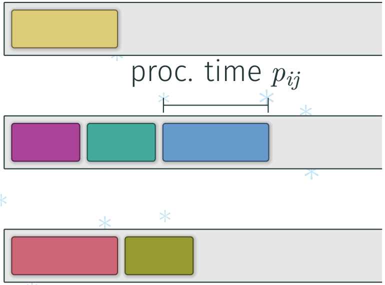
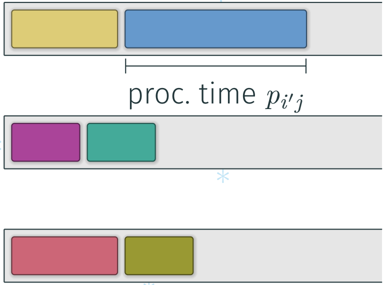
Better Trees for Santa Claus

Étienne Bamas Lars Rohwedder

MAPSP'24


proc. time p_{ij}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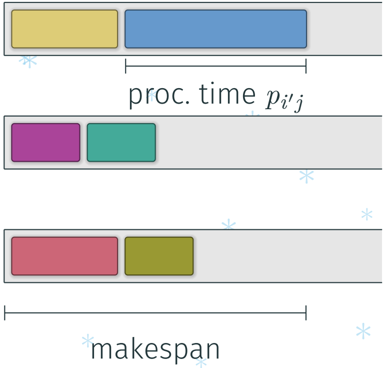
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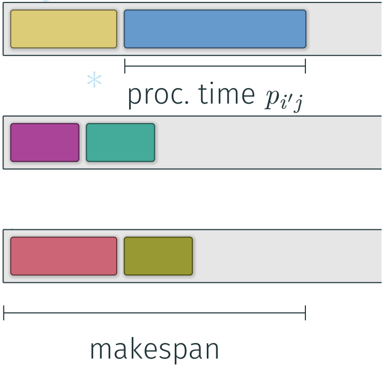
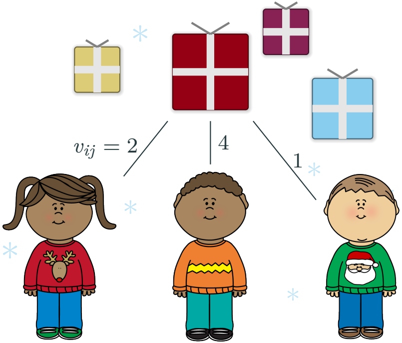
makespan

Makespan scheduling



Max-min fair allocation (Santa Claus problem)



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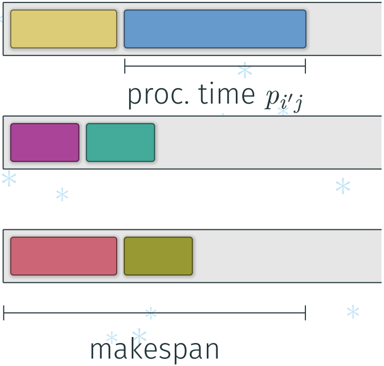

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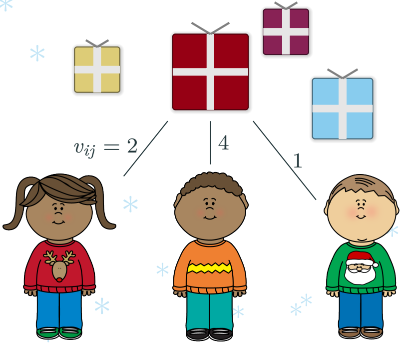

Big open question:

$(2 - 1/\alpha)\mathrm{OPT}$?

