K-Algebra

Vector space

A vector space V over a field $\mathbb{K}$ is a set of vectors, together with two operations:

- Vector addition: $(+) : V \times V \to V$
- Scalar multiplication: $(\cdot) : \mathbb{K} \times V \to V$

such that

- The set V forms an abelian group under the operation of vector addition
  - Associativity: $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V : (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - Commutativity: $\forall \mathbf{u}, \mathbf{v} \in V : \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - Identity: $\exists \mathbf{0} \in V$ s.t. $\forall \mathbf{v} \in V : \mathbf{v} + \mathbf{0} = \mathbf{v}$
  - Inverse: $\forall \mathbf{v} \in V, \exists (-\mathbf{v}) \in V$ s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
- Scalar multiplication distributes over vector addition: $\forall a \in \mathbb{K}, \mathbf{u}, \mathbf{v} \in V : a \cdot (\mathbf{u} + \mathbf{v}) = (a \cdot \mathbf{u}) + (a \cdot \mathbf{v})$
- Scalar multiplication distributes over addition in $\mathbb{K}$: $\forall a, b \in \mathbb{K}, \mathbf{v} \in V : (a + b) \cdot \mathbf{v} = (a \cdot \mathbf{v}) + (b \cdot \mathbf{v})$
- Scalar multiplication is *compatible* with multiplication in $\mathbb{K}$: $\forall a, b \in \mathbb{K}, \mathbf{v} \in V : a \cdot (b \cdot \mathbf{v}) = (ab) \cdot \mathbf{v}$
- The following identity law holds for scalar multiplication: $\forall \mathbf{v} \in V : 1_{\mathbb{K}} \cdot \mathbf{v} = \mathbf{v}$ (where $1_{\mathbb{K}}$ is the multiplicative identity in $\mathbb{K}$)

Compact definition

These last four axioms can be expressed more compactly, by instead requiring a ring homomorphism

$$f : \mathbb{K} \to \mathrm{End}(V)$$

from $\mathbb{K}$ to the *endomorphism ring* formed by the group V. This ring $\mathrm{End}(V)$ is defined as

1. The set of endomorphisms of V
2. Addition, defined as pointwise addition of functions; $[\varphi + \psi](\mathbf{v}) = \varphi(\mathbf{v}) + \psi(\mathbf{v})$ and
3. Multiplication, defined as function composition; $\varphi\psi = \varphi \circ \psi$.

We then define scalar multiplication as

$$a \cdot \mathbf{v} = [f(a)](\mathbf{v})$$

for which we will use f_a to denote $f(a)$, so that we can write, e.g., $a \cdot \mathbf{v} = f_a(\mathbf{v})$

f is a ring homomorphism. This means that

$$(1)\ f_{ab} = f_a \circ f_b \qquad (2)\ f_{a+b} = f_a + f_b \qquad (3)\ f_1 = 1$$

Given any $a \in \mathbb{K}, f(a)$ is a homomorphism $V \to V$

$$f_a(\mathbf{u} + \mathbf{v}) = f_a(\mathbf{u}) + f_a(\mathbf{v})$$

Generalization to rings

The notion of a *module* is a generalization of vector spaces, in which the scalars are elements of any ring (i.e., not necessarily a field).

Linear map

If V and W are vector spaces defined over $\mathbb{K}$, a function $f : V \to W$ is a *linear map* if it preserves the vector space structure under the two operations, i.e,

1. Vector addition: $\forall \mathbf{u}, \mathbf{v} \in V : f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$; and
2. Scalar multiplication: $\forall a \in \mathbb{K}, \mathbf{v} \in V : f(a \cdot \mathbf{v}) = a \cdot f(\mathbf{v})$.

In other words, a linear map is a *homomorphism* between vector spaces.

Bilinear map

Given three vector spaces, U, V, and W, all defined over a a field $\mathbb{K}$, a *bilinear* map is a function $b : U \times V \to W$ which is linear in both arguments. That is, b is linear with respect to;

- Addition, in the first argument: $\forall \mathbf{u}, \mathbf{x} \in U, \mathbf{v} \in V : b(\mathbf{u} + \mathbf{x}, \mathbf{v}) = b(\mathbf{u}, \mathbf{v}) + b(\mathbf{x}, \mathbf{v})$
- Addition, in the second argument: $\forall \mathbf{u} \in U, \mathbf{v}, \mathbf{y} \in V : b(\mathbf{u}, \mathbf{v} + \mathbf{y}) = b(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{y})$
- Scalar multiplication: $\forall c \in \mathbb{K}, \mathbf{u} \in U, \mathbf{v} \in V : b(c \cdot \mathbf{u}, \mathbf{v}) = c \cdot b(\mathbf{u}, \mathbf{v}) = b(\mathbf{u}, c \cdot \mathbf{v})$

Algebra over a field

If $\mathbb{K}$ is a field, and V a vector space over $\mathbb{K}$ equipped with a bilinear map $(\cdot) : V \times V \to V$, then V is called an *algebra over* $\mathbb{K}$ (or K-algebra for short).

TODO