0.1 Introduksjon

0.1 Brøk som omskriving av delestykke

A fraction is a different way of writing a divison. In a fraction the dividend is called the *numerator* and the divisor the denominator.

$$1 : 4 = \frac{1 \longleftarrow \text{Tellar}}{4 \longleftarrow \text{Nemnar}}$$

The language box

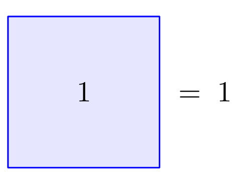
Common ways of saying $\frac{1}{4}$ are[1]

- "one fourth"
- "1 of 4"
- "1 over 4"

[1] We also have the expressions from the Language Box on page **??**.

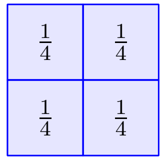
Brøk som mengde

Let us present $\frac{1}{4}$ as a figure. We then think of the number 1 as a box[1]:



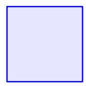
[1] For practical reasons we choose a one-box larger than the one used in *Kapittel* **??**.

Further on we divide this box into four smaller equal-sized boxes. The sum of these boxes equals 1.



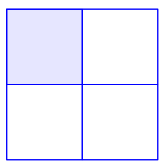
$= 1$

One such box equals $\frac{1}{4}$:



$$= \frac{1}{4}$$

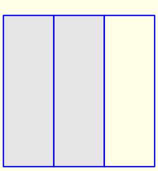
However, if one from a figure only shall see how large a fraction is, the size of 1 must be known, and to make this more apparent we'll also include the "empty" boxes:

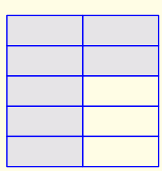


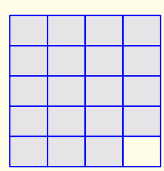
$$= \frac{1}{4}$$

In this way, the blue and the empty boxes tells us how many pieces 1 is divided into, while the blue boxes alone tells how many of these boxes wich are *actually* present. In other words,

$$\begin{aligned} \text{number of blue boxes} &= \text{numerator} \\ \text{number of blue boxes} + \text{number of empty boxes} &= \text{denomiator} \end{aligned}$$

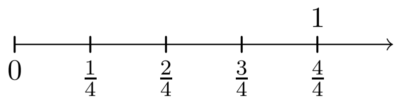
 $= \frac{2}{3}$

 $= \frac{7}{10}$

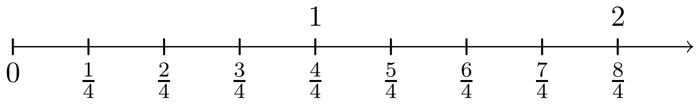
 $= \frac{19}{20}$

Fractions on the number line

On the number line we divide the length between 0 and 1 into as many pieces as the denominator indicates. In the case of a fraction with denominator 4, we seperate the length between 0 og 1 into 4 equal lengths:



Moreover, fractions larger than 1 are easily presented on the number line:



Numerator and denominator summarized

Although already mentioned, the interpretation of the numerator and denominator is of such an importance that we shortly summarize it:

- The denominator tells how many pieces 1 is divided into.
- The numerator tells how many of these pieces are present.

0.2 Verdi, utviding og forkorting av brøk

0.2 The value of a fraction

The value of a fraction is given by dividing the numerator by the denominator.

Example

Find the value of $\frac{1}{4}$.

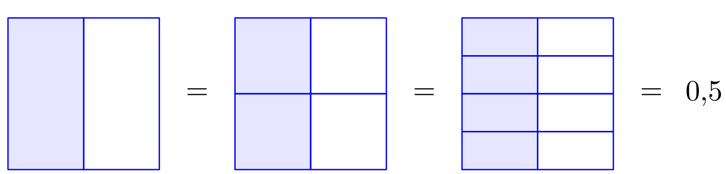
Answer:

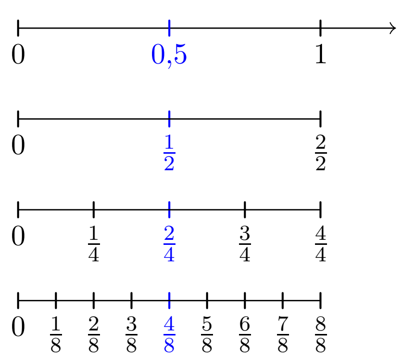
$$\frac{1}{4} = 0.25$$

Fractions with equal value

Fractions can have the same value even though they look different. If you calculate 1 : 2, 2 : 4 and 4 : 8, you will in all cases end up with 0.5 as the answer. This means that

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = 0{,}5$$



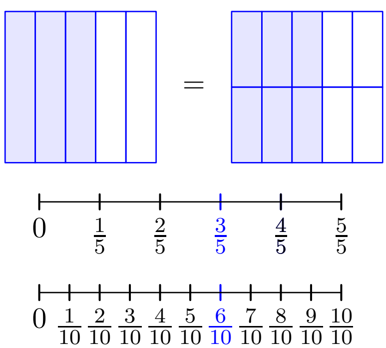


Expanding

The fact that fraction can look different but have the same vaue, implies that we can change a fraction's look without changing its value. Let's as an expample change $\frac{3}{5}$ into a fraction of equal value but with 10 as its denominator:

- We can make $\frac{3}{5}$ into a fraction with 10 as its denominator if we divide each fifth into 2 equal pieces. In that case, 1 is divided into $5 \cdot 2 = 10$ pieces in total.
- The numerator of $\frac{3}{5}$ indicates that there are 3 fifths. When these 3 pieces are divided by 2, they make up $3 \cdot 2 = 6$ tenths. Hence $\frac{3}{5}$ equals $\frac{6}{10}$.

$$\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10}$$



Simplifying

Note that we can also go "the opposite way". We can change $\frac{6}{10}$ into a fraction with 5 as denominator by dividing both the numerator and the denominator by 2:

$$\frac{6}{10} = \frac{6:2}{10:2} = \frac{3}{5}$$

> **0.3 Utviding av brøk**
>
> We can either multiply or divide both the numerator and the denominator by the same number without alternating the fractions value.
>
> Multiplying by a number larger than 1 is called *expanding* the fraction. Dividing by a number larger than 1 is called *simplifying* the fraction.

Example 1

Expand $\frac{3}{5}$ into a fraction with 20 as its denominator.

Answer:

Since $5 \cdot 4 = 20$, we multiply both the numerator and the denominator by 4:

$$\begin{aligned}\frac{3}{5} &= \frac{3 \cdot 4}{5 \cdot 4} \\ &= \frac{12}{20}\end{aligned}$$

Example 2

Expand $\frac{150}{50}$ into a fraction with 100 as its denominator.

Answer:

Since $50 \cdot 2 = 100$, we multiply both the numerator and the denominator by 2:

$$\begin{aligned}\frac{150}{50} &= \frac{150 \cdot 2}{50 \cdot 2} \\ &= \frac{300}{100}\end{aligned}$$

Example 3

Simplify $\frac{18}{30}$ into a fraction with 5 as its denominator.

Answer:

Since $30 : 6 = 5$, we divide both the numerator and the denominator by 5:

$$\begin{aligned}\frac{18}{30} &= \frac{18 : 6}{30 : 6} \\ &= \frac{3}{5}\end{aligned}$$

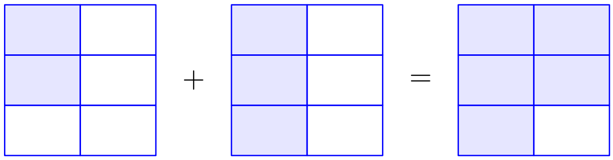
0.3 Addisjon og subtraksjon

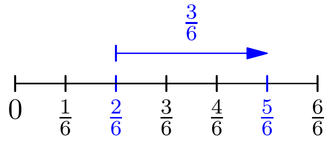
Addition and subtraction of fractions is in large parts focused around the denominators. Recall that the denominators indicates the partitioning of 1. If fractions have equal denominators, they represent amounts of equal-sized pieces. In this case it makes sense calculating addition or subtraction of the numerators. If fractions have unequal denominators, however, they represent amounts of different-sized pieces, and hence addition and subtraction of the numerators makes no sense directly.