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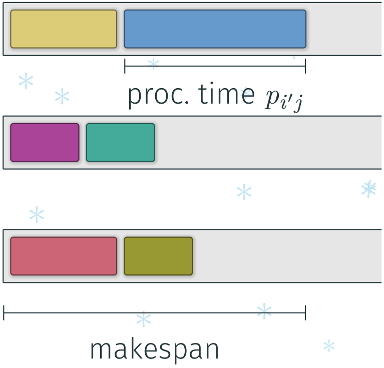

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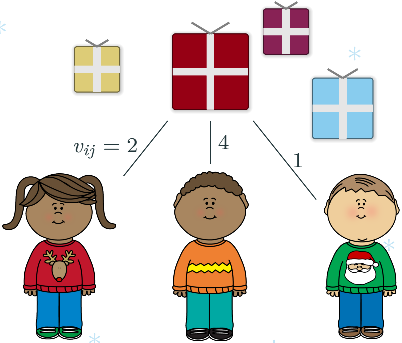

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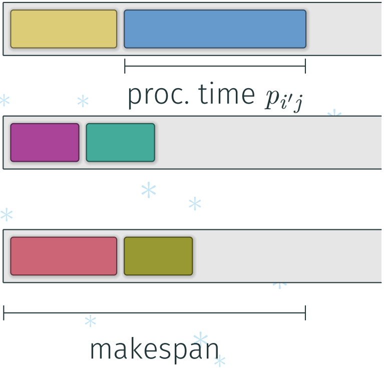

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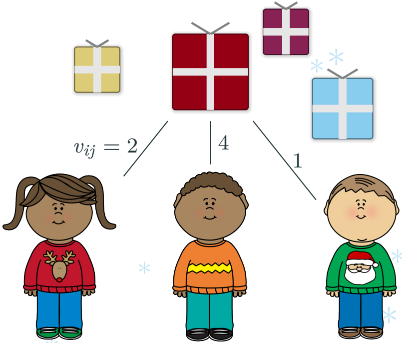

Big open question: Bamas, Lindermayr, Megow, **R.**, Schlöter [SODA'24]

$(2 - 1/\alpha)\mathrm{OPT}$? $\Longrightarrow$ $\alpha \cdot \mathrm{OPT}$?

Status of the Santa Claus problem

Open: constant approximation

Chakrabarty, Chuzhoy, Khanna [FOCS'09]: $(\log n)^{O(1)}$-approximation (quasi-polynomial time)

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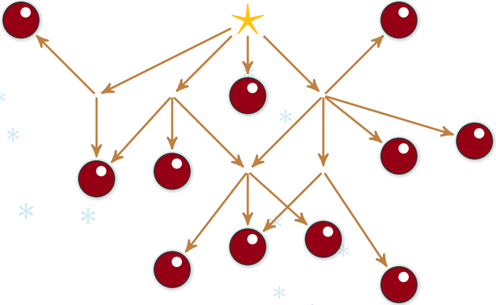
Our contribution: $(\log\log n)^{O(1)}$-approximation for max-min degree arborescence (quasi-polynomial time)

Max-min degree arborescence

Input: directed graph $G = (V, E)$, source ★$\in V$ and sinks ●$\subseteq V \setminus \{s\}$

Output: arborescence $A \subseteq G$ with root ★, leaves in ●, and outdegree k for all non-sink vertices

Goal: maximize k



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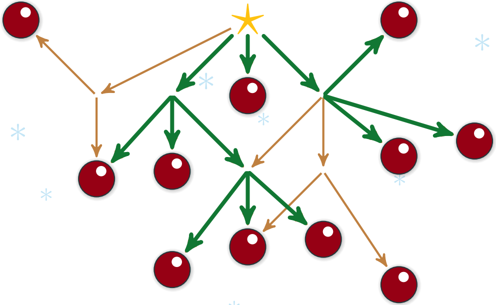
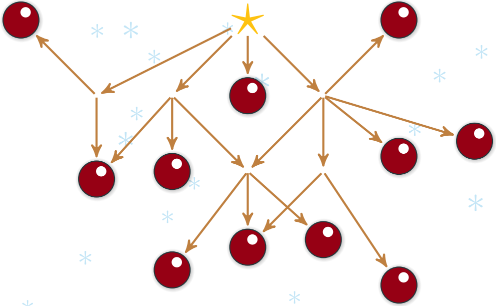


Randomized rounding by Bateni, Charikar, Guruswami

randomly select children of root, then recurse on children...



