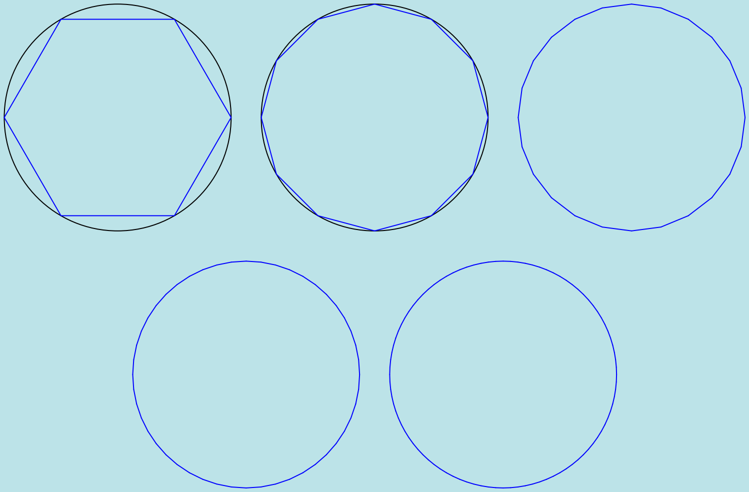
First Principles of Math



Sindre Sogge Heggen

*"Wahrlich es ist nicht das Wissen, sondern das Lernen,
nicht das Besitzen, sondern das Erwerben,
nicht das Da-Seyn, sondern das Hinkommen,
was den grössten Genuss gewährt"*

*"It is not knowing, rather learning,
not possessing, rather obtaining,
not being present, rather reaching there,
which serves the greatest joy."*

— Carl Friedrich Gauss

This document is created by Sindre Sogge Heggen. The text is written in LaTeX and the figures are made using Asymptote.

07.04.2021

Foreword

The extent and applications of mathematics are enormous, but a fair share of it is founded on a manageable amount of principles: I wish to present these in this book. I have chosen to call a principle in summarized form a *rule*. You will find the rules in blue text boxes, usually followed by an example of its usage. One of the main targets of this book is presenting the logical justification for the rules. In Chapter 1 - 5 you will find explanations[1] preceding every rule, while in chapter 6 some explanations are found directly after stating the rules (and eventual examples). As of chapter 7, some explanations are found in a concluding section named *Explanations*. This indicates that they are rather intricate or are so intuitively true that many will find the explanation superfluous.

The structure of the book

The book consists of a *Part I* and a *Part II*. *Part I* focuses on the basic understanding of the numbers and operations of calculation. *Part II* introduces the concept of algebra and the closely related topics of powers, equations, and functions. In addition, both *Part I* and *Part II* end with a chapter on geometry.

Notice! You will not find practice problems and applications of mathematics in real life in this book. These are two very important elements to come, either integrated in this book or as an independent document.

A note on convention

Although I am very much aware of the convention of writing commas and dots in center-aligned equations, I opted against this[2]. In this way, a center-aligned equation is a grammatical hybrid; it can end with both an invisible comma or dot, or nothing at all.

[1] To *explain* the rules rather than *proving* them is a deliberate decision. A proof demands mathematical rigor that often forces a lot of assumptions and definitions along the way. This can make the main insight disappear in the crowd of details. However, some of the explanations are valid as proofs.

[2] I've never liked the looks of it.

Dear reader.

This book is free of charge; however, I've invested a lot of time and resources in creating it. I really want to continue creating books which makes mathematics available for free, but it can turn out to be quite difficult if there is no income connected to it. Therefore, if you like this book, I hope you can donate a small sum using PayPal. Thank you in advance!

The book is updated as soon as possible when errors are discovered. Please download the latest version.

Contact: `sindre.heggen@gmail.com`

Symbols

$=$	”equals”
$<$	”less than”
$>$	”greater than”
$\leq$	”less than or equal to”
$\geq$	”greater than or equal to”
$\in$	”included in”
$\vee$	”or”
$\wedge$	”and”
$[a, b]$	”closed interval from a to b”
$\|a\|$	”length/absolute value of a”
$\perp$	”perpendicular to”
$\parallel$	”parallel with”
$\triangle$	”triangle”
$\square$	”quadrilateral”

Contents

Part I

Numbers, calculations, and geometry

Chapter 1

The numbers

1.1 The equal sign, amounts, and number lines

The equal sign

As the name implies, the *equal sign* $=$ refers to things that are the same. In what sense some things are the same is a philosophical question and initially we are bound to this: What equality $=$ points to must be understood by the context in which the sign is used. With this understanding of $=$ we can study some basic properties of our numbers and then later return to more precise meanings of the sign.

> **The language box**
>
> Common ways of expressing $=$ is
>
> - ”equals”
> - ”is the same as”

Amounts and number lines

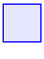
There are many ways a number can be defined, however, in this book we shall stick to two ways of interpreting a number; a number as an *amount* and a number as a *placement on a line*. All representations of numbers rely on the understanding of 0 and 1.

Numbers as amounts

Talking about an amount, the number 0 is[1] connected to ”nothing”. A figure showing nothing will therefore equal 0:

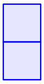
$$= 0$$

1 we’ll draw like a box:

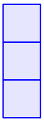


$$= 1$$

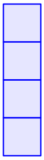
In this way, other numbers are defined by how many one-boxes (ones/units) we have:



$= 2$



$= 3$

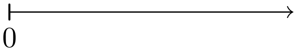


$= 4$

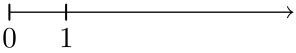
[1]In *Chapter 2* we’ll see that there are other interpretations of 0.

Numbers as placements on a line

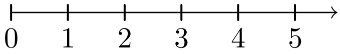
When placing numbers on a line, 0 is our starting point:



Now we place 1 a set length to the right of 0:



Other numbers are now defined by how many one-lengths (ones/units) we are away from 0:



Positive integers

We'll soon see that numbers do not necessarily have to be a *whole* amount of ones, but those which *are* have their own name:

> **1.1 Positive integers**
>
> Numbers which are a whole amount of ones are called *positive*[1] *integers*. The positive integers are
>
> $$1, 2, 3, 4, 5 \text{ and so on.}$$
>
> Positive integers are also called *natural numbers*.

> **What about 0?**
>
> Some authors also include 0 in the definition of natural numbers. This is in some cases beneficial, in others not.

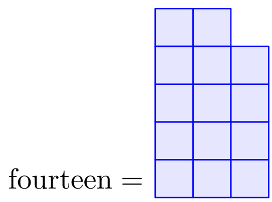
[1] We'll see what the word *positive* refers to in chapter *chapter 5*.

1.2 Numbers, digits and value

Our numbers consist of the *digits* $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9 along with their *positions*. The digits and their positions defines[1] the *value* of numbers.

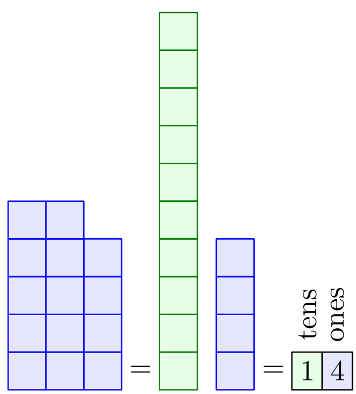
Integers larger then 10

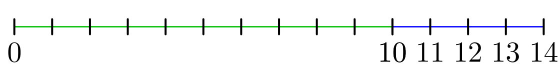
Let's, as an example, write the number *fourteen* by our digits.



We can now make a group of 10 ones, then we also have 4 ones. By this, we write fourteen as

$$\text{fourteen} = 14$$



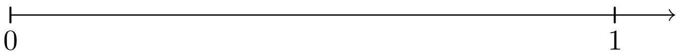


[1]Later on, we'll also see that *signs* have an impact on a numbers value (see *Chapter 5*).

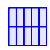
Decimal numbers

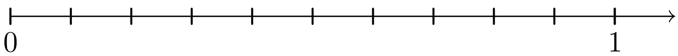
Sometimes we don't have a whole amount of ones, and this brings about the need to divide "ones" into smaller pieces. Let's start off by drawing a one:

 $= 1$



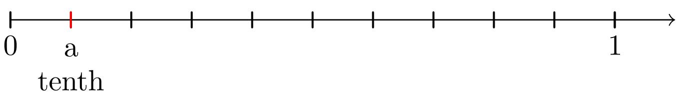
Now we divide our one into 10 smaller pieces:

 $= 1$



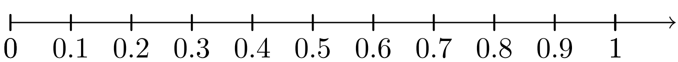
Since we have divided 1 into 10 pieces, we name one such piece *a tenth*:

 = a tenth

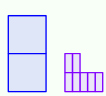


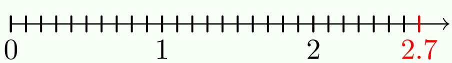
We indicate tenths by using the *decimal mark*: .

 $= 0.1$



Example

 $= 2.7$



The language box

In a lot of countries, a comma is used in place of the period for the decimal mark.

3,5 (*other*)

3.5 (*English*)

Base-10 positional notation

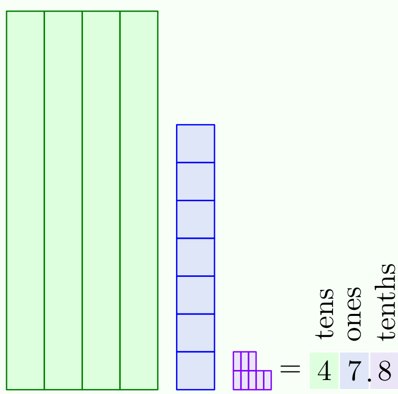
So far, we have seen how we can express the value of a number by placing digits according to the amount of tens, ones and tenths. The pattern continues:

1.2 Base-10 positional notation

The value of a number is given by the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9 and their position. In respect to the digit indicating ones,

- digits to the left indicate amounts of tens, hundeds, thousands etc.
- digits to the left indicate amounts of tenths, hundredths, thousandths etc.

Example 1



Example 2

thousands hundreds tens ones tenths hundredths

3805.72

1.3 Coordinate systems

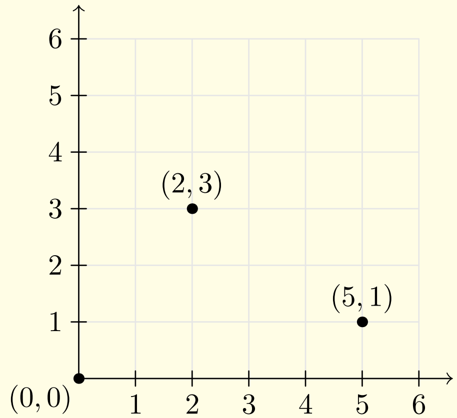
Two number lines can be put together to form a *coordinate system*. In that case we place one number line *horizontally* and one *vertically*. A position in a coordinate system is called a *point*.

In fact, there are many types of coordinate systems, but we'll use the *cartesian coordinate system*. It is named after the French mathematician and philosopher, René Descartes.

A point is written as two numbers inside a bracket. We shall call these two numbers the *first coordinate* and the *second coordinate*.

- The first coordinate tells how many units to move along the horizontal axis.
- The second coordinate tells how many units to move along the vertical axis.

In the figure, the points $(2, 3)$, $(5, 1)$ and $(0, 0)$ are shown. The point where the axes intersect, $(0, 0)$, is called *origo*.



Chapter 2

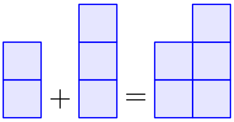
The four elementary operations

2.1 Addition

Addition with amounts

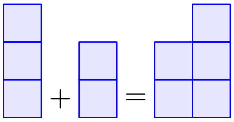
When we have an amount and wish to add more, we use the symbol $+$. If we have 2 and want to add 3, we write

$$2 + 3 = 5$$



The order in which we add have no impact on the results; starting off with 2 and adding 3 is the same as starting off with 3 and adding 2:

$$3 + 2 = 5$$



The language box

A calculation involving addition includes two or more *terms* and one *sum*. In the calculation

$$2 + 3 = 5$$

both 2 and 3 are terms while 5 is the sum.

Common ways of saying $2 + 3$ include

- ”2 plus 3”
- ”2 added to 3”
- ”2 and 3 added”

2.1 Addition is commutative

The order of the terms has no impact on the sum.

Example

$$2 + 5 = 7 = 5 + 2$$

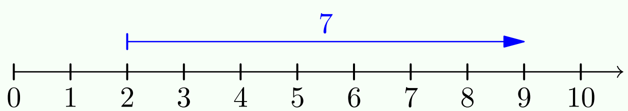
$$6 + 3 = 9 = 3 + 6$$

Addition on the number line: moving to the right

On a number line, addition with positive numbers involves moving *to the right*:

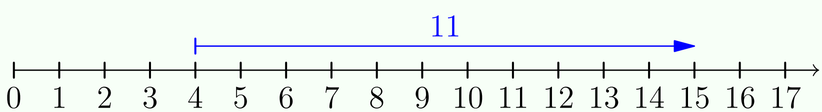
Example 1

$$2 + 7 = 9$$



Example 2

$$4 + 11 = 15$$



Interpretation of $=$

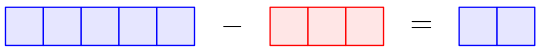
$+$ brings the possibility of expressing numbers in different ways, for example is $5 = 2 + 3$ and $5 = 1 + 4$. In this context, $=$ means ”has the same value as”. This is also the case regarding subtraction, multiplication and division which we’ll look at in the next three sections.

2.2 Subtraction

Subtraction with amounts

When removing a part of an amount, we use the symbol $-$:

$$5 - 3 = 2$$



> **The language box**
>
> A calculation involving subtraction includes one or more *terms* and one *difference*. In the calculation
>
> $$5 - 3 = 2$$
>
> both 5 and 3 are terms while 2 is the difference.
>
> Common ways of saying $5 - 3$ include
>
> - ”5 minus 3”
> - ”3 subtracted from 5”

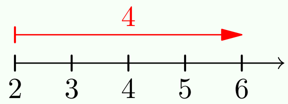
> **A new interpretation of 0**
>
> As mentioned earlier in this book, 0 can be interpreted as ”nothing”. However, subtraction brings the possibility of expressing 0 by other numbers, for example $7 - 7 = 0$ and $19 - 19 = 0$. In many practical situations, 0 indicates some form of equilibrium, like two equal but opposite forces.

Subtraction on the number line: Moving to the left

In *Section 2.1*, we have seen that $+$ (with positive numbers) involves moving *to the right* on the number line. With $-$ it's the opposite, we move *to the left*[1]:

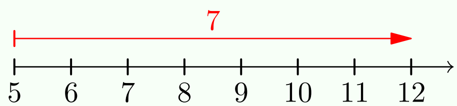
Example 1

$$6 - 4 = 2$$



Example 2

$$12 - 7 = 5$$



Notice

At first it may seem a bit odd moving in the opposite direction of the way in which the arrows point, as in *Example 1* and *2*. However, in *Chapter 5* this will turn out to be useful.

[1] In figures with number lines, the red colored arrows indicates that you shall start at the arrowhead and move to the other end.

2.3 Multiplication

Multiplication by integers: initial definition

When adding equal numbers, we can use the multiplication symbol $\cdot$ to write our calculations more compact:

Example

$$4 + 4 + 4 = 4 \cdot 3$$
$$8 + 8 = 8 \cdot 2$$
$$1 + 1 + 1 + 1 + 1 = 1 \cdot 5$$

The language box

A calculation involving multiplication includes several *factors* and one *product*. In the calculation

$$4 \cdot 3 = 12$$

both 4 and 3 are factors, while 12 is the product.

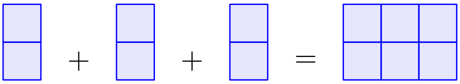
Common ways of saying $4 \cdot 3$ include

- ”4 times 3”
- ”4 multiplied by 3”
- ”4 and 3 multiplied together”

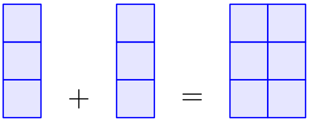
A lot of texts use $\times$ instead of $\cdot$. In computer programming, * is the most common symbol for multiplication.

Multiplication involving amounts

Let us illustrate $2 \cdot 3$:

$2 \cdot 3 =$ 

Now notice the product of $3 \cdot 2$:

$3 \cdot 2 =$ 

2.2 Multiplication is commutative

The order of the factors has no impact on the product.

Example

$$3 \cdot 4 = 12 = 4 \cdot 3$$

$$6 \cdot 7 = 42 = 7 \cdot 6$$

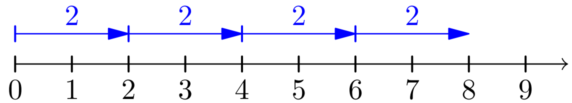
$$8 \cdot 9 = 72 = 9 \cdot 8$$

Multiplication on the number line

We can also use the number line to calculate multiplications. In the case of $2 \cdot 4$ we can think like this:

"$2 \cdot 4$ means moving 2 places to the right, 4 times."

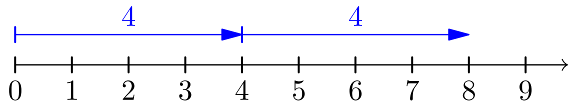
$$2 \cdot 4 = 8$$



We can also use the number line to prove to ourselves that multiplication is commutative:

"$4 \cdot 2$ means moving 4 places to the right, 2 times."

$$4 \cdot 2 = 8$$



Final definition of multiplication by positive integers

It may be the most intuitive to interpret "2 times 3" as "3, 2 times". Then it follows:

$$\text{"2 times 3"} = 3 + 3$$

In this section we introduced $2 \cdot 3$, that is "2 times 3", as $2 + 2 + 2$. With this interpretation, $3 + 3$ corresponds to $3 \cdot 2$, but the fact that multiplication is a commutative operation (*Rule 2.2*) ensures that the one interpretation does not exclude the other; $2 \cdot 3 = 2 + 2 + 2$ and $2 \cdot 3 = 3 + 3$ are two expressions of same value.

2.3 Multiplication as repeated addition

Multiplication involving a positive integer can be expressed as repeated addition.

Example 1

$$4 + 4 + 4 = 4 \cdot 3 = 3 + 3 + 3 + 3$$

$$8 + 8 = 8 \cdot 2 = 2 + 2 + 2 + 2 + 2 + 2 + 2$$

$$1 + 1 + 1 + 1 + 1 = 1 \cdot 5 = 5$$

Notice

The fact that multiplication with positive integers can be expressed as repeated addition does not exclude other expressions. There's nothing wrong with writing $2 \cdot 3 = 1 + 5$.

2.4 Division

$:$ is the symbol for division. Division has three different interpretations:

2.4 The three interpretations of division

- **Distribution of amounts**

  $12 : 3 =$ "The number in each group when evenly distributing 12 into 3 groups"

- **Number of equal terms**

  $12 : 3 =$ "The number of 3's added to make 12"

- **The inverse operation of multiplication**

  $12 : 3 =$ "The number which yields 12 when multiplied by 3"

The language box

A calculation involving division includes a *dividend*, a *divisor* and a *quotient*. In the calculation

$$12 : 3 = 4$$

12 is the dividend, 3 is the divisor and 4 is the quotient.

Common ways of saying $12 : 3$ include

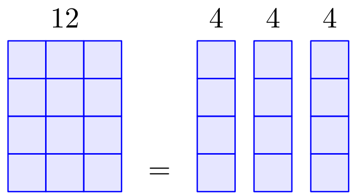
- "12 divided by 3"
- "12 to 3"

In a lot of contexts, $/$ is used instead of $:$, especially in computer programming.

Sometimes $12 : 3$ is called "the *ratio* of 12 to 3".

Distribution of amounts

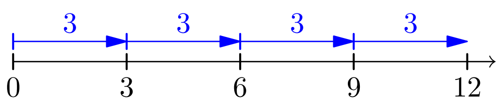
The calculation $12 : 3$ tells that we shall distribute 12 into 3 equal groups:



We observe that each group contains 4 boxes, which means that

$$12 : 3 = 4$$

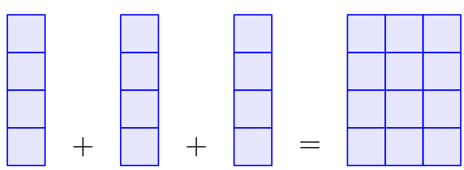
Number of equal terms



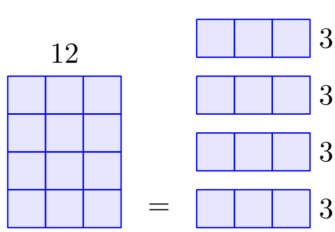
12 equals the sum of 4 instances of 3, that is $12 : 3 = 4$.

The inverse operation of multiplication

We have just seen that if we divide 12 into 3 equal groups, we get 4 in each group. Hence $12 : 3 = 4$. The sum of these groups makes 12:



However, this is the same as multiplying 4 by 3, in other words: If we know that $4 \cdot 3 = 12$, we also know that $12 : 3 = 4$. As well we know that $12 : 4 = 3$.



Example 1

Since $6 \cdot 3 = 18$,

$$18 : 6 = 3$$

$$18 : 3 = 6$$

Example 2

Since $5 \cdot 7 = 35$,

$$35 : 5 = 7$$

$$35 : 7 = 5$$

Chapter 3

Factorization and order of operations

3.1 Factorization

If an integer dividend and an integer divisor results in an integer quotient, we say that the dividend is *divisible* by the divisor. For example is 6 divisible with 3 because $6 : 3 = 2$, and 40 is divisible with 10 because $40 : 10 = 4$. The concept of divisibility contributes to the definition of *prime numbers*:

3.1 Primtal

A natural number larger than 1, and only divisible by itself and 1, is a prime number.

Example

The first five prime numbers are $2, 3, 5, 7$ and 11.

3.2 Factorization

Factorization involves writing a number as the product of other numbers.

Example

Factorize 24 in three different ways.

Answer

$$24 = 2 \cdot 12$$

$$24 = 3 \cdot 8$$

$$24 = 2 \cdot 3 \cdot 4$$

3.3 Prime factorization

Factorization involving prime factors only is called prime factorization.

Example

Prime factorize 12.

Answer

$$12 = 2 \cdot 2 \cdot 3$$

3.2 Order of operations

Priority of the operations

Look at the following calculation:

$$2 + 3 \cdot 4$$

This *could* have been interpreted in two ways:

1. "2 plus 3 equals 5. 5 times 4 equals 20. The answer is 20."
2. "3 times 4 equals 12. 2 plus 12 equals 14. The answer is."

