Diff Übungen 4

1) $e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \ldots = \sum_{n=0}^{\infty}\frac{A^n}{n!}$

$e^{At} = \sum_{n=0}^{\infty}\frac{(At)^n}{n!} = I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \ldots$

$\frac{d}{dt}e^{At} = 0 + A + A^2 t + \frac{1}{2}A^3 t^2$

$= A(I + At + \frac{1}{2}A^2 t^2 + \ldots)$

$\underline{\underline{= Ae^{At}}}$

2) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 6-\lambda & 2 \\ 2 & 2 & 6-\lambda \end{vmatrix}$

$(1-\lambda)((6-\lambda)(6-\lambda)-4) - 2(2(6-\lambda)-4) + 2(4-(6-\lambda)(2))$

$= (1-\lambda)(6-\lambda)^2 - 4(1-\lambda) - 4(6-\lambda) + 8 + 8 - 4(6-\lambda)$

$= (1-\lambda)(6-\lambda)^2 - 4(1-\lambda) - 4(6-\lambda) - 4(6-\lambda) + 16$

$= (1-\lambda)(6-\lambda)^2 - 4(1-\lambda + 6 - \lambda + 6 - \lambda) + 16$

$= (1-\lambda)(6-\lambda)^2 - 4(13 - 3\lambda) + 16$

$= (1-\lambda)(6-\lambda)^2 - 52 + 12\lambda + 16$

$= (1-\lambda)(36 - 12\lambda + \lambda^2) - 36 + 12\lambda$

$= \cancel{36} - \cancel{12\lambda} + \lambda^2 - 36\lambda + 12\lambda^2 - \lambda^3 - \cancel{36} + \cancel{12\lambda}$

$= -\lambda^3 + 13\lambda^2 - 36\lambda = 0$

$\lambda(-\lambda^2 + 13\lambda - 36) = 0 \qquad \underline{\lambda_1 = 0}$

$-\lambda^2 + 13\lambda - 36 = 0$

$\lambda^2 - 13\lambda + 36 = 0$

$\lambda = \frac{13 \pm \sqrt{169 - 144}}{2} = \frac{13 \pm 5}{2}$

$\underline{\lambda_2 = 4} \quad \underline{\lambda_3 = 9}$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 6 & 2 \\ 2 & 6 & -2 \end{bmatrix}$$

$\lambda_1 = 0$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 6 & 2 & 0 \\ 2 & 2 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -4 x_3 \qquad x_2 = t$

$x_2 = x_3$

$v_1 = s \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = s \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 4$

$$\begin{bmatrix} -3 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} -5 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$x_1 = 0$

$x_2 = -x_3$

$x_3 = s$

$v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\lambda_1 = 9$

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} -8 & 2 & 2 \\ 2 & -3 & 2 \\ 0 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} -8 & 2 & 2 \\ 2 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -8 & 0 & 4 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$2x_1 = x_2$

$2x_1 = x_3$

$x_1 = t$

$v_3 = s \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix} \quad V = [v_1 \mid v_2 \mid v_3] = \begin{bmatrix} 4 & 0 & 1 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

$$AV = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 1 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-2-2 & 0+2-2 & 1+4+4 \\ 8-6-2 & 0+6-2 & 2+12+4 \\ 8-4-6 & 0+2-6 & 2+4+12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 9 \\ 0 & 4 & 18 \\ 0 & -4 & 18 \end{bmatrix} = [v_1 \mid d_2 v_2 \mid d_3 v_3]$$

$$AV = [d_1 v_1 \mid d_2 v_2 \mid d_3 v_3] = V \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \quad \Lambda = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}$$

$$\underline{AV = V \Lambda}$$

3) Diagonaliserbar om en matrise i $\mathbb{R}^{n \times n}$ har n egenvektorer

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$(2-\lambda)\left((2-\lambda)^2 - 1\right) - 1(2-\lambda+1) + 1(1-2+\lambda)$$

$$= (2-\lambda)^3 - 2 + \lambda - 2 + \lambda - 1 + 1 - 2 + \lambda$$

$$= (2-\lambda)^3 - 6 + 3\lambda$$

$$= (2-\lambda)^3 - 3(2-\lambda)$$

$$= (2-\lambda)\left((2-\lambda)^2 - 3\right)$$

$$= (2-\lambda)\left(4 - 4\lambda + \lambda^2 - 3\right)$$

$$= (2-\lambda)\left(\lambda^2 - 4\lambda + 1\right)$$

$$\lambda = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$\underline{\underline{\text{Er diagonaliserbar}}}$$

