Geometry and the Kato square root problem

Lashi Bandara

Centre for Mathematics and its Applications
Australian National University

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Outline

- Brief overview of the Kato square root problem on $\mathbb{R}^n$.

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- Kato square root problem on smooth manifolds with non-smooth metrics, connection to geometric flows and PDEs.

The Kato square root problem

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$$\mathrm{Re}\, a(x) \geq \kappa_1 \quad \text{and} \quad \mathrm{Re}\, \langle A\nabla u, \nabla u \rangle \geq \kappa_2 \|u\|^2.$$

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The Kato