K-Algebras

Vector spaces

A vector space V over a field $\mathbb{K}$ is a set of vectors, together with two operations;

- vector addition; $(+) : V \times V \to V$; and
- scalar multiplication; $(\cdot) : \mathbb{K} \times V \to V$;

such that

- The set V forms an abelian group under vector addition;
  - Associativity; $\forall u, v, w \in V : (u + v) + w = u + (v + w)$
  - Commutativity; $\forall u, v \in V : u + v = v + u$
  - Identity; $\exists 0 \in V$ s.t. $\forall v \in V : v + 0 = v$
  - Inverse; $\forall v \in V, \exists (-v) \in V$ s.t. $v + (-v) = 0$
- Scalar multiplication distributes over vector addition; $\forall a \in \mathbb{K}, \vec{u}, \vec{v} \in V : a \cdot (\vec{u} + \vec{v}) = (a \cdot \vec{u}) + (a \cdot \vec{v})$
- Scalar multiplication distributes over addition in $\mathbb{K}$; $\forall a, b \in \mathbb{K}, \vec{v} \in V : (a + b) \cdot \vec{v} = (a \cdot \vec{v}) + (b \cdot \vec{v})$
- Scalar multiplication is *compatible* with multiplication in $\mathbb{K}$; $\forall a, b \in \mathbb{K}, \vec{v} \in V : a \cdot (b \cdot \vec{v}) = (ab) \cdot \vec{v}$
- The identity law for scalar multiplication holds: $\forall v \in V : 1_{\mathbb{K}} \cdot v = v$

Linear map

If V and W are vector spaces defined over $\mathbb{K}$, a function $f : V \to W$ is a *linear map* if the following holds

- $\forall \vec{u}, \vec{v} \in V : f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$
- $\forall a \in \mathbb{K}, v \in V : f(a \cdot \vec{v}) = a \cdot f(\vec{v})$

Bilinear map

Given three vector spaces, U, V, and W, defined over a a field $\mathbb{K}$,

Algebra over a field

If $\mathbb{K}$ is a field, and V a vector space over $\mathbb{K}$ equipped with a bilinear map $(\cdot) : V \times V \to V$, then V is called an *algebra over* $\mathbb{K}$ (or K-algebra for short).

TODO