Introduction to Fourier Transforms

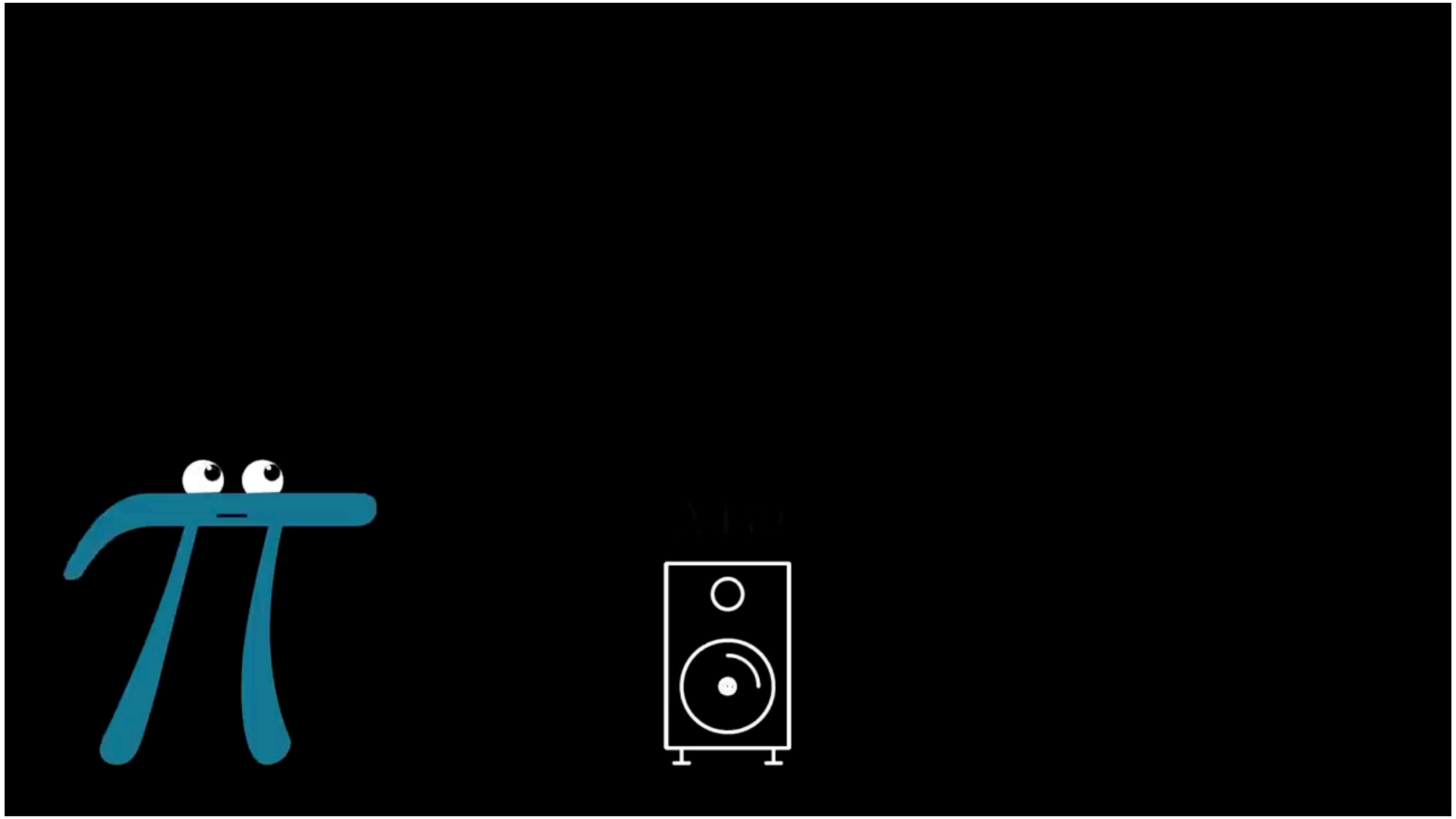
FYS-2010-1 25V Image Analysis
by Elisabeth Wetzer

Overview

- Intuition
- Formal Definition
- Applications and Visualization
- Useful Properties
- Application Examples
- Implementation
- Summing Up

1-D Signals

- **Intuition**
  - Connection to signals



Source: youtube.com/@3blue1brown

Definition

- **Fourier Transform** $$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\ e^{-i2\pi\xi x}\ dx, \qquad \forall \xi \in \mathbb{R}.$$

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- **Inverse Transform** $$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{i2\pi\xi x}\, d\xi, \quad \forall\, x \in \mathbb{R}.$$
- **Fourier Transform Pair** $$f(x) \xleftrightarrow{\mathcal{F}} \widehat{f}(\xi)$$

Definition

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- **Inverse Transform**

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{i2\pi\xi x}\, d\xi, \quad \forall\, x \in \mathbb{R}.$$

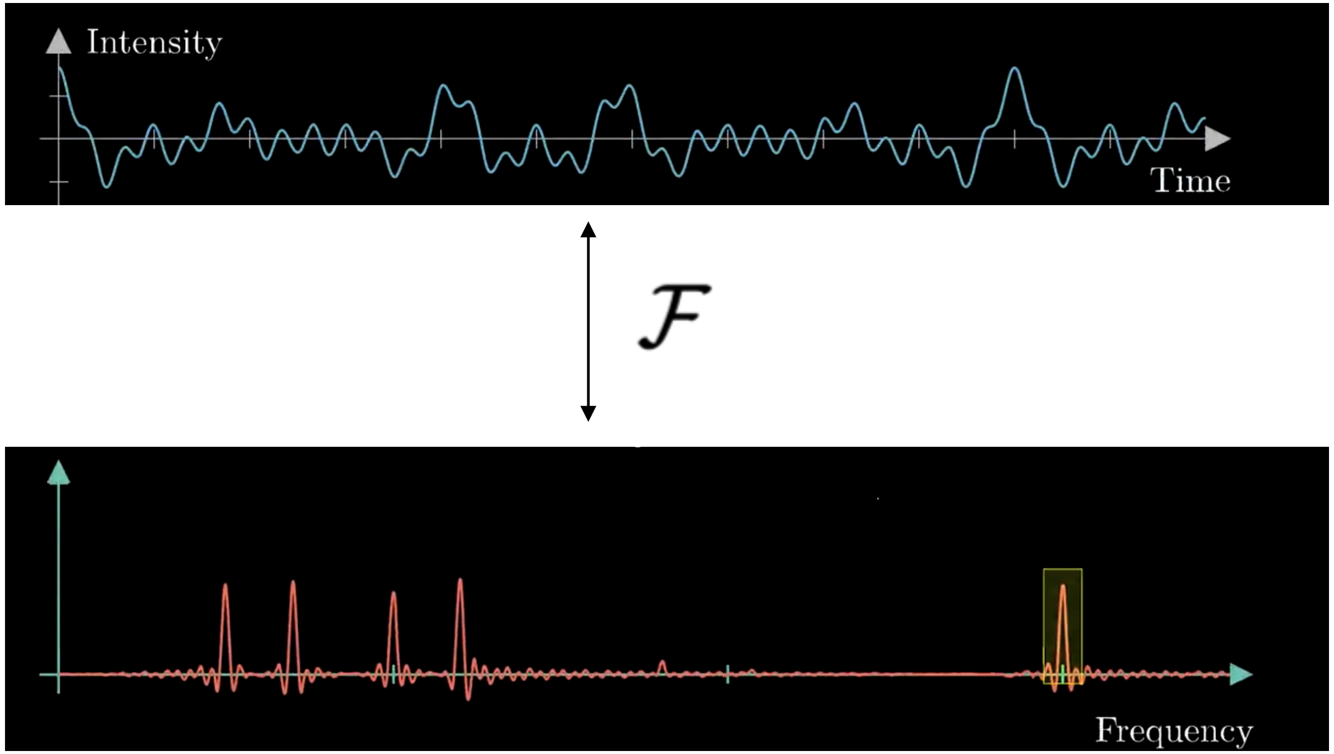
$$\widehat{f}(\xi) \mapsto \mathbb{C} \qquad f(x) \mapsto \mathbb{R}$$

Applications in Image Analysis

- Image Compression
- Noise Reduction
- Feature Extraction
- Image Reconstruction
- Edge Detection
- Image Filtering
- Template Matching
- …. many more!

