Cases of Surjectivity of the Matrix Exponential into Lie Groups: $\mathfrak{so}(3) \to \mathbf{SO}(3)$ and $M_{n \times n}(\mathbb{C}) \to \mathbf{GL}_n(\mathbb{C})$

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1 Introduction

In this short paper, we prove some properties of the matrix exponential and logarithm, guiding us to two examples of when the exponential map is surjective: $\mathfrak{so}(3) \to \mathrm{SO}(3)$ and $M_{n \times n}(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$.

2 Matrix Lie Groups

Let $M_{n \times m}(\Omega)$ denote the set of $n \times m$ matrices over the set Ω. Following the convention of [1], we define a matrix lie group as a topologically closed subgroup of $\mathrm{GL}_n(\mathbb{C})$. All norms induce the same open sets in finite-dimensional vector spaces (see §1.4 of [2]), so we choose the operator norm: $\|\mathbf{X}\|_{\mathrm{op}} := \sup\{\|\mathbf{X}\mathbf{v}\|_2 : \|\mathbf{v}\|_2 = 1\}$, where $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{C}^n$.

3 The Matrix Exponential

Given a matrix $\mathbf{A} \in M_{n \times n}(\mathbb{C})$, we define the matrix exponential as

$$\exp(\mathbf{A}) := e^{\mathbf{A}} := \sum_{k=0}^{\infty} \frac{\mathbf{X}^k}{k!}.$$

This extends the familiar definition of exponentiation from $\mathbb{C}$ to square matrices over $\mathbb{C}$. Before we can prove that this power series converges, we need two facts. The first is Theorem 1.3.9 of *An Introduction to Banach Space Theory* [3]. We restate the theorem below but skip the proof for brevity.

Proposition 1. *Consider any series* $\mathbf{s}_m = \sum_{k=0}^{m} \mathbf{x}_k$ *in a complete normed vector space* $(X, \|\cdot\|)$. *If* $\sum_{k=0}^{\infty} \|\mathbf{x}_k\| < \infty$ *(i.e. the series converges absolutely), then* $\mathbf{s}_m$ *converges in* $(X, \|\cdot\|)$.

This allows us to prove the convergence of $e^{\mathbf{A}}$ by showing that it is absolutely convergent. Absolute convergence follows from our next proposition.

Proposition 2. *If* $\mathbf{X}, \mathbf{Y}$ *are any two elements of* $M_{n \times n}(\mathbb{C})$, *then* $\|\mathbf{X}\mathbf{Y}\|_{op} \leq \|\mathbf{X}\|_{op}\|\mathbf{Y}\|_{op}$.

Proof: Recall that $\|\mathbf{X}\mathbf{Y}\|_{\mathrm{op}} = \sup\{\|\mathbf{X}\mathbf{Y}\mathbf{v}\|_2 : \|\mathbf{v}\|_2 = 1\}$. Let $\mathbf{u} = \mathbf{Y}\mathbf{v}$. Then $\|\mathbf{u}\|_2 \leq \|\mathbf{Y}\|_{\mathrm{op}}\|\mathbf{v}\|_2 = \|\mathbf{Y}\|_{\mathrm{op}}$ by the definition of the operator norm. Likewise, $\|\mathbf{X}\mathbf{u}\|_2 \leq \|\mathbf{X}\|_{\mathrm{op}}\|\mathbf{u}\|_2 \leq \|\mathbf{X}\|_{\mathrm{op}}\|\mathbf{Y}\|_{\mathrm{op}}$. Combining the previous two inequalities yields $\|\mathbf{X}\mathbf{Y}\mathbf{v}\|_2 \leq \|\mathbf{X}\|_{\mathrm{op}}\|\mathbf{Y}\|_{\mathrm{op}}$ for all $\mathbf{v} \in \mathbb{C}^n$ with unit norm. Thus, the supremum over all such $\mathbf{v}$ will be less than or equal to $\|\mathbf{X}\|_{\mathrm{op}}\|\mathbf{Y}\|_{\mathrm{op}}$. $\square$

We can now easily show that $e^{\mathbf{A}}$ is convergent for any square matrix over $\mathbb{C}$. Note that $\|\mathbf{A}^m\|_{\mathrm{op}} \leq \|\mathbf{A}\|_{\mathrm{op}}^m$ by Proposition 2, so

$$\sum_{k=0}^{\infty} \left\|\frac{\mathbf{A}^k}{k!}\right\|_{\mathrm{op}} \leq \sum_{k=0}^{\infty} \frac{1}{k!}\|\mathbf{A}\|_{\mathrm{op}}^k = e^{\|\mathbf{A}\|_{\mathrm{op}}} < \infty.$$

