Fourier Transforms: CFT, DFT, FFT

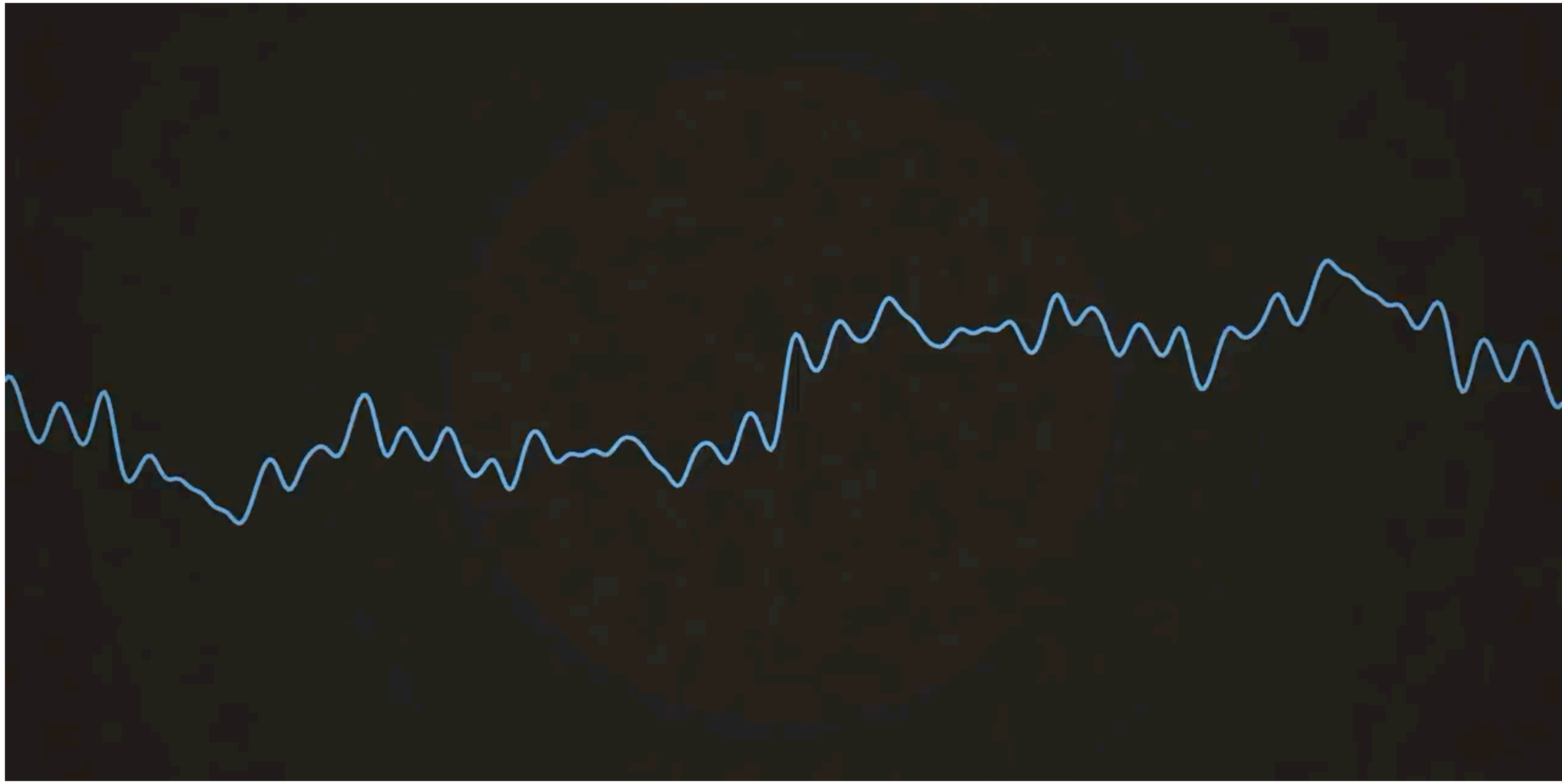
FYS-2010-1 25V Image Analysis
by Elisabeth Wetzer

Overview

- **Recap**
- **Continuous Fourier Transform**
  - Notation
  - Euler’s Formula
- **Discrete Fourier Transform**
  - The effect of sampling
  - How to compute
  - Fast Fourier Transform
  - Periodicity and Frequency leakage
- **2D Discrete Fourier Transform**
  - Definition
  - Separability
- **Small Quiz**

Recap

- **Fourier Transform**
  - A signal can has a representation in time/spatial domain and a corresponding representation in frequency domain



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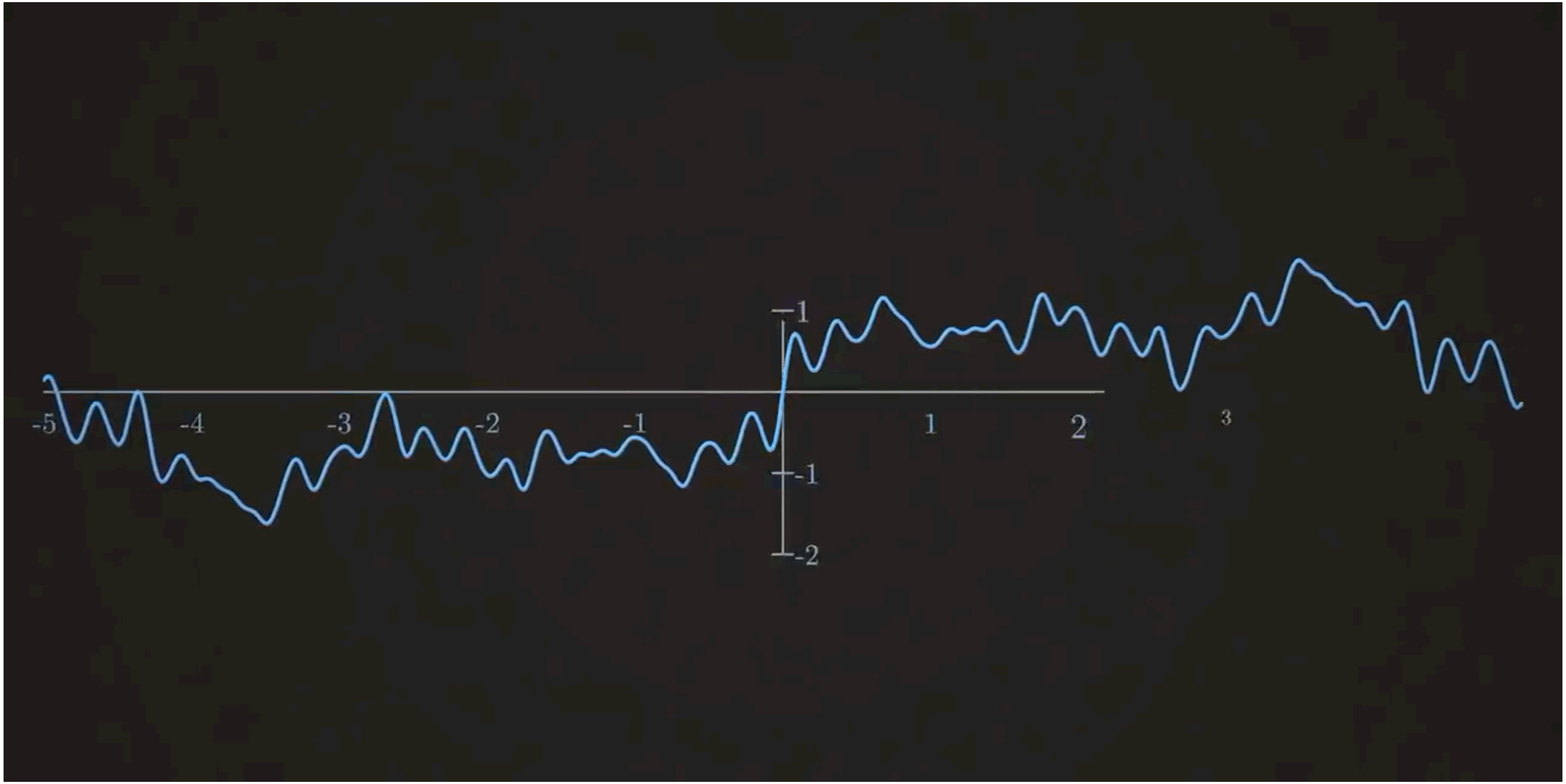
  - We can map back and forth using the Fourier Transform and its Inverse
  - The frequency domain represents how much of the signal lies within each given frequency band and decomposes the signal into the underlying signals that together make up the observed signal

Continuous Fourier Transform (CFT)

- **Fourier Transform** $$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x}\, dx, \quad \forall \xi \in \mathbb{R}.$$
- **Inverse Transform** $$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{i2\pi\xi x}\, d\xi, \quad \forall\, x \in \mathbb{R}.$$
- **Fourier Transform Pair** $$f(x) \xleftrightarrow{\mathcal{F}} \widehat{f}(\xi)$$

Continuous Fourier Transform (CFT)



Continuous Fourier Transform (CFT)

- **Obtaining the Frequency Spectrum (Intuition)**
  - To know how much of a particular sine wave is in a signal, multiply the signal by the sine wave at each point and add up the area under the curve
  - If they are uncorrelated, i.e. if the particular sine wave is not part of the signal, the area will be zero
  - If they are correlated, i.e. if the particular sine wave is part of the signal, the area will be non-zero
  - The size of the area gives the relative amplitude of that frequency sine wave in the signal
  - Doing this for all frequencies results in the corresponding frequency spectrum

FT $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x}\, dx, \quad \forall \xi \in \mathbb{R}.$

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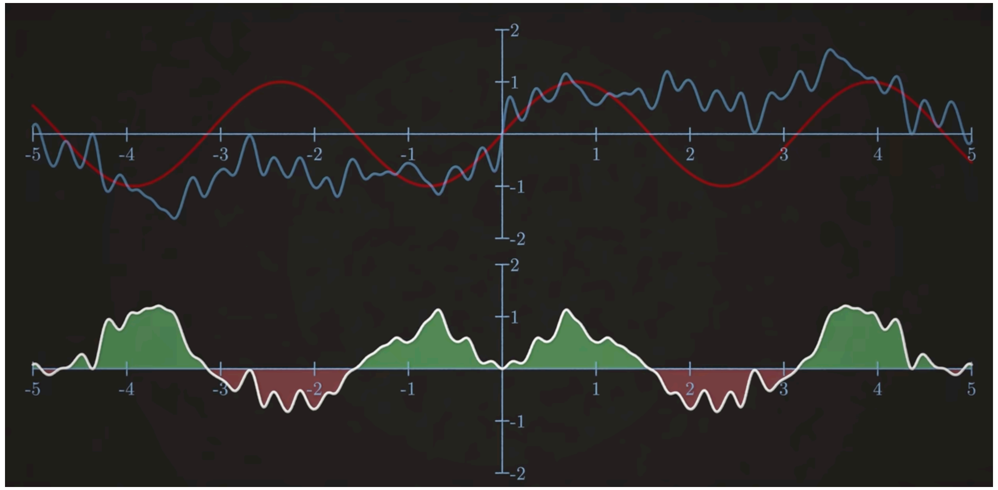

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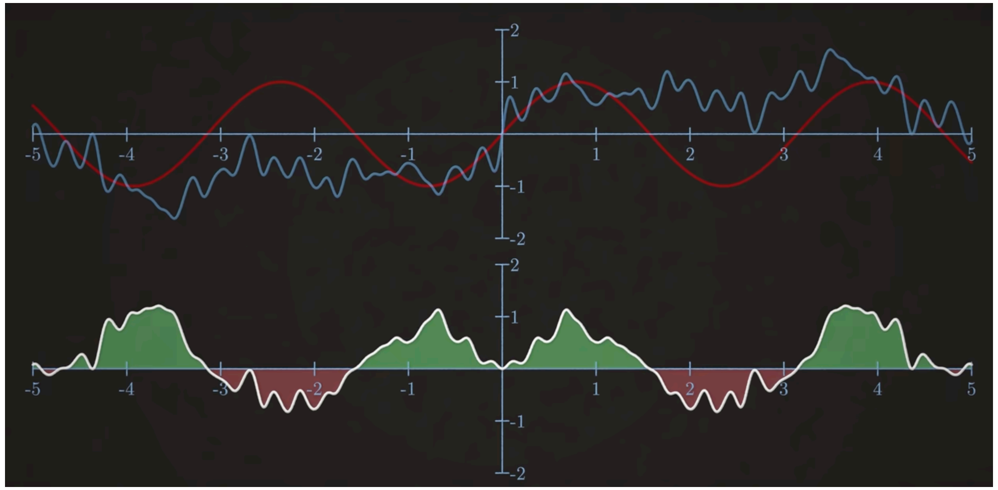

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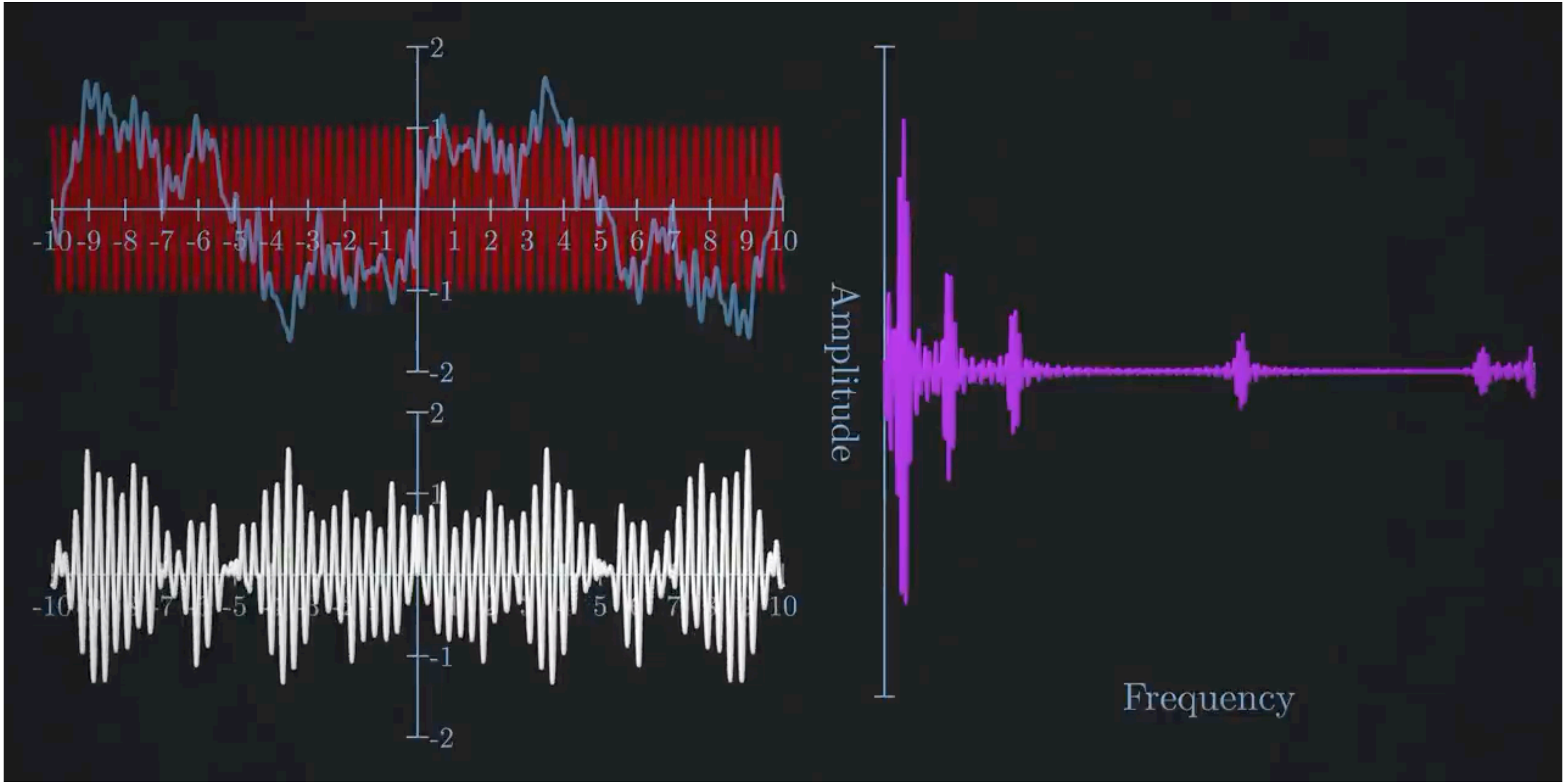
Almost! Sine is not enough