But the answers are not the same! This points out the need to have rules for what to calculate first. One of these rules is that multiplication and division is to be calculated *before* addition or subtraction, which means that

$$\begin{aligned} 2 + 3 \cdot 4 &= \text{"Calculate } 3 \cdot 4\text{, then add 2"} \\ &= 2 + 12 \\ &= 14 \end{aligned}$$

But what if we wanted to calculate $2 + 3$ first, then multiply the sum by 4? We use parentheses to tell that something is to be calculated first:

$$\begin{aligned} (2 + 3) \cdot 4 &= \text{"Calculate } 2 + 3\text{, multiply by 4 afterwards"} \\ &= 5 \cdot 4 \\ &= 20 \end{aligned}$$

3.4 Order of operations

1. Expressions with parentheses
2. Multiplication or division
3. Addition or subtraction

Example 1

Calculate

$$23 - (3+9) + 4 \cdot 7$$

Answer

$$\begin{aligned} 23 - (3+9) + 4 \cdot 7 &= 23 - 12 + 4 \cdot 7 && \text{Parentheses} \\ &= 23 - 12 + 28 && \text{Multiplication} \\ &= 39 && \text{Addition and subtraction} \end{aligned}$$

Example 2

Calculate

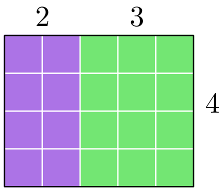
$$18 : (7-5) - 3$$

Answer

$$\begin{aligned} 18 : (7-5) - 3 &= 18 : 2 - 3 && \text{Parentheses} \\ &= 9 - 3 && \text{Division} \\ &= 6 && \text{Addition and subtraction} \end{aligned}$$

Multiplication involving parentheses

How many boxes are present in this figure?



Two correct interpretations include:

1. It is $2 \cdot 4 = 8$ purple boxes and $3 \cdot 4 = 12$ green boxes. In total there are $8 + 12 = 20$ boxes. This we can write as

$$2 \cdot 4 + 3 \cdot 4 = 20$$

2. It is $2 + 3 = 5$ boxes horizontally and 4 boxes vertically, so there are $5 \cdot 4 = 20$ boxes in total. This we can write as

$$(2+3) \cdot 4 = 20$$

From these two calculations it follows that

$$(2+3)\cdot 4 = 2\cdot 4 + 3\cdot 4$$

3.5 Distributive law

When an expression enclosed by parentheses is a factor, we can multiply the other factors with each term inside the parentheses.

Example 1

$$(4+7)\cdot 8 = 4\cdot 8 + 7\cdot 8$$

Example 2

$$\begin{aligned}(10-7)\cdot 2 &= 10\cdot 2 - 7\cdot 2\\ &= 20 - 14\\ &= 6\end{aligned}$$

Notice: Obviously, it would be easier to calculate like this:

$$(10-7)\cdot 2 = 3\cdot 2 = 6$$

Example 2

Calculate $12\cdot 3$.

Answer

$$\begin{aligned}12\cdot 3 &= (10+2)\cdot 3\\ &= 10\cdot 3 + 2\cdot 3\\ &= 30 + 6\\ &= 36\end{aligned}$$

Notice

We introduced parentheses as an indicator of what to calculate first, but *Rule 3.5* gives an alternative and equivalent interpretation of parentheses . The rule is especially useful when working with algebra (see *Part II*).

Multiplying by 0

Earlier we have seen that 0 can be expressed as the difference between two numbers, and this can help us calculate when multiplying by 0. Let's look at the calculation

$$(2-2)\cdot 3$$

By *Rule 3.5*, we get

$$\begin{aligned}(2-2)\cdot 3 &= 2\cdot 3-2\cdot 3\\ &= 6-6\\ &= 0\end{aligned}$$

Since $0=2-2$, this means that

$$0\cdot 3=0$$

3.6 Multiplication by 0

If 0 is a factor, the product equals 0.

Example 1

$$7\cdot 0=0$$

$$0\cdot 219=0$$

Associative laws

3.7 Associative law for addition

The placement of parentheses between terms has no impact on the sum.

Example

$$(2+3)+4=5+4=9$$

$$2+(3+4)=2+7=9$$

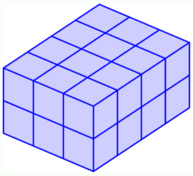
3.8 Associative law for multiplication

The placement of parentheses between factors has no impact on the product.

Example

$$(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24$$

$$2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24$$



Opposite to addition and multiplication, neither subtraction nor divison is associative:

$$(12 - 5) - 4 = 7 - 4 = 3$$
$$12 - (5 - 4) = 12 - 1 = 11$$

$$(80 : 10) : 2 = 8 : 2 = 4$$
$$80 : (10 : 2) = 80 : 5 = 16$$

We have seen how parentheses helps indicating the *priority* of operations, but the fact that subtraction and division are non-associative brings the need of having a rule of in which *direction* to calculate.

3.9 Direction of calculations

Operations which by *Rule 3.4* have equal priority, are to be calculated from left to right.

Example 1

$$\begin{aligned} 12 - 5 - 4 &= (12 - 5) - 4 \\ &= 7 - 4 \\ &= 3 \end{aligned}$$

Example 2

$$\begin{aligned} 80 : 10 : 2 &= (80 : 10) : 2 \\ &= 8 : 2 \\ &= 4 \end{aligned}$$

Example 3

$$\begin{aligned} 6 : 3 \cdot 4 &= (6 : 3) \cdot 4 \\ &= 2 \cdot 4 \\ &= 8 \end{aligned}$$

Chapter 4

Fractions

4.1 Introduction

4.1 Fractions as rewriting of division

A fraction is a different way of writing a division. In a fraction the dividend is called the *numerator* and the divisor the denominator.

$$1:4 = \frac{1 \longleftarrow \text{Numerator}}{4 \longleftarrow \text{Denominator}}$$

The language box

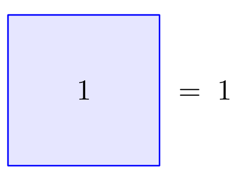
Common ways of saying $\frac{1}{4}$ are[1]

- "one fourth"
- "1 of 4"
- "1 over 4"

[1] We also have the expressions from the language box on page 23.

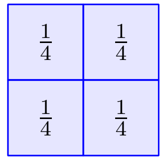
Fractions as amounts

Let us present $\frac{1}{4}$ as an amount. We then think of the number 1 as a box[1]:



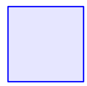
[1] For practical reasons, we choose a unit box larger than the one used in *Chapter 1*.

Further, we divide this box into four smaller, equal-sized boxes. The sum of these boxes equals 1.



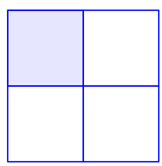
$$= 1$$

One such box equals $\frac{1}{4}$:



$$= \frac{1}{4}$$

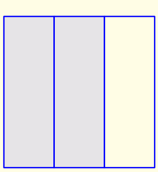
However, if you from a figure only are to see how large a fraction is, the size of 1 must be known, and to make this more apparent we'll also include the "empty" boxes:



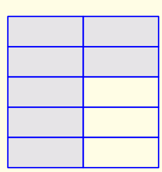
$$= \frac{1}{4}$$

In this way, the blue and the empty boxes tell us how many pieces 1 is divided into, while the blue boxes alone tells how many of these boxes are *actually* present. In other words,

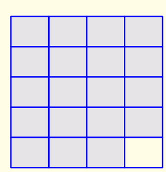
$$\begin{aligned} \text{number of blue boxes} &= \text{numerator} \\ \text{number of blue boxes} + \text{number of empty boxes} &= \text{denomiator} \end{aligned}$$



$$= \frac{2}{3}$$



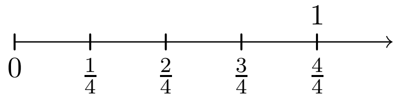
$$= \frac{7}{10}$$



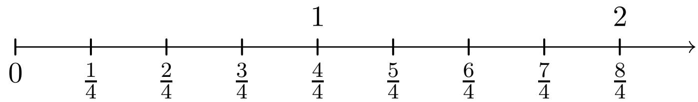
$$= \frac{19}{20}$$

Fractions on the number line

On the number line, we divide the length between 0 and 1 into as many pieces as the denominator indicates. In the case of a fraction with denominator 4, we divide the length between 0 and 1 into 4 equal lengths:



Moreover, fractions larger than 1 are easily presented on the number line:



Numerator and denominator summarized

Although already mentioned, the interpretations of the numerator and the denominator are of such importance that we shortly summarize them:

- The denominator tells how many pieces 1 is divided into.
- The numerator tells how many of these pieces are present.

4.2 Values, expanding and simplifying

4.2 The value of a fraction

The value of a fraction is given by dividing the numerator by the denominator.

Example

Find the value of $\frac{1}{4}$.

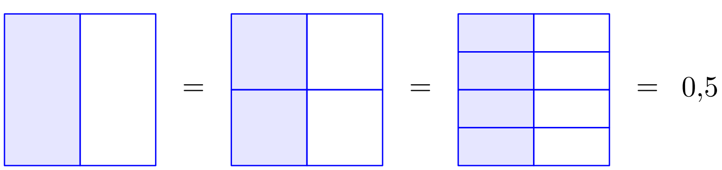
Answer

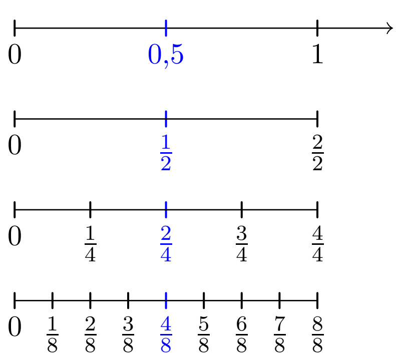
$$\frac{1}{4} = 0.25$$

Fractions with equal value

Fractions can have the same value even though they look different. If you calculate 1 : 2, 2 : 4 and 4 : 8, you will in every case end up with 0.5 as the answer. This means that

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = 0{,}5$$





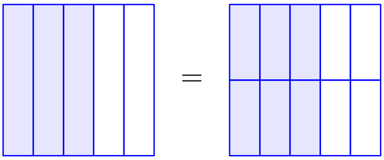
Expanding

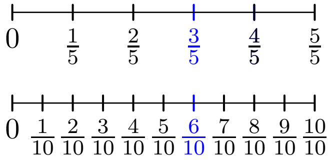
The fact that fractions can look different but have the same value, implies that we can change a fraction's look without changing its value. Let's, as an example, change $\frac{3}{5}$ into a fraction of equal value but with denominator 10:

- We can make $\frac{3}{5}$ into a fraction with denominator 10 if we divide each fifth into 2 equal pieces. In that case, 1 is divided into $5 \cdot 2 = 10$ pieces in total.

- The numerator of $\frac{3}{5}$ indicates that there are 3 fifths. When these are divided by 2, they make up $3 \cdot 2 = 6$ tenths. Hence $\frac{3}{5}$ equals $\frac{6}{10}$.

$$\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10}$$





Simplifying

Notice that we can also go "the opposite way". We can change $\frac{6}{10}$ into a fraction with denominator 5 by dividing both the numerator and the denominator by 2:

$$\frac{6}{10} = \frac{6:2}{10:2} = \frac{3}{5}$$

> ### 4.3 Expanding of fractions
>
> We can either multiply or divide both the numerator and the denominator by the same number without alternating the fractions value.
>
> Multiplying by a number larger than 1 is called *expanding* the fraction. Dividing by a number larger than 1 is called *simplifying* the fraction.

Example 1

Expand $\frac{3}{5}$ into a fraction with denominator 20.

Answer

Since $5 \cdot 4 = 20$, we multiply both the numerator and the denominator by 4:

$$\begin{aligned}\frac{3}{5} &= \frac{3 \cdot 4}{5 \cdot 4} \\ &= \frac{12}{20}\end{aligned}$$

Example 2

Expand $\frac{150}{50}$ into a fraction with denominator 100.

Answer

Since $50 \cdot 2 = 100$, we multiply both the numerator and the denominator by 2:

$$\begin{aligned}\frac{150}{50} &= \frac{150 \cdot 2}{50 \cdot 2} \\ &= \frac{300}{100}\end{aligned}$$

Example 3

Simplify $\frac{18}{30}$ into a fraction with denominator 5.

Answer

Since $30 : 6 = 5$, we divide both the numerator and the denominator by 6:

$$\begin{aligned}\frac{18}{30} &= \frac{18 : 6}{30 : 6} \\ &= \frac{3}{5}\end{aligned}$$

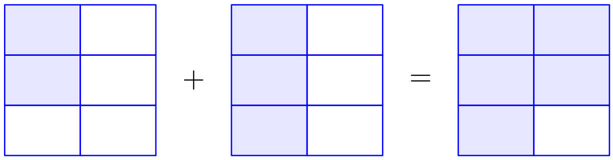
4.3 Addition and subtraction

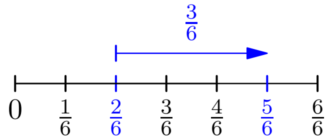
Addition and subtraction of fractions are in large parts focused around the denominators. Recall that the denominators indicate the partitioning of 1. If fractions have equal denominators, they represent amounts of equal-sized pieces. In this case it makes sense calculating addition or subtraction of the numerators. However, if fractions have unequal denominators, they represent amounts of different-sized pieces, and hence addition and subtraction of the numerators makes no sense directly.

Equal denominators

If we, for example, have 2 sixths and add 3 sixths, the sum is 5 sixths:

$$\frac{2}{6} + \frac{3}{6} = \frac{5}{6}$$





4.4 Addition/subtraction of fractions with equal denominators

When adding/subtracting fractions with equal denominators, we find the sum/difference of the numerators and keep the denominator.

Example 1

$$\begin{aligned}\frac{2}{7} + \frac{8}{7} &= \frac{2+8}{7} \\ &= \frac{10}{7}\end{aligned}$$

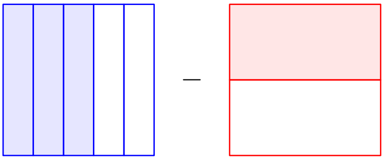
Example 2

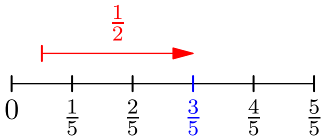
$$\frac{7}{9} - \frac{5}{9} = \frac{7-5}{9}$$
$$= \frac{2}{9}$$

Unequal denominators

Let's examine the calculation[1]

$$\frac{3}{5} - \frac{1}{2}$$



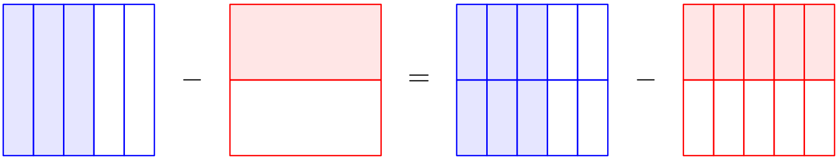


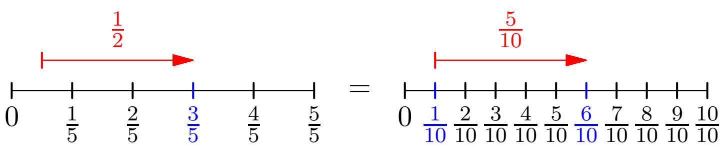
To write the difference as a single fraction, the two terms need to have denominators of equal value. Both of the fractions can have denominator 10:

$$\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10} \qquad\qquad \frac{1}{2} = \frac{1 \cdot 5}{2 \cdot 5} = \frac{5}{10}$$

Hence

$$\frac{3}{5} - \frac{1}{2} = \frac{6}{10} - \frac{5}{10}$$





[1] Recall that the red-colored arrow indicates that you shall start at the arrowhead and then move to the other end.

Summarized, we have expanded the fractions such that they have denominators of equal value, that is 10. When the denominators are equal, we can calculate the difference of the numerators:

$$\begin{aligned}\frac{3}{5}-\frac{1}{2}&=\frac{6}{10}-\frac{5}{10}\\&=\frac{1}{10}\end{aligned}$$

4.5 Addition/subtraction of fractions with unequal denominators

When calculating addition/subtraction of fractions, we must expand the fractions such that they have denominators of equal value, and then apply *Rule 4.4*.

Example 1

Calculate

$$\frac{2}{9}+\frac{6}{7}$$

Both denominators can be transformed into 63 if multiplied by a fitting integer. Therefore, we expand the fractions as follows:

$$\begin{aligned}\frac{2\cdot 7}{9\cdot 7}+\frac{6\cdot 9}{7\cdot 9}&=\frac{14}{63}+\frac{54}{63}\\&=\frac{68}{63}\end{aligned}$$

Common denominator

In *Example 1* above, 63 is called a *common denominator* because there exists integers which, when multiplied by the original denominators, results in 63:

$$9 \cdot 7 = 63$$
$$7 \cdot 9 = 63$$

Multiplying together the original denominators always results in a common denominator but one can avoid large numbers by finding the *smallest* common denominator. Take, for example,

$$\frac{7}{6} + \frac{5}{3}$$

$6 \cdot 3 = 18$ is a common denominator, but it's worth noticing that $6 \cdot 1 = 3 \cdot 2 = 6$ is too. Hence,

$$\begin{aligned}\frac{7}{6} + \frac{5}{3} &= \frac{7}{6} + \frac{5 \cdot 2}{3 \cdot 2} \\ &= \frac{7}{6} + \frac{10}{6} = \frac{17}{6}\end{aligned}$$

Example 2

Calculate

$$\frac{3}{2} - \frac{5}{8} + \frac{10}{4}$$

Answer

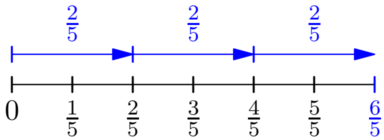
All denominators can be transformed into 8 if multiplied by a fitting integer. Therefore, we expand the fractions as follows:

$$\begin{aligned}\frac{3}{2} - \frac{5}{8} + \frac{10}{4} &= \frac{3 \cdot 4}{2 \cdot 4} - \frac{5}{8} + \frac{10 \cdot 2}{4 \cdot 2} \\ &= \frac{12}{8} - \frac{5}{8} + \frac{20}{8} \\ &= \frac{27}{8}\end{aligned}$$

4.4 Fractions multiplied by integers

In *Section 2.3* we observed that multiplying by an integer corresponds to repeated addition. Hence, if we are to calculate $\frac{2}{5} \cdot 3$, we can write

$$\begin{aligned} \frac{2}{5} \cdot 3 &= \frac{2}{5} + \frac{2}{5} + \frac{2}{5} \\ &= \frac{2+2+2}{5} \\ &= \frac{6}{5} \end{aligned}$$



Noticing that $2+2+2 = 2 \cdot 3$, we get

$$\begin{aligned} \frac{2}{5} \cdot 3 &= \frac{2 \cdot 3}{5} \\ &= \frac{6}{5} \end{aligned}$$

Multiplication of integers and fractions are also commutative[1]:

$$\begin{aligned} 3 \cdot \frac{2}{5} &= 3 \cdot 2 : 5 \\ &= 6 : 5 \\ &= \frac{6}{5} \end{aligned}$$

4.6 Fractions multiplied by integers

When multiplying a fraction by an integer, we multiply the numerator by the integer.

[1]Recall that $\frac{2}{5}$ corresponds to $2 : 5$.

Example 1

$$\begin{aligned}\frac{1}{3}\cdot 4 &= \frac{1\cdot 4}{3}\\ &= \frac{4}{3}\end{aligned}$$

Example 2

$$\begin{aligned}3\cdot\frac{2}{5} &= \frac{3\cdot 2}{5}\\ &= \frac{6}{5}\end{aligned}$$

An interpretation of multiplying by a fraction

By *Rule 4.6* we can make an interpretation of multiplying by a fraction. For example, multiplying 3 by $\frac{2}{5}$ can be interpreted in these two following ways:

- We multiply 3 by 2 and divide by 5:

$$(3\cdot 2):5 = \frac{3\cdot 2}{5} = \frac{6}{5}$$

- We divide 3 by 5 and multiply the quotient by 2:

$$3:5 = \frac{3}{5} \quad , \quad \frac{3}{5}\cdot 2 = \frac{3\cdot 2}{5} = \frac{6}{5}$$

4.5 Fractions divided by integers

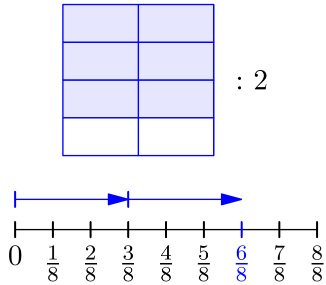
It is now important to recall two things:

- Division can be interpreted as an equal distribution of amounts
- In a fraction, it is the numerator which indicates the amount (the denominator indicates the partitioning of 1)

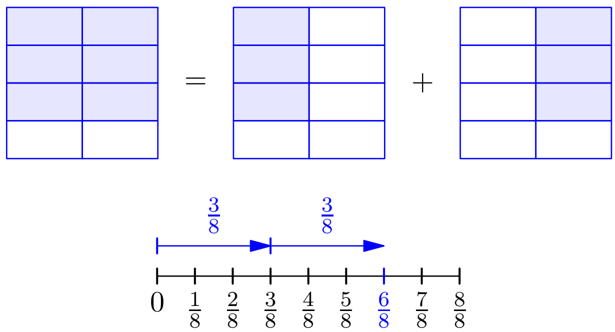
When the numerator is divisible by the divisor

Let's calculate

$$\frac{6}{8} : 2$$



We have 6 eights which are to be equally distributed into 2 groups. This results in $6 : 2 = 3$ eights.



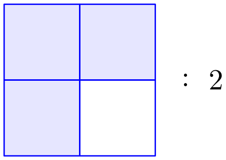
Thus

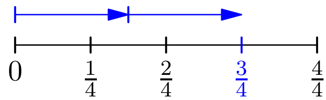
$$\frac{6}{8} : 2 = \frac{3}{8}$$

When the numerator is not divisible by the denominator

What if we are to divide $\frac{3}{4}$ by 2?

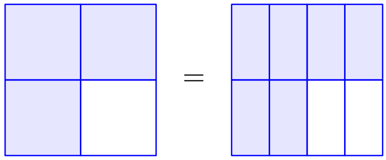
$$\frac{3}{4} : 2$$

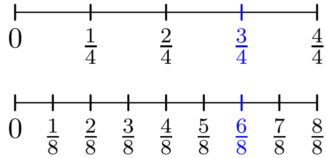




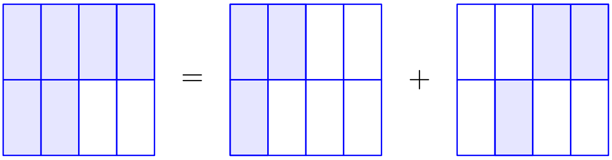
Thing is, we can always expand the fraction such that the numerator becomes divisible by the divisor. Since 2 is the divisor, we expand the fraction by 2:

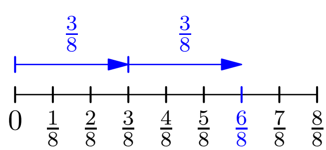
$$\frac{3}{4} = \frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8}$$





Now we have 6 eights. 6 eights divided by 2 equals 3 eights:





Hence

$$\frac{3}{4} : 2 = \frac{3}{8}$$

In effect, we have multiplied $\frac{3}{4}$ by 2:

$$\begin{aligned} \frac{3}{4} : 2 &= \frac{3}{4 \cdot 2} \\ &= \frac{3}{8} \end{aligned}$$

4.7 Fractions divided by integers

When dividing a fraction by an integer, we multiply the denominator by the integer.

