

# Diff. Equations 4

$$1) e^A = I + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \dots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \dots$$

$$\begin{aligned} \frac{d}{dt} e^{At} &= 0 + A + A^2 t + \frac{1}{2} A^3 t^2 + \dots \\ &= A(I + At + \frac{1}{2} A^2 t^2 + \dots) \\ &= \underline{\underline{A e^{At}}} \end{aligned}$$

$$2) A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 6-\lambda & 2 \\ 2 & 2 & 6-\lambda \end{vmatrix}$$

$$(1-\lambda)(6-\lambda)(6-\lambda) - 4 - 2(2(6-\lambda) - 4) + 2(4 - (6-\lambda)(2))$$

$$= (1-\lambda)(6-\lambda)^2 - 4(1-\lambda) - 4(6-\lambda) + 8 + 8 - 4(6-\lambda)$$

$$= (1-\lambda)(6-\lambda)^2 - 4(1-\lambda) - 4(6-\lambda) - 4(6-\lambda) + 16$$

$$= (1-\lambda)(6-\lambda)^2 - 4(1-\lambda + 6-\lambda + 6-\lambda) + 16$$

$$= (1-\lambda)(6-\lambda)^2 - 4(13-3\lambda) + 16$$

$$= (1-\lambda)(6-\lambda)^2 - 52 + 12\lambda + 16$$

$$= (1-\lambda)(36 - 12\lambda + \lambda^2) - 36 + 12\lambda$$

$$= \cancel{36} - \cancel{12\lambda} + \lambda^2 - 36\lambda + 12\lambda^2 - \lambda^3 - \cancel{36} + \cancel{12\lambda}$$

$$= -\lambda^3 + 13\lambda^2 - 36\lambda = 0$$

$$\lambda(-\lambda^2 + 13\lambda - 36) = 0 \quad \underline{\lambda_1 = 0}$$

$$-\lambda^2 + 13\lambda - 36 = 0$$

$$\lambda^2 - 13\lambda + 36 = 0$$

$$\lambda = \frac{13 \pm \sqrt{169 - 144}}{2} = \frac{13 \pm 5}{2}$$

$$\underline{\lambda_2 = 4} \quad \underline{\lambda_3 = 9}$$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix}$$

$$\lambda_1 = 0$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = -4x_2$$

$$x_2 = s$$

$$u_1 = s \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = s \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

$$x_2 = x_3$$

$$\lambda_2 = 4$$

$$\begin{bmatrix} -3 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} -5 & 0 & 0 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} -5 & 0 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0$$

$$x_2 = -x_3$$

$$x_3 = s$$

$$u_2 = s \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\lambda_3 = 9$$

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} -8 & 2 & 2 \\ 2 & -3 & 2 \\ 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{R_1 + 4R_3} \begin{bmatrix} -8 & 2 & 2 \\ 2 & -3 & 2 \\ 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 = x_3$$

$$2x_1 = x_3$$

$$x_1 = s$$

$$u_3 = s \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix} \quad V = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

$$AV = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-2-2 & 0+2-2 & 1+4+4 \\ 8-6-2 & 0+6-2 & 2+1+2+4 \\ 8-4-6 & 0+2-6 & 2+4+1+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 9 \\ 0 & 4 & 18 \\ 0 & -4 & 18 \end{bmatrix} = \begin{bmatrix} | & | & | \\ d_1 v_1 & d_2 v_2 & d_3 v_3 \\ | & | & | \end{bmatrix}$$

$$AV = \begin{bmatrix} | & | & | \\ d_1 v_1 & d_2 v_2 & d_3 v_3 \\ | & | & | \end{bmatrix} = V \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \quad \Delta = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}$$

$$\underline{AV = V\Delta}$$

3) Diagonalisierbar ein reelles  $\mathbb{R}^{n \times n}$  hat  $n$  Eigenwerte

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$(2-\lambda)((2-\lambda)^2 - 1) - 1(2-\lambda+1) + 1(1-2+\lambda)$$

$$= (2-\lambda)^3 - 2 + \lambda - 2 + \lambda - 1 + 1 - 2 + \lambda$$

$$= (2-\lambda)^3 - 6 + 3\lambda$$

$$= (2-\lambda)^3 - 3(2-\lambda)$$

$$= (2-\lambda)((2-\lambda)^2 - 3)$$

$$= (2-\lambda)(4 - 4\lambda + \lambda^2 - 3)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 1)$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

Er diagonalisierbar

$$7) \quad \dot{x} = Ax \quad x = Vz \quad \dot{x} = V \dot{z}$$

$$A = V \Lambda V^{-1}$$

$$V \dot{z} = Ax = AVz$$

$$V \dot{z} = V \Lambda V^{-1} Vz$$

$$V \dot{z} = V \Lambda z$$

$$V^{-1} V \dot{z} = V^{-1} V \Lambda z$$

$$\underline{\dot{z} = \Lambda z}$$

8)