Equal denominators

For example, if we have 2 sixths and add 3 sixths, the sum is 5 sixths:

$$\frac{2}{6} + \frac{3}{6} = \frac{5}{6}$$





0.4 Addistion/subtraction of fractions with equal denominators

When adding/subtracting fractions with equal denominators, we find the sum/difference of the numerators and keep the denominator.

Example 1

$$\begin{aligned} \frac{2}{7} + \frac{8}{7} &= \frac{2+8}{7} \\ &= \frac{10}{7} \end{aligned}$$

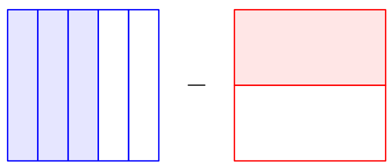
Example 2

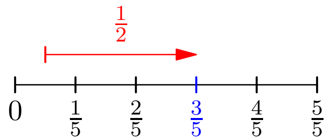
$$\frac{7}{9} - \frac{5}{9} = \frac{7-5}{9} = \frac{2}{9}$$

Ulike nemnarar

Let's examine the calculation[1]

$$\frac{3}{5} - \frac{1}{2}$$



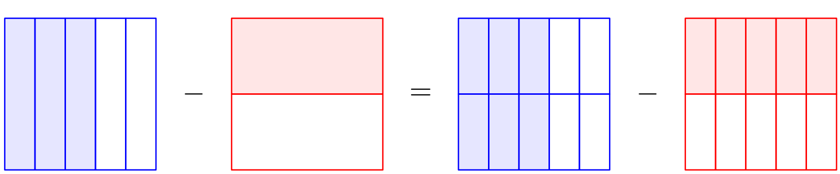


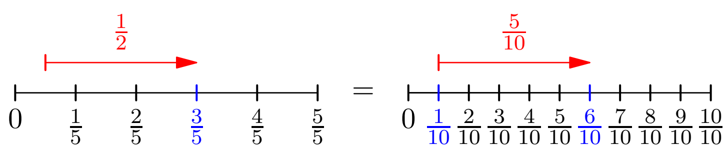
To write the difference as a fraction, the two terms needs a common denominator. Both of the fractions can have 10 as its denominator:

$$\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10} \qquad \frac{1}{2} = \frac{1 \cdot 5}{2 \cdot 5} = \frac{5}{10}$$

Hence

$$\frac{3}{5} - \frac{1}{2} = \frac{6}{10} - \frac{5}{10}$$





[1]Recall that the red-colored arrow indicates that one shall start at the arrow head and then move to the other end.

To summarize, we have expanded the fractions such that they have a common denominator, that is 10. When the denominators are equal, we can calculate the difference of the numerators:

$$\begin{aligned}\frac{3}{5} - \frac{1}{2} &= \frac{6}{10} - \frac{5}{10} \\ &= \frac{1}{10}\end{aligned}$$

0.5 Addisjon/subtraksjon av brøkar med ulik nemnar

When calculating addition/subtraction of fractions, we must expand the fractions such that they have a common denominator and then apply *Rule 0.4*.

Example 1

Calculate

$$\frac{2}{9} + \frac{6}{7}$$

Both denominators can be transformed into 63 if multiplied by a fitting integer. Therefore, we expand the fractions followingly:

$$\begin{aligned}\frac{2 \cdot 7}{9 \cdot 7} + \frac{6 \cdot 9}{7 \cdot 9} &= \frac{14}{63} + \frac{54}{63} \\ &= \frac{68}{63}\end{aligned}$$

Fellesnemnar

In *Example 1* above 63 is called a *common denominator* because there exists integers which, when multiplied by the denominators, results in 63:

$$9 \cdot 7 = 63$$

$$7 \cdot 9 = 63$$

Multiplying together the denominators always results in a common denominator but one can avoid large numbers by finding the *smallast* common denominator. Take, for example,

$$\frac{7}{6} + \frac{5}{3}$$

$6 \cdot 3 = 18$ is a common denominator, but it's worth noticing that $6 \cdot 1 = 3 \cdot 2 = 6$ is too.