Continuous Fourier Transform (CFT)



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Continuous Fourier Transform (CFT)

- **Equivalent Forms (via Euler's formula)**

$$\widehat{f}(\xi) = \underbrace{Ae^{i\theta}}_{\text{polar coordinate form}} = \underbrace{A\cos(\theta) + iA\sin(\theta)}_{\text{rectangular coordinate form}}.$$

$$\begin{aligned}
\widehat{f}(\xi) \cdot e^{i2\pi\xi x} &= Ae^{i\theta} \cdot e^{i2\pi\xi x} \\
&= \underbrace{Ae^{i(2\pi\xi x+\theta)}}_{\text{polar coordinate form}} \\
&= \underbrace{A\cos(2\pi\xi x + \theta) + iA\sin(2\pi\xi x + \theta)}_{\text{rectangular coordinate form}}.
\end{aligned}$$

Continuous Fourier Transform (CFT)

- **Obtaining the Frequency Spectrum (Intuition)**
  - For each frequency, we have to multiply by a sine wave and a cosine wave and find the amplitudes of each
  - The ratio of those amplitudes indicates the phase
- **Euler’s Formula**
  - Allows to calculate them simultaneously using the exponential instead of sweeping all frequencies individually

Discrete Fourier Transform (DFT)

- **Discrete Fourier Transform (DFT)**
  - Images are not actually continuous signals, but sampled on a grid
  - Integral turns into a series

Discrete Fourier transform

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi \frac{k}{N} n}$$

Inverse transform

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot e^{i2\pi \frac{k}{N} n}$$

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  - $\frac{1}{N}$ may be multiplied to either DFT or IDFT, or alternatively $\frac{1}{\sqrt{N}}$ can be multiplied to both transforms (less common, but all versions are ok and equivalent!)

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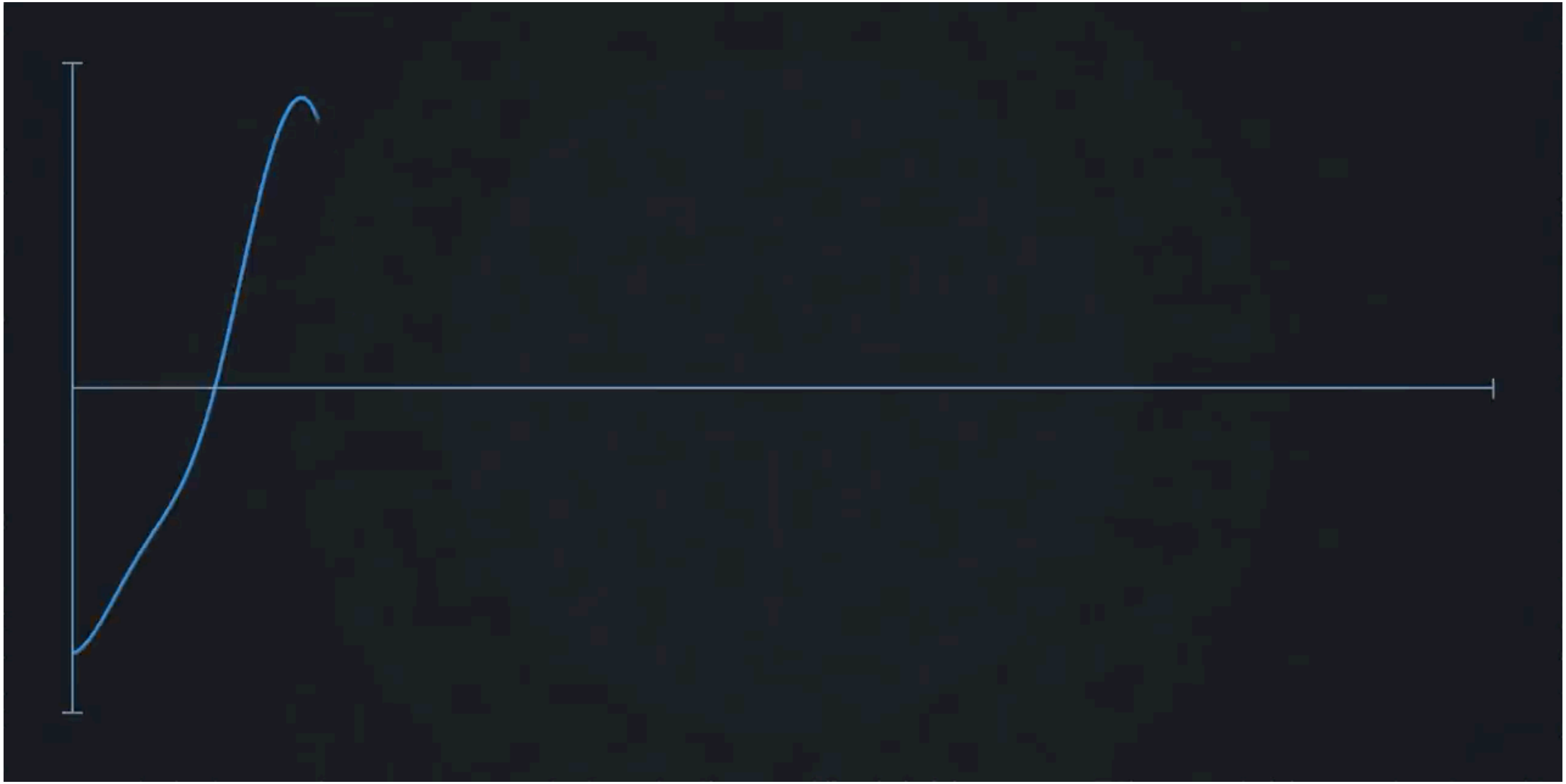
DFT - Sampling



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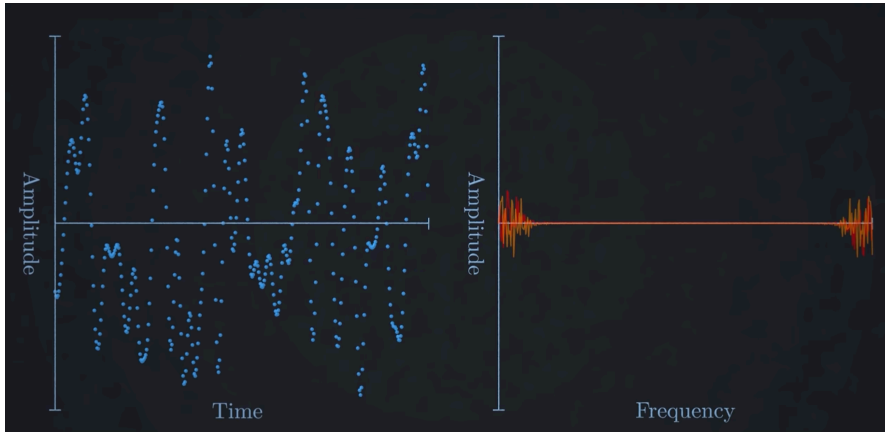
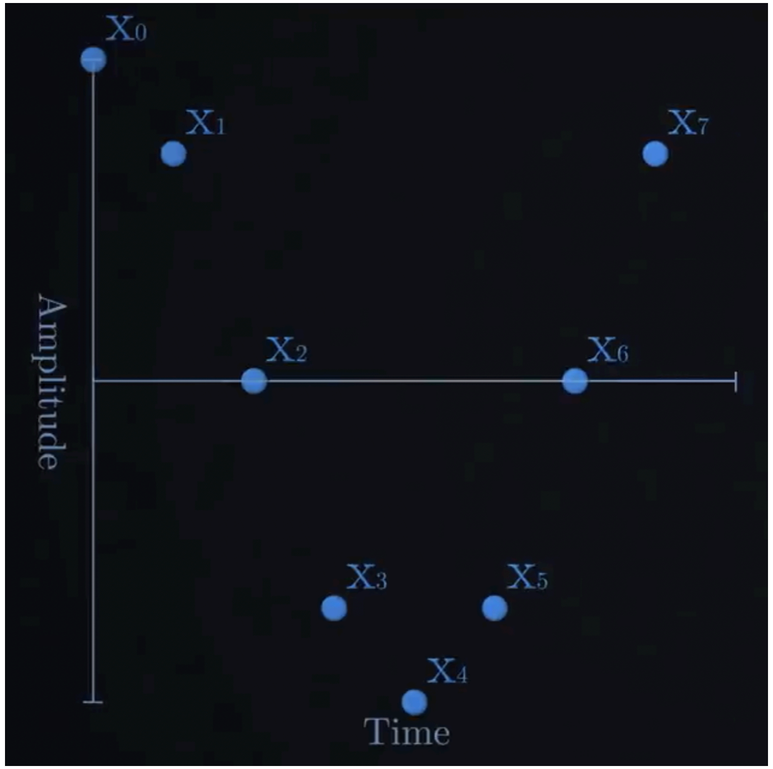


DFT - Sampling

- **Duality between signal length and resolution**
  - Discrete Signal $\Longleftrightarrow$ Discrete Frequency Spectrum
  - Number of Samples $=$ Number of Frequency Bins
  - The coarser sampled the data points, the lower the maximum frequency that can be determined, i.e.:
    Low Spatial/Time Resolution $\Longleftrightarrow$ "Short" Frequency Spectrum
  - The shorter the signal, the harder it is to tell similar frequencies apart, i.e.:
    Short Signal $\Longleftrightarrow$ Low Frequency Resolution

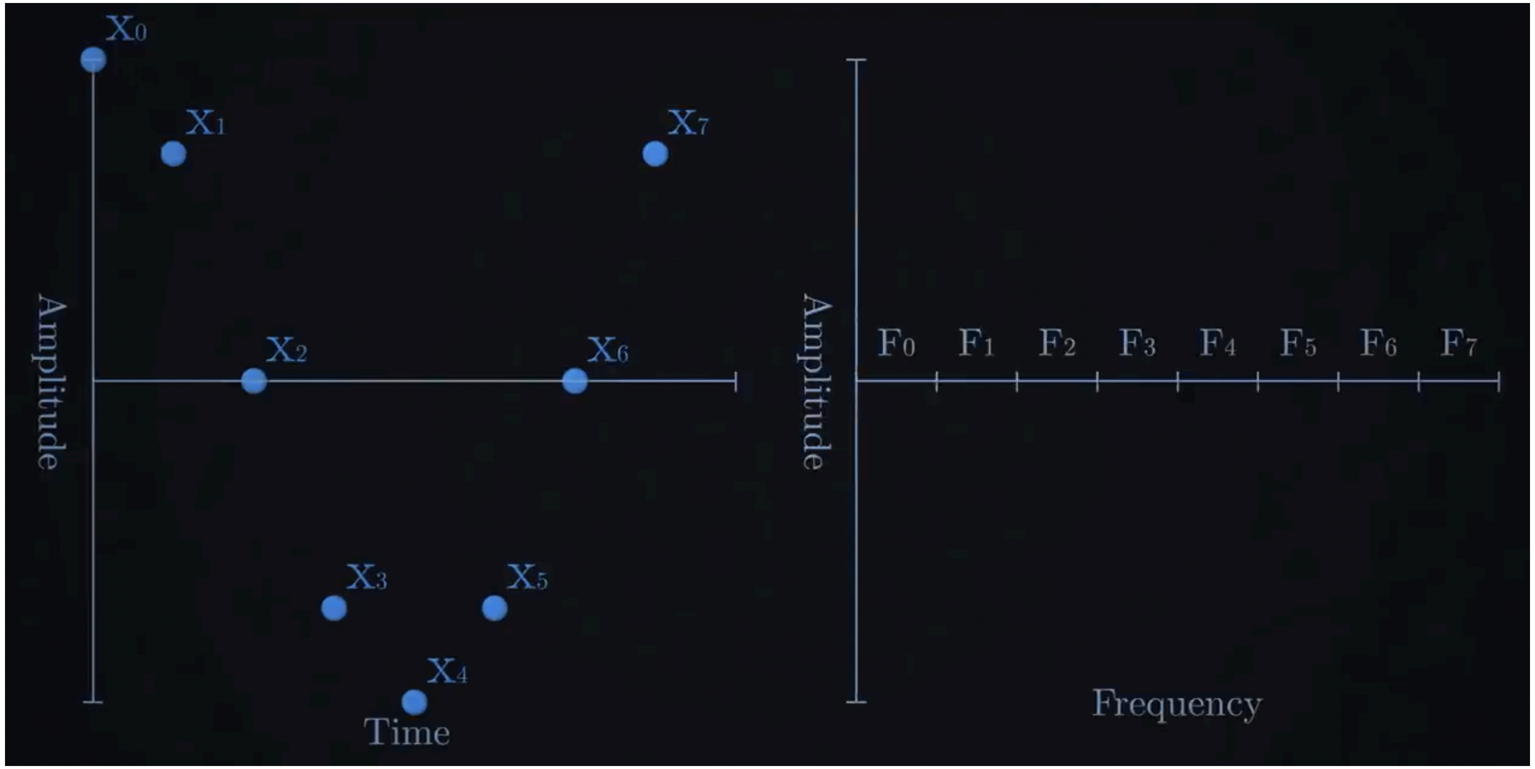
DFT - How to Compute

- Example
- Assume signal consisting of 8 data points
- How many frequency bins?