Example 1

$$\begin{aligned} \frac{5}{3} : 6 &= \frac{5}{3 \cdot 6} \\ &= \frac{5}{18} \end{aligned}$$

Notice

At the start of this section we found that

$$\frac{4}{8} : 2 = \frac{2}{8}$$

In that case, there were no need to multiply the denominator by 2, such as *Rule 4.7* implies. However, if we do, we have

$$\frac{4}{8} : 2 = \frac{4}{8 \cdot 2} = \frac{4}{16}$$

Now,

$$\frac{2}{8} = \frac{2 \cdot 2}{8 \cdot 2} = \frac{4}{16}$$

Hence, unsurprisingly, the two answers are of equal value.

4.6 Fractions multiplied by fractions

We have seen that[1] multiplying a number by a fraction involves multiplying the number by the numerator and then dividing the product by the denominator. Let us apply this to calculate

$$\frac{5}{4} \cdot \frac{3}{2}$$

Firstly, we multiply $\frac{5}{4}$ by 3, then we divide the resulting product by 2. By *Rule 4.6*, we have

$$\frac{5}{4} \cdot 3 = \frac{5 \cdot 3}{4}$$

And by *Rule 4.7*, we get

$$\frac{5 \cdot 3}{4} : 2 = \frac{5 \cdot 3}{4 \cdot 2}$$

Hence

$$\frac{5}{4} \cdot \frac{3}{2} = \frac{5 \cdot 3}{4 \cdot 2}$$

4.8 Fractions multiplied by fractions

When multiplying a fraction by a fraction, we multiply numerator by numerator and denominator by denominator.

Example 1

$$\begin{aligned} \frac{4}{7} \cdot \frac{6}{9} &= \frac{4 \cdot 6}{7 \cdot 9} \\ &= \frac{24}{63} \end{aligned}$$

Example 2

$$\begin{aligned} \frac{1}{2} \cdot \frac{9}{10} &= \frac{1 \cdot 9}{2 \cdot 10} \\ &= \frac{9}{20} \end{aligned}$$

[1] Look at the text box titled "*An interpretation of multiplying by a fraction*" on page 46.

4.7 Cancelation of fractions

When the numerator and the denominator are of equal value, the fractions value always equals 1. For example, $\frac{3}{3} = 1$, $\frac{25}{25} = 1$ etc. We can exploit this fact to simplify expressions involving fractions.

Let us simplify the expression

$$\frac{8 \cdot 5}{9 \cdot 8}$$

Since $8 \cdot 5 = 5 \cdot 8$, we can write

$$\frac{8 \cdot 5}{9 \cdot 8} = \frac{5 \cdot 8}{9 \cdot 8}$$

And, as recently seen (*Rule 4.8*), we have

$$\frac{5 \cdot 8}{9 \cdot 8} = \frac{5}{9} \cdot \frac{8}{8}$$

Since $\frac{8}{8} = 1$,

$$\begin{aligned} \frac{5}{9} \cdot \frac{8}{8} &= \frac{5}{9} \cdot 1 \\ &= \frac{5}{9} \end{aligned}$$

When multiplication is exclusively present in a fraction, you can always shuffle the way we did in the above expressions. However, when you have understood the outcome of the shuffling, it is better to apply *cancellation*. You then draw a line across two and two equal factors, thus indicating that they constitute a fraction which equals 1. Hence, our most recent example can be simplified to

$$\frac{\cancel{8} \cdot 5}{9 \cdot \cancel{8}} = \frac{5}{9}$$

4.9 Cancellation of factors

When multiplication is exclusively present in a fraction, we can cancel pair of equal factors in numerator and denominator.

Example 1

Cancel as many factors as possible in the fraction.

$$\frac{3 \cdot 12 \cdot 7}{7 \cdot 4 \cdot 12}$$

Answer

$$\frac{3 \cdot \cancel{12} \cdot \cancel{7}}{\cancel{7} \cdot 4 \cdot \cancel{12}} = \frac{3}{4}$$

Example 1

Simplify the fraction $\frac{12}{42}$.

Answer

$$\begin{aligned} \frac{12}{42} &= \frac{\cancel{6} \cdot 2}{\cancel{6} \cdot 7} \\ &= \frac{2}{7} \end{aligned}$$

Example 2

Simplify the fraction $\frac{48}{16}$.

Answer

$$\begin{aligned} \frac{48}{16} &= \frac{3 \cdot \cancel{16}}{\cancel{16}} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

Notice: If all factors are canceled in the numerator or the denominator, 1 takes their place.

Fractions simplify calculations

The decimal number 0.125 can be written as the fraction $\frac{1}{8}$. The calculation

$$0.125 \cdot 16$$

is, for the most of us, rather strenuous to carry out. However, exploiting the nature of fractions, we have

$$\begin{aligned} 0.125 \cdot 16 &= \frac{1}{8} \cdot 16 \\ &= \frac{2 \cdot \cancel{8}}{\cancel{8}} \\ &= 2 \end{aligned}$$

"Cancelling zeros"

A number such as 3000 equals $3 \cdot 10 \cdot 10 \cdot 10$, while 700 equals $7 \cdot 10 \cdot 10$. Hence, we can simplify $\frac{3000}{700}$ like this:

$$\begin{aligned} \frac{3000}{700} &= \frac{3 \cdot \cancel{10} \cdot \cancel{10} \cdot 10}{7 \cdot \cancel{10} \cdot \cancel{10}} \\ &= \frac{3 \cdot 10}{7} \\ &= \frac{30}{7} \end{aligned}$$

In practice, this is the same as "cancelling zeros":

$$\frac{30\cancel{00}}{7\cancel{00}} = \frac{30}{7}$$

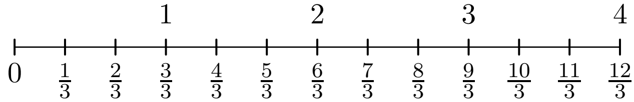
Aware! Zeros are the only digits we can "cancel" this way. For example, $\frac{123}{13}$ cannot be simplified in any way. Also, we can only "cancel" zeros which are right-most situated, e.g. we cannot "cancel" zeros in the fraction $\frac{101}{10}$.

4.8 Division by fractions

Division by studying the number line

Let's calculate $4 : \frac{2}{3}$. Since the fraction have denominator 3, it could be wise to transform also 4 into a fraction with denominator 3.

$$4 = \frac{12}{3}$$

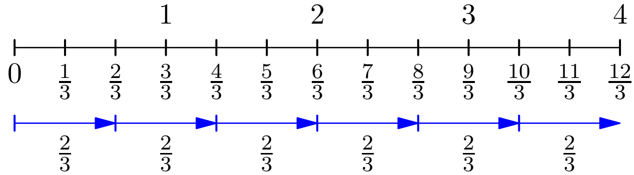


Recall that one of the interpretations of $4 : \frac{2}{3}$ is

"The number of $\frac{2}{3}$'s added to make 4."

By studying a number line, we find that 6 instances of $\frac{2}{3}$ added together equals 4. Hence

$$4 : \frac{2}{3} = 6$$



A general method

We can't study the number line every time we are to divide by a fraction, so here we shall find a general method, again with $4 : \frac{2}{3}$ as our example. In this case, we apply the following interpretation of division:

$$4 : \frac{2}{3} = \text{"The number to multiply } \tfrac{2}{3} \text{ by to make 4."}$$

We begin the search of this number by multiplying $\frac{2}{3}$ by the number which results in the product equal to 1. This number is the *inverted fraction* of $\frac{2}{3}$, namely $\frac{3}{2}$:

$$\frac{2}{3} \cdot \frac{3}{2} = 1$$

Now we only have to multiply by 4 to make 4:

$$\frac{2}{3} \cdot \frac{3}{2} \cdot 4 = 4$$

Therefore, to make 4 we must multiply $\dfrac{2}{3}$ by $\dfrac{3}{2} \cdot 4$. Consequently,

$$\begin{aligned} 4 : \frac{2}{3} &= \frac{3}{2} \cdot 4 \\ &= 6 \end{aligned}$$

4.10 Fractions divided by fractions

When dividing a number by a fraction, we multiply the number by the inverted fraction.

Example 1

$$\begin{aligned} 6 : \frac{2}{9} &= 6 \cdot \frac{9}{2} \\ &= 27 \end{aligned}$$

Example 2

$$\begin{aligned} \frac{4}{3} : \frac{5}{8} &= \frac{4}{3} \cdot \frac{8}{5} \\ &= \frac{32}{15} \end{aligned}$$

Example 3

$$\begin{aligned}\frac{3}{5} : \frac{3}{10} &= \frac{3}{5} \cdot \frac{10}{3} \\ &= \frac{30}{15}\end{aligned}$$

In this case we should also observe that the fraction can be simplified:

$$\begin{aligned}\frac{30}{15} &= \frac{2 \cdot \cancel{15}}{\cancel{15}} \\ &= 2\end{aligned}$$

Notice: Canceling factors along the way saves the labor of working with large numbers:

$$\begin{aligned}\frac{3}{5} \cdot \frac{10}{3} &= \frac{\cancel{3} \cdot 2 \cdot \cancel{5}}{\cancel{5} \cdot \cancel{3}} \\ &= 2\end{aligned}$$

4.9 Rational numbers

4.11 Rational numbers

Any number which can be expressed as a fraction is a *rational number*.

Merk

Rational numbers is a collective name of

- **Integers**

  For example $4 = \frac{4}{1}$.

- **Decimal numbers with a finite number of digits**

  For example $0{,}2 = \frac{1}{5}$.

- **Decimal numbers with infinite digits in a repeating manner**

  For example [1] $0.08\bar{3} = \frac{1}{12}$.

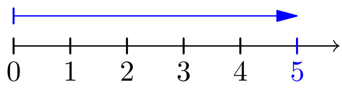
[1] $\bar{3}$ indicates that 3 repeats infinite. Another way of expressing this is by using ... . That is, $0.08\bar{3} = 0.08333333...$

Chapter 5

Negative numbers

5.1 Introduction

Earlier we have seen that e.g. 5 on a number line is placed 5 units to the right of 0.

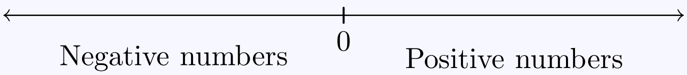


But what if we move in the other direction, that is to the left? The question is answered by introducing *negative numbers*.

5.1 Positive and negative numbers

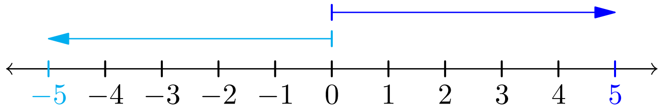
On a number line, the following applies:

- Numbers placed to *the right* of 0 are positive numbers.
- Numbers placed to *the left* of 0 are negative numbers.



However, relying on the number line every time negative numbers are involved would be very inconvenient, therefore we use a symbol to indicate negative numbers. This is $-$, simply the same as the symbol of subtraction. From this it follows that 5 is a positive number, while -5 is a negative number. On the number line,

- 5 is placed 5 units to *the right* of 0.
- -5 is placed 5 units to *the left* of 0.



Hence, the big difference between 5 and -5 is on which side of 0 the numbers are placed. Since 5 and -5 have the same distance from 0, we say that 5 and -5 have equal *length*.

5.2 Length (absolute value/modulus/magnitude)

The length of a number is expressed by the symbol $|\,|$.

The length of a positive number equals the value of the number.

The length of a negative number equals the value of the positive number with corresponding digits.

Example 1

$$|27| = 27$$

Example 2

$$|-27| = 27$$

Sign

Sign is a collective name of $+$ and $-$. $+$ is the sign of 5 and $-$ is the sign of -5.

5.2 The elementary operations

The introduction of negative numbers bring new aspects to the elementary operations. When adding, subtracting, multiplying or dividing by negative numbers, we'll frequently, for clarity, enclose negative numbers by parentheses. Then we'll write e.g. -4 as (-4).

Addition

When adding in *Section 2.1* $+$ implied moving to *the right*. Negative numbers bring an alternation of the interpretation of $+$:

$+$ ”As long and in *the same* direction as”

Let's study the calculation

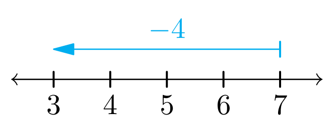
$$7 + (-4)$$

Our alternated definition of $+$ implies that

$$7 + (-4) = \text{”7 and as long and in the } \textit{same} \text{ direction as } (-4)\text{”}$$

(-4) has length 4 and direction to *the left*. Hence, the calculation tells us to start at 7 and then move the length of 4 to *the left*.

$$7 + (-4) = 3$$



5.3 Addition involving negative numbers

Adding a negative number is the same as subtracting the number of equal magnitude.

Example 1

$$4 + (-3) = 4 - 3 = 1$$

Example 2

$$-8 + (-3) = -8 - 3 = -11$$

Notice

Rule 2.1 declares that addition is commutative. This also applies after introducing negative numbers, for example is

$$7 + (-3) = 4 = -3 + 7$$

Subtraction

In *Section 2.2*, $-$ implied moving to *the left*. The interpretation of $-$ also needs an alternation when working with negative numbers:

$-$ "As long and in the *opposite* direction as"

Let's study the calculation

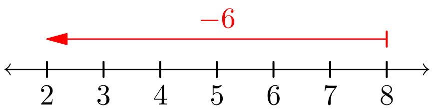
$$2 - (-6)$$

Our alternated definition of $-$ implies that

$$2 - (-6) = \text{"2 and as long and in the } \textit{opposite} \text{ direction as } (-6)\text{"}$$

-6 have length 6 and direction to *the left*. When moving an equal length, but in the *opposite* direction, we have to move the length of 6 to *the right*[1]. This is equivalent to adding 6:

$$2 - (-6) = 2 + 6 = 8$$



5.4 Subtraction involving negative numbers

Subtracting a negative number is the same as adding the positive number of equal magnitude.

Example 1

$$11 - (-9) = 11 + 9 = 20$$

[1]Once again, recall that the red colored arrow indicates starting at the arrowhead, then moving to the other end.

Example 2

$$-3 - (-7) = -3 + 7 = 4$$

Multiplication

In *Section 2.3*, multiplication by positive integers were introduced as repeated addition. By our alternated interpretations of addition and subtraction we can now also alternate the interpretation of multiplication:

5.5 Multiplication by positive and negative integers

- Multiplication by a positive integer corresponds to repeated addition.
- Multiplication by a negative integer corresponds to repeated subtraction.

Example 1

$$\begin{aligned} 2 \cdot 3 &= \text{"As long and in the \textit{same} direction as 2, 3 times"} \\ &= 2 + 2 + 2 \\ &= 6 \end{aligned}$$

Example 2

$$\begin{aligned} (-2) \cdot 3 &= \text{"As long and in the \textit{same} direction as } (-2)\text{, 3 times"} \\ &= -2 - 2 - 2 \\ &= -6 \end{aligned}$$

Example 3

$$\begin{aligned} 2 \cdot (-3) &= \text{"As long and in the \textit{opposite} direction as 2, 3 times"} \\ &= -2 - 2 - 2 \\ &= -6 \end{aligned}$$

Example 4

$$\begin{aligned}(-3)\cdot(-4) &= \text{"As long and in the \textit{opposite} direction as } -3,\ 4 \text{ times"}\\ &= 3+3+3+3\\ &= 12\end{aligned}$$

Multiplication is commutative

Example 2 and *Example 3* on page 63 illustrates that *Rule 2.2* also implies after introducing negative numbers:

$$(-2)\cdot 3 = 3\cdot(-2)$$

It would be laborious to calculate multiplication by repeated addition/subtraction every time a negative number were involved, however, as a direct consequence of *Rule 5.5* we can make the two following rules:

5.6 Multiplication by negative numbers I

The product of a negative number and a positive number is a negative number.

The magnitude of the factors multiplied together yields the magnitude of the product.

Example 1

Calculate $(-7)\cdot 8$

Answer

Since $7\cdot 8 = 56$, we have $(-7)\cdot 8 = -56$

Example 2

Calculate $3\cdot(-9)$.

Answer

Since $3\cdot 9 = 27$, we have $3\cdot(-9) = -27$

5.7 Multiplication by negative numbers II

The product of two negative numbers is a positive number.

The magnitude of the factors multiplied together yields the value of the product.

Example 1

$$(-5) \cdot (-10) = 5 \cdot 10 = 50$$

Example 2

$$(-2) \cdot (-8) = 2 \cdot 8 = 16$$

Division

From the definition of division (see *Section 2.4*), combined with what we now know about multiplication involving negative numbers, it follows that

$$-18 : 6 = \text{"The number which yields } -18 \text{ when multiplied by } 6\text{ "}$$

$$6 \cdot (-3) = -18, \text{ hence } -18 : 6 = -3$$

$$42 : (-7) = \text{"The number which yields } 42 \text{ when multiplied by } -7\text{"}$$

$$(-7) \cdot (-8) = 42, \text{ hence } 42 : (-7) = -8$$

$$-45 : (-5) = \text{"The number which yields } -45 \text{ when multiplied by } -5\text{"}$$

$$(-5) \cdot 9 = -45, \text{ hence } -45 : (-5) = 9$$

5.8 Division involving negative numbers

Division between a positive number and a negative number yields a negative number.

Division between two negative numbers yields a positive number.

The magnitude of the dividend divided by the magnitude of the divisor yields the magnitude of the quotient.

Example 1

$$-24 : 6 = -4$$

Example 2

$$24 : (-2) = -12$$

Example 3

$$-24 : (-3) = 8$$

Example 4

$$\frac{2}{-3} = -\frac{2}{3}$$

Example 5

$$\frac{-10}{7} = -\frac{10}{7}$$

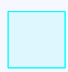
5.3 Negative numbers as amounts

Notice: This view of negative numbers will first come into use in Section 8.2, a section a lot of readers can skip without loss of understanding.

So far, we have studied negative number by the aid of number lines. Studying negative numbers as amounts is at first difficult because negative amounts makes no sense! To make an interpretation of negative numbers through the perspective of amounts, we'll use what we shall call the *weight principle*. Then we look upon the numbers as amounts of forces. The positive numbers are amounts of forces acting downwards while the negative numbers are amounts of forces working upwards[1]. In this way, the results of calculations involving positive and negative numbers can be looked upon as the result of weighing the amounts. Hence, a positive number and a negative number of equal magnitude will cancel each other.

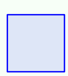
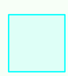
5.9 Negative numbers as amounts

Negative numbers will be illustrated as a light blue amount:

 $= -1$

Example

$$1 + (-1) = 0$$

 $+$  $= 0$

[1]From reality one can look upon the positive and the negative numbers as balloons filled with air and helium, respectively. Balloons filled with air acts with a force downwards (they fall), while balloons filled with helium acts with a force upwards (they rise).

Chapter 6

Geometry

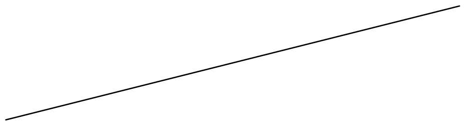
6.1 Terms

Point

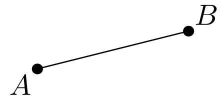
A given position is called a[1] *point*. We mark a point by drawing a dot, which we preferably name by a letter. Below we have drawn the points A and B.

Line and segment

A straight dash with infinite length (!) is called a *line*. The fact that a line has infinite length, makes *drawing* a line impossible, we can only *imagine* a line. Imagining a line can be done by drawing a straight dash and think of its ends as wandering out in each direction.



A straight dash between two points is called a *segment*.

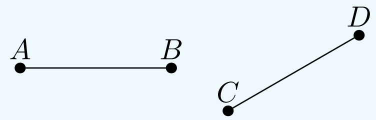


We write the segment between the points A and B as AB.

> **Notice**
>
> A segment is a part of a line, therefore a line and a segment have a lot of attributes in common. When writing about lines, it will be up to the reader to confirm whether the same applies for segments. Hence we avoid the need of writing "lines/segments".

[1] See also *Section 1.3*.

Segment or length?

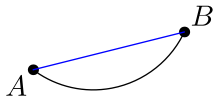


The segments AB and CD have equal length, but they are not the same segment. Still we'll write $AB = CD$. That is, we'll use the same names for the line segments and their lengths (the same applies for angles and their values, see page 72 - 74). We'll do this by the following reasons:

- The context will make it clear weather we are talking about a segment or a length.
- Finding it necessary to write e.g. "the length of AB" would make sentences less readable.

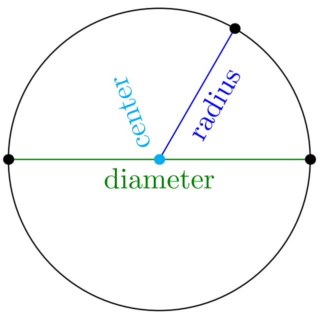
Distance

There are infinite ways one can move from one point to another and some ways will be longer than others. When talking about a distance in geometry, we usually mean the *shortest* distance. For geometries studied in this book the shortest distance between two points will always equal the length of the segment (blue in the below figure) connecting them.



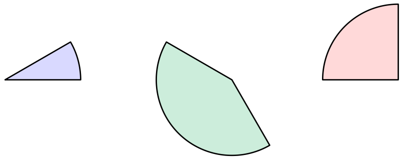
Circle; center, radius and diameter

If we make an enclosed curve where all points on this curve have the same distance to a given point, we have a *circle*. The point which all the points on the curve have an equal distance to is the *center* of the circle. A segment between a point on the curve and the center is called a *radius*. A segment between two points on the curve, passing through the circle center, is called a *diameter*[1].



Arcs and sectors

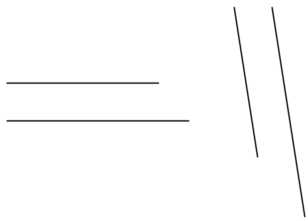
A part of a circular curve is called an *arc*. The shape formed by an arc and two associated radii is called a *sector*. The below figure shows three different sectors.



[1] As mentioned, *radius* and *diameter* can just as well indicate the length of the segments.

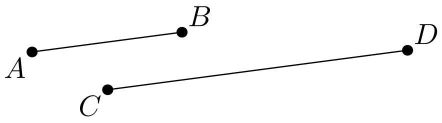
Parallel lines

Lines aligned in the same direction are *parallel*. The below figure shows two pairs of parallel lines.



