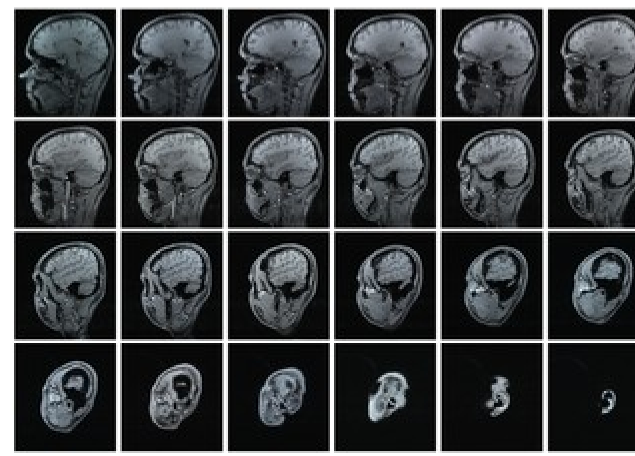
Computed Tomography
Principle

Waves are sent through the body at many different angles and collected.
(CT Xrays, MRI radiofrequency)

CT scan
Xrays

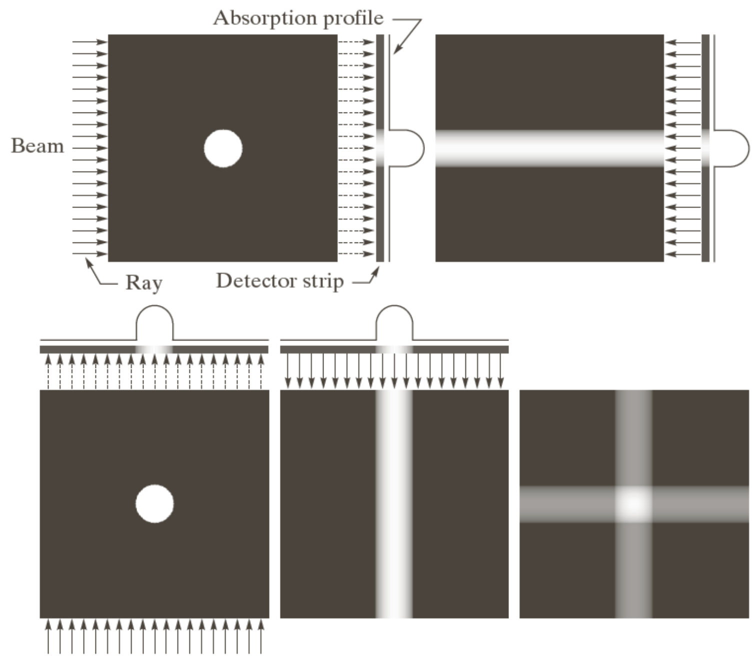
MRI imaging



Wikimedia

Images need to be reconstructed

General idea for reconstruction

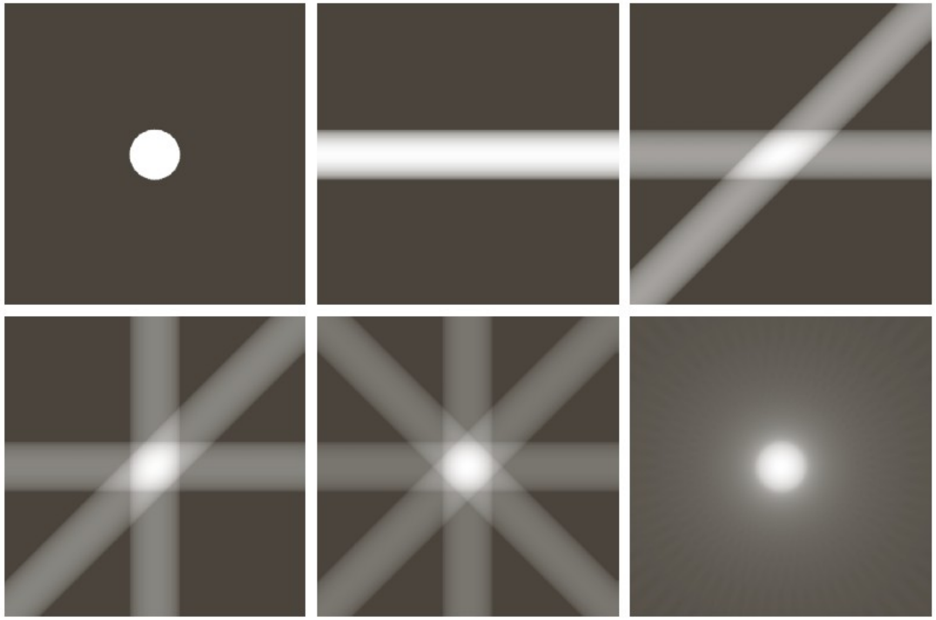


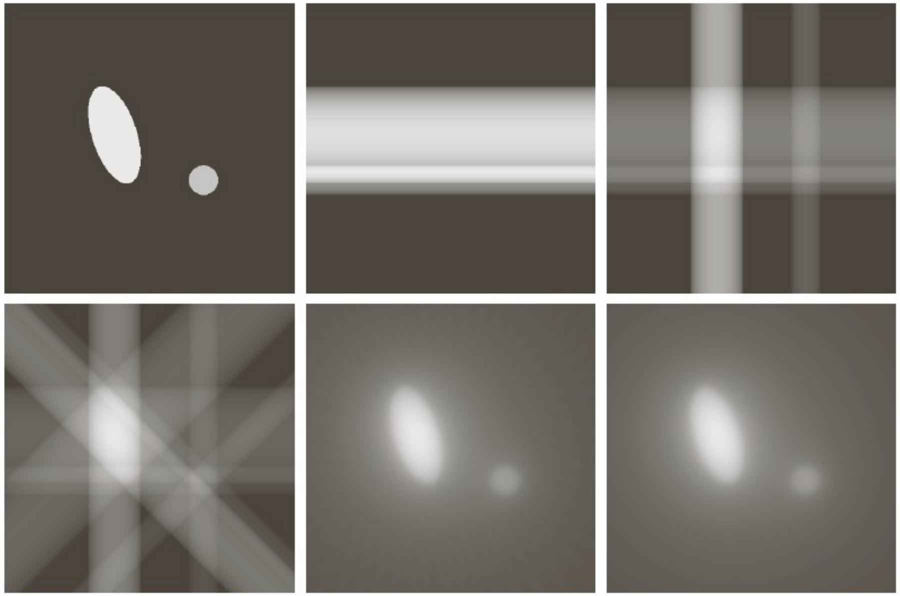
a b
c d e

FIGURE 5.32 (a) Flat region showing a simple object, an input parallel beam, and a detector strip. (b) Result of back-projecting the sensed strip data (i.e., the 1-D absorption profile). (c) The beam and detectors rotated by 90°. (d) Back-projection. (e) The sum of (b) and (d). The intensity where the back-projections intersect is twice the intensity of the individual back-projections.

a b c
d e f

FIGURE 5.33
(a) Same as Fig. 5.32(a).
(b)–(e) Reconstruction using 1, 2, 3, and 4 backprojections 45° apart.
(f) Reconstruction with 32 backprojections 5.625° apart (note the blurring).





a b c
d e f

FIGURE 5.34 (a) A region with two objects. (b)–(d) Reconstruction using 1, 2, and 4 backprojections 45° apart. (e) Reconstruction with 32 backprojections 5.625° apart. (f) Reconstruction with 64 backprojections 2.8125° apart.