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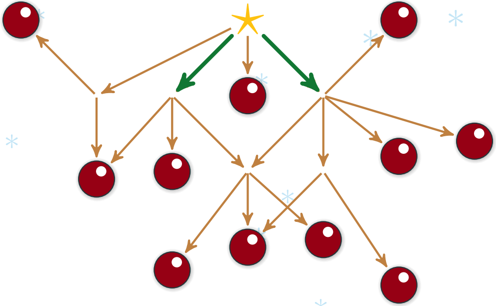

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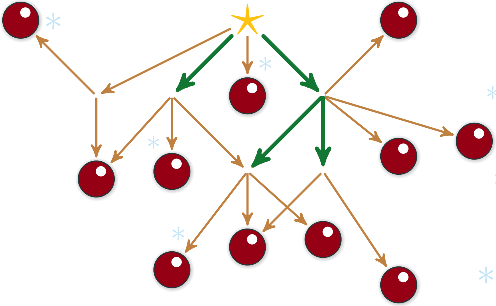

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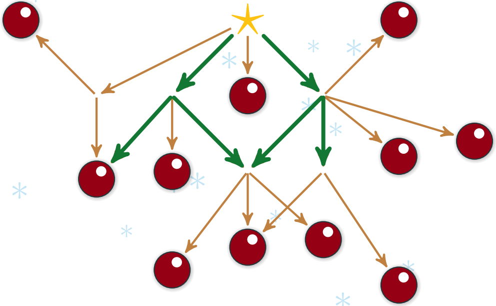

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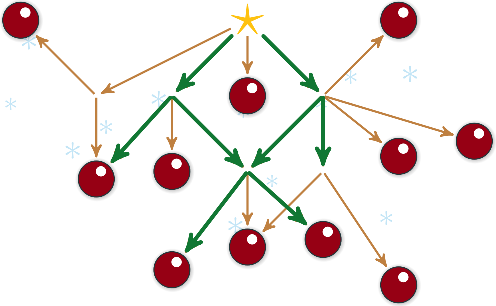

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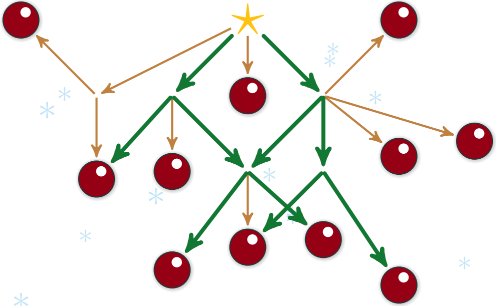

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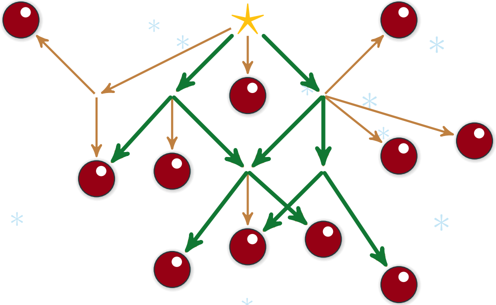

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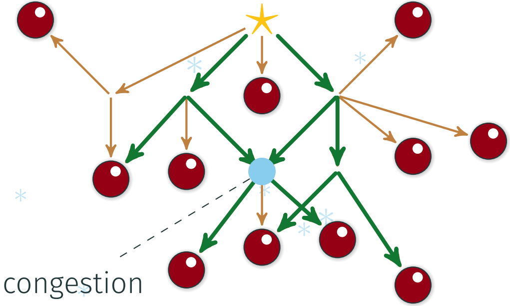

Bound on congestion can be turned into approximation rate

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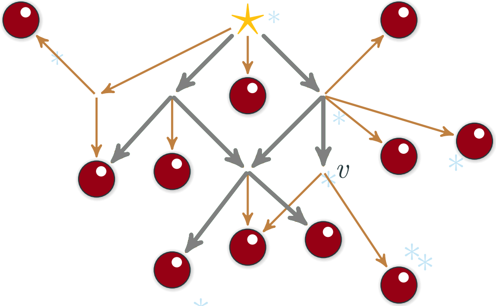

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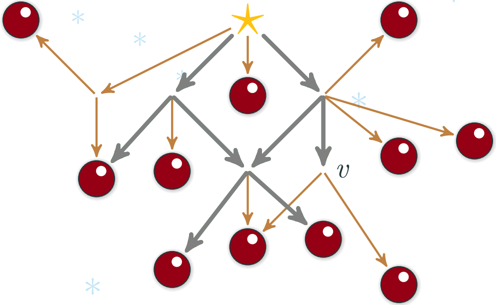

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Naive: at vertex v, for $e \in \delta^+(v)$ use probability $x_e/x(\delta^-(v))$.

