Geometry and the Kato square root problem

Lashi Bandara

Centre for Mathematics and its Applications
Australian National University

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Outline

- Brief overview of the Kato square root problem on $\mathbb{R}^n$.
- A motivating application to hyperbolic PDE.
- Recent progress on the Kato square root problem on smooth manifolds by McIntosh and B.
- Recent progress on subelliptic Kato square root problems on Lie groups by ter Elst, McIntosh, and B.
- Kato square root problem on smooth manifolds with non-smooth metrics, connection to geometric flows and PDEs.

The Kato square root problem

Let $A \in \mathrm{L}^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N))$ and $a \in \mathrm{L}^\infty(\mathbb{R}^n)$. Suppose that there exists $\kappa_1, \kappa_2 > 0$ such that for all $u \in \mathrm{W}^{1,2}(\mathbb{R}^n)$,

$$\mathrm{Re}\, a(x) \geq \kappa_1 \quad \text{and} \quad \mathrm{Re}\, \langle A\nabla u, \nabla u\rangle \geq \kappa_2 \|u\|^2.$$

The Kato square root problem on $\mathbb{R}^n$ is the statement that

$$\begin{aligned} &\mathcal{D}(\sqrt{-a\, \mathrm{div}\, A\nabla}) = \mathrm{W}^{1,2}(\mathbb{R}^n) \\ &\|\sqrt{-a\, \mathrm{div}\, A\nabla} u\| \simeq \|\nabla u\|. \end{aligned} \tag{K1}$$

This was answered in the positive in 2002 by Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh and Phillipe Tchamitchian in [AHLMcT].

- If further $A = A^*$, (K1) is a trivial consequence of the Lax-Milgram Theorem.
- Solution to (K1) implies that $\mathcal{D}(\sqrt{-\operatorname{div} A\nabla}) = \mathcal{D}(\sqrt{-\operatorname{div} A^*\nabla})$. We can ask a more abstract question for *accretive* operators L on a Hilbert space $\mathscr{H}$. There, the question is whether $\mathcal{D}(\sqrt{L^*}) = \mathcal{D}(\sqrt{L})$. In general, this is not true by a counterexample of McIntosh in 1972 in [Mc72].
- A second related question is the following. Suppose that J_t is a family of *closed, densely-defined, Hermitian* forms on $\mathscr{H}$ with domain $\mathcal{W}$ and $L(t)$ the associated self-adjoint operators to J_t with domain $\mathcal{W}$. If $t \mapsto J_t$ extends to holomorphic family (for small z), then is $\partial_t \sqrt{L(t)} : \mathcal{V} \to \mathscr{H}$ a bounded operator?
- Counterexample to this second question by McIntosh in 1982 in [Mc82].

Motivations from PDE

For $k = 1, 2$, let $L_k = -\operatorname{div} A_k \nabla$ where $A_k \in \mathrm{L}^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^n))$ non-negative self-adjoint and L_k uniformly elliptic.

As aforementioned, $\mathcal{D}(\sqrt{L_k}) = \mathrm{W}^{1,2}(\mathbb{R}^n)$ and $\|\sqrt{L_k} u\| \simeq \|\nabla u\|$ for $u \in \mathrm{W}^{1,2}(\mathbb{R}^n)$.

Let u_k be solutions to the wave equation with respect to L_k with the same initial data. That is,

$$\begin{aligned} \partial_t^2 u_k + L_k u_k &= 0 \\ \partial_t u_k|_{t=0} &= g \in \mathrm{L}^2(\mathbb{R}^n) \\ u_k(0) &= f \in \mathrm{W}^{1,2}(\mathbb{R}^n). \end{aligned}$$

Suppose there exists a $C > 0$

$$\|\sqrt{L_1}u - \sqrt{L_2}u\| \leq C\|A_1 - A_2\|_\infty \|\nabla u\|. \tag{P}$$

Then, whenever $t > 0$, the following estimate holds:

$$\|u_1(t) - u_2(t)\| + \|\int_0^t \nabla(u_1(s) - u_2(s))\ ds\| \\ \leq Ct\|A_1 - A_2\|_\infty(\|\nabla f\| + \|g\|).$$

See [Aus].

