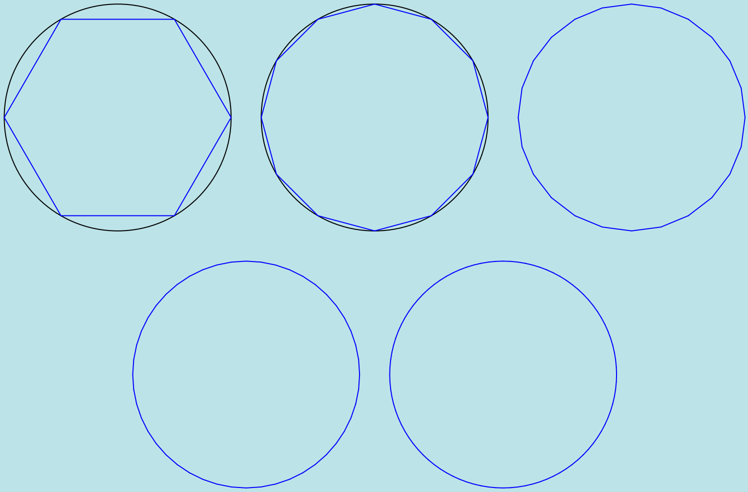
First Principles of Math



Sindre Sogge Heggen

*"Wahrlich es ist nicht das Wissen, sondern das Lernen,
nicht das Besitzen, sondern das Erwerben,
nicht das Da-Seyn, sondern das Hinkommen,
was den grössten Genuss gewährt"*

*"It is not knowing, rather learning,
not possessing, rather obtaining,
not being present, rather reaching there,
which serves the greatest joy."*

— Carl Friedrich Gauss

This document is created by Sindre Sogge Heggen. The text is written in LaTeX and the figures are made using Asymptote.

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Foreword

The extent and applications of mathematics are enormous, but a fair share of it is founded on a manageable amount of principles: I wish to present these in this book. I have chosen to call a principle in summarized form a *rule*. You will find the rules in blue text boxes, usually followed by an example of its usage. One of the main targets of this book is presenting the logical justification for the rules. In Chapter 1 - 5 you will find explanations[1] preceding every rule, while in chapter 6 some explanations are found directly after stating the rules (and eventual examples). As of chapter 7, some explanations are found in a concluding section named *Explanations*. This indicates that they are rather intricate or are so intuitively true that many will find the explanation superfluous.

The structure of the book

The book consists of a *Part I* and a *Part II*. *Part I* focuses on the basic understanding of the numbers and operations of calculation. *Part II* introduces the concept of algebra and the closely related topics of powers, equations, and functions. In addition, both *Part I* and *Part II* end with a chapter on geometry.

Notice! You will not find practice problems and applications of mathematics in real life in this book. These are two very important elements to come, either integrated in this book or as an independent document.

A note on convention

Although I am very much aware of the convention of writing commas and dots in center-aligned equations, I opted against this[2]. In this way, a center-aligned equation is a grammatical hybrid; it can end with both an invisible comma or dot, or nothing at all.

[1]To *explain* the rules rather than *proving* them is a deliberate decision. A proof demands mathematical rigour that often forces a lot of assumptions and definitions along the way. This can make the main insight disappear in the crowd of details. However, some of the explanations are valid as proofs.

[2]I've never liked the looks of it.

Dear reader.

This book is free of charge; however, I've invested a lot of time and resources in creating it. I really want to continue creating books which makes mathematics available for free, but it can turn out to be quite difficult if there is no income connected to it. Therefore, if you like this book, I hope you can donate a small sum using PayPal. Thank you in advance!

The book is updated as soon as possible when errors are discovered. Please download the latest version.

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Symbols

$=$	"equals"
$<$	"less than"
$>$	"greater than"
$\leq$	"less than or equal to"
$\geq$	"greater than or equal to"
$\in$	"included in"
$\vee$	"or"
$[a, b]$	"closed interval from a to b"
$\|a\|$	"length/absolute value of a"
$\perp$	"perpendicular to"
$\parallel$	"parallel with"
$\triangle$	"triangle"
$\square$	"quadrilateral"

Contents

Part I

Numbers, calculations, and geometry

Chapter 1

The numbers

1.1 The equal sign, amounts, and number lines

The equal sign

As the name implies, the *equal sign* $=$ refers to things that are the same. In what sense some things are the same is a philosophical question and initially we are bound to this: What equality $=$ points to must be understood by the context in which the sign is used. With this understanding of $=$ we can study some basic properties of our numbers and then later return to more precise meanings of the sign.