What does it look like?

- **Examples**



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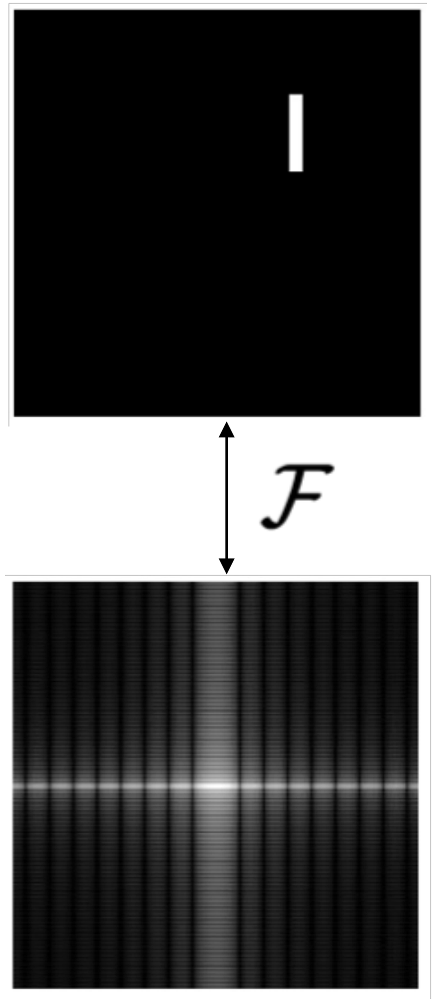

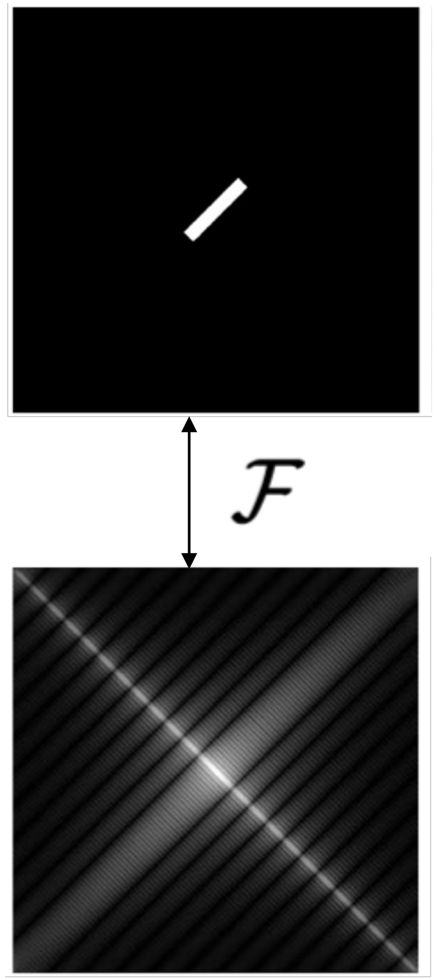

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- **Examples**

$$f(x) \xleftrightarrow{\mathcal{F}} \widehat{f}(\xi)$$

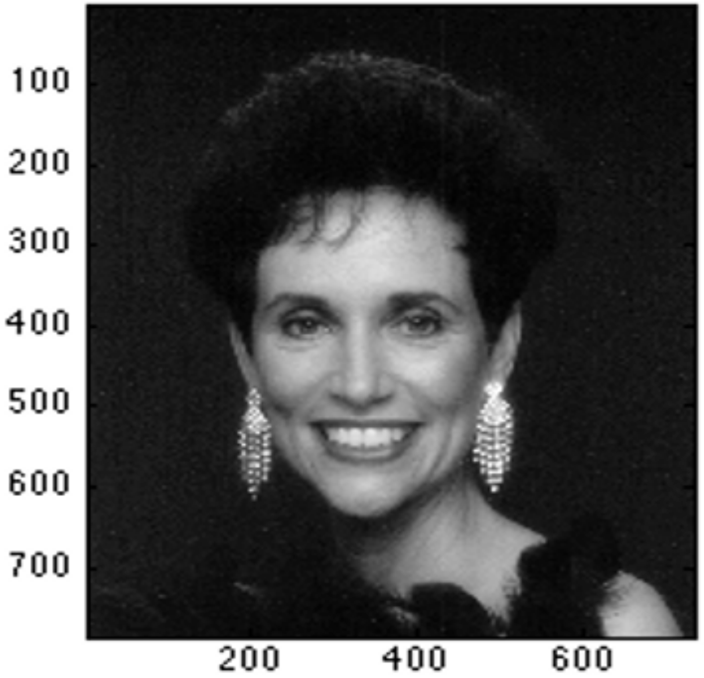

Image in Spatial Domain

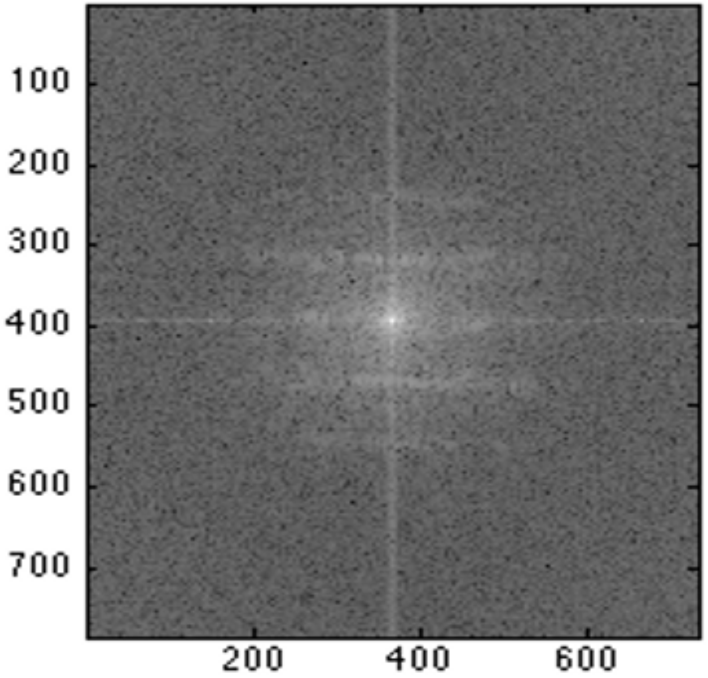

Image in Frequency Domain

FT $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x}\, dx, \quad \forall \xi \in \mathbb{R}.$

Inv.-FT $f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{i2\pi\xi x}\, d\xi, \quad \forall\, x \in \mathbb{R}.$

Useful Properties

- **Equivalent Forms (via Euler's formula)**

$$\widehat{f}(\xi) = \underbrace{Ae^{i\theta}}_{\text{polar coordinate form}} = \underbrace{A\cos(\theta) + iA\sin(\theta)}_{\text{rectangular coordinate form}}.$$

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$$\begin{aligned}
\widehat{f}(\xi) \cdot e^{i2\pi\xi x} &= Ae^{i\theta} \cdot e^{i2\pi\xi x} \\
&= \underbrace{Ae^{i(2\pi\xi x + \theta)}}_{\text{polar coordinate form}} \\
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Useful Properties

- **Linearity**

$$a\, f(x) + b\, h(x) \overset{\mathcal{F}}{\Longleftrightarrow} a\, \widehat{f}(\xi) + b\, \widehat{h}(\xi); \qquad a, b \in \mathbb{C}$$

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- **Translation**

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Useful Properties

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- **Frequency Shift** $e^{i2\pi\xi_0 x} f(x) \overset{\mathcal{F}}{\Longleftrightarrow} \widehat{f}(\xi - \xi_0); \quad \xi_0 \in \mathbb{R}$

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Useful Properties

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- **Scaling**

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Useful Properties

- … and now my favorite property in application!