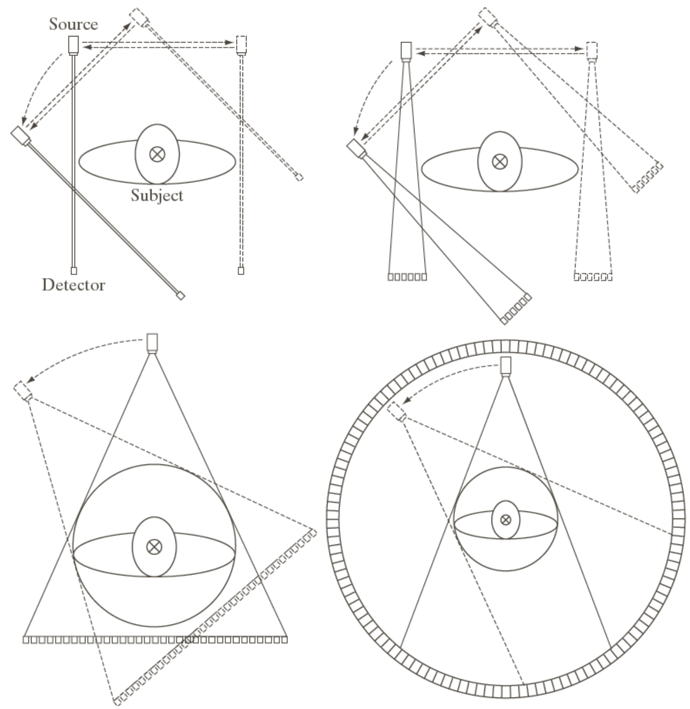
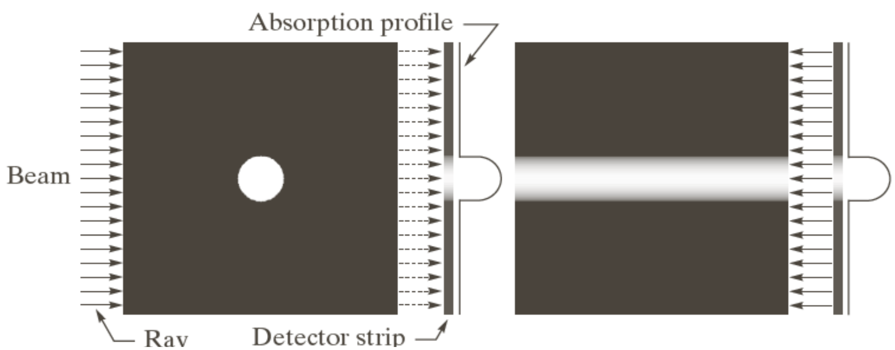


FIGURE 5.35 Four generations of CT scanners. The dotted arrow lines indicate incremental linear motion. The dotted arrow arcs indicate incremental rotation. The cross-mark on the subject's head indicates linear motion perpendicular to the plane of the paper. The double arrows in (a) and (b) indicate that the source/detector unit is translated and then brought back into its original position.