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \dots$$

$$e^{\Lambda t} = \sum_{n=0}^{\infty} \frac{(\Lambda t)^n}{n!} = I + \Lambda t + \frac{1}{2} \Lambda^2 t^2 + \frac{1}{3!} \Lambda^3 t^3 + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \lambda_1^2 t^2 & 0 \\ 0 & \frac{1}{2} \lambda_2^2 t^2 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$9) \quad A = V \Lambda V^{-1} \Rightarrow e^A = V e^{\Lambda} V^{-1}$$

$$A^n = V \Lambda^n V^{-1}$$

$$e^A = I + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \dots$$

$$= I + V \Lambda V^{-1} + \frac{1}{2} V \Lambda^2 V^{-1} + \frac{1}{3!} V \Lambda^3 V^{-1}$$

$$= V (I + \Lambda + \frac{1}{2} \Lambda^2 + \frac{1}{3!} \Lambda^3) V^{-1}$$

$$= V e^{\Lambda} V^{-1}$$

1 0)

$$\vec{x}_1 = -x_1$$

$$\vec{x}_2 = x_1 - x_2$$

$$x^2 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$\begin{vmatrix} -1-\lambda & 0 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)^2 = 0$$

$$\lambda = -1$$

$$\begin{bmatrix} -1+1 & 0 \\ 1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{matrix} x_1 = 0 \\ x_2 = s \end{matrix} \quad q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

1 the diagonalisierbar, base ein eigenvektor

1 3)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = I + z + \frac{1}{2} z^2 + \frac{1}{3!} z^3 + \dots$$

$$e^{z+w} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{z^n w^m}{n! m!}$$

$$e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{l=0}^n \frac{n!}{l!(n-l)!} z^l w^{n-l} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^n \frac{1}{n!} \frac{n!}{l!(n-l)!} z^l w^{n-l}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^n \frac{n!}{l!(n-l)!} z^l w^{n-l} \quad ?$$

14)

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = -I + N$$

$$IN = NI \quad ?$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+1 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 1+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$IN = NI$$

15)

$$e^{(\lambda I + N)t} = e^{\lambda I t} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \quad ?$$

$$e^{\lambda I t + N t} = e^{\lambda I t} e^{N t}$$

$$e^{N t} = I + N t + \frac{1}{2} (N t)^2 + \frac{1}{3!} (N t)^3 + \dots$$

$$N^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^{N t} = I + N t + \underbrace{\frac{1}{2} (N t)^2 + \dots}_{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

$$e^{N t} = I + N t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

$$e^{\lambda I t + N t} = e^{\lambda I t} e^{N t} = \underline{\underline{e^{\lambda I t} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}}}$$

16)

$$\lambda = -1$$

$$e^{At} x_0 = e^{(I+N)t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} 1 \\ t+1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ e^{-t}(t+1) \end{bmatrix}$$

17)

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ 1 & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$N \in \mathbb{R}^{(n+1) \times (n+1)} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$N \cdot N = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \vdots \\ \vdots & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & 1 & 0 & 0 \end{bmatrix} \Rightarrow N^n = \begin{bmatrix} \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots \\ 1 & 0 & \vdots \end{bmatrix} \Rightarrow N^{n+1} = \begin{bmatrix} 0 & \vdots & \vdots \\ 0 & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{Nt} = I + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \dots + \frac{1}{n!}(Nt)^n$$

$$= \begin{bmatrix} 1 & \vdots & \vdots \\ \vdots & 1 & \vdots \\ \vdots & \vdots & 1 \end{bmatrix} + \begin{bmatrix} 0 & \vdots & \vdots \\ t & 0 & \vdots \\ \vdots & t & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \vdots \\ \frac{1}{2}t^2 & 0 & \vdots \\ \vdots & \frac{1}{2}t^2 & 0 \end{bmatrix} + \dots = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \dots \\ \vdots & 1 & t & \vdots \\ \vdots & \vdots & 1 & t & \vdots \end{bmatrix}$$

18)

$$\dot{x} = Ax \quad A = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \quad \lambda = 1$$

$$e^{At} = e^{(\lambda I + N)t} \quad N = \begin{pmatrix} 0 & 1 & 1 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$$

$$e^{At} = e^{\lambda I t} e^{Nt} = e^{t} \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

19)

$$\dot{x} = Ax \quad A = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \quad \lambda = 1$$

$$e^{At} = e^{(\lambda I + N)t} = e^{t} \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

20)  $\dot{x} = Ax \quad A = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \quad \lambda = 1$

$$e^{At} = e^{\lambda t} e^{Nt} = N = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$e^{Nt} = I + Nt + \frac{1}{2}(Nt)^2 = I + Nt$$

$$e^{At} = e^{t} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$2.1) \quad \dot{x} = Ax \quad A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad \lambda = 1$$

$$e^{At} = e^{\lambda I t} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} t$$

$$2.2) \quad \dot{x} = Ax \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda I)^k = 0$$

$$\begin{bmatrix} -\lambda & 0 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{bmatrix}$$

$$(-\lambda)(-1-\lambda)(-1-\lambda) - (0) + (0) = 0$$

$$(-\lambda)(-1-\lambda)(-1-\lambda) = 0$$

$$\lambda_1 = 0, \lambda_2 = -1$$

$$\lambda = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim$$

$$2x_1 = -x_2$$

$$2x_2 = x_3$$

$$\xi_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$$\lambda = -1$$

$$\lambda_{2,3} = -1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & & \\ & -1 & \\ & & -1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 0 & u_{13} \\ -2 & 0 & u_{23} \\ 4 & 1 & u_{33} \end{bmatrix}$$

$$(A - \lambda I)^2 = \begin{bmatrix} -\lambda & 0 & 0 & -\lambda \\ 2 & -1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 0 & -1-\lambda \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (-\lambda)^2 & 0 & 0 & 0 \\ -2\lambda & (-1-\lambda)^2 & 0 & 2(-\lambda) \\ 4 & 2(-1-\lambda) & 2(-1-\lambda) & (-1-\lambda)^2 \end{bmatrix}$$