Square roots of perturbed sub-elliptic operators on Lie groups

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We let $d\mu$ denote the left invariant *Haar* measure.

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The vectorfields $\{A_i\}$ are linearly independent and *global*.

Distance

Theorem of Carathéodory-Chow tells us that for any two points $x, y \in \mathcal{G}$, we can find a curve $\gamma : [0,1] \to \mathcal{G}$ such that

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The measure $d\mu$ is Borel-regular with respect to d and we consider $(\mathcal{G}, d, d\mu)$ as a measure metric space.

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This is a densely-defined, self-adjoint operator on $\mathrm{L}^2(\mathcal{G})$.

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$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}],\ \mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1],\ \mathfrak{g}_3 = [\mathfrak{g}_1, \mathfrak{g}_2], \dots$$

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On such spaces, we consider the uniformly elliptic second order operator

$$\mathrm{D}_H = -b \sum_{i,j} A_i b_{ij} A_j$$

where $b, b_{ij} \in \mathrm{L}^\infty(\mathcal{G})$.

The main theorem for nilpotent Lie groups

Theorem (B.-E.-Mc)

Let $\mathcal{G}$ be a connected nilpotent and suppose there exist $\kappa_1, \kappa_2 > 0$ such that

$$\operatorname{Re} b(x) \geq \kappa_1 \quad \textit{and} \quad \operatorname{Re} \int_{\mathcal{G}} \sum_{i,j} b_{ij} A_i u \overline{A_j u} \geq \kappa_2 \sum_i \|A_i u\|^2$$

for almost all $x \in \mathcal{G}$ and $u \in \mathrm{H}^1(\mathcal{G})$. Then,

(i) $\mathcal{D}(\sqrt{\mathrm{D}_H}) = \cap_{i=1}^m \mathcal{D}(A_i) = \mathrm{H}^1(\mathcal{G})$, *and*

(ii) $\|\sqrt{\mathrm{D}_H} u\| \simeq \sum_{i=1}^m \|A_i u\|$ *for all $u \in \mathrm{H}^1(\mathcal{G})$.*

Stability

Theorem (B.-E.-Mc)

Let $0 < \eta_i < \kappa_i$ *and suppose that* $\tilde{b}, \tilde{b}_{ij} \in \mathrm{L}^\infty(\mathcal{G})$ *such that* $\|\tilde{b}\|_\infty \leq \eta_1$ *and* $\|(\tilde{b}_{ij})\|_\infty \leq \eta_2$. *Then,*

$$\|\sqrt{\mathrm{D}_H} u - \sqrt{\tilde{\mathrm{D}}_H} u\| \lesssim (\|\tilde{b}\|_\infty + \|(\tilde{b}_{ij})\|_\infty) \sum_{i=1}^{k} \|A_i u\|,$$

for $u \in \mathrm{H}^1(\mathcal{G})$ *and where*

$$\tilde{\mathrm{D}}_H = (b + \tilde{b}) \sum_{i,j=1}^{k} A_i (b_{ij} + \tilde{b}_{ij}) A_j.$$

Operator theory

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(H2) The operators $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ satisfy

$$\begin{aligned} \operatorname{Re}\langle B_1 u, u\rangle &\geq \kappa_1 \|u\| && \text{whenever } u \in \mathcal{R}(\Gamma^*) \\ \operatorname{Re}\langle B_2 u, u\rangle &\geq \kappa_2 \|u\| && \text{whenever } u \in \mathcal{R}(\Gamma) \end{aligned}$$

where $\kappa_1, \kappa_2 > 0$ are constants.

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Let $\Gamma_B^* = B_1 \Gamma^* B_2$, $\Pi_B = \Gamma + \Gamma_B^*$, and $\Pi = \Gamma + \Gamma^*$.

Harmonic analysis and Kato square root type estimates

Theorem (Kato square root type estimate)

Suppose that (Γ, B_1, B_2) *satisfy (H1)-(H3) and*

$$\int_0^\infty \|t\Pi_B(1 + t^2\Pi_B^2)^{-1}u\|^2 \ \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathscr{H}$. *Then,*

(i) *There is a spectral decomposition* $\mathscr{H} = \mathcal{N}(\Pi_B) \oplus E_B^+ \oplus E_B^-$, *where* $E_B^\pm$ *are spectral subspaces and the sum is in general non-orthogonal, and*

(ii) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B^2})$ *with*

$$\|\Gamma u\| + \|\Gamma_B u\| \simeq \|\Pi_B u\| \simeq \|\sqrt{\Pi_B^2} u\| \text{ for all } u \in \mathcal{D}(\Pi_B).$$

Homogeneous conditions

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(H4) Let $\mathcal{X}$ be a complete, connected metric space and μ a Borel-regular measure on $\mathcal{X}$ that is *doubling*. Then set $\mathscr{H} = \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N; d\mu)$.