> **The language box**
>
> Common ways of expressing $=$ is
>
> - "equals"
> - "is the same as"

Amounts and number lines

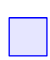
There are many ways a number can be defined, however, in this book we shall stick to two ways of interpreting a number; a number as an *amount* and a number as a *placement on a line*. All representations of numbers relies on the understanding of 0 and 1.

Numbers as amounts

Talking about an amount, the number 0 is[1] connected to "nothing". A figure showing nothing will therefore equal 0:

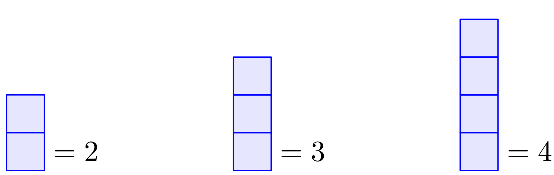
$$= 0$$

1 we'll draw like a box:



$$= 1$$

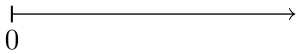
In this way, other numbers are defined by how many one-boxes (ones/units) we have:



[1]In *Chapter 2* we'll se that there are other interpretations of 0.

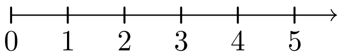
Numbers as placements on a line

When placing numbers on a line, 0 is our starting point:



Now we place 1 a set length to the right of 0:

Other numbers are now defined by how many one-lengths (ones/units) we are away from 0:



Positive integers

We'll soon see that numbers do not necessarily have to be a *whole* amount of ones, but those which *are* have their own name:

1.1 Positive integers

Numbers which are a whole amount of ones are called *positive*[1] *integers*. The positive integers are

$$1, 2, 3, 4, 5 \text{ and so on.}$$

Positive integers are also called *natural numbers*.

What about 0?

Some authors also include 0 in the definition of positive integers/natural numbers. This is in some cases beneficial, in others not.

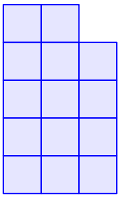
[1]We'll see what the the word *positive* refers to in chapter *chapter* **??**.

1.2 Numbers, digits and value

Our numbers consists of the *digits* $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9 along with their *positions*. The digits and their positions defines[1] the *value* of numbers.

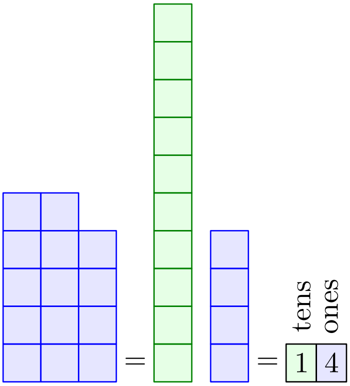
Integers larger then 10

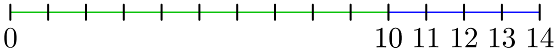
Let's, as an example, write the number *fourteen* by our digits.

fourteen =


We can now make a group of 10 ones, then we also have 4 ones. By this, we write fourteen as

$$\text{fourteen} = 14$$

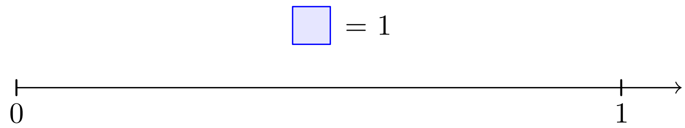




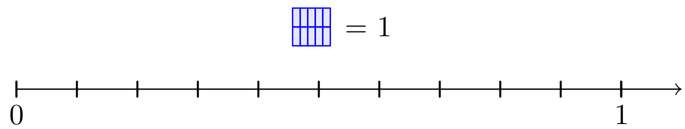
[1] Later on, we'll also see that *signs* have an impact on a numbers value (see *Chapter* **??**).