So $e^{\mathbf{A}}$ converges by Proposition 1. Furthermore, this function is continuous since for any closed ball of radius $r > 0$ centered around $\mathbf{0}$ in $M_{n \times n}(\mathbb{C})$,

$$\left\|e^{\mathbf{X}} - \sum_{k=0}^{n} \frac{\mathbf{X}^k}{k!}\right\|_{\mathrm{op}} = \left\|\sum_{k=0}^{\infty} \frac{\mathbf{X}^k}{k!} - \sum_{k=0}^{n} \frac{\mathbf{X}^k}{k!}\right\|_{\mathrm{op}} = \left\|\sum_{k=n+1}^{\infty} \frac{\mathbf{X}^k}{k!}\right\|_{\mathrm{op}} \leq \sum_{k=n+1}^{\infty} \frac{1}{k!}\|\mathbf{X}\|_{\mathrm{op}}^k \leq \sum_{k=n+1}^{\infty} \frac{r^k}{k!}.$$

Since $\sum_{k=0}^{\infty} \frac{r^k}{k!}$ converges to e^r, the tail end of the partial sums must approach zero, meaning that for any $\varepsilon > 0$, we can choose an n_ε large enough that $\sum_{k=n_\varepsilon+1}^{\infty} \left(\frac{r^k}{k!}\right) < \varepsilon$. Since this is true for all $\mathbf{X}$ with $\|\mathbf{X}\|_{\mathrm{op}} \leq r$, the partial sums converge uniformly to $e^{\mathbf{X}}$ on this set. Since any compact set in $M_{n\times n}(\mathbb{C})$ can be covered by a closed ball of large enough radius, this means the series converges uniformly on all compact subsets of $M_{n\times n}(\mathbb{C})$. That means that the series is normally convergent, so the fact that the partial sums are continuous means that the limit function is continuous (See §5.2 of [4]).

Proposition 3. *For any* $\mathbf{X}, \mathbf{Y} \in M_{n\times n}(\mathbb{C})$, *the following facts hold.*

(a) *if* $\mathbf{X}$ *and* $\mathbf{Y}$ *commute, then* $e^{\mathbf{X}+\mathbf{Y}} = e^{\mathbf{X}}e^{\mathbf{Y}}$.

(b) *if* $\mathbf{C}$ *is invertible, then* $e^{\mathbf{C}\mathbf{X}\mathbf{C}^{-1}} = \mathbf{C}e^{\mathbf{X}}\mathbf{C}^{-1}$.

Proof of part (a): Because the matrix exponential converges absolutely, we can multiply the series term by term according to Merten's Theorem (Theorem 3.50 of [5]), so

$$e^{\mathbf{X}}e^{\mathbf{Y}} = (\mathbf{I} + \mathbf{X} + \frac{\mathbf{X}^2}{2!} + \frac{\mathbf{X}^3}{3!} + \cdots)(\mathbf{I} + \mathbf{Y} + \frac{\mathbf{Y}^2}{2!} + \frac{\mathbf{Y}^3}{3!} + \cdots).$$

The terms of combined degree m are the product of an $\mathbf{X}$ term of degree $k \leq m$ and a $\mathbf{Y}$ term of degree $m - k$, so the Cauchy product simplifies to

$$e^{\mathbf{X}}e^{\mathbf{Y}} = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{\mathbf{X}^k\mathbf{Y}^{m-k}}{k!(m-k)!} = \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{m}\frac{m!\mathbf{X}^k\mathbf{Y}^{m-k}}{k!(m-k!)} = \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}\mathbf{X}^k\mathbf{Y}^{m-k}. \qquad (3.1)$$

In (3.1) we have what looks like the binomial theorem. Note that the binomial coefficients assume that we can choose products in any order, i.e. that $\mathbf{X}$ and $\mathbf{Y}$ commute. Luckily, they do, so

$$e^{\mathbf{X}}e^{\mathbf{Y}} = \sum_{m=0}^{\infty}\frac{1}{m!}(\mathbf{X}+\mathbf{Y})^m = e^{\mathbf{X}+\mathbf{Y}}.$$

Proof of part (b): The key is to recognize that $(\mathbf{C}\mathbf{X}\mathbf{C}^{-1})^n = \mathbf{C}\mathbf{X}\mathbf{C}^{-1}\cdots\mathbf{C}\mathbf{X}\mathbf{C}^{-1} = \mathbf{C}\mathbf{X}^n\mathbf{C}^{-1}$. Thus,

$$
\begin{aligned}
e^{\mathbf{C}\mathbf{X}\mathbf{C}^{-1}} &= \lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{n!}(\mathbf{C}\mathbf{X}\mathbf{C}^{-1})^n \\
&= \lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{n!}\mathbf{C}\mathbf{X}^n\mathbf{C}^{-1} \\
&= \mathbf{C}\Big(\lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{n!}\mathbf{X}^n\Big)\mathbf{C}^{-1} \\
&= \mathbf{C}e^{\mathbf{X}}\mathbf{C}^{-1}. \quad \square
\end{aligned}
\qquad (3.2)
$$

Proposition 3 is a powerful tool for computing exponentials and will aid us in proving the following results.