Useful Properties

- **Convolution Theorem**

$$\widehat{fg} = \hat{f} * \hat{g}$$

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Useful Properties

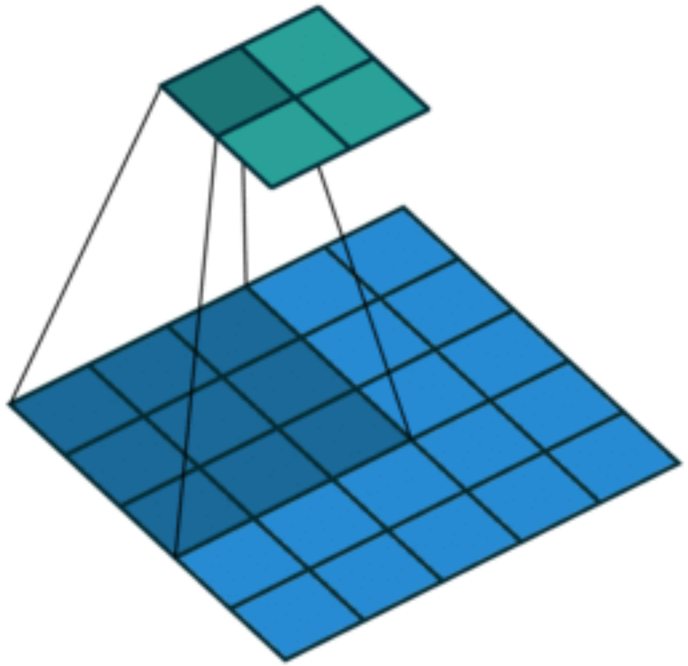
- **Convolution Theorem**

$$\widehat{fg} = \hat{f} * \hat{g}$$

- **Awesome in many signal processing applications, but more than that, it can make solving PDEs like the heat equation really easy 🎉**

Source: Towards Data Science

Application Examples



- Filtering in Spatial domain means convolving a filter with the entire image

- Thanks to Convolution Theorem in frequency domain it is just a multiplication with a binary mask!

Application Examples

- **Filtering**

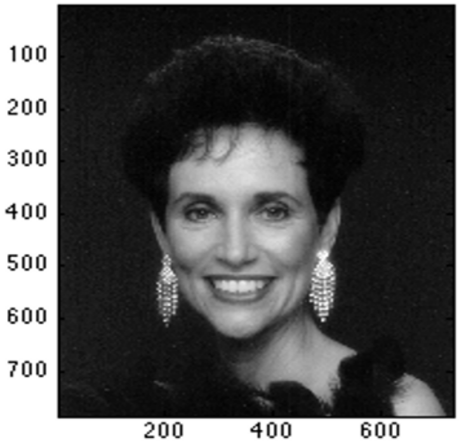
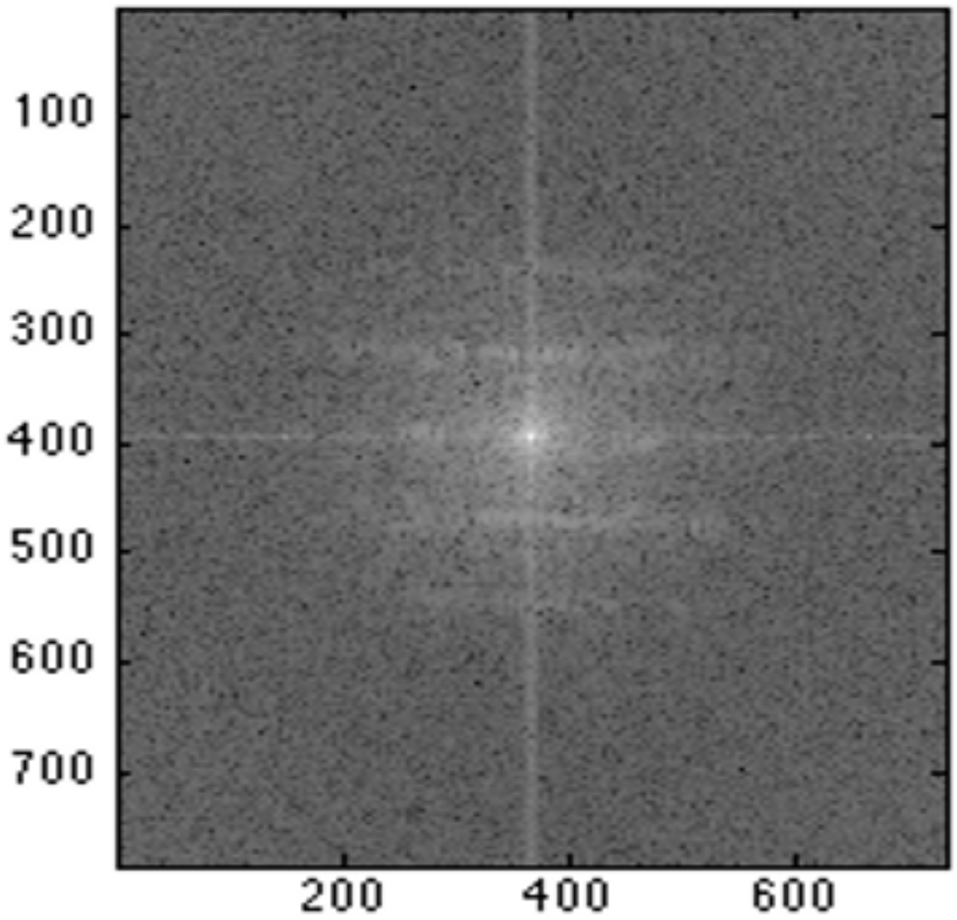


Image in Spatial Domain

Image in Frequency Domain



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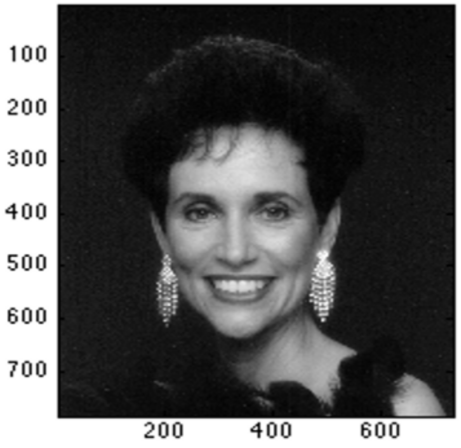