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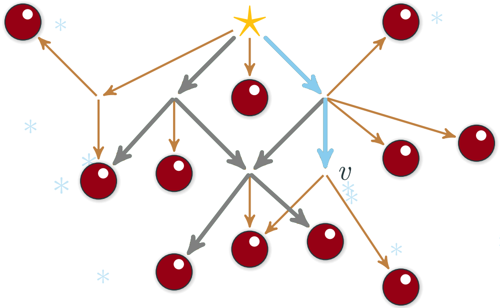

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Smart: use Sherali-Adams hierarchy (on naive relaxation) and condition on path to v
$\rightsquigarrow$ congestion $O(\log n)$ w.h.p.

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Works if initial solution has "locally low congestion" (new concept).

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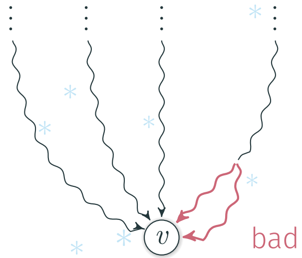

Using Lovász Local Lemma (LLL) we can assume that congestion comes from paths that are disjoint on last $O(\log \log n)$ edges*

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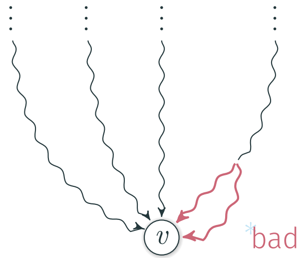

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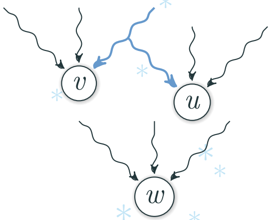

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Limited dependencies $\rightsquigarrow$ LLL applicable

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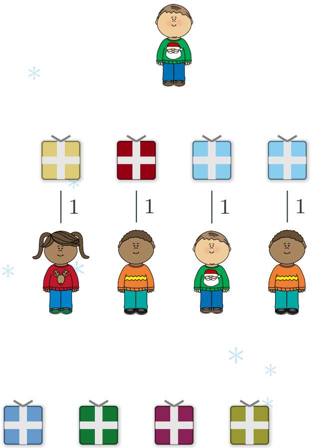
Future: possible implications for Santa Claus

Algorithm by Chakrabarty, Chuzhoy, Khanna

Suppose we build a solution by iteratively growing set of happy children (w.l.o.g. $\Leftrightarrow$ that have value at least 1).



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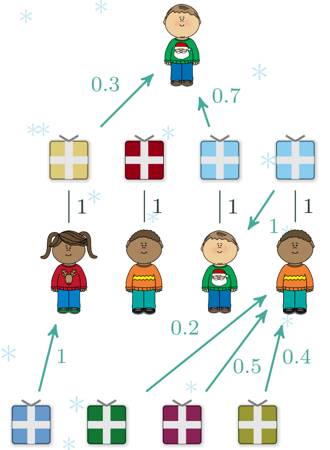

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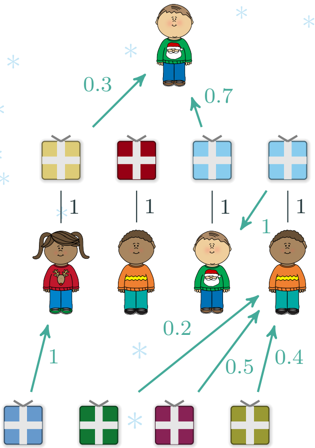

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still $(\log n)^{O(1)}$-approximation possible, (*) can be established with loss of $(\log n)^{O(1)}$
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