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(H6) For every bounded Lipschitz function $\xi : \mathcal{X} \to \mathbb{C}$, multiplication by ξ preserves $\mathcal{D}(\Gamma)$ and $\mathrm{M}_\xi = [\Gamma, \xi I]$ is a multiplication operator. Furthermore, there exists a constant $m > 0$ such that $|\mathrm{M}_\xi(x)| \leq m \, |\mathrm{Lip}\, \xi(x)|$ for almost all $x \in \mathcal{X}$.

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(H7) For each open ball B, we have

$$\int_B \Gamma u \; d\mu = 0 \quad \text{and} \quad \int_B \Gamma^* v \; d\mu = 0$$

for all $u \in \mathcal{D}(\Gamma)$ with $\mathrm{spt}\ u \subset B$ and for all $v \in \mathcal{D}(\Gamma^*)$ with $\mathrm{spt}\ v \subset B$.

(H8) -1 (Poincaré hypothesis)
There exists $C' > 0$, $c > 0$ and an operator $\Xi : \mathcal{D}(\Xi) \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N) \to \mathrm{L}^2(\mathcal{X}, \mathbb{C}^M)$ such that $\mathcal{D}(\Pi) \cap \mathcal{R}(\Pi) \subset \mathcal{D}(\Xi)$ and

$$\int_B |u - u_B|^2 \, d\mu \leq C' r^2 \int_B |\Xi u|^2 \ d\mu$$

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This is slightly different from (H8) in [Bandara].

Theorem (B.)

Let $\mathcal{X}$, (Γ, B_1, B_2) satisfy (H1)-(H8). Then, Π_B satisfies the quadratic estimate
$$\int_0^\infty \|t\Pi_B(1 + t^2\Pi_B^2)^{-1}u\|^2 \, \frac{dt}{t} \simeq \|u\|^2$$
for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N)$.

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$$\nabla f = A_k f A^k$$

where $A^k = {A_k}^* \in \mathcal{W}^*$.

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We have that $\mathcal{W} \cong \mathbb{C}^k$ and $\mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{W}) \cong \mathrm{L}^2(\mathbb{C}^{k+1})$.

Operator setup

Define: $\Gamma : \mathcal{D}(\Gamma) \subset \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{W}^*) \to \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{W}^*)$ by

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Then,

$$\Gamma^* = \begin{pmatrix} 0 & -\operatorname{div} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Pi = \begin{pmatrix} 0 & -\operatorname{div} \\ \nabla & 0 \end{pmatrix},$$

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Let $B = (b_{ij})$. Then, define

$$B_1 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

Proof of the homogeneous problem

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(H2) By accretivity assumptions.

(H3) By construction.

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(H7) By the left invariance of the measure $d\mu$.

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$$\int_B |f - f_B|^2 \; d\mu \lesssim r^2 \int_B |\nabla f|^2 \; d\mu$$

for all balls B, and $f \in \mathrm{C}^\infty(B)$. See [SC, (P.1), p118].

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-2 The crucial fact needed here is the regularity result [ERS, Lemma 4.2] which gives

$$\|A_i A_j f\| \lesssim \|\Delta f\|$$

for $f \in \mathrm{H}^2(\mathcal{G}) = \mathcal{D}(\Delta)$.

Inhomogeneous problem

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Let $b, b_{ij}, c_k, d_k, e \in \mathrm{L}^\infty(\mathcal{G})$. Define the following uniformly elliptic second order operator

$$\mathrm{D}_I = -b \sum_{ij=1}^{m} A_i b_{ij} A_j u - b \sum_{i=1}^{m} A_i c_i u - b \sum_{i=1}^{m} d_i A_i u - beu.$$

Theorem (B.-E.-Mc)

