Fourier transform, key concepts for image analysis

1. Convolution
2. Aliasing

Fourier Transform (FT)

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2i\pi kx} dx$$

Discrete Time Fourier Transform (DTFT)

$$F[k] = \sum_{n=-\infty}^{\infty} f[n] e^{-2i\pi kn}$$

Periodic
F[k] =F[k+m]
m: integer

or

$$F[\mu] = \sum_{-\infty}^{\infty} f[n] e^{-i2\pi\mu n\Delta T}$$

$$\mu = \frac{k}{\Delta T}$$

Discrete Fourier Transform (DFT)

$$F[k] = \sum_{n=0}^{N-1} f[n] e^{-2i\pi\frac{kn}{N}}$$

Periodic
F[k] =F[k+m*N]

Definition up to a normalization term!

The convolution theorem

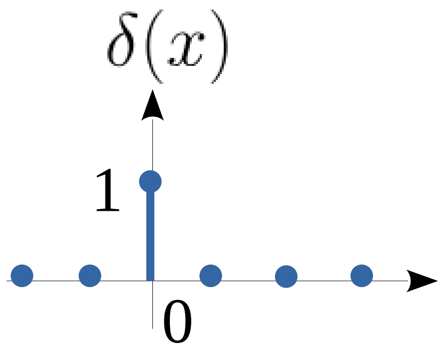
Convolution

Reminder

$$(f*w)(x)=\sum_{k\in\mathbb{Z}} f(k)\,w(x-k)=\sum_{s\in\mathbb{Z}} f(x-s)\,w(s)$$

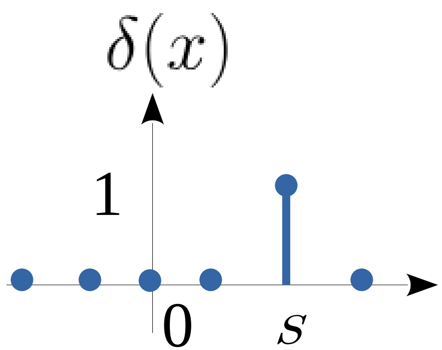
Neutral element:
Kronecker delta

$$(f*\delta)(x)=f(x)$$



Shifted delta:

$$\delta_s(x)=\delta(x-s)$$



$$(f*\delta_s)(x)=f(x-s)$$

A convolution with a shifted delta shifts the function

Illustrating the convolution theorem with a first example: translation in space

- Discrete Time Fourier Transform $$F(k) = \sum_n f(n) e^{-2i\pi nk}$$
- Discrete Time Fourier Transform of the shifted kronecker delta
$$\sum_n \delta_s(n) e^{-2i\pi nk} = e^{-2i\pi sk}$$

$$\sum_{k=-\infty}^{\infty} \underbrace{F(k)}_{\text{FT of f}} \underbrace{e^{-j2\pi ks}}_{\text{FT of }\delta} e^{j2\pi kn} = \sum_{k=-\infty}^{\infty} F(k) e^{j2\pi k(n-s)} = f(n-s) = \underbrace{f * \delta_s(n)}_{\text{Convolution}}$$

FT of f FT of δ Convolution

Multiplication of 2 FTs

- Shift in time (or space) is a multiplication by a phase factor in Fourier

$$f * \delta_s \Leftrightarrow F(k)\, e^{-j2\pi k s}$$

Convolution Multiplication

- Vice versa: Fourier transform of $f(n)\, e^{j2\pi s n}$

- A shift in Frequency:

$$\sum_{n=-\infty}^{\infty} f(n)\, e^{j2\pi k s} e^{-j2\pi k n} = \sum_{n=-\infty}^{\infty} f(t)\, e^{-j2\pi(k-s)t} = F(k-s)$$

Multiplication

Convolution with δ
In the Fourier space!

Convolution Theorem

- General property, for any function f and g

$$(f * h)(t) \longleftrightarrow F(\mu)H(\mu)$$

$$f(t)h(t) \longleftrightarrow (F * H)(\mu)$$

Time or space

Fourier space

It works for continuous FT and DTFT
and DFT (with a slight adjustement)!

Important concept! Remember spatial filtering: filter using convolution
We can now filter using a simple multiplication!

