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Teaching Children to be Mathematicians Versus Teaching About Mathematics*

SEYMOUR PAPERT

Massachusetts Institute of Technology, Artificial Intelligence Laboratory, Cambridge, U.S.A.

Summary

The important difference between the work of a child in an elementary mathematics class and that of a mathematician is not in the subject matter (old fashioned numbers versus groups or categories or whatever) but in the fact that the mathematician is creatively engaged in the pursuit of a personally meaningful project. In this respect a child's work in an art class is often close to that of a grown-up artist. The paper presents the results of some mathematical research guided by the goal of producing mathematical concepts and topics to close this gap. The prime example used here is 'Turtle Geometry', which is concerned with programming a moving point to generate geometric forms. By embodying the moving point as a 'cybernetic turtle' controlled by an actual computer, the constructive aspects of the theory come out sufficiently to capture the minds and imaginations of almost all the elementary school children with whom we have worked—including some at the lowest levels of previous mathematical performance.

Introduction

Being a mathematician is no more definable as 'knowing' a set of mathematical facts than being a poet is definable as knowing a set of linguistic facts. Some modern mathematical education reformers will give this statement a too easy assent with the comment: 'Yes, they must understand, not merely know'. But this misses the capital point that being a mathematician, again like being a poet, or a composer or an engineer, means *doing*, rather than knowing or understanding. This paper is an attempt to explore some ways in which one might be able to put children in a better position to *do* mathematics rather than merely to learn *about it*.

The plan of the essay is to develop some examples of new kinds of mathematical activity for children, and then to discuss the general issues alluded to in the preceding paragraph. Without the examples, abstract statements about 'doing', 'knowing' and 'understanding' mathematics cannot be expected to have more than a suggestive meaning. On the other hand the description of the examples will be easier to follow if the reader has a prior idea of their intention. And so the author will first sketch, very impressionistically, his position on some of the major issues. In doing so he will exploit the dialectical device employed in the previous paragraph to obtain a little more precision of statement by explicitly excluding the most likely misinterpretation.

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It is generally assumed in our society that every child should, and can, have experience of creative work in language and plastic arts. It is equally generally assumed that very few people can work creatively in mathematics. The author believes that there has been an unwitting conspiracy of psychologists and mathematicians in maintaining this assumption. The psychologists contribute to it out of genuine ignorance of what creative mathematical work might be like. The mathematicians, very often, do so out of elitism, in the form of a deep conviction that mathematical creativity is the privilege of a tiny minority.

Here again, if we want any clarity, it is necessary to ward off a too easy, superficial assent from mathematical education reformers who say, 'Yes, that's why we must use the method of discovery'. For, when 'discovery' means discovery this is wonderful, but in reality 'discovery' usually means something akin to the following fantasy about a poetry class: the discovery method teacher has perfected a series of questions that lead the class to discover the line 'Mary had a little lamb'. The author's point is not that this would be good or bad, but that no one would confuse it with creative work in poetry.

Is it possible for children to do creative mathematics (that is to say: to *do* mathematics) at all stages of their scholastic (and even adult!) lives? The author will argue that the answer is: yes, but a great deal of creative mathematical work by adult mathematicians is necessary to make it possible. The reason for the qualification is that the traditional branches of mathematics do not provide the most fertile ground for the easy, prolific growth of mathematical traits of mind. We may have to develop quite new branches of mathematics with the special property that they allow beginners more space to romp creatively, than does number theory or modernistic algebra. In the following pages will be found some specific examples which it would be pretentious to call 'new pedagogical oriented branches of mathematics' but which will suggest to co-operative readers what this phrase could mean.

Obstreperous readers will have no trouble finding objections. Mathematical elitists will say: 'How dare you bring these trivia to disturb our contemplation of the true mathematical structures'. Practical people will say: 'Romping? Pomping? Who needs it? What about practical skills in arithmetic?'

The snob and the anti-snob are expressing the same objection in different words. Let me paraphrase it, 'Traditional schools have found mathematics hard to teach to so-called average children. Someone brings along a new set of activities, which seem to be fun and easy to learn. He declares them to be mathematics! Well, that does not make them mathematics, and it does not turn them into solutions to any of the hard problems facing the world of mathematical education.'

