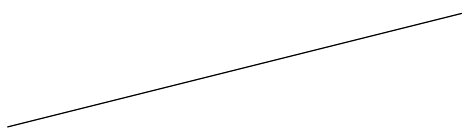
0.1 Terms

Point

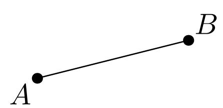
A given position is called a[1] *point*. We mark a point by drawing a dot, which we preferably name by a letter. Below we have drawn the points A and B.

Line and segment

A straight dash with infinite length (!) is called a *line*. The fact that a line has infinite length, makes *drawing* a line impossible, we can only *imagine* a line. Imagining a line can be done by drawing a straight dash and think of its ends as wandering out in each direction.



A straight dash between two points is called a *segment*.

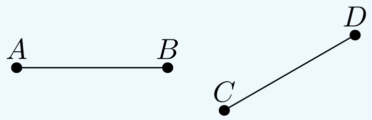


We write the segment between the points A and B as AB.

> **Notice**
>
> A segment is a part of a line, therefore a line and a segment have a lot of attributes in common. When writing about lines, it will be up to the reader to confirm whether the same applies for segments. Hence we avoid the need of writing "lines/segments".

[1] See also *Section* **??**.

Segment or length?

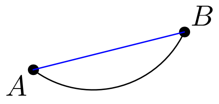


The segments AB and CD have equal length, but they are not the same segment. Still we'll write $AB = CD$. That is, we'll use the same names for the line segments and their lengths (the same applies for angles and their values, see page 4 - 6). We'll do this by the following reasons:

- The context will make it clear weather we are talking about a segment or a length.
- Finding it necessary to write e.g. "the length of AB" would make sentences less readable.

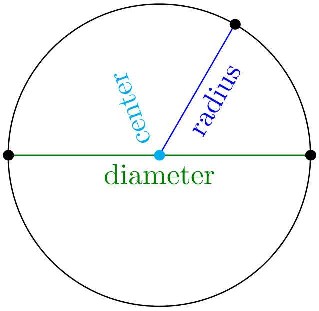
Distance

There are infinite ways one can move from one point to another and some ways will be longer than others. When talking about a distance in geometry, we usually mean the *shortest* distance. For geometries studied in this book the shortest distance between two points will always equal the length of the segment (blue in the below figure) connecting them.



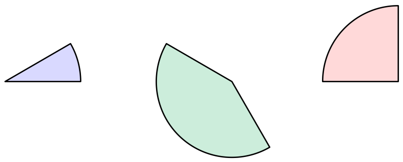
Circle; center, radius and diameter

If we make an enclosed curve where all points on this curve have the same distance to a given point, we have a *circle*. The point which all the points on the curve have an equal distance to is the *center* of the circle. A segment between a point on the curve and the center is called a *radius*. A segment between two points on the curve, passing through the circle center, is called a *diameter*[1].



Arcs and sectors

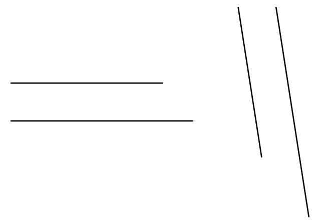
A part of a circular curve is called an *arc*. The shape formed by an arc and two associated radii is called a *sector*. The below figure shows three different sectors.



[1] As mentioned, *radius* and *diameter* can just as well indicate the length of the segments.

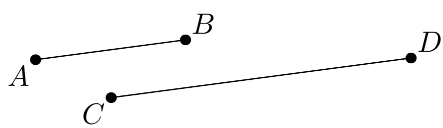
Parallel lines

Lines aligned in the same direction are *parallel*. The below figure shows two pairs of parallel lines.