Total absorption of Xrays along the line:
Integral over a line in the 2D plane



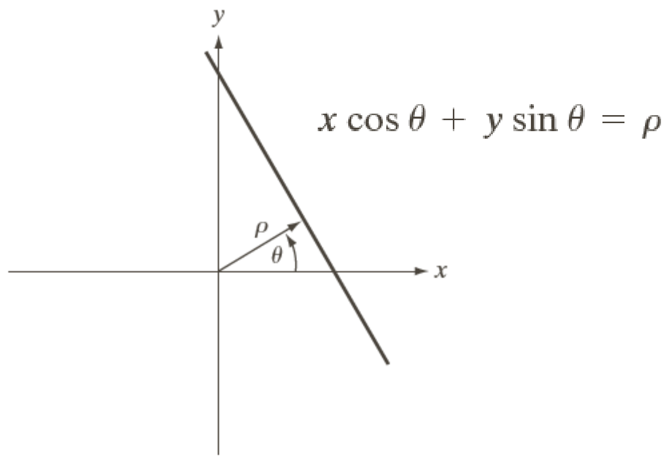
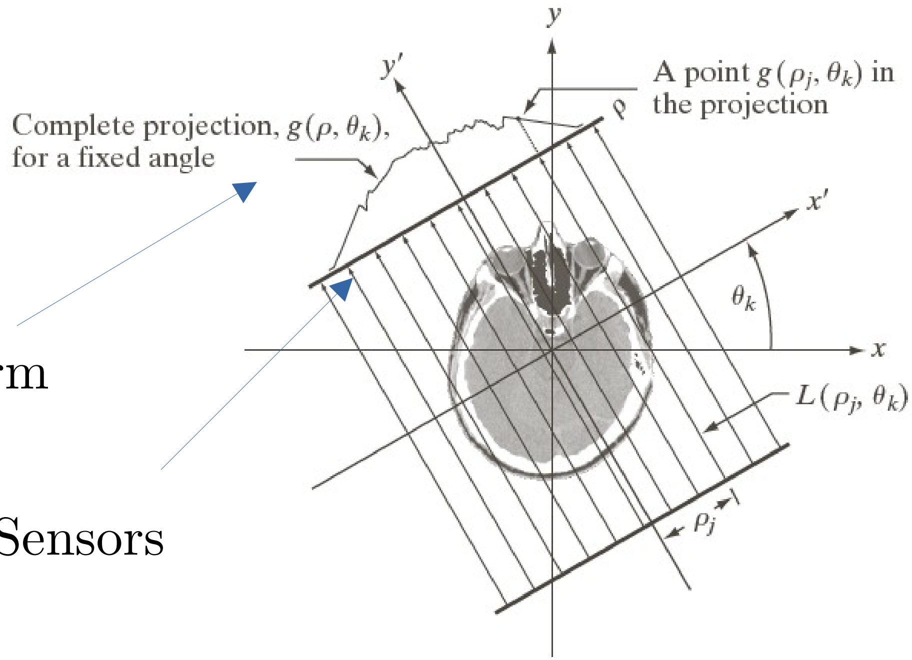


FIGURE 5.36 Normal representation of a straight line.

$$Rf(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)dxdy$$

Rf is called the Radon transform of f
This is the output of the scanner sensors, for a range of (ρ, θ)

FIGURE 5.37 Geometry of a parallel-ray beam.



Radon transform $Rf(\theta, \rho)$ of f

Sensors

How to obtain $f(x,y)$?

