Rekker 2

2)

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

$$e^x = 1 + \int_0^x e^s \, ds$$

$u(s) = u'(s) = e^x \qquad v(s) = x - s$

$$e^x = 1 + \left(-\int_0^x e^s \, ds\right)$$

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int u'(x)\, v(x)\, dx$$

$$= 1 + \left(-\left([(x-s)e^s]_0^x - \int_0^x (x-s)e^s \, ds\right)\right)$$

$$= 1 + \left(-\left((x-x)e^x - (x-0)e^0\right) - \int_0^x (x-s)e^s \, ds\right)$$

$$= 1 + \left(-\left((-x) - \int_0^x \dots\right)\right)$$

$$\underline{\underline{= 1 + x + \int_0^x (x-s)e^s \, ds}}$$

Greit ?

3)

$$e^x = 1 + x + \int_0^x (x-s)e^s \, ds$$

$u(s) = u'(s) = e^s$

$v(s) = \frac{1}{2}(x-s)^2$

$v'(s) = -(x-s)$

$$e^x = 1 + x - \int_0^x u(s)\, v'(s)\, ds$$

$$e^x = 1 + x - [u(s)\, v(s)]_0^x + \int_0^x v(s)\, u'(s)\, ds$$

$$e^x = 1 + x - \left(e^x \tfrac{1}{2}(x-x)^2\right) + \left(e^0 \tfrac{1}{2}(x-0)^2\right) + \int_0^x \tfrac{1}{2}(x-s)^2 e^s \, ds$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-s)^2 e^s \, ds$$

4) $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-s)^2 e^s \, ds$

$v'(s) = -(x-s)^2$

$v(s) = \frac{1}{3}(x-s)^3$

$u(s) = u'(s) = e^s$

$e^x = 1 + x + \frac{1}{2}x^2 - \frac{1}{2}\int_0^x u(s)v'(s)\, ds$

$e^x = 1 + x + \frac{1}{2}x^2 - \frac{1}{2}\left(e^x \frac{1}{3}(x-x)^3\right) + \frac{1}{2}\left(e^0 \frac{1}{3}(x-0)^3\right) + \int_0^x \frac{1}{3}(x-s)^3 e^s \, ds$

$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{3}\int_0^x (x-s)^3 e^s \, ds$

Vrech

$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$p_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots + \frac{1}{n!}x^n$

7) $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$

$+ \frac{1}{6}f'''(a)(x-a)^3 + \ldots + \frac{1}{n!}f^n(a)(x-a)^n + \frac{1}{n!}\int_a^x (x-s)^{n+1} f^{(n+1)}(s)\, ds$

Feilledd

$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^n(a)(x-a)^n$

8) $a = 0 \quad f(x) = \sin x$

$f(a) = 0$

$f'(x) = \cos x \quad f'(a) = 1$

$f''(x) = -\sin x \quad f''(a) = 0$

$f^3(x) = -\cos x \quad f^3(a) = -1$

$p(x) = x - \frac{1}{6}x^3$

$a = 0 \quad f(x) = \cos x$

$f(a) = 1$

$f'(x) = -\sin x \quad f'(a) = 0$

$f''(x) = -\cos x \quad f''(a) = -1$

$f^3(x) = \sin x \quad f^3(a) = 0$

$p(x) = 1 - \frac{1}{2}x^2$

9) $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a) + \frac{1}{6} f^{3}(a)(x-a) + \dots + \frac{1}{n!} f^{n}(x-a)^{n}$

$+ \frac{1}{(n+1)!} f^{n+1}(s)(x-a)^{n+1}$

$x_{n+1} = g(x_n)$

$n = 0$

$g(x) = g(r) + g'(r)(x_n - r)$

$x_{n+1} = g(r) + g'(r)(x_n - r)$

$x_{n+1} - g(r) = g'(r)(x_n - r)$

10)

om $|g'(r)| > 1$, så vil $\{x_n\}$ ikke konvergere.

Om $|g'(r)|$ er veldig liten så vil den konvergere fort

L'Hôpital

$\lim_{x\to a} f(x) = \lim_{x \to a} g(x) = 0$, $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ eksisterer $\to$ $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Bevis

$f'(x) = \lim_{x\to h} \frac{f(x+h) - f(x)}{h} = \frac{f(x+h) - f(x)}{h} - \delta(h)$, $\lim_{h\to 0} \delta(h) = 0$

$h f'(x) = f(x+h) - f(x) - h\,\delta(h)$

$f(x+h) = h f'(x) + f(x) + h\,\delta(h)$

og

$g(x+h) = h\, g'(x) + g(x) + h\,\varepsilon(h)$

$\lim_{x\to a} f(x+h) = \lim_{x\to a} h f'(x) + \lim_{x\to a} f(x) + \lim_{x\to a} h\,\delta(h)$