PROOF

We will do that in the exercise session.
(hint: you need to make a change of variables)

But you can see a proof of the convolution theorem for example here:

https://tinyurl.com/4kuzuwe9

Discrete Fourier Transform and Convolution

Convolution theorem holds with a *cyclic convolution* for the DFT

DFT(f*w)[k]= F[k]W[k]   DFT(fw) = (F * W)[k]

with the cyclic convolution:

$$(f * w)[n] = \sum_{m=0}^{N-1} f[m]w\left[(n-m)_N\right]$$

$(n-m)_N$ is (n-m) modulo N (stays in the range [0,N])

DFT **periodizes** the function/filter, «live on a circle»

Discrete Fourier Transform and periodicity

DFT **periodizes** the function/filter, «live on a circle»

DFT and inverse DFT:

$$F[k] = \sum_{n=0}^{N-1} f[n] e^{-2i\pi \frac{kn}{N}} \qquad\qquad f[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k] e^{i2\pi k \frac{n}{N}}$$

The complex exponential periodize the function
The variable n can have values > N

- Remark: Zero-padding reduces the effect of periodization

We will see now the link between discrete FTs and periodization and its consequences

Aliasing

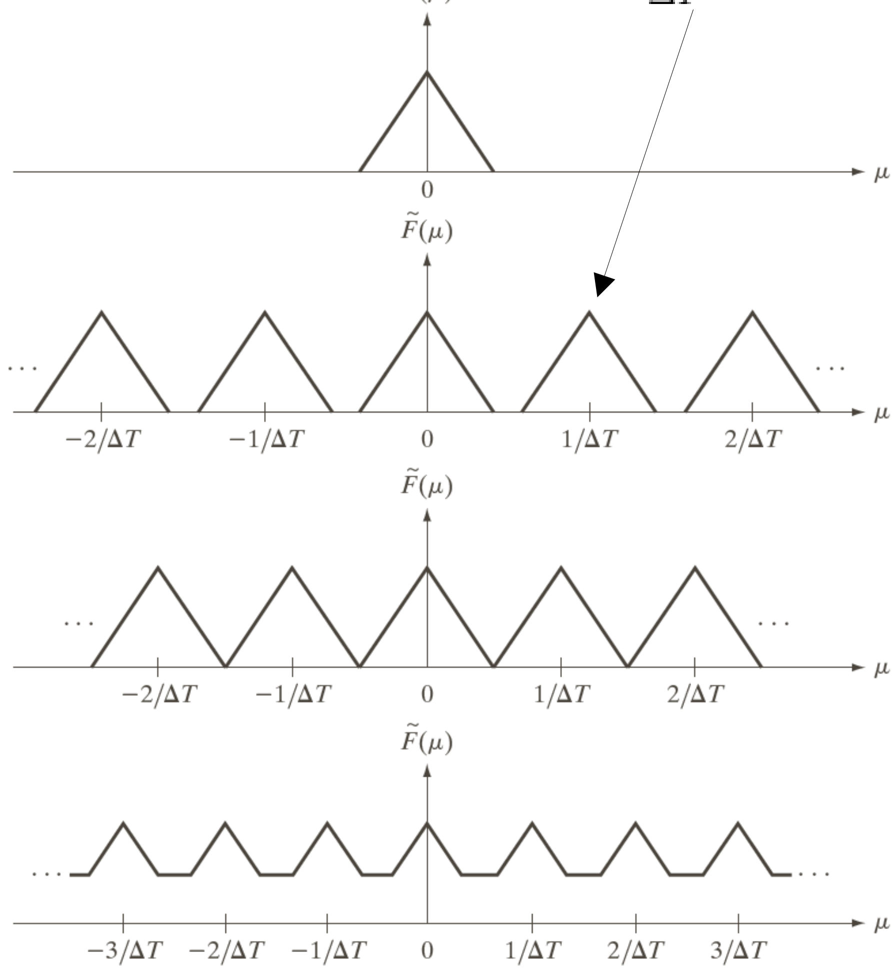
Discretization -> frequency periodization

