Chapter 4
Filtering in the Frequency Domain

Gaussian distribution has a bell-shape curve. Figure 1 shows examples of Gaussian distribution curves or Gaussian probability density function (PDF).

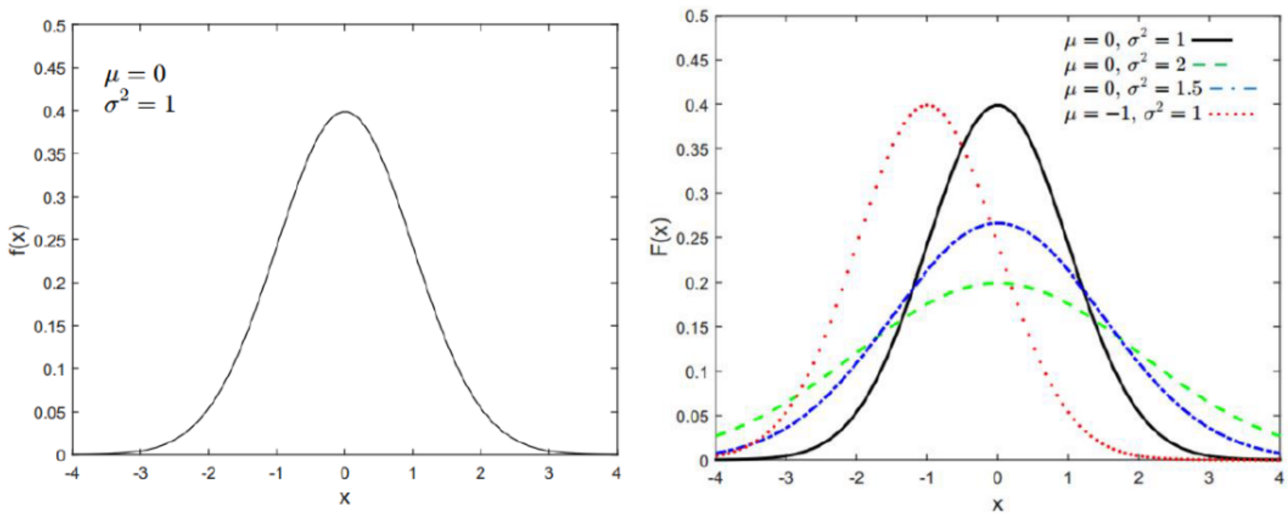


Figure 1: Examples of different Gaussian curves (PDF) with different parameters. (left) is the basic Gaussian distribution curve (PDF) with mean = 0 and variance = 1 and (right) different types of Gaussian curve (PDF).

[TUTORIAL: PYTHON for fitting Gaussian distribution on data (wasyresearch.com)]

[Python - Gaussian fit - GeeksforGeeks]

Discrete Fourier Transform (2D)

- Everything the same, except two coordinates in space *x*, *y* and two frequency components *u*, *v*
- 2D discrete Fourier transform of (*M* x *N*) image

$$\boxed{F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f\,(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}}$$

for $u = 0,\ldots, M - 1,\quad v = 0,\ldots, N - 1$

2D Inverse discrete Fourier transform

$$\boxed{f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F\,(u,v)e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}}$$

for $x = 0,\ldots, M-1, \quad y = 0,\ldots, N - 1$

Weighting

2D cosine & sine functions/images

Discrete Fourier Transform (2D)

Same kind of properties as 1D with linerity, similarity, periodicity and shift

- Linerity:

$$\alpha f(x,y) + \beta g(x,y) \Leftrightarrow \alpha F(u,v)$$

- Similarity:

$$f(ax,by) \Leftrightarrow \frac{1}{ab} F\left(\frac{u}{a},\frac{v}{b}\right)$$

- Periodicity:

$$F(u, v) = \mathrm{DFT}\{ f(x, y) \} = F(u + k_1M, v + k_2N)$$
$$f(x, y) = \mathrm{IDFT}\{ F(u, v) \} = f(x + k_1M, y + k_2N)$$

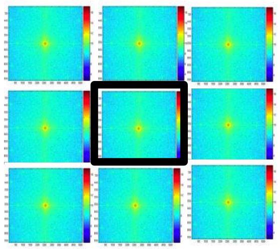
- Shift:

$$f(x-a, y-b) \Leftrightarrow e^{j2\pi(au+bv)} F(u,v)$$

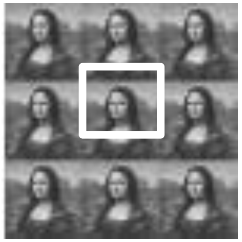
- Discrete Fourier Transform (2D): periodicity