Let $\mathcal{G}$ be a connected Lie group and suppose there exists $\kappa_1, \kappa_2 > 0$ such that

$$\operatorname{Re} b(x) \geq \kappa_1,$$

$$\operatorname{Re} \int_{\mathcal{G}} \left(eu + \sum_{i=1}^{m} d_i A_i u \right) \overline{u} + \sum_{i=1}^{m} \left(c_i u + \sum_{j=1}^{m} b_{ij} A_j u \right) \overline{A_i u} \, d\mu \geq \kappa_2 \left(\|u\|^2 + \sum_{i=1}^{m} \|A_i u\|^2 \right)$$

for almost all $x \in \mathcal{G}$ and $u \in \mathrm{H}^1(\mathcal{G})$. Then,

(i) $\mathcal{D}(\sqrt{\mathrm{D}_I}) = \cap_{i=1}^{m} \mathcal{D}(A_i) = \mathrm{H}^1(\mathcal{G})$, *and*

(ii) $\|\sqrt{\mathrm{D}_I} u\| \simeq \|u\| + \sum_{i=1}^{m} \|A_i u\|$ *for all $u \in \mathrm{H}^1(\mathcal{G})$.*

Spaces of exponential growth

$(\mathcal{X}, d, \mu)$ an exponentially locally doubling measure metric space. That is: there exist $\kappa, \lambda \geq 0$ and constant $C \geq 1$ such that

$$0 < \mu(B(x, tr)) \leq C t^{\kappa} e^{\lambda t r} \mu(B(x, r))$$

for all $x \in \mathcal{X}$, $r > 0$ and $t \geq 1$.

Changes to (H7) and (H8)

The following (H7) from [Morris]:

(H7) There exist $c > 0$ such that for all open balls $B \subset \mathcal{X}$ with $r \leq 1$,

$$\left| \int_B \Gamma u \, d\mu \right| \leq c\mu(B)^{\frac{1}{2}} \|u\| \quad \text{and} \quad \left| \int_B \Gamma^* v \, d\mu \right| \leq c\mu(B)^{\frac{1}{2}} \|v\|$$

for all $u \in \mathcal{D}(\Gamma),\ v \in \mathcal{D}(\Gamma^*)$ with $\mathrm{spt}\ u,\ \mathrm{spt}\ v \subset B$.

We introduce the following *local* (H8):

(H8) -1 (Local Poincaré hypothesis)
There exists $C' > 0$, $c > 0$ and an operator $\Xi : \mathcal{D}(\Xi) \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N) \to \mathrm{L}^2(\mathcal{X}, \mathbb{C}^M)$ such that $\mathcal{D}(\Pi) \cap \mathcal{R}(\Pi) \subset \mathcal{D}(\Xi)$ and

$$\int_B |u - u_B|^2 \, d\mu \leq C' r^2 \int_B (|\Xi u|^2 + |u|^2) \, d\mu$$

for all balls $B = B(x, r)$ and for $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$.

-2 (Coercivity hypothesis)
There exists $\tilde{C} > 0$ such that for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$,

$$\|\Xi u\| + \|u\| \leq \tilde{C} \|\Pi u\|.$$

Theorem (Morris)

Let $\mathcal{X}$, (Γ, B_1, B_2) *satisfy (H1)-(H8). Then,* Π_B *satisfies the quadratic estimate*

$$\int_0^\infty \|t\Pi_B(1 + t^2\Pi_B^2)^{-1}u\|^2 \ \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N)$.

Setup

Set $\mathcal{X} = \mathcal{G}$ and $\mathscr{H} = \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{W}) \cong \mathrm{L}^2(\mathbb{C}^{k+2})$.

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$$\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \ \Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix}, \text{ and } \Pi^* = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}.$$

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Let $\tilde{B}_{00} = e$, $\tilde{B}_{10} = (c_1, \dots, c_m)$, $\tilde{B}_{01} = (d_1, \dots, d_m)^{\mathrm{tr}}$, $\tilde{B}_{11} = (b_{ij})$, and $B = (\tilde{B}_{ij})$.

Then, we can write

$$B_1 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

Proof

The proofs of (H1)-(H6) are similar to the homogeneous situation.

(H7) The proof is the same as the homogeneous situation except the lower order term introduces the term $\mu(B)^{\frac{1}{2}}\|u\|$ on the right.

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(H8) -1 The existence of a local Poincaré inequality is guaranteed by [ER2, Proposition 2.4]:

$$\int_B |f - f_B|^2 \ d\mu \lesssim r^2 \int_B (|\nabla f|^2 + |f|^2) \ d\mu$$

for all balls $B = B(x, r)$ and where $f \in \mathrm{C}^\infty(B)$.

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-2 The crucial fact needed here is the regularity result in [ER, Theorem 7.2],

$$\|A_i A_j u\|^2 \lesssim \|\Delta u\|^2 + \|u\|^2$$

for $u \in \mathrm{H}^2(\mathcal{G}) = \mathcal{D}(\Delta)$.

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