Sinogram

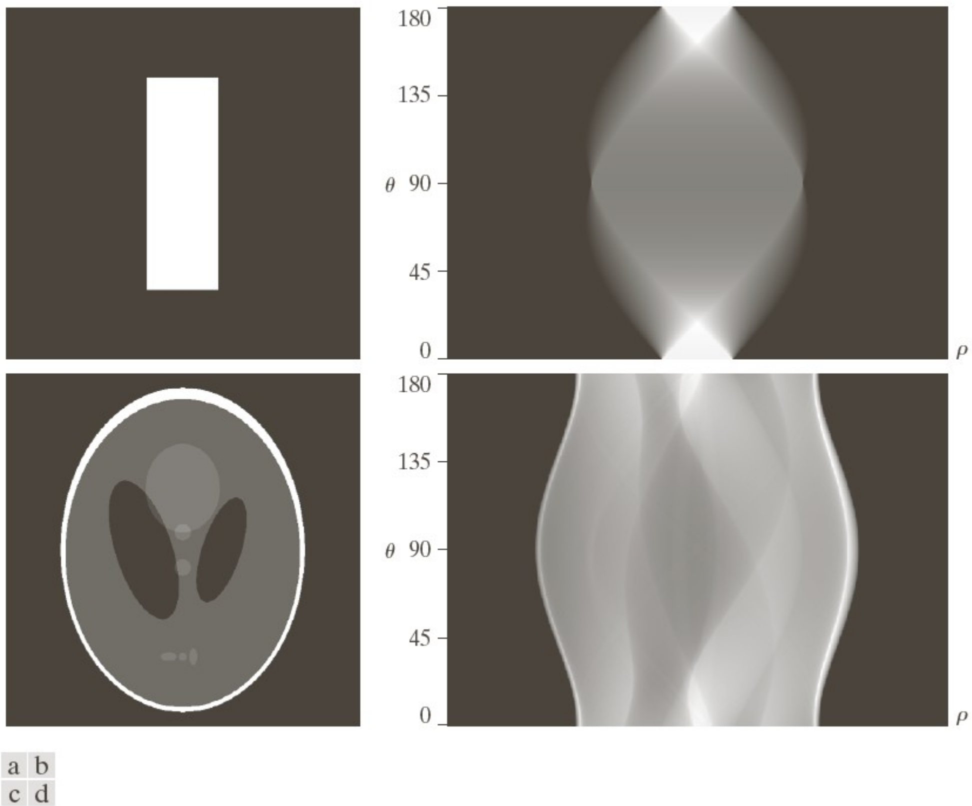
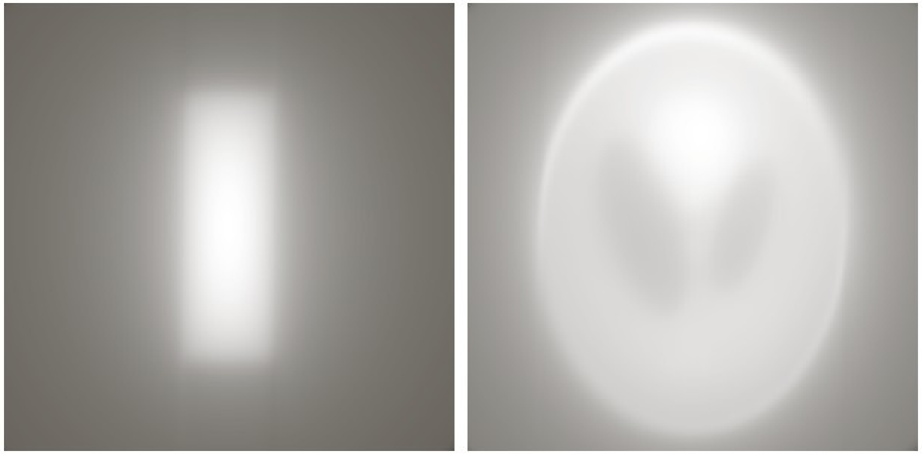


FIGURE 5.39 Two images and their sinograms (Radon transforms). Each row of a sinogram is a projection along the corresponding angle on the vertical axis. Image (c) is called the *Shepp-Logan phantom*. In its original form, the contrast of the phantom is quite low. It is shown enhanced here to facilitate viewing.

First approach summing all the projections

$$f(x,y) = \sum_{\theta=0}^{\pi} f_\theta(x,y) = \sum_{\theta=0}^{\pi} Rf(x\cos\theta + y\sin\theta, \theta)$$



a b

FIGURE 5.40 Backprojections of the sinograms in Fig. 5.39.

Result: Blurry image
Proof of concept: it is doable, but let us do it properly now

Fourier-slice theorem

The Fourier transform of the Radon transform at angle θ :

$$G(\omega,\theta)=\int_{-\infty}^{\infty} Rf(\theta,\rho)e^{-j2\pi\omega\rho}d\rho$$

$$G(\omega,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta+y\sin\theta-\rho)e^{-j2\pi\omega\rho}dxdyd\rho$$

$$G(\omega,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi\omega(x\cos\theta+y\sin\theta)}dxdy$$

is the Fourier transform of the **image** f at frequencies $u=\omega\cos\theta$, $v=\omega\sin\theta$