The estimate (P) is related to the second question of Kato.

By solving the Kato square root problem (K1) for *complex* coefficients A, we are able to automatically obtain (P) from (K1).

Kato square root problem on manifolds

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with metric g, Levi-Civita connection ∇, and volume measure μ.

Write $\mathrm{div} = -\nabla^*$ in L^2 and let $S = (\mathrm{I}, \nabla)$.

Assume $a \in \mathrm{L}^\infty(\mathcal{M})$ and $A = (A_{ij}) \in \mathrm{L}^\infty(\mathcal{M}, \mathcal{L}(\mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M}))$.

Consider the following second order differential operator $\mathrm{L}_A : \mathcal{D}(\mathrm{L}_A) \subset \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ defined by

$$\mathrm{L}_A u = aS^*ASu = -a\,\mathrm{div}(A_{11}\nabla u) - a\,\mathrm{div}(A_{10}u) + aA_{01}\nabla u + aA_{00}u.$$

The main theorem on manifolds

Theorem (B.-Mc, 2012)

Let $\mathcal{M}$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$. *Suppose the following ellipticity condition holds: there exists* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2$$
$$\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2 \|u\|^2_{\mathrm{W}^{1,2}}$$

for $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{W}^{1,2}(\mathcal{M})$. *Then,* $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{M})$ *and* $\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{M})$.

Lipschitz estimates

Since we allow the coefficients a and A to be *complex*, we obtain the following stability result as a consequence:

Theorem (B.-Mc, 2012)

Let $\mathcal{M}$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(\mathcal{M}) \geq \kappa > 0$. *Suppose that there exist* $\kappa_1, \kappa_2 > 0$ *such that* $\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2$ *and* $\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2 \|u\|^2_{\mathrm{W}^{1,2}}$ *for* $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{W}^{1,2}(\mathcal{M})$. *Then for every* $\eta_i < \kappa_i$, *whenever* $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, *the estimate*

$$\|\sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\|u\|_{\mathrm{W}^{1,2}}$$

holds for all $u \in \mathrm{W}^{1,2}(\mathcal{M})$. *The implicit constant depends in particular on* A, a *and* η_i.

The Hodge-Dirac operator

Let $\mathbf{\Omega}(\mathcal{M})$ denote the algebra of differential forms over $\mathcal{M}$ under the exterior product $\wedge$.

Let d be the exterior derivative as an operator on $\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}))$ and d^* its adjoint, both of which are *nilpotent* operators.

The Hodge-Dirac operator is then the self-adjoint operator $\mathrm{D} = \mathrm{d} + \mathrm{d}^*$. The Hodge-Laplacian is then $\mathrm{D}^2 = \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d}$.

For an invertible $A \in \mathrm{L}^\infty(\mathcal{L}(\mathbf{\Omega}(\mathcal{M})))$, we consider perturbing D to obtain the operator $\mathrm{D}_A = \mathrm{d} + A^{-1}\mathrm{d}^* A$.

Curvature endomorphism for forms

Let $\{\theta^i\}$ be an orthonormal frame at x for $\mathbf{\Omega}^1(\mathcal{M}) = \mathrm{T}^*\mathcal{M}$.

Denote the components of the curvature tensor in this frame by Rm_{ijkl}. The curvature endomorphism is then the operator

$$\mathrm{R}\,\omega = \mathrm{Rm}_{ijkl}\,\theta^i \wedge (\theta^j \llcorner (\theta^k \wedge (\theta^l \llcorner \omega)))$$

for $\omega \in \mathbf{\Omega}_x(\mathcal{M})$.

This can be seen as an extension of Ricci curvature for forms, since $\mathrm{g}(\mathrm{R}\,\omega, \eta) = \mathrm{Ric}(\omega^\flat, \eta^\flat)$ whenever $\omega, \eta \in \mathbf{\Omega}^1_x(\mathcal{M})$ and where $\flat : \mathrm{T}^*\mathcal{M} \to \mathrm{T}\mathcal{M}$ is the flat isomorphism through the metric g.