Decimal numbers

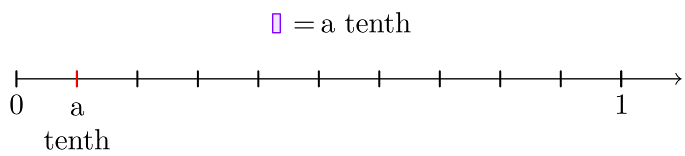
Sometimes we don't have a whole amount of ones, and this brings about the need to divide "ones" into smaller pieces. Let's start off by drawing a one:



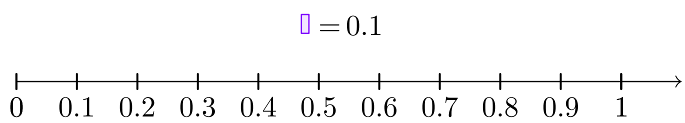
Now we divide our one into 10 smaller pieces:



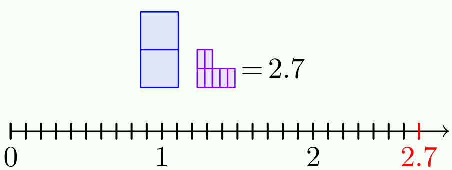
Since we have divided 1 into 10 pieces, we name one such piece *a tenth*:



We indicate tenths by using the *decimal mark*: .



Example



The language box

In a lot of countires, a comma is used in place of the period for the decimal mark.

3,5 (*other*)

3.5 (*english*)

Base-10 positional notation

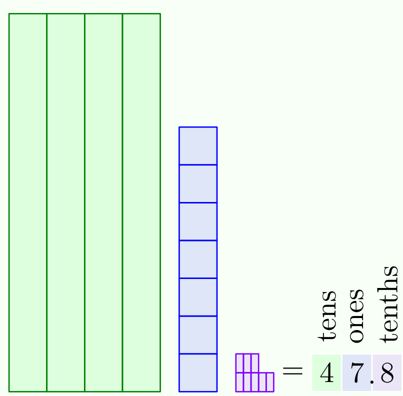
So far, we have seen how we can express the value of a number by placing digits according to the amount of tens, ones and tenths. The pattern continues:

1.2 Base-10 positional notation

The value of a number is given by the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9 and their position. In respect to the digit indicating ones,

- digits to the left indicate amounts of tens, hundeds, thousands etc.
- digits to the left indicate amounts of tenths, hundredths, thousandths etc.

Example 1



Example 2

thousands hundreds tens ones tenths hundredths

3805.72

1.3 Coordinate systems

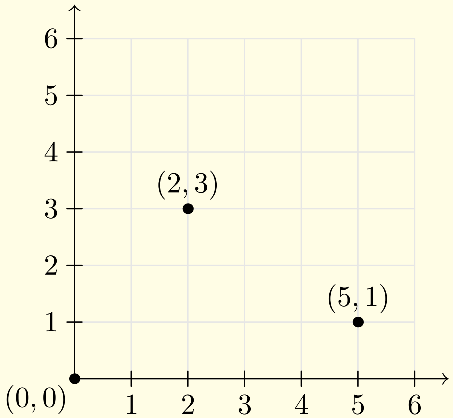
Two number lines can be put together to form a *coordinate system*. In that case we place one number line *horizontally* and one *vertically*. A position in a coordinate system is called a *point*.

In fact, there are many types of coordinate systems, but we'll use the *cartesian coordinate system*. It is named after the french mathematician and philosopher René Descartes.

A point is written as two numbers inside a bracket. We shall call these two numbers the *first coordinate* and the *second coordinate*.

- The first coordinate tells how many units to move along the horizontal axis.
- The second coordinate tells how many units to move along the vertical axis.

In the figure, the points (2,3), (5,1) and $(0,0)$ are shown. The point where the axes intersect, $(0,0)$, is called *origo*.



Chapter 2

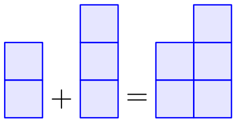
The four elementary operations

2.1 Addition

Addition with amounts

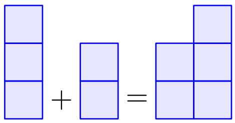
When we have an amount and wish to add more, we use the symbol $+$. If we have 2 and want to add 3, we write

$$2 + 3 = 5$$



The order in which we add have no impact on the results; starting off with 2 and adding 3 is the same as starting off with 3 and adding 2:

$$3 + 2 = 5$$



The language box

A calculation involving addition includes two or more *terms* and one *sum*. In the calculation

$$2 + 3 = 5$$

both 2 and 3 are terms while 5 is the sum.