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$f(x, y)$



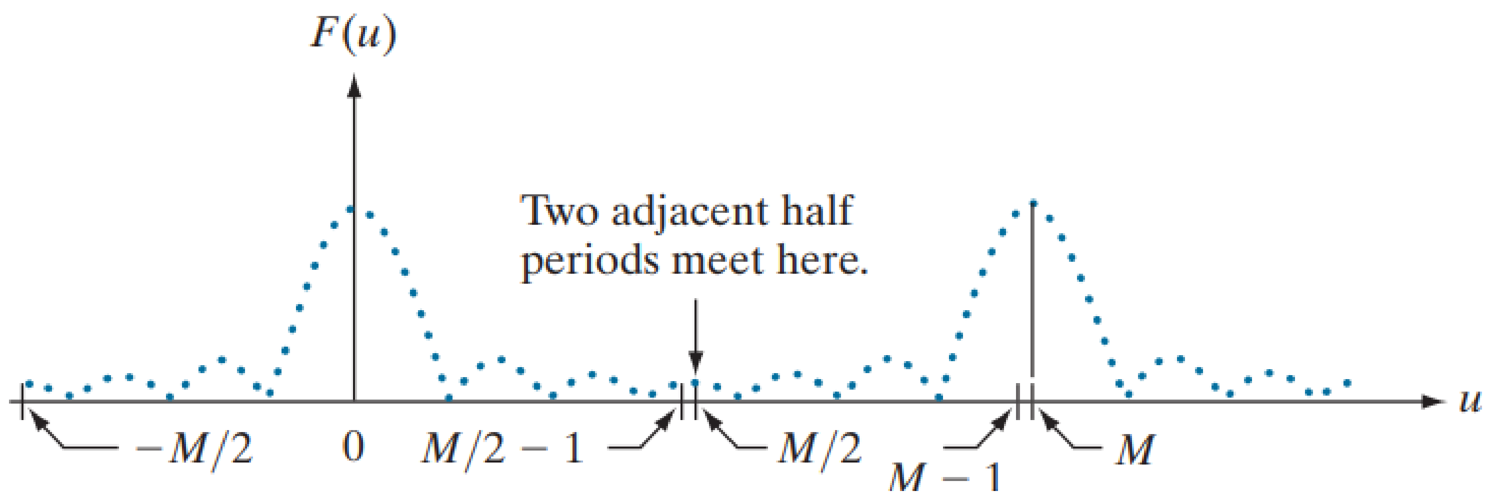
$f(x, y) = \mathrm{IDFT}\{F(u, v)\}$



Periodicities!

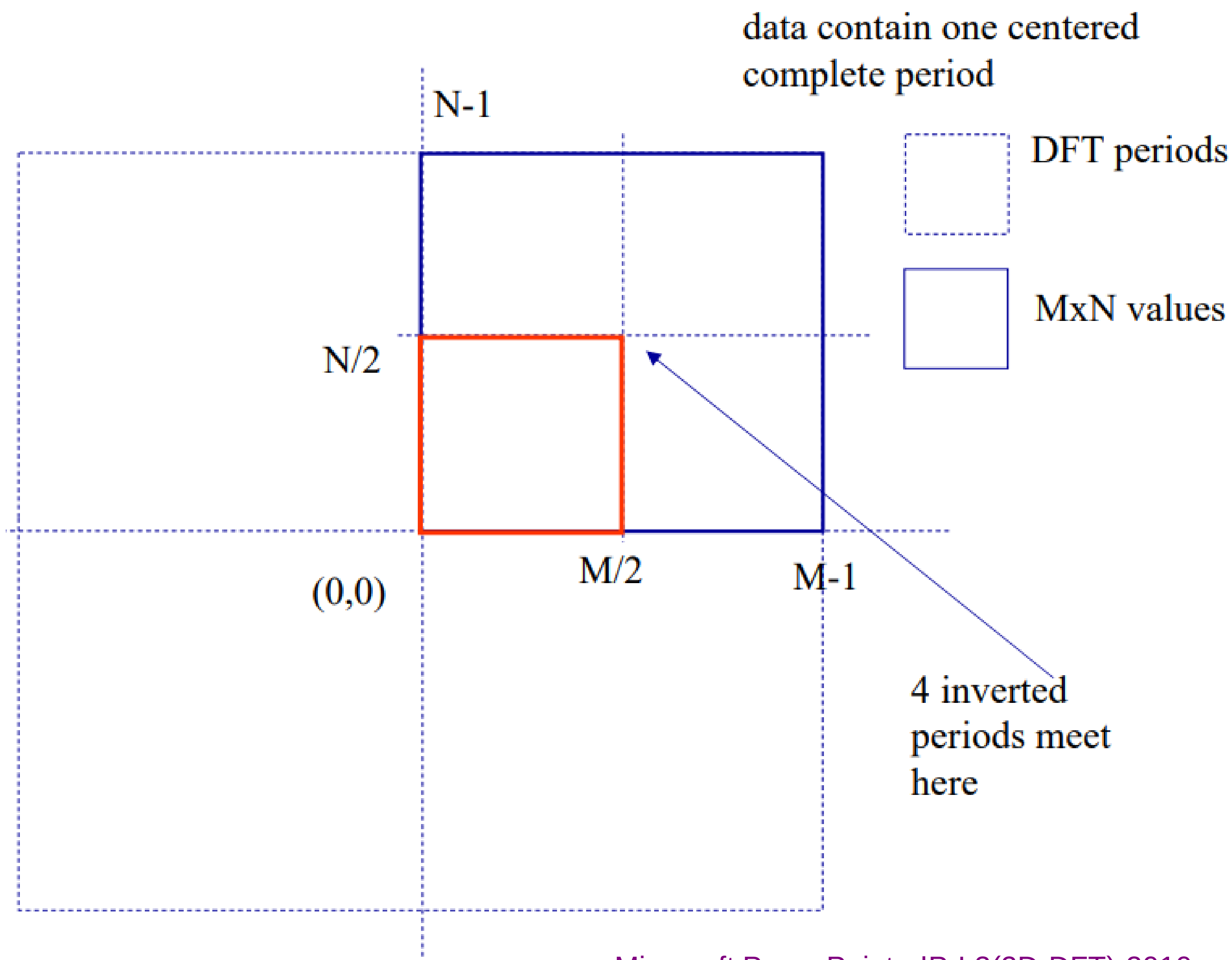
$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