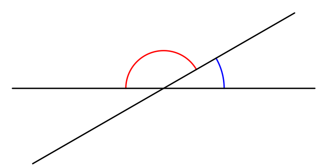
We use the symbol $\parallel$ to indicate that two lines are parallel.

$$AB \parallel CD$$



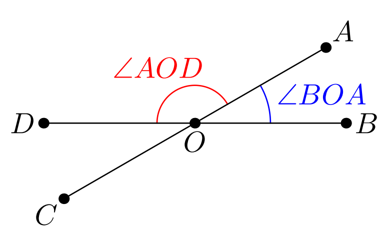
Angles

Non-parallel lines will sooner or later intersect. The gap formed by two non-parallel lines is called an *angle*. We draw angles as small circular curves:



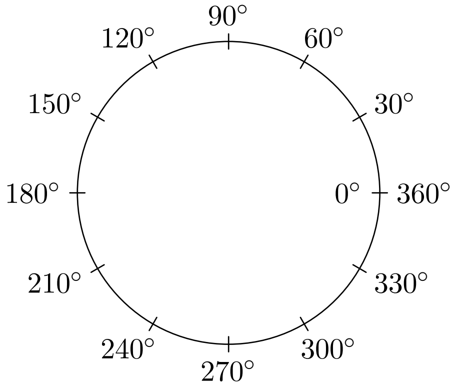
Lines creating an angle are called the *sides* of the angle. The intersection point of the lines are called the *vertex* of the angle. It is common to use the symbol $\angle$ to underline the angle in question. In the below figure we have the following:

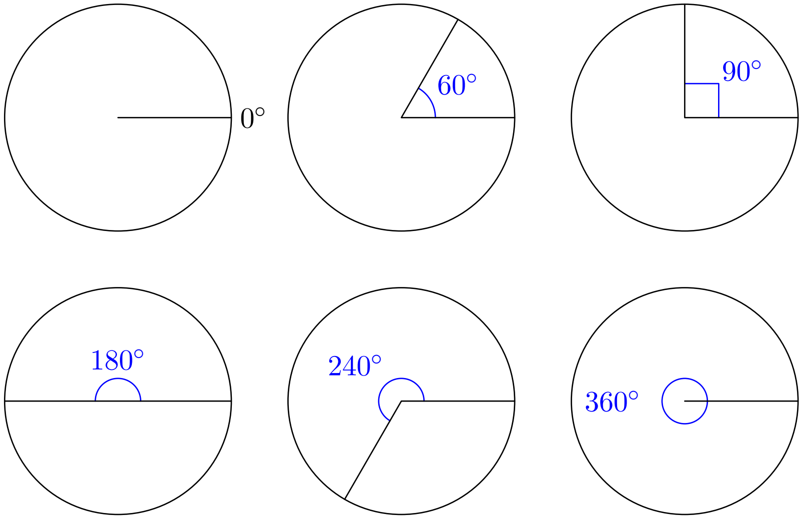
- the angle $\angle BOA$ has angle sides OB and OA and vertex O.
- the angle $\angle AOD$ has angle sides OA and OD and vertex O.



Measure of angles in degrees

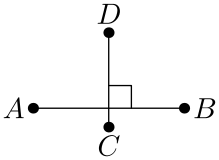
When measuring an angle in degrees, we imagine a circular curve divided into 360 equally long pieces. We call one such piece 1 *degree*, indicated by the symbol $^\circ$.





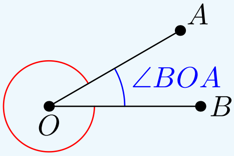
Notice that an angle with measure 90° is indicated by the symbol $\square$. Such an angle is called a *right* angle. Lines which form right angles are said to be *perpendicular* to one another, indicated by the symbol $\perp$.

$$AB \perp CD$$

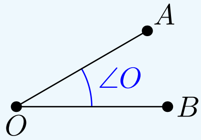


What angle?

Strictly speaking, when two segments (or lines) intersect, they form two angles; the one larger than 180° and the other smaller than 180°. Usually it is the smaller angle we wish to study, therefore it is common to define $\angle AOB$ as the *smaller* angle formed by the segments OA and OB.