4   The Matrix Logarithm

Just like for the exponential, we can use our existing definitions from the complex numbers to extend the logarithm to $M_{n\times n}(\mathbb{C})$. For $\mathbf{A} \in M_{n\times n}(\mathbb{C})$, we define the matrix logarithm as

$$\log(\mathbf{A}) = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{(\mathbf{A}-\mathbf{I})^m}{m}.$$

If we consider the absolute convergence of $\log\mathbf{A}$, we can see that

$$\sum_{m=0}^{\infty}\left\|(-1)^{m+1}\frac{(\mathbf{A}-\mathbf{I})^m}{m}\right\|_{\mathrm{op}} = \sum_{m=0}^{\infty}\frac{1}{m}\left\|(\mathbf{A}-\mathbf{I})^m\right\|_{\mathrm{op}} \leq \sum_{m=0}^{\infty}\|\mathbf{A}-\mathbf{I}\|_{\mathrm{op}}^m. \qquad (4.1)$$

Since $\|\mathbf{A}-\mathbf{I}\|_{\mathrm{op}} < 1$, (4.1) converges by the geometric series test. So $\log\mathbf{A}$ converges by Proposition 1.

Proposition 4. *log is continuous on the set* $\{\mathbf{A} \in M_{n \times n}(\mathbb{C}) : \|\mathbf{A} - \mathbf{I}\|_{op} < 1\}$.

Proof: Let $\mathbf{X} = \mathbf{A} - \mathbf{I}$. Then by assumption, $\|\mathbf{X}\|_{op} < 1$, and

$$\log \mathbf{A} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(\mathbf{A} - \mathbf{I})^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\mathbf{X}^n.$$

Let $R \in (0, 1)$. Let $\|\mathbf{X}\|_{op} \leq R$. Then

$$
\begin{aligned}
\left\| \log \mathbf{A} - \sum_{n=1}^{m} \frac{(-1)^{n+1}}{n}\mathbf{X}^n \right\|_{op} &= \left\| \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\mathbf{X}^n - \sum_{n=1}^{m} \frac{(-1)^{n+1}}{n}\mathbf{X}^n \right\|_{op} \\
&= \left\| \sum_{n=m+1}^{\infty} \frac{(-1)^{n+1}}{n}\mathbf{X}^n \right\|_{op} \\
&\leq \sum_{n=m+1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| \|\mathbf{X}\|_{op}^n \\
&\leq \sum_{n=m+1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| R^n.
\end{aligned}
\tag{4.2}
$$

Since $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} R^n = \log(R + 1)$, we know that the tail of the sum approaches 0, so as $m \to \infty$, the last line of (4.1) gets arbitrarily small. That is, for any $\varepsilon > 0$, there is some positive integer M such that $\sum_{n=M+1}^{\infty} \frac{(-1)^{n+1}}{n} R^n < \varepsilon$. Since this bound does not depend on which $\mathbf{X}$ we choose, the series converges uniformly to $\log \mathbf{A}$ on $\{\|\mathbf{A} - \mathbf{I}\|_{op} \leq R\}$. This is true for all $R \in (0, 1)$, so the series converges normally on $\{\|\mathbf{A} - \mathbf{I}\|_{op} < 1\}$. Since each of the partial sums is a continuous polynomial and the convergence is normal, the limit $(\log \mathbf{A})$ is continuous on $\{\|\mathbf{A} - \mathbf{I}\|_{op} < 1\}$. $\quad\square$

Proposition 5. *For all* $\mathbf{A}, \mathbf{B} \in M_{n \times n}(\mathbb{C})$:

  (a) *If* $\|\mathbf{A} - \mathbf{I}\|_{op} < 1$, *then* $\exp(\log \mathbf{A}) = \mathbf{A}$.

  (b) *If* $\mathbf{B}$ *has* $\|\mathbf{B}\|_{op} < \log 2$, *then* $\log(\exp \mathbf{B}) = \mathbf{B}$.

Proof of part (a): Let $\mathbf{X} = \mathbf{A} - \mathbf{I}$. Then $\exp(\log \mathbf{A}) = \mathbf{A}$ if and only if $\exp(\log(\mathbf{I} + \mathbf{X})) = \mathbf{I} + \mathbf{X}$. We break this proof down into two cases.