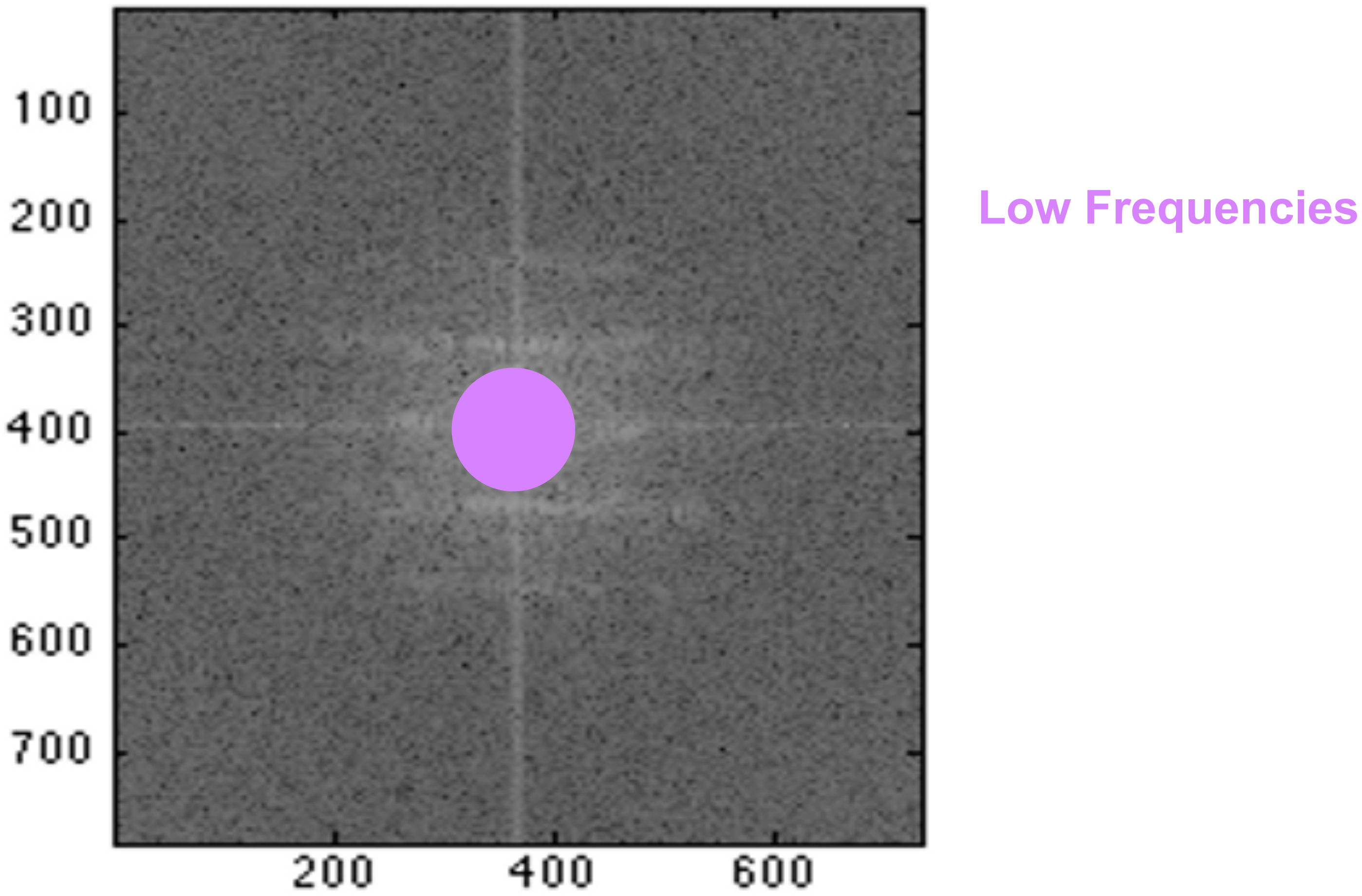

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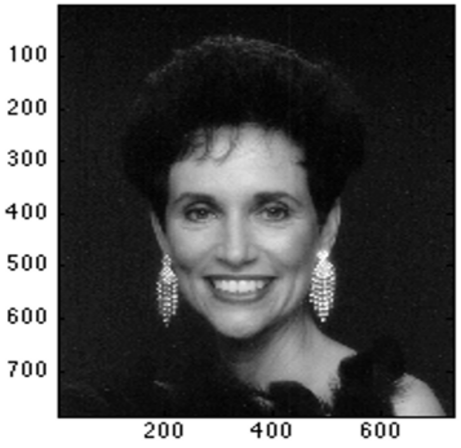

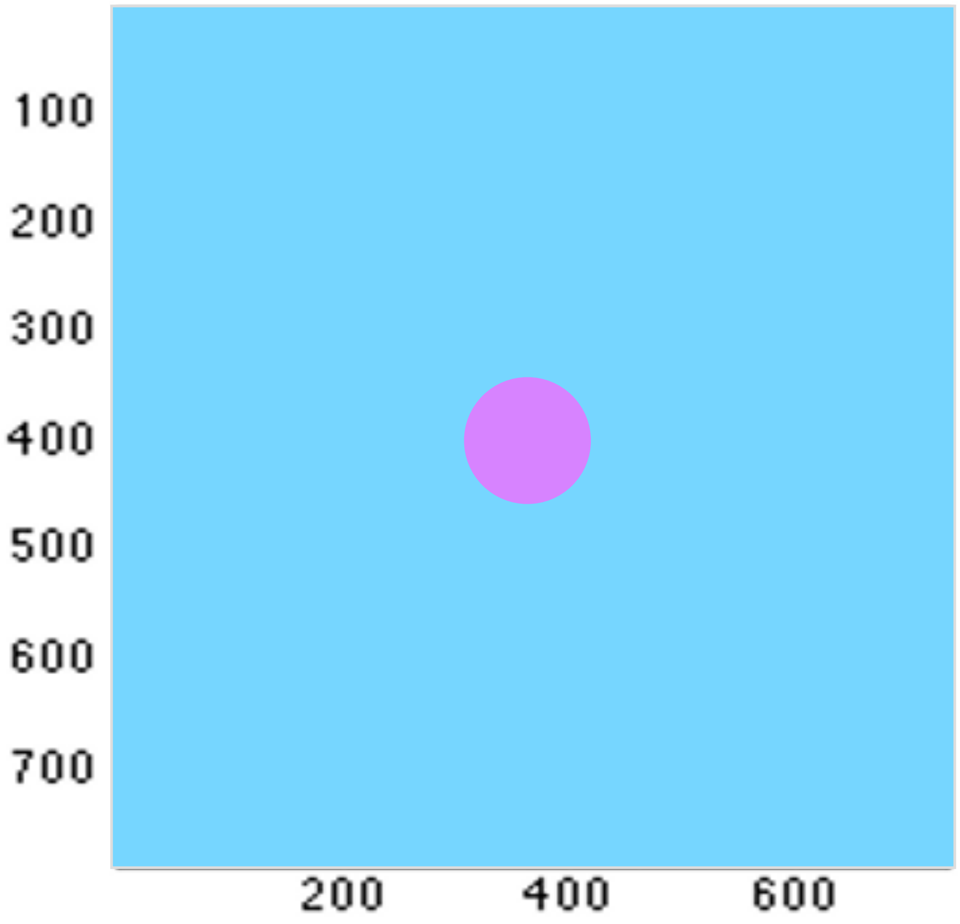

Low Frequencies

High Frequencies

Application Examples

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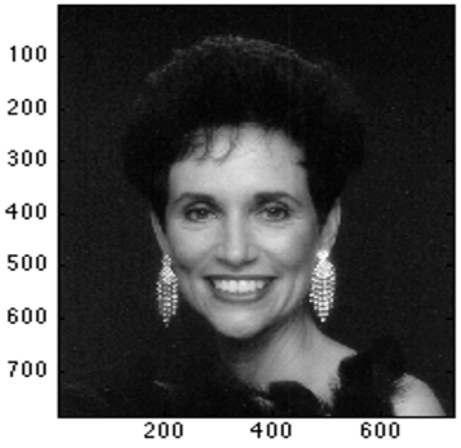

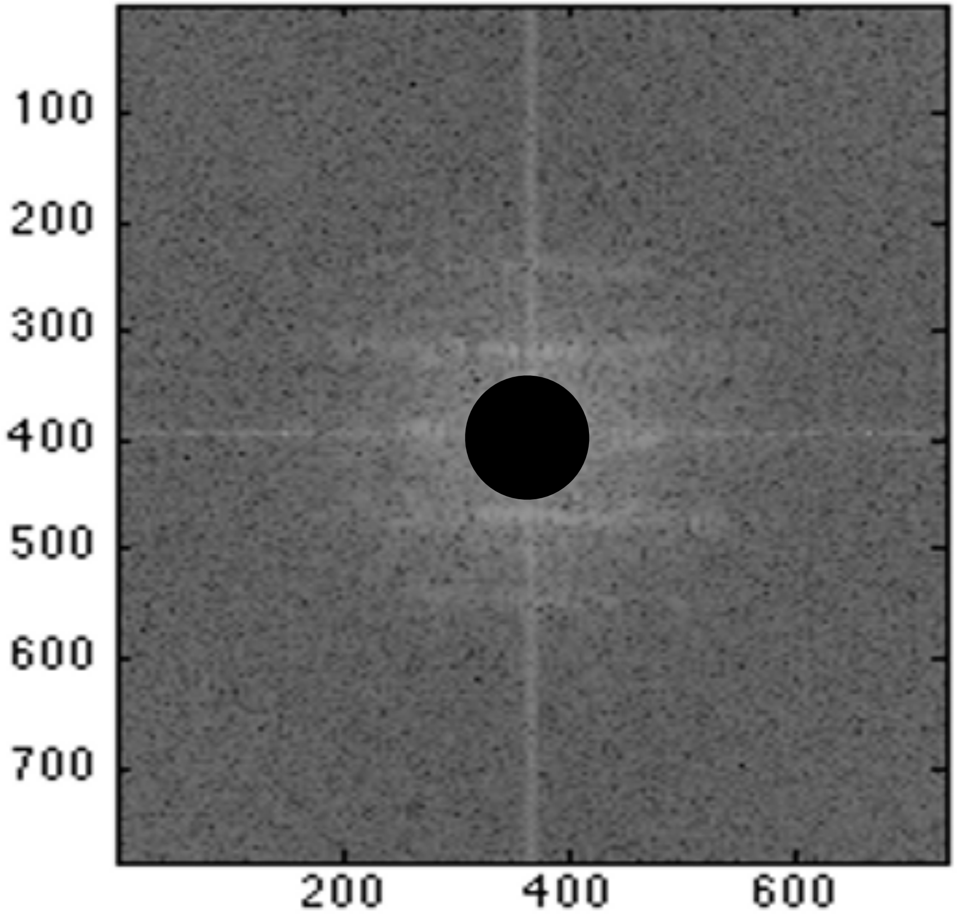

High Pass Filter:

Set the center to zero!