This argument raises serious issues, from which the author will single out a question which he will ask in a number of different forms:

> In becoming a mathematician does one learn something other and more general than the specific content of particular mathematical topics? Is there such a thing as a Mathematical Way of Thinking? Can this be learned and taught? Once one has acquired it, does it then become quite easy to learn *particular* topics—like the ones that obsess our elitist and practical critics?

Psychologists sometimes react by saying, 'Oh, you mean the transfer problem'. But the author does not mean anything analogous to experiments on whether students who were taught algebra last year *automatically* learn geometry more easily than students who spent last year doing gymnastics. He is asking whether one can identify and teach (or foster the growth of) something *other* than algebra or geometry, which, once learned, will make it easy to learn algebra and geometry. No doubt, this other thing (let us call it the MWOT) can only be taught by using particular topics as vehicles. But the 'transfer' experiment is profoundly changed if the question is whether one can *use* algebra as a *vehicle for deliberately teaching transferable general concepts and skills.* The conjecture underlying this essay is a very qualified affirmative answer to this question. Yes, one *can* use algebra as a vehicle for initiating students to the mathematical way of thinking. But, to do so effectively one should first identify as far as possible components of the general intellectual skills one is trying to teach; and when this is done it will appear that algebra (in any traditional sense) is not a particularly good vehicle.

The alternative choices of vehicle described below all involve using computers, but in a way that is very different from the usual suggestions of using them either as 'teaching machines' or as 'super-slide-rules'. In our ideal of a school mathematical laboratory the computer is used as a means to control physical processes in order to achieve definite goals—for example as part of an auto-pilot system to fly model airplanes, or as the 'nervous system' of a model animal with balancing reflexes, walking ability, simple visual ability and so on. To achieve these goals, mathematical principles are needed; conversely, in this context, mathematical principles become sources of power, thereby acquiring meaning for large categories of students who fail to see any point or pleasure in bookish mathematics and who, under prevailing school conditions, simply drop out by labelling themselves 'not mathematically minded'.

The too easy acceptance of this takes the form: 'Yes, applications are motivating'. But 'motivation' fails to distinguish alienated work for a material or social reward from a true personal involvement. To develop this point the author needs to separate a number of aspects of the way the child relates to his work.

A simple, and important one, is the time scale. A child interested in flying model airplanes under computer control will work at this project over a long period. He will have time to try different approaches to sub-problems. He will have time to talk about it, to establish a common language with a collaborator or an instructor, to relate it to other interests and problems. This *project-oriented* approach contrasts with the *problem* approach of most mathematics teaching: a bad feature of the typical problem is that the child does not stay with it long enough to benefit much from success or from failure.

Along with time scale goes structure. A project is long enough to have recognizable phases—such as planning, choosing a strategy of attempting a very simple case first, finding the simple solution, *debugging it* and so on. And if the time scale is long enough, and the structures clear enough, the child can develop a vocabulary for articulate discussion of the process of working towards his goals.

The author believes in articulate discussion (in monologue or dialogue) of how one solves problems, of why one fails that one, of what gaps or deformations exist in one's knowledge and of what could be done about it. The author will defend this belief against two quite distinct objections. One objection says: '*it is impossible to verbalize;* problems are solved by intuitive acts of insight and these cannot be articulated'. The other objection says: '*it is bad to verbalize;* remember the centipede who was paralysed when the toad asked which leg came after which'.

One must beware of quantifier mistakes when discussing these objections. For example, Bruner tells us (in his book *Towards a Theory of Instruction*) that he finds words and diagrams 'impotent' in getting a child to ride a bicycle. But while his evidence shows (at best) that *some* words and diagrams are impotent, he suggests the conclusion that *all* words and diagrams are impotent. The interesting conjecture is this: the impotence of words and diagrams used by Bruner is explicable by his cultural origins; the vocabulary and conceptual framework of classical psychology is simply inadequate for the description of such dynamic processes as riding a bicycle! To push the rhetoric further, the author suspects that if Bruner tried to write a program to make an IBM 360 drive a radio controlled motorcycle, he would have to conclude (for the sake of consistency) that the order code of the 360 was impotent for this task. Now, in our laboratory we have studied how people balance bicycles and more complicated devices such as unicycles and circus balls. There is nothing complex or mysterious or indescribable about these processes. We can describe them in a non-impotent way *provided that* a suitable descriptive system has been set up in advance. Key components of the descriptive system rest on concepts like: the idea of a 'first order' or 'linear' theory in which control variables can be assumed to act independently; or the idea of feedback.