Discrete Fourier Transform (2D), phase



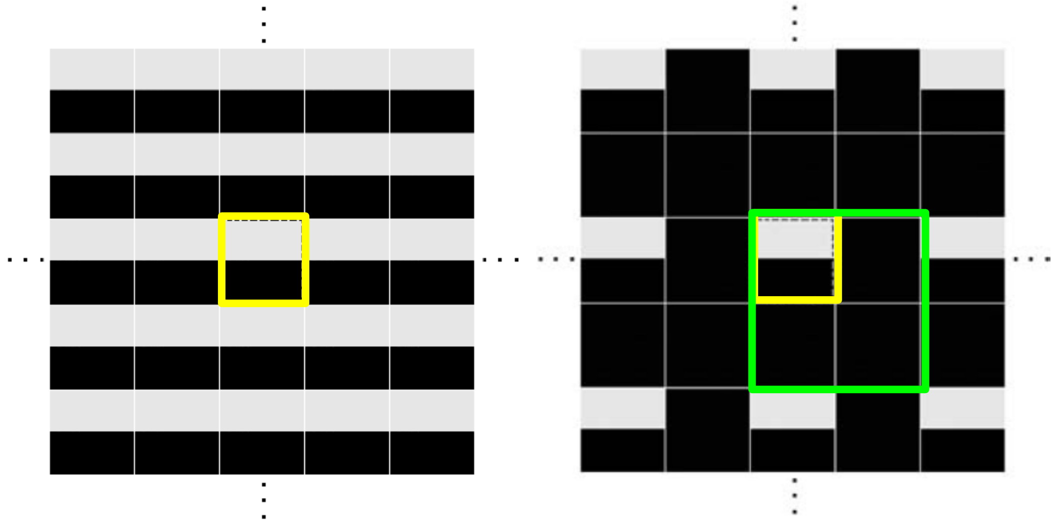
Centering the Fourier transform.
A 1-D DFT showing an infinite number of periods.

- Discrete Fourier Transform (2D): periodicity



Discrete Fourier Transform (2D)

The period can be increased by 2D zero-padding.



a b

FIGURE 4.33 2-D image periodicity inherent in using the DFT. (a) Periodicity without image padding. (b) Periodicity after padding with 0s (black). The dashed areas in the center correspond to the image in Fig. 4.32(a). (The thin white lines in both images are superimposed for clarity; they are not part of the data.)

- **Discrete Fourier Transform (2D), spectrum and phase**
- $F(u, v)$ is complex

$$F(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

Amplitudes of cosines composing the image: ***Spectrum***

Phase angle: Displacement of cosines – location of objects

$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

$$\phi(u,v) = arctan\frac{I(u,v)}{R(u,v)}$$

Python:
Numpy.abs(F)

- Taking the log for visualization

- DC component dominating:
  - Proportional to average intensity of image
  - Do a log-transformation of spectrum (S)

$$F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

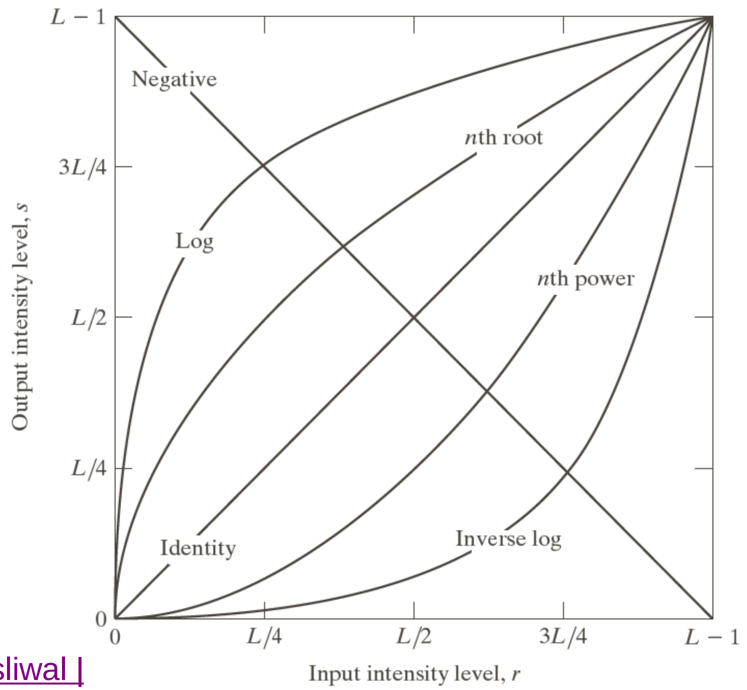
S_log = log (1 + S);

- Visualization
  - Scale S_log to [0, 255] and use **imshow** from matplotlib
  - Should scale automatically

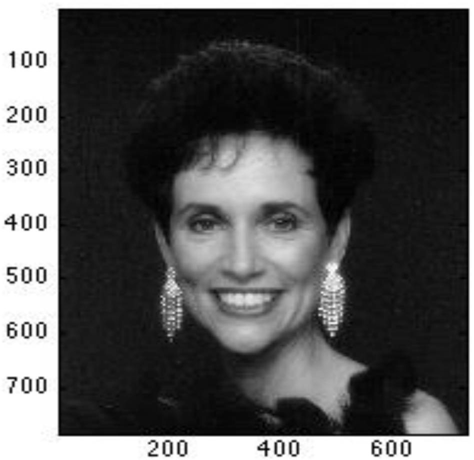
[matplotlib.pyplot.imshow — Matplotlib 3.6.2 documentation]

[Imshow in Python (plotly.com)]