Application Examples

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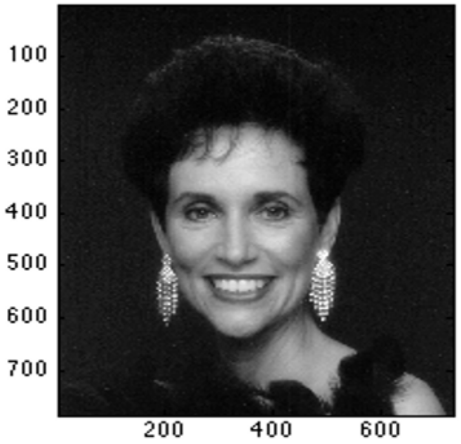
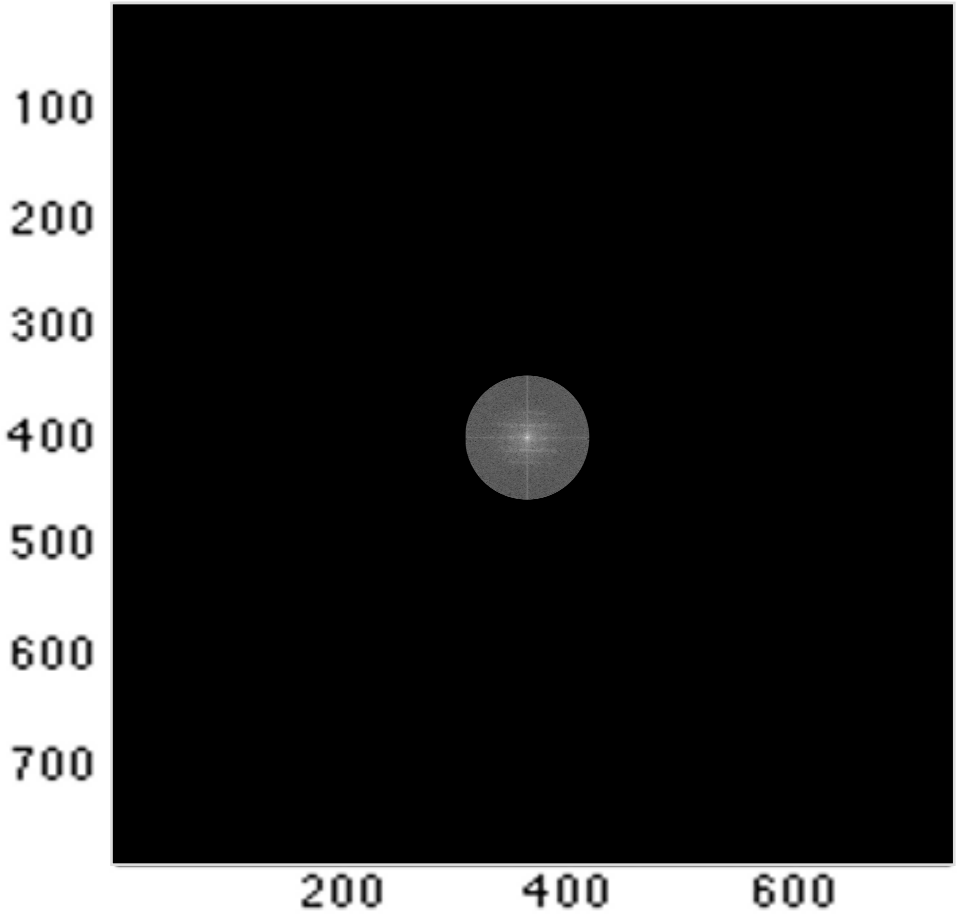




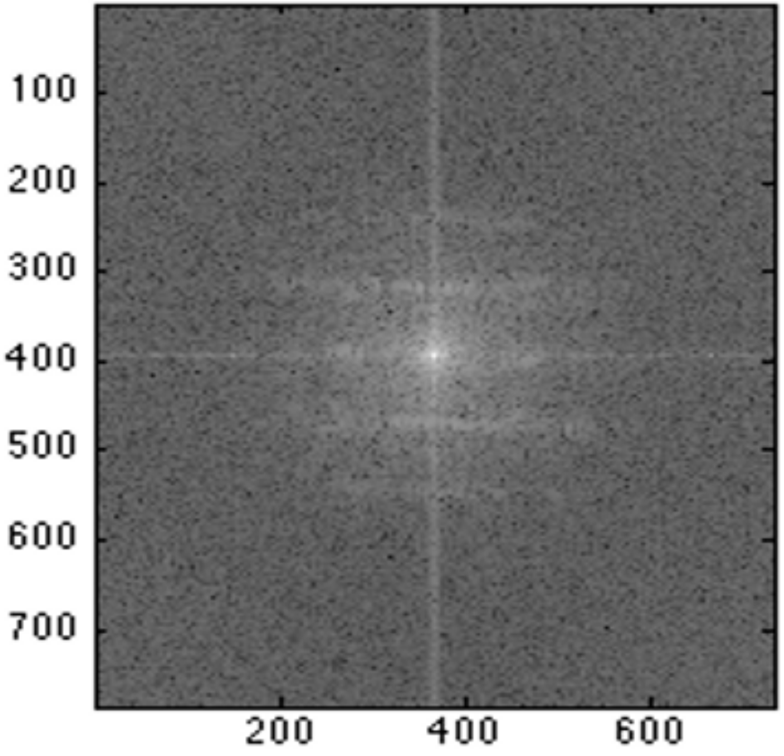
Low Pass Filter:

Set everything but the center to zero!

