} = \begin{bmatrix}e^{-t}\\ e^{-t}(t+1)\end{bmatrix}$$

17)

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ 1 & 1 & \cdot \end{bmatrix}\begin{bmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ 1 & 1 & \cdot \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$N \in \mathbb{R}^{(n+1)\times(n+1)} = \begin{bmatrix} 0 & 0 & \cdots & & \\ 1 & 0 & & & \\ \vdots & 1 & 0 & & \\ & & 1 & 0 & \\ & & & \ddots & 0 \\ & & & & 1 \; 0 \end{bmatrix}$$

$$N \cdot N = \begin{bmatrix} 0 & \cdots & & & \\ 0 & 0 & & & \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & \\ \vdots & 0 & \ddots & & 0 \\ \vdots & \vdots & \cdots & 1\,0 & 0 \end{bmatrix} \Rightarrow N^n = \begin{bmatrix} \cdot & & \\ 0 & & \\ 1 & 0 & \cdots \end{bmatrix} \Rightarrow N^{n+1} = \begin{bmatrix} 0 & \cdots & & \\ 0 & & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$e^{Nt} = I + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \dots + \frac{1}{n!}(Nt)^n$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ & & & \ddots \end{bmatrix} + \begin{bmatrix} 0 & & \\ t & 0 & \\ & t & 0 \\ & & & \ddots \end{bmatrix} + \begin{bmatrix} 0 & & \\ 0 & 0 & \\ \frac{1}{2}t^2 & 0 & \\ & \frac{1}{2}t^2 & \ddots \end{bmatrix} + \dots = \underline{\begin{bmatrix} 1 & & \\ t & 1 & \\ \frac{1}{2}t^2 & t & 1 \\ \vdots & & & \ddots \\ \frac{1}{n!}t^n & & & & 1 \end{bmatrix}}$$

18)

$$\dot{x} = Ax \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \qquad \lambda = 1$$

$$e^{At} = e^{(\lambda I + N)t} \qquad N = \begin{bmatrix} \cdot & 1 & 1 \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$e^{At} = e^{\lambda I t} e^{Nt} = e^{\lambda t} \begin{bmatrix} 0 & t & \cancel{t+\frac{1}{2}t^2} \\ \cdot & 0 & t \\ \cdot & \cdot & 0 \end{bmatrix}$$

19)

$$\dot{x} = A\,x \qquad A = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix} \qquad \lambda = 1$$

$$e^{At} = e^{(\lambda I + N)t} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ & 1 & t \\ & & 1 \end{bmatrix}$$

20)

$$\dot{x} = Ax \qquad A = \begin{bmatrix} 1 & 1 & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \qquad \lambda = 1$$

$$e^{At} = e^{\lambda t} e^{Nt} \qquad = N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{Nt} = I + Nt + \frac{1}{2}(Nt)^2 = I + Nt$$

$$e^{At} = e^{\lambda t} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2 1)

$$\dot{x} = Ax \qquad A = \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix} \qquad \lambda = 1$$

$$e^{At} = e^{\lambda I t} \begin{bmatrix} 1 & & \\ & 1 & t \\ & & 1 \end{bmatrix}$$

2 2)

$$\dot{x} = Ax \qquad A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda I)^k \xi = 0$$

$$\begin{bmatrix} -\lambda & 0 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{bmatrix}$$

$$(-\lambda)((-1-\lambda)(-1-\lambda)) - (0) + (0) = 0$$

$$(-\lambda)(-1-\lambda)(-1-\lambda) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = -1$$

$$\lambda_1 = 0$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right] \sim$$

$$2x_1 = -x_2 \qquad \xi_1 = s\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$$

$$2x_2 = x_3$$

$$x = s$$

$$\lambda_{2,3} = -1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = s \end{matrix} \qquad \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & & \\ & -1 & 1 \\ & & -1 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 0 & v_{13} \\ -2 & 0 & v_{23} \\ -4 & 1 & v_{33} \end{bmatrix}$$

$$(A - \lambda I)^2 = \begin{bmatrix} -\lambda & 0 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{bmatrix} \begin{bmatrix} -\lambda & 0 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} (-\lambda)^2 & 0 & 0 \\ -2\lambda + (-1-\lambda)^2 & 0 + (-1-\lambda)^2 & 2(-\lambda) \\ 4 + 0 & 2(-1-\lambda) + 2(-1-\lambda) & (-1-\lambda) \end{bmatrix}$$