Case 1: X is diagonalizable. Then $\mathbf{X} = \mathbf{C}\mathbf{D}\mathbf{C}^{-1}$ where $\mathbf{D}$ is a diagonal matrix whose entries are the (not necessarily distinct) eigenvalues of $\mathbf{A}$. Since $\|\mathbf{X}\|_{op} < 1$ by assumption, each eigenvalue λ_i has modulus less than 1 (since otherwise $\mathbf{X}$ scales some vector by more than $\|\mathbf{X}\|_{op}$, which is impossible). Thus, $\lambda_i + 1$ is contained in the ball of radius 1 around 1 in $\mathbb{C}$, so $\log(\lambda_i + 1)$ converges. So

$$
\begin{aligned}
\log(\mathbf{X} + \mathbf{I}) &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\mathbf{X}^m \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}(\mathbf{C}\mathbf{D}\mathbf{C}^{-1})^m \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\mathbf{C} \begin{pmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^m \end{pmatrix} \mathbf{C}^{-1} \\
&= \mathbf{C} \begin{pmatrix} \log(\lambda_1 + 1) & 0 & \dots & 0 \\ 0 & \log(\lambda_2 + 1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \log(\lambda_n + 1) \end{pmatrix} \mathbf{C}^{-1}.
\end{aligned}
\tag{4.3}
$$

Since $|(\lambda_i + 1) - 1| < 1$, $e^{\log(\lambda_i + 1)} = \lambda_i + 1$. Thus,

$$
\exp(\log(\mathbf{X} + \mathbf{I})) = \mathbf{C}
\begin{pmatrix}
\lambda_1 + 1 & 0 & \ldots & 0 \\
0 & \lambda_2 + 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n + 1
\end{pmatrix}
\mathbf{C}^{-1}
$$
$$
= \mathbf{C}(\mathbf{D} + \mathbf{I})\mathbf{C}^{-1}
$$
$$
= \mathbf{C}\mathbf{D}\mathbf{C}^{-1} + \mathbf{I}
$$
$$
= \mathbf{X} + \mathbf{I}. \tag{4.4}
$$

Case 2: X is not diagonalizable. Then since the characteristic polynomial of $\mathbf{X}$ splits over $\mathbb{C}$, it has a Jordan normal decomposition $\mathbf{X} = \mathbf{B}\mathbf{J}\mathbf{B}^{-1}$, where $\mathbf{J}$ has the eigenvalues of $\mathbf{X}$ on its diagonal. Let $\mathbf{D}$ be the diagonal matrix with diagonal entries $1, 2, \ldots, n$. Then let $\mathbf{X}_k = \mathbf{B}(\mathbf{J} + \frac{1}{k}\mathbf{D})\mathbf{B}^{-1}$ define a sequence that converges to $\mathbf{X}$. $\mathbf{X}_k$ is written in Jordan normal form (since the addition of $\frac{1}{k}\mathbf{D}$ only changes the diagonal entries). Thus, the ith eigenvalue of $\mathbf{X}_k$ is $\mathbf{J}_{ii} + \frac{i}{k}$. For sufficiently large k, each of these values will be distinct, meaning $\mathbf{X}_k$ will be diagonalizable. Furthermore, since $\mathbf{X}_k \to \mathbf{X}$ and $\|\mathbf{X}\|_{\text{op}} < 1$, eventually $\|\mathbf{X}_k\|_{\text{op}} < 1$. By the continuity of log when $\|\mathbf{X}\|_{\text{op}} < 1$ and the global continuity of exp, we can swap limits for this sequence for large k, so

$$
\exp(\log(\mathbf{X} + \mathbf{I})) = \exp(\log(\lim_{k \to \infty}[\mathbf{X}_k + \mathbf{I}])) = \lim_{k \to \infty}\exp(\log(\mathbf{X}_k + \mathbf{I})) = \lim_{k \to \infty}\mathbf{X}_k + \mathbf{I} = \mathbf{X} + \mathbf{I}. \tag{4.5}
$$

Where we were able to cancel exp and log in the fourth expression using the diagonalizable case.

Proof of part (b): First we note that if $\|\mathbf{B}\|_{\text{op}} < \log 2$, then

$$
\|e^{\mathbf{B}} - \mathbf{I}\|_{\text{op}} = \|(\mathbf{I} + \mathbf{B} + \frac{\mathbf{B}^2}{2!} + \frac{\mathbf{B}^3}{3!} + \cdots) - \mathbf{I}\|_{\text{op}}
$$
$$
\leq \|\mathbf{B}\|_{\text{op}} + \left\|\frac{\mathbf{B}^2}{2!}\right\|_{\text{op}} + \left\|\frac{\mathbf{B}^3}{3!}\right\|_{\text{op}} + \cdots
$$
$$
= e^{\|\mathbf{B}\|_{\text{op}}} - 1
$$
$$
< e^{\log 2} - 1
$$
$$
= 1. \tag{4.6}
$$