Inverse Fourier transform to recover the image:

$$f(x,y) = \int_0^{2\pi} \int_0^{\infty} G(\omega,\theta) e^{j2\pi\omega(x\cos\theta + y\sin\theta)} \omega d\omega d\theta$$

$$u = \omega \cos\theta$$

$$v = \omega \sin\theta$$

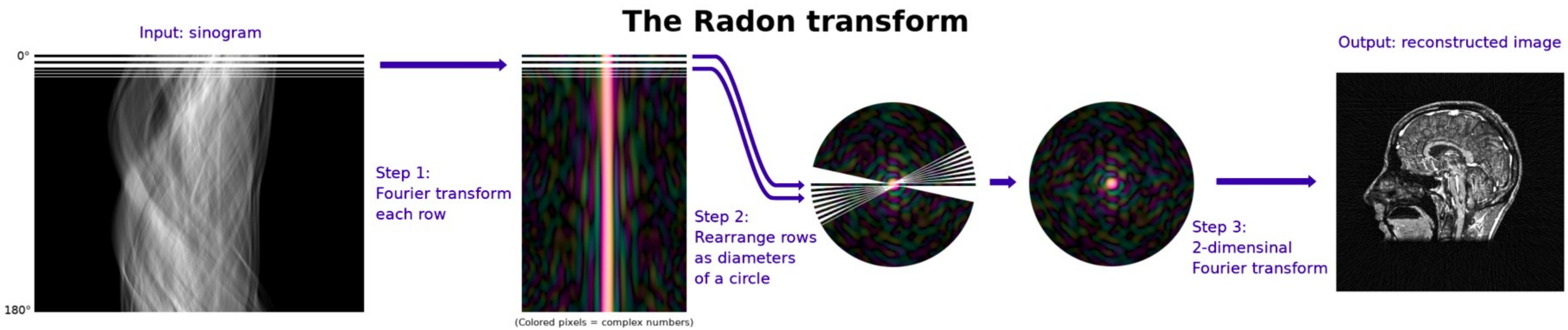
$$dudv = \omega d\omega d\theta$$

Symmetry of the Radon Transform: $G(\omega, \theta + \pi) = G(-\omega, \theta)$

$$f(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} |\omega| G(\omega,\theta) e^{j2\pi\omega(x\cos\theta + y\sin\theta)} d\omega d\theta$$

Can be seen as high pass filter

Tomography reconstruction:



Wikipedia, Radon Transform

$$G(\omega,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi\omega(x\cos\theta+y\sin\theta)}dxdy$$

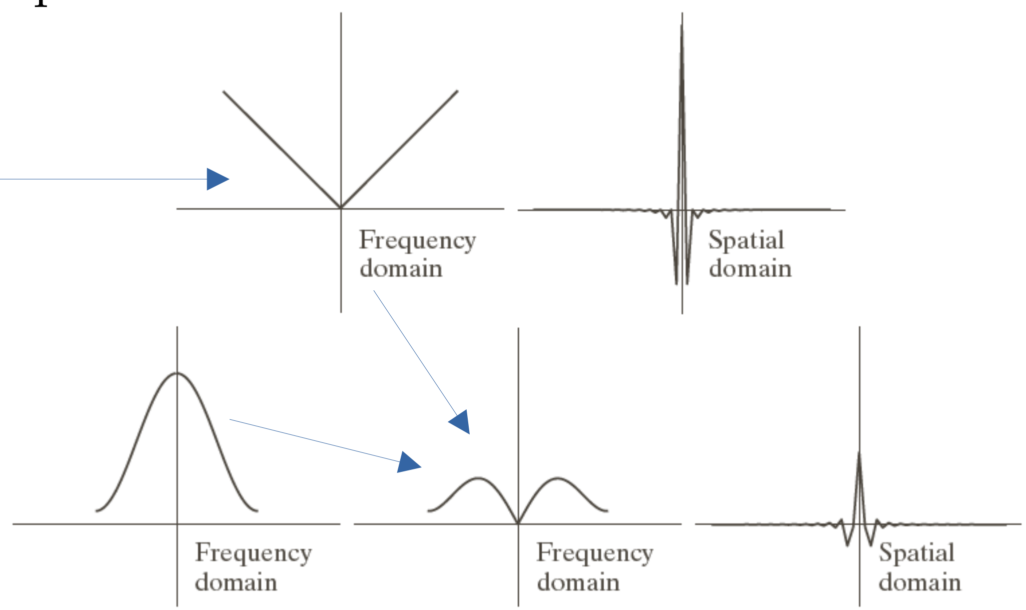
Or equivalently, 1D IDFT in ω with highpass filter and sum over θ:

$$f(x,y)=\int_{0}^{\pi}\int_{-\infty}^{\infty}|\omega|G(\omega,\theta)e^{j2\pi\omega(x\cos\theta+y\sin\theta)}d\omega d\theta$$

$$f(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} |\omega| G(\omega,\theta) e^{j2\pi\omega(x\cos\theta + y\sin\theta)} d\omega d\theta$$

Discrete measurements: FT replaced by Discrete FT
! Aliasing!
Need to low-pass filter

|ω|



a b
c d e

FIGURE 5.42 (a) Frequency domain plot of the filter $|\omega|$ after band-limiting it with a box filter. (b) Spatial domain representation. (c) Hamming windowing function. (d) Windowed ramp filter, formed as the product of (a) and (c). (e) Spatial representation of the product (note the decrease in ringing).

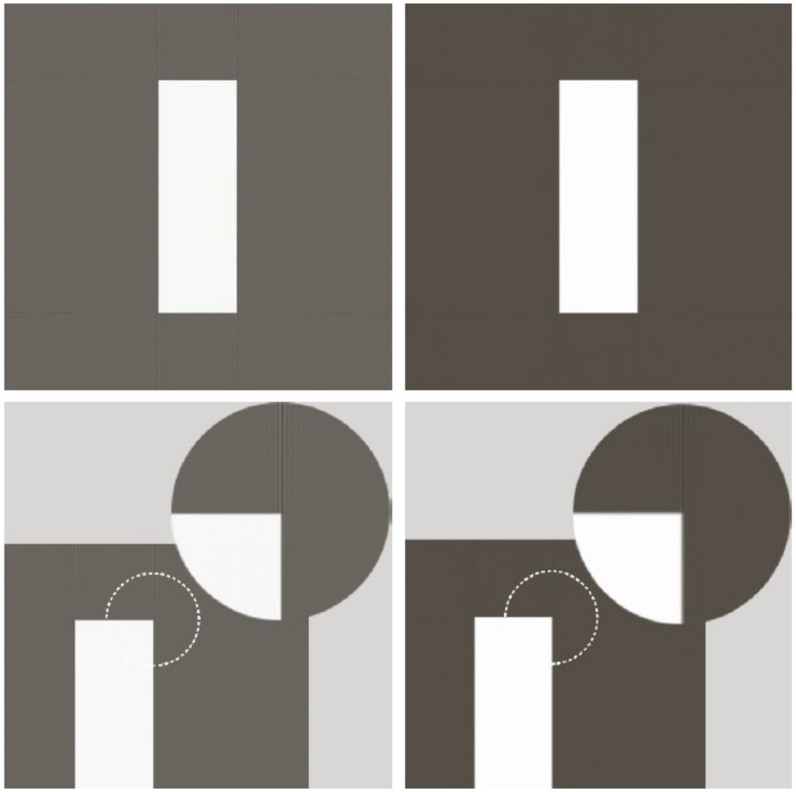
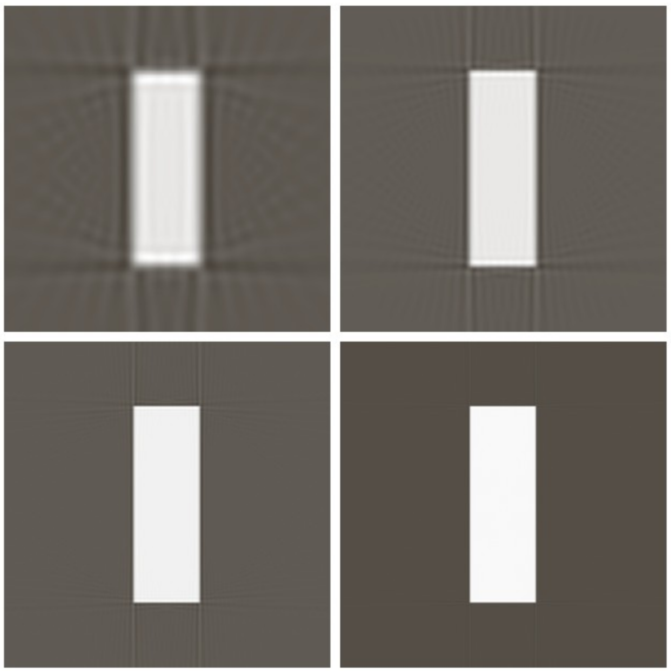