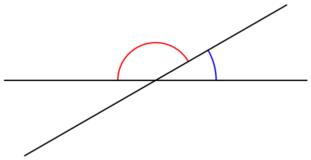
We use the symbol $\parallel$ to indicate that two lines are parallel.

$$AB \parallel CD$$



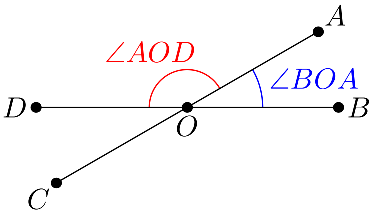
Angles

Non-parallel lines will sooner or later intersect. The gap formed by two non-parallel lines is called an *angle*. We draw angles as small circular curves:



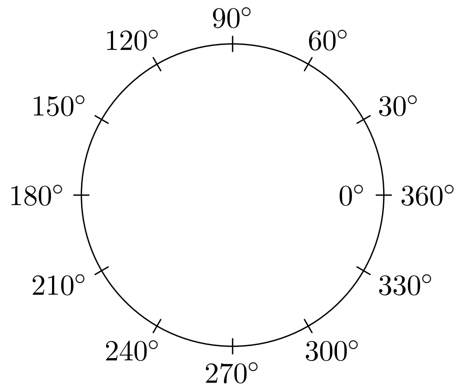
Lines creating an angle are called the *sides* of the angle. The intersection point of the lines are called the *vertex* of the angle. It is common to use the symbol $\angle$ to underline the angle in question. In the below figure we have the following:

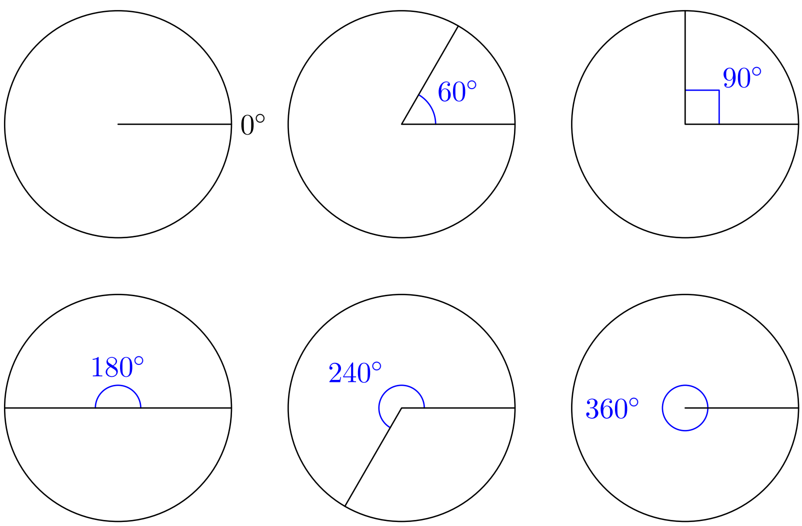
- the angle $\angle BOA$ has angle sides OB and OA and vertex O.
- the angle $\angle AOD$ has angle sides OA and OD and vertex O.



Measure of angles in degrees

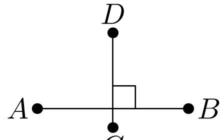
When measuring an angle in degrees, we imagine a circular curve divided into 360 equally long pieces. We call one such piece 1 *degree*, indicated by the symbol $^\circ$.





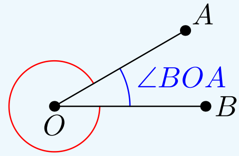
Notice that an angle with measure 90° is indicated by the symbol $\square$. Such an angle is called a *right* angle. Lines which form right angles are said to be *perpendicular* to one another, indicated by the symbol $\perp$.

$AB \perp CD$

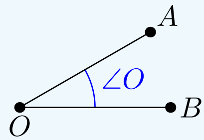


What angle?

Strictly speaking, when two segments (or lines) intersect, they form two angles; the one larger than 180° and the other smaller than 180°. Usually it is the smaller angle we wish to study, therefore it is common to define $\angle AOB$ as the *smaller* angle formed by the segments OA and OB.

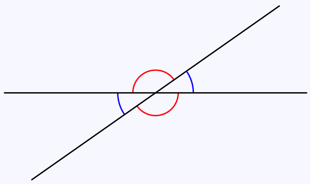


As long as there are only two segments/lines present, it is common using only one letter to indicate the angle:

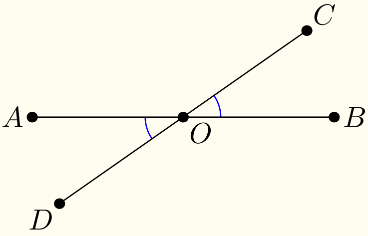


6.1 Vertical angles

Two opposite angles with a common vertex is called *vertical angles*. Vertical angles are of equal measure.



6.1 Vertical angles (explanation)



We have

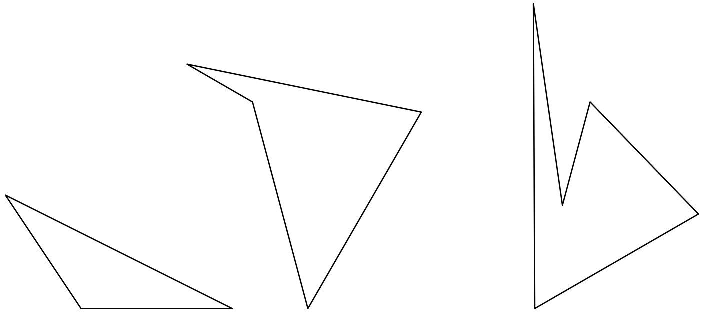
$$\angle BOC + \angle DOB = 180^\circ$$

$$\angle AOD + \angle DOB = 180^\circ$$

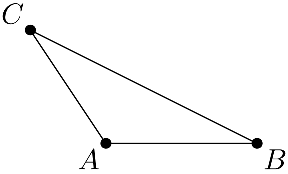
Hence, $\angle BOC = \angle AOD$. Similarly, $\angle COA = \angle DOB$.

Sides and vertices

When line segments form an enclosed shape, they form a *polygon*. The below figure shows, from left to right, a triangle (3-gon), a quadrilateral (4-gon) and a pentagon (5-gon).

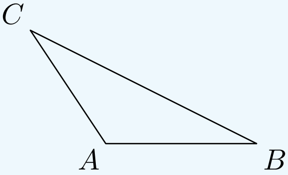


The segments of a polygon are called *edges* or *sides*. The respective intersection points of the segments are the *vertices* of the polygon. That is, the triangle below has vertices A, B and C and sides (edges) AB, BC and AC.



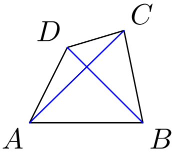
Notice

Often we'll write a letter only to indicate a vertex of a polygon.



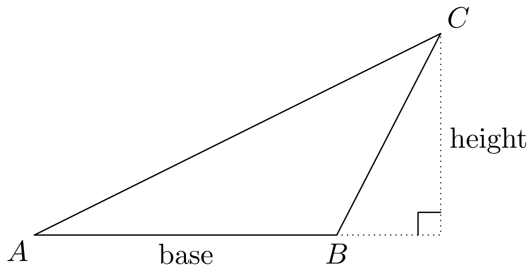
Diagonals

Segments between two vertices not belonging to the same side of a polygon is called a *diagonal*. The below figure shows the diagonals AC and BD.

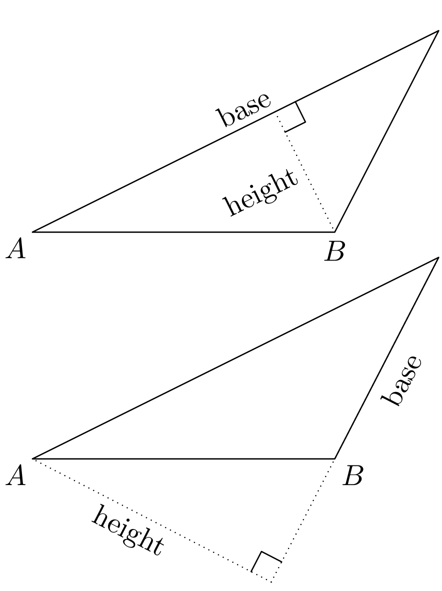


Altitudes and base lines

In *Section 6.4*, the terms *base* and *height* (*altitude*) play an important role. To find the height of a triangle, we choose one of the sides to be the base. In the below figure, let's start with AB as the base. Then the height is the segment from AB (potentially, as is the case here, the extension of AB) to C, perpendicular to AB.



Since there are three sides which can be bases, a triangle has three heights.



> **Notice**
>
> The terms altitude and base also applies to other polygons.

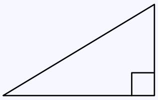
6.2 Attributes of triangles and quadrilaterals

In addition to having a certain number of sides and vertices, polygons have other attributes, such as sides or angles of equal measure, or parallel sides. There are specific names of polygons with special attributes, and these names can be put into an overview where some ”inherit”[1] attributes from others.

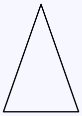
6.2 Triangles

$$\begin{tikzcd} & \text{Right triangle} & \\ \text{Triangle} \arrow[ur] \arrow[dr] & & \\ & \text{Isosceles triangle} \arrow[r] & \text{Equilateral triangle} \end{tikzcd}$$

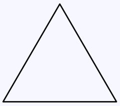
Triangle
Have three sides and three vertices.



Right triangle
Have an angle of 90°.



Isosceles triangle
At least two sides are of equal length.
At least two angles are of equal measure.



Equilateral triangle
The sides are of equal length.
Each of the angles equals 60°.

Example

Since an equilateral triangle have three sides of equal length and three angles equal to 60°, it is also an isosceles triangle.

The language box

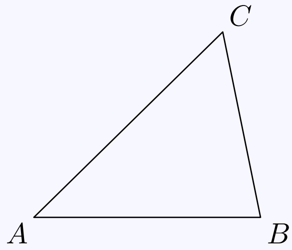
The longest side of a right triangle is called the *hypotenuse* and the shorter sides are called *legs*.

[1] In *Rule 6.2* and *Rule 6.4* this is indicated by arrows.

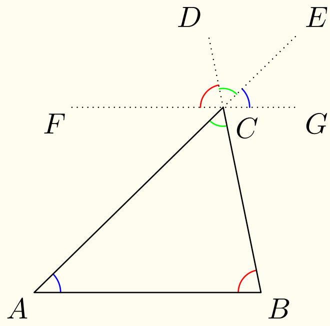
6.3 The sum of angles in a triangle

In a triangle, the sum of the angles equals 180°.

$$\angle A + \angle B + \angle C = 180^\circ$$



6.3 The sum of angles in a triangle (explanation)



We draw a segment FG passing through C and parallel to AB. Moreover, we place E and D on the extension of AC and BC, respectively. Then $\angle A = \angle GCE$ and $\angle B = \angle DCF$. $\angle ACB = \angle ECD$ because they are vertical angles. Now

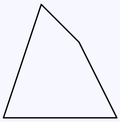
$$\angle DCF + \angle ECD = \angle GCE = 180^\circ$$

Hence

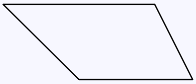
$$\angle CBA + \angle ACB + \angle BAC = 180^\circ$$

6.4 Quadrilaterals

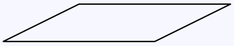
Quadrilateral ⟶ Trapezoid ⟶ Parallelogram ⟶ Rhombus / Rectangle ⟶ Square



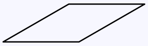
Quadrilateral
Have four sides and four vertices.



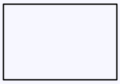
Trapezoid
Have at least one pair of parallel sides.



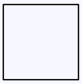
Parallelogram
Have two pairs of parallel sides.
Have two pairs of equal angles.



Rhombus
All sides are of equal length.



Rectangle
All angles equals 90°.



Square

Example

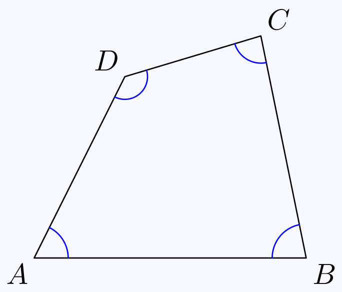
The square is both a rhombus and a rectangle, which means it "inherits" their attributes. From this it follows that, in a square,

- all sides are of equal length.
- all angles equals 90°.

6.5 The sum of angles in a quadrilateral

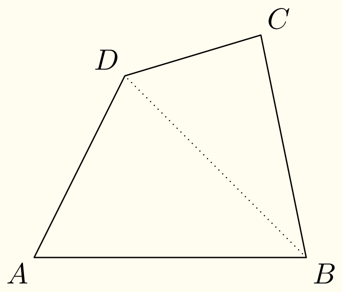
In a quadrilateral, the sum of the angles equals 360°.

$$\angle A + \angle B + \angle C + \angle D = 360^\circ$$



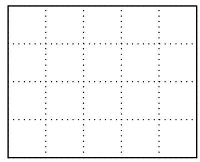
6.5 The sum of angles in a quadrilateral (explanation)

The total sum of angles of $\triangle ABD$ and $\triangle BCD$ equals the sum of the angles in $\square ABCD$. By *Rule 6.3*, the sum of angles of triangles 180°, therefore the sum of the angles of $\square ABCD$ equals $2 \cdot 180^\circ = 360^\circ$.

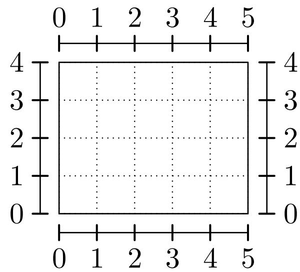


6.3 Perimeter

When we measure the length around an enclosed shape, we find its *perimeter*. Let's find the perimeter of this rectangle:



The rectangle has two sides of length 4 and two sides of length 5.



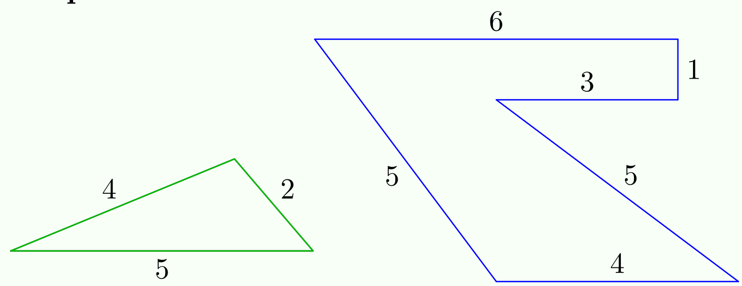
Hence

$$\begin{aligned}\text{The perimeter of the rectangle} &= 4 + 4 + 5 + 5 \\ &= 18\end{aligned}$$

6.6 Perimeter

A perimeter is the length around a closed shape.

Example



(a) *(b)*

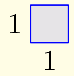
In figure *(a)* the perimeter equals $5 + 2 + 4 = 11$.

In figure *(b)* the perimeter equals $4 + 5 + 3 + 1 + 6 + 5 = 24$.

6.4 Area

Our surroundings are full of *surfaces*, for example on a floor or a sheet. When measuring surfaces, we find their *area*. The concept of area is the following:

We imagine a square with sides of length 1. We call this the *unit square*.

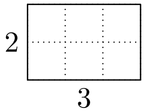


Then, regarding the surface for which we seek the area of, we ask:

”How many unit squares does this surface contain?”

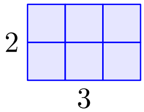
The area of a rectangle

Let’s find the area of a rectangle with baseline 3 and altitude 2.



Simply by counting, we find that the rectangle contains 6 unit squares:

$$\text{The area of the rectangle} = 6$$



Looking back at *Section 2.3*, we notice that

$$\begin{aligned}\text{The area of the rectangle} &= 3 \cdot 2 \\ &= 6\end{aligned}$$

6.7 The area of a rectangle

$$\text{Area} = \text{baseline} \cdot \text{altitude}$$

height

base

Width and length

In a rectangle, the baseline and the altitude are also referred to as (in random order) the *width* and the *length*.

Example 1

Find the area of the rectangle[1].

2

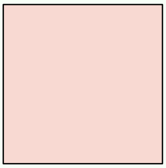
4

Answer

$$\text{The area of the rectangle} = 4 \cdot 2 = 8$$

Example 2

Find the area of the square.



3

Answer

$$\text{The area of the square} = 3 \cdot 3 = 9$$

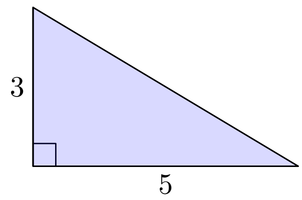
[1] *Notice*: The lengths used in one figure will not necessarily correspond to the lengths in another figure. That is, a side of length 1 in one figure can might as well be shorter than a side of length 1 in another figure.

The area of a triangle

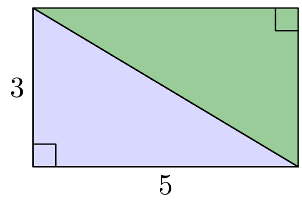
Concerning triangles, there are three different cases to study:

1) The baseline and the altitude have a common end point

Let's find the area of a right triangle with baseline 5 and altitude 3.



We can make a rectangle by copying our triangle and joining the hypotenuses:



By *Rule 6.7*, the area of the rectangle equals $5 \cdot 3$. The area of one of the triangles makes up half the area of the rectangle, so

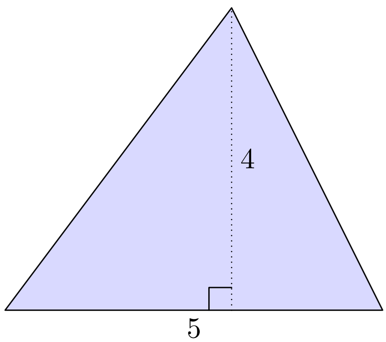
$$\text{The area of the blue triangle} = \frac{5 \cdot 3}{2}$$

Regarding the blue triangle we have

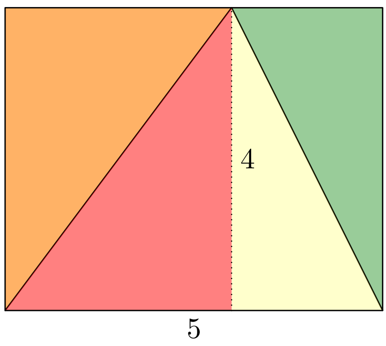
$$\frac{5 \cdot 3}{2} = \frac{\text{baseline} \cdot \text{height}}{2}$$

2) The altitude is placed inside the triangle, but have no common end point with the baseline

The below triangle has baseline 5 and altitude 4.



We make a rectangle containing the blue triangle (split into the red triangle and the yellow triangle):



Observe that

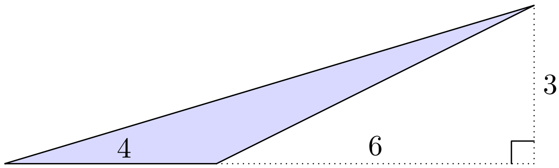
- the area of the red triangle makes up half the area of the rectangle formed by the red and the yellow triangle.
- the area of the yellow triangle makes up half the area of the rectangle formed by the yellow and the green triangle.

It now follows that the sum of the areas of the yellow triangle and the red triangle makes up half the area of the rectangle formed by the four colored triangles. The area of this rectangle equals $5 \cdot 4$, and since our original triangle (the blue) includes the red triangle and the orange triangle, we have

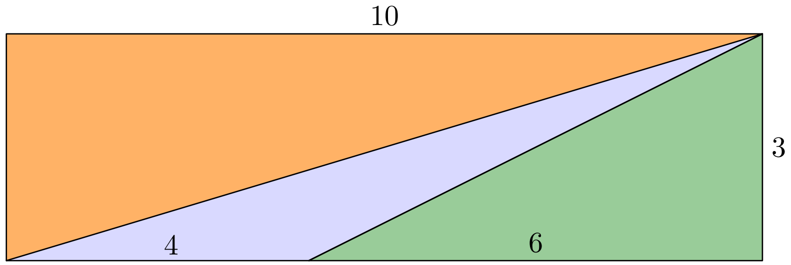
$$\text{The area of the blue triangle} = \frac{5 \cdot 4}{2} = \frac{\text{baseline} \cdot \text{height}}{2}$$

2) The altitude is placed outside the triangle

The below triangle has baseline 4 and altitude 3.



We now make a rectangle containing the blue triangle:



Now we introduce the following names:

$$\begin{aligned} \text{The area of the rectangle} &= R \\ \text{The area of the blue triangle} &= B \\ \text{The area of the orange triangle} &= O \\ \text{The area of the green triangle} &= G \end{aligned}$$

We have

$$R = 3 \cdot 10 = 30$$

$$O = \frac{3 \cdot 10}{2} = 15$$

$$G = \frac{3 \cdot 6}{2} = 9$$

Moreover,

$$\begin{aligned} B &= R - O - G \\ &= 30 - 15 - 9 \\ &= 6 \end{aligned}$$

Observe that we can write

$$6 = \frac{4 \cdot 3}{2}$$

Regarding the blue triangle we recognize this as

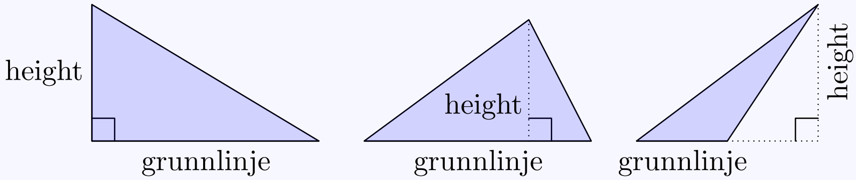
$$\frac{4 \cdot 3}{2} = \frac{\text{base} \cdot \text{height}}{2}$$

The three cases summarized

For a chosen baseline in a triangle, one of the cases discussed will always be valid. All cases resulted in the same expression for the area of the triangle.

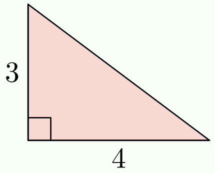
6.8 The area of a triangle

$$\text{Area} = \frac{\text{base} \cdot \text{height}}{2}$$



Example 1

Find the area of the triangle.

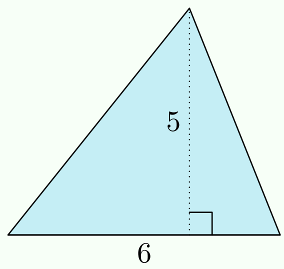


Answer

$$\begin{aligned}\text{The area of the triangle} &= \frac{4 \cdot 3}{2} \\ &= 6\end{aligned}$$

Example 2

Find the area of the triangle.

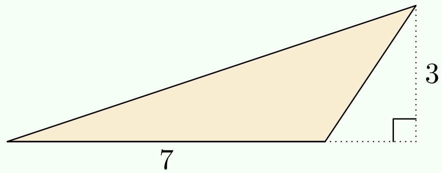


Answer

$$\text{The area of the triangle} = \frac{6 \cdot 5}{2} = 15$$

Example 3

Find the area of the triangle.



Answer

$$\text{The area of the triangle} = \frac{7 \cdot 3}{2} = \frac{21}{2}$$

Part II

Algebra and geometry

Chapter 7

Algebra

7.1 Introduction

Simply said, *algebra* is mathematics where letters represent numbers. This makes it easier working with *general* cases. For example, $3 \cdot 2 = 2 \cdot 3$ and $6 \cdot 7 = 7 \cdot 6$ but these are only two of the infinitely many examples of the commutative property of multiplication! One of the aims of algebra is giving *one* example that explains *all* cases, and since our digits (0-9) are inevitably connected to specific numbers, we apply letters to reach this target.