Example 2

Calculate

$$\frac{3}{2} - \frac{5}{8} + \frac{10}{4}$$

Answer:

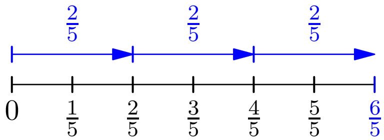
All denominator can be transformed into 8 if multiplied by a fitting integer. Therefore we expand the fractions followingly:

$$\begin{aligned}
\frac{3}{2} - \frac{5}{8} + \frac{10}{4} &= \frac{3 \cdot 4}{2 \cdot 4} - \frac{5}{8} + \frac{10 \cdot 2}{4 \cdot 2} \\
&= \frac{12}{8} - \frac{5}{8} + \frac{20}{8} \\
&= \frac{27}{8}
\end{aligned}$$

0.4 Brøk gonga med heiltal

In *Section ??* we observed that multiplying by an integer corresponds to repeated addition. Hence, if we are to calculate $\frac{2}{5} \cdot 3$, we can write

$$\begin{aligned}\frac{2}{5} \cdot 3 &= \frac{2}{5} + \frac{2}{5} + \frac{2}{5} \\ &= \frac{2+2+2}{5} \\ &= \frac{6}{5}\end{aligned}$$



Noticing that $2+2+2 = 2 \cdot 3$, we get

$$\begin{aligned}\frac{2}{5} \cdot 3 &= \frac{2 \cdot 3}{5} \\ &= \frac{6}{5}\end{aligned}$$

Multiplication of integers and fractions are also commutative[1]:

$$\begin{aligned}3 \cdot \frac{2}{5} &= 3 \cdot 2 : 5 \\ &= 6 : 5 \\ &= \frac{6}{5}\end{aligned}$$

0.6 Brøk gonga med heiltal

When multiplying a fraction by an integer, we multiply the integer by the fractions numerator.

[1]Recall that $\frac{2}{5}$ corresponds to $2 : 5$.

Example 1

$$\begin{aligned}\frac{1}{3} \cdot 4 &= \frac{1 \cdot 4}{3} \\ &= \frac{4}{3}\end{aligned}$$

Example 2

$$\begin{aligned}3 \cdot \frac{2}{5} &= \frac{3 \cdot 2}{5} \\ &= \frac{6}{5}\end{aligned}$$

An interpretation of multiplying by a fraction

By *Rule 0.6* we can make an interpretation of multiplying by a fraction. For example, multiplying 3 by $\frac{2}{5}$ can be interpreted in these two following ways:

- We multiply 3 by 2 and divide the product by 5:

$$3 \cdot 2 = 6 \quad , \quad 6 : 5 = \frac{6}{5}$$

- We divide 3 by 5 and multiply the quotient by 2:

$$3 : 5 = \frac{3}{5} \quad , \quad \frac{3}{5} \cdot 2 = \frac{3 \cdot 2}{5} = \frac{6}{5}$$

0.5 Brøk delt med heiltal

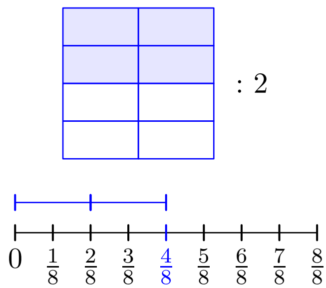
It is now important to recall two things:

- Division can be interpreted as an distribution of equal amounts
- In a fraction it is the numerator which indicates the amount (the denominator indicates the division of 1)

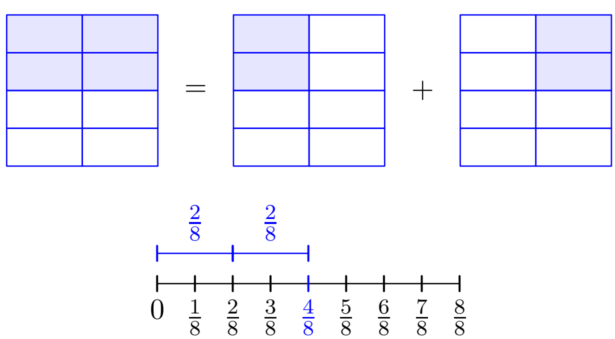
Tilfellet der tellaren er deleleg med divisoren

Let's calculate

$$\frac{4}{8} : 2$$



We have 4 eights which are to be equally distributed into 2 groups. This results in $4 : 2 = 2$ eights.



