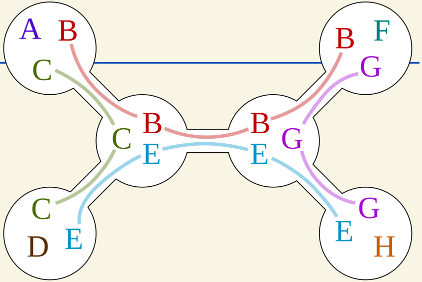
Parameterized Complexity Theory II

DM898: Parameterized Algorithms
Lars Rohwedder



Today's lecture

- Proof of Sparsification Lemma

Overview

Assume for all constant $\epsilon > 0$ there is an algorithm $\mathrm{Sparse}_\epsilon(V, \mathcal{C})$ that solves a 3-SAT instance with n variables V and m clauses $\mathcal{C}$ in time

$$O\left(2^{\epsilon(n+m)}\right)$$

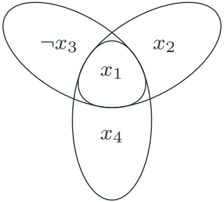
We show that if this is the case, then for each $\epsilon > 0$ there is also an algorithm that solves 3-SAT in time $O(2^{\epsilon n})$ using Sparse as a subroutine, therefore refuting ETH.

The algorithm is defined for each $0 < \delta < 1/10$ and we analyze the running time based on δ. We specify δ as a function in ϵ later to achieve our goal of $O(2^{\epsilon n})$.

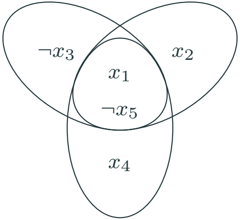
Sunflowers

In a branching algorithm we will exploit sunflower structures formed by clauses over the literals. More precisely, we are interested in **pseudo**-sunflowers, where the petals may overlap, and we require the core to be non-empty

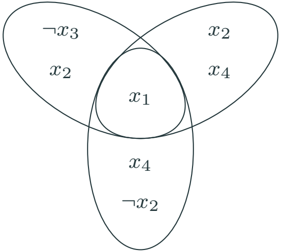
Formally: a set of equal-size clauses (sets of literals) $S_1, \ldots, S_k$ with intersection $Y \neq \emptyset$. Recall that Y is the core and $S_1 \setminus Y, \ldots, S_k \setminus Y$ are the petals. Only the following three cases can occur:



Small sunflower
(core size 1, petal size 1)



Medium sunflower
(core size 2, petal size 1)



Large sunflower
(core size 1, petal size 2)

For any such sunflower $Y, S_1, \ldots, S_k$, the following is a valid branching rule based on whether Y is satisfied:

Core branch (Y satisfied) Add Y to clause set
Petal branch (Y not satisfied) Add $S_1 \setminus Y, \ldots, S_k \setminus Y$ to clause set

Afterwards we can remove redundant clauses, in particular, $S_1, \ldots, S_k$

Algorithm

The branching algorithm is very carefully defined. Both order of cases and the required sizes of the sunflowers are **very important** for the analysis to work

Alg(V, $\mathcal{C}$)

While there are clauses $C, C' \in \mathcal{C}$ with $C \subsetneq C'$

- $\mathcal{C} \leftarrow \mathcal{C} \setminus \{C'\}$ // C implies C'

If $\{x_i\} \in \mathcal{C}$ and $\{\neg x_i\} \in \mathcal{C}$ for some $x_i \in V$

- return No

If there is a small sunflower $Y, S_1, \ldots, S_k$ with $k \geq 1/\delta$

- return Alg(V, $\mathcal{C} \cup \{Y\}$) or Alg(V, $\mathcal{C} \cup \{S_1 \setminus Y, \ldots, S_k \setminus Y\}$)

else if there is a medium sunflower $Y, S_1, \ldots, S_k$ with $k \geq 1/\delta$

- return Alg(V, $\mathcal{C} \cup \{Y\}$) or Alg(V, $\mathcal{C} \cup \{S_1 \setminus Y, \ldots, S_k \setminus Y\}$)

else if there is a large sunflower $Y, S_1, \ldots, S_k$ with $k \geq \mathbf{1/\delta^2}$

- return Alg(V, $\mathcal{C} \cup \{Y\}$) or Alg(V, $\mathcal{C} \cup \{S_1 \setminus Y, \ldots, S_k \setminus Y\}$)

else // at most $6/\delta^2 \cdot n$ clauses

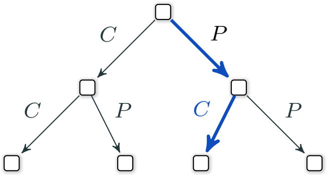
- return $\mathrm{Sparse}_{\delta^3}(V, \mathcal{C})$

An analysis that almost works

If we could show **(unfortunately this is not true)** that the depth of the enumeration tree, i.e., the **maximum length of a root-leaf path**, is bounded by δn, then the overal running time would be

$$n^{O(1)} \cdot 2^{\delta n} \cdot 2^{\delta^3 (n + 6/\delta^2 \cdot n)} \leq n^{O(1)} \cdot 2^{8 \delta n} .$$

So for any $\epsilon > 0$, to solve 3-SAT in $O(2^{\epsilon n})$ we could run the algorithm with $\delta = \epsilon/9$.