The value of the numbers represented by letters will often vary, in that case we call the letter-numbers *variables*. If letter-numbers on the other hand have a specific value, they are called *constants*.

In *Part I*, we studied calculations through examples with specific numbers, however, most of these rules are *general*; they are valid for all numbers. On page 92 - 95, many of these rules are reproduced in a general form. A good way of getting acquainted with algebra is comparing the rules here presented by the way they are expressed in[1] *Part I*.

7.1 Addition is commutative (2.1)

$$a + b = b + a$$

Example

$$7 + 5 = 5 + 7$$

7.2 Multiplication is commutative (2.2)

$$a \cdot b = b \cdot a$$

Example 1

$$9 \cdot 8 = 8 \cdot 9$$

Example 2

$$8 \cdot a = a \cdot 8$$

[1]The number of the rules as found in *Part I* are written inside parentheses.

Multiplication involving letters

When multiplication involves letters, it is common to omit the symbol of multiplication. If a specific number and a letter are multiplied together, the specific number is written first. For example,

$$a \cdot b = ab$$

and

$$a \cdot 8 = 8a$$

We also write

$$1 \cdot a = a$$

In addition, it is common to omit the symbol of multiplication when an expression with parentheses is involved:

$$3 \cdot (a + b) = 3(a + b)$$

7.3 Fractions as rewriting of division (4.1)

$$a : b = \frac{a}{b}$$

Example

$$a : 2 = \frac{a}{2}$$

7.4 Fractions multiplied by fractions (4.8)

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Example 1

$$\frac{2}{11} \cdot \frac{13}{21} = \frac{2 \cdot 13}{11 \cdot 21} = \frac{26}{231}$$

Example 2

$$\frac{3}{b} \cdot \frac{a}{7} = \frac{3a}{7b}$$

7.5 Division by fractions (4.10)

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Example 1

$$\frac{1}{2} : \frac{5}{7} = \frac{1}{2} \cdot \frac{7}{5}$$

Example 2

$$\begin{aligned} \frac{a}{13} : \frac{b}{3} &= \frac{a}{13} \cdot \frac{3}{b} \\ &= \frac{3a}{13b} \end{aligned}$$

7.6 Distributive law (3.5)

$$(a+b)c = ac + bc$$

Example 1

$$(2+a)b = 2b + ab$$

Example 2

$$a(5b-3) = 5ab - 3a$$

7.7 Multiplication by negative numbers I (5.6)

$$a \cdot (-b) = -(a \cdot b)$$

Example 1

$$\begin{aligned} 3 \cdot (-4) &= -(3 \cdot 4) \\ &= -12 \end{aligned}$$

Example 2

$$\begin{aligned}(-a)\cdot 7 &= -(a\cdot 7)\\ &= -7a\end{aligned}$$

7.8 Multiplication by negative numbers II (5.7)

$$(-a)\cdot(-b) = a\cdot b$$

Example 1

$$\begin{aligned}(-2)\cdot(-8) &= 2\cdot 8\\ &= 16\end{aligned}$$

Example 2

$$(-a)\cdot(-15) = 15a$$

Extensions of the rules

One of the strengths of algebra is that we can express compact rules which are easily extended to apply for other cases. Let's, as an example, find another expression of

$$(a+b+c)d$$

Rule 7.6 does not directly imply how to calculate between the expression inside the parentheses and d, but there is no wrongdoing in defining $a+b$ as k:

$$a+b=k$$

Then

$$(a+b+c)d=(k+c)d$$

Now, by *Rule 7.6*, we have

$$(k+c)d=kd+cd$$

Inserting the expression for k, we have

$$kd+cd=(a+b)d+cd$$

By applying *Rule 7.6* once more we can write

$$(a+b)d+cd=ad+bc+cd$$

Then

$$(a+b+c)d=ad+bc+cd$$

Notice! This example is not meant to show how to handle expressions not directly covered by Rule *7.1 - 7.8, but to emphasize why it's always sufficient to write rules with the least amount of terms, factors etc. Usually you apply extension of the rules without even thinking about it, and surely not in such meticulous manner as here provided.*

7.2 Powers

$$\text{base} \longrightarrow 2^{3 \longleftarrow \text{exponent}}$$

A power is composed by a *base* and an *exponent*. For example, 2^3 is a power with base 2 and exponent 3. An exponent which is a positive integer indicates the amount of instances of the base to be multiplied together, that is

$$2^3 = 2 \cdot 2 \cdot 2$$

7.9 Powers

a^n is a power with base a and exponent n.

If n is a natural number, a^n corresponds to n instances of a multiplied together.

Notice: $a^1 = a$

Example 1

$$\begin{aligned} 5^3 &= 5 \cdot 5 \cdot 5 \\ &= 125 \end{aligned}$$

Example 2

$$c^4 = c \cdot c \cdot c \cdot c$$

Example 3

$$\begin{aligned} (-7)^2 &= (-7) \cdot (-7) \\ &= 49 \end{aligned}$$

The language box

Common ways of saying 2^3 are[1]

- ”2 to the power of 3”
- ”2 to the third power”

In computer programming, the symbol `^` or the symbols `**` is usually written between the base and the exponent.

[1] Attention! The examples illustrate a paradox in the English language; *power* is also a synonym for *exponent*.

Notice

The next pages declares rules concerning powers with corresponding explanations. Even though one wish to have these explanations as general as possible, we choose to use, mostly, specific numbers as exponents . Using variables as exponents would lead to less reader-friendly expressions, and it is our claim that the general cases are well illustrated by the specific cases.

7.10 Multiplication by powers

$$a^m \cdot a^n = a^{m+n}$$

Example 1

$$\begin{aligned} 3^5 \cdot 3^2 &= 3^{5+2} \\ &= 3^7 \end{aligned}$$

Example 2

$$\begin{aligned} b^4 \cdot b^{11} &= b^{3+11} \\ &= b^{14} \end{aligned}$$

Example 3

$$\begin{aligned} a^5 \cdot a^{-7} &= a^{5-7} \\ &= a^{-2} \end{aligned}$$

(See *Rule 7.13* regarding how powers with negative exponents can be interpreted.)

7.10 Multiplication by powers (explanation)

Let's study the case

$$a^2 \cdot a^3$$

We have

$$a^2 = 2 \cdot 2$$

$$a^3 = 2 \cdot 2 \cdot 2$$

Hence we can write

$$\begin{aligned} a^2 \cdot a^3 &= \overbrace{a \cdot a}^{a^2} \cdot \overbrace{a \cdot a \cdot a}^{a^3} \\ &= a^5 \end{aligned}$$

7.11 Division by powers

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 1

$$\frac{3^5}{3^2} = 3^{5-2} = 3^3$$

Example 2

$$\begin{aligned} \frac{2^4 \cdot a^7}{a^6 \cdot 2^2} &= 2^{4-2} \cdot a^{7-6} \\ &= 2^2 a \\ &= 4a \end{aligned}$$

7.11 Division by powers (explanation)

Let's examine the fraction $\frac{a^5}{a^2}$. Expanding the powers, we get

$$\begin{aligned}\frac{a^5}{a^2} &= \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} \\ &= \frac{\cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a}{\cancel{a} \cdot \cancel{a}} \\ &= a \cdot a \cdot a \\ &= a^3\end{aligned}$$

The above calculations are equivalent to writing

$$\begin{aligned}\frac{a^5}{a^2} &= a^{5-2} \\ &= a^3\end{aligned}$$

7.12 The special case of a^0

$$a^0 = 1$$

Example 1

$$1000^0 = 1$$

Example 2

$$(-b)^0 = 1$$

7.12 The special case of a^0 (explanation)

A number divided by itself always equals 1, therefore

$$\frac{a^n}{a^n} = 1$$

From this, and *Rule 7.11*, it follows that

$$\begin{aligned}1 &= \frac{a^n}{a^n} \\ &= a^{n-n} \\ &= a^0\end{aligned}$$

7.13 Powers with negative exponents

$$a^{-n} = \frac{1}{a^n}$$

Example 1

$$a^{-8} = \frac{1}{a^8}$$

Example 2

$$(-4)^{-3} = \frac{1}{(-4)^3} = -\frac{1}{64}$$

7.13 Powers with negative exponents (explanation)

By *Rule 7.12*, we have $a^0 = 1$. Thus

$$\frac{1}{a^n} = \frac{a^0}{a^n}$$

By *Rule 7.11*, we obtain

$$\begin{aligned}\frac{a^0}{a^n} &= a^{0-n}\\ &= a^{-n}\end{aligned}$$

7.14 Fractions as base

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

Example 1

$$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$$

Example 2

$$\left(\frac{a}{7}\right)^3 = \frac{a^3}{7^3} = \frac{a^3}{343}$$

7.14 Fractions as base (explanation)

Let's study

$$\left(\frac{a}{b}\right)^3$$

We have

$$\begin{aligned}\left(\frac{a}{b}\right)^3 &= \frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\\ &= \frac{a\cdot a\cdot a}{b\cdot b\cdot b}\\ &= \frac{a^3}{b^3}\end{aligned}$$

7.15 Factors as base

$$(ab)^m = a^m b^m$$

Example 1

$$\begin{aligned}(3a)^5 &= 3^5 a^5\\ &= 243a^5\end{aligned}$$

Example 2

$$(ab)^4 = a^4 b^4$$

7.15 Factors as base (explanation)

Let's use $(a\cdot b)^3$ as an example. We have

$$\begin{aligned}(a\cdot b)^3 &= (a\cdot b)\cdot(a\cdot b)\cdot(a\cdot b)\\ &= a\cdot a\cdot a\cdot b\cdot b\cdot b\\ &= a^3 b^3\end{aligned}$$

7.16 Powers as base

$$(a^m)^n = a^{m \cdot n}$$

Example 1

$$\begin{aligned}\left(c^4\right)^5 &= c^{4 \cdot 5}\\ &= c^{20}\end{aligned}$$

Example 2

$$\begin{aligned}\left(3^{\frac{5}{4}}\right)^8 &= 3^{\frac{5}{4} \cdot 8}\\ &= 3^{10}\end{aligned}$$

7.16 Powers as base (explanation)

Let's use $(a^3)^4$ as an example. We have

$$\left(a^3\right)^4 = a^3 \cdot a^3 \cdot a^3 \cdot a^3$$

By *Rule 7.10*, we get

$$\begin{aligned}a^3 \cdot a^3 \cdot a^3 \cdot a^3 &= a^{3+3+3+3}\\ &= a^{3 \cdot 4}\\ &= a^{12}\end{aligned}$$

7.17 n-rot

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The symbol $\sqrt{\ }$ is called the *radical sign*. In the case of an exponent equal to $\frac{1}{2}$, it is common to omit 2 from the radical:

$$a^{\frac{1}{2}} = \sqrt{a}$$

Example

By *Rule 7.16*, we have

$$\begin{aligned} \left(a^b\right)^{\frac{1}{b}} &= a^{b \cdot \frac{1}{b}} \\ &= a \end{aligned}$$

For example is

$$9^{\frac{1}{2}} = \sqrt{9} = 3, \text{ since } 3^2 = 9$$

$$125^{\frac{1}{3}} = \sqrt[3]{125} = 5, \text{ since } 5^3 = 125$$

$$16^{\frac{1}{4}} = \sqrt[4]{16} = 2, \text{ since } 2^4 = 16$$

The language box

$\sqrt{9}$ is called " the square (the 2nd) root of 9"

$\sqrt[3]{8}$ is called "the cube (the 3th) root of 8"

7.3 Irrational numbers

7.18 Irrational numbers

A number which is *not* a rational number, is an irrational number[1].

The value of an irrational number are decimal numbers with infinite digits in a non-repeating manner.

Example 1

$\sqrt{2}$ is an irrational number.

$$\sqrt{2} = 1.414213562373...$$

[1] Strictly speaking, irrational numbers are all *real* numbers which are not rational numbers. But to explain what *real* numbers are, we have to mention *imaginary* numbers, and this we choose not to do in this book.

Comment (for the particularly interested)

Mathematics is *axiomatically* founded. This means we declare[1] some propositions to be true, and these are called *axioms* or *postulates*. For the subject of calculations we have 12 axioms[2], but in this book we have confined ourselves to explicitly mention the following 6:

Axioms

For the numbers a, b and c we have

$$a + (b + c) = (a + b) + c \tag{A1}$$

$$a + b = b + a \tag{A2}$$

$$a(bc) = (ab)c \tag{A3}$$

$$ab = ba \tag{A4}$$

$$a(b + c) = ab + ac \tag{A5}$$

$$a \cdot \frac{1}{a} = 1 \qquad (a \neq 0) \tag{A6}$$

(A1)	Associative law for addition
(A2)	Commutative law for addition
(A3)	Associative law for multiplication
(A4)	Commutative law for multiplication
(A5)	Distributive law
(A6)	Existence of a multiplicative identitity

By applying axioms, we can derive more complex contexts which we call *theorems*. In this book we chose to let *rules* be the collective name for definitions, theorems and axioms. This is because alle three, in practice, draws up guidelines (rules) inside the mathematical system in which we wish to operate.

---

[1] Preferably, as few as possible.

[2] The number can slightly vary, depending on how the axioms are expressed.

In *Part I* we have tried to present the *motivation* behind the axioms, because (obviously) they are not randomly selected. The train of thoughts that leads us to them is the following:

1. Vi define positive numbers as representations of either an amount or a placement on a number line.

2. We define what addition, subtraction, multiplication and division entail for positive integers (and 0).

3. From the marks above, it's as good as self-evident that (A1) - (A6) is valid for all positive integers.

4. We define also fractions as representations of either an amount or a placement on a number line. What the elementary operations entail for fractions rests upon what is valid for the positive integers.

5. From the remarks above, we conclude that (A1) - (A6) is valid for all rational numbers.

6. We introduce negative numbers and an extended interpretation of addition and subtraction. This in turn leads to the interpretations of multiplication and division involving negative numbers.

7. (A1) - (A6) is still valid after the introduction of negative integers. Deriving that they are also valid for negative rational numbers is a formality (omitted in the book).

8. We can never write the value of an irrational number exact, but it can be approximated by a rational number[1]. Therefore, all calculations involving irrational umbers is, in practice, calculations involving rational numbers, and in this way we can conclude that[2] (A1) - (A6) is also valid for irrational numbers.

A similar train of thoughts can be applied concerning the power-rules found in *Section 7.2*.

[1]For example, we can write $\sqrt{2} = 1.414213562373... \approx \frac{1414213562373}{1000000000000}$

[2]*Attention!* This explanation is good enough for the aim of this book but is a rather extreme simplification. Irrational numbers are a very complex subject, in fact, many books presenting advanced mathematics utilize several chapters to cover the subject in full depth .

Chapter 8

Equations

8.1 Introduction

Even though every mathematical expression involving $=$ is an *equation*, the word is, traditionally, closely linked to the presence of an *unknown* number.

Say we want to find the number which when added by 4 results in 7. The name of this unknown number is free to choose but commonly it's called x. Our equation can now be written as

$$x + 4 = 7$$

The x-value[1] which results in the same values on each side of the equal sign is the *solution* of the equation. It is nothing wrong done by simply observing what the value of x must be. Probably you have already realized that $x = 3$ is the solution of the equation, since

$$3 + 4 = 7$$

However, most equations are difficult to solve simply by observing, and it is therefore vice to take the advantage of more general methods. In reality, there is only one principle to follow:

> We can always carry out a mathematical operation on one of the sides of the equal sign, as long as we carry out the operation on the other side too.

The mathematical operations presented in this book is the four elementary operations. Concerning these the principal sounds:

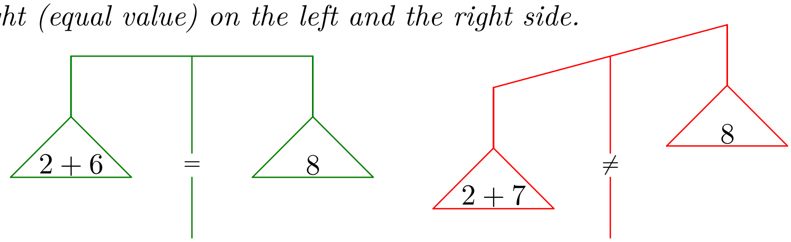
> We can always add, subtract, multiply or divide by a number on one side of the equal sign, as long as we also do it on the other side.

The principle follows from the meaning of $=$. When two expressions are of equal value, their values are necessarily still equal as long as we carry out identical mathematical operations on them. Anyways, in the coming section we'll specify this principle for every single elementary operation. If you already feel things make sense, you can, without no great loss of insight, skip to *Section 8.3*.

[1] In other cases it can be several values.

8.2 Solving with the elementary operations

In the figures of this section we'll understand equations from what we call the weight principle. In that case, $=$ *indicates*[1] *there is equal weight (equal value) on the left and the right side.*

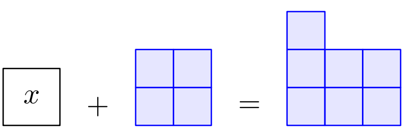


Addition and subtraction; moving terms

First example

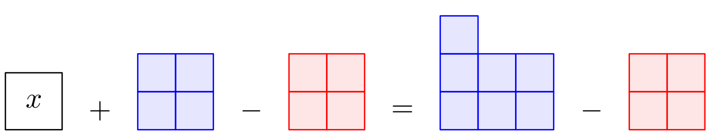
We have already found the solution of this equation, but let's now solve it in a different way[2]:

$$x + 4 = 7$$



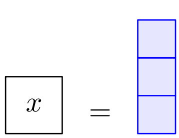
The value of x becomes clear if x is alone on one of the sides, and we can isolate x on the left side by removing 4. But if we are to remove 4 from the left side, we must also remove 4 from the right side, in order to preserve equal values on both sides.

$$x + 4 - {\color{red}4} = 7 - {\color{red}4}$$



Since $4 - {\color{red}4} = 0$ and $7 - {\color{red}4} = 3$, we get

$$x = 3$$



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[1] $\neq$ symbols "not equal".

[2] *Notice*: In earlier figures, there have been a correspondence between the size of the boxes and the (absolute) value of the number they represent. This does not apply to the boxes representing x.

In a more abbreviated way this can be written as

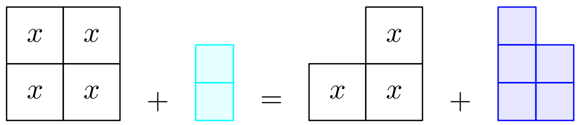
$$
\begin{aligned}
x + 4 &= 7 \\
x &= 7 - 4 \\
x &= 3
\end{aligned}
$$

Between the first and second line it is common to say that *4 has shifted side and therefore also sign (from $+$ to $-$).*

Second example

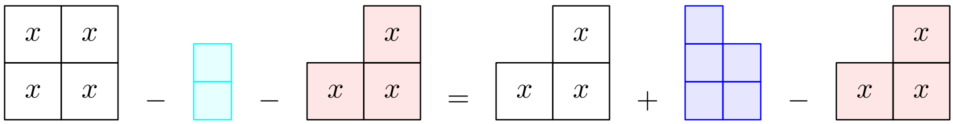
Let's move on to a somehow more complex equation[1]:

$$4x - 2 = 3x + 5$$



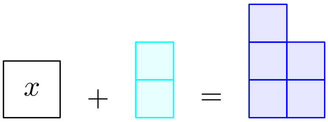
To get an expression with x exclusively on one side, we remove $3x$ on both sides:

$$4x - 2 - {\color{red}3x} = 3x + 5 - {\color{red}3x}$$



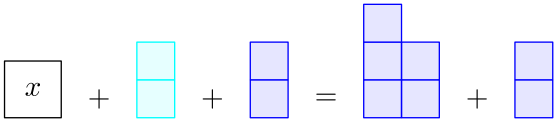
Now,

$$x - 2 = 5$$



To isolate x we add 2 on the left side. Then we must also add 2 on the right side:

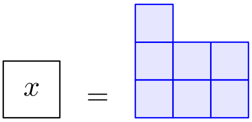
$$x - 2 + {\color{blue}2} = 5 + {\color{blue}2}$$



[1]Notice that the figure illustrates $4x + (-2)$ (see *Section 5.3*) on the left side. However, $4x + (-2)$ equals $4x - 2$ (see *Section 5.2*).

Hence

$$x = 7$$



The steps we have made can be summarized in this way:

$$\begin{aligned}
4x - 2 &= 3x + 5 && \text{1. figure}\\
4x - {\color{red}3x} - 2 &= 3x - {\color{red}3x} + 5 && \text{2. figure}\\
x - 2 &= 5 && \text{3. figure}\\
x - 2 + {\color{blue}2} &= 5 + {\color{blue}2} && \text{4. figure}\\
x &= 7 && \text{5. figure}
\end{aligned}$$

In a more abbreviated way we can write

$$\begin{aligned}
4x - 2 &= 3x + 5\\
4x - {\color{red}3x} &= 5 + {\color{blue}2}\\
x &= 7
\end{aligned}$$

8.1 Moving numbers across the equal sign

To solve an equation, we gather all x-terms and all known terms on respective sides of the equal sign. A term which shifts side, also shifts sign.

Example 1

Solve the equation

$$3x + 3 = 2x + 5$$

Answer

$$\begin{aligned}
3x - 2x &= 5 - 3\\
x &= 2
\end{aligned}$$

Example 2

Solve the equation

$$-4x - 3 = -5x + 12$$

Answer

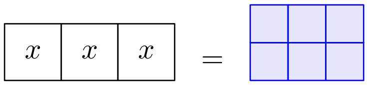
$$\begin{aligned}
-4x + 5x &= 12 + 3\\
x &= 15
\end{aligned}$$

Multiplication and division

Division

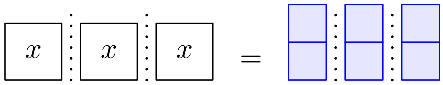
So far we have studied equations resulting in a single instance of x on one side of the equal sign. Often there are several instances of x, as, for example, in the equation

$$3x = 6$$



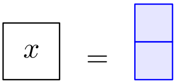
If we separate the left side into three equal groups, we get a single x in each group. And by separating the right side into three equal groups, all groups present are of equal value

$$\frac{3x}{3} = \frac{6}{3}$$



Therefore

$$x = 2$$



Let's summarize our calculations:

$$3x = 6 \qquad \text{1. figure}$$

$$\frac{\cancel{3}x}{\cancel{3}} = \frac{6}{3} \qquad \text{2. figure}$$

$$x = 2 \qquad \text{3. figure}$$

8.2 Division on both sides of an equation

We can divide both sides of an equation by the same number.