$$F[\mu] = \sum_{-\infty}^{\infty} f[n] e^{-i2\pi\mu n\Delta T}$$

$$\mu = \frac{k}{\Delta T}$$

$$F[\mu + \frac{m}{\Delta T}] = F[\mu] \qquad m \in \mathbb{N}$$

Decreasing
Sampling frequency
$1/\Delta T$



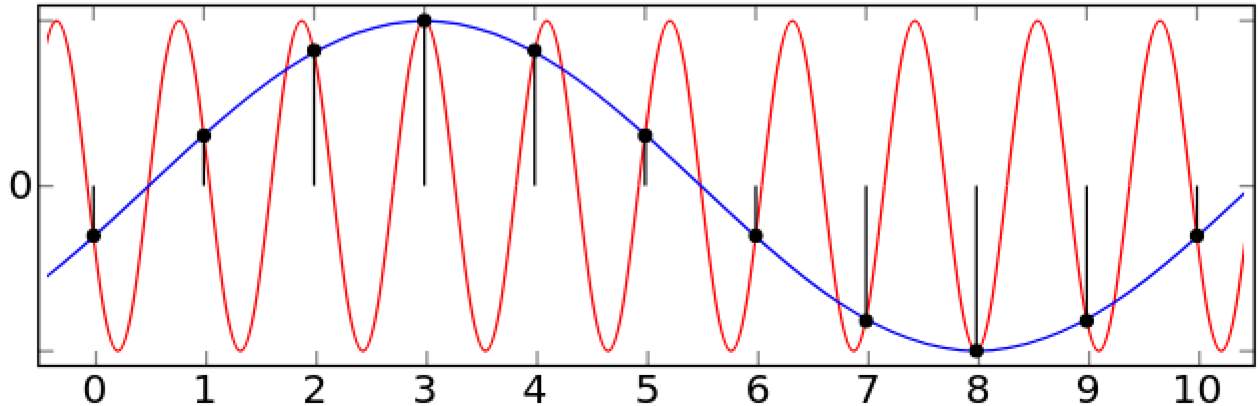
a
b
c
d

FIGURE 4.6 (a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

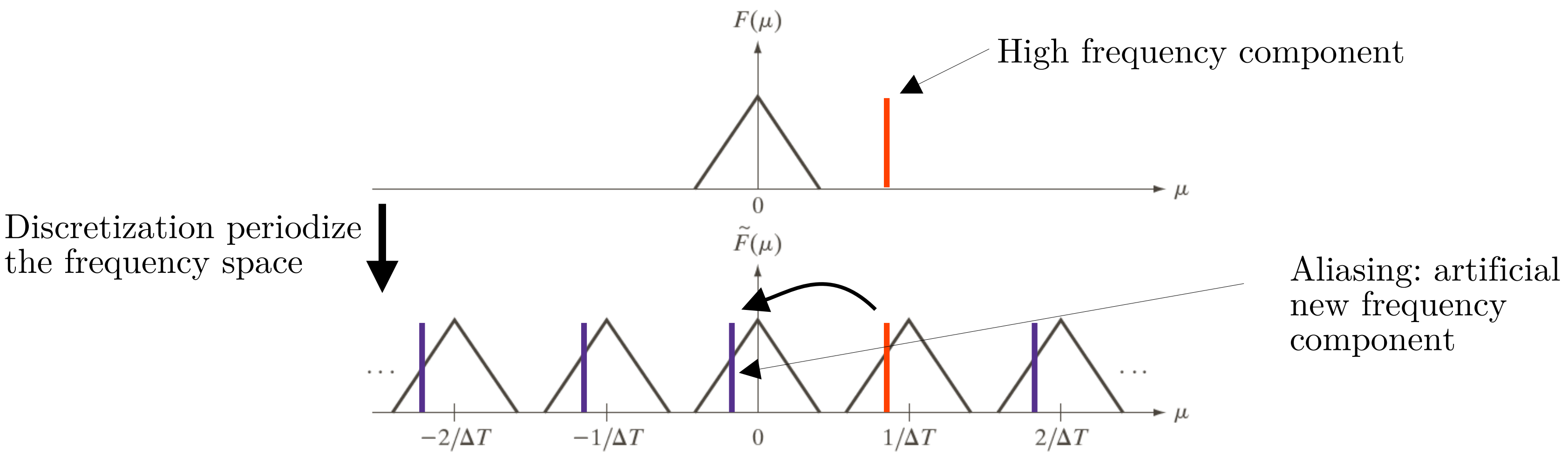
Problem!