A fundamental problem for the theory of mathematical education is to identify and name the concepts needed to enable the beginner to discuss his mathematical thinking in a clear articulate way. And when we know such concepts we may want to seek out (or invent!) areas of mathematical work which exemplify these concepts particularly well. The next section of this paper will describe a new piece of mathematics with the property that it allows clear discussion and simple models of heuristics that are foggy and confusing for beginners when presented in the context of more traditional elementary mathematics.

Turtle Geometry: A Piece of Learnable and Lovable Mathematics

The physical context for the following discussion is a quintuple consisting of a child, a teletype machine, a computer, a large flat surface and an apparatus called a *turtle*. A turtle is a cybernetic toy capable of moving forward or back in a particular direction (relative to itself) and of rotating about its central axis. It has a *pen*, which can be in two states called PENUP and PENDOWN. The turtle is made to act by typing commands whose effect is illustrated in Figure 1.

At any time the turtle is at a particular *place* and facing in a particular *direction*. The place and direction together are the turtle's geometric *state*. Figure 1 shows the turtle in a field, used here only to give the reader a frame of reference.

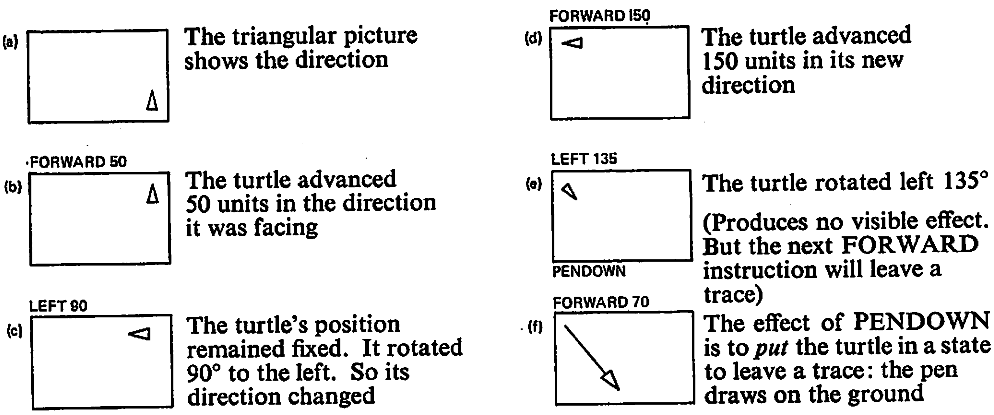


Figure 1. *Turtle language*

Direct Commands

The following commands will cause the turtle to draw Figure 2.

```
TO PEACE
PENDOWN
FORWARD 100
RIGHT 60
FORWARD 100
BACK 100
LEFT 120
FORWARD 100
END
```

PEACE

Figure 2. *A turtle procedure and the resulting diagram*

Defining a Procedure

The computer is assumed to accept the language LOGO (which we have developed expressly for the purpose of teaching children, not programming but mathematics). The LOGO idiom for asserting the fact that we are about to define a procedure is illustrated by the following example. We first decide on a *name* for the procedure. Suppose we choose 'PEACE'. Then we type:

```
TO PEACE
1 FORWARD 100
2 RIGHT 60
3 FORWARD 100
4 BACK 100
5 LEFT 120
6 FORWARD 100
END
```

These are directions telling the computer how to PEACE. The word 'TO' informs the computer that the next word, 'PEACE', is being defined and that the numbered lines constitute its definition

The turtle does not move while we are typing this. The word 'TO' and the line numbers indicated that we were not telling it to go forward and so on; rather we were telling it *how* to execute the new command. When we have indicated by the word 'END' that our definition is complete the machine echoes back:

```
PEACE DEFINED
```

and now if we type

```
PENDOWN
PEACE
```

the turtle will carry out the commands and draw Figure 2. Were we to omit the command 'PENDOWN' it would go through the motions of drawing it without leaving a visible trace.

The peace sign in Figure 2 lacks a circle. How can we describe a circle in turtle language?