7) $\dot{x} = A x \qquad x = V y \qquad \dot{x} = V \dot{y}$

$A = V \Lambda V^{-1}$

$V \dot{y} = A x = A V y$

$V \dot{y} = V \Lambda V^{-1} V y$

$V \dot{y} = V \Lambda y$

$V^{-1} V \dot{y} = V^{-1} V \Lambda y$

$\underline{\underline{\dot{y} = \Lambda y}}$

8)

$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \dots$

$e^{\Lambda t} = \sum_{n=0}^{\infty} \frac{(\Lambda t)^n}{n!} = I + \Lambda t + \frac{1}{2} \Lambda^2 t^2 + \frac{1}{3!} \Lambda^3 t^3 + \dots$

$$= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \ddots \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & \\ & \lambda_2 t & \\ & & \ddots \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\lambda_1 t^2 & & 0 \\ & \frac{1}{2}\lambda_2 t^2 & \\ & & \ddots \end{bmatrix} + \dots$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_3 t} \\ & & & \ddots \end{bmatrix}$$

9) $A = V \Lambda V^{-1} \Rightarrow e^A = V e^{\Lambda} V^{-1}$

$A^m = V \Lambda^m V^{-1}$

$e^A = I + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \dots$

$= I + V \Lambda V^{-1} + \frac{1}{2} V \Lambda^2 V^{-1} + \frac{1}{3!} V \Lambda^3 V^{-1}$

$= V \left(I + \Lambda + \frac{1}{2} \Lambda^2 + \frac{1}{3!} \Lambda^3 \right) V^{-1}$

$= V e^{\Lambda} V^{-1}$

1 ①)

$$\dot{x_1} = -x_1$$

$$\dot{x_2} = x_1 - x_2$$

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\begin{vmatrix} -1 - \lambda & 0 \\ 1 & -1 - \lambda \end{vmatrix} = 0$$

$$(-1 - \lambda)^2 = 0$$

$$\lambda = -1$$

$$\begin{bmatrix} -1 + 1 & 0 \\ 1 & -1 + 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ 0 \end{bmatrix} \quad x_1 = 0 \quad x_2 = 8 \qquad q_1 = 8 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1 lke diagonaliseerbar, brace én egenvektor

① ③)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = I + z + \frac{1}{2} z^2 + \frac{1}{3!} z^3 + \dots$$

$$e^z w = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{z^n}{n!} \frac{w^m}{m!} \qquad ?$$

$$e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{l=0}^{n} \frac{n!}{l!(n-l)!} z^l w^{n-l} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{1}{n!} \frac{n!}{l!(n-l)!} z^l w^{n-l}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} z^l w^{n-l} \qquad ?$$

14)

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \lambda I + N$$

$$IN = NI \quad ?$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+1 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 1+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$IN = NI$$

15)

$$e^{(\lambda I + N)t} = e^{\lambda I t}\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \quad ?$$

$$e^{\lambda I t + Nt} = e^{\lambda I t} e^{Nt}$$

$$e^{Nt} = I + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \dots$$

$$N^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^{Nt} = I + Nt + \underbrace{\frac{1}{2}(Nt)^2 + \dots}_{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

$$e^{Nt} = I + Nt = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

$$e^{\lambda I t + Nt} = e^{\lambda I t} e^{Nt} = e^{\lambda I t} + \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

16)

$\lambda = -1$

$$e^{At}x_0 = e^{(\lambda I + N)t}\begin{bmatrix}1\\1\end{bmatrix} = e^{-t}\begin{bmatrix}1 & 0\\t & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = e^{-t}\begin{bmatrix}1\\t+1\end{bmatrix} = \begin{bmatrix}e^{-t}\\e^{-t}(t+1)\end{bmatrix}$$

17)

$$\begin{bmatrix}1 & \vdots & \\ 1 & \vdots & \end{bmatrix}\begin{bmatrix}1 & \vdots & \\ 1 & \vdots & \end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\1 & 0 & 0\end{bmatrix}$$