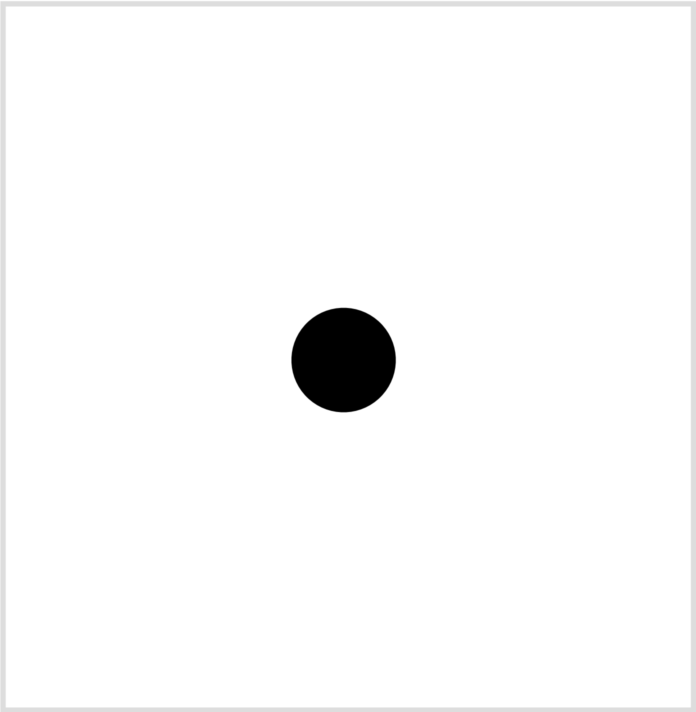
Application Examples

$$\widehat{fg} = \hat{f} * \hat{g}$$

- **Filtering**



$*$



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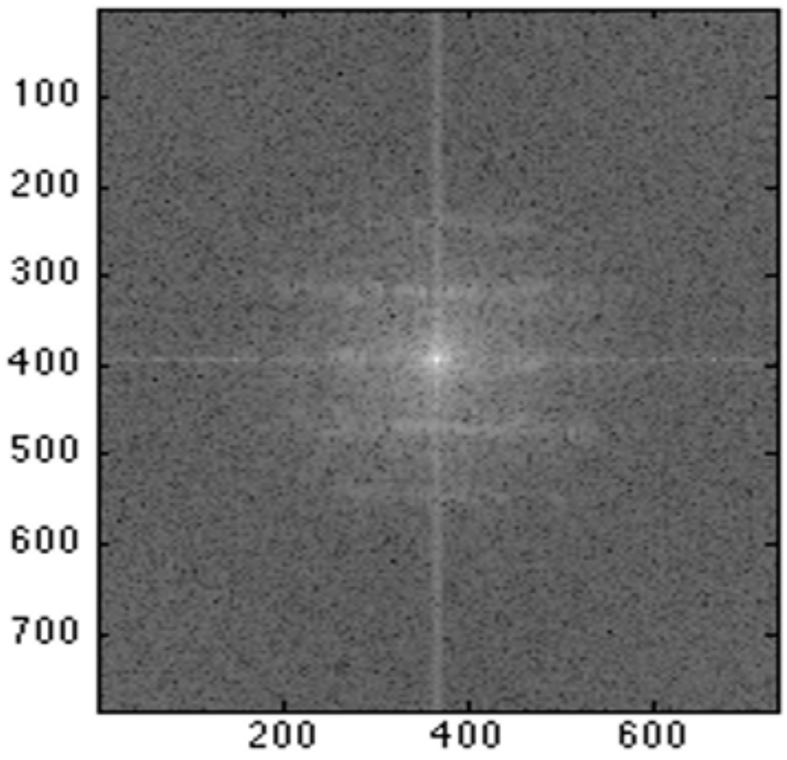

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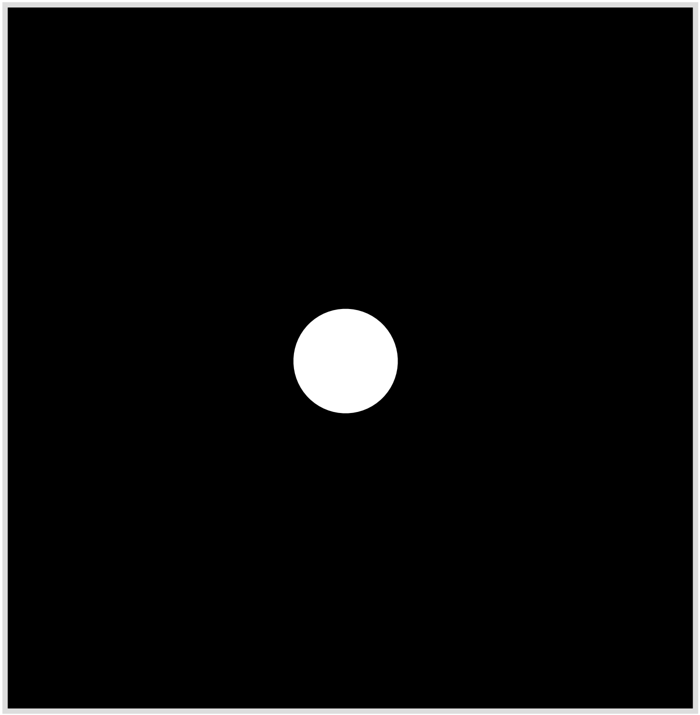

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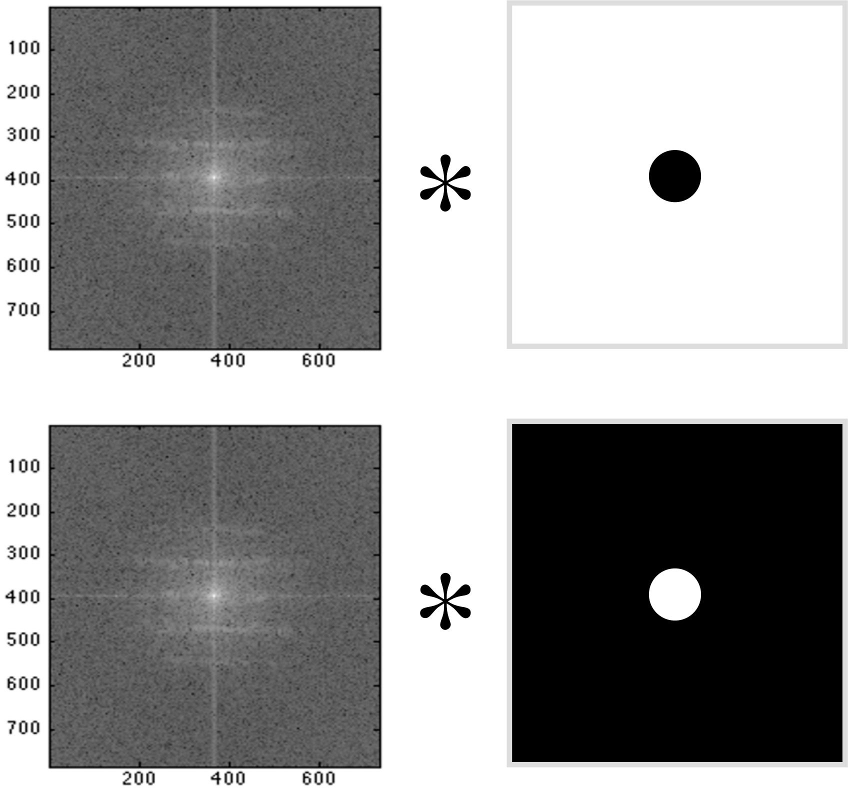
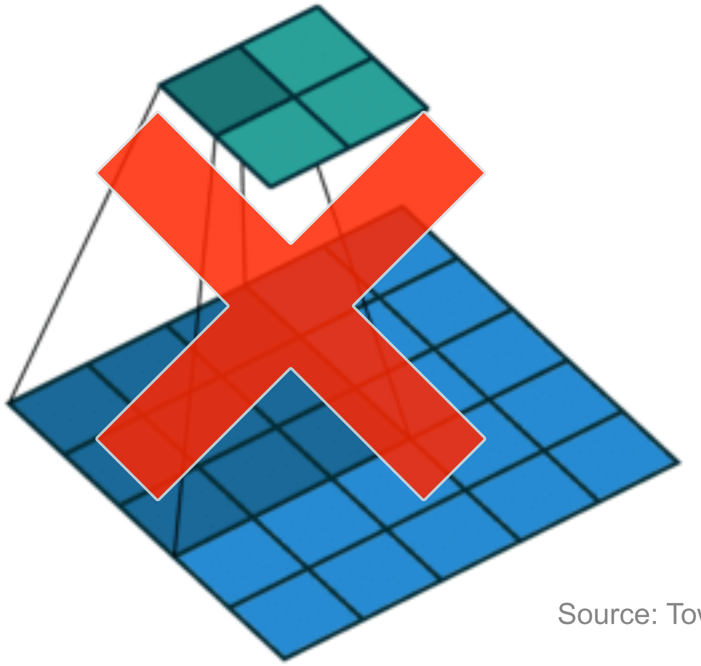
Application Examples