Sampling frequency is twice the max signal frequency

Aliasing phenomenon

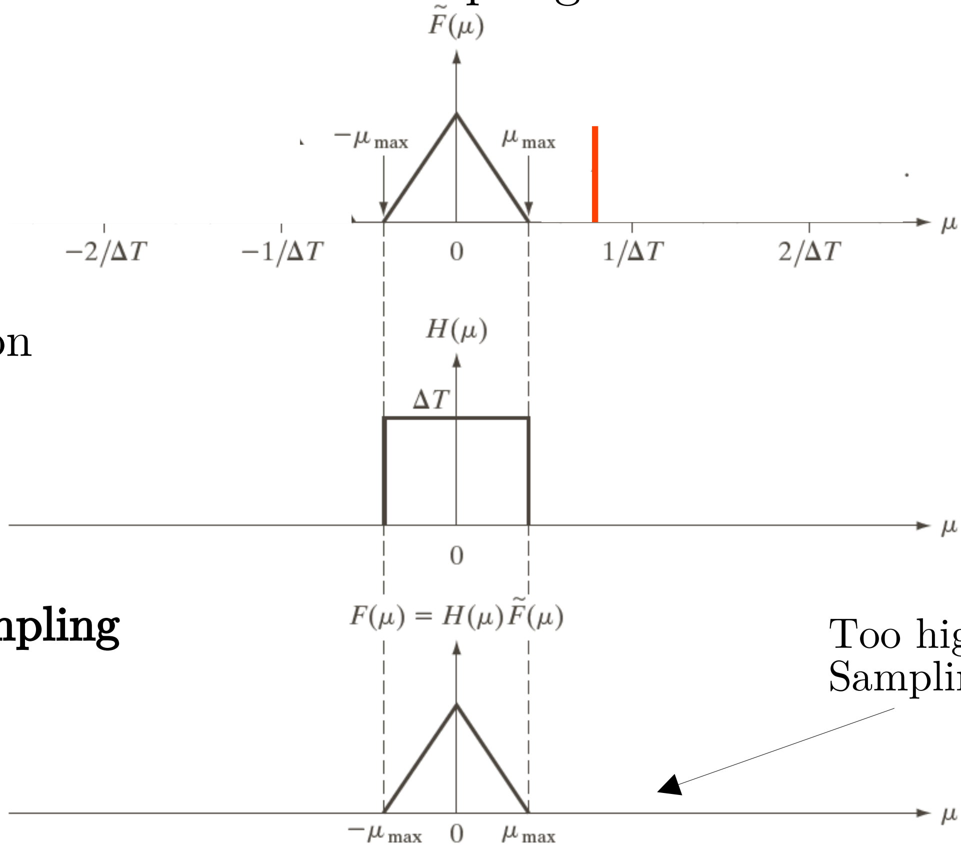


Variations of the function are faster than the sampling



Solution, anti-aliasing:

Apply a low-pass filter before sampling



a
b
c

FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

Need clean separation between copies of $F(\mu)$:

$$1/\Delta T > 2\mu_{max}$$

Nyquist–Shannon sampling theorem

Too high frequencies removed
Sampling at $1/\Delta T$ is ok

Aliasing on images

Sampling:

- natural (continuous) image -> pixels
- resizing image, high number of pixels -> low number of pixels