$N \in \mathbb{C}^{(n+1)\times(n+1)} = \begin{bmatrix}0 & 0 & & & \vdots\\t & 0 & & & \vdots\\ & 1 & 0 & & \vdots\\ & & 1 & 0 & \\ & & & & 0\\ & & & & 1 & 0\end{bmatrix}$

$N \cdot N = \begin{bmatrix}0 & & & \\ 0 & 0 & & \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & 0\\ & 0 & 1 & 0 & 0\\ & & & 1 & 0 & 0\end{bmatrix} \implies N^n = \begin{bmatrix} & & \vdots & \\ 0 & & \vdots & \\ 1 & 0 & \cdots & \end{bmatrix} \implies N^{n+1} = \begin{bmatrix}0 & & \vdots & \\ 0 & & \vdots & \\ 0 & 0 & 0 & 0\end{bmatrix}$

$$e^{Nt} = I + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \ldots + \frac{1}{n!}(Nt)^n$$

$$= \begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix} + \begin{bmatrix}0 & & \\ t & 0 & \\ & t & 0\end{bmatrix} + \begin{bmatrix}0 & & \\ 0 & 0 & \\ \frac{1}{2}t^2 & 0 & \frac{1}{2}t^2\end{bmatrix} + \ldots = \begin{bmatrix}1 & & \\ t & 1 & \\ \frac{1}{2}t^2 & t & \\ \frac{1}{n!}t^n & & 1\end{bmatrix}$$

18)

$$\dot{x} = A x \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \qquad \lambda = 1$$

$$e^{At} = e^{(\lambda I + N)t} \qquad N = \begin{bmatrix} & 1 & 1 \\ & & 1 \\ & & \end{bmatrix}$$

$$e^{At} = e^{\lambda I t} e^{N t} = e^{\lambda t} \begin{bmatrix} 0 & t & t + \tfrac{1}{2}t^2 \\ & 0 & t \\ & & 0 \end{bmatrix}$$

19)

$$\dot{x} = A x \qquad A = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix} \qquad \lambda = 1$$

$$e^{At} = e^{(\lambda I + N)t} = e^{\lambda t} \begin{bmatrix} 1 & t & \tfrac{1}{2}t^2 \\ & 1 & t \\ & & 1 \end{bmatrix}$$

20) $$\dot{x} = A x \qquad A = \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad \lambda = 1$$

$$e^{At} = e^{\lambda t} e^{N t} = N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{N t} = I + N t + \tfrac{1}{2}(N t)^2 = I + N t$$

$$e^{At} = e^{\lambda t} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.1)

$$\dot{x} = A x \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ & & 1 \\ & & 2 \end{bmatrix} \qquad \lambda = 1$$

$$e^{At} = e^{\lambda I t} \begin{bmatrix} 1 & 1 & t \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

2.2)

$$\dot{x} = A x \qquad A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda I)\xi = 0$$

$$\begin{bmatrix} -\lambda & 0 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{bmatrix}$$

$$(-\lambda)((-1-\lambda)(-1-\lambda)) - (0) + (0) = 0$$

$$(-\lambda)(-1-\lambda)(-1-\lambda) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = -1$$

$$\lambda_1 = 0$$

$$\begin{array}{ccc|c} -0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{array} \curvearrowright$$

$$2x_1 = -x_2$$

$$2x_2 = x_3 \qquad \qquad \xi_1 = s\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$$

$$\lambda = 1$$

$$\lambda_{2,3} = -1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad \begin{array}{c} x_1 = 0 \\ x_2 = 0 \\ x_3 = s \end{array} \qquad \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & -1 & \\ & & -1 \\ & & -1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & U_{13} \\ -2 & 0 & U_{23} \\ -4 & 1 & U_{33} \end{bmatrix}$$

$$(A - \lambda I)^2 = \begin{bmatrix} -\lambda & 0 \\ 2 & \\ 0 & -1 - \lambda & -1 - \lambda \end{bmatrix} \begin{bmatrix} -\lambda & 0 \\ 2 & \\ 0 & -1 - \lambda & -1 - \lambda \end{bmatrix}$$

$$= \begin{bmatrix} (-\lambda)^2 & 0 \\ -2\lambda + (-1 - \lambda)^2 & 0 + (-1 - \lambda)^2 & 2(-\lambda) \\ 4 + 0 & 2(-1 - \lambda) + 2(-1 - \lambda) & (-1 - \lambda)^2 \end{bmatrix}$$