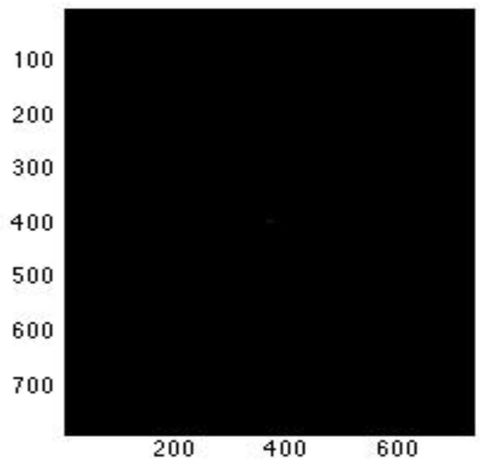
[Image Enhancement. In this article, we'll learn different… | by Avi Kasliwal | AlphaVIsion | Medium]



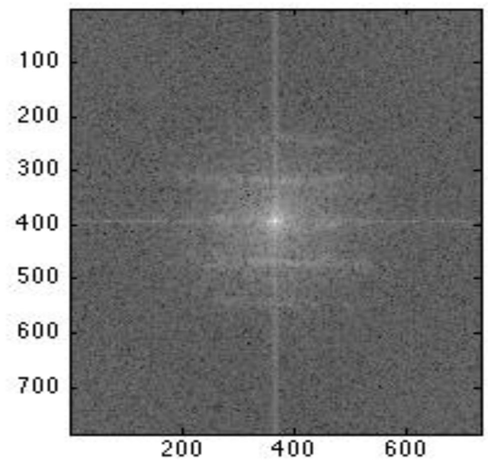
Discrete Fourier Transform (2D)



Spectrum



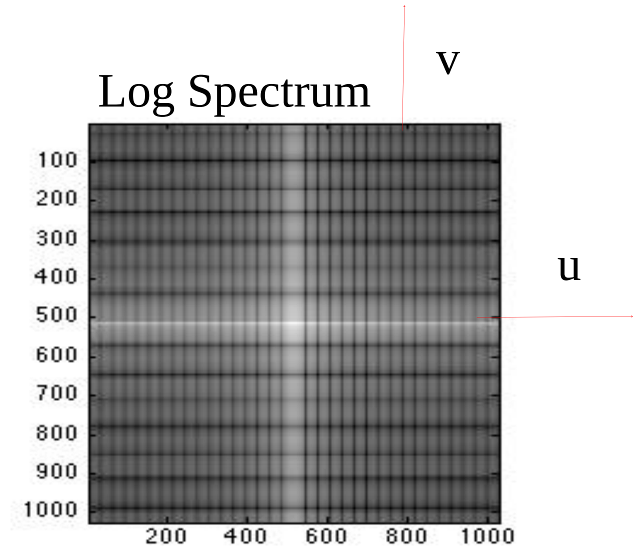
Log Spectrum



Peak with high value at $F(0, 0)$,
the rest is dark compared to it

- Discrete Fourier Transform (2D)
- DFT of 2D box/rectangle

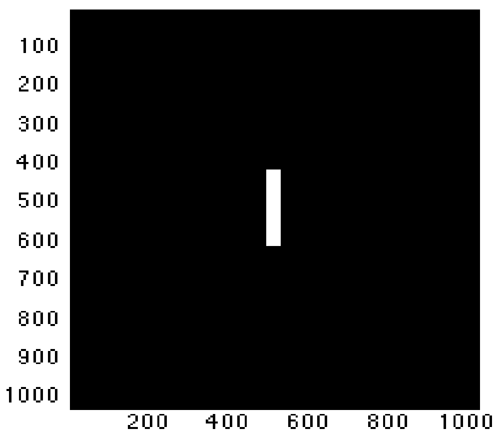
Log Spectrum

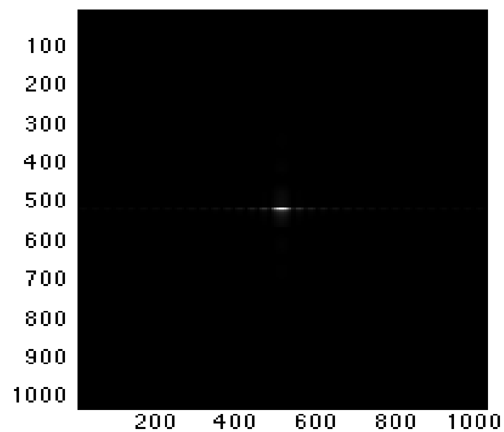


$|F\ (u, v)|$

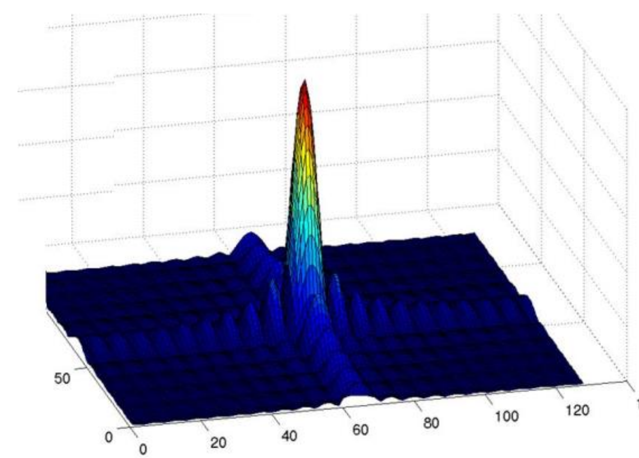
Rectangle centred at origin with sides of length X and Y