DFT - How to Compute

- **Example**
- **Assume signal consisting of 8 data points**
- **I.e., 8 frequency bins**



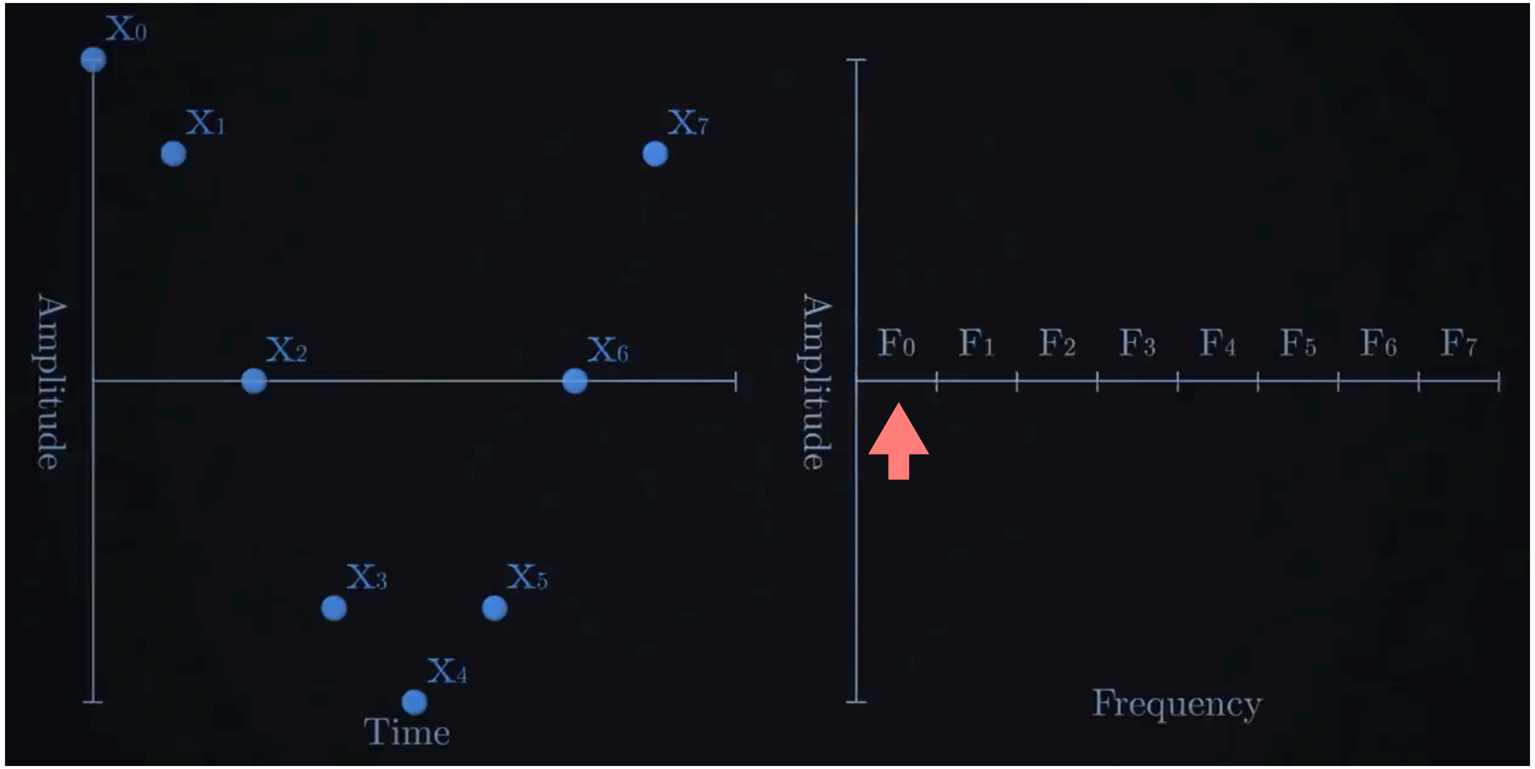
Discrete Fourier transform

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi \frac{k}{N} n}$$

each term of this series is called “Frequency component” (“bin”)

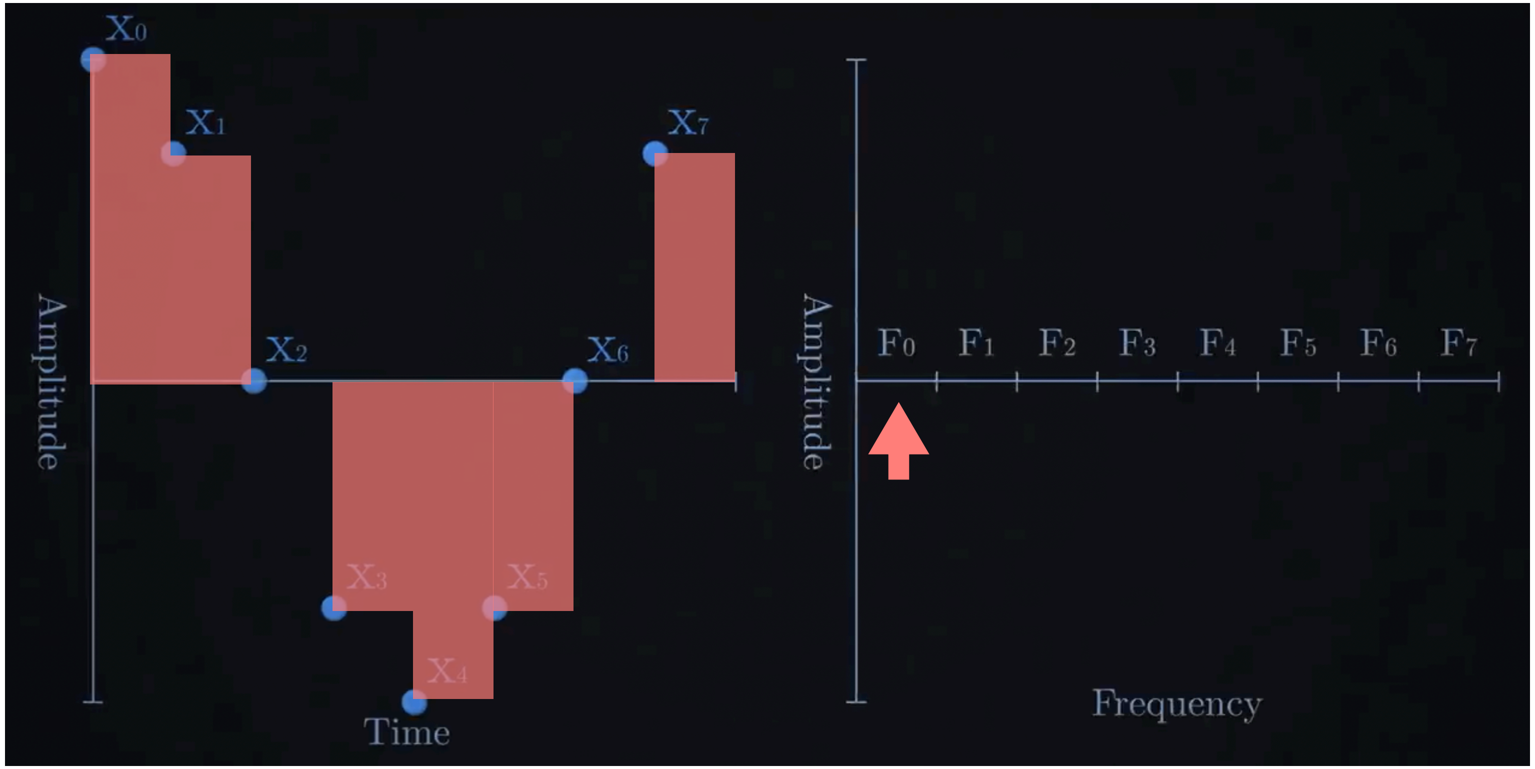
DFT - How to Compute

- F_0
  - corresponds to frequency zero
  - measures systematic shift off the x-axis
  - “measuring DC offset”



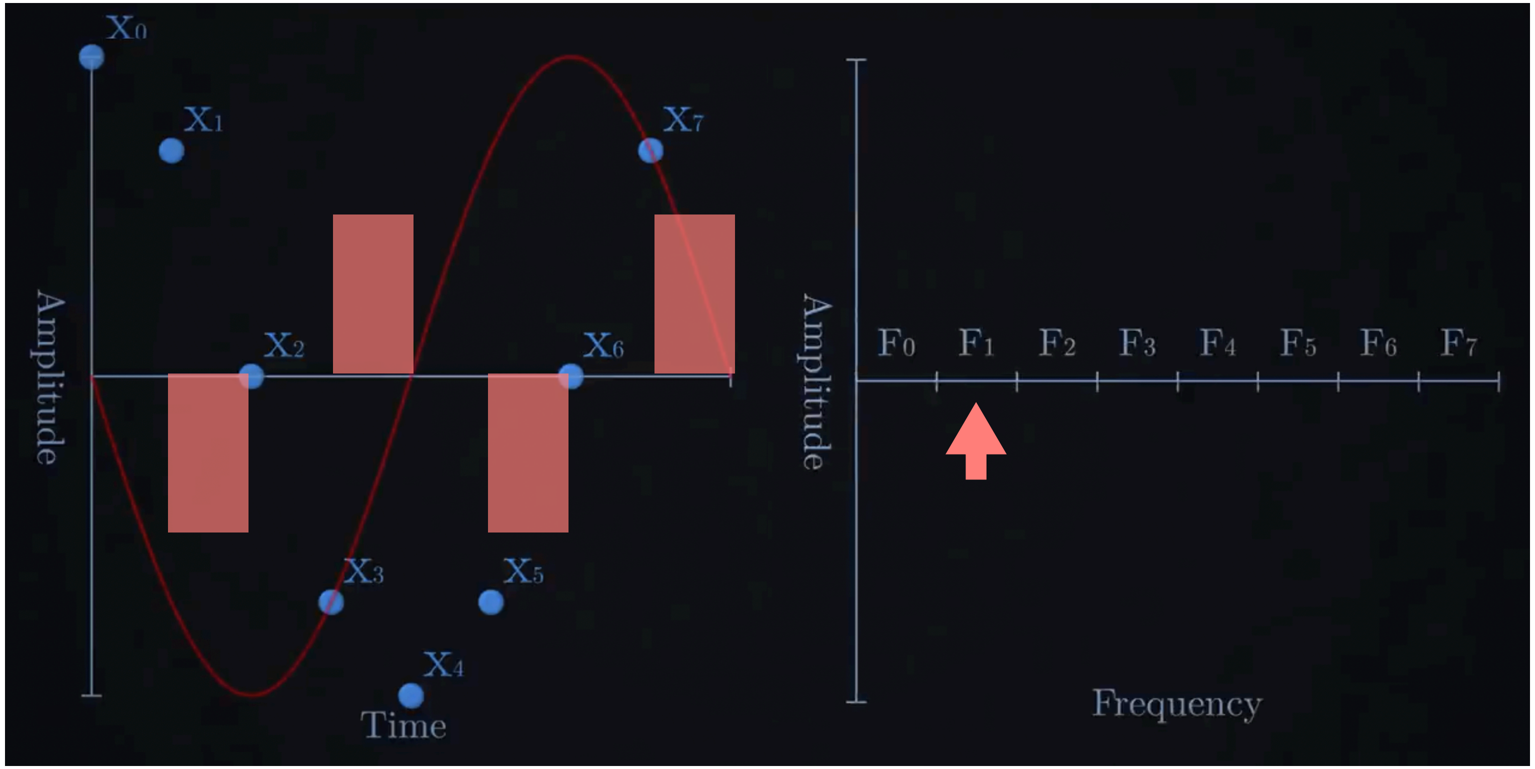
DFT - How to Compute

- F_0
  - Multiply all data points by 1 and add them up
  - In this case zero



DFT - How to Compute

- $\mathbf{F_1}$
  - Frequency that fits one period into the duration of the signal
  - Multiply each data point by the sine wave of that frequency, in this case 1 Hz and add them up



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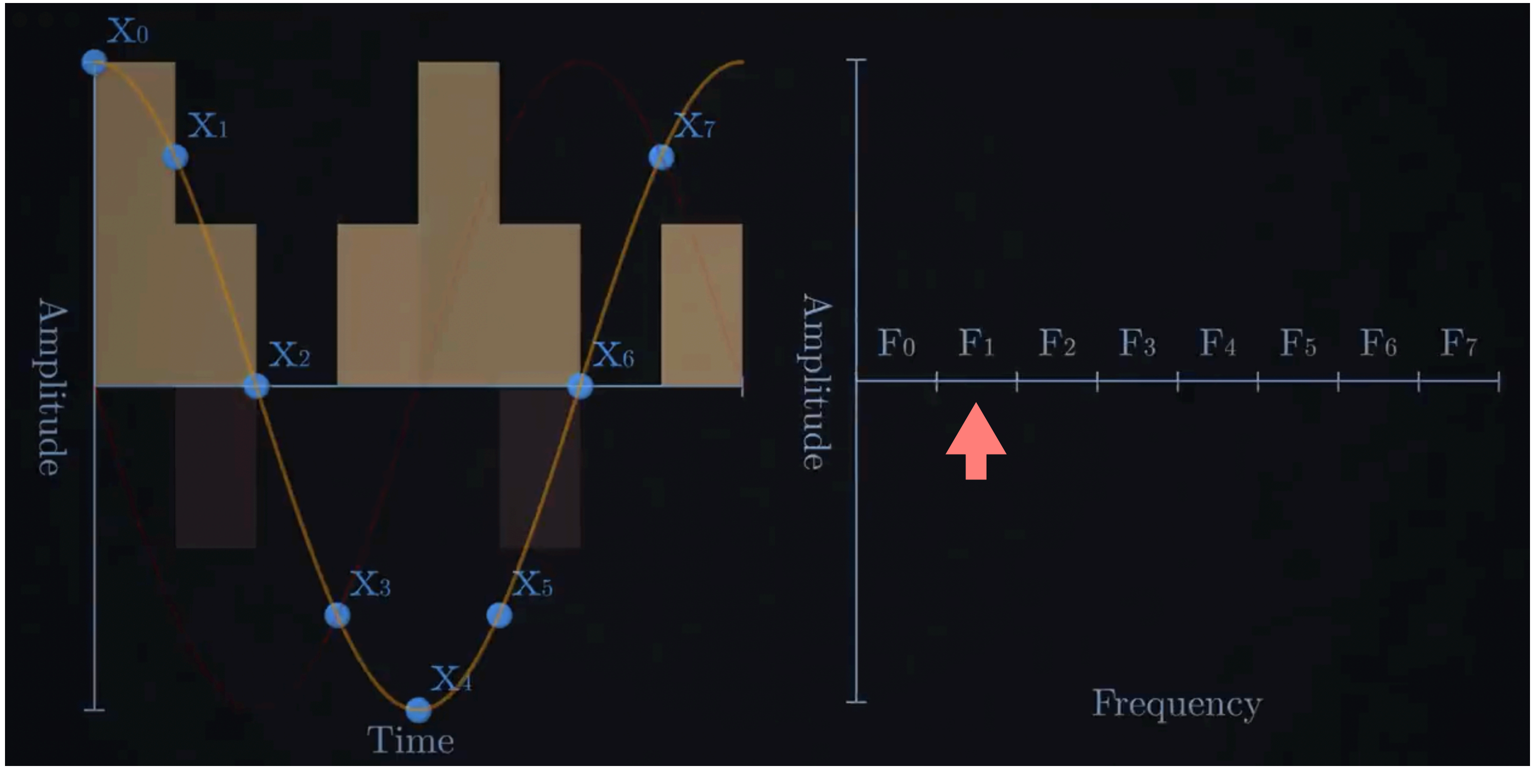