Common ways of saying $2 + 3$ include

- ”2 plus 3”
- ”2 added to 3”
- ”2 and 3 added”

2.1 Addition is commutative

The order of the terms has no impact on the sum.

Example

$$2 + 5 = 7 = 5 + 2$$

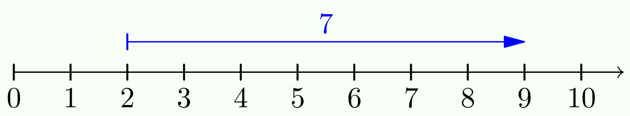
$$6 + 3 = 9 = 3 + 6$$

Addition on the number line: moving to the right

On a number line, addition with positive numbers involves moving *to the right*:

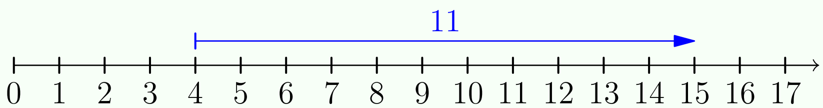
Example 1

$$2 + 7 = 9$$



Example 2

$$4 + 11 = 15$$



Interpretation of $=$

$+$ brings the possibility of expressing numbers in different ways, for example is $5 = 2 + 3$ and $5 = 1 + 4$. In this context, $=$ means "has the same value as". This is also the case regarding subtraction, multiplication and division which we'll look at in the next three sections.

2.2 Subtraction

Subtraction with amounts

When removing a part of an amount, we use the symbol $-$:

$$5 - 3 = 2$$

The language box

A calculation involving subtraction includes one or more *terms* and one *difference*. In the calculation

$$5 - 3 = 2$$

both 5 and 3 are terms while 2 is the difference.

Common ways of saying $5 - 3$ include

- ”5 minus 3”
- ”3 subtracted from 5”

A new interpretation of 0

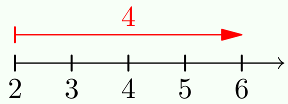
As mentioned earlier in this book, 0 can be interpreted as ”nothing”. However, subtraction brings the possibility of expressing 0 by other numbers, for example $7 - 7 = 0$ and $19 - 19 = 0$. In many practical situations, 0 indicates some form of equilibrium, like two equal but opposite forces.

Subtraction on the number line: Moving to the left

In *Section 2.1*, we have seen that $+$ (with positive numbers) involves moving *to the right* on the number line. With $-$ it's the opposite, we move *to the left*[1]:

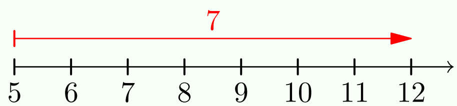
Example 1

$$6 - 4 = 2$$



Example 2

$$12 - 7 = 5$$



Notice

At first it may seem a bit odd moving in the opposite direction of the way in wich the arrows point, as in *Example 1* and *2*. However, in *Chapter* **??** this will turn out to be useful.

[1]In figures with number lines, the red colored arrows indicates that you shall start at the arrow head and move to the other end.

2.3 Multiplication

Multiplication by integers: inital definition

When adding equal numbers, we can use the multiplication symbol $\cdot$ to write our calculations more compact:

Example

$$4 + 4 + 4 = 4 \cdot 3$$

$$8 + 8 = 8 \cdot 2$$

$$1 + 1 + 1 + 1 + 1 = 1 \cdot 5$$

The language box

A calculation involving multiplication includes severeal *factors* and one *product*. In the calculation

$$4 \cdot 3 = 12$$

both 4 and 3 are factors, while 12 is the product.