The Weitzenböck formula then asserts that $\mathrm{D}^2 = \mathrm{tr}_{12}\,\nabla^2 + \mathrm{R}\,.$

Theorem (B., 2012)

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold and let $\beta \in \mathbb{C} \setminus \{0\}$. Suppose there exist $\eta, \kappa > 0$ such that $|\mathrm{Ric}| \leq \eta$ and $\mathrm{inj}(\mathcal{M}) \geq \kappa$. Furthermore, suppose there is a $\zeta \in \mathbb{R}$ satisfying $\mathrm{g}(\mathrm{R}\, u, u) \geq \zeta\, |u|^2$, for $u \in \mathbf{\Omega}_x(\mathcal{M})$ and $A \in \mathrm{L}^\infty(\mathcal{L}(\mathbf{\Omega}(\mathcal{M})))$ and $\kappa_1 > 0$ satisfying

$$\mathrm{Re}\,\langle Au, u\rangle \geq \kappa_1 \|u\|^2.$$

Then, $\mathcal{D}(\sqrt{\mathrm{D}_A^2 + |\beta|^2}) = \mathcal{D}(\mathrm{D}_A) = \mathcal{D}(\mathrm{d}) \cap \mathcal{D}(\mathrm{d}^* A)$ and $\|\sqrt{\mathrm{D}_A^2 + |\beta|^2} u\| \simeq \|\,\mathrm{D}_A\, u\| + \|u\|$.

Lie groups

Let $\mathcal{G}$ be a Lie group of dimension n with Lie algebra $\mathfrak{g}$ and equipped with the left-invariant Haar measure μ.

We say that a linearly independent $\mathfrak{a} = \{a_1, \ldots, a_k\} \subset \mathfrak{g}$ is an *algebraic basis* if we can recover a basis for $\mathfrak{g}$ through multi-commutation.

Let A_i denote the right-translation of a_i and $A^i = A_i^*$. Let $\mathrm{span}\,\{A_1, \ldots, A_k\} = \mathcal{A} \subset \mathrm{T}\mathcal{G}$ be the bundle obtained through the right-translation of $\mathfrak{a}$ and $\mathcal{A}^* = \left\{A^1, \ldots, A^k\right\}$ the dual of $\mathcal{A}$.

Subelliptic distance

Theorem of Carathéodory-Chow tells us that for any two points $x, y \in \mathcal{G}$, we can find an absolutely continuous curve $\gamma : [0,1] \to \mathcal{G}$ such that

$$\dot{\gamma}(t) = \sum_i \dot{\gamma}^i(t) A_i(\gamma(t)) \quad \in \mathcal{A}.$$

The length of such a curve then is given by

$$\ell(\gamma) = \int_0^1 (\sum_i \left|\dot{\gamma}^i(t)\right|^2)^{\frac{1}{2}} \, dt$$

Define distance $d(x,y)$ as the infimum over the length of all such curves.

The measure μ is Borel-regular with respect to d.

Subelliptic operators

For $f \in \mathrm{C}^\infty(\mathcal{G})$, define
$$\nabla f = A_i f \ A^i.$$
This defines an *sub-connection* on $\mathrm{C}^\infty(\mathcal{M})$.

Each vector field A_i is a skew-adjoint differential operator. We consider it as a unbounded operator on $\mathrm{L}^2(\mathcal{G})$ with domain $\mathcal{D}(A_i)$.

By also considering ∇ as a closed, densely-defined operator on $\mathrm{L}^2(\mathcal{M})$, we obtain the first-order Sobolev space $\mathrm{W}^{1,2}(\mathcal{G})' = \mathcal{D}(\nabla) = \cap_{i=1}^k \mathcal{D}(A_i)$.

We write the divergence as $\mathrm{div} = -\nabla^*$. Then, the subelliptic Laplacian associated to $\mathcal{A}$ is
$$\Delta = -\mathrm{div}\,\nabla = -\sum_{i=1}^{k} A_i^2.$$

Nilpotent Lie groups

The Lie group $\mathcal{G}$ is *nilpotent* if the inductively defined sequence $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}],\ \mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}], \ldots$ is eventually zero.