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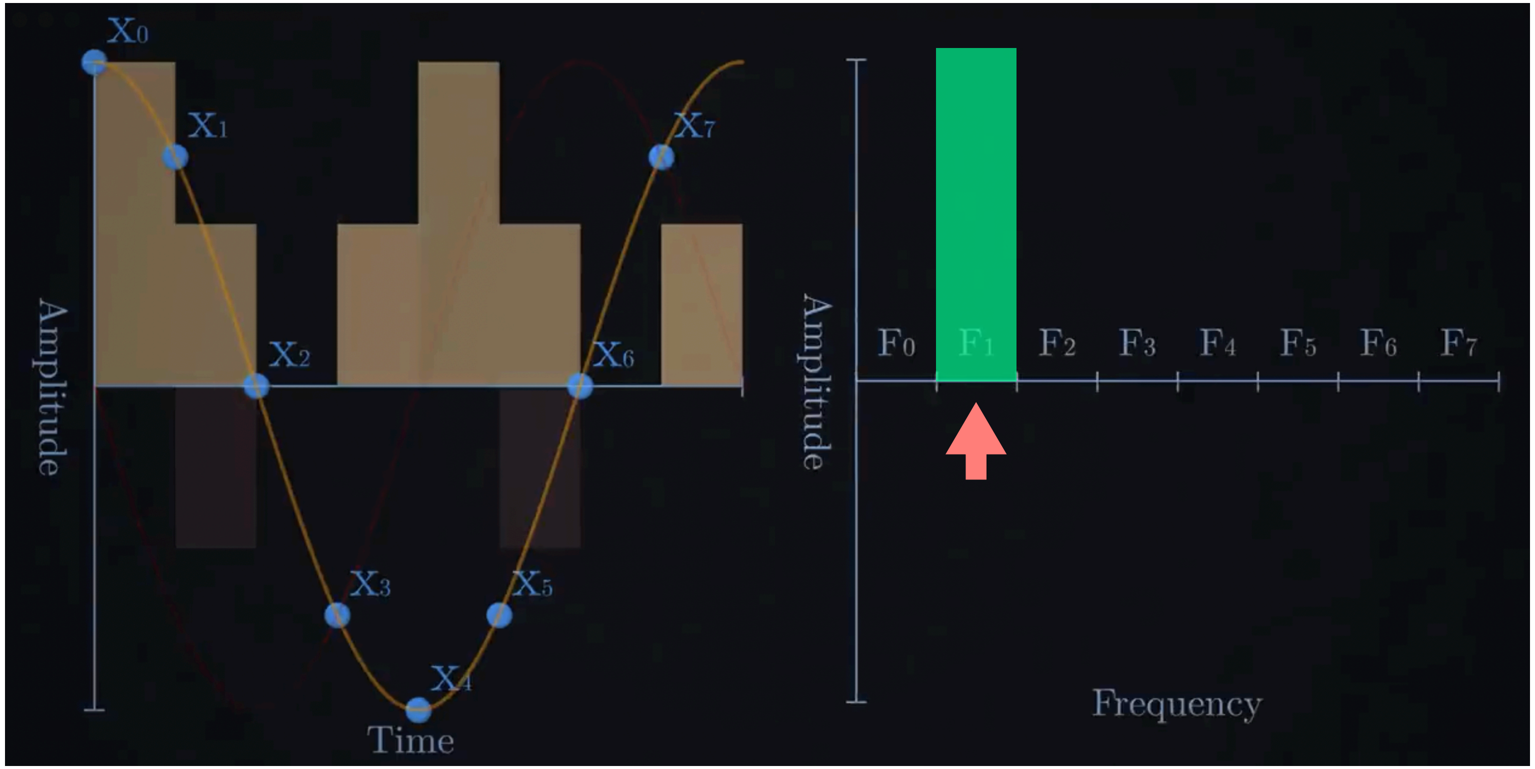

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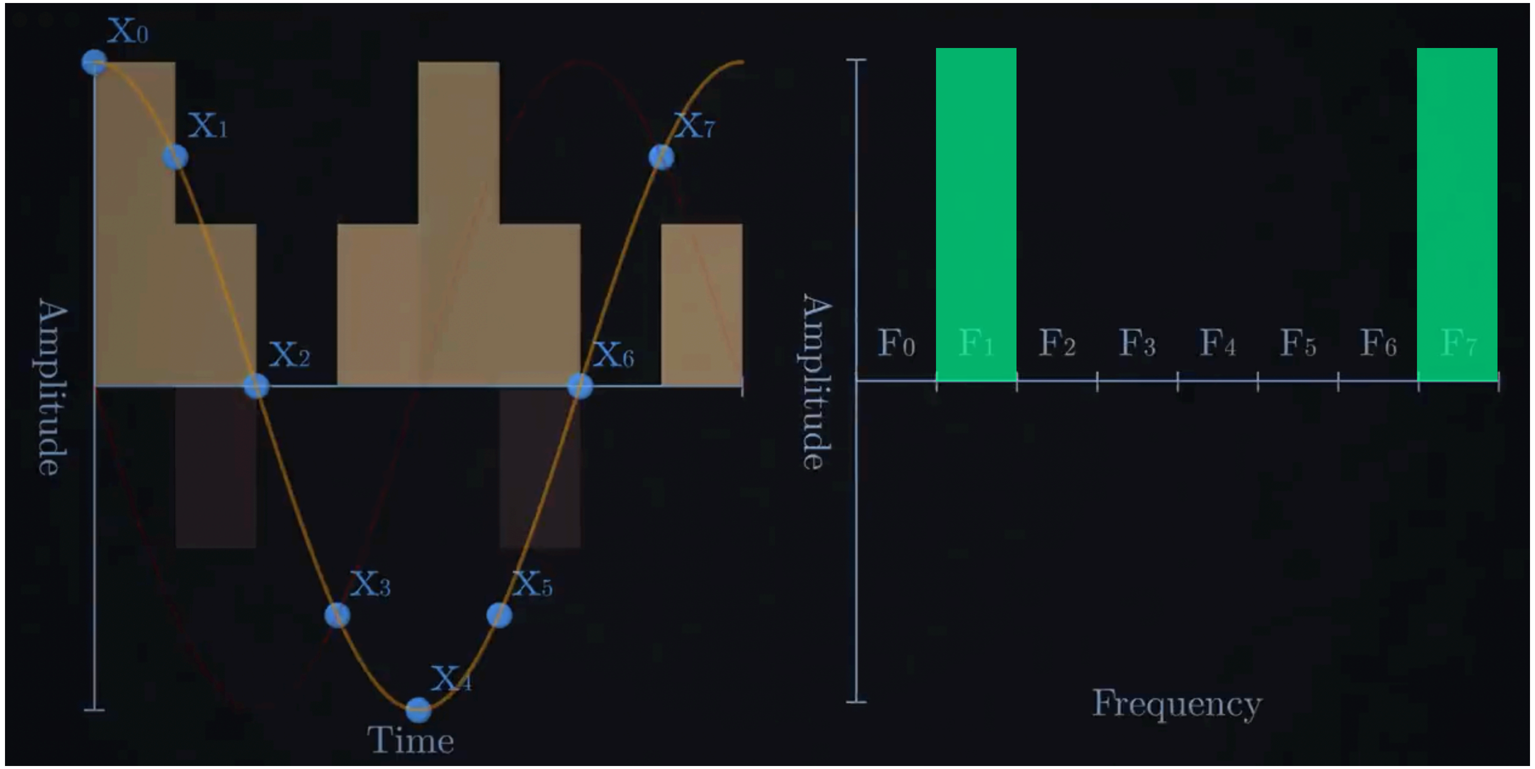

…. and so on for all frequency components!

Fast Fourier Transform (FFT)

DFT - Notation

- **Usually simplified notation**
  - $f(x)$ for $x = 0,1,2,\ldots N-1$ represents N samples from its continuous counterpart
  - Not necessarily taken at integer values, but at equally spaced, arbitrary points
  - x_0 is the location of the first sampled point $f(x_0)$
  - The next point gives $f(x_0 + \Delta x)$, the k-th point gives $f(x_0 + k\Delta x)$, but often it is written as $f(k)$ or x_k
- **Similar for FT:**
  - $\hat{f}(\xi)$ most often means $\hat{f}(\xi) \mathrel{\hat{=}} \hat{f}(\xi\Delta\xi)$ for $\xi = 0,1,2,\ldots,N-1$ and the sequence starts at the true zero frequency, often written as X_k

Periodicity and Frequency Leakage

- **Multiples of the Fundamental Frequency**
  - Lowest non-zero frequency that can be measured is one whose period is equal to the duration of the entire signal
  - Higher frequency bins are integer multiples of that frequency so that they fit 2,3,4,… periods in the duration of the signal

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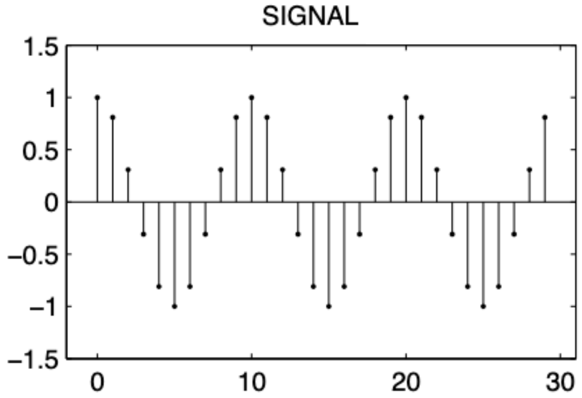

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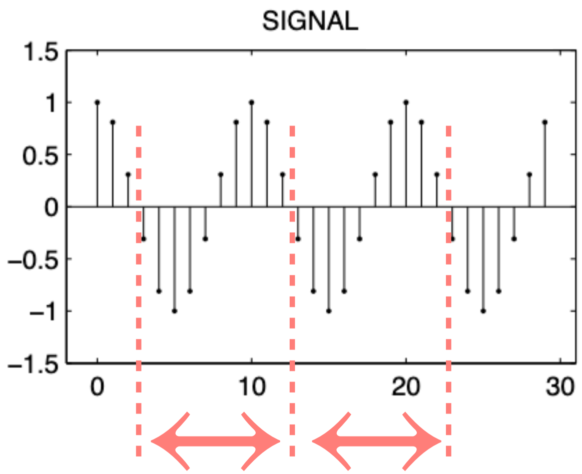

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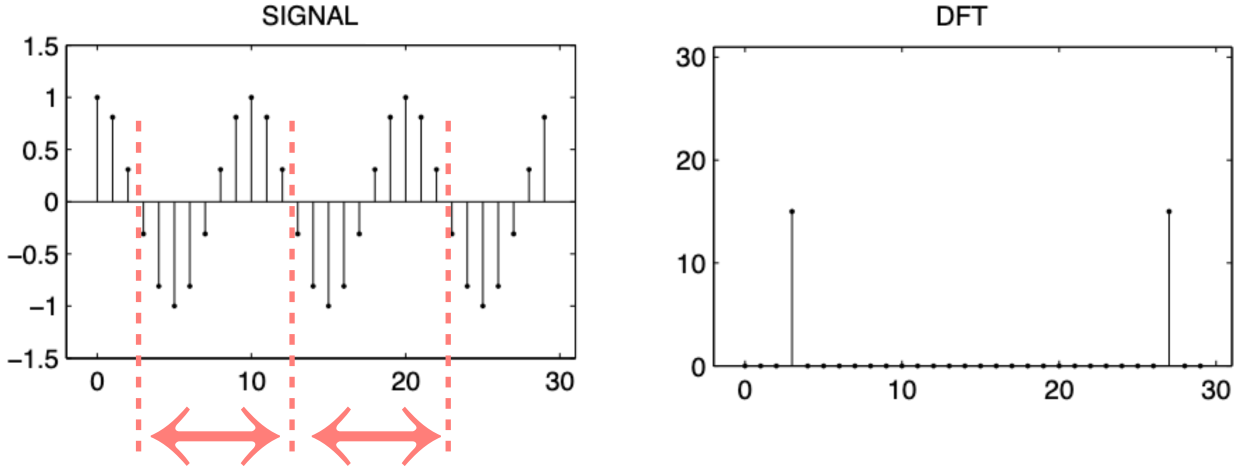

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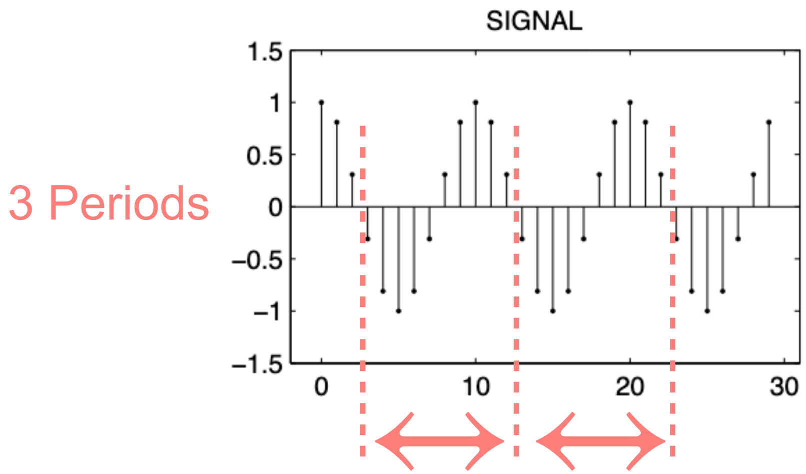

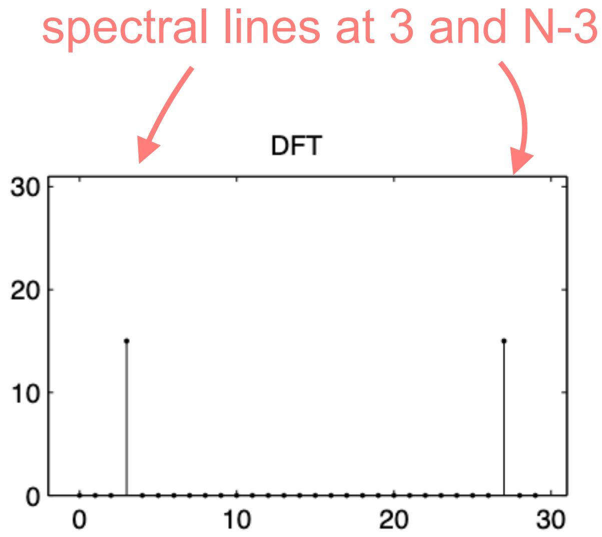

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But what if the signal period doesn’t fit as an integer multiple in the signal duration?