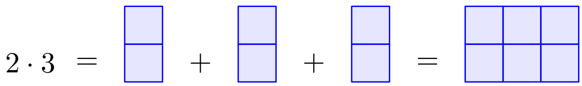
Common ways of saying $4 \cdot 3$ include

- ”4 times 3”
- ”4 multiplied by 3”
- ”4 and 3 multiplied together”

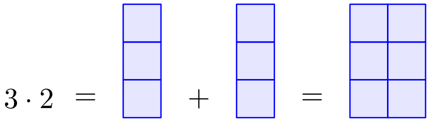
A lot of texts use $\times$ instead of $\cdot$. In computer programming, * is the most common symbol for multiplication.

Multiplication involving amounts

Let us illustrate $2 \cdot 3$:



Now notice the product of $3 \cdot 2$:



2.2 Multiplication is commutative

The order of the factors has no impact on the product.

Example

$$3 \cdot 4 = 12 = 4 \cdot 3$$

$$6 \cdot 7 = 42 = 7 \cdot 6$$

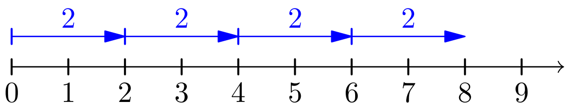
$$8 \cdot 9 = 72 = 9 \cdot 8$$

Multiplication on the number line

We can also use the number line to calculate multiplications. In the case of $2 \cdot 4$ we can think like this:

"$2 \cdot 4$ means moving 2 places to the right, 4 times."

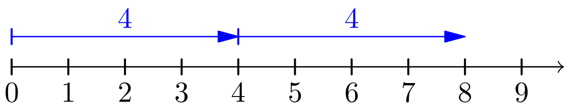
$$2 \cdot 4 = 8$$



We can also use the number line to prove to ourselves that multiplication is commutative:

"$4 \cdot 2$ means moving 4 places to the right, 2 times."

$$4 \cdot 2 = 8$$



Final definition of multiplication by positive integers

It may be the most intuitive to interpret "2 times 3" as "3, 2 times". Then it follows:

$$\text{"2 times 3"} = 3 + 3$$

In this section we introduced $2 \cdot 3$, that is "2 times 3", as $2 + 2 + 2$. With this interpretation, $3 + 3$ corresponds to $3 \cdot 2$, but the fact that multiplication is a commutative operation (*Rule 2.2*) ensures that the one interpretation does not exclude the other; $2 \cdot 3 = 2 + 2 + 2$ and $2 \cdot 3 = 3 + 3$ are two expressions of same value.

2.3 Multiplication as repeated addition

Multiplication involving a positive integer can be expressed as repeated addition.

Example 1

$$4+4+4=4\cdot 3=3+3+3+3$$

$$8+8=8\cdot 2=2+2+2+2+2+2+2$$

$$1+1+1+1+1=1\cdot 5=5$$

Notice

The fact that multiplication with positive integers can be expressed as repeated addition does not exclude other expressions. There's nothing wrong with writing $2\cdot 3=1+5$.

2.4 Division

$:$ is the symbol for divison. Division has three different interpretations:

2.4 The three interpretations of division

- **Distribution of amounts**

  $12 : 3 =$ "The number in each group when evenly distributing 12 into 3 groups"

- **Number of equal terms**

  $12 : 3 =$ "The number of 3's added to make 12"

- **The inverse operation of multiplication**

  $12 : 3 =$ "The number which yields 12 when multiplied by 3"

The language box

A calculation involving division includes a *dividend*, a *divisor* and a *quotient*. In the calculation

$$12 : 3 = 4$$

12 is the dividend, 3 is the divisor and 4 is the quotient.

Common ways of saying $12 : 3$ include

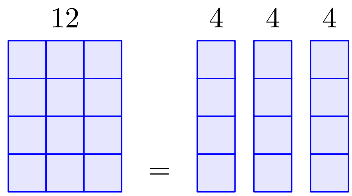
- "12 divided by 3"
- "12 to 3"

In a lot of contexts, $/$ is used instead of $:$, especially in computer programming.

Sometimes $12 : 3$ is called "the *ratio* of 12 to 3".