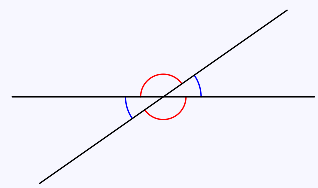


As long as there are only two segments/lines present, it is common using only one letter to indicate the angle:

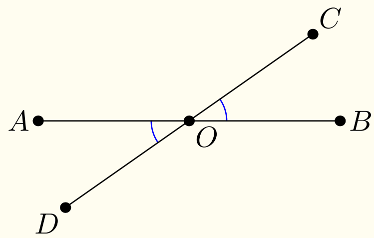


0.1 Vertical angles

Two opposite angles with a common vertex is called *vertical angles*. Vertical angles are of equal measure.



0.1 Vertical angles (explanation)



We have

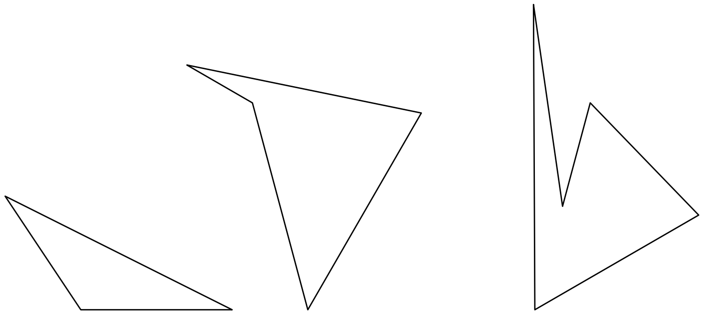
$$\angle BOC + \angle DOB = 180^\circ$$

$$\angle AOD + \angle DOB = 180^\circ$$

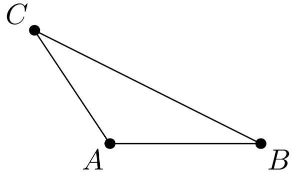
Hence, $\angle BOC = \angle AOD$. Similarly, $\angle COA = \angle DOB$.

Sides and vertices

When line segments form an enclosed shape, they form a *polygon*. The below figure shows, from left to right, a triangle (3-gon), a quadrilateral (4-gon) and a pentagon (5-gon).

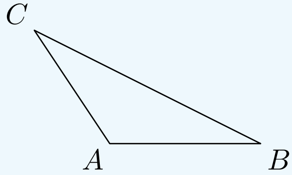


The segments of a polygon are called *edges* or *sides*. The respective intersection points of the segments are the *vertices* of the polygon. That is, the triangle below has vertices A, B and C and sides (edges) AB, BC and AC.



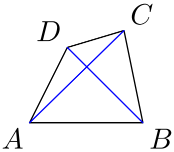
Notice

Often we'll write a letter only to indicate a vertex of a polygon.



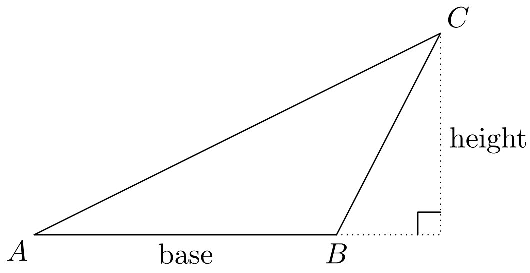
Diagonals

Segments between two vertices not belonging to the same side of a polygon is called a *diagonal*. The below figure shows the diagonals AC and BD.

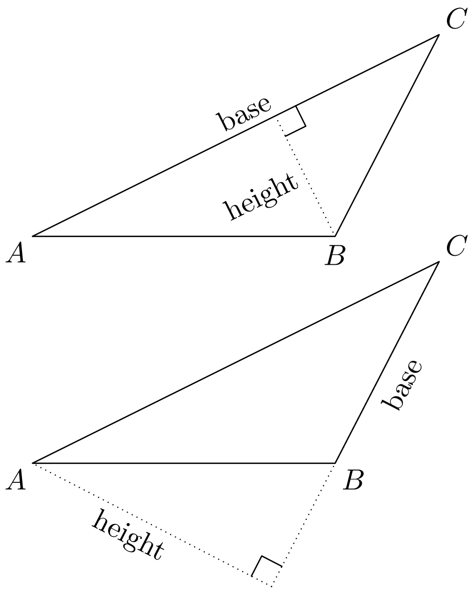


Altitudes and base lines

In *Section 0.4*, the terms *base* and *height* (*altitude*) play an important role. To find the height of a triangle, we choose one of the sides to be the base. In the below figure, let's start with AB as the base. Then the height is the segment from AB (potentially, as is the case here, the extension of AB) to C, perpendicular to AB.