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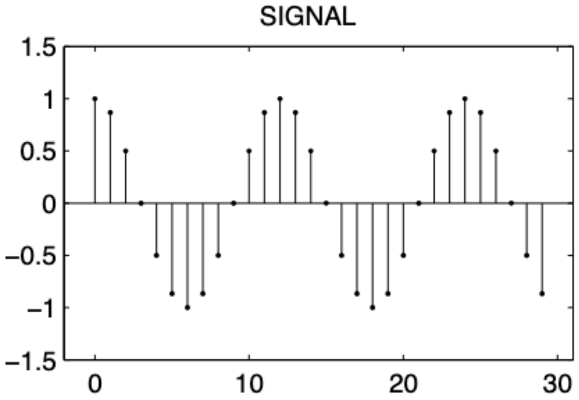

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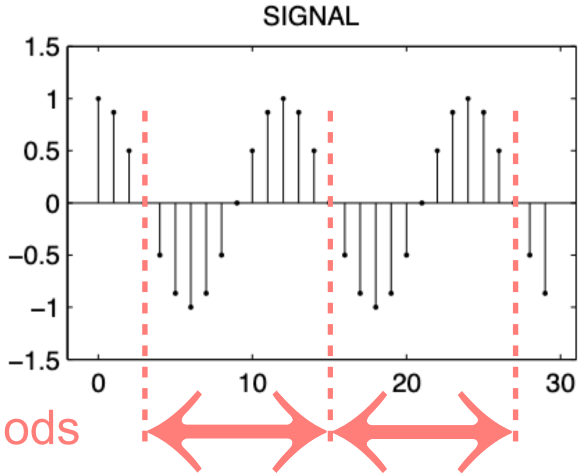

2.5 Periods

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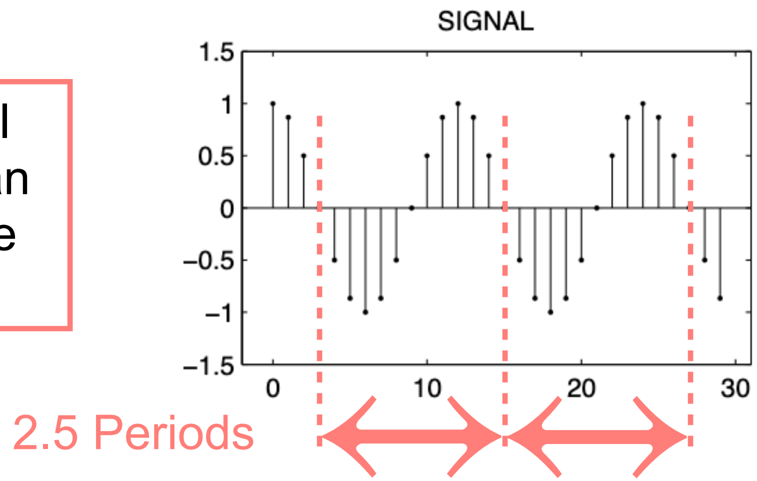

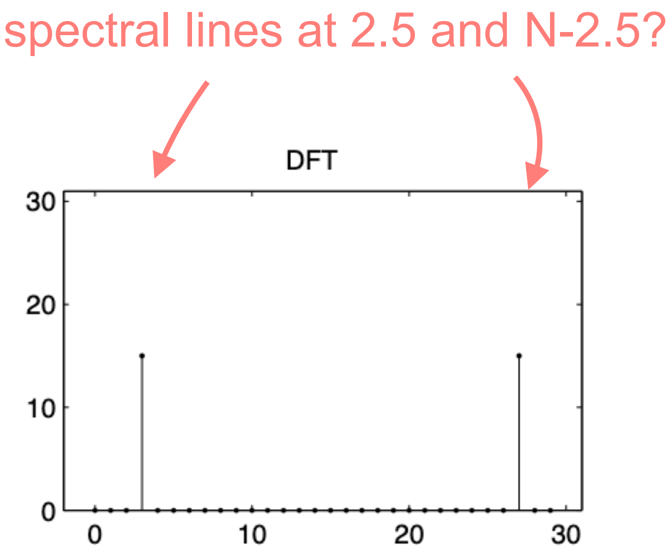

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spectral lines at 2.5 and N-2.5?

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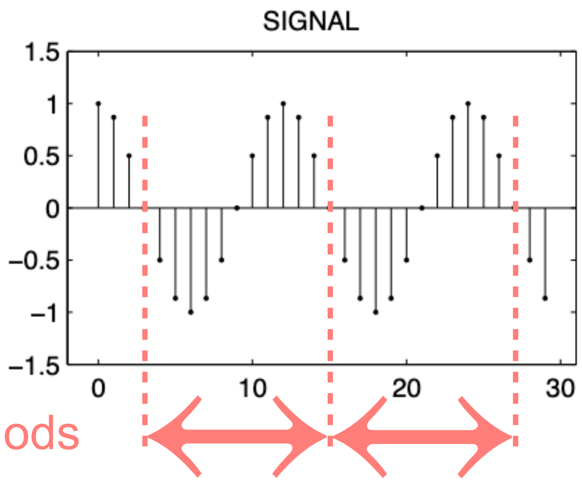

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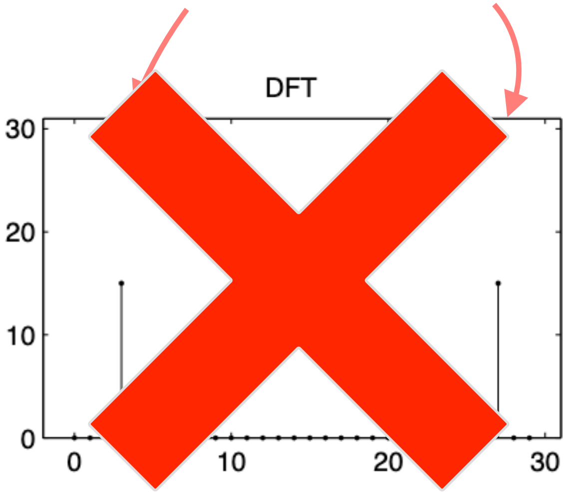

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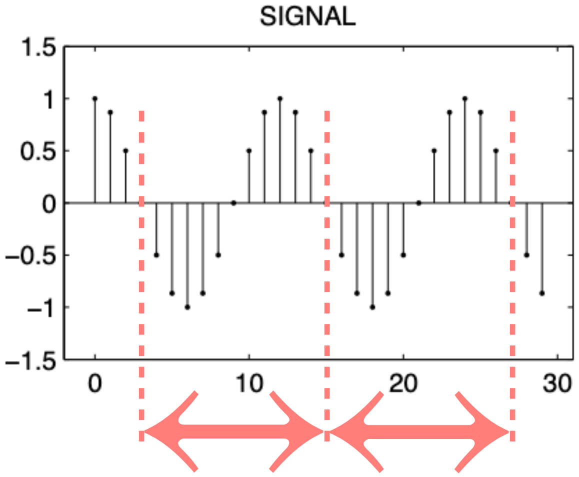
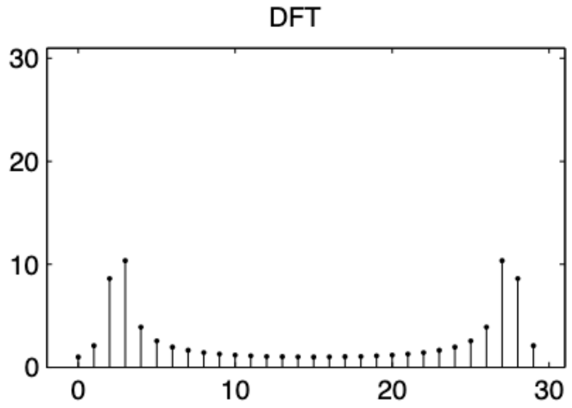
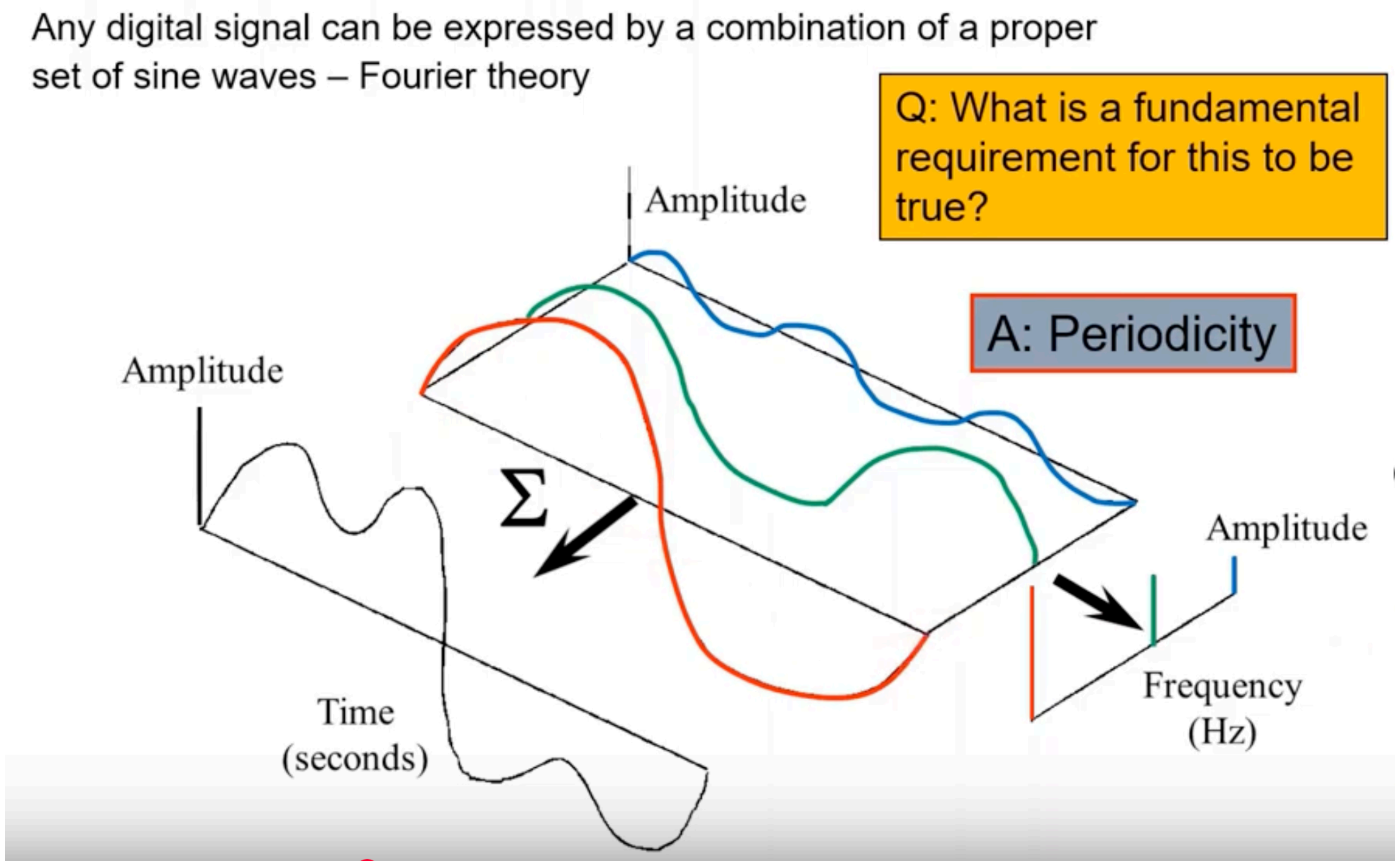


smeared out over entire frequency spectrum - "spectral leakage"



Periodicity and Frequency Leakage

- **Reason**
  - Fourier Transform requires signal to be periodic
  - Meaning it has the same value in the beginning and the end
- **Assumption**
  - We sample from an infinite signal
  - Sample can be stitched to itself to give true, infinite signal



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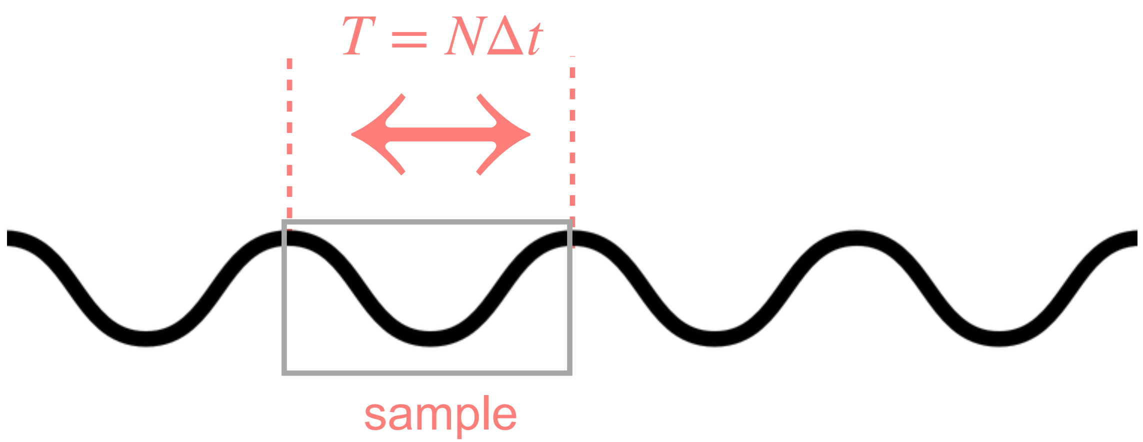

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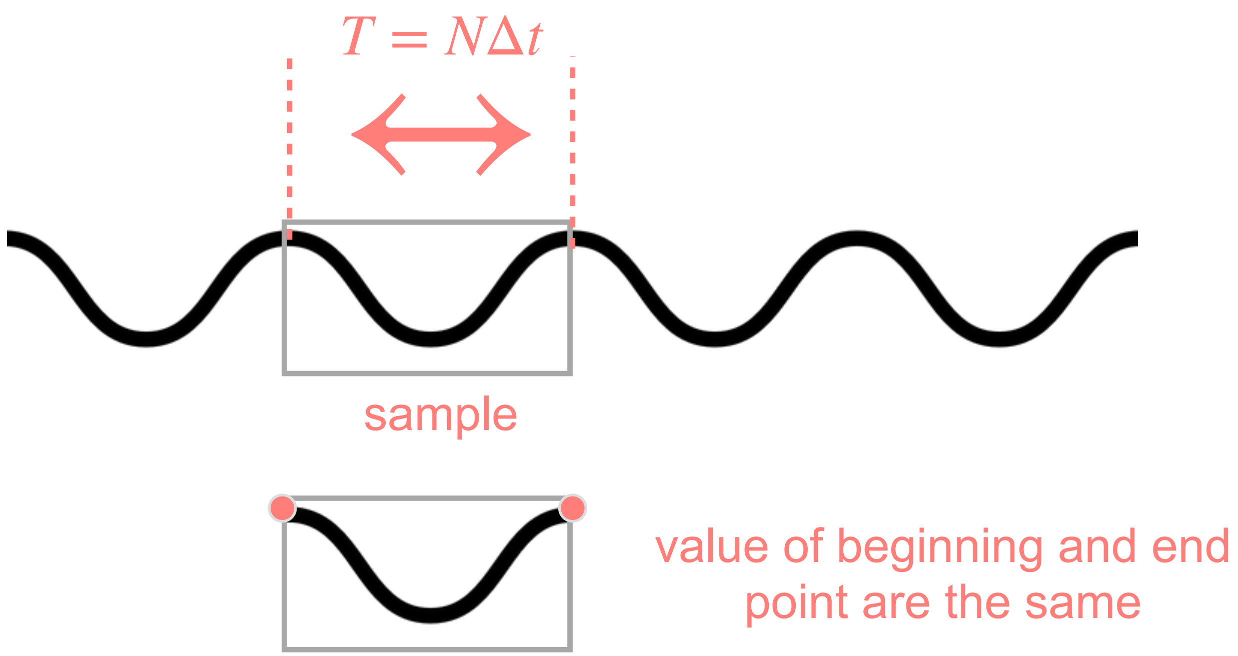