Distribution of amounts

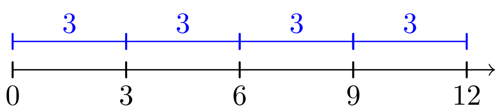
The calculation $12 : 3$ tells that we shall distribute 12 into 3 equal groups:



We observe that each group contains 4 boxes, which means that

$$12 : 3 = 4$$

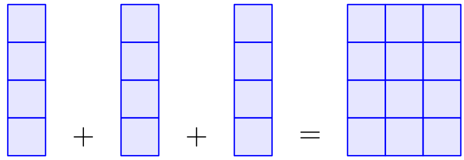
Number of equal terms



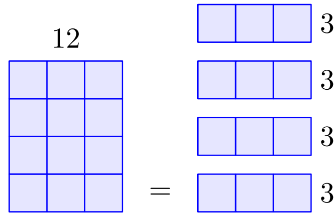
12 equals the sum of 4 instances of 3, that is $12 : 3 = 4$.

The inverse operation of multiplication

We have just seen that if we divide 12 into 3 equal groups, we get 4 in each group. Hence $12 : 3 = 4$. The sum of these groups makes 12:



However, this is the same as multiplying 4 by 3, in other words: If we know that $4 \cdot 3 = 12$, we also know that $12 : 3 = 4$. As well we know that $12 : 4 = 3$.



Example 1

Since $6 \cdot 3 = 18$,

$$18 : 6 = 3$$

$$18 : 3 = 6$$

Example 2

Since $5 \cdot 7 = 35$,

$$35 : 5 = 7$$

$$35 : 7 = 5$$

Chapter 3

Factorization and order of operations

3.1 Factorization

If an integer dividend and an integer divisor results in an integer quotient, we say that the dividend is *divisible* by the divisor. For example is 6 divisible with 3 because $6 : 3 = 2$, and 40 is divsible with 10 because $40 : 10 = 4$. The concept of divisibility contributes to the definition of *prime numbers*:

3.1 Primtal

A natural number larger than 1, and only divisible by itself and 1, is a prime number.

Example

The first five prime numbers are $2, 3, 5, 7$ and 11.

3.2 Factorization

Factorization involves writing a number as the product of other numbers.

Example

Factorize 24 in three different ways.

Answer

$$24 = 2 \cdot 12$$

$$24 = 3 \cdot 8$$

$$24 = 2 \cdot 3 \cdot 4$$

3.3 Prime factorization

Factorization involving prime factors only is called prime factorization.

Example

Prime factorize 12.

Answer

$$12 = 2 \cdot 2 \cdot 3$$

3.2 Order of operations

Priority of the operations

Look at the following calculation:

$$2 + 3 \cdot 4$$

This *could* have been interpreted in two ways:

1. ”2 plus 3 equals 5. 5 times 4 equals 20. The answer is 20.”
2. ”3 times 4 equals 12. 2 plus 12 equals 14. The answer is.”

But the answers are not the same! This points out the need of having rules for what to calculate first. One of these rules is that multiplication and division is to be calculated *before* addition or subtraction, which means that

$$\begin{aligned} 2 + 3 \cdot 4 &= \text{”Calculate } 3 \cdot 4\text{, then add 2”} \\ &= 2 + 12 \\ &= 14 \end{aligned}$$

But what if we wanted to calculate $2 + 3$ first, then multiplying the sum by 4? We use parentheses to tell that something is to be calculeted first:

$$\begin{aligned} (2 + 3) \cdot 4 &= \text{”Calculate } 2 + 3\text{, multiply by 4 afterwards”} \\ &= 5 \cdot 4 \\ &= 20 \end{aligned}$$

3.4 Order of operations

1. Expressions with parentheses
2. Multiplication or division
3. Addition or subtraction

Example 1

Calculate

$$23 - (3 + 9) + 4 \cdot 7$$

Answer

$$\begin{aligned} 23 - (3+9) + 4 \cdot 7 &= 23 - 12 + 4 \cdot 7 && \text{Paranthesis} \\ &= 23 - 12 + 28 && \text{Multiplication} \\ &= 39 && \text{Addition and subtraction} \end{aligned}$$

Example 2

Calculate

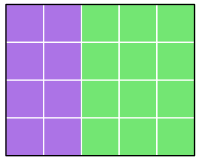
$$18 : (7 - 5) - 3$$

Answer

$$\begin{aligned} 18 : (7 - 5) &= 18 : 2 - 3 && \text{Paranthesis} \\ &= 9 - 3 && \text{Division} \\ &= 6 && \text{Addition and subtraction} \end{aligned}$$

Multiplication involving paranthesis

How many boxes are present in this figure?