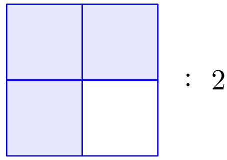
Thus

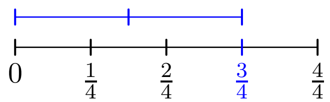
$$\frac{4}{8} : 2 = \frac{2}{8}$$

When the numerator is not divisible by the denominator

What if we are to divide $\frac{3}{4}$ by 2?

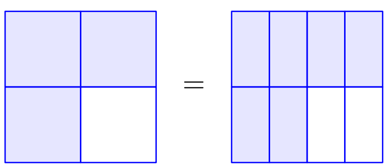
$$\frac{3}{4} : 2$$

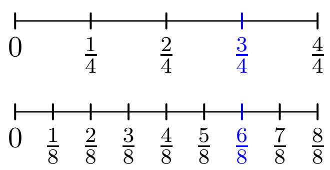




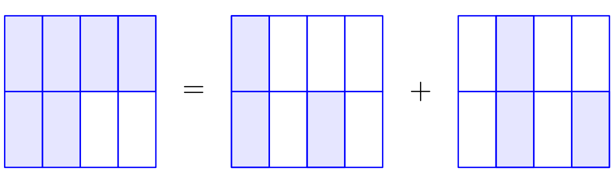
Thing is that we can always expand the fraction, such that the numerator becomes divisible by the divisor. Since 2 is the divisor, we expand the fraction by 2:

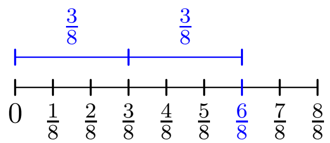
$$\frac{3}{4} = \frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8}$$





Now we have 6 eights. 6 eights divided by 2 equals 3 eights:





Hence
$$\frac{3}{4} : 2 = \frac{3}{8}$$
In effect, we have multiplied $\frac{3}{4}$ by 2:
$$\begin{aligned}\frac{3}{4} : 2 &= \frac{3}{4 \cdot 2} \\ &= \frac{3}{8}\end{aligned}$$

0.7 Fraction divided by integers

When dividing a fraction by an integer, we multiply the denominator by the integer.

Example 1
$$\begin{aligned}\frac{5}{3} : 6 &= \frac{5}{3 \cdot 6} \\ &= \frac{5}{18}\end{aligned}$$

Exception

At the start of this section we found that
$$\frac{4}{8} : 2 = \frac{2}{8}$$
In that case we did not multiply the denominator by 2, such as *Rule 0.7* implies. However, if we do that we have
$$\frac{4}{8} : 2 = \frac{4}{8 \cdot 2} = \frac{4}{16}$$
But
$$\frac{2}{8} = \frac{2 \cdot 2}{8 \cdot 2} = \frac{4}{16}$$
Hence, the two answers are of equal value. Thing is that when dividing a fraction by an integer, and the numerator is divisible by the integer, we can divide the numerator by the integer directly. Therefore, in such cases it is not wrong but neither necessary to apply *Rule 0.7*.

0.6 Brøk gonga med brøk

We have seen that[1] multiplying a fraction by number involves multiplying the number by the numerator and then dividing the product by the denominator. Let us apply this to calculate

$$\frac{5}{4} \cdot \frac{3}{2}$$

Firstly, we multiply $\frac{5}{4}$ by 3, then we divide the resulting product by 2. By *Rule 0.6*, we have

$$\frac{5}{4} \cdot 3 = \frac{5 \cdot 3}{4}$$

And by *Rule 0.7*, we get

$$\frac{5 \cdot 3}{4} : 2 = \frac{5 \cdot 3}{4 \cdot 2}$$

Hence

$$\frac{5}{4} \cdot \frac{3}{2} = \frac{5 \cdot 3}{4 \cdot 2}$$

0.8 Brøk gonga med brøk

When multiplying a fraction by a fraction, we multiply numerator by numerator and denominator by denominator.