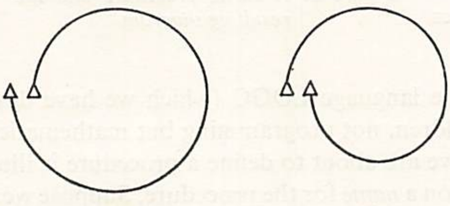
An idea that easily presents itself to mathematicians is: let the turtle take a tiny step forward, then turn a tiny amount and keep doing this. This might not quite produce a circle, but it is a good first plan, so let us begin to work on it. So we define a procedure:

```
TO CIRCUS
1 FORWARD 5
2 RIGHT 7
3 CIRCUS
END
```

Notice two features

1. The procedure refers to itself in line 3. This looks circular (though not in the sense we require) but really is not. The effect is merely to set up a never-ending process by getting the computer into the tight spot *you* would be in if you were the kind of person who cannot fail to keep a promise and you had been tricked into saying, 'I promise to repeat the sentence I just said'.

2. We selected the numbers 5 and 7 because they seemed small, but without a firm idea of what would happen. However, an advantage of having a computer is that we can try our procedure to see what it does. If an undesirable effect follows we can always *debug* it; in this case, perhaps, by choosing different numbers. If, for example, the turtle drew something like Figure 3(a), we would say to ourselves, 'It is not turning enough' and replace 7 by 8; on the other hand if it drew Figure 3(b) we might replace 7 by 6.



(a) (b)

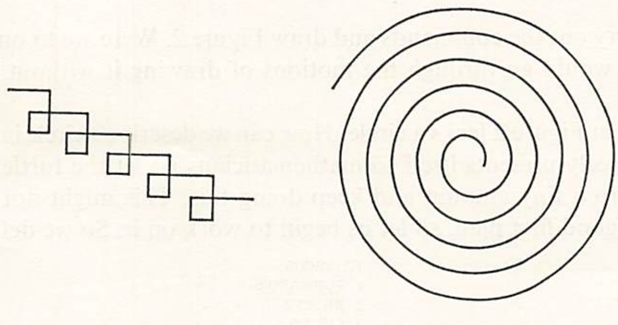
Figure 3. *Which kind of spiral is made by repeating FORWARD 5, RIGHT 7 ad int?*

The author wishes he could collect statistics about how many mathematically sophisticated readers fell into my trap! Experience shows that a large proportion of mathematics graduate students will do so. In fact, the procedure cannot generate either Figure 3(a) or 3(b)! If it did, it would surely go on to produce an infinite spiral. And one can easily see that this is impossible since the same sequence of commands would have to produce parts of the curve that are almost flat, and other parts that are very curved. More technically, one can see that the procedure CIRCUS *must* produce a close approximation to a circle (i.e. what is, for all practical purposes, a circle) because it must produce a curve of constant curvature.

One can come to the same conclusion from a more general theorem. We call procedures like CIRCUS 'fixed instruction procedures' because they contain no *variables*.

Theorem. Any figure generated by a fixed instruction procedure can be bounded either by a circle or by two parallel straight lines.

Examples of figures that can and that cannot be so bounded are shown in Figure 4.



(a) (b).

Figure 4. (*a*) *A figure bounded by parallel lines;* (*b*) *A figure bounded neither by parallel lines nor by a circle*

We now show how to make procedures with inputs in the sense that the command FORWARD has a number, called an input, associated with it. The next example shows how we do so. (The words on the title line preceded by ':' are names of the inputs, rather like the x's in school algebra.) In the fifth grade class we read *:NUMBER* as *dots*

NUMBER or as *the thing of 'NUMBER'*, emphasizing that what is being discussed is not the word 'NUMBER' but a thing of which this word is the name.

```
TO POLY :STEP :ANGLE
1 FORWARD :STEP
2 LEFT :ANGLE
3 POLY :STEP :ANGLE
END
```

This procedure generates a rather wonderful collection of pictures as we give it different inputs.

Although POLY has provision for inputs it is really a fixed instruction procedure. To create one that is not, we change the last line of POLY. We also change the title, though we do not need to do so.

Old Procedure

```
TO POLY :STEP :ANGLE
1 FORWARD :STEP
2 LEFT :ANGLE
3 POLY :STEP :ANGLE
END
```

New Procedure

```
TO POLYSPI :STEP :ANGLE
1 FORWARD :STEP
2 LEFT :ANGLE
3 POLYSPI :STEP+20 :ANGLE
END
```

The effect of POLYSPI is shown in Figure 5.