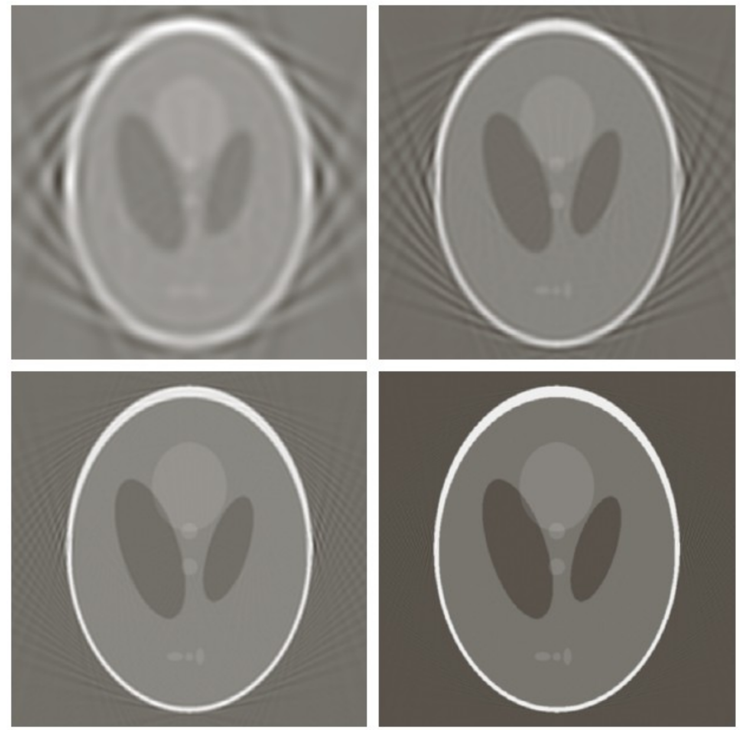
FIGURE 5.43 Filtered back-projections of the rectangle using (a) a ramp filter, and (b) a Hamming-windowed ramp filter. The second row shows zoomed details of the images in the first row. Compare with Fig. 5.40(a).

The sampling of angles is important.
Spatial sampling too low → aliasing



a b
c d

FIGURE 5.48 Reconstruction of the rectangle image from filtered fan backprojections. (a) 1° increments of α and β. (b) 0.5° increments. (c) 0.25° increments. (d) 0.125° increments. Compare (d) with Fig. 5.43(b).



a b
c d

FIGURE 5.49 Reconstruction of the head phantom image from filtered fan backprojections. (a) 1° increments of α and β. (b) 0.5° increments. (c) 0.25° increments. (d) 0.125° increments. Compare (d) with Fig. 5.44(b).

Reduce the number of samples: faster and cheaper,
But aliasing appears