Since there are three sides which can be bases, a triangle has three heights.



> **Notice**
>
> The terms altitude and base also applies to other polygons.

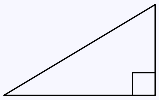
0.2 Attributes of triangles and quadrilaterals

In addition to having a certain number of sides and vertices, polygons have other attributes, such as sides or angles of equal measure, or parallel sides. There are specific names of polygons with special attributes, and these names can be put into an overview where some "inherit"[1] attributes from others.

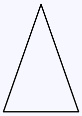
0.2 Triangles

$$\begin{tikzcd}[row sep=tiny] & \text{Right triangle} & \\ \text{Triangle} \arrow[ur] \arrow[r] & \text{Isosceles triangle} \arrow[r] & \text{Equilateral triangle} \end{tikzcd}$$

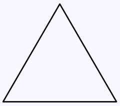
Triangle
Have three sides and three vertices.



Right triangle
Have an angle of 90°.



Isosceles triangle
At least two sides are of equal length.
At least two angles are of equal measure.



Equilateral triangle
The sides are of equal length.
Each of the angles equals 60°.

Example

Since an equilateral triangle have three sides of equal length and three angles equal to 60°, it is also an isosceles triangle.

The language box

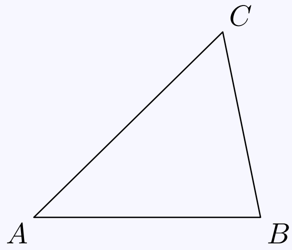
The longest side of a right triangle is called the *hypotenuse* and the shorter sides are called *legs*.

[1] In *Rule 0.2* and *Rule 0.4* this is indicated by arrows.

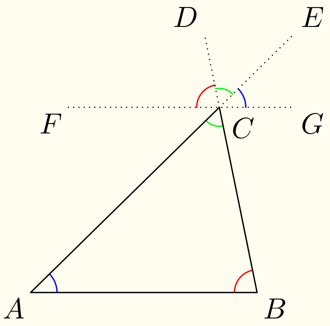
0.3 The sum of angles in a triangle

In a triangle, the sum of the angles equals 180°.

$$\angle A + \angle B + \angle C = 180^\circ$$



0.3 The sum of angles in a triangle (explanation)



We draw a segment FG passing through C and parallel to AB. Moreover, we place E and D on the extension of AC and BC, respectively. Then $\angle A = \angle GCE$ and $\angle B = \angle DCF$. $\angle ACB = \angle ECD$ because they are vertical angles. Now

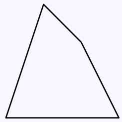
$$\angle DCF + \angle ECD = \angle GCE = 180^\circ$$

Hence

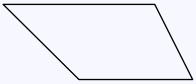
$$\angle CBA + \angle ACB + \angle BAC = 180^\circ$$

0.4 Quadrilaterals

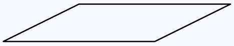
$$\begin{tikzcd}[row sep=tiny] & & & \text{Rhombus} \arrow[dr] & \\ \text{Quadrilateral} \arrow[r] & \text{Trapezoid} \arrow[r] & \text{Parallelogram} \arrow[ur] \arrow[dr] & & \text{Square} \\ & & & \text{Rectangle} \arrow[ur] & \end{tikzcd}$$



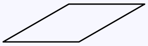
Quadrilateral
Have four sides and four vertices.



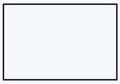
Trapezoid
Have at least one pair of parallel sides.



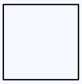
Parallelogram
Have two pairs of parallel sides.
Have two pairs of equal angles.