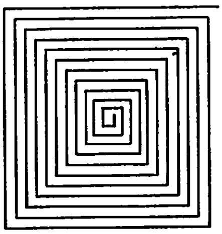


Figure 5. *POLYSPI 5 90 or squiral*

We have seen we can use POLY to draw a circle. Can we now use it to draw our PEACE sign? We could, but will do better to make a procedure, here called ARC whose effect will be to draw any circular segment given the diameter and the angle to be drawn as in Figure 6. The procedure is as follows where in line 2 a special constant called 'PIE' is used and the asterisk sign is used for multiplication. (Do not assume that :PIE is what its name suggests.)

```
TO ARC :DIAM :SECTOR
1 IF :SECTOR=0 STOP
2 FORWARD :PIE*:DIAM
3 RIGHT 1
4 ARC :DIAM :SECTOR-1
END
```

We can now make a procedure using the old procedure PEACE as a *sub-procedure*:

```
TO SUPERPEACE
1 ARC 200 360
2 RIGHT 90
3 PEACE
END
```

Figure 6. *A better peace sign using the old one as sub-procedure*

Better yet we could rewrite PEACE to have inputs. For example:

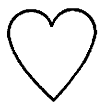
```
TO PEACE :SIZE
1 FORWARD :SIZE
2 RIGHT 60
3 FORWARD :SIZE
4 BACK :SIZE
5 LEFT 120
6 FORWARD :SIZE
7 RIGHT 90
8 ARC 2*:SIZE 360
```

Then peace signs of different sizes can be made by the commands:

```
PEACE 100
PEACE 20
```

and so on.
We can use the command ARC to draw a heart:

```
TO HEART :SIZE
1 ARC :SIZE/2 180
2 RIGHT 180
3 ARC :SIZE/2 180
4 ARC :SIZE*2 60
5 RIGHT 60
6 ARC :SIZE*2 60
END
```



Minitheorem: A heart can be made of four circular arcs.

We can also use it to draw a flower. Notice in the following the characteristic building of new definitions on old ones.

A computer program to draw this flower uses the geometric observation that petals can be decomposed (rather surprisingly!) as two quarter circles. So let us assume we have a procedure called TO QCIRCLE whose effect is shown by the examples. Some of them show initial and final positions of the turtle, some do not.

```
QCIRCLE 50
```

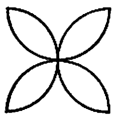
```
QCIRCLE 100
```

Now let us see how to make a petal.

```
TO PETAL :SIZE
1 QCIRCLE :SIZE
2 RIGHT 90
3 QCIRCLE :SIZE
END
```

PETAL 100

```
TO FLOWER :SIZE
1 PETAL :SIZE
2 PETAL :SIZE
3 PETAL :SIZE
4 PETAL :SIZE
END
```


FLOWER 100


STEM 100

```
TO STEM :SIZE
1 RIGHT 180
2 FORWARD 2*:SIZE
3 RIGHT 90
4 PETAL :SIZE/2
5 FORWARD :SIZE
END
```

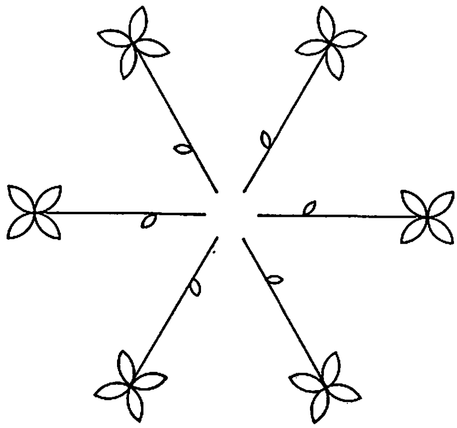

PLANT 50

```
TO PLANT :SIZE
1 PENDOWN
2 FLOWER :SIZE
3 STEM :SIZE
4 PENUP
END
```

Now let us play a little.