Spectrum



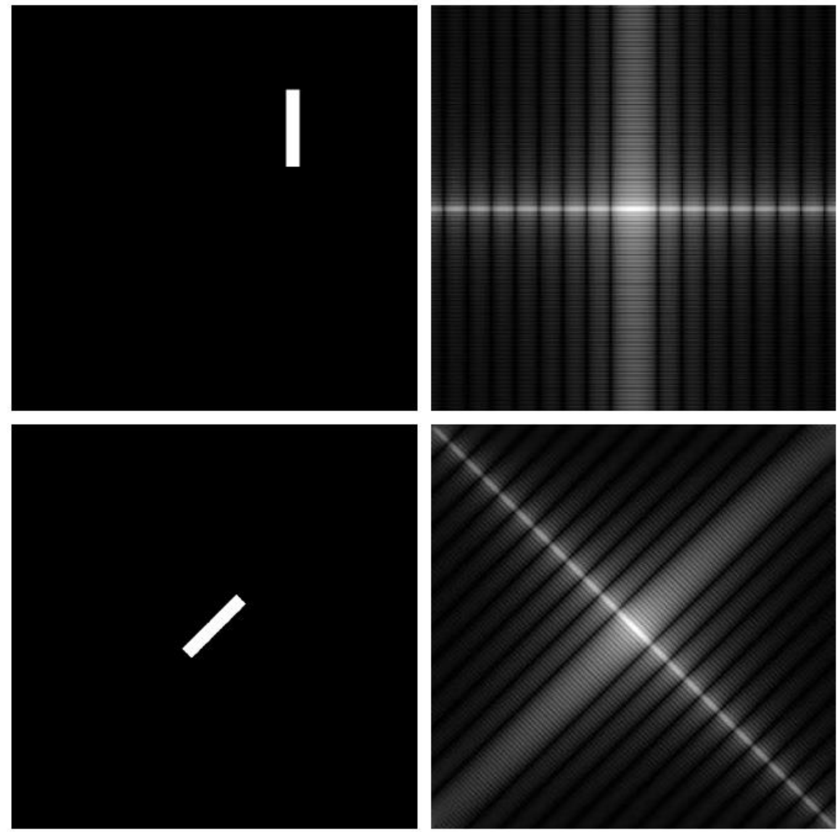


$f\ (x, y)$



- Discrete Fourier Transform (2D)

- Spectrum is invariant to translation of image
- Spectrum is not invariant to rotation of image

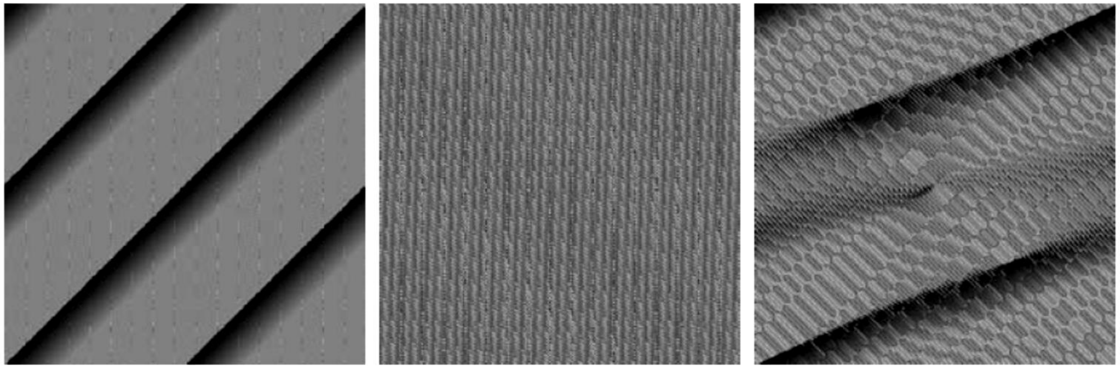


a b
c d

FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

- Discrete Fourier Transform (2D), phase

- Even though spectrum is invariant to translation, phase angle is not



a b c

FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).

- But phase rarely used in practice

Discrete Fourier Transform (2D), phase

- translation

$$f[k,l]e^{j2\pi\left(\frac{m}{M}k+\frac{n}{N}l\right)} \leftrightarrow F[u-m,v-l]$$

$$f[k-m,l-n] \leftrightarrow F[u,v]^{-j2\pi\left(\frac{m}{M}k+\frac{n}{N}l\right)}$$

- rotation

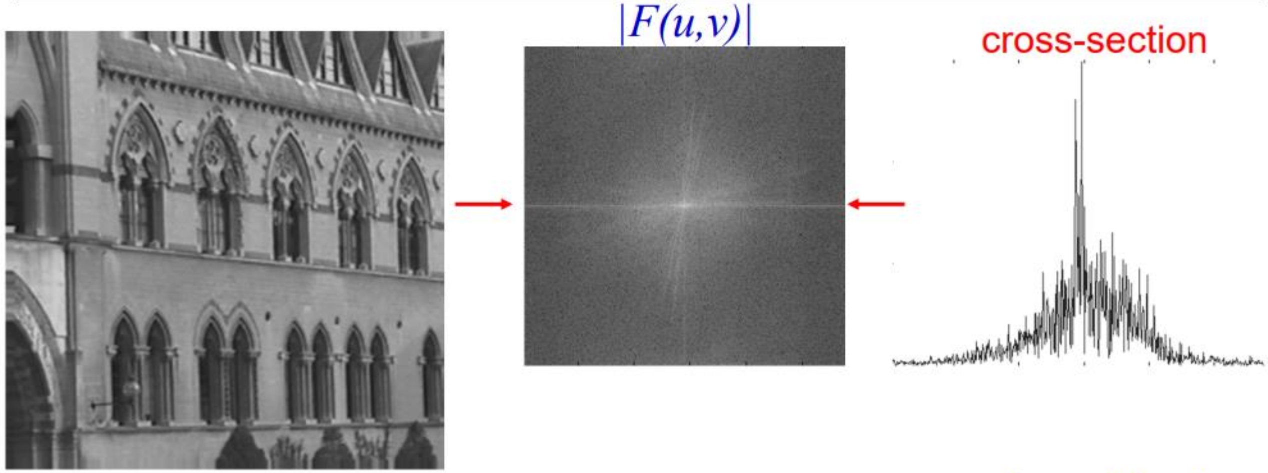
$$\begin{cases} k = r\cos\vartheta \\ l = r\sin\vartheta \end{cases} \qquad \begin{cases} u = \omega\cos\varphi \\ l = \omega\sin\varphi \end{cases}$$