Example 1

Solve the equation

$$4x = 20$$

Answer

$$\frac{\cancel{4}x}{\cancel{4}} = \frac{20}{4}$$
$$x = 5$$

Example 2

Solve the equation

$$2x + 6 = 3x - 2$$

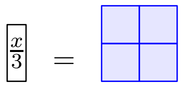
Answer

$$2x - 3x = -2 - 6$$
$$-x = -8$$
$$\frac{\cancel{-1}x}{\cancel{-1}} = \frac{-8}{-1} \qquad (-x = -1x)$$
$$x = 8$$

Multiplication

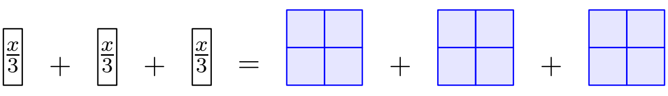
Let's solve the equation

$$\frac{x}{3} = 4$$



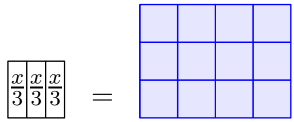
We can get a unit x on the left side if we add two more instances of $\frac{x}{3}$. The equation informs that $\frac{x}{3}$ equals 4, this implies that for every instance of $\frac{x}{3}$ we add to the left side, we must add 4 to the right side, in order to keep the balance.

$$\frac{x}{3} + \frac{x}{3} + \frac{x}{3} = 4 + 4 + 4$$



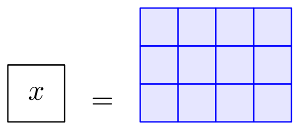
Now we notice that $\frac{x}{3} + \frac{x}{3} + \frac{x}{3} = \frac{x}{3} \cdot 3$ and that $4 + 4 + 4 = 4 \cdot 3$:

$$\frac{x}{3} \cdot 3 = 4 \cdot 3$$



Since $\frac{x}{3} \cdot 3 = x$ and $4 \cdot 3 = 12$, we have

$$x = 12$$



Our steps can be summarized in the following way:

$$\frac{x}{3} = 4 \qquad \text{1. figure}$$

$$\frac{x}{3} + \frac{x}{3} + \frac{x}{3} = 4 + 4 + 4 \qquad \text{2. figure}$$

$$\frac{x}{3} \cdot 3 = 4 \cdot 3 \qquad \text{3. figure}$$

$$x = 12 \qquad \text{4. figure}$$

In a more abbreviated form this can be written as

$$\frac{x}{3} = 4$$

$$\frac{x}{\cancel{3}} \cdot \cancel{3} = 4 \cdot 3$$

$$x = 12$$

8.3 Multiplication on both sides of an equation

We can multiply both sides of an equation by the same number.

Example 1

Solve the equation

$$\frac{x}{5} = 2$$

Answer

$$\frac{x}{\cancel{5}} \cdot \cancel{5} = 2 \cdot 5$$
$$x = 10$$

Example 2

Solve the equation

$$\frac{7x}{10} - 5 = 13 + \frac{x}{10}$$

Answer

$$\frac{7x}{10} - \frac{x}{10} = 13 + 5$$
$$\frac{6x}{10} = 18$$
$$\frac{6x}{\cancel{10}} \cdot \cancel{10} = 18 \cdot 10$$
$$6x = 180$$
$$\frac{\cancel{6}x}{\cancel{6}} = \frac{180}{6}$$
$$x = 30$$

8.3 Solving with elementary operations summarized

8.4 Solving methods with elementary operations

We can always

- add or subtract both sides of an equation by the same number. This is equivalent to shifting a term from one side to the other, also shifting the terms sign.
- multiply or divide both sides of an equation by the same number.

Example 1

Solve the equation

$$3x - 4 = 6 + 2x$$

Answer

$$\begin{aligned} 3x - 2x &= 6 + 4 \\ x &= 10 \end{aligned}$$

Example 2

Solve the equation

$$9 - 7x = 8x + 3$$

Answer

$$\begin{aligned} 9 - 7x &= -8x + 3 \\ 8x - 7x &= 3 - 9 \\ x &= -6 \end{aligned}$$

Example 3

Solve the equation

$$10x - 20 = 7x - 5$$

Answer

$$10x - 20 = 7x - 5$$
$$10x - 7x = 20 - 5$$
$$3x = 15$$
$$\frac{\cancel{3}x}{\cancel{3}} = \frac{15}{3}$$
$$x = 5$$

Example 4

Solve the equation

$$15 - 4x = x + 5$$

Answer

$$15 - 5 = x + 4x$$
$$10 = 5x$$
$$\frac{10}{5} = \frac{\cancel{5}x}{\cancel{5}}$$
$$2 = x$$

Example 5

Solve the equation

$$\frac{4x}{9} - 20 = 8 - \frac{3x}{9}$$

Answer

$$\frac{4x}{9} + \frac{3x}{9} = 20 + 8$$
$$\frac{\cancel{7}x}{9 \cdot \cancel{7}} = \frac{28}{7}$$
$$\frac{x}{\cancel{9}} \cdot \cancel{9} = 4 \cdot 9$$
$$x = 36$$

Example 6

Solve the equation

$$\frac{1}{3}x + \frac{1}{6} = \frac{5}{12}x + 2$$

Answer

To avoid fractions, we multiply both sides by the common denominator 12:

$$\left(\frac{1}{3}x + \frac{1}{6}\right)12 = \left(\frac{5}{12}x + 2\right)12$$

$$\frac{1}{3}x \cdot 12 + \frac{1}{6} \cdot 12 = \frac{5}{12}x \cdot 12 + 2 \cdot 12$$

$$4x + 2 = 5x + 24$$

$$4x - 5x = 24 - 2$$

$$-x = 22$$

$$\frac{\cancel{-1}x}{\cancel{-1}} = \frac{22}{-1}$$

$$x = -22$$

Tip

Some like to make the rule that "we can multiply or divide all terms by the same number". In that case, we could have jumped to the second line in the calculations of the example above.

Example 7

Solve the equation

$$3 - \frac{6}{x} = 2 + \frac{5}{2x}$$

Answer

We multiply both sides by the common denominator $2x$:

$$2x\left(3 - \frac{6}{x}\right) = 2x\left(2 + \frac{5}{2x}\right)$$

$$\begin{aligned} 6x - 12 &= 4x + 5 \\ 6x - 4x &= 5 + 12 \\ 2x &= 17 \\ x &= \frac{17}{2} \end{aligned}$$

8.4 Power equations

Let's solve the equation

$$x^2 = 9$$

This is called a *power equation*. In general, power equations are difficult to solve applying the four elementary operations only. Applying power-rules, we raise both sides to the power of the inverse of the exponent associated with x:

$$\left(x^2\right)^{\frac{1}{2}} = 9^{\frac{1}{2}}$$

By *Rule 7.16*, we have

$$\begin{aligned} x^{2\cdot\frac{1}{2}} &= 9^{\frac{1}{2}} \\ x &= 9^{\frac{1}{2}} \end{aligned}$$

Since $3^2 = 9$, we have $9^{\frac{1}{2}} = 3$. Now observe this:

The principle stated on page 109 says we can, like we just did, carry out a mathematical operation on both sides of an equation. However, sticking to this principle does not guarantee that all solutions are found.

Concerning our equation, we know that $x = 3$ is a solution. For the sake of it, we can confirm this by the calculation

$$3^2 = 3 \cdot 3 = 9$$

But we also have

$$(-3)^2 = (-3)(-3) = 9$$

Hence, -3 is also a solution of our original equation!

8.5 Power equations

An equation which can be written as

$$x^a = b$$

where a and b are constants, is a *power* equation.

The equation has a distinct solutions.

Example 1

Solve the equation

$$x^2 + 5 = 21$$

Answer

$$\begin{aligned} x^2 + 5 &= 21 \\ x^2 &= 21 - 5 \\ x^2 &= 16 \end{aligned}$$

Since $4 \cdot 4 = 16$ and $(-4) \cdot (-4) = 16$, we have

$$x = 4 \qquad \vee \qquad x = -4$$

Example 2

Solve the equation

$$3x^2 + 1 = 7$$

Answer

$$\begin{aligned} 3x^2 &= 7 - 1 \\ 3x^2 &= 6 \\ \frac{\cancel{3}x^2}{\cancel{3}} &= \frac{6}{3} \\ x^2 &= 2 \end{aligned}$$

Hence,

$$x = \sqrt{2} \qquad \vee \qquad x = -\sqrt{2}$$

Notice

Although the equation

$$x^a = b$$

has a solutions, they are not necessarily all *real*[1]. Concerning this book, it means we settle with finding all rational or irrational numbers which solves the equation. For example,

$$x^3 = 8$$

has 3 solutions, but we settle with the solution $x = 2$.

[1]As earlier mentioned, *real* and *imaginary* numbers lie outside the scope of this book.

Chapter 9

Functions

9.1 Introduction

Variables are values that change. A value which changes in compliance with a variable is called a *function*.

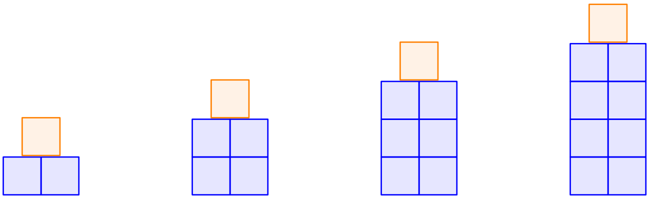


Figure 1 *Figure 2* *Figure 3* *Figure 4*

In the above figure, the amount of boxes follows a specific pattern. Mathematically we can describe the pattern like this:

$$\text{Number of boxes in } \textit{Figure 1} = 2 \cdot 1 + 1 = 3$$
$$\text{Number of boxes in } \textit{Figure 2} = 2 \cdot 2 + 1 = 5$$
$$\text{Number of boxes in } \textit{Figure 3} = 2 \cdot 3 + 1 = 7$$
$$\text{Number of boxes in } \textit{Figure 4} = 2 \cdot 4 + 1 = 9$$

Hence, for a figure of a random number x, we have

$$\text{Number of boxes in } \textit{Figure } x = 2x + 1$$

The amount of boxes changes in compliance with the change of x, in this case we say that

> ”Number of boxes in *Figure* x” is a function of x.

> $2x + 1$ is the expression of the function ”Number of boxes in *Figure* x”.

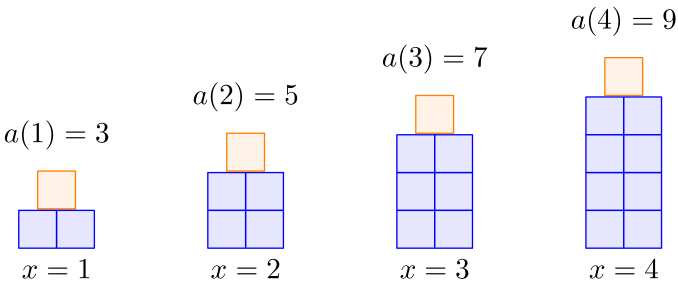
General expressions

If we were to continue working with the function just studied, writing "Number of boxes in *Figure* x" all the time would be very cumbersome. It is common to let letters indicate functions and to write the associated variable inside parentheses. Let's rename "Number of boxes in *Figure* x" to $a(x)$. Then

$$\text{Number of boxes in } \textit{Figure } x = a(x) = 2x + 1$$

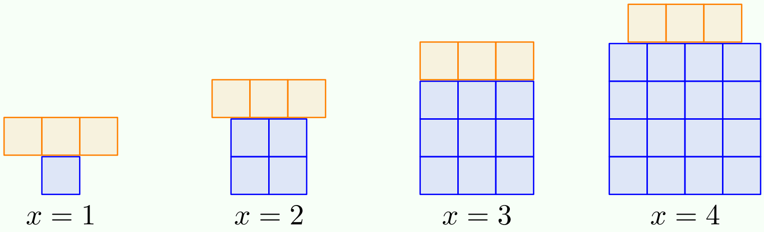
If we write $a(x)$, but substitute x by a specific number, we substitute x by this number in the expression of our function:

$$\begin{aligned} a(1) &= 2 \cdot 1 + 1 = 3 \\ a(2) &= 2 \cdot 2 + 1 = 5 \\ a(3) &= 2 \cdot 3 + 1 = 7 \\ a(4) &= 2 \cdot 4 + 1 = 9 \end{aligned}$$



Example

Let the number of boxes in the below pattern be given by $a(x)$.



a) Find the expression of $a(x)$.

b) How many boxes are there when $x = 10$?

c) What is the value of x when $a(x) = 628$?

Answer

a) We notice that

- When $x = 1$, there are $1 \cdot 1 + 3 = 4$ boxes.
- When $x = 2$, there are $2 \cdot 2 + 3 = 7$ boxes.
- When $x = 3$, there are $3 \cdot 3 + 3 = 12$ boxes.
- When $x = 4$, there are $4 \cdot 4 + 3 = 17$ boxes.

Therefore

$$a(x) = x \cdot x + 3 = x^2 + 3$$

b)

$$a(10) = 10^2 + 3 = 100 + 3 = 103$$

When $x = 10$, there are 103 boxes.

c) We have the equation

$$\begin{aligned} x^2 + 3 &= 628 \\ x^2 &= 625 \end{aligned}$$

Hence

$$x = 15 \qquad \vee \qquad x = -15$$

Since we seek a positive value of x, we have $x = 15$.

9.2 Linear functions and graphs

When a variable x and a function $f(x)$ are present, we have two values; the value of x and the associated value of $f(x)$. These pairs of values can be put into a coordinate system (see *Section 1.3*) to form the *graph* of $f(x)$.

Let's use the function

$$f(x) = 2x - 1$$

as an example. We have

$$f(0) = 2 \cdot 0 - 1 = -1$$

$$f(1) = 2 \cdot 1 - 1 = 1$$

$$f(2) = 2 \cdot 2 - 1 = 3$$

$$f(3) = 2 \cdot 3 - 1 = 5$$

These pairs of values can be put into a table:

x	0	1	2	3
$f(x)$	-1	1	3	5

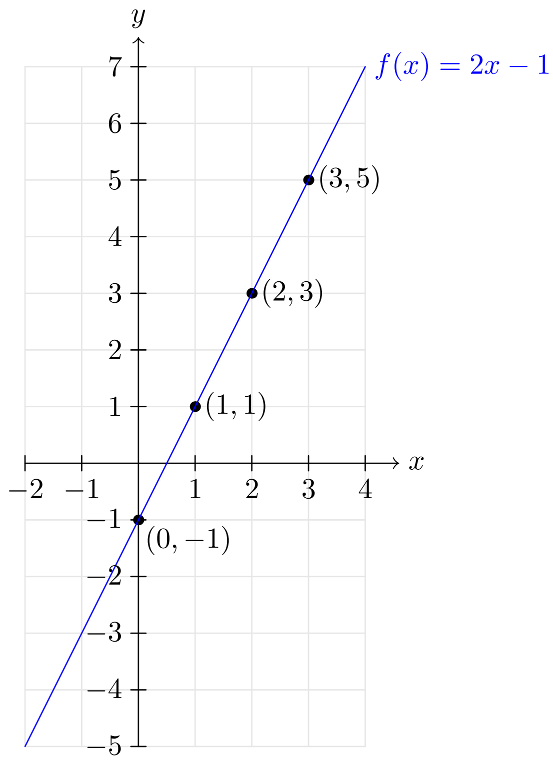
The above table yields the points

$$(0, -1) \qquad (1, 1) \qquad (2, 3) \qquad (3, 5)$$

Now we place these points into a coordinate system (see the figure on page 128). Concerning functions, it is common to name the horizontal and the vertical axis the x-axis and the y-axis, respectively. Now the graph of $f(x)$ is the curve passing through all the infinite many points we can create by the x-values and their associated $f(x)$-values. Our function is a *linear* function, which means its graph is a straight line. Hence, the graph is created by drawing the line going through the points we found.

As earlier mentioned, we can never draw a line, only a part of it. This also applies to graphs. In the figure on page 128 we have drawn the graph of $f(x)$ for x-values in the range -2 to 4. That x is included in this *interval* we write as[1] $-2 \leq x \leq 4$ or $x \in [-2, 4]$.

[1]Consult the list of symbols on page 4.


y
x
$f(x) = 2x - 1$
$(3, 5)$
$(2, 3)$
$(1, 1)$
$(0, -1)$
-2 -1 1 2 3 4
7 6 5 4 3 2 1 -1 -2 -3 -4 -5

9.1 Linear functions

A function with the expression

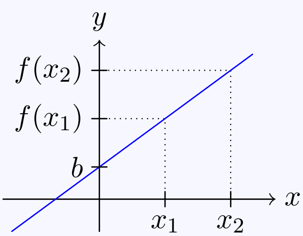
$$f(x) = ax + b$$

where a and b are constants, is a *linear* function with *slope* a and *intercept* b.

The graph of a linear function is a straight line passing through the point $(0, b)$.

For two distinct x-values, x_1 and x_2, we have

$$a = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Example 1

Find the slope and the intercept of the functions.

$$f(x) = 2x + 1$$

$$g(x) = -3 + \frac{7}{2}$$

$$h(x) = \frac{1}{4}x - \frac{5}{6}$$

$$j(x) = 4 - \frac{1}{2}x$$

Answer

- $f(x)$ have slope 2 and intercept 1.
- $g(x)$ have slope -3 and intercept $\frac{7}{2}$.
- $h(x)$ have slope $\frac{1}{4}$ and intercept $-\frac{5}{6}$.
- $j(x)$ have slope $-\frac{1}{2}$ and intercept 4.

Example 2

Draw the graph of

$$f(x) = \frac{3}{4}x - 2$$

for $x \in [-5, 6]$.

Answer

To draw the graph of a linear function, we only need to know two points lying on it. The points are free to choose, therefore, in order to make calculations as simple as possible, we start off by finding the point where $x = 0$:

$$f(0) = \frac{3}{4} \cdot 0 - 2 = -2$$

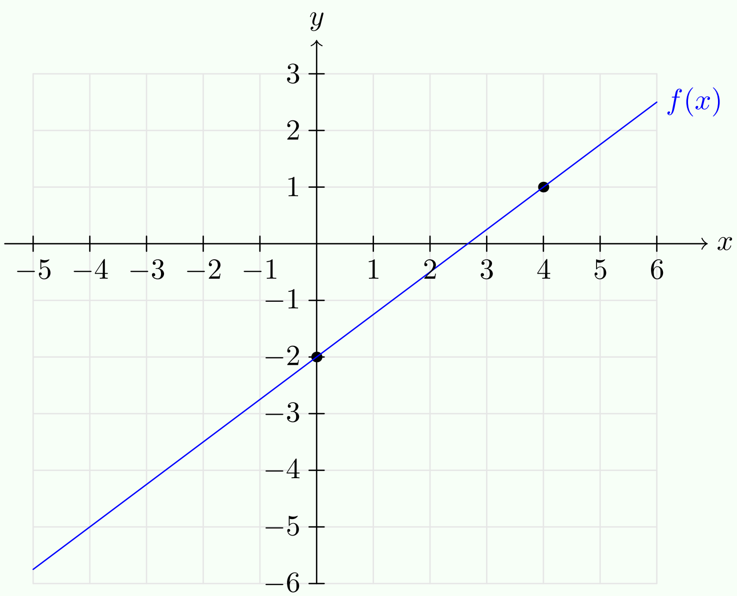
Further on, we choose $x = 4$, since this also results in easy calculations:

$$f(4) = \frac{3}{4} \cdot 4 - 2 = 1$$

Now we have all the information we need and for the record we put it into a table:

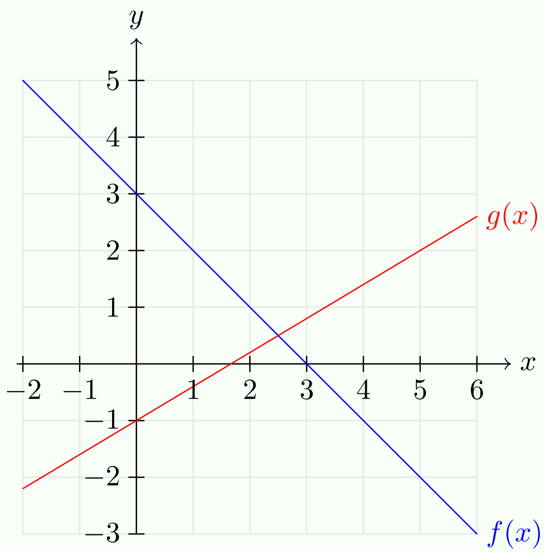
x	0	4
$f(x)$	-2	1

Now we place the points in a coordinate system and draw the line passing through them:



Example 3

Find the respective expressions of $f(x)$ and $g(x)$.



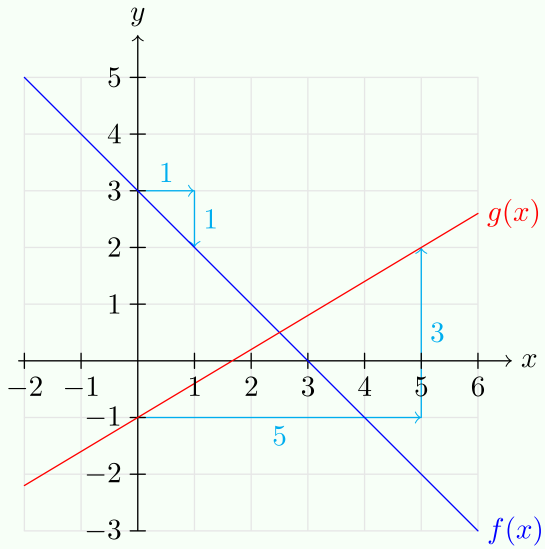
Answer

Firstly, we find the expression of $f(x)$. The point $(0, 3)$ lies on the graph of $f(x)$ (also see the figure on the next page). It then follows that $f(0) = 3$, and hence 3 is the intercept of $f(x)$. Moreover, we observe that $(1, 2)$ also lies on the graph of $f(x)$. The slope of $f(x)$ is then expressed by the fraction

$$\frac{2-3}{1-0} = -1$$

Therefore

$$f(x) = -x + 3$$



We now move our attention to finding the expression of $g(x)$. The point $(0, -1)$ lies on the graph of $g(x)$. It then follows that $f(0) = -1$, and hence -1 is the intercept of $g(x)$. Moreover, we observe that $(5, 2)$ also lies on the graph of $g(x)$. The slope $g(x)$ is then expressed by the fraction

$$\frac{2-(-1)}{5-0} = \frac{3}{5}$$

Therefore

$$g(x) = \frac{3}{5}x + 1$$

9.1 Linear functions (explanation)

The expression of *a*

Given a linear function

$$f(x) = ax + b$$

For two distinct x-values, x_1 and x_2, we have

$$f(x_1) = ax_1 + b \tag{9.18}$$

$$f(x_2) = ax_2 + b \tag{9.19}$$

Subtracting (9.2) from (9.19), we get

$$f(x_2) - f(x_1) = ax_2 + b - (ax_1 + b)$$

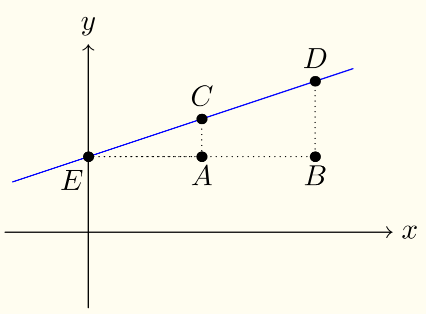
$$f(x_2) - f(x_1) = ax_2 - ax_1$$

$$f(x_2) - f(x_1) = a(x_2 - x_1)$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = a \tag{9.20}$$

The graph of a linear function is a straight line

Given a linear function $f(x) = ax + b$ and two distinct x-values x_1 and x_2. Let $A = (x_1, b)$, $B = (x_2, b)$, $C = (b, f(x_1))$, $D = (0, f(x_2))$ and $E = (0, b)$.



