Maths : Differential equations cheat sheet

1 First order

Let be I an interval of $\mathbb{R}$.

$$(E) : y' = a(x)y + b(x)$$

with $a, b \in \mathbb{R}^I$.

1.1 Homogenous equation

$$(E_0) : y' = a(x)y$$

The solution set of (E_0) is, with $A \in \mathcal{D}(I) \mid A' = a$:

$$S_0 = \left\{x \longmapsto \lambda \mathrm{e}^{A(x)} \mid \lambda \in \mathbb{R}\right\}$$

1.2 Particular solution

1.2.1 Constant variation

Let be $\lambda \in \mathcal{D}(I)$.
Let be $f(x) = \lambda(x)\mathrm{e}^{A(x)}$. $f \in \mathcal{D}(I)$ by product and composition.

$$\begin{aligned} f \text{ solution of } (E) \quad &\Leftrightarrow \quad f' = a(x)f(x) + b(x) \\ &\Leftrightarrow \quad \lambda'(x)\mathrm{e}^{A(x)} + a(x)\lambda(x)\mathrm{e}^{A(x)} = a(x)f(x) + b(x) \\ &\Leftrightarrow \quad \lambda'(x) = b(x)\mathrm{e}^{-A(x)} \end{aligned}$$

Therefore $f(x) = \lambda(x)\mathrm{e}^{A(x)}$ is solution of $(E) \;\Leftrightarrow\; \lambda'(x) = b(x)\mathrm{e}^{-A(x)}$.

1.2.2 Constant coef and exponential second member

$$(E_1) : y' - ay = \lambda \mathrm{e}^{\alpha x}$$

If $\alpha = -a$, then $x \longmapsto Cx\mathrm{e}^{\alpha x}$, $C \in \mathbb{R}$ is solution ;
Else $(\alpha \neq -a)$, $x \longmapsto C\mathrm{e}^{\alpha x}$ is solution.

1.3 Full solutions

The solution set of (E) is, with f a solution,

$$\{f + h_0 \mid h_0 \in S_0\}$$

So the solutions on I are, for $x_0 \in I$,

$$\left\{x \longmapsto \lambda \mathrm{e}^{A(x)} + \mathrm{e}^{A(x)} \int_{x_0}^{x} b(t)\mathrm{e}^{-A(t)}\mathrm{d}t \;\middle|\; \lambda \in \mathbb{R}\right\}$$

2 Second order with constant coefficients

$$(E') : ay'' + by' + cy = f(x)$$

with : $(a, b, c) \in \mathbb{C}^* \times \mathbb{C}$ and $f \in \mathbb{C}^{\mathbb{R}}$.

2.1 Homogenous equation

$$(E'_0) : ay'' + by' + cy = 0$$
$$(E'_c) : ax^2 + bx + c = 0$$

If (E'_c) has one double root r, the solution set of (E'_0) is :

$$\{x \longmapsto (\lambda x + \mu)\mathrm{e}^{rx} \mid \lambda, \mu \in \mathbb{C}\}$$

Else if (E'_c) has two roots $(r_1, r_2) \in \mathbb{C}$, the solution set of (E'_0) is :

$$\{x \longmapsto \lambda \mathrm{e}^{r_1 x} + \mu \mathrm{e}^{r_2 x} \mid \lambda, \mu \in \mathbb{C}\}$$

If $(r_1, r_2) \in \mathbb{C}\backslash\mathbb{R}$, then $\exists(\alpha, \beta) \in \mathbb{R} \mid r_1 = \overline{r_2} = \alpha + i\beta$, and the real solution set for (E'_0) is :

$$\{x \longmapsto (\lambda \sin(\beta x) + \mu \cos(\beta x))\mathrm{e}^{\alpha x} \mid \lambda, \mu \in \mathbb{R}\}$$

2.2 Particular solution

With S_0 the solution set of (E'_0), and g a solution of (E'), the solution set of (E') is of the form :

$$\{g + h_0 \mid h_0 \in S_0\}$$

2.2.1 Exponential second member

$$(E'_1) : ay'' + by' + cy = A\mathrm{e}^{\alpha x}$$

If α is not solution of (E'_c), then $x \longmapsto C\mathrm{e}^{\alpha x}$, $C \in \mathbb{R}$ is solution of (E') ;
Else if α is a simple root of (E'_c), then $x \longmapsto Cx\mathrm{e}^{\alpha x}$, $C \in \mathbb{R}$ is solution of (E') ;
Else if α is the double root of (E'_c), then $x \longmapsto Cx^2\mathrm{e}^{\alpha x}$, $C \in \mathbb{R}$ is solution of (E') ;