HEXAFLOWER 60

```
TO HEXAFLOWER :SIZE
1 RIGHT 90
2 FORWARD 4*:SIZE
3 PLANT :SIZE
4 FORWARD :SIZE
5 RIGHT 30
6 HEXAFLOWER :SIZE
END
```

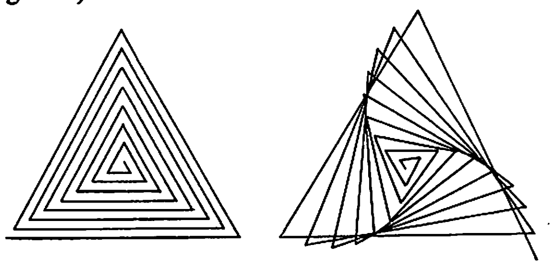


Creativity? Mathematics?

In classes run by members of the M.I.T. Artificial Intelligence Laboratory we have taught this kind of geometry to fifth graders, some of whom were in the lowest categories of performance in 'mathematics'. Their attitude towards mathematics as normally taught was well expressed by a fifth-grade girl who said firmly, 'There ain't nothing fun in math!' She did not classify working with the computer as mathematics, and we saw no reason to disabuse her. There will be time for her to discover that what she is learning to do in an exciting and personal way will elucidate those strange rituals she meets in the mathematics class.

Typical activities in early stages of work with children of this age is exploring the behaviour of the procedure POLY by giving it different inputs. There is inevitable challenge—and competition—in producing beautiful or spectacular, or just different, effects. One gets ahead in the game by discovering a new phenomenon and by finding out what classes of angles will produce it.

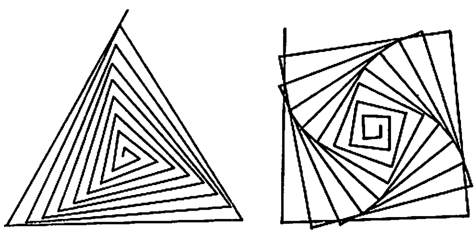
The real excitement comes when one becomes courageous enough to change the procedure itself. For example making the change to POLYSPI occurs to some children and, in our class, led to a great deal of excitement around the truly spontaneous discovery of the figure now called a *squiral* (Figure 5). (Note: By spontaneous the author means, amongst other things, to exclude the situation of the discovery teacher standing in front of the class soliciting pseudo-randomly generated suggestions. The squiral was found by a child sitting all alone at his computer terminal!) By no means all the children will take this step—indeed once a few have done so it becomes derivative for the others. Nevertheless, we might encourage them to explore inputs to POLYSPI. There is room here for the discovery of more phenomena. For example, taking :ANGLE as 120 produces a neat triangular spiral. But 123 produces a very different phenomena (as shown in Figure 7).



POLYSPI 5 120 POLYSPI 5 123

Figure 7. *Exploring some spiral effects*

What else produces similar effects? (See Figure 8.)



POLYSPI 5 121 POLYSPI 5 93

Figure 8. *Understanding Figure 7 better by exploring two related effects*

The possibilities for original minor discoveries are great. One girl became excited for the first time about mathematics by realizing how easy it was to make a program for Figure 9 by

1. Observing herself draw a similar figure.
2. *Naming* the elements of her figure—'BIG' and 'SMALL'—so that she could talk about them and so describe what she was doing.
3. Describing it in LOGO.

```
TO GROWSHRINK :BIG :SMALL
1 FORWARD :BIG
2 RIGHT 90
3 FORWARD :SMALL
4 RIGHT 90
5 GROWSHRINK :BIG-10 :SMALL+10
END
```

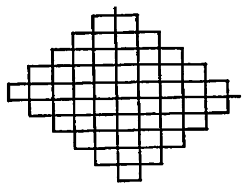


Figure 9. *Another direction of generalization of POLY. This figure shows the intention of the child who wrote the procedure GROWSHRINK*

The possibilities are endless. These are small discoveries. But perhaps one is already closer to mathematics in doing this than in learning new formal manipulations, transforming bases, intersecting sets and drifting through misty lessons on the difference between fractions, rationals and equivalence classes of pairs of integers. Perhaps learning to make small discoveries puts one more surely on a path to making big ones than does faultlessly learning any number of sound algebraic concepts.

Some Physical Mathematics

The turtle language is appropriate for many important physical problems. Consider, for example, the problem of understanding planetary orbits as if one were a junior high school student. One would find conceptual barriers of varying degrees of difficulty. Certainly the idea of the inverse square law is simple enough. Somewhat harder is the representation of velocities, accelerations and forces as vectors. But the insuperable difficulty in reading a text on the subject comes from the role of differential equations. The really elegant and intelligible physical ideas give rise to local differential descriptions of orbits; translating those into global ones usually involves going through the messy business called 'solving' differential equations.