Example 1

$$\begin{aligned} \frac{4}{7} \cdot \frac{6}{9} &= \frac{4 \cdot 6}{7 \cdot 9} \\ &= \frac{24}{63} \end{aligned}$$

Example 2

$$\begin{aligned} \frac{1}{2} \cdot \frac{9}{10} &= \frac{1 \cdot 9}{2 \cdot 10} \\ &= \frac{9}{20} \end{aligned}$$

[1] Look at the text box with the title *An interpretation of multiplying by a fraction* on page 12.

0.7 Kansellering av faktorar

When the numerator and the denomerator are of equal value, the fractions value always equals 1. For example, $\frac{3}{3} = 1$, $\frac{25}{25} = 1$ etc. We can exploit this fact to simplify expressions involving fractions.

Let us simplify the expression

$$\frac{8 \cdot 5}{9 \cdot 8}$$

Since $8 \cdot 5 = 5 \cdot 8$, we can write

$$\frac{8 \cdot 5}{9 \cdot 8} = \frac{5 \cdot 8}{9 \cdot 8}$$

And, as recently seen (*Rule 0.8*), we have

$$\frac{5 \cdot 8}{9 \cdot 8} = \frac{5}{9} \cdot \frac{8}{8}$$

Since $\frac{8}{8} = 1$,

$$\begin{aligned}\frac{5}{9} \cdot \frac{8}{8} &= \frac{5}{9} \cdot 1 \\ &= \frac{5}{9}\end{aligned}$$

When multiplication is exclusively, present in a fraction, we can always shuffle the way we did in the expressions above, however, when you have understood the outcome of the shuffling it is better to apply *cancellation*. You then draw a line across two and two equal factors, by this indicating that they constitute a fraction which equals 1. Thus, our most recent example can be simplified as

$$\frac{\cancel{8} \cdot 5}{9 \cdot \cancel{8}} = \frac{5}{9}$$

0.9 Cancellation of factors

When multiplication is exclusively present in a fraction, we can cancel pair of equal factors in numerator and denominator.

Example 1

Cancel as many factors as possible in the fraction

$$\frac{3 \cdot 12 \cdot 7}{7 \cdot 4 \cdot 12}$$

Answer:

$$\frac{3 \cdot \cancel{12} \cdot \cancel{7}}{\cancel{7} \cdot 4 \cdot \cancel{12}} = \frac{3}{4}$$

Example 1

Simplify the fraction $\frac{12}{42}$.

Answer:

$$\begin{aligned} \frac{12}{42} &= \frac{\cancel{6} \cdot 2}{\cancel{6} \cdot 7} \\ &= \frac{2}{7} \end{aligned}$$

Example 2

Simplify the fraction $\frac{48}{16}$.

Answer:

$$\begin{aligned} \frac{48}{16} &= \frac{3 \cdot \cancel{16}}{\cancel{16}} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

Notice: If all factors are canceled in the numerator or the denominator, 1 takes their place.

Fractions simplify calculations

The decimal number 0.125 can be written as the fraction $\frac{1}{8}$. The calculation

$$0.125 \cdot 16$$

is, for the most of us, rather strenuous to perform by hand. But, exploiting the nature of fractions, we have

$$\begin{aligned} 0.125 \cdot 16 &= \frac{1}{8} \cdot 16 \\ &= \frac{2 \cdot \cancel{8}}{\cancel{8}} \\ &= 2 \end{aligned}$$

"Cancelling zeros"

A number such as 3000 equals $3 \cdot 10 \cdot 10 \cdot 10$, while 700 equals $7 \cdot 10 \cdot 10$. Hence, we can simplify $\frac{3000}{700}$ like this:

$$\begin{aligned} \frac{3000}{700} &= \frac{3 \cdot \cancel{10} \cdot \cancel{10} \cdot 10}{7 \cdot \cancel{10} \cdot \cancel{10}} \\ &= \frac{3 \cdot 10}{7} \\ &= \frac{30}{7} \end{aligned}$$

In practice, this is the same as "cancelling zeros":

$$\frac{30\cancel{00}}{7\cancel{00}} = \frac{30}{7}$$

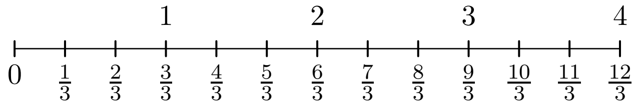
Aware! Zeros are the only digits which can be "canceled" this way, for example, $\frac{123}{13}$ cannot be simplified in any way. Also, we can only "cancel" zeros which are right-most situated, for example, we cannot "cancel" zeros in the fraction $\frac{101}{10}$.