To ways of thinking are these:

1. It is $2 \cdot 4 = 8$ purple boxes and $3 \cdot 4 = 12$ green boxes. In total there are $8 + 12 = 20$ boxes. This we can write as

$$2 \cdot 4 + 3 \cdot 4 = 20$$

2. It is $2 + 3 = 5$ boxes horizontally and 4 boxes vertically, so there are $5 \cdot 4 = 20$ boxes in total. This we can write as

$$(2 + 3) \cdot 4 = 20$$

From these two calculations it follows that

$$(2+3)\cdot 4 = 2\cdot 4 + 3\cdot 4$$

3.5 Distributive law

When an expression enclosed by a parenthesis is a factor, we can multiply the other factors with each term inside the parenthesis.

Example 1

$$(4+7)\cdot 8 = 4\cdot 8 + 7\cdot 8$$

Example 2

$$\begin{aligned}(10-7)\cdot 2 &= 10\cdot 2 - 7\cdot 2\\ &= 20-14\\ &= 6\end{aligned}$$

Notice: Obviously, it would be easier to calculate like this:

$$(10-7)\cdot 2 = 3\cdot 2 = 6$$

Example 2

Calculate $12\cdot 3$.

Answer

$$\begin{aligned}12\cdot 3 &= (10+2)\cdot 3\\ &= 10\cdot 3 + 2\cdot 3\\ &= 30+6\\ &= 36\end{aligned}$$

Notice

We introduced parenthesis as an indicator of what to calculate first, but *Rule 3.5* gives an alternative and equivalent interpretation of parenthesis . The rule is especially useful when working with algebra (see *Part* **??**).

Multiplying by 0

Earlier whe have seen that 0 can be expressed as the difference between two numbers, and this can help us calculate when multiplying by 0. Let's look at the calculation

$$(2-2)\cdot 3$$

By *Rule 3.5*, we get

$$\begin{aligned}(2-2)\cdot 3 &= 2\cdot 3 - 2\cdot 3\\ &= 6-6\\ &= 0\end{aligned}$$

Since $0 = 2-2$, this means that

$$0\cdot 3 = 0$$

3.6 Multiplication by 0

If 0 is a factor,the product equals 0.

Example 1

$$7\cdot 0 = 0$$

$$0\cdot 219 = 0$$

Associative laws

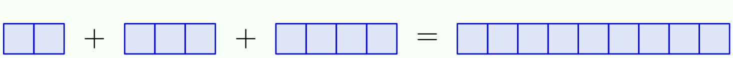
3.7 Associative law for addition

The placement of parentheses between terms have no impact on the sum.

Example

$$(2+3)+4 = 8$$

$$2+(3+4) = 8$$



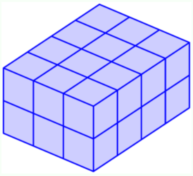
3.8 Associative law for multiplication

The placement of parentheses between factors have no impact on the product.

Example

$$(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24$$

$$2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24$$



Opposite to addition and multiplication, neither subtraction nor divison is associative:

$$(12 - 5) - 4 = 7 - 4 = 3$$
$$12 - (5 - 4) = 12 - 1 = 11$$

$$(80 : 10) : 2 = 8 : 2 = 4$$
$$80 : (10 : 2) = 80 : 5 = 16$$

We have seen how parentheses hjelps indicating the *priority* of operations, but the fact that subtraction and divison is non-associative brings the need of having a rule of in which *direction* to calculate.

3.9 Direction of calculations

Operations which by *Rule 3.4* have equal priority, are to be calculated from left to right.

Example 1

$$\begin{aligned} 12 - 5 - 4 &= (12 - 5) - 4 \\ &= 7 - 4 \\ &= 3 \end{aligned}$$

Example 2

$$\begin{aligned} 80 : 10 : 2 &= (80 : 10) : 2 \\ &= 8 : 2 \\ &= 4 \end{aligned}$$

Example 3

$$\begin{aligned} 6 : 3 \cdot 4 &= (6 : 3) \cdot 4 \\ &= 2 \cdot 4 \\ &= 8 \end{aligned}$$