Turtle geometry helps at all these points. The use of vectors is extremely natural. And the local differential description takes the form of a procedure that can be run so as to produce a drawing of a solution or studied using theorems and analytic concepts about procedures.

Control Theory as a Grade School Subject or Physics in the Finger Tips

We begin by inviting the reader to carry out the experiments illustrated in Figure 10—or to recall doing something similar.

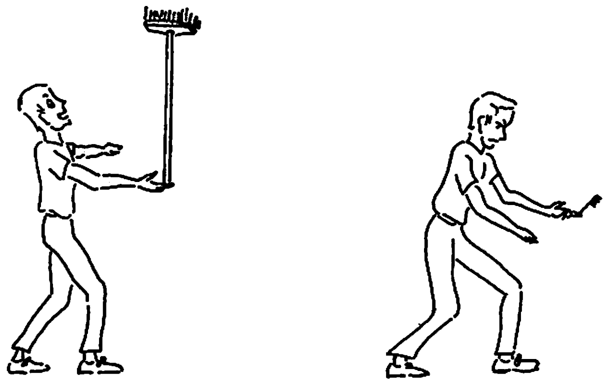


Figure 10. *Explaining this phenomenon leads into physics, mathematics, biology and the methodology of building formal models*

One of the goals of this unit of study will be to understand how people do this and particularly to understand what properties of a human being determine what objects he can and what objects he cannot balance.

A 'formal physical' model of the stick balancing situation is provided by the apparatus illustrated in Figure 11.

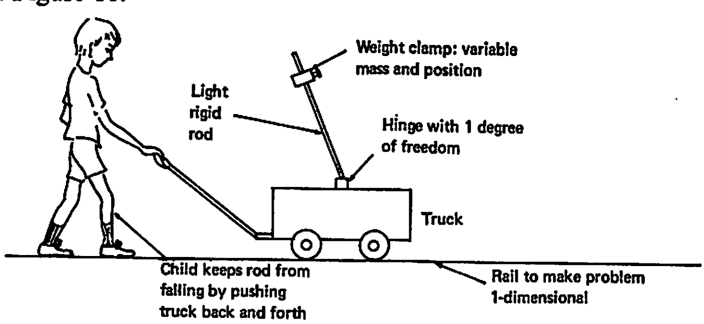


Figure 11. *A first step towards a model*

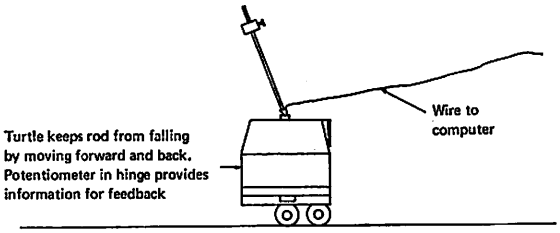


Figure 12. *A more formal model. The wire gives angle information. The child writes the program to balance the inverted pendulum*

A computer-controlled version replaces the track and the child by a turtle with the angle sensor plugged into its sensor socket (see Figure 12). A simple-minded procedure will do a fair amount of balancing (provided that the turtle is fast!):

```
TO BALANCE
1 TEST ANGLE > 10
2 IFTRUE FORWARD 8
3 TEST ANGLE < -10
4 IFTRUE BACK 8
5 WAIT 1
6 BALANCE
END
```

This procedure is written as part of a project plan that begins by saying: neglect all complications, try something. Complications that have been neglected include:

1. The end of the line bug.
2. The overshoot bug. (Perhaps in lines 2 and 4 the value 8 is too much or too little.)
3. The wobbly bug. The TEST in the procedure might catch the rod over to the left while it is in rapid motion towards the right. When this happens we should leave well alone!

One by one these bugs, and others, can be eliminated. It is not hard to build a program and choose constants so that with a given setting of the movable weight, balance will be maintained for long periods of time.

What are the Primitive Concepts of Mathematics?