Theorem (B.-E.-Mc., 2012)

Let $(\mathcal{G}, \mathrm{d}, \mu)$ *be a connected, nilpotent Lie group with* $\mathfrak{a}$ *an algebraic basis,* d *the associated sub-elliptic distance, and* μ *the left Haar measure. Suppose that* $a, A \in \mathrm{L}^\infty$ *and that there exist* $\kappa_1, \kappa_2 > 0$ *satisfying*

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2, \quad \textit{and} \quad \mathrm{Re}\,\langle A\nabla u, \nabla u\rangle \geq \kappa_2 \|\nabla u\|^2.$$

for every $v \in \mathrm{L}^2(\mathcal{G})$ *and* $u \in \mathrm{W}^{1,2}(\mathcal{G})'$. *Then,* $\mathcal{D}(\sqrt{-a\,\mathrm{div}\,A\nabla}) = \mathrm{W}^{1,2}(\mathcal{G})'$ *and* $\|\sqrt{-a\,\mathrm{div}\,A\nabla}u\| \simeq \|\nabla u\|$ *for* $u \in \mathrm{W}^{1,2}(\mathcal{G})'$.

General Lie groups

Let $S = (\mathrm{I}, \nabla)$ as in the manifold case.

Theorem (B.-E.-Mc., 2012)

Let $(\mathcal{G}, \mathrm{d}, \mu)$ *be a connected Lie group,* $\mathfrak{a}$ *an algebraic basis,* d *the associated sub-elliptic distance, and* μ *the left Haar measure. Let* $a, A \in \mathrm{L}^\infty$ *such that*

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2, \quad \textit{and} \quad \mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2 \|u\|_{\mathrm{W}^{1,2\prime}}$$

for every $v \in \mathrm{L}^2(\mathcal{G})$ *and* $u \in \mathrm{W}^{1,2}(\mathcal{G})'$. *Then,* $\mathcal{D}(\sqrt{aS^*AS}) = \mathrm{W}^{1,2}(\mathcal{G})'$ *with* $\|\sqrt{aS^*AS}u\| \simeq \|u\|_{\mathrm{W}^{1,2\prime}} = \|u\| + \|\nabla u\|$.

Operator theory

We adapt the framework due to Axelsson (Rosén), Keith, McIntosh in [AKMc].

Let $\mathscr{H}$ be a Hilbert space and $\Gamma : \mathscr{H} \to \mathscr{H}$ a closed, densely-defined, nilpotent operator.

Suppose that $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ such that here exist $\kappa_1, \kappa_2 > 0$ satisfying

$$\mathrm{Re}\,\langle B_1 u, u\rangle \geq \kappa_1 \|u\|^2 \quad \text{and} \quad \mathrm{Re}\,\langle B_2 v, v\rangle \geq \kappa_2 \|v\|^2$$

for $u \in \mathcal{R}(\Gamma^*)$ and $v \in \mathcal{R}(\Gamma)$.

Furthermore, suppose the operators B_1, B_2 satisfy $B_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$.

The primary operator we consider is $\Pi_B = \Gamma + B_1 \Gamma^* B_2$.

If the quadratic estimates

$$\int_0^\infty \|t\Pi_B(1+t^2\Pi_B^2)^{-1}u\|^2 \simeq \|u\| \qquad \text{(Q)}$$

hold for every $u \in \overline{\mathcal{R}(\Pi_B)}$, then, $\mathscr{H}$ decomposes into the spectral subspaces of Π_B as $\mathscr{H} = \mathcal{N}(\Pi_B) \oplus E_+ \oplus E_-$ and

$$\mathcal{D}(\sqrt{\Pi_B^2}) = \mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma^* B_2)$$
$$\|\sqrt{\Pi_B^2}u\| \simeq \|\Pi_B u\| \simeq \|\Gamma u\| + \|\Gamma^* B_2 u\|.$$

The Kato problems are then obtained by letting $\mathscr{H} = \mathrm{L}^2(\mathcal{M}) \oplus (\mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M}))$ and letting

$$\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix},\ \Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix},\ B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

Geometry and harmonic analysis

Harmonic analytic methods are used to prove quadratic estimates (Q).