$$f[r,\vartheta+\vartheta_0] \leftrightarrow F[\omega,\varphi+\vartheta_0]$$

Rotations in spatial domain correspond equal rotations in Fourier domain

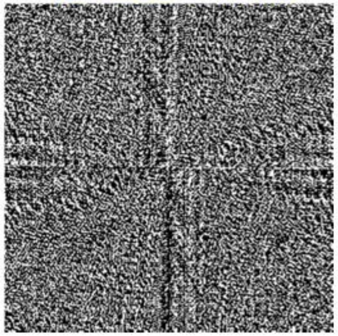
[Microsoft PowerPoint - IP-L3(2D-DFT)-2010.ppt (univr.it)]

Discrete Fourier Transform (2D), phase



f(x,y)

phase F(u,v)

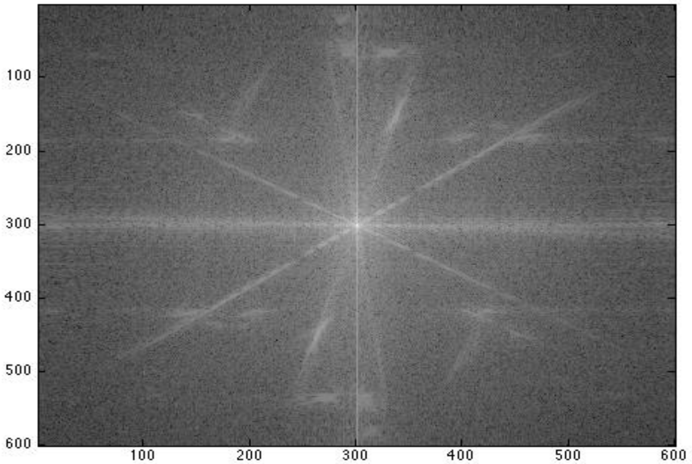


Phase 2D DFT
is less informative

- Interpretation of the Fourier transform of images (spectrum)

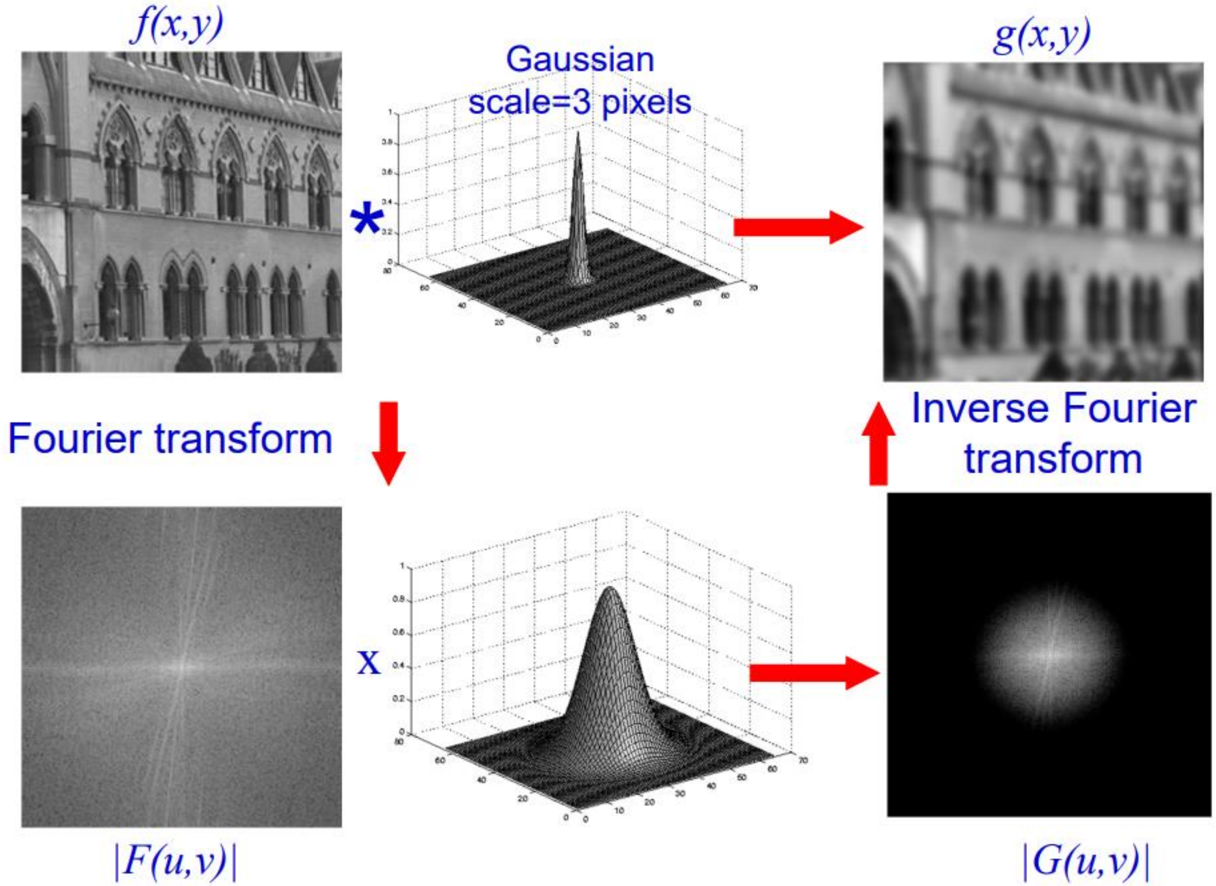
- Area around center ($M/2$, $N/2$) show importance of low frequencies corresponding to smooth areas
- Edges require many cosines of high frequency