Rhombus
All sides are of equal length.



Rectangle
All angles equals 90°.



Square

Example

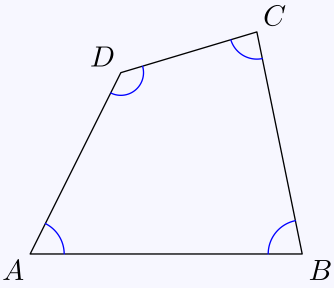
The square is both a rhombus and a rectangle, which means it ”inherits” their attributes. From this it follows that, in a square,

- all sides are of equal length.
- all angles equals 90°.

0.5 The sum of angles in a quadrilateral

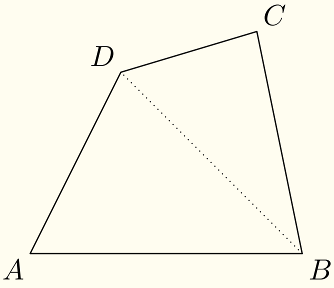
In a quadrilateral, the sum of the angles equals 360°.

$$\angle A + \angle B + \angle C + \angle D = 360^\circ$$



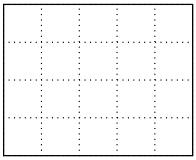
0.5 The sum of angles in a quadrilateral (explanation)

The total sum of angles of $\triangle ABD$ and $\triangle BCD$ equals the sum of the angles in $\square ABCD$. By *Rule 0.3*, the sum of angles of triangles 180°, therefore the sum of the angles of $\square ABCD$ equals $2 \cdot 180^\circ = 360^\circ$.

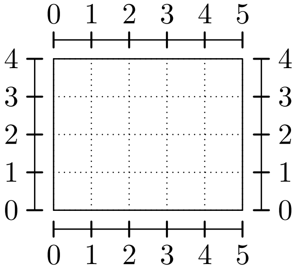


0.3 Perimeter

When we measure the length around an enclosed shape, we find its *perimeter*. Let's find the perimeter of this rectangle:



The rectangle has two sides of length 4 and two sides of length 5.



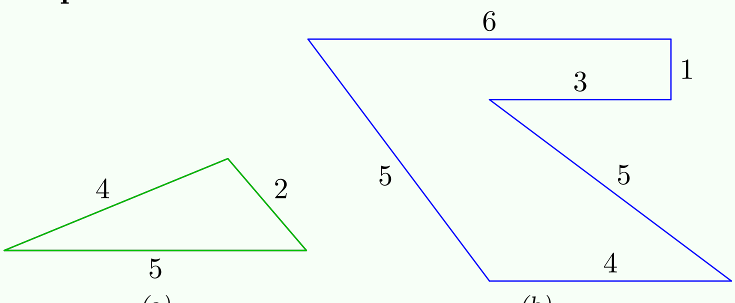
Hence

$$\begin{aligned}\text{The perimeter of the rectangle} &= 4+4+5+5\\ &= 18\end{aligned}$$

0.6 Perimeter

A perimeter is the length around a closed shape.

Example



(a) *(b)*

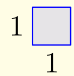
In figure *(a)* the perimeter equals $5+2+4=11$.

In figure *(b)* the perimeter equals $4+5+3+1+6+5=24$.

0.4 Area

Our surroundings are full of *surfaces*, for example on a floor or a sheet. When measuring surfaces, we find their *area*. The concept of area is the following:

We imagine a square with sides of length 1. We call this the *unit square*.

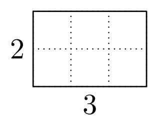


Then, regarding the surface for which we seek the area of, we ask:

”How many unit squares does this surface contain?”

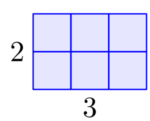
The area of a rectangle

Let’s find the area of a rectangle with baseline 3 and altitude 2.