0.8 Deling med brøk

Divison by studying the number line

Let's calculate $4 : \frac{2}{3}$. Since the fraction have 3 as its denominator, it could be wise to transform 4 into a fraction which also have 3 as its denominator.

$$4 = \frac{12}{3}$$

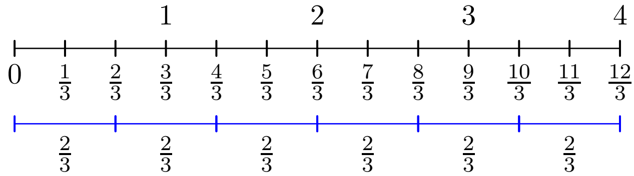


Recall that one of the interpretations of $4 : \frac{2}{3}$ is

"The number of $\frac{2}{3}$'s added to make 4."

By studying a number line, we find that 6 instances of $\frac{2}{3}$ added equals 4. Hence

$$4 : \frac{2}{3} = 6$$



Ein generell metode

We can't study the number line every time we are to divide by a fraction, so here we shall find a general method, again with $4 : \frac{2}{3}$ as our example. Then we apply this interpretation of divison:

$$4 : \frac{2}{3} = \text{"The number to multiply } \tfrac{2}{3} \text{ by to make 4."}$$

We begin the search of this number by multiplying $\frac{2}{3}$ by the number which results in the product equal to 1. This number is the *inverted fraction* of $\frac{2}{3}$, which is $\frac{3}{2}$:

$$\frac{2}{3} \cdot \frac{3}{2} = 1$$

Now we only have to multiply by 4 to make 4:

$$\frac{2}{3} \cdot \frac{3}{2} \cdot 4 = 4$$

Therefore, to make 4 we must multiply $\dfrac{2}{3}$ by $\dfrac{3}{2} \cdot 4$. Consequently,

$$\begin{aligned} 4 : \frac{2}{3} &= \frac{3}{2} \cdot 4 \\ &= 6 \end{aligned}$$

0.10 Brøk delt på brøk

When dividing a number by a fraction, we multiply the number by the inverted fraction.

Example 1

$$\begin{aligned} 6 : \frac{2}{9} &= 6 \cdot \frac{9}{2} \\ &= 27 \end{aligned}$$

Example 2

$$\begin{aligned} \frac{4}{3} : \frac{5}{8} &= \frac{4}{3} \cdot \frac{8}{5} \\ &= \frac{32}{15} \end{aligned}$$

Example 3

$$\frac{3}{5} : \frac{3}{10} = \frac{3}{5} \cdot \frac{10}{3}$$
$$= \frac{30}{15}$$

In this case we should also observe that the fraction can be simplified:

$$\frac{30}{15} = \frac{2 \cdot \cancel{15}}{\cancel{15}}$$
$$= 2$$

Notice: Canceling factors along the way saves the labour of working with large numbers:

$$\frac{3}{5} \cdot \frac{10}{3} = \frac{\cancel{3} \cdot 2 \cdot \cancel{5}}{\cancel{5} \cdot \cancel{3}}$$
$$= 2$$

0.9 Rasjonale tal

0.11 Rational numbers

Any number which can be expressed a fraction is a *rational number*.

Merk

Rational numbers gives a collective name for

- **Heiltal**

  For example $4 = \frac{4}{1}$.

- **Decimal numbers with a finite number of digits**

  For example $0{,}2 = \frac{1}{5}$.

- **Decimal numbers with repeating digits**

  For example [1] $0.08\bar{3} = \frac{1}{12}$.

[1] $\bar{3}$ indicates that 3 repeats infinite. Another way of expressing this is by using ... . That is, $0.08\bar{3} = 0.08333333...$