Vector spaces

A *vector space* defined over a field $\mathbb{K}$ is a set V of elements, known as *vectors*, together with two operations;

- vector addition, $+ : V \times V \to V$; and
- scalar multiplication, $\cdot : \mathbb{K} \times V \to V$;

such that the following eight axioms are satisfied:

1. Addition of vectors is associative:
$$\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V : (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$
2. Addition of vectors is commutative:
$$\forall \mathbf{u}, \mathbf{v} \in V : \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$
3. There exists an identity vector in V:
$$\exists \mathbf{0} \in V \text{ s.t. } \forall \mathbf{v} \in V : \mathbf{v} + \mathbf{0} = \mathbf{v}$$
4. Every vector has an additive inverse:
$$\forall \mathbf{v} \in V, \exists (-\mathbf{v}) \in V \text{ s.t. } \mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$
5. Scalar multiplication distributes over vector addition:
$$\forall a \in \mathbb{K}, \mathbf{u}, \mathbf{v} \in V : a \cdot (\mathbf{u} + \mathbf{v}) = (a \cdot \mathbf{u}) + (a \cdot \mathbf{v})$$
6. Scalar multiplication is *compatible* with multiplication in $\mathbb{K}$:
$$\forall a, b \in \mathbb{K}, \mathbf{v} \in V : a \cdot (b \cdot \mathbf{v}) = (ab) \cdot \mathbf{v}$$
7. Scalar multiplication satisfies the identity law:
$$\forall \mathbf{v} \in V : 1_{\mathbb{K}} \cdot \mathbf{v} = \mathbf{v} \text{ (where } 1_{\mathbb{K}} \text{ is the multiplicative identity in } \mathbb{K})$$
8. Scalar multiplication distributes over addition:
$$\forall a, b \in \mathbb{K}, \mathbf{v} \in V : (a + b) \cdot \mathbf{v} = (a \cdot \mathbf{v}) + (b \cdot \mathbf{v})$$

Compact definition

The first four axioms can be replaced by stating that the set V forms an abelian group under the operation of vector addition, with the zero vector ($\mathbf{0}$) as the identity element. Axiom 5–8 are identical to the requirement that there exists a ring homomorphism
$$f : \mathbb{K} \to \mathrm{End}(V)$$
where $\mathrm{End}(V)$ is the endomorphism ring induced by the the group V, that is:

1. The set of endomorphisms; $\varphi : V \to V$
2. Addition, defined as pointwise addition of functions; $[\varphi + \psi](\mathbf{v}) = \varphi(\mathbf{v}) + \psi(\mathbf{v})$
3. Multiplication, defined as function composition; $\varphi\psi = \varphi \circ \psi$

We can then define scalar multiplication in terms of this homomorphism, as
$$a \cdot \mathbf{v} = [f(a)](\mathbf{v}).$$

To simplify notation, we allow ourselves to write, e.g., f_k to mean $f(k)$. Now, as it turns out, law 5–7 above correspond directly to the ring homomorphism axioms:
$$f_{a+b} = f_a + f_b \ (5) \qquad f_{ab} = f_a \circ f_b \ (6) \qquad f_1 = \mathrm{id} \ (7)$$

From this first identity, and by the definition of addition in $\mathrm{End}(V)$, we get
$$f_{a+b}(\mathbf{v}) = [f_a + f_b](\mathbf{v}) = f_a(\mathbf{v}) + f_b(\mathbf{v})$$
which gives us the eighth and last axiom as well.

So, to summarize, a vector space is:

- A field $\mathbb{K}$
- An abelian group V
- A homomorphism $f : \mathbb{K} \to \mathrm{End}(V)$

Generalization to rings

The notion of a *module* is a generalization of vector spaces, in which the scalars are elements of any unital ring (i.e., not necessarily a field).

Linear map

If V and W are vector spaces defined over $\mathbb{K}$, a function $f : V \to W$ is a *linear map* if it preserves the vector space structure under the two operations; i.e,

1. Vector addition: $\forall \mathbf{u}, \mathbf{v} \in V : f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$; and
2. Scalar multiplication: $\forall a \in \mathbb{K}, \mathbf{v} \in V : f(a \cdot \mathbf{v}) = a \cdot f(\mathbf{v})$.

In other words, a linear map is a homomorphism between vector spaces.

Bilinear map

Given three vector spaces, U, V, and W, all defined over a a field $\mathbb{K}$, a *bilinear* map is a function $b : U \times V \to W$ which is linear in both arguments. That is, b is linear with respect to;

- Addition, in the first argument: $\forall \mathbf{u}, \mathbf{x} \in U, \mathbf{v} \in V : b(\mathbf{u} + \mathbf{x}, \mathbf{v}) = b(\mathbf{u}, \mathbf{v}) + b(\mathbf{x}, \mathbf{v})$
- Addition, in the second argument: $\forall \mathbf{u} \in U, \mathbf{v}, \mathbf{y} \in V : b(\mathbf{u}, \mathbf{v} + \mathbf{y}) = b(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{y})$
- Scalar multiplication: $\forall c \in \mathbb{K}, \mathbf{u} \in U, \mathbf{v} \in V : b(c \cdot \mathbf{u}, \mathbf{v}) = c \cdot b(\mathbf{u}, \mathbf{v}) = b(\mathbf{u}, c \cdot \mathbf{v})$

Algebra over a field

If $\mathbb{K}$ is a field, and V a vector space over $\mathbb{K}$ equipped with a bilinear map $\cdot : V \times V \to V$, then V is called an *algebra over* $\mathbb{K}$, or K-algebra for short.

Example

The complex numbers form an algebra over $\mathbb{R}$, with the bilinear map given as multiplication of two complex numbers, defined as:

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$