Even symmetry, as images have real values



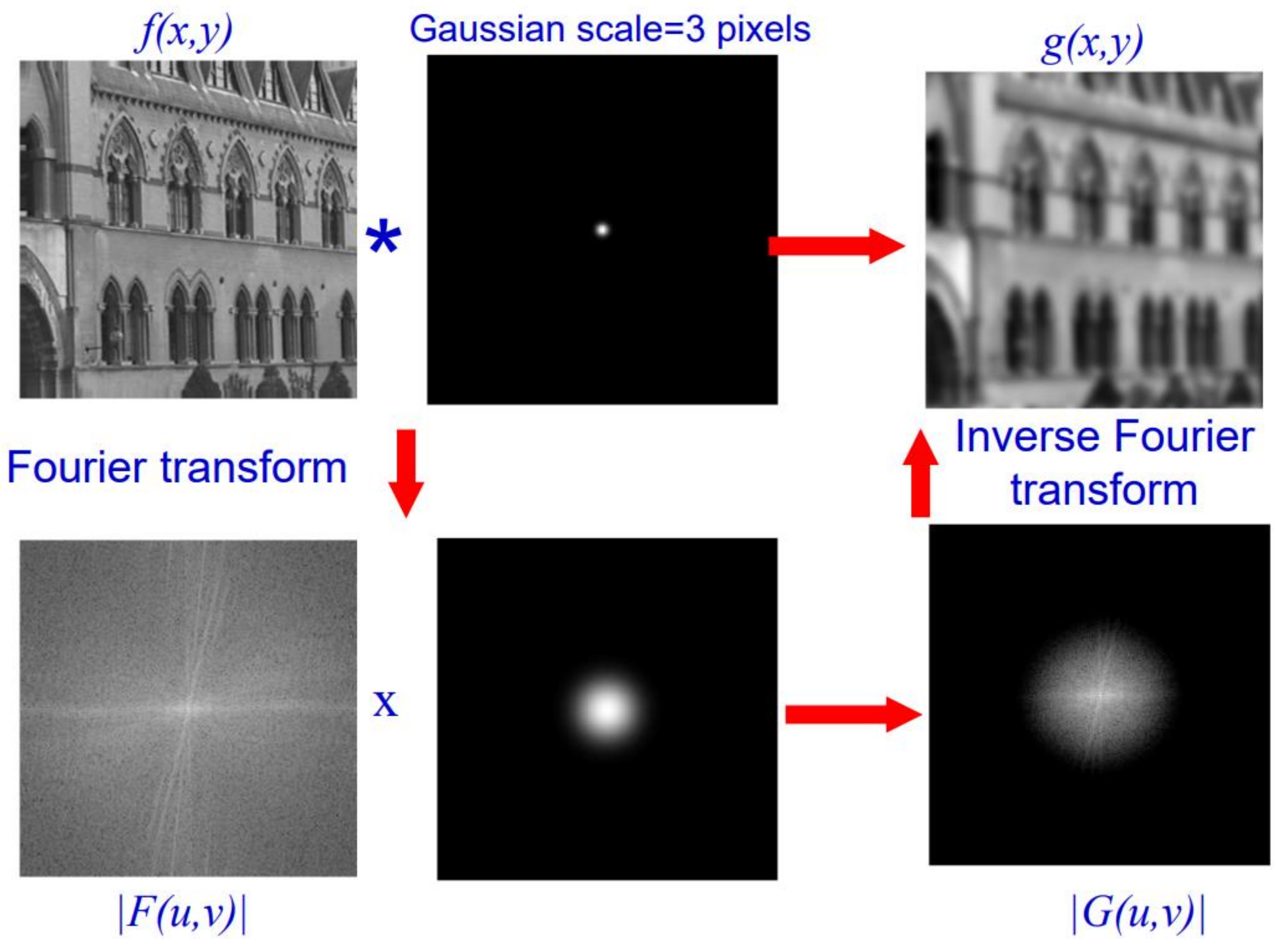
lab06_Fourier2D (unioviedo.es)

