

## Vector spaces

A vector space defined over a field  $\mathbb{K}$  is a set  $V$  of elements, known as *vectors*, together with two operations;

- vector addition,  $+: V \times V \rightarrow V$ ; and
- scalar multiplication,  $\cdot: \mathbb{K} \times V \rightarrow V$ ;

such that the following eight axioms are satisfied:

1. Addition of vectors is associative:

$$\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V : (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

2. Addition of vectors is commutative:

$$\forall \mathbf{u}, \mathbf{v} \in V : \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

3. There exists an identity vector in  $V$ :

$$\exists \mathbf{0} \in V \text{ s.t. } \forall \mathbf{v} \in V : \mathbf{v} + \mathbf{0} = \mathbf{v}$$

4. Every vector has an additive inverse:

$$\forall \mathbf{v} \in V, \exists (-\mathbf{v}) \in V \text{ s.t. } \mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

5. Scalar multiplication distributes over vector addition:

$$\forall a \in \mathbb{K}, \mathbf{u}, \mathbf{v} \in V : a \cdot (\mathbf{u} + \mathbf{v}) = (a \cdot \mathbf{u}) + (a \cdot \mathbf{v})$$

6. Scalar multiplication is *compatible* with multiplication in  $\mathbb{K}$ :

$$\forall a, b \in \mathbb{K}, \mathbf{v} \in V : a \cdot (b \cdot \mathbf{v}) = (ab) \cdot \mathbf{v}$$

7. Scalar multiplication satisfies the identity law:

$$\forall \mathbf{v} \in V : 1_{\mathbb{K}} \cdot \mathbf{v} = \mathbf{v} \text{ (where } 1_{\mathbb{K}} \text{ is the multiplicative identity in } \mathbb{K})$$

8. Scalar multiplication distributes over addition:

$$\forall a, b \in \mathbb{K}, \mathbf{v} \in V : (a + b) \cdot \mathbf{v} = (a \cdot \mathbf{v}) + (b \cdot \mathbf{v})$$

### Compact definition

The first four axioms can be replaced by saying that the set  $V$  forms an abelian group under the operation of vector addition, with the zero vector ( $\mathbf{0}$ ) acting as the identity element. Axiom 5–8 are identical to the requirement that there exists a ring homomorphism

$$f : \mathbb{K} \rightarrow \text{End}(V)$$

where  $\text{End}(V)$  is the endomorphism ring induced by the the group  $V$ , that is:

1. The set of endomorphisms;  $\varphi : V \rightarrow V$
2. Addition, defined as pointwise addition of functions;  $[\varphi + \psi](\mathbf{v}) = \varphi(\mathbf{v}) + \psi(\mathbf{v})$
3. Multiplication, defined as function composition;  $\varphi\psi = \varphi \circ \psi$

We can then define scalar multiplication in terms of this homomorphism, as

$$a \cdot \mathbf{v} = [f(a)](\mathbf{v}).$$

To simplify notation, we allow ourselves to write, e.g.,  $f_k$  to mean  $f(k)$ . Now, as it turns out, law 5–7 above correspond directly to the ring homomorphism axioms:

$$f_{a+b} = f_a + f_b \quad (5) \quad f_{ab} = f_a \circ f_b \quad (6) \quad f_1 = \text{id} \quad (7)$$

From this first identity, and by the definition of addition in  $\text{End}(V)$ , we get

$$f_{a+b}(\mathbf{v}) = [f_a + f_b](\mathbf{v}) = f_a(\mathbf{v}) + f_b(\mathbf{v})$$

which gives us the eighth and last axiom as well.

So, to summarize, a vector space is:

- A field  $\mathbb{K}$
- An abelian group  $V$
- A homomorphism  $f : \mathbb{K} \rightarrow \text{End}(V)$

## Generalization to rings

The notion of a *module* is a generalization of vector spaces, in which the scalars are elements of any unital ring (i.e., not necessarily a field).

## Linear map

If  $V$  and  $W$  are vector spaces defined over  $\mathbb{K}$ , a function  $f : V \rightarrow W$  is a *linear map* if it preserves the vector space structure under the two operations; i.e.,

1. Vector addition:  $\forall \mathbf{u}, \mathbf{v} \in V : f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ ; and
2. Scalar multiplication:  $\forall a \in \mathbb{K}, \mathbf{v} \in V : f(a \cdot \mathbf{v}) = a \cdot f(\mathbf{v})$ .

In other words, a linear map is a homomorphism between vector spaces.

## Bilinear map

Given three vector spaces,  $U, V$ , and  $W$ , all defined over a field  $\mathbb{K}$ , a *bilinear map* is a function  $b : U \times V \rightarrow W$  which is linear in both arguments. That is,  $b$  is linear with respect to;

- Addition, in the first argument:  $\forall \mathbf{u}, \mathbf{x} \in U, \mathbf{v} \in V : b(\mathbf{u} + \mathbf{x}, \mathbf{v}) = b(\mathbf{u}, \mathbf{v}) + b(\mathbf{x}, \mathbf{v})$
- Addition, in the second argument:  $\forall \mathbf{u} \in U, \mathbf{v}, \mathbf{y} \in V : b(\mathbf{u}, \mathbf{v} + \mathbf{y}) = b(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{y})$
- Scalar multiplication:  $\forall c \in \mathbb{K}, \mathbf{u} \in U, \mathbf{v} \in V : b(c \cdot \mathbf{u}, \mathbf{v}) = c \cdot b(\mathbf{u}, \mathbf{v}) = b(\mathbf{u}, c \cdot \mathbf{v})$

## Algebra over a field

If  $\mathbb{K}$  is a field, and  $V$  a vector space over  $\mathbb{K}$  equipped with a bilinear map  $\cdot : V \times V \rightarrow V$ , then  $V$  is called an *algebra over  $\mathbb{K}$* , or  $K$ -algebra for short.