The idea is to reduce the quadratic estimate (Q) to a *Carleson measure* estimate. This is achieved via a *local* $T(b)$ argument.

Geometry enters the picture precisely in the harmonic analysis. We need to perform harmonic analysis on vector fields, not just functions.

One can show that this is *not* artificial - the Kato problem on functions immediately provides a solution to the dual problem on vector fields.

Elements of the proofs

Similar in structure to the proof of [AKMc] which is inspired from the proof in [AHLMcT].

- A dyadic decomposition of the space
- A notion of averaging (in an integral sense)
- Poincaré inequality - on both functions and vector fields
- Control of ∇^2 in terms of Δ.

The case of non-smooth metrics on manifolds

We let $\mathcal{M}$ be a smooth, complete manifold as before but now let g be a C^0 metric. Let μ_{g} denote the volume measure with respect to g.

Let $\mathrm{h} \in \mathrm{C}^0(\mathcal{T}^{(2,0)}\mathcal{M})$. Then, define

$$\|\mathrm{h}\|_{\mathrm{op},\mathrm{g}} = \sup_{x \in \mathcal{M}} \sup_{|u|_{\mathrm{g}} = |v|_{\mathrm{g}} = 1} |\mathrm{h}_x(u, v)| .$$

If $\tilde{\mathrm{g}}$ is another C^0 metric satisfying $\|\mathrm{g} - \tilde{\mathrm{g}}\|_{\mathrm{op},\mathrm{g}} \leq \delta < 1$, then $\mathrm{L}^2(\mathcal{M}, \mathrm{g}) = \mathrm{L}^2(\mathcal{M}, \tilde{\mathrm{g}})$ and $\mathrm{W}^{1,2}(\mathcal{M}, \mathrm{g}) = \mathrm{W}^{1,2}(\mathcal{M}, \tilde{\mathrm{g}})$ with comparable norms.

Π_B under a change of metric

The operator Γ_{g} does not change under the change of metric. However,

$$\Gamma_{\mathrm{g}}^* = C^{-1}\Gamma_{\tilde{\mathrm{g}}}^* C$$

where C is the bounded, invertible, multiplication operator on $\mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$.

Thus,

$$\Pi_{B,\mathrm{g}} = \Gamma_{\mathrm{g}} + B_1\Gamma_{\mathrm{g}}^* B_2 = \Gamma_{\tilde{\mathrm{g}}} + B_1 C^{-1}\Gamma_{\tilde{\mathrm{g}}}^* C B_2.$$

This allows us to reduce the study of $\Pi_{B,\mathrm{g}}$ for a C^0 metric g to the study of $\Pi_{\tilde{B},\tilde{\mathrm{g}}} = \Gamma_{\tilde{\mathrm{g}}} + \tilde{B}_1\Gamma_{\tilde{\mathrm{g}}}^*\tilde{B}_2$ where $\tilde{B}_1 = B_1 C^{-1}$ and $\tilde{B}_2 = CB_2$, but now with a smooth metric $\tilde{\mathrm{g}}$.

Connection to geometric flows

Given a C^0 metric g on a smooth *compact* manifold, we are able to always find C^∞ metric $\tilde{\mathrm{g}}$.

The metric $\tilde{\mathrm{g}}$ has $\mathrm{inj}(\mathcal{M}, \tilde{\mathrm{g}}) > \kappa$ and $|\mathrm{Ric}(\tilde{\mathrm{g}})|_{\tilde{\mathrm{g}}} \leq \eta$ so we obtain a corresponding Kato square root estimate in this setting.

The non-compact situation poses issues.

Smooth the metric via mean curvature flow for, say, a C^2 imbedding?

Smooth the metric via Ricci flow in the general case? Regularity of the initial metric?

Application to PDE

In the case we are able to find a suitable C^∞ metric near the C^0 one, then we have Lipschitz estimates.

Possible application to hyperbolic PDE?

"Stability" of geometries with Ricci bounds and injectivity radius bounds?

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