Discrete Fourier Transform (2D), phase

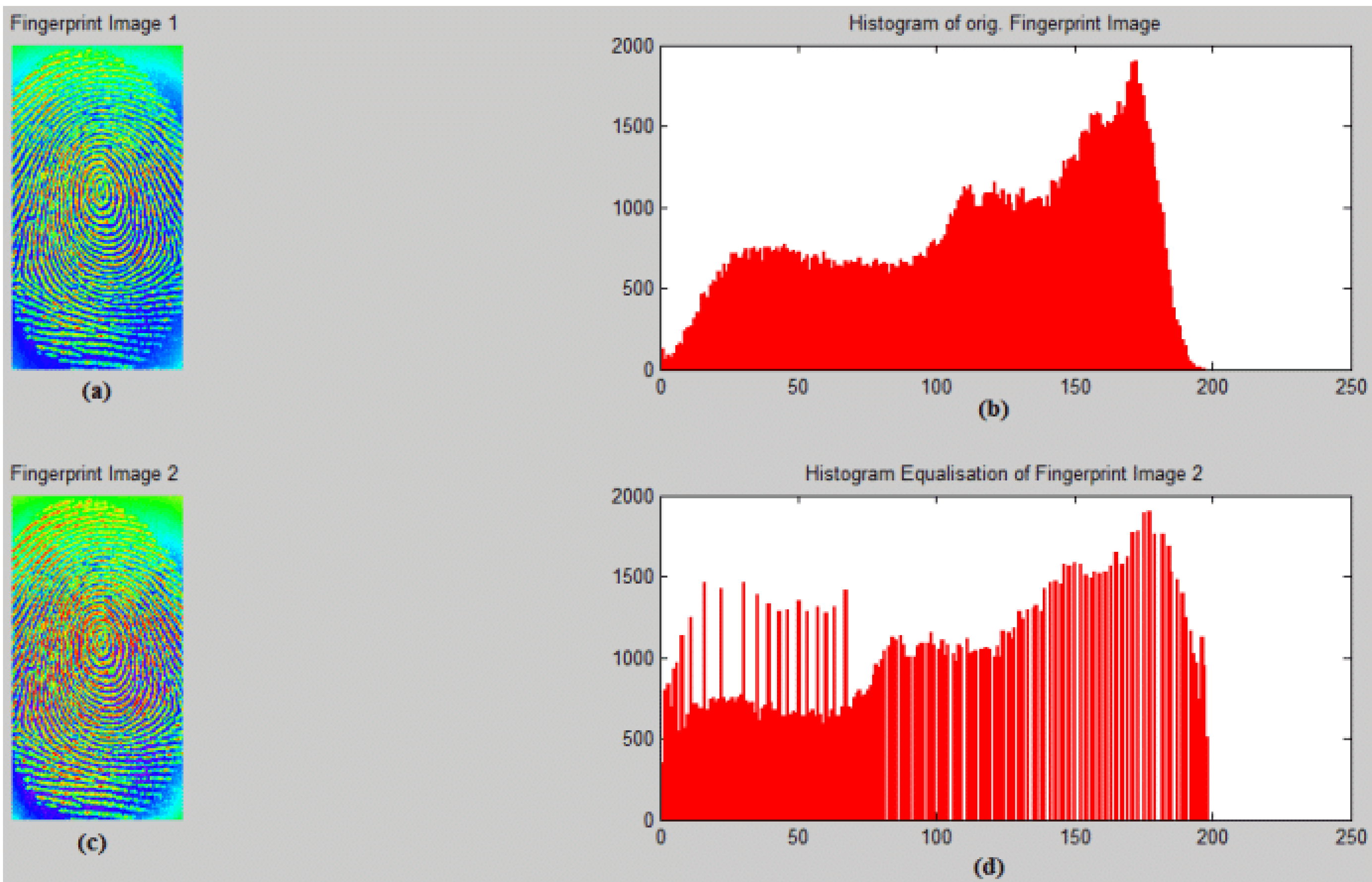


Convolution theorem

Space convolution = frequency multiplication

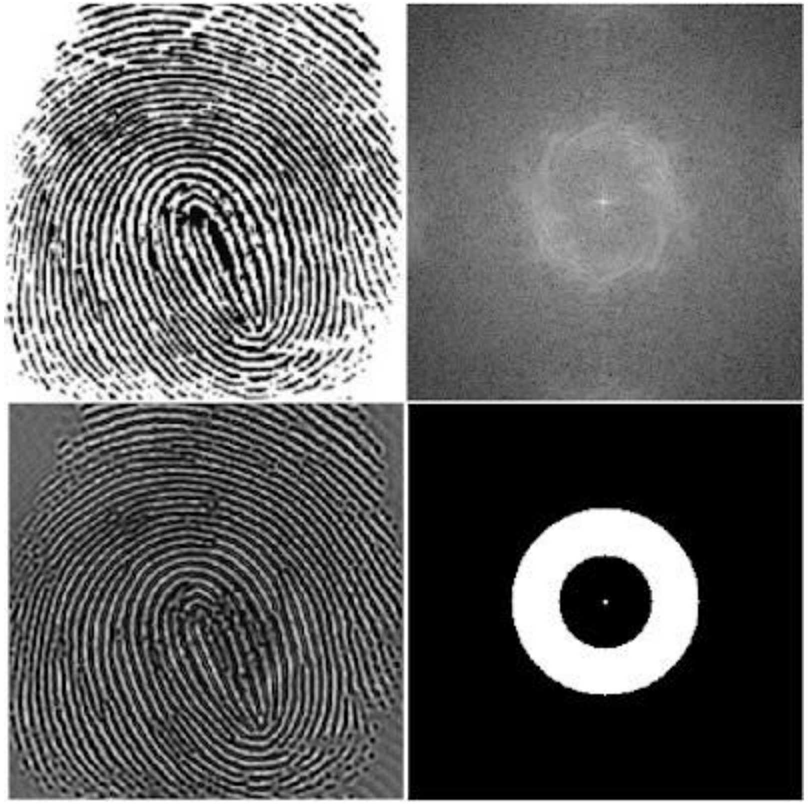


Effect of histogram equalization Fingerprint Enhancement by Fourier Transform



https://www.google.com/url?sa=i&url=https%3A%2F%2Fwww.researchgate.net%2Ffigure%2FEffect-of-histogram-equalization-Fingerprint-Enhancement-by-Fourier-Transform-Fourier_fig3_281449451&psig=AOvVaw1kPMddJvN1usb_IX5c2Sq-&ust=1706705021440000&source=images&cd=vfe&opi=89978449&ved=0CBMQjhxqFwoTCKj_rfORhYQDFQAAAAAdAAAAABAL

Fourier transform



Wanna Eat Six Kamatis: ACTIVITY 7 Enhancement in the Frequency Domain (winsomerara.blogspot.com)