Again, if $\mathbf{B}$ is diagonalizable, then $\mathbf{B} = \mathbf{C}\mathbf{D}\mathbf{C}^{-1}$, where $\mathbf{D}$ is a diagonal matrix of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{B}$ and $\mathbf{C}$ is invertible. Since the exponential of a diagonal matrix is taken entry-wise, $e^{\mathbf{D}}_{ii} = e^{\lambda_i}$. Furthermore, these are the eigenvalues of $e^{\mathbf{B}}$ because $e^{\mathbf{B}} = \mathbf{C}e^{\mathbf{D}}\mathbf{C}^{-1}$ is a diagonalization of $e^{\mathbf{B}}$. To get a bound on the modulus of these eigenvalues, consider that subtracting $\mathbf{I}$ from a matrix decreases all its eigenvalues by 1, so the eigenvalues of $e^{\mathbf{B}} - \mathbf{I}$ are $e^{\lambda_i} - 1$. Since $\|e^{\mathbf{B}} - \mathbf{I}\|_{\text{op}} < 1$, each eigenvalue must have modulus less than 1 as shown in case 1 of part (a). Thus, $|e^{\lambda_i} - 1| < 1$, so $\log e^{\lambda_i} = \lambda_i$. Since the log of a diagonal matrix is also taken entry-wise [as we showed in equation (4.2)], that means $\log(e^{\mathbf{D}}) = \mathbf{D}$. In (4.2), we also showed that $\log(\mathbf{C}\mathbf{D}\mathbf{C}^{-1}) = \mathbf{C}\log(\mathbf{A})\mathbf{C}^{-1}$ for invertible $\mathbf{C}$ and diagonal $\mathbf{A}$. Thus,

$$
\log e^{\mathbf{B}} = \log(\mathbf{C}e^{\mathbf{D}}\mathbf{C}^{-1})
$$
$$
= \mathbf{C}\log(e^{\mathbf{D}})\mathbf{C}^{-1}
$$
$$
= \mathbf{C}\mathbf{D}\mathbf{C}^{-1}
$$
$$
= \mathbf{B}. \tag{4.7}
$$

In the case that $\mathbf{B}$ is not diagonalizable, we can approximate it by a sequence of diagonalizable matrices like in part (a), using the continuity of log and exp to get the desired result. $\square$

5 Cases of Surjectivity of The Exponential

We now have the tools needed to investigate cases where the exponential map is surjective.

5.1 Surjectivity of $\exp : M_{n \times n}(\mathbb{C}) \to \mathbf{GL}_n(\mathbb{C})$

To prove that this map is surjective, we follow the outline of exercises 2.8 and 2.9 from *Lie Groups, Lie Algebras, and Representations* [1]. First we require some definitions. A square matrix $\mathbf{N}$ is said to be nilpotent if there exists a positive integer k such that $\mathbf{N}^k = \mathbf{0}$. A square matrix $\mathbf{U}$ is unipotent if $\mathbf{U} - \mathbf{I}$ is nilpotent.

Proposition 6. *If $\mathbf{U}$ is unipotent, then $\log(\mathbf{U})$ is nilpotent, and if $\mathbf{N}$ is nilpotent, then $\exp(\mathbf{N})$ is unipotent.*

Proof: We first must show that finite sums of commuting nilpotent matrices are nilpotent. Let $\mathbf{N}, \mathbf{M}$ be nilpotent matrices that commute with eachother. Then there exists some $k \in \mathbb{N}$ such that $\mathbf{N}^k = \mathbf{M}^k = \mathbf{0}$. By the binomial theorem: $(\mathbf{N} + \mathbf{M})^p = \sum_{j=1}^{p} \binom{p}{j} \mathbf{N}^j \mathbf{M}^{p-j}$ (this is where we needed commuting matrices). If we choose $p \geq 2k$, then $p - j \geq k$ or $j \geq k$ since $(p-j) + j = p \geq 2k$. Thus, for every term in $(\mathbf{N} + \mathbf{M})^p$, either $\mathbf{N}^j$ or $\mathbf{M}^{p-j}$ is $\mathbf{0}$, meaning the whole sum is $\mathbf{0}$. So, $\mathbf{N} + \mathbf{M}$ is nilpotent. Continuing inductively, we can see this is true for any finite sum (since any term in the sum will commute).

Now we can prove the proposition. Since $\mathbf{U}$ is unipotent, $\mathbf{U} - \mathbf{I}$ is nilpontent, so there are finitely many $k \in \mathbb{N}$ such that $(\mathbf{U} - \mathbf{I})^k \neq \mathbf{0}$. The power series of $\log(\mathbf{U})$ becomes

$$\log(\mathbf{U}) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (\mathbf{U} - \mathbf{I})^m = \sum_{m=1}^{k} \frac{(-1)^{m+1}}{m} (\mathbf{U} - \mathbf{I})^m. \tag{5.1}$$

Where for the last equality we used the fact that eventually every summand is $\mathbf{0}$. Since each term in the sum is a scaled positive power of $\mathbf{U} - \mathbf{I}$, the terms all commute. Thus, this sum is nilpotent, proving part one of the proposition.

