

Vector spaces

A vector space defined over a field \mathbb{K} is a set V of elements, known as *vectors*, together with two operations;

- vector addition, $+: V \times V \rightarrow V$; and
- scalar multiplication, $\cdot: \mathbb{K} \times V \rightarrow V$;

such that the following eight axioms are satisfied:

1. Addition of vectors is associative:

$$\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V : (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

2. Addition of vectors is commutative:

$$\forall \mathbf{u}, \mathbf{v} \in V : \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

3. There exists an identity vector in V :

$$\exists \mathbf{0} \in V \text{ s.t. } \forall \mathbf{v} \in V : \mathbf{v} + \mathbf{0} = \mathbf{v}$$

4. Every vector has an additive inverse:

$$\forall \mathbf{v} \in V, \exists (-\mathbf{v}) \in V \text{ s.t. } \mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

5. Scalar multiplication distributes over vector addition:

$$\forall a \in \mathbb{K}, \mathbf{u}, \mathbf{v} \in V : a \cdot (\mathbf{u} + \mathbf{v}) = (a \cdot \mathbf{u}) + (a \cdot \mathbf{v})$$

6. Scalar multiplication is *compatible* with multiplication in \mathbb{K} :

$$\forall a, b \in \mathbb{K}, \mathbf{v} \in V : a \cdot (b \cdot \mathbf{v}) = (ab) \cdot \mathbf{v}$$

7. Scalar multiplication satisfies the identity law:

$$\forall \mathbf{v} \in V : 1_{\mathbb{K}} \cdot \mathbf{v} = \mathbf{v} \text{ (where } 1_{\mathbb{K}} \text{ is the multiplicative identity in } \mathbb{K})$$

8. Scalar multiplication distributes over addition:

$$\forall a, b \in \mathbb{K}, \mathbf{v} \in V : (a + b) \cdot \mathbf{v} = (a \cdot \mathbf{v}) + (b \cdot \mathbf{v})$$

Compact definition

The first four axioms can be replaced by saying that the set V forms an abelian group under the operation of vector addition, with the zero vector ($\mathbf{0}$) acting as the identity element. Axiom 5–8 are identical to the requirement that there exists a ring homomorphism

$$f : \mathbb{K} \rightarrow \text{End}(V)$$

where $\text{End}(V)$ is the endomorphism ring induced by the the group V , that is:

1. The set of endomorphisms; $\varphi : V \rightarrow V$
2. Addition, defined as pointwise addition of functions; $[\varphi + \psi](\mathbf{v}) = \varphi(\mathbf{v}) + \psi(\mathbf{v})$
3. Multiplication, defined as function composition; $\varphi\psi = \varphi \circ \psi$

We can then define scalar multiplication in terms of this homomorphism, as

$$a \cdot \mathbf{v} = [f(a)](\mathbf{v}).$$

To simplify notation, we allow ourselves to write, e.g., f_k to mean $f(k)$. Now, as it turns out, law 5–7 above correspond directly to the ring homomorphism axioms:

$$f_{a+b} = f_a + f_b \quad (5) \quad f_{ab} = f_a \circ f_b \quad (6) \quad f_1 = \text{id} \quad (7)$$

From this first identity, and by the definition of addition in $\text{End}(V)$, we get

$$f_{a+b}(\mathbf{v}) = [f_a + f_b](\mathbf{v}) = f_a(\mathbf{v}) + f_b(\mathbf{v})$$

which gives us the eighth and last axiom as well.

So, to summarize, a vector space is:

- A field \mathbb{K}
- An abelian group V
- A homomorphism $f : \mathbb{K} \rightarrow \text{End}(V)$

Generalization to rings

The notion of a *module* is a generalization of vector spaces, in which the scalars are elements of any unital ring (i.e., not necessarily a field).

Linear map

Bilinear map

Algebra over a field