By (9.20), we obtain

$$\frac{f(x_1) - f(0)}{x_1 - 0} = a$$

$$\frac{ax_1 + b - b}{x_1} = a$$

$$\frac{ax_1}{x_1} = a \tag{9.21}$$

Similarly,

$$\frac{ax_2}{x_2} = a \tag{9.22}$$

Moreover,

$$AC = f(x_1) - b = ax_1$$

$$BD = f(x_2) - b = ax_2$$

$$EA = x_1$$

$$EB = x_2$$

From (9.21) and (9.22) it follows that

$$\frac{ax_1}{x_1} = \frac{ax_2}{x_2}$$

Hence

$$\frac{AC}{BD} = \frac{EA}{EB}$$

In addition, $\angle A = \angle B$, so $\triangle EAC$ and $\triangle EBD$ satisfy term iii from *Rule 10.12*, and hence the triangles are similar. Consequently, C and D lies on the same line, which must be the graph of $f(x)$.

Chapter 10

Geometry

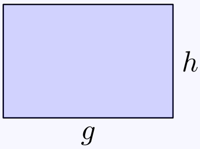
10.1 Formulas of area and perimeter

A *formula* is an equation where (usually) one variable is isolated on one side of the equal sign. In *Section 6.4* we have already looked at the formulas for the area of rectangles and triangles, but there using words instead of symbols. Here we shall reproduce these two formulas, followed by other classical formulas for area and perimeter.

10.1 The area of a rectangle (6.4)

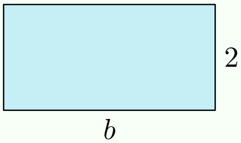
The area A of a rectangle with base g and height h is

$$A = gh$$



Example 1

Find the area of the rectangle.



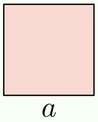
Answer

The area A of the rectangle is

$$A = b \cdot 2 = 2b$$

Example 2

Find the area of the square.



Answer

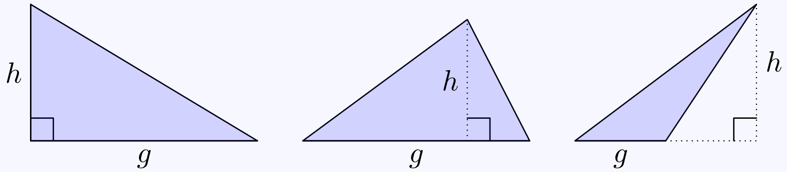
The area A of the square is

$$A = a \cdot a = a^2$$

10.2 The area of a triangle (6.4)

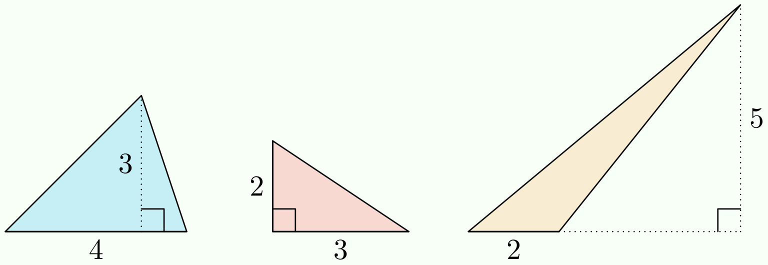
The area A of a triangle with base g and height h is

$$A = \frac{gh}{2}$$



Example

Which one of the triangles have the largest area?



Answer

Let A_1, A_2 and A_3 donate the areas of, respectively, the triangle to the left, in the middle and to the right. Then

$$A_1 = \frac{4 \cdot 3}{2} = 6$$

$$A_2 = \frac{2 \cdot 3}{2} = 3$$

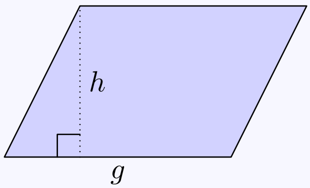
$$A_3 = \frac{2 \cdot 5}{2} = 5$$

Hence, it is the triangle to the left which has the largest area.

10.3 The area of a parallelogram

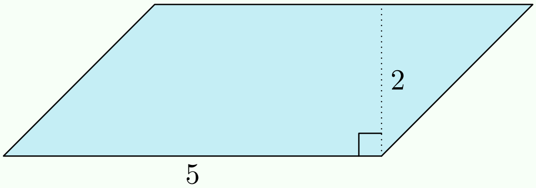
The area A of a parallelogram with base g and height h is

$$A = gh$$



Example

Find the area of the parallelogram



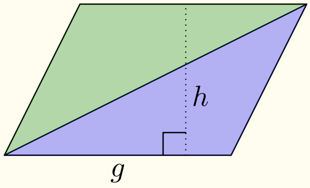
Answer

The area A of the parallelogram is

$$A = 5 \cdot 2 = 10$$

10.3 The area of a parallelogram (explanation)

From a parallelogram we can always, by drawing one of its diagonals, form two triangles which both have base g and height h.



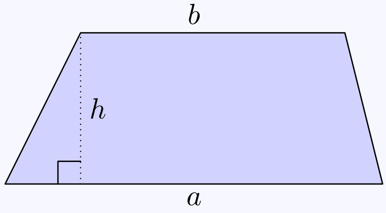
Hence, both triangles have an area equal to $\frac{gh}{2}$. Therefore, the area A of the parallelogram is

$$\begin{aligned} A &= \frac{gh}{2} + \frac{gh}{2} \\ &= g \cdot h \end{aligned}$$

10.4 The area of a trapezoid

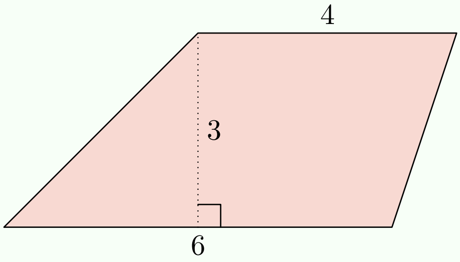
The area A of a trapezoid with parallel sides a and b and height h is

$$A = \frac{h(a+b)}{2}$$



Example

Find the area of the trapezoid.



Answer

The area A of the trapezoid is

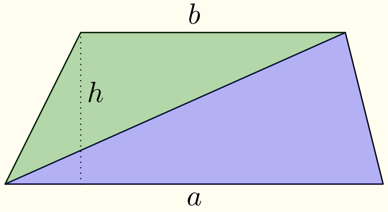
$$\begin{aligned} A &= \frac{3(6+4)}{2} \\ &= \frac{3 \cdot 10}{2} \\ &= 15 \end{aligned}$$

Notice

In respect of a base and a height, the area formulas for a parallelogram and a rectangle are identical. Applying *Rule 10.4* on a parallelogram also results in an expression equal to gh. This follows from the fact that a parallelogram is just a special case of a trapezoid (and a rectangle is just a special case of a parallelogram).

10.4 The area of a trapezoid (explanation)

In a trapezoid, we can, by drawing one of the diagonals, create two triangles:



In the above figure we have

$$\text{The area of the blue triangle} = \frac{ah}{2}$$

$$\text{The area of the green triangle} = \frac{bh}{2}$$

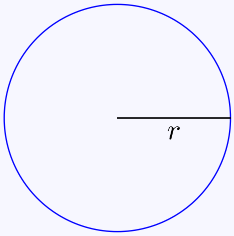
Therefore, the area A of the trapezoid is

$$\begin{aligned} A &= \frac{ah}{2} + \frac{bh}{2} \\ &= \frac{h(a+b)}{2} \end{aligned}$$

10.5 The perimeter of a circle (and the value of π)

The perimeter (the circumference) O of a circle with radius r is

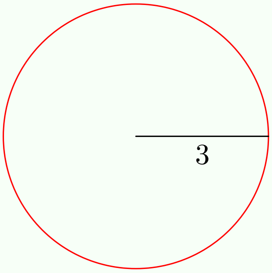
$$O = 2\pi r$$



$\pi = 3.141592653589793....$

Example 1

Find the circumference of the circle.



Answer

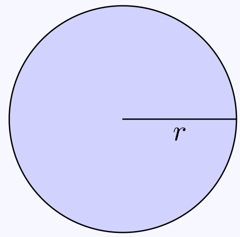
The circumference O is

$$\begin{aligned} O &= 2\pi \cdot 3 \\ &= 6\pi \end{aligned}$$

10.6 The area of a circle

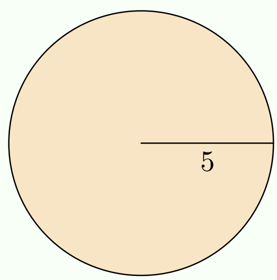
The area A of a circle with radius r is

$$A = \pi r^2$$



Example

Find the area of the circle.



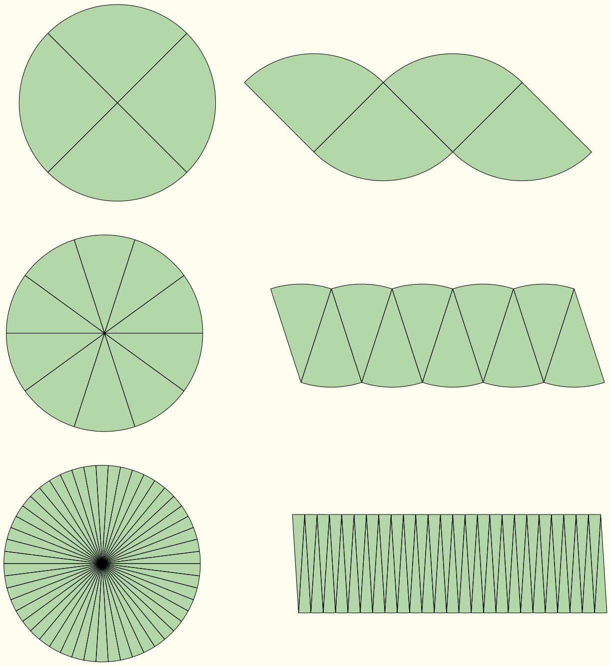
Answer

The area A of the circle is

$$A = \pi \cdot 5^2 = 25\pi$$

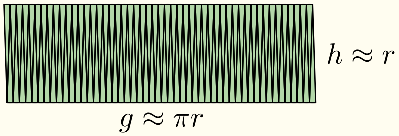
10.6 The area of a circle (explanation)

In the below figure, we have divided a circle into 4, 10 and 50 (equal-sized) sectors, and placed them consecutively.



In each case, the arcs make up the circumference of the circle. If the circle has radius r, the sum of the arcs equals $2\pi r$. And when there are equally many sectors turned upwards as downwards, the total length of the arcs equals πr on both the bottom and the top.

The more sectors the circle is divided into, the more the composition takes the form of a rectangle (in the figure below there are 100 sectors). The base g of this "rectangle" approximately equals πr, while the height h approximately equals r.



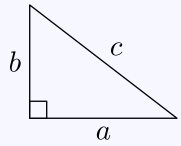
Hence, the area A of the "rectangle", that is, the circle, is

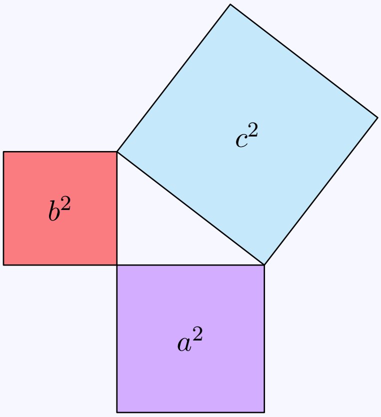
$$A \approx gh \approx \pi r \cdot r = \pi r^2$$

10.7 Pythagoras's theorem

In a right triangle, the area of the square formed by the hypotenuse equals the sum of the areas of the squares formed by the legs.

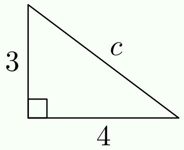
$$a^2 + b^2 = c^2$$





Example 1

Find the length of c.



Answer

We know that

$$c^2 = a^2 + b^2$$

where a and b are the legs of the right triangle. Therefore

$$\begin{aligned} c^2 &= 4^2 + 3^2 \\ &= 16 + 9 \\ &= 25 \end{aligned}$$

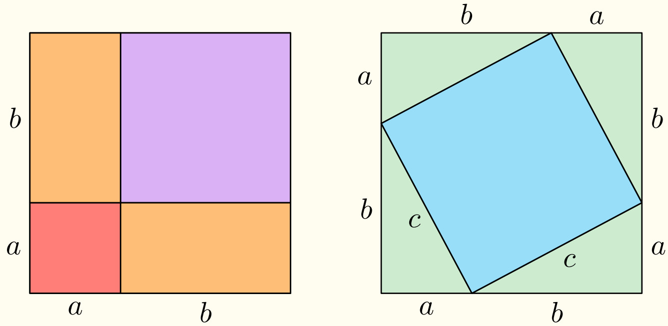
Hence,

$$c = 5 \qquad \vee \qquad c = -5$$

Since c is a length, $c = 5$.

10.7 Pythagoras's theorem (explanation)

The below figure shows equal-sized squares divided into different shapes.

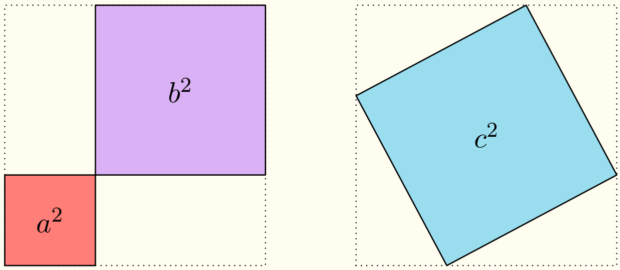


We observe the following:

1. The area of the red square is a^2, the area of the purple square is b^2 and the area of the blue square is c^2.

2. The area of an orange square is ab and the area of a green triangle is $\frac{ab}{2}$.

3. If we remove the two orange rectangles and the four green triangles, the remaining area to the left equals the remaining area to the right (by remark 2).



Hence

$$a^2 + b^2 = c^2 \tag{10.1}$$

Given a triangle with sides of length a, b and c, of which c is the longest. As long as the triangle is right, we can always form two squares with sides of length $a + b$, as in the initial figure. Therefore, (10.1) is valid for alle right triangles.

10.2 Congruent and similar triangles

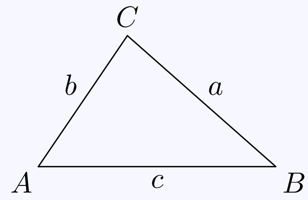
10.8 Unique construction of triangles

A triangle $\triangle ABC$, as shown in the below figure, can be uniquely constructed if one of the following terms are satisfied:

i) c, $\angle A$ and $\angle B$ are known.

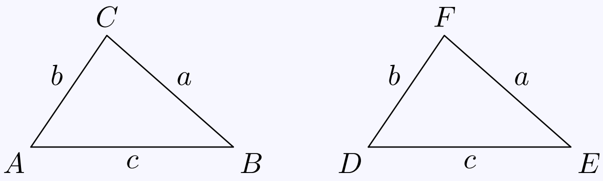
ii) a, b and c are known.

iii) b, c and $\angle A$ are known.



10.9 Congruent triangles

Two triangles of equal shape and size are congruent.

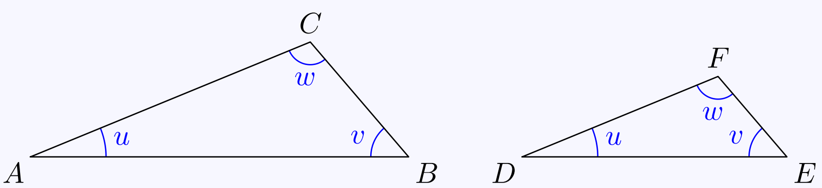


The congruence in the above figure is written

$$\triangle ABC \cong \triangle DEF$$

10.10 Similar triangles

Similar triangles constitute three pairs of equal angles.

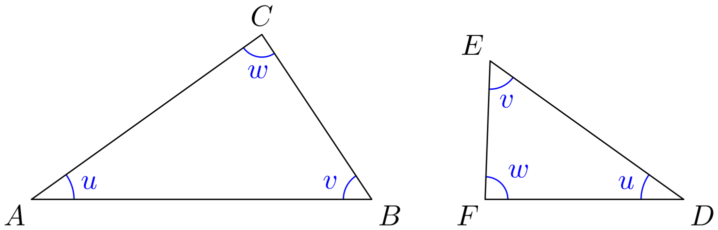


The similarity in the above figure is written

$$\triangle ABC \sim \triangle DEF$$

Corresponding sides

When studying similar triangles, *corresponding* sides plays an important role. Corresponding sides are sides in similar triangles adjacent to the same angle.



Regarding the similar triangles $\triangle ABC$ and $\triangle DEF$ we have

In $\triangle ABC$ is

- BC adjacent to u.
- AC adjacent to v
- AB adjacent to w.

I $\triangle DEF$ is

- FE adjacent to u.
- FD adjacent to v
- ED adjacent to w.

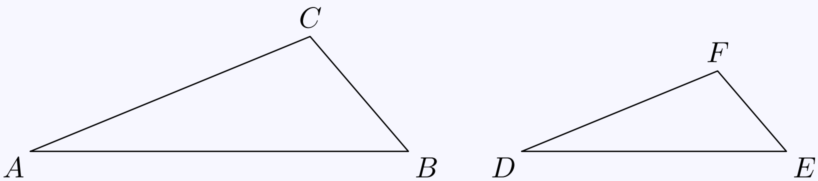
This means that these are corresponding sides:

- BC and FE
- AC and FD
- AB and ED

10.11 Ratios in similar triangles

If two triangles are similar, the ratios of corresponding sides are equal[1].

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$



[1] Here, we take it for granted that corresponding sides are apparent from the figure.

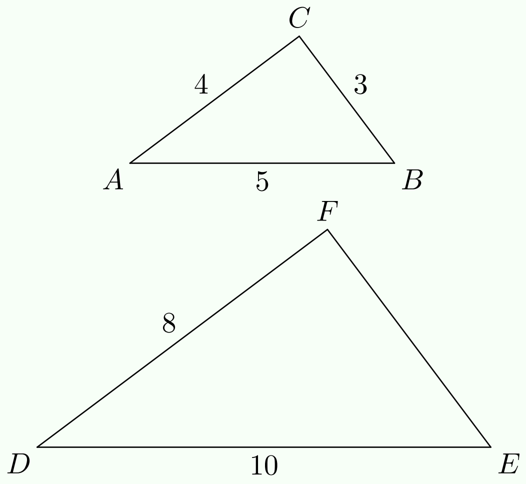
Notice

From *Rule 10.11* it follows that

$$\frac{AB}{BC} = \frac{DE}{EF} \quad , \quad \frac{AB}{AC} = \frac{DE}{DF} \quad , \quad \frac{BC}{AC} = \frac{EF}{DF}$$

Example

The triangles are similar. Find the length of EF.



Answer

We observe that AB corresponds to DE, BC to EF and AC to DF. Therefore

$$\frac{DE}{AB} = \frac{EF}{BC}$$
$$\frac{10}{5} = \frac{EF}{3}$$
$$2 \cdot 3 = \frac{EF}{\cancel{3}} \cdot \cancel{3}$$
$$6 = EF$$

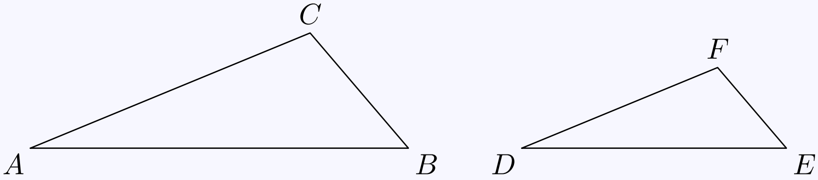
10.12 Terms of similar triangles

Two triangles $\triangle ABC$ and $\triangle DEF$ are similar if one of these terms are satisfied:

i) They constitute two pairs of equal angles.

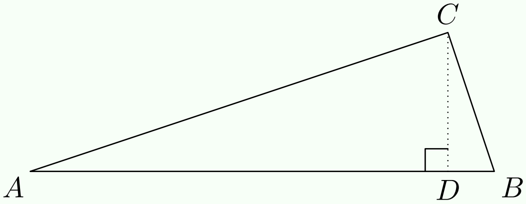
ii) $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$

iii) $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ and $\angle A = \angle D$.



Example 1

$\angle ACB = 90°$. Show that $\triangle ABC \sim ACD$.



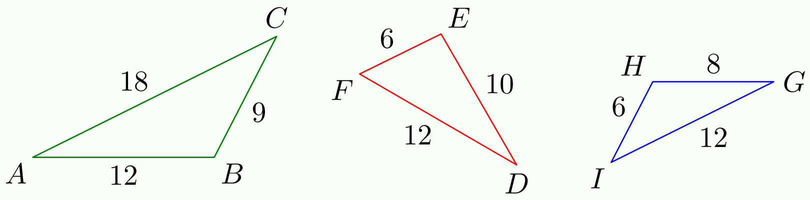
Answer

$\triangle ABC$ and $\triangle ACD$ are both right and they have $\angle DAC$ in common. Hence, the triangles satisfy term *i* from *Rule 10.12*, and therefore they are similar.

Notice: Similarly it can be shown that $\triangle ABC \sim CBD$.

Example 2

Examine whether the triangles are similar.



Answer

We have

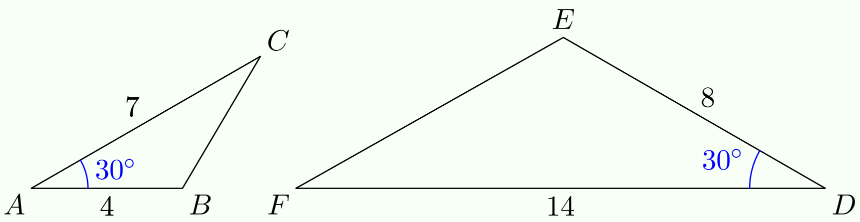
$$\frac{AC}{FD} = \frac{18}{12} = \frac{3}{2} \quad , \quad \frac{BC}{FE} = \frac{9}{6} = \frac{3}{2} \quad , \quad \frac{AB}{DE} = \frac{12}{10} = \frac{6}{5}$$

$$\frac{AC}{IG} = \frac{18}{12} = \frac{3}{2} \quad , \quad \frac{BC}{IH} = \frac{9}{6} = \frac{3}{2} \quad , \quad \frac{AC}{IG} = \frac{18}{12} = \frac{3}{2}$$

Hence, $\triangle ABC$ and $\triangle GHI$ satisfy term *ii* from *Rule 10.12*, and therefore they are similar. (Hence, $\triangle GHI$ and $\triangle FED$ are not similar.)