For part two, since $\mathbf{N}$ is nilpotent, there are finitely many $k \in \mathbb{N}$ such that $\mathbf{N}^k \neq \mathbf{0}$. The power series expansion of $\exp(\mathbf{N})$ can be rewritten as

$$\exp(\mathbf{N}) = \sum_{m=0}^{\infty} \frac{1}{m!} \mathbf{N}^m = \sum_{m=0}^{k} \frac{1}{m!} \mathbf{N}^m = \mathbf{I} + \sum_{m=1}^{k} \frac{1}{m!} \mathbf{N}^m. \tag{5.2}$$

Since the terms of the summation are all scaled positive powers of $\mathbf{N}$, they commute, so the summation is nilpotent. Thus, $\exp(\mathbf{N})$ is unipotent. $\square$

These facts help us expand the cases where log is the inverse of exp, as the next proposition will show.

Proposition 7. *If $\mathbf{U}$ is unipotent, then $\exp(\log \mathbf{U}) = \mathbf{U}$, and if $\mathbf{N}$ is nilpotent, then $\log(\exp \mathbf{N}) = \mathbf{N}$.*

Proof:. Let $\gamma(t) = \mathbf{I} + t(\mathbf{U} - \mathbf{I})$ be a smooth curve in $M_{n \times n}(\mathbb{C})$ parameterized by $t \in \mathbb{R}$. By Proposition 6, since $\mathbf{U}$ is unipotent, the power series of $\log \mathbf{U}$ is finite, hence a polynomial of t. Since $\log \mathbf{U}$ is nilpotent, the power series of $\exp \log(\mathbf{U})$ is finite by Proposition 6, so it is a polynomial over t. Furthermore, $\|\gamma(t) - \mathbf{I}\|_{\text{op}} = \|t(\mathbf{U} - \mathbf{I})\|_{\text{op}}$, which approaches zero as t does. Thus, we can pick some $\varepsilon > 0$ such that for all $t \in [0, \varepsilon] : \|\gamma(t) - \mathbf{I}\|_{\text{op}} < 1$. Thus, $\exp(\log \gamma(t)) = \gamma(t)$ on $[0, \varepsilon]$ by Proposition 5. If we consider $[0, \varepsilon]$ to be a subset of $\mathbb{C}$, then for the ith row and j^{th} column of our matrices, $\exp \log(\gamma(t))_{i,j}$ and $\gamma(t)_{i,j}$ are two polynomials over $\mathbb{C}$ that agree on $[0, \varepsilon]$, which has a nonisolated point. By the identity principle (See §5.7 of [4]), this means $\exp \log(\mathbf{U}(t))_{i,j} = \mathbf{U}(t)_{i,j}$ on all of $\mathbb{C}$. Since all entries agree, $\exp(\log(\gamma(t)) = \gamma(t)$ on all of $\mathbb{C}$, namely at $t = 1$. Since $\gamma(1) = \mathbf{U}$, we have $\exp(\log \mathbf{U}) = \mathbf{U}$ for all unipotent matrices.

The proof proceeds similarly for $\mathbf{N}$. If we let $\alpha(t) = t\mathbf{N}$, then as a consequence of Proposition 7, $\log(\exp \alpha(t))$ depends polynomially on t. Since $\|\mathbf{N}(t)\|_{\text{op}} = |t| \|\mathbf{N}\|_{\text{op}}$, we can again choose a small $\varepsilon > 0$ such that for all $t \in [0, \varepsilon] : \|\alpha(t)\|_{\text{op}} < \log(2)$. Thus, on $[0, \varepsilon]$, $\log(\exp \alpha(t)) = \alpha(t)$. Applying the identity principle element-wise as done above will yield the desired result. $\square$

With Propositions 5 and 6 under our belts, we are ready to prove $\exp : M_{n \times n}(\mathbb{C}) \to \text{GL}_n(\mathbb{C})$ is surjective. From the fundamental theorem of algebra, the characteristic polynomial of any matrix $\mathbf{A} \in \text{GL}_n(\mathbb{C})$ will split over $\mathbb{C}$. Thus, there is a Jordan normal decomposition of the form $\mathbf{A} = \mathbf{CJC}^{-1}$ where $\mathbf{C}$ is an invertible matrix and $\mathbf{J}$ is a Jordan block matrix (see §7 of [6]). Jordan block matrices are of the form