Simply by counting, we find that the rectangle contains 6 unit squares:

$$\text{The area of the rectangle} = 6$$

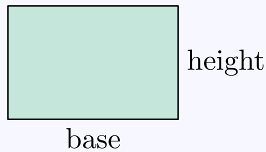


Looking back at *Section* **??**, we notice that

$$\begin{aligned}\text{The area of the rectangle} &= 3 \cdot 2 \\ &= 6\end{aligned}$$

0.7 The area of a rectangle

$$\text{Area} = \text{baseline} \cdot \text{altitude}$$

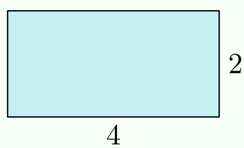


Width and length

In a rectangle, the baseline and the altitude are also referred to as (in random order) the *width* and the *length*.

Example 1

Find the area of the rectangle[1].

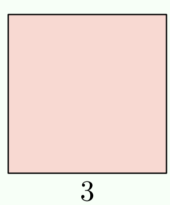


Answer

$$\text{The area of the rectangle} = 4 \cdot 2 = 8$$

Example 2

Find the area of the square.



Answer

$$\text{The area of the square} = 3 \cdot 3 = 9$$

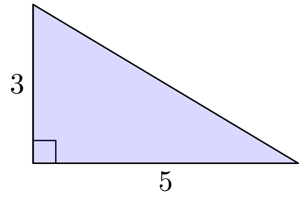
[1] *Notice*: The lengths used in one figure will not necessarily correspond to the lengths in another figure. That is, a side of length 1 in one figure can might as well be shorter than a side of length 1 in another figure.

The area of a triangle

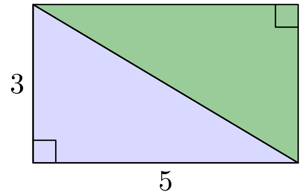
Concerning triangles, there are three different cases to study:

1) The baseline and the altitude have a common end point

Let's find the area of a right triangle with baseline 5 and altitude 3.



We can make a rectangle by copying our triangle and joining the hypotenuses:



By *Rule 0.7*, the area of the rectangle equals $5 \cdot 3$. The area of one of the triangles makes up half the area of the rectangle, so

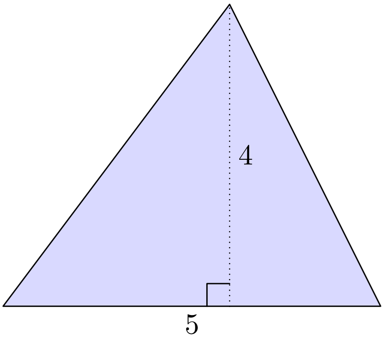
$$\text{The area of the blue triangle} = \frac{5 \cdot 3}{2}$$

Regarding the blue triangle we have

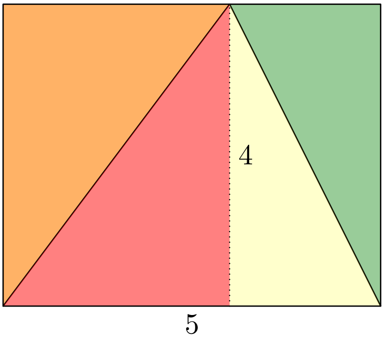
$$\frac{5 \cdot 3}{2} = \frac{\text{baseline} \cdot \text{height}}{2}$$

2) The altitude is placed inside the triangle, but have no common end point with the baseline

The below triangle has baseline 5 and altitude 4.



We make a rectangle containing the blue triangle (split into the red triangle and the yellow triangle):



Observe that

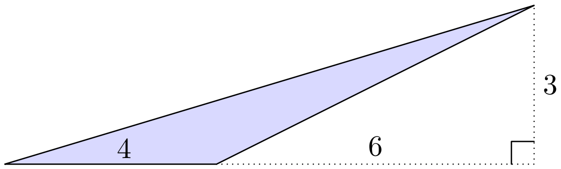
- the area of the red triangle makes up half the area of the rectangle formed by the red and the yellow triangle.
- the area of the yellow triangle makes up half the area of the rectangle formed by the yellow and the green triangle.