Example 3

Examine whether the triangles are similar.



Answer

We have $\angle BAC = \angle EDF$. Also,

$$\frac{ED}{AB} = \frac{8}{4} = 2 \quad , \quad \frac{FD}{AC} = \frac{14}{7} = 2$$

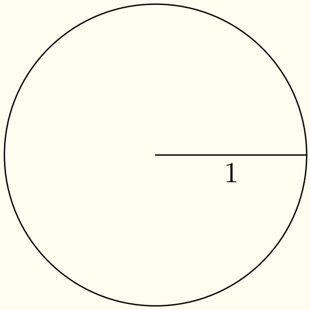
Hence, term *iii* from *Rule 10.12* is satisfied, and therefore the triangles are similar.

10.3 Explanations

10.5 The perimeter of a circle (and the value of π) (explanation)

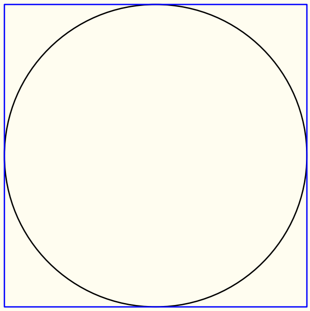
Here we shall use regular polygons along the path to our wanted result. In regular polygons, all sides are of equal length. Since all polygons here to be mentioned are regular, we'll mention them simply as polygons.

We'll start off by examining some approximations of the circumference O_1 of a circle with radius 1.



Upper and lower boundary

When seeking a value, it is a good habit to conclude how large or small you *expect* it to be. With this target, we enclose the circle by a square with sides of length 2:

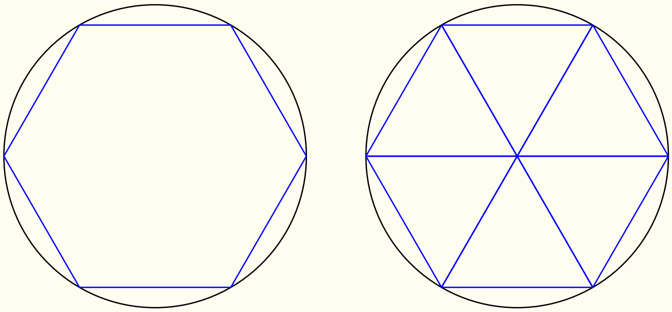


Clearly, the circumference of the circle is smaller than the perimeter of the square, therefore

$$\begin{aligned} O_1 &< 2 \cdot 4 \\ &< 8 \end{aligned}$$

Now we inscribe a 6-gon (hexagon). The hexagon can be divided into 6 equilateral triangles with, necessarily, sides of length 1. The circumference of the circle must be larger than the perimeter of the hexagon, so

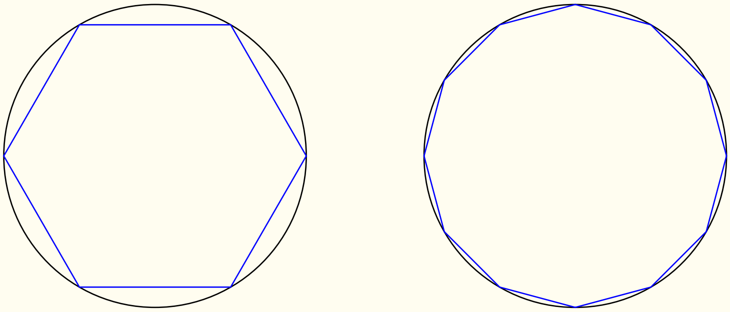
$$\begin{aligned} O_1 &> 6 \cdot 1 \\ &> 6 \end{aligned}$$



Now advancing to a more sophisticated hunt for the circumference, we know that we seek a value between 6 and 8.

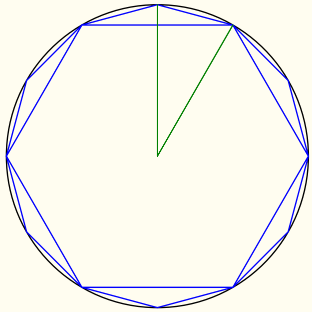
Increasingly better approximations

The idea of inscribing polygons carries on. We let the below figures work as a sufficient prove of the fact that the more sides of the polygon, the better estimate its perimeter makes of the circumference of the circle.

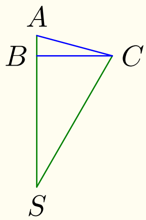


(a) 6-gon *(b)* 12-gon

Since a 6-gon has sides of length 1, it is tempting to examine if this can help us find the side lengths of other polygons. By inscribing both a 6-gon and a 12-gon (and drawing a triangle) we have a figure like this:



(a) A 6-gon and a 12-gon together with a triangle formed by the circle center and one side of the 12-gon.



(b) The triangle from figure *(a)*.

Let s_{12} and s_6 denote the side lengths of the 12-gon and the 6-gon, respectively. Moreover, we observe that both A and C lies on the circular arc and that both $\triangle ABC$ and $\triangle BSC$ are right-angled (explain to yourself why!). We have

$$\begin{aligned} SC &= 1 \\ BC &= \frac{s_6}{2} \\ SB &= \sqrt{SC^2 - BC^2} \\ BA &= 1 - SB \\ AC &= s_{12} \\ s_{12}^2 &= BA^2 + BC^2 \end{aligned}$$

To find s_{12}, we need to know BA, and to find BA we need to know SB. Hence, we start off finding SB. Since $SC = 1$ and $BC = \frac{s_6}{2}$,

$$\begin{aligned} SB &= \sqrt{1 - \left(\frac{s_6}{2}\right)^2} \\ &= \sqrt{1 - \frac{s_6^2}{4}} \end{aligned}$$

Now we focus on finding s_{12}:

$$\begin{aligned} s_{12}^2 &= (1 - SB)^2 + \left(\frac{s_6}{2}\right)^2 \\ &= 1^2 - 2SB + SB^2 + \frac{s_6^2}{4} \end{aligned}$$

At first, it looks like the expression to the right cannot be simplified, but a small operation can change this. If -1 was a term present, we could have combined -1 and $\frac{s_6^2}{4}$ to become $-SB^2$. We obtain -1 by both adding and subtracting it on the right side of the equation:

$$\begin{aligned} s_{12}^2 &= 1 - 2SB + SB^2 + \frac{s_6^2}{4} - 1 + 1 \\ &= 2 - 2SB + SB^2 - \left(1 - \frac{s_6^2}{4}\right) \\ &= 2 - 2SB + SB^2 - SB^2 \\ &= 2 - 2SB \\ &= 2 - 2\sqrt{1 - \frac{s_6^2}{4}} \\ &= 2 - \sqrt{4}\sqrt{1 - \frac{s_6^2}{4}} \\ &= 2 - \sqrt{4 - s_6^2} \end{aligned}$$

Hence

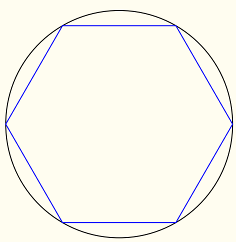
$$s_{12} = \sqrt{2 - \sqrt{4 - s_6^2}}$$

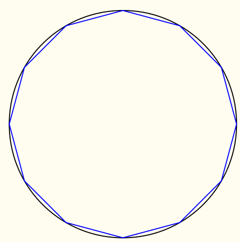
Even though we have derived a relation between the side lengths s_{12} and s_6, this relation is valid for all pairs of side lengths where one is the side length of a polygon with twice as many sides as the other. Now let s_n and s_{2n}, respectively, denote the side lengths of a polygon and a polygon with twice as many sides. Then

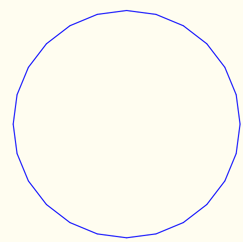
$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}} \tag{10.2}$$

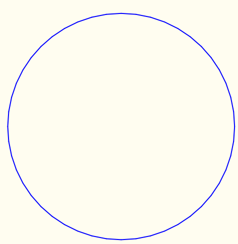
The perimeter of a polygon inscribed in the circle is an estimate of the circumference. Applying (10.2), we can successively find the side length of a polygon with twice as many sides as the previous. The below table shows the side length and the associated estimate of the circumference up to a 96-gon:

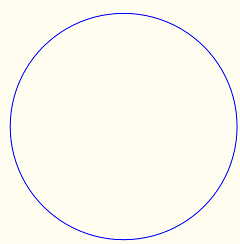
Side length formula	*Side length*	*Estimate, circumference*
	$s_6 = 1$	$6 \cdot s_6 \; = 6$
$s_{12} = \sqrt{2 - \sqrt{4 - s_6^2}}$	$s_{12} = 0.517...$	$12 \cdot s_{12} = 6.211...$
$s_{24} = \sqrt{2 - \sqrt{4 - s_{12}^2}}$	$s_{24} = 0.261...$	$24 \cdot s_{24} = 6.265...$
$s_{48} = \sqrt{2 - \sqrt{4 - s_{24}^2}}$	$s_{48} = 0.130...$	$48 \cdot s_{48} = 6.278...$
$s_{96} = \sqrt{2 - \sqrt{4 - s_{48}^2}}$	$s_{96} = 0.065...$	$96 \cdot s_{96} = 6.282...$


(a) 6-gon


(b) 12-gon


(c) 24-gon


(d) 48-gon


(e) 96-gon

In fact, the mathematician Archimedes reached as far as the above calculation approximately 250 b.c!

A computer has no problems performing calculations[1] on a polygon with extremely many sides. Calculating the perimeter of a 201 326 592-gong yields

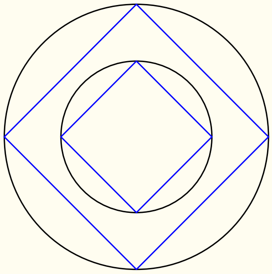
$$\text{Circumference of a circle with radius 1} = 6.283185307179586...$$

(With the aid of more advanced mathematics it can be proved that the circumference of a circle with radius 1 is an irrational number, but that the digits shown above are correct, thereby the equal sign.)

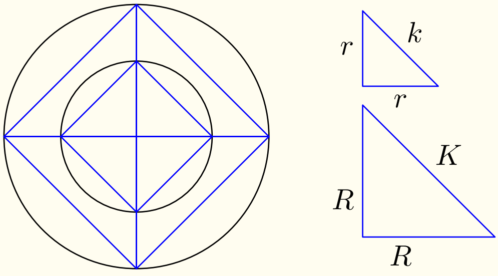
The formula and π

We shall now derive the famous formula for the circumference of any circle. Here as well, we take it for granted that the perimeter of an inscribed polygon yields an estimate of the circumference which gets more accurate the more sides the polygon has.

For the sake of simplicity, we shall use inscribed squares to illustrate the outline. We draw two circles of random size, but the one larger than the other, and inscribe a square in both. Let R and r denote the radius of the larger and the smaller circle, respectively. Also, let K and k denote the side length of the larger and the smaller square, respectively.



Both squares can be divided into four isosceles triangles:



Since these triangles are similar,

$$\frac{K}{R} = \frac{k}{r} \tag{10.3}$$

Let $\tilde{O} = 4K$ and $\tilde{o} = 4k$ denote the estimated circumferences of the larger and the smaller circle, respectively. Multiplying both sides of (10.3) by 4 yields

$$\frac{4A}{R} = \frac{4a}{r}$$
$$\frac{\tilde{O}}{R} = \frac{\tilde{o}}{r} \tag{10.4}$$

Now we observe this:

If we were to inscribe polygons with 4, 100 or any number of sides, the polygons could still be divided into triangles obeying (10.3). *And in the same way as we did in the above example, we could then rewrite* (10.3) *into* (10.4).

Let's therefore imagine polygons with such a large number of sides that we accept their respective perimeters as equal to the respective circumferences of the circles. Letting O and o denote the circumferences of the larger and smaller circle respectively, we have

$$\frac{O}{R} = \frac{o}{r}$$

Since the circles are randomly chosen, we conclude that *all circles have the same ratio of the circumference to the radius*. An equivalent statement is that *all circles have the same ratio of the circumference to the diameter.*

The ratio of the circumference O to the diameter d in a circle is named π (pronounced "pi"):

$$\frac{O}{d} = \pi$$

The above equation yields the formula for the circumference of a circle with diameter d and radius r:

$$\begin{aligned} O &= \pi d \\ &= 2\pi r \end{aligned}$$

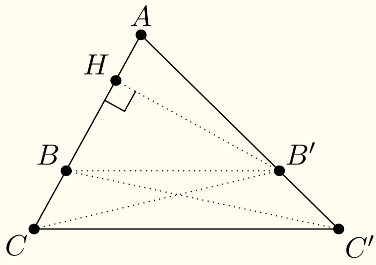
Earlier we found that the circumference of a circle with radius 1 (and diameter 2) equals 6.283185307179586... . Hence

$$\begin{aligned} \pi &= \frac{6.283185307179586...}{2} \\ &= 3.141592653589793... \end{aligned}$$

[1]For those interested in computer programming, the iteration algorithm must be alternated in order to avoid instabilities when the number of sides are large.

10.11 Ratios in similar triangles (explanation)

Here, we shall write the area of a triangle $\triangle ABC$ as ABC.



In the figure above, we have $BB'||CC'$. With BB' as base, HB' is the height of both $\triangle CBB'$ and $\triangle CBB'$. Therefore

$$CBB' = C'BB' \tag{10.5}$$

Moreover,

$$ABB' = AB \cdot HB'$$

$$CBB' = BC \cdot HB'$$

Hence

$$\frac{ABB'}{CBB'} = \frac{AB}{BC} \tag{10.6}$$

Similarly,

$$\frac{ABB'}{C'BB'} = \frac{AB'}{B'C'} \tag{10.7}$$

From (10.5), (10.6) and (10.7) it follows that

$$\frac{AB}{BC} = \frac{ABB'}{CBB'} = \frac{ABB'}{C'BB'} = \frac{AB'}{B'C'} \tag{10.8}$$

For the similar triangles $\triangle ACC'$ and $\triangle ABB'$,

$$\begin{aligned}\frac{AC}{AB} &= \frac{AB + BC}{AB}\\ &= 1 + \frac{BC}{AB}\end{aligned}$$

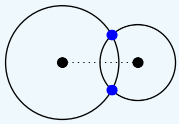
$$\begin{aligned}\frac{AC'}{AB'} &= \frac{AB' + B'C'}{AB'}\\ &= 1 + \frac{B'C'}{AB'}\end{aligned}$$

By (10.8), the ratio of corresponding sides in the two triangles are equal.

Notice

In the following explanations of term *ii* and *iii* from *Rule 10.8* we assume this:

- Two circles intersect in maximum two points.
- Given a coordinate system placed in the center of one of the circles, such that the horizontal axis passes through both circle centers. If (a, b) is one of the intersection points, $(a, -b)$ is the other.

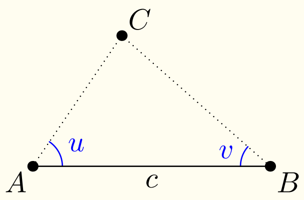


The remarks above are quite easy to prove, but since they are largely intuitively true, we hold them as granted. This implies that the triangle formed by the two centers and one of the intersection points is congruent to the triangle formed by the two centers and the other intersection point. By this, we can study attributes of triangles with the aid of semi-circles.

10.8 Unique construction of triangles (explanation)

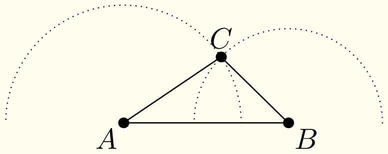
Term i

Given a length c and two angles u and v. We make a segment AB with length c. Then we dot two angle sides, such that $\angle A = u$ and $B = v$. As long as these angle sides are not parallel, they must intersect in one, and one only, point (C in the figure). Together with A and B, this point will form a triangle uniquely determined by c, u and v.



Term ii

Given three lengths a, b and c. We make a segment AB with length c. Then we make two semi-circles with respective radii a and b and centers B and A. If a triangle $\triangle ABC$ is to have sides of length a, b and c, C must lie on both of the semi-circles. Since the semi-circles intersect in one point only, $\triangle ABC$ is uniquely determined by a, b and c.



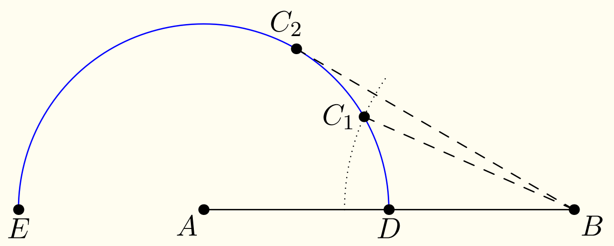
Term iii

Given two lengths b and c and an angel u. We begin as follows:

1. We make a segment AB with length c.

2. In A we draw a semi-circle with radius b.

By placing C randomly on the arc of the semi-circle, we get all instances of a triangle $\triangle ABC$ with sides of length $AB = c$ and $AC = b$. Specifically placing C on the arc of the semi-circle is equivalent to setting a specific value of $\angle A$. Now it remains to show that every placement of C implies a unique length of BC.



Let C_1 and C_2 denote two potential placements of C, where C_2, along the semicircle, lies closer to E than C_1. Now we dot a circular arc with radius BC_1 and center B. Since the dotted arc and the semi-circle only intersects in C_1, other points will either lie inside or outside the dotted arc. Necessarily, C_2 lies outside the dotted arc, and therefore BC_2 is longer than BC_1. From this we can conclude that the length of BC increases as C moves against E along the semi-circle. Therefore, specifying $\angle A = u$ yields a unique value of BC, and hence a unique triangle $\triangle ABC$ where $AC = b$, $c = AB$ and $\angle BAC = u$.

10.12 Terms of similar triangles (explanation)

Term i

Given two triangles $\triangle ABC$ and $\triangle DEF$. By *Rule 6.3*,

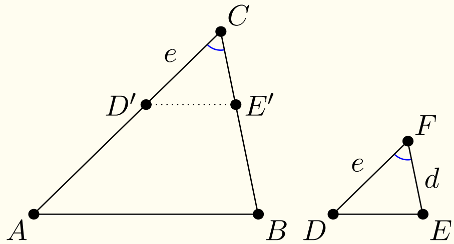
$$\angle A + \angle B + \angle C = \angle D + \angle E + \angle F$$

If $\angle A = \angle D$ and $\angle B = \angle E$, it follows that $\angle C = \angle E$.

Term ii

Given two triangles $\triangle ABC$ and $\triangle DEF$, where

$$\frac{AC}{DF} = \frac{BC}{EF} \quad , \quad \angle C = \angle F \tag{10.9}$$



Let $a = BC$, $b = AC$, $d = EF$ and $e = DF$. We place D' and E' on AC and BC, respectively, such that $D'C = e$ and $AB \parallel D'E'$. Then $\triangle ABC \sim \triangle D'E'C$, and hence

$$\frac{E'C}{BC} = \frac{D'C}{AC}$$

$$E'C = \frac{ae}{b}$$

By (10.9),

$$EF = \frac{ae}{b}$$

Hence $E'C = EF$. From term *ii* of *Rule 10.8* it now follows that $\triangle D'E'C \cong \triangle DEF$. This implies that $\triangle ABC \sim \triangle DEF$.

Term iii

Given two triangles $\triangle ABC$ and $\triangle DEF$, where

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} \tag{10.10}$$

We place D' and E' on AC and BC, respectively, such that $D'C = e$ and $E'C = d$. From term *i* of *Rule 10.12* we have $\triangle ABC \sim \triangle D'E'C$. Therefore

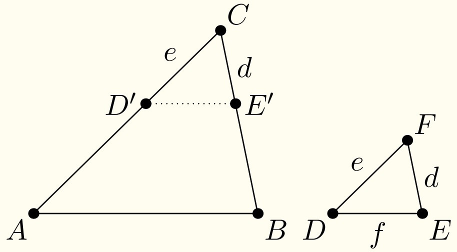
$$\frac{D'E'}{AB} = \frac{D'C}{AC}$$

$$D'E' = \frac{ae}{c}$$

By (10.10),

$$f = \frac{ae}{c}$$

Hence, the side lengths of $\triangle D'E'C$ and $\triangle DEF$ are pairwise equal, and then, from term *i* of *Rule 10.8*, they are congruent. This implies that $\triangle ABC \sim \triangle DEF$.



Comment (for the particularly interested)

Also in geometry, axioms (see comment on page 106) lays the ground of the mathematical system we create, but the axiomatic structure of geometry is quite extensive and intricate. In addition, some theorems are such intuitively true that it, at least in a book like this, would be more confusing than clarifying to explain them all in detail.

However, it is worth noticing that *Rule 10.8* states three terms regarding the unique construction in a triangle, and *Rule 10.9* states a term regarding congruence. In more advanced texts on geometry, chances are that you will recognize the content of these rules as axioms and theorems of congruence:

Congruence

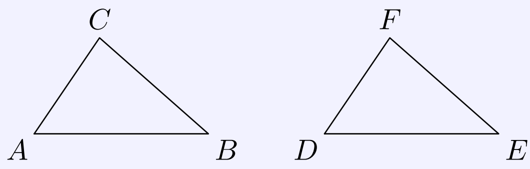
Two triangles $\triangle ABC$ and $\triangle DEF$ are congruent if one of the following terms are satisfied:

i) $AB = DE$, $BC = EF$ and $\angle A = \angle D$.

ii) $\angle A = \angle D$, $\angle B = \angle E$ and $AB = DE$.

iii) $AB = DE$, $BC = EF$ and $AC = FD$.

iv) $\angle A = \angle D$ and $\angle B = \angle E$ and, in addition, $AB = DE$ or $BC = EF$ or $AC = FD$.



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i) The Side-angle-side (SAS) axiom
ii) The Angle-side-angle (ASA) theorem
iii) The Side-side-side (SSS) theorem
iv) The Side-angle-angle (SAA) theorem

In the text-box on the previous side, term i) - iii) brings sufficient information regarding the unique construction of a triangle. However, in this book we have chosen to separate the concepts of congruence and unique construction. This is done under the presumption that most people will have a good intuition about congruent triangles, while having more difficulties stating terms of unique construction − and it is not necessarily easy to observe this directly from the terms of congruence.

Also, observe that iv) is just term ii) in a wider sense, but it cannot be used as a term of unique construction. Therefore, this term is not found in either *Rule 10.8* or *Rule 10.9*.

Litterature

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Lindstrøm, T. (2006). *Kalkulus* (2nd ed.). Oslo, Universitetsforlaget AS.

Moise, E. E. (1974). *Elementary geometry from an advanced standpoint.* Reading, Addison-Wesley Publishing Company.

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Note: The text, at least a very similar one, about Pythagoras's theorem on page 145 was first printed in Skage Hansen's book Tempelgeometri (2020).

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