Original image

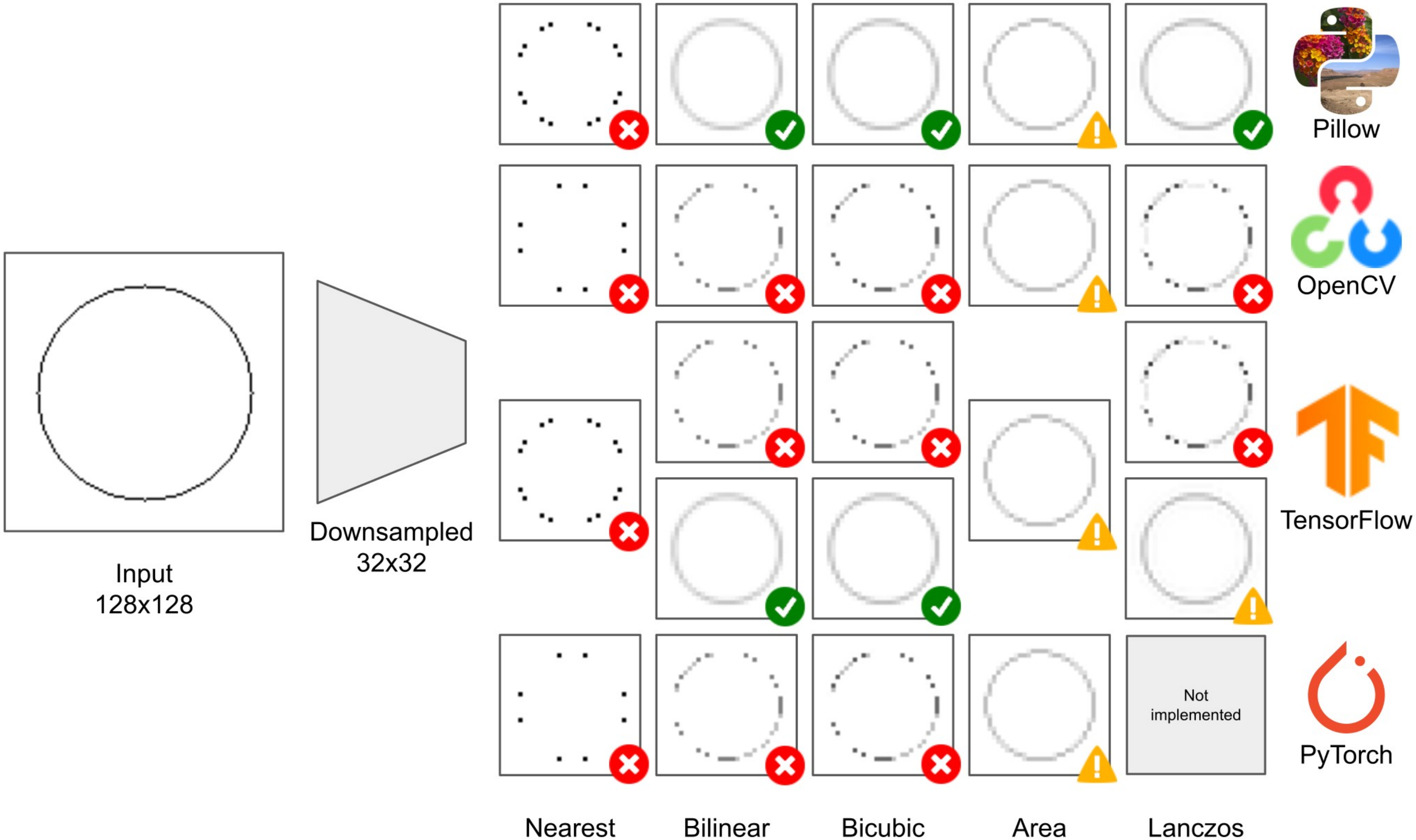
Resizing to 50%
by removing pixels
Aliasing!

Bluring before
resizing

Other example:
https://www.adobe.com/creativecloud/photography/discover/anti-aliasing.html
Visual example (aliasing in time):
https://www.youtube.com/watch?v=yr3ngmRuGUc

Aliasing on images

- Not so long ago in machine learning...

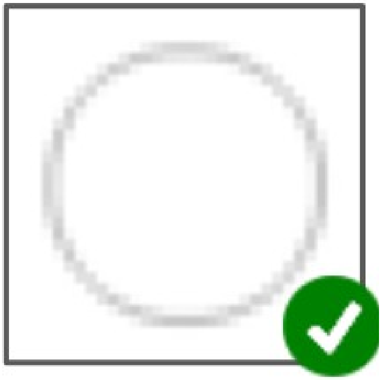


From blog post:
https://blog.zuru.tech/machine-learning/2021/08/09/the-dangers-behind-image-resizing

Aliasing on images

- Solution:

1) low-pass filter -> blur the image, the thin line becomes much thicker
2) downsample the blurry image



Deep learning and aliasing on images

- See also: https://richzhang.github.io/antialiased-cnns/
  (Results from 2019)