$\lim_{x\to a} f(x+h) = h f'(a) \quad + \quad 0 \quad + h\,\delta(h)$

$\lim_{x\to a} g(x+h) = h\, g'(a) \quad + \quad 0 \quad + h\,\varepsilon(g)$

$\lim_{x\to a} \frac{f(x+h)}{g(x+h)} = \frac{h f'(a) + h\,\delta(h)}{h\, g'(a) + h\,\varepsilon(h)} = \frac{f'(a) + \delta(h)}{g'(a) + \varepsilon(h)}$

$h \to 0 \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ $\square$

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Taylors teorem

La $f(x)$ være n ganger deriverbar. Da finnes en $h_k(x)$ slik at

$$f(x) = f(a) + f'(a)(x-a) + \dots + f^n(a)\frac{1}{n!}(x-a) + h_n(x)(x-a)^n$$

$$= \left(\sum_{k=0}^{n} \frac{f^n(a)}{n!}(x-a)^n\right) + h_n(x)(x-a)^n$$

og $\lim_{x \to a} h_k(a) = 0$

Bevis:

$$h_n(x) = \begin{cases} \frac{p(x) - f(x)}{(x-a)} & x \neq a \\ 0 & x = a \end{cases}$$

$$p_n(x) = \sum_{k=0}^{n} \frac{f^n(a)}{k!}(x-a)^k$$

$$f(x) + h_n(x)(x-a)^n = f(x) + (x-a)^n \cdot \begin{cases} \frac{p(x) - f(x)}{(x-a)} & x \neq a \\ 0 & x = a \end{cases}$$

$$= f(x) + \begin{cases} p(x) - f(x) & x \neq a \\ 0 & x = a \end{cases}$$

$$= \begin{cases} p(x) & x \neq a \\ f(x) & x = a \end{cases}$$

$$= \begin{cases} p(x) & x \neq a \\ f(a) & x = a \end{cases}$$

$$f(a) = p(a)$$

$$= \begin{cases} p(x) & x \neq a \\ p(a) & x = a \end{cases} = p(x)$$

$$p(x) = f(x) + h_k(x-a)^n$$

$$\lim_{x\to a} h_n(x) = 0 \ ?$$

$$\lim_{x\to a} \begin{cases} \frac{p(x) - f(x)}{(x-a)^n} & x \neq a \\ 0 & x = a \end{cases}$$

Grenseverdien $x \to a$ sier at x går mot a, men blir aldri a

$$= \lim_{x\to a} \frac{p(x) - f(x)}{(x-a)^n}$$

$p(x), f(x), (x-a)^n$ er n-ganger deriverbare $\Rightarrow$ L'Hôpital

$$p'(x) = f'(a) + f''(a)(x-a) + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}$$

$$p^{(i)}(x) = f^{(i)}(a) + f^{(i+1)}(a) + \frac{f^{(n-i)}(a)}{(n-i)!}(x-a)^{n-i}$$

$$0 \leq i \leq n$$

$$p^{(i)}(a) = f^{(i)}(a) \Rightarrow p(a) = f(a)$$

$$\lim_{x\to a} \frac{p(x) - f(x)}{(x-a)^n} = \frac{p(a) - f(a)}{(a-a)^n} = \frac{0}{0}$$

$$\lim_{x\to a} \frac{p'(x) - f'(x)}{n(x-a)^n} = \lim_{x\to a} \frac{p''(x) - f''(x)}{n(n-1)(x-a)^n} = \ldots = \lim_{x\to a} \frac{p^{(n-1)}(a) - f^{(n-1)}(a)}{n!(x-a)}$$

$$= \lim_{x\to a} \frac{p^{(n)}(a) - f^{(n)}(a)}{n!} = \frac{1}{n!} \cdot \underbrace{\lim_{x\to a} p^{(n)}(a) - f^{(n)}(a)}_{0}$$

$$\underline{\underline{\lim_{x\to a} h_n(x) = 0}} \qquad \square$$

1.1) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = g(x_n)$ Taylors teorem

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{n!}\int_a^x (x-s)^n f^{(n+1)}(s)\, ds$$

