K-Algebra

Vector space

A vector space V over a field $\mathbb{K}$ is a set of vectors, together with two operations;

- vector addition; $(+) : V \times V \to V$; and
- scalar multiplication; $(\cdot) : \mathbb{K} \times V \to V$

such that

- The set V forms an abelian group under the operation of vector addition;
  - Associativity; $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V : (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - Commutativity; $\forall \mathbf{u}, \mathbf{v} \in V : \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - Identity; $\exists \mathbf{0} \in V$ s.t. $\forall \mathbf{v} \in V : \mathbf{v} + \mathbf{0} = \mathbf{v}$
  - Inverse; $\forall \mathbf{v} \in V, \exists (-\mathbf{v}) \in V$ s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
- Scalar multiplication distributes over vector addition; $\forall a \in \mathbb{K}, \mathbf{u}, \mathbf{v} \in V : a \cdot (\mathbf{u} + \mathbf{v}) = (a \cdot \mathbf{u}) + (a \cdot \mathbf{v})$
- Scalar multiplication distributes over addition in $\mathbb{K}$; $\forall a, b \in \mathbb{K}, \mathbf{v} \in V : (a + b) \cdot \mathbf{v} = (a \cdot \mathbf{v}) + (b \cdot \mathbf{v})$
- Scalar multiplication is *compatible* with multiplication in $\mathbb{K}$; $\forall a, b \in \mathbb{K}, \mathbf{v} \in V : a \cdot (b \cdot \mathbf{v}) = (ab) \cdot \mathbf{v}$
- The following identity law holds for scalar multiplication; $\forall \mathbf{v} \in V : 1_{\mathbb{K}} \cdot \mathbf{v} = \mathbf{v}$ (where $1_{\mathbb{K}}$ is the multiplicative identity in $\mathbb{K}$)

Compact definition

TODO

Generalization to rings

The notion of a *module* is a generalization of vector spaces, in which the scalars are elements of any ring (i.e., not necessarily a field).

Linear map

If V and W are vector spaces defined over $\mathbb{K}$, a function $f : V \to W$ is a *linear map* if it preserves the vector space structure under the two operations, i.e,

1. vector addition; $\forall \mathbf{u}, \mathbf{v} \in V : f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$; and
2. scalar multiplication; $\forall a \in \mathbb{K}, \mathbf{v} \in V : f(a \cdot \mathbf{v}) = a \cdot f(\mathbf{v})$

In other words, a linear map is a *homomorphism* between vector spaces.

Bilinear map

Given three vector spaces, U, V, and W, all defined over a a field $\mathbb{K}$, a *bilinear map* is a function $b : U \times V \to W$ which is linear in both arguments. That is, b is linear with respect to;

- addition, in the first argument; $\forall \mathbf{u}, \mathbf{x} \in U, \mathbf{v} \in V : b(\mathbf{u} + \mathbf{x}, \mathbf{v}) = b(\mathbf{u}, \mathbf{v}) + b(\mathbf{x}, \mathbf{v})$
- addition, in the second argument; $\forall \mathbf{u} \in U, \mathbf{v}, \mathbf{y} \in V : b(\mathbf{u}, \mathbf{v} + \mathbf{y}) = b(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{y})$
- scalar multiplication; $\forall c \in \mathbb{K}, \mathbf{u} \in U, \mathbf{v} \in V : b(c \cdot \mathbf{u}, \mathbf{v}) = c \cdot b(\mathbf{u}, \mathbf{v}) = b(\mathbf{u}, c \cdot \mathbf{v})$

Algebra over a field

If $\mathbb{K}$ is a field, and V a vector space over $\mathbb{K}$ equipped with a bilinear map $(\cdot) : V \times V \to V$, then V is called an *algebra over* $\mathbb{K}$ (or K-algebra for short).

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