To see points and lines as *the* primitive concepts of geometry is to forget not only the *logical* primitives (such as quantifiers) but especially the *epistemological* primitives, such as the notion of a mathematical system itself. For most children at school the problem is not that they do not understand *particular* mathematical structures or concepts. Rather, they do not understand what kind of thing a mathematical structure is: they do not see the point of the whole enterprise. Asking them to learn it is like asking them to learn poetry in a completely unknown foreign language.

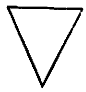
It is sometimes said that in teaching mathematics we should emphasize the process of mathematization. The author says: 'Excellent!' But on condition that the child should have the experience of mathematizing for himself. Otherwise the word 'mathematizing' is just one more scholastic term. The thrust of the explorations which have been described is to allow the child to have living experiences of mathematizing as an introduction to mathematics. We have seen how he mathematizes a heart, a squiral, his own behaviour in drawing a GROWSHRINK, the process of balancing a stick and so on. When mathematizing familiar processes is a fluent, natural and enjoyable activity, then is the time to talk about mathematizing mathematical structures, as in a good pure course on modern algebra.

But what are the ingredients of the process of mathematizing? Is it possible to formulate and teach knowledge about how one is to tackle, for example, the problem of setting up a mathematical representation of an object such as the hearts and flowers we discussed earlier?

Our answer is very definitely affirmative, especially in the context of the kind of work described above. Consider, for example, how we would teach children to go about problems like drawing a heart. First step we say: if you cannot solve the problem as it stands, try simplifying it; if you cannot find a complete solution, find a partial one. No doubt everyone gives similar advice. The difference is that in this context the advice is concrete enough to be followed by children who seem quite impervious to the usual mathematics.

A simplification of the heart problem is to settle, as a first approximation, on a triangle; which we then consider to be a very primitive heart.

```
TO TRI
1 FORWARD 100
2 RIGHT 120
3 FORWARD 100
4 RIGHT 120
5 FORWARD 100
END
```



TRI

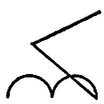
Now that we have this construction firmly in hand we can allow ourselves to modify it so as to make it a better heart. The obvious plan is to replace the horizontal line by a structure line. So we write a procedure to make this. First choose it a name, say 'TOP', then write:

```
TO TOP :SIZE
1 ARC :SIZE/2 180
2 RIGHT 180
3 ARC :SIZE/2 180
END
```

TOP

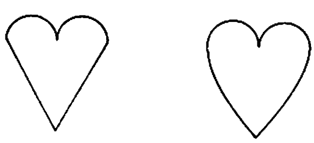
Replacing line 1 in TO TRI by TOP we get:

```
TO TRI
1 TOP 100
2 RIGHT 120
etc.
```



HEART WITH BUG

The effect is as shown! Is this a failure? We might have so classified it (and ourselves!) if we did not have another heuristic concept: BUGS and DEBUGGING. Our procedure did not fail. It has a perfectly intelligible *bug*. To find the bug we follow the procedure through in a very FORMAL way. (Formal is another concept we try to teach.) We soon find that the trouble is in line 2. Also we can see why. Replacing line 1 by TOP did what we wanted, but it also produced a SIDE-EFFECT. (Another important concept.) It left the turtle facing in a different direction. Correcting it is a mere matter of changing line 2 to RIGHT 30. And then we can go on to make the fully curved heart. Unless we decide that a straight-sided one is good enough for our purposes (see Figure 13).



(a) (b)

Figure 13. (*a*) *Straight-sided heart;* (*b*) *Curved heart*

Our image of teaching mathematics concentrates on teaching concepts and terminology to enable children to be articulate about the process of developing a mathematical analysis. Part of doing so is studying good models (such as the heart anecdote) and getting a lot of practice in describing one's own attempts at following the pattern of the model in other problems. It seems quite paradoxical that in developing mathematical curricula, whole conferences of superb mathematicians are devoted to discussing the appropriate language for expressing the *formal* part of mathematics, while the individual teacher or writer of textbooks is left to decide how (and even *whether*) to deal with heuristic concepts.

In summary, we have advanced three central theses:

1. The non-formal mathematical primitives are neglected in most discussions of mathematical curricula.
2. That the choice of content material, especially for the early years, should be made primarily as a function of its suitability for developing heuristic concepts.
3. Computational mathematics, in the sense illustrated by turtle geometry, has strong advantages in this respect over 'classical' topics.