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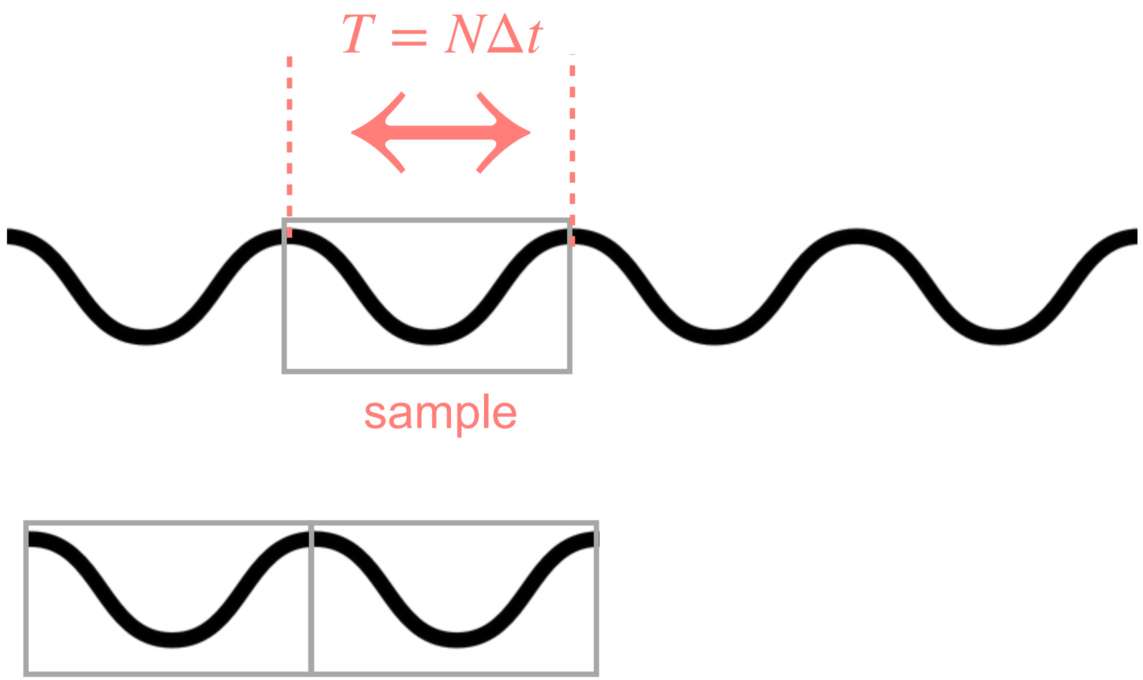

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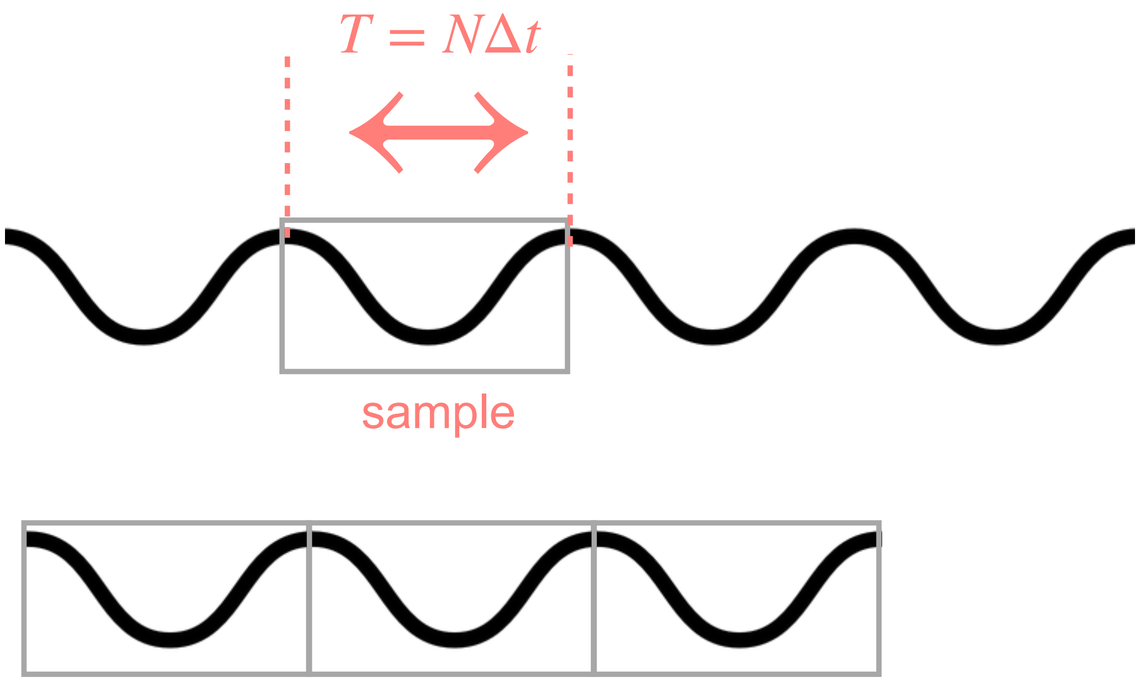

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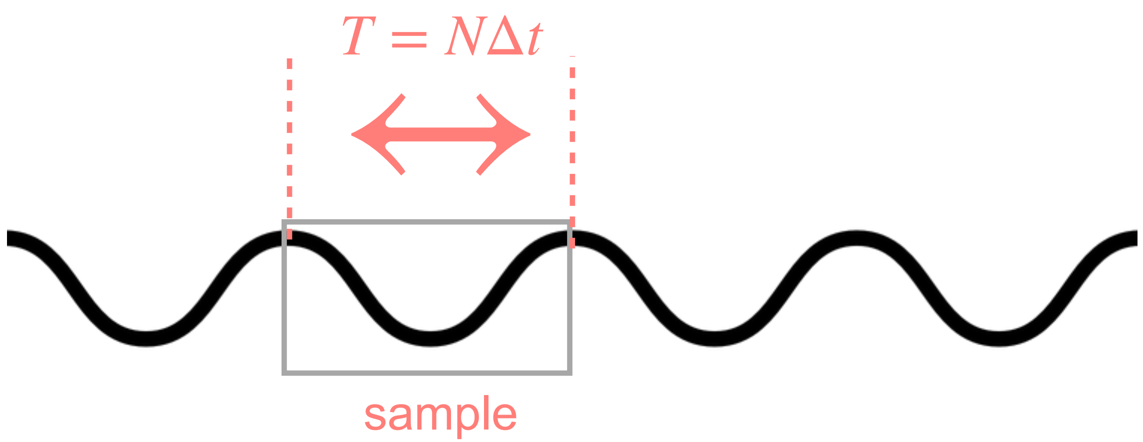

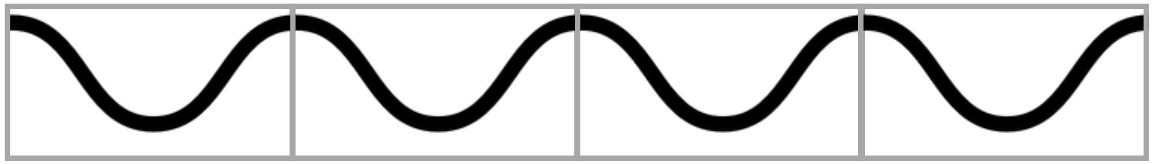

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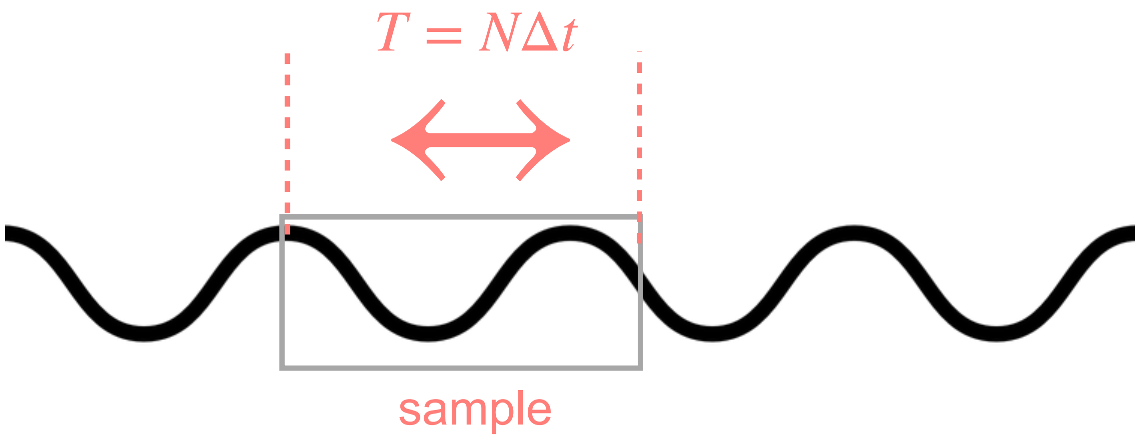

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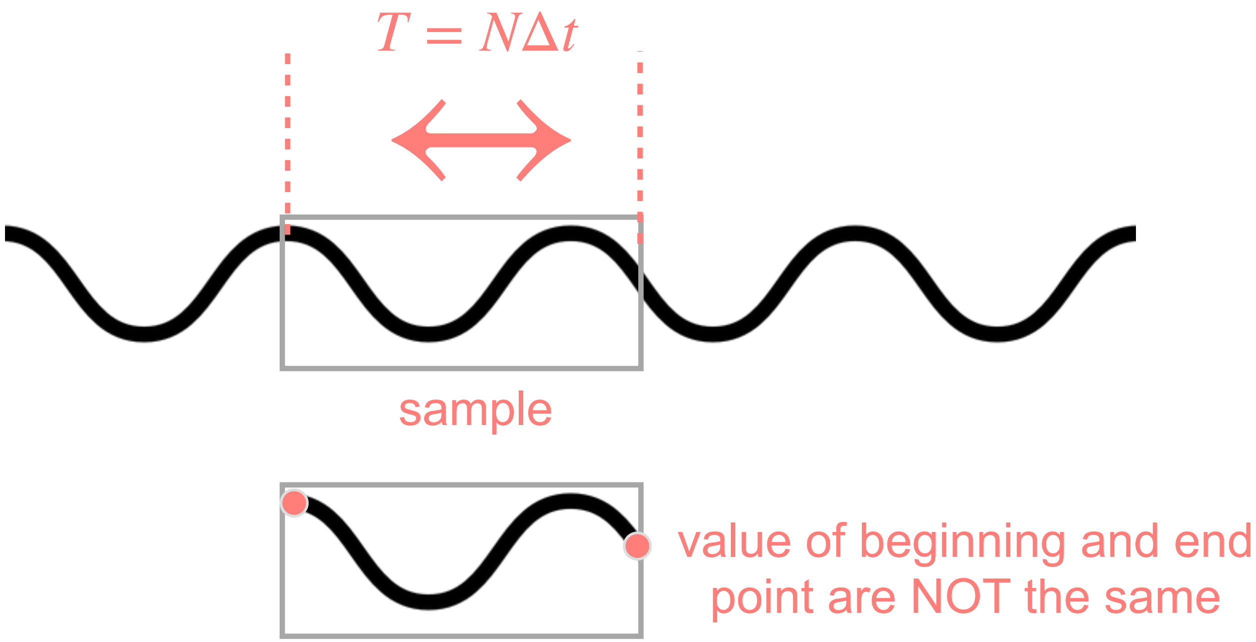
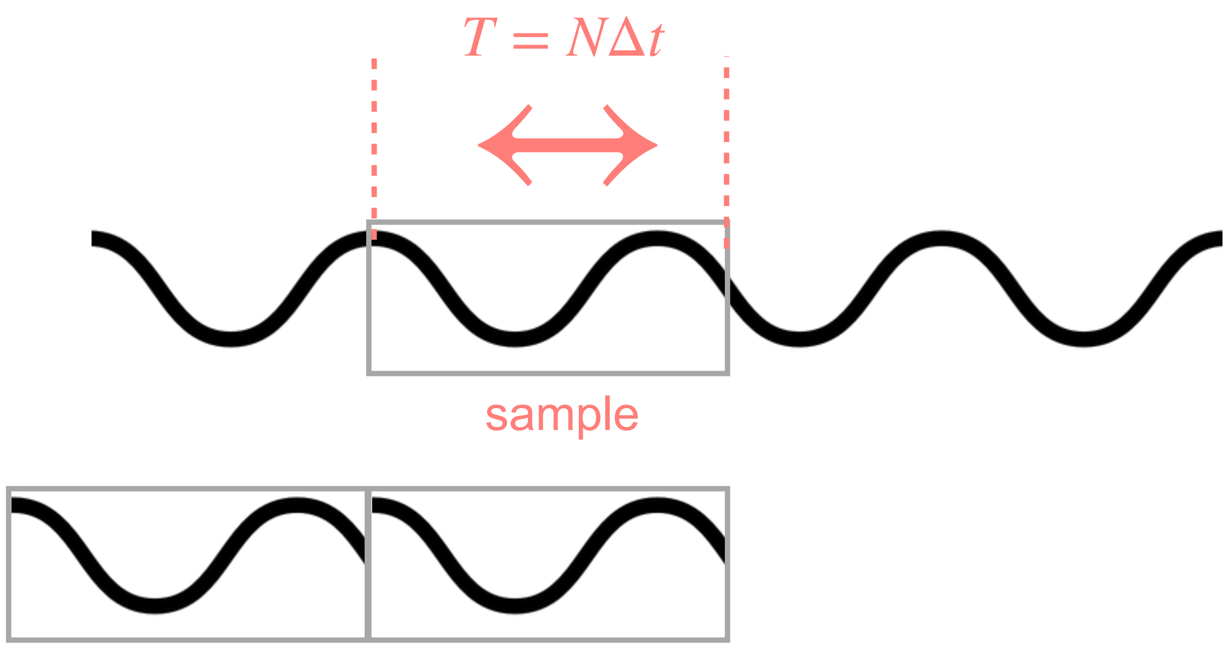