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{J}_k \end{bmatrix}. \tag{5.3}$$

Where each $\mathbf{J}_i = \lambda_i \mathbf{I} + \mathbf{N}_i$ for a nilpotent matrix $\mathbf{N}_i$ and a nonzero complex number λ_i (the λ_i are nonzero since $\mathbf{A}$ is invertible). We first show that any Jordan block is in the image of the exponential, then show that combining the Jordan block preimages results in a preimage for $\mathbf{J}$. Let $\mathbf{B} = \lambda \mathbf{I} + \mathbf{N} = \lambda \mathbf{I}(\mathbf{I} + \mathbf{N}/\lambda)$ be any Jordan block. Since $\lambda \neq 0$, there exists some $z \in \mathbb{C}$ such that $\lambda = e^z$. Thus, $\lambda \mathbf{I} = \exp(z\mathbf{I})$ (This is because exponentials of diagonal matrices are taken entry-wise). So we have $\mathbf{B} = \exp(z\mathbf{I})(\mathbf{I} + \mathbf{N}/\lambda)$. Since $(\mathbf{I} + \mathbf{N}/\lambda)$ is unipotent, $\exp(\log(\mathbf{I} + \mathbf{N}/\lambda)) = \mathbf{I} + \mathbf{N}/\lambda$ by Proposition 7. Thus, $\mathbf{B} = \exp(z\mathbf{I})\exp(\log(\mathbf{I} + \mathbf{N}/\lambda))$. Scalar multiples of the identity matrix commute with any other matrix , so $\mathbf{B} = \exp(z\mathbf{I} + \log(\mathbf{I} + \mathbf{N}/\lambda))$ by Proposition 3. Thus, any invertible Jordan block is in the image of the exponential.

If we let $\mathbf{L}_i = z_i\mathbf{I} + \log(\mathbf{I} + \mathbf{N}_i/\lambda_i)$ be the preimage of $\mathbf{J}_i$, then substituting $\mathbf{L}_i$ into the position of $\mathbf{J}_i$ for each $i \in \{1, \dots, k\}$ yields a matrix $\mathbf{L}$ whose image under exp is $\mathbf{J}$. This is because every power of $\mathbf{L}$ looks like

$$\mathbf{L}^m = \begin{bmatrix} \mathbf{L}_1^m & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2^m & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{L}_k^m \end{bmatrix}. \tag{5.4}$$

Taking the limit of the partial sums of $\exp(\mathbf{L})$ yields

$$\exp(\mathbf{L}) = \lim_{m \to \infty} \begin{bmatrix} \mathbf{L}_1^m/m! & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2^m/m! & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{L}_k^m/m! \end{bmatrix} = \begin{bmatrix} \exp(\mathbf{L}_1) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \exp(\mathbf{L}_2) & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \exp(\mathbf{L}_k) \end{bmatrix}. \tag{5.5}$$

Where we used the fact that limits of matrices are taken element wise. Since $\exp(\mathbf{L_i}) = \mathbf{J}_i$, this matrix is equal to $\mathbf{J}$. Finally, since $\mathbf{C}$ is invertible, $\exp(\mathbf{CLC}^{-1}) = \mathbf{C}\exp(\mathbf{L})\mathbf{C}^{-1} = \mathbf{CJC}^{-1} = \mathbf{A}$ by Proposition 3, concluding the proof that $\exp : M_{n \times n}(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ is surjective $\quad \square$

5.2 Surjectivity of $\exp : \mathfrak{so}(3) \to \mathbf{SO}(3)$

The following definition is taken from §2.5 of *Lie Groups, Lie Algebras, and Representations* [1]. Given a matrix Lie Group G, its Lie algebra, denoted $\mathfrak{g}$, is the set of matrices $\mathbf{X}$ such that for all $t \in \mathbb{R}$: $e^{t\mathbf{X}} \in G$. The Lie algebra is of special interest in group theory because it is a vector space under the bracket operation, enabling us to study Lie groups via their Lie algebras using the tools of linear algebra [1]. While we won't explore Lie algebras deeply, we *are* concerned with the image of the exponential on the lie algebra of SO(3), which we denote $\mathfrak{so}(3)$. As shown in section 5.4 of [7], the lie algebra of $\mathfrak{so}(3)$ is the set of skew-symmetric real matrices.