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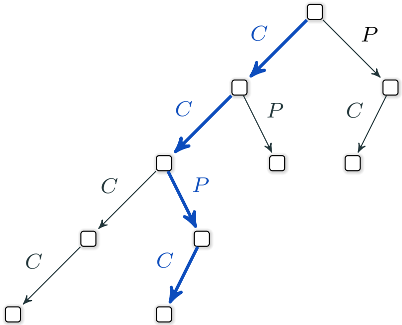

Instead we will prove that while paths can be fairly long, on each path the number of petal branches is very small. As we will see, this suffices to bound the number of leafs.

Simple bounds

Fix one root-leaf path P in the enumeration tree

Reminder

Types of branches based on sunflower $Y, S_1, \dots, S_k$:

Small branch: $|Y| = 1,\ |S_i| = 2,\ k \geq 1/\delta$

Medium branch: $|Y| = 2,\ |S_i| = 3,\ k \geq 1/\delta$

Large branch: $|Y| = 1,\ |S_i| = 3,\ k \geq 1/\delta^2$

Core branch: Alg$(V, \mathcal{C} \cup \{Y\})$

Petal branch: Alg$(V, \mathcal{C} \cup \{S_1 \setminus Y, \dots, S_k \setminus Y\})$

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Petal branch			

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Small and large core branches: Each fixes one variable (by inserting a singleton clause)

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	Small	Medium	Large
Core branch	$\leq n$		$\leq n$
Petal branch			

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Small and medium petal branches: Each fixes $1/\delta$ variables (by inserting singleton clauses)

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For the other bounds we need some good ideas...

New 2-clauses

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We take a closer look at **new** 2-clauses: 2-clauses that are added during path P, i.e., were not in the original clause set.

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Invariant

There is no small sunflower $Y, S_1, \ldots, S_k$ with $k > 2/\delta$ consisting only of new 2-clauses

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- While there is such a sunflower with $k \geq 1/\delta$, we only make small branches that do not add more 2-clauses
- Consider such a sunflowers of size $< 1/\delta$. **Medium core branch** adds ≤ 1 petal. **Large petal branch** adds $\leq 1/\delta$ petals: otherwise, there is medium branch that has priority. No other branches add 2-clauses

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Lemma

Throughout a root-leaf path, $\leq 2n/\delta + 2/\delta \cdot n \leq 4/\delta \cdot n$ new 2-clauses are added

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At the leaf there are $\leq 2n/\delta$ many 2-clauses. All new 2-clauses beyond this, also need to be removed. Only new singleton clauses can remove new 2-clauses and each only $\leq 2/\delta$ many (by invariant).

Remaining bounds

Small branch: $|Y| = 1,\ |S_i| = 2,\ k \geq 1/\delta$
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	Small	Medium	Large	Total
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Petal branch	$\leq \delta n$	$\leq \delta n$	$\leq 4\delta n$	$\leq 6\delta \cdot n$

Running time

Number of root-leaf paths or simply leafs

$$\leq \sum_{i=0}^{\lfloor 6/\delta \cdot n \rfloor} \sum_{k=0}^{\lfloor 6\delta n \rfloor} \binom{i+k}{k}$$

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$$\binom{a}{b} \leq \left(\frac{ea}{b}\right)^b \quad \leftarrow \text{monotone for } \mathbf{b} \in [\mathbf{0}, \mathbf{ea/2}]$$

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Thus, each element in the sum is

$$\binom{i+k}{k} \leq \left(\frac{e(i+k)}{k}\right)^{k} \leq \left(\frac{e \cdot 7/\delta \cdot n}{6\delta n}\right)^{6\delta n} \leq \left(\frac{3}{\delta^2}\right)^{6\delta n} \leq 2^{12\log(3/\delta)\cdot\delta\cdot n}$$

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Total running time:

$$n^{O(1)} \cdot 2^{12 \log(3/\delta) \cdot \delta \cdot n} \cdot 2^{\delta^3 (n + 6n/\delta^2)} \leq 2^{19 \log(3/\delta) \cdot \delta \cdot n}$$

Running time

Number of root-leaf paths or simply leafs

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Total running time:

$$n^{O(1)} \cdot 2^{12\log(3/\delta)\cdot\delta\cdot n} \cdot 2^{\delta^3(n+6n/\delta^2)} \leq 2^{19\log(3/\delta)\cdot\delta\cdot n} \leq \mathbf{O(2^{\epsilon n})}$$

Set $\delta > 0$ small enough so that $19\log(3/\delta)\cdot\delta < \epsilon$