$f(x) = f(x_n) + f'(x_n)(x - x_n) + R_1$ $\quad R_1 =$ Restledd lagrange

$R_1 = \frac{1}{2} f''(s)(x - x_n)^2$ for en s

$f(r) = f(x_n) + f'(x_n)(r - x_n) + \frac{1}{2} f''(s)(r - x_n)^2 \mid \cdot \frac{1}{f'(x_n)}$ $\quad f(r) = 0$

$$0 = \frac{f(x_n)}{f'(x_n)} + (r - x_n) + \frac{1}{2} \cdot \frac{f''(s)}{f'(x_n)}(r - x_n)^2$$

$$x_n - \frac{f(x_n)}{f'(x_n)} - r = \frac{f''(s)}{2f'(x_n)}(r - x_n)^2$$

$$\underline{\underline{x_{n+1} - r = \frac{f^{(2)}(s)}{2f'(x_n)}(r - x_n)^2}}$$

1.2) $f(x) = \frac{1}{1-x}$ $\quad x \in (-1, 1)$

$$\frac{d}{dx^n}\frac{1}{1-x} = \frac{1}{(1-x)^{(n)}} = \frac{1}{\frac{1}{n!}(1-x)^n} = \frac{n!}{(1-x)^n}$$

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} \cdot f^{(k)}(0) \cdot x^k = \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{k!}{(1-0)^k} \cdot x^k = \sum_{k=0}^{n} x^k = \underline{\underline{1 + x + x^2 + \dots + x^n}}$$

1.3)

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ Konvegerer

$$\lim_{n\to\infty} \left|\frac{(x^{n+1})/(n+1)}{(x^n)/n}\right| = \lim_{n\to\infty} \left|\frac{n\, x^{n+1}(n+1)}{x^n}\right| = \lim_{n\to\infty} \left|\frac{n\, x^{n+1}}{(n+1)\, x^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| \cdot \left|\frac{n}{n+1}\right| = \lim_{n\to\infty} \left|x^{n+1-n}\right| \cdot \left|\frac{n}{n+1}\right| = |x| \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \underline{\underline{|x|}}$$

Konvegerer $x = [-1, 1)$

14)

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

$$\lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)^2}{x^n/n^2}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| \cdot \left|\frac{n^2}{n^2+2n+1}\right| = |x| \cdot \lim_{n\to\infty} \left|\frac{n^2}{n^2+2n+1}\right|$$

$= |x|$ Konvergence $x = \langle -1, 1 \rangle$

15)

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} \quad \lim_{n\to\infty} \left|\frac{x^{n+1}/2^{n+1}}{x^n/2^n}\right| = |x| \cdot \lim_{n\to\infty} \left|\frac{2^n}{2^{n+1}}\right| = |x| \cdot \frac{1}{2} = \frac{|x|}{2} < 1$$

$\underline{\underline{|x| < 2}}$

16)

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n} \to \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{(-1)^n} \frac{x^{n+1} \cdot \frac{1}{2^{n+1}}}{x^n \cdot \frac{1}{2^n}}\right| = \lim_{n\to\infty} \left|(-1) \cdot x \cdot \frac{2^n}{2^{n+1}}\right|$$

$= \left|-x \cdot \frac{1}{2}\right| < 1$

$\underline{\underline{= |-x| < 2}}$

17)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n} \cdot \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty} \left|x \cdot \frac{n!}{(n+1)!}\right| = \underline{|x| < 1}$$

18)

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0} \frac{\cos\theta}{1} = \lim_{\theta\to 0} \cos\theta = \underline{\underline{1}}$$

19) $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{1} = \underline{\underline{0}}$

20)

$$\lim_{x\to 0} \frac{e^{x^3}-1}{x-\sin x} = \lim_{x\to 0} \frac{3x^2e^{x^3}}{1-\cos x} = \lim_{x\to 0} \frac{6xe^{x^3}-9x^4e^{x^4}}{\sin x}$$

$$= \lim_{x\to 0} \frac{6e^{x^3}-18x^2e^{x^3}-36x^3e^{x^3}+27x^7e^{x^4}}{\cos x} = \frac{6}{1} = \underline{\underline{6}}$$

21)

$$\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-\frac{1}{2}x^2}\,dx$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^{-\frac{1}{2}x^2} = \sum_{n=0}^{\infty}\frac{(-\frac{1}{2}x^2)^n}{n!} = \sum \frac{-\frac{1^n}{2^n}x^{2n}}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n n!}$$

$$\frac{1}{\sqrt{2\pi}}\int_0^a e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}}\int_0^a \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n n!}$$

$$\frac{1}{\sqrt{2\pi}}\int_0^1 \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n n!} = \frac{1}{\sqrt{2\pi}}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\int_0^1 x^{2n}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\left[\frac{1}{2n+1}x^{2n+1}\right]_0^1 = \underline{\underline{\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\cdot\frac{1}{2n+1}}}$$

Heng deg rela ☺

23)

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \ldots = e^1$$

24)

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} = e^i \Rightarrow x = i$$

25)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$

$$\cos 1 = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1^{2n}}{2n!}$$

26) Vraħ

27) Vrall !!

28) $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{1}{2} A^2 + \frac{1}{6} A^3 + \ldots$