It now follows that the sum of the areas of the yellow triangle and the red triangle makes up half the area of the rectangle formed by the four colored triangles. The area of this rectangle equals $5 \cdot 4$, and since our original triangle (the blue) includes the red triangle and the orange triangle, we have

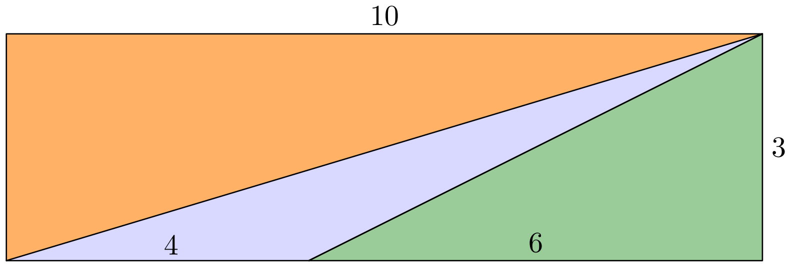
$$\text{The area of the blue triangle} = \frac{5 \cdot 4}{2} = \frac{\text{baseline} \cdot \text{height}}{2}$$

2) The altitude is placed outside the triangle

The below triangle has baseline 4 and altitude 3.



We now make a rectangle containing the blue triangle:



Now we introduce the following names:

$$\begin{aligned}\text{The area of the rectangle} &= R\\ \text{The area of the blue triangle} &= B\\ \text{The area of the orange triangle} &= O\\ \text{The area of the green triangle} &= G\end{aligned}$$

We have

$$R = 3 \cdot 10 = 30$$

$$O = \frac{3 \cdot 10}{2} = 15$$

$$G = \frac{3 \cdot 6}{2} = 9$$

Moreover,

$$\begin{aligned}B &= R - O - G\\ &= 30 - 15 - 9\\ &= 6\end{aligned}$$

Observe that we can write

$$6 = \frac{4 \cdot 3}{2}$$

Regarding the blue triangle we recognize this as

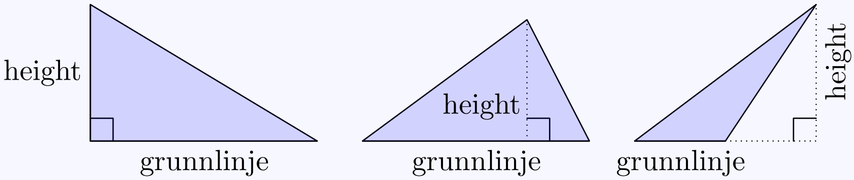
$$\frac{4 \cdot 3}{2} = \frac{\text{base} \cdot \text{height}}{2}$$

The three cases summarized

For a chosen baseline in a triangle, one of the cases discussed will always be valid. All cases resulted in the same expression for the area of the triangle.

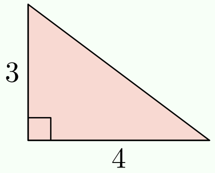
0.8 The area of a triangle

$$\text{Area} = \frac{\text{base} \cdot \text{height}}{2}$$



Example 1

Find the area of the triangle.

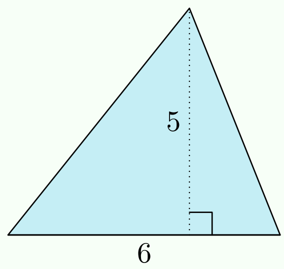


Answer

$$\begin{aligned}\text{The area of the triangle} &= \frac{4 \cdot 3}{2} \\ &= 6\end{aligned}$$

Example 2

Find the area of the triangle.

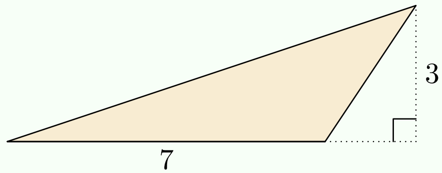


Answer

$$\text{The area of the triangle} = \frac{6 \cdot 5}{2} = 15$$

Example 3

Find the area of the triangle.



Answer

$$\text{The area of the triangle} = \frac{7 \cdot 3}{2} = \frac{21}{2}$$