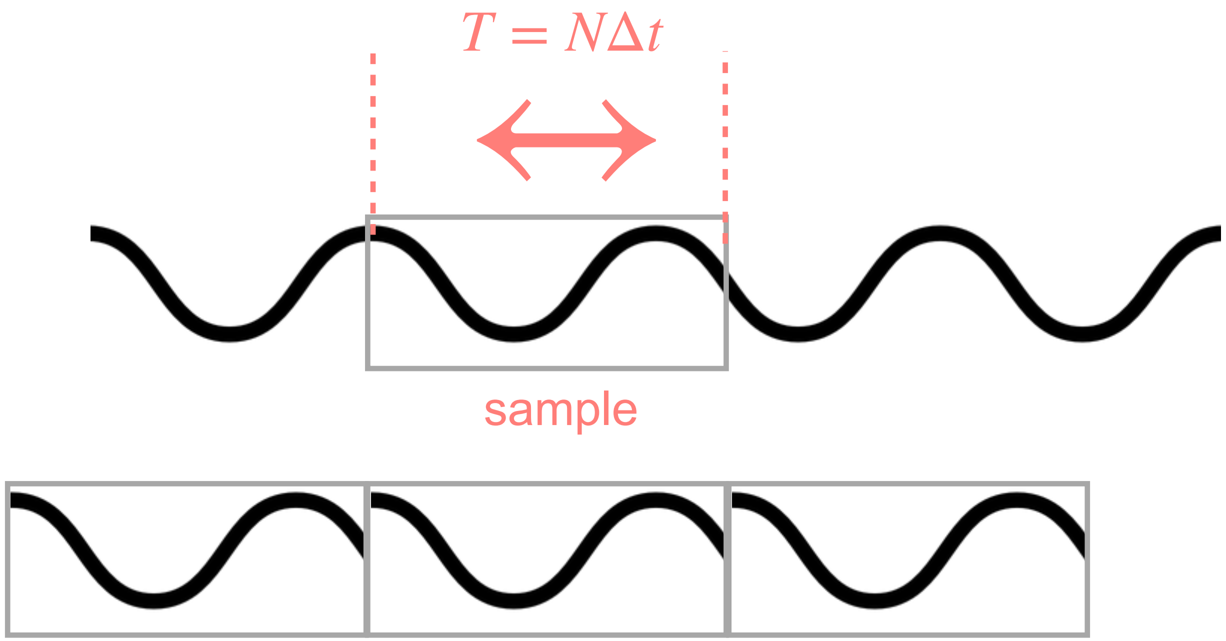
Periodicity and Frequency Leakage

- **Reason**
  - Fourier Transform requires signal to be periodic
  - Meaning it has the same value in the beginning and the end
- **Assumption**
  - We sample from an infinite signal
  - Sample can be stitched to itself to give true, infinite signal



Periodicity and Frequency Leakage

- **Causes**
  - Sampling
- **Effects**
  - Aliasing
  - 1D and 2D signals



Periodicity and Frequency Leakage

- **Causes**
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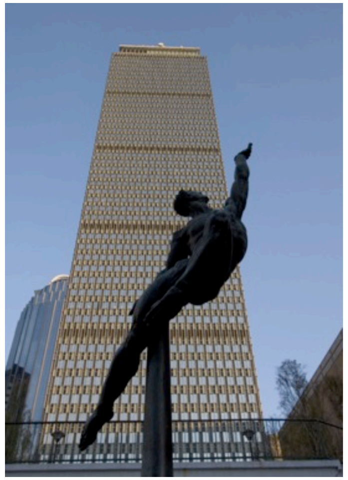
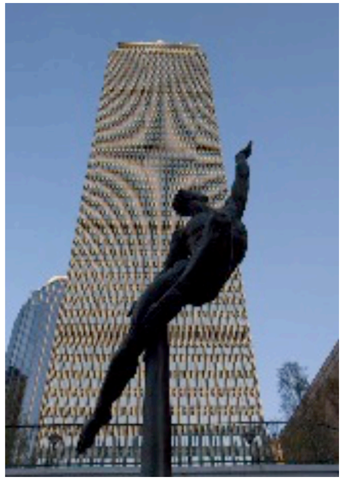
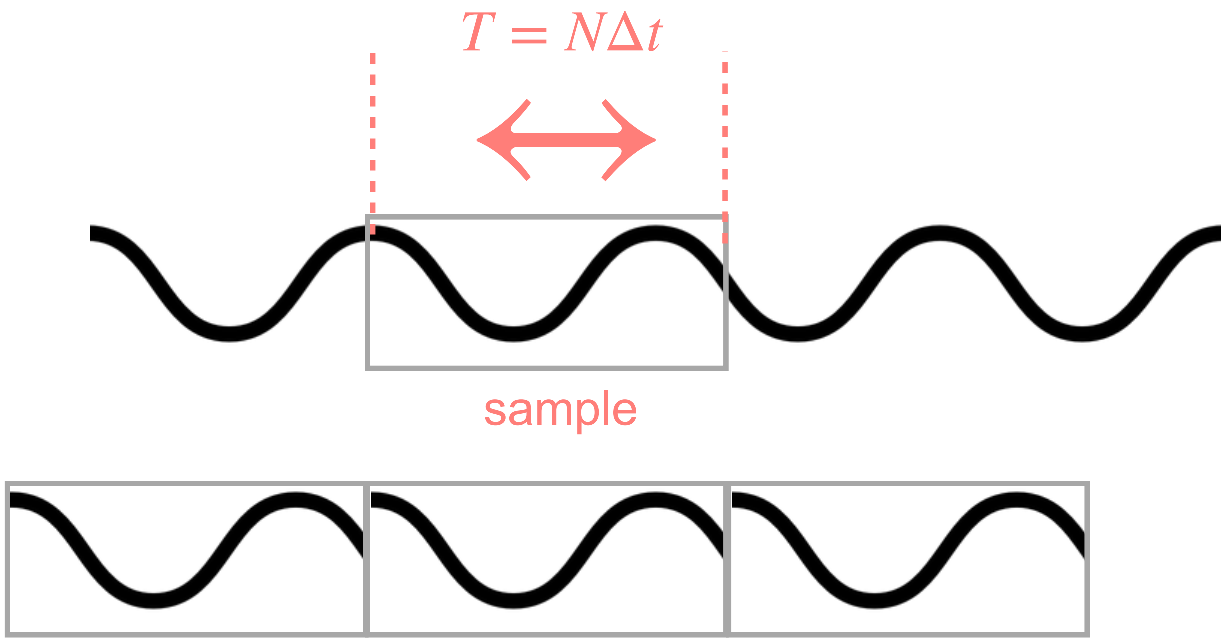
  - 1D and 2D signals

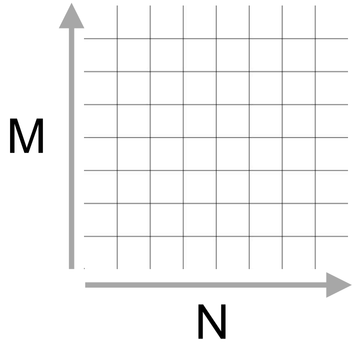






Two-Dimensional DFT and its Inverse

2D Image $f(x, y)$



- **Definition**

$$\hat{f}(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \exp\left(-i2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)\right)$$

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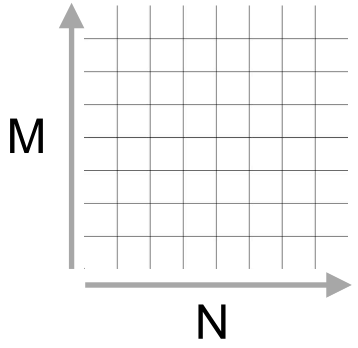

- **Definition**

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- **Scaling factor**
  - As before the scaling factor $\frac{1}{MN}$ may be multiplied either with the FT or IFT, or $\frac{1}{\sqrt{MN}}$ is multiplied to both

Two-Dimensional DFT and its Inverse

- **Separability**

$$\hat{f}(u,v) = \frac{1}{M}\sum_{x=0}^{M-1} \exp\left(\frac{-i2\pi ux}{M}\right) \frac{1}{N}\sum_{y=0}^{N-1} \exp\left(\frac{-i2\pi vy}{N}\right)$$

$$= \frac{1}{M}\sum_{x=0}^{M-1} \hat{f}(x,v)\exp\left(\frac{-i2\pi ux}{M}\right)$$

where

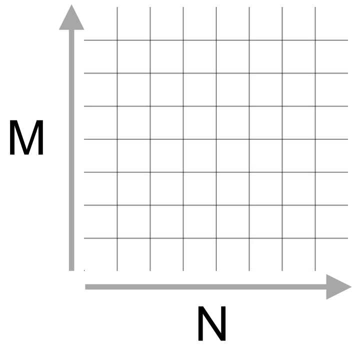
$$\hat{f}(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp\left(\frac{-i2\pi vy}{N}\right)$$

- For each value of x, and for values $v = 0,1,2,\ldots,N-1$, this equation is a complete 1-D Fourier transform

- Meaning $\hat{f}(x,v)$ is the Fourier transform along one row of $f(x,y)$ and by varying x from 0 to $M-1$, we compute the Fourier transform along all rows of $f(x,y)$

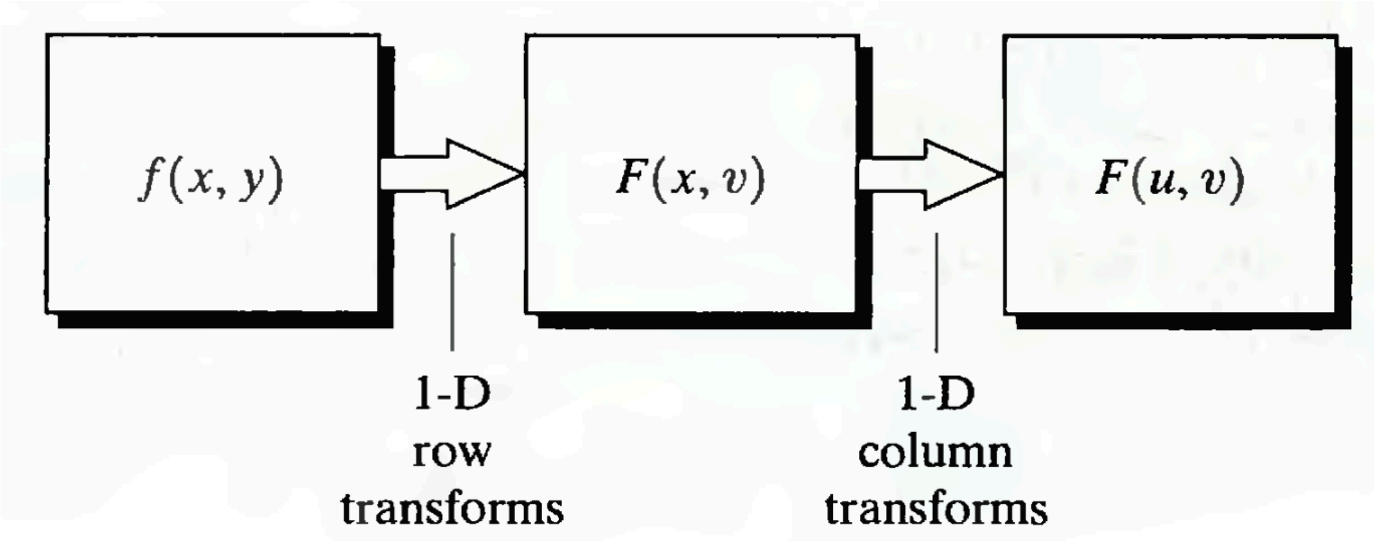
- To get the full 2D-FT, u also has to be varied from 0 to $M-1$, i.e. a 1-D FT along each column of $\hat{f}(x,v)$