Proposition 8. *The map* $\exp : \mathfrak{so}(3) \to SO(3)$ *is surjective.*

Proof: In the standard basis, a rotation through the x-axis by angle θ can be represented in the form

$$\mathbf{R}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{5.6}$$

By Euler's Rotation Theorem (Euler 1775, [8]), $\mathbf{R}$ is equivalent to a rotation about an axis spanned by a unit vector $\mathbf{v} \in \mathbb{R}^3$. If we consider any set of linearly independent vectors where $\mathbf{v}$ is our first vector, then the Graham Schmidt process will generate an orthonormal basis $\{\mathbf{v}, \mathbf{u}, \mathbf{w}\}$. In this basis, $\mathbf{v}$ remains fixed under transformation by $\mathbf{R}$, and $\mathbf{u}, \mathbf{v}$ span the plane which $\mathbf{R}$ induces a rotation of angle θ through. Thus, in our new basis, we may represent $\mathbf{R}$ in the form of (5.6). In the language of similarity,

$\mathbf{R} = \mathbf{B}\mathbf{R}_x\mathbf{B}^{-1}$ for an invertible change-of-basis matrix $\mathbf{B}$. We would now like to show that $\mathbf{R}_x$ is in the image of exp. Consider the matrix

$$\mathbf{X} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\theta \\ 0 & \theta & 0 \end{pmatrix}.$$

The powers of $\mathbf{X}$ are cyclic, since

$$\mathbf{X}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\theta^2 & 0 \\ 0 & 0 & -\theta^2 \end{pmatrix}$$

$$\mathbf{X}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta^3 \\ 0 & -\theta^3 & 0 \end{pmatrix}$$

$$\mathbf{X}^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \theta^4 & 0 \\ 0 & 0 & \theta^4 \end{pmatrix}$$

$$\mathbf{X}^5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\theta^5 \\ 0 & \theta^5 & 0 \end{pmatrix} = \theta^4\mathbf{X}.$$

Using this fact we can show that the explicit formula for $e^{\mathbf{X}}$ is

$$e^{\mathbf{X}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix}. \tag{5.7}$$

Where the 1 in the first entry comes from the identity term $\mathbf{X}^0 = \mathbf{I}$, and the remaining terms come from inspecting the entries of successive powers of $\mathbf{X}$. Computing them yields

$$
\begin{aligned}
\alpha &= 1 + \frac{0}{1!} - \frac{\theta^2}{2!} + \frac{0}{3!} + \frac{\theta^4}{4!} + \cdots = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots = \cos(\theta), \\
\beta &= 0 - \frac{\theta}{1!} + \frac{0}{2!} + \frac{\theta^3}{3!} + \frac{0}{4!} + \cdots = -\theta + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \cdots = -\sin(\theta), \\
\gamma &= 0 + \frac{\theta}{1!} + \frac{0}{2!} - \frac{\theta^3}{3!} + \frac{0}{4!} + \cdots = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots = \sin(\theta), \\
\delta &= 1 + \frac{0}{1!} - \frac{\theta^2}{2!} + \frac{0}{3!} + \frac{\theta^4}{4!} + \cdots = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots = \cos(\theta).
\end{aligned}
\tag{5.8}
$$

Which shows that $e^{\mathbf{X}} = \mathbf{R}_x$. Thus,

$$e^{(\mathbf{B}\mathbf{X}\mathbf{B}^{-1})} = \mathbf{B}e^{\mathbf{X}}\mathbf{B}^{-1} = \mathbf{B}\mathbf{R}_x\mathbf{B}^{-1} = \mathbf{R}. \tag{5.9}$$

If we can show that $\mathbf{B}\mathbf{X}\mathbf{B}^{-1}$ is skew-symmetric, then we are done. Recall that $\mathbf{B}$ is the change of basis matrix from the standard basis to the orthonormal basis $\{\mathbf{v}, \mathbf{u}, \mathbf{w}\}$. These are two orthonormal bases, so $\mathbf{B}$ is orthogonal (see §14.3 of [9]). So $\mathbf{B}^{\mathsf{T}} = \mathbf{B}^{-1}$ and

$$
\begin{aligned}
\mathbf{B}\mathbf{X}\mathbf{B}^{-1} &= \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\theta \\ 0 & \theta & 0 \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \\
&= \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -\theta c & -\theta f & -\theta i \\ \theta b & \theta e & \theta h \end{pmatrix} \\
&= \begin{pmatrix} \theta bc + \theta bc & -\theta bf + \theta ec & -\theta bi + \theta ch \\ -\theta ec + \theta bf & -\theta ef + \theta ef & -\theta ei + \theta fh \\ -\theta ch + \theta bi & -\theta fh + \theta ei & -\theta hi + \theta hi \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\theta bf + \theta ec & -\theta bi + \theta ch \\ -(-\theta bf + \theta ec) & 0 & -\theta ei + \theta fh \\ -(-\theta bc + \theta ch) & -(\theta ei + \theta fh)ei & 0 \end{pmatrix}.
\end{aligned}
\tag{5.10}
$$

So $\mathbf{B}\mathbf{X}\mathbf{B}^{-1}$ is skew-symmetric, concluding the proof that $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$ is surjective. $\quad\square$

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