- **Filtering in frequency domain** $\widehat{fg} = \hat{f} * \hat{g}$





Source: Towards Data Science

Application Examples

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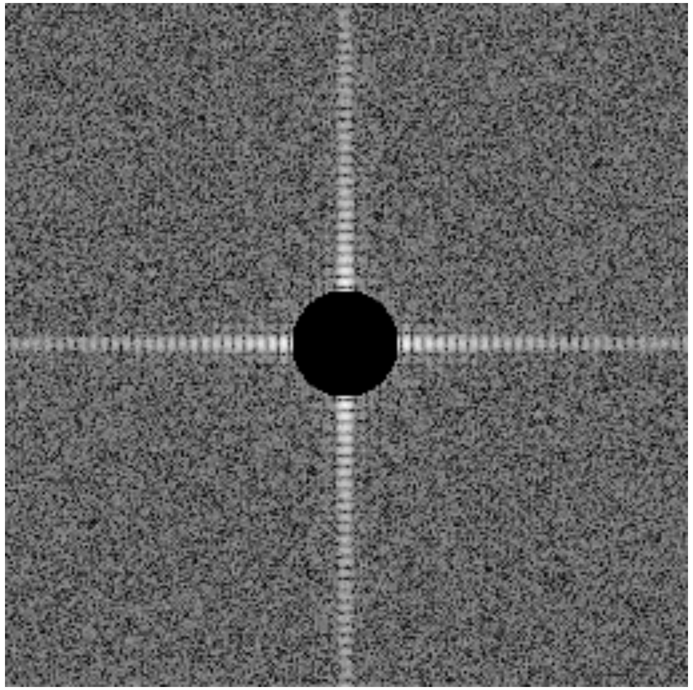
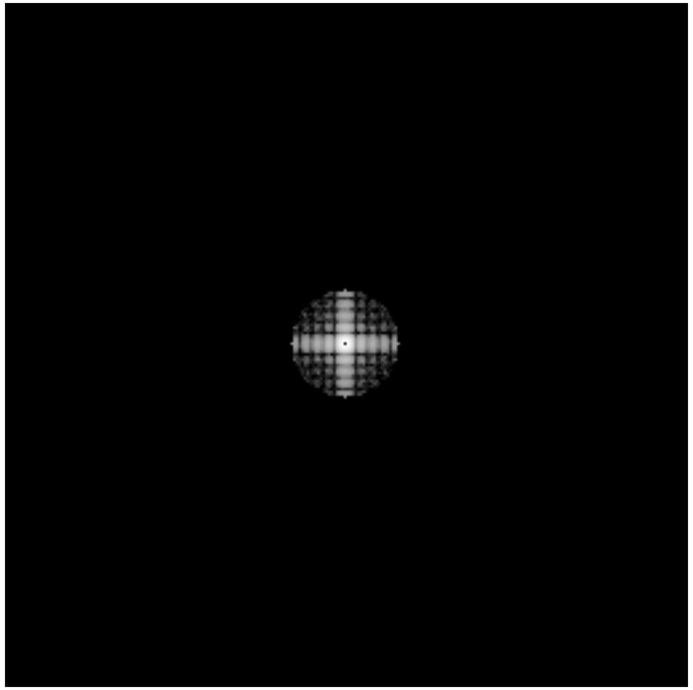
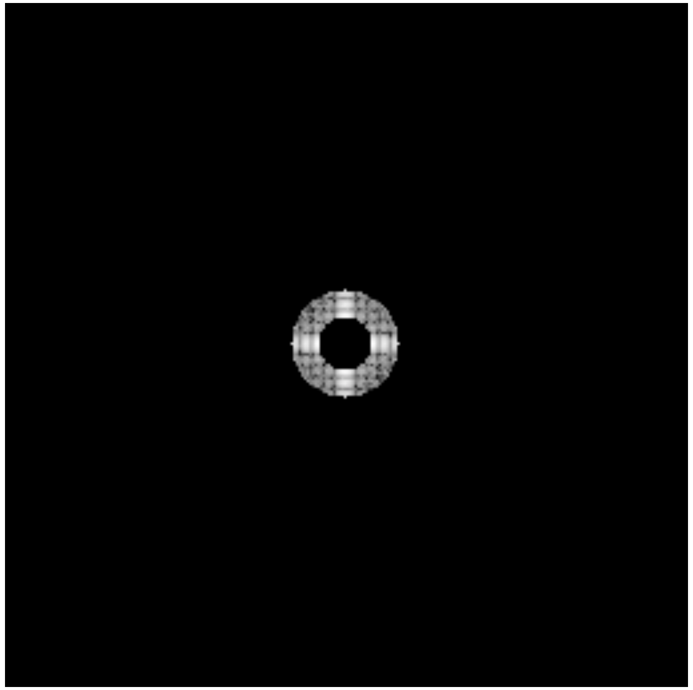


High-Pass



Low-Pass



Band-Pass

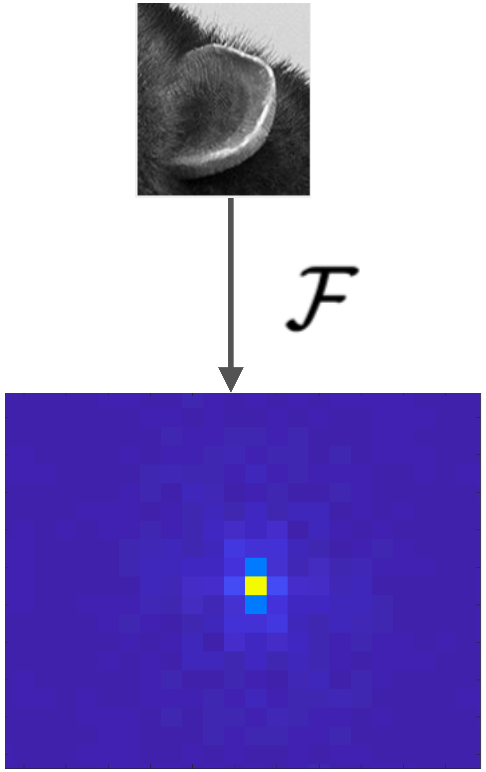
Application Examples

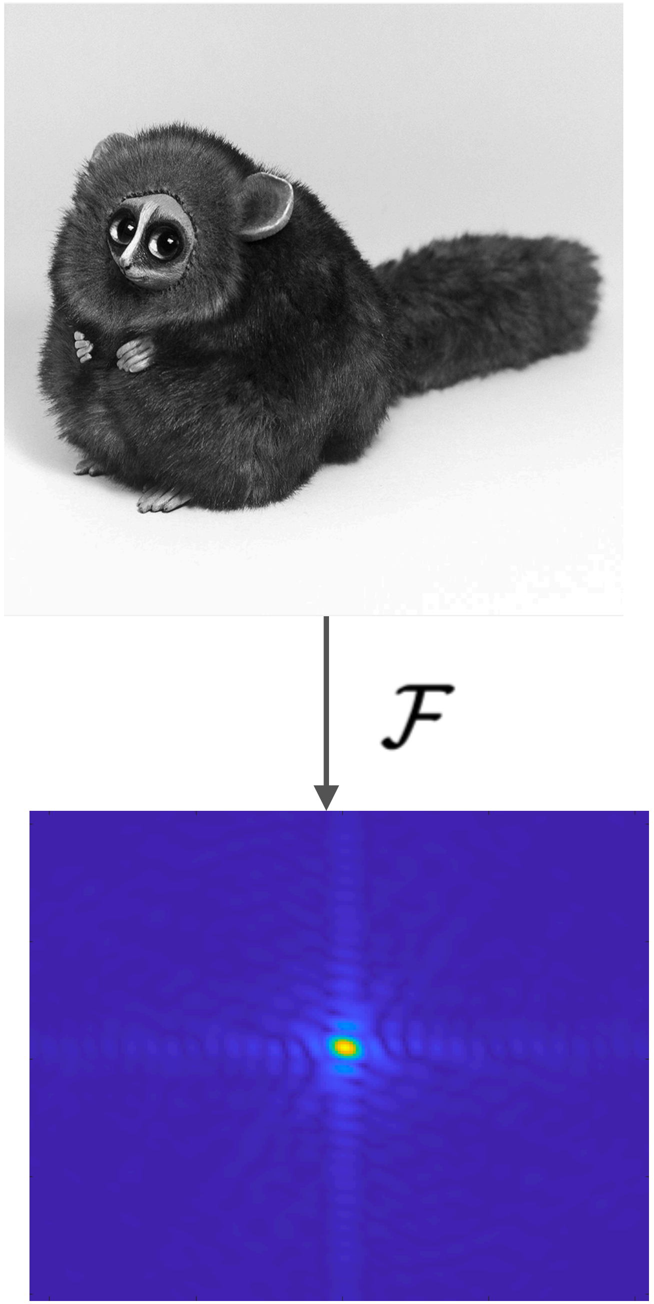
- **Template Matching**

Where in the image is this template?

Application Examples

- **Template Matching**





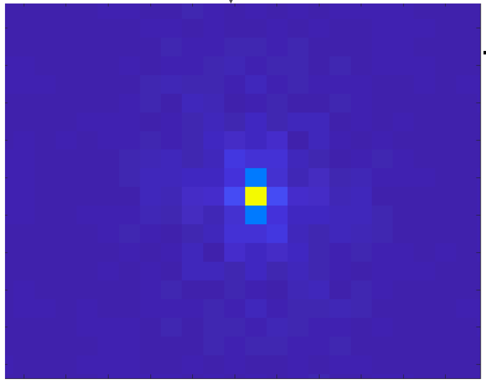
Application Examples

- **Template Matching**

* here means conjugate;
multiplication is element-wise

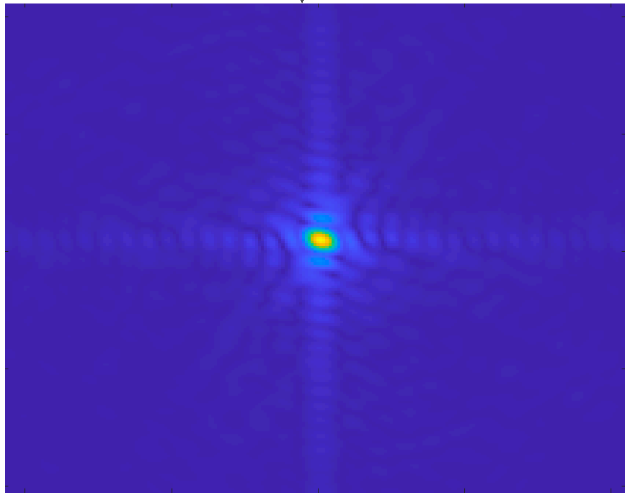
$\mathcal{F}$

$\mathcal{F}$



$$\frac{\widehat{f_{Template}} \circ \widehat{f^{*}_{background}}}{|\widehat{f_{Template}} \circ \widehat{f^{*}_{background}}|}$$

Phase Correlation



Application Examples

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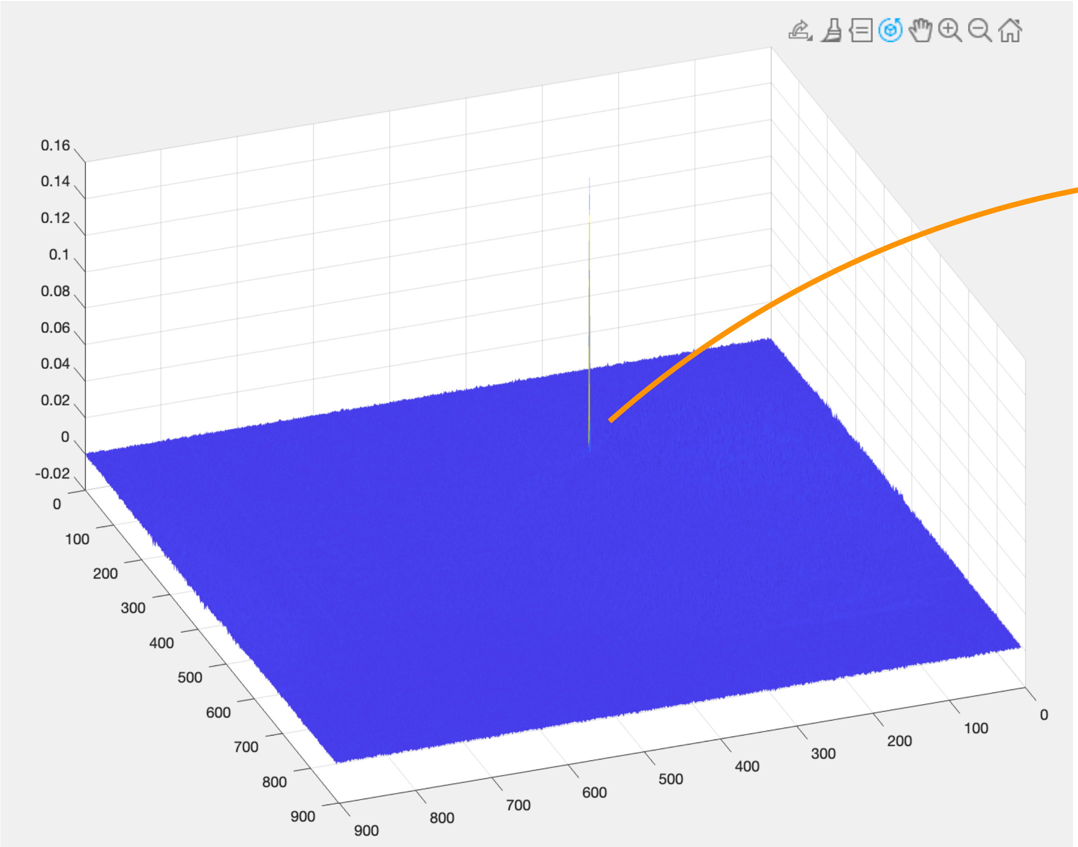

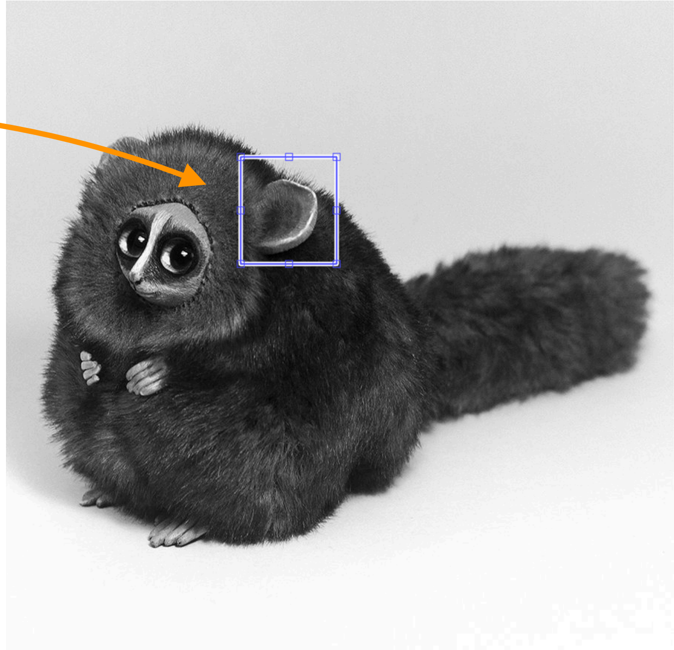

Implementation

- **Discrete Fourier Transform**
  - Images are not actually continuous signals, but sampled on a grid
  - Integral turns into a series

Implementation

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+ some few other caveats :)

Implementation

- **Discrete Fourier Transform (DFT)**
  - Images are not actually continuous signals, but sampled on a grid
  - Integral turns into a series

Discrete Fourier transform

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi \frac{k}{N} n}$$

Inverse transform

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot e^{i2\pi \frac{k}{N} n}$$

- **Fast Fourier Transform (FFT)**
  - Very fast algorithm to compute the DFT, easy to use in Python, Matlab, Julia,…

Implementation

```
import numpy as np
import matplotlib.pyplot as plt

# Load an image (grayscale)
image = plt.imread('your_image.png')

# Apply Fourier Transform
f_transform = np.fft.fft2(image)
f_transform_shifted = np.fft.fftshift(f_transform)

# Get magnitude spectrum
magnitude_spectrum = np.log(np.abs(f_transform_shifted) + 1)

# Plot magnitude spectrum
plt.subplot(1, 2, 2)
plt.title('Magnitude Spectrum')
plt.imshow(magnitude_spectrum, cmap='gray')
plt.axis('off')

plt.show()
```

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Using it is really easy!

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For visualization of the spectrum

Summing Up

- **Definition**
  - A mathematical operation that decomposes a function into its constituent frequencies
- **Frequency Domain**
  - Represents how much of the signal lies within each given frequency band
- **Inverse Fourier Transform**
  - Converts the frequency domain back to the original signal
- **Continuous vs. Discrete**
  - The Continuous Fourier Transform (CFT) is for continuous signals, while the Discrete Fourier Transform (DFT) is for discrete signals (like digital images)
- **Fast Fourier Transform (FFT)**
  - Efficient algorithm to compute the DFT
- **Magnitude and Phase**
  - Each frequency component has a magnitude (how strong it is) and a phase (where it starts)