2D Image $f(x,y)$

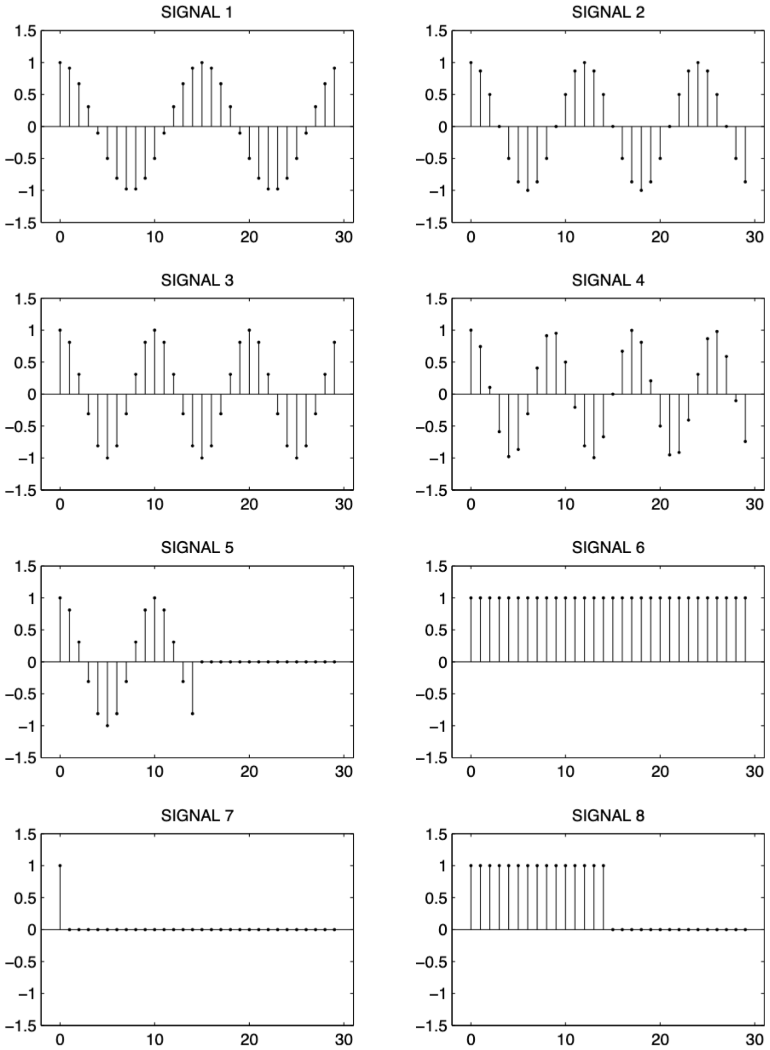


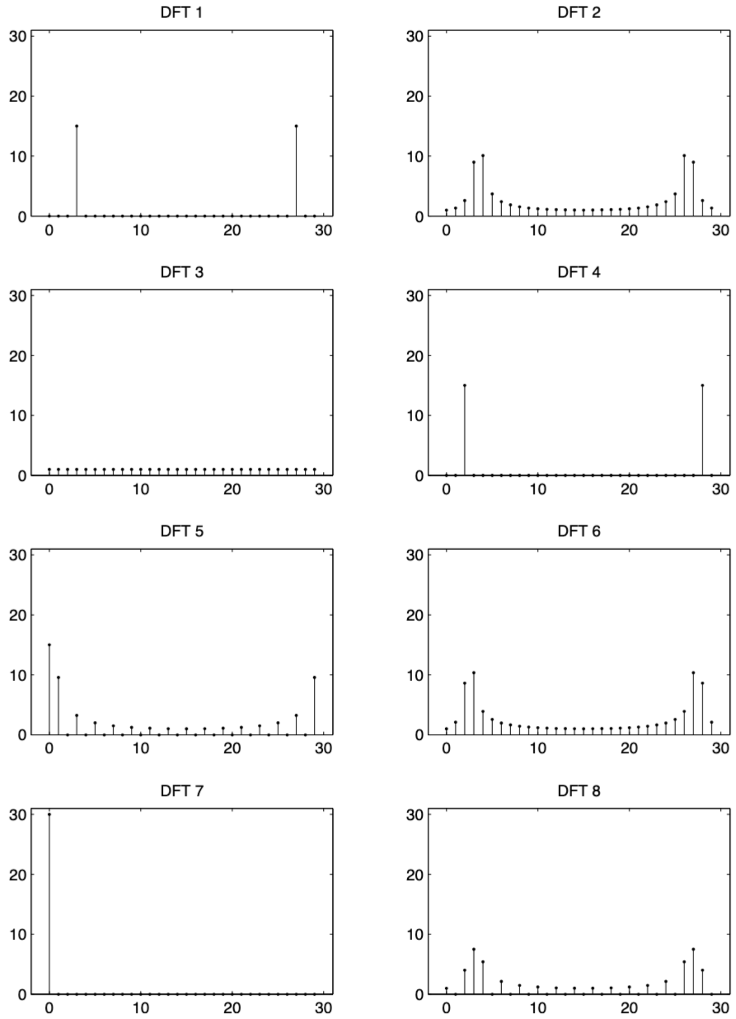
Two-Dimensional DFT and its Inverse

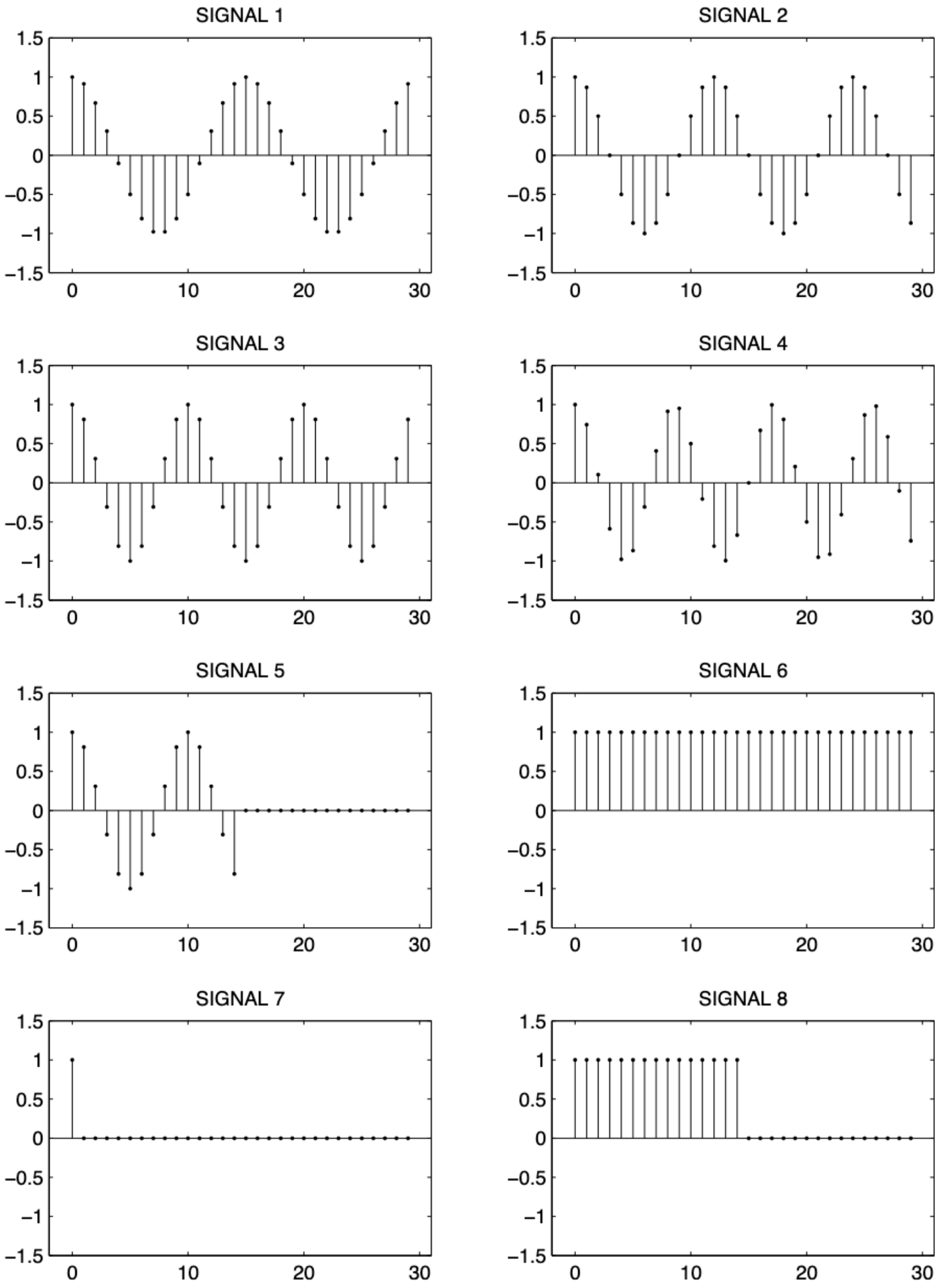
- **Separability**
  - This mean we can compute the 2-D transform by first computing a 1-D transform along each row of the input image, and then computing a 1-D transform along each column of this intermediate result.
  - The same holds for reverse the order of computation: columns first, followed by rows.

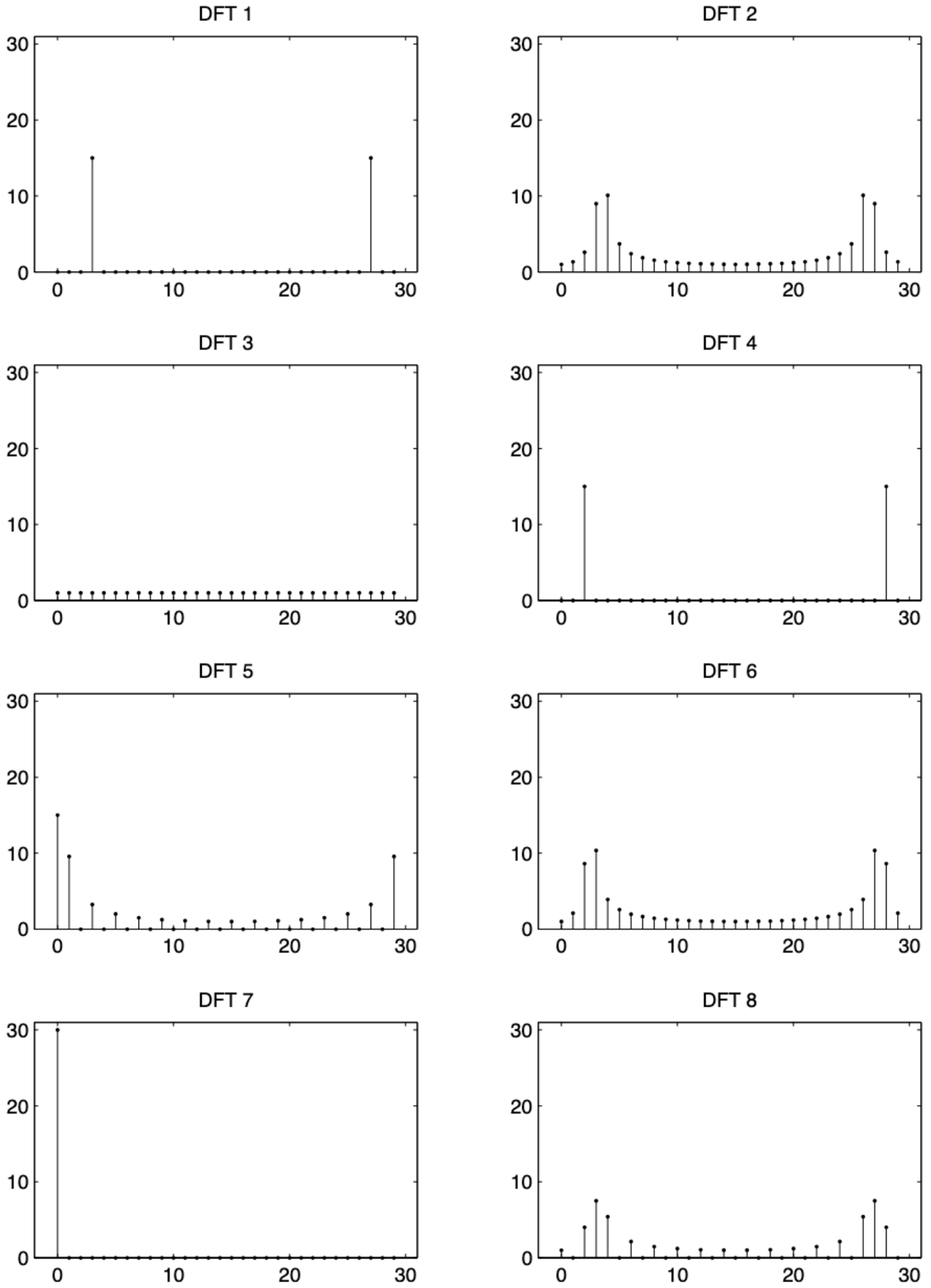


Quiz: Match the DFT!

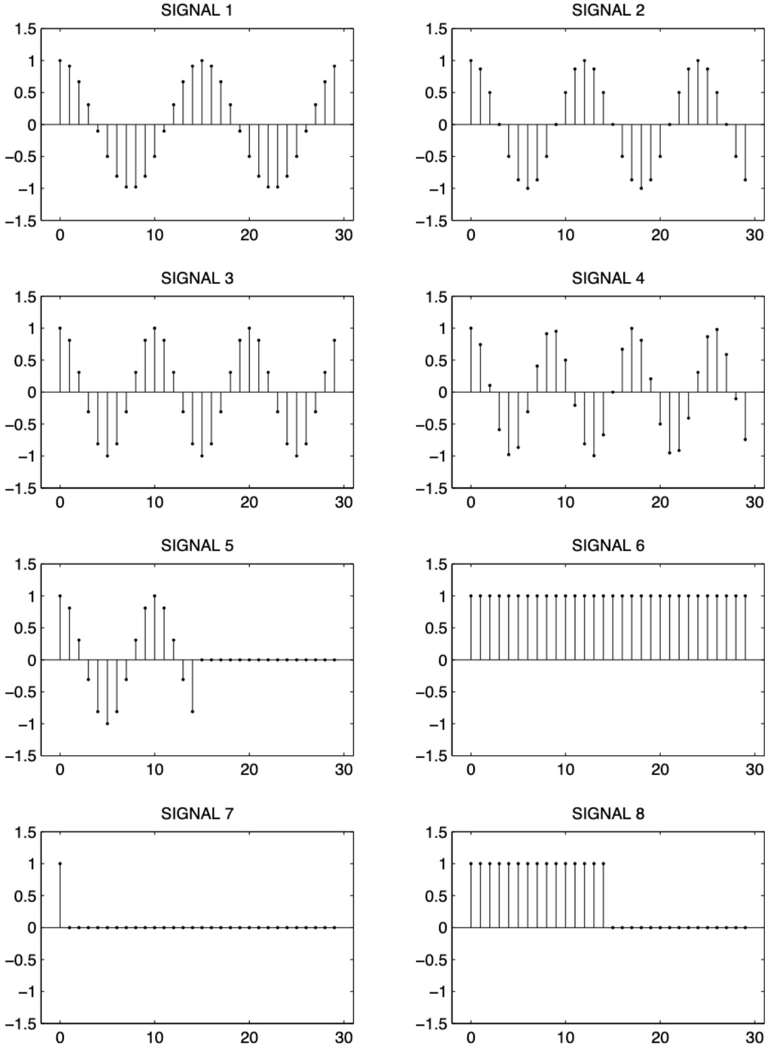





SIGNAL 1
SIGNAL 2
SIGNAL 3
SIGNAL 4
SIGNAL 5
SIGNAL 6
SIGNAL 7
SIGNAL 8

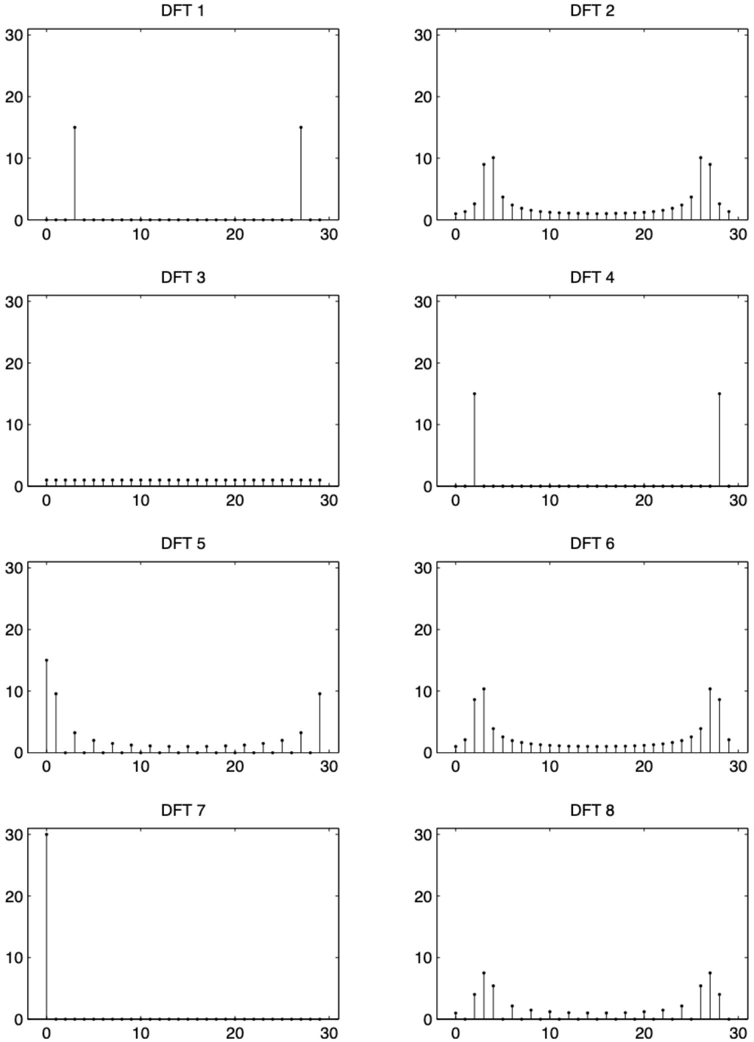

DFT 1
DFT 2
DFT 3
DFT 4
DFT 5
DFT 6
DFT 7
DFT 8

Quiz: Match the DFT!



Solution:

Signal	DFT
1	4
2	6
3	1
4	2
5	8
6	7
7	3
8	5



Resources

- **Animations**
  - Veritasium: https://www.youtube.com/watch?v=nmgFG7PUHfo
  - 3Blue1Brown: https://www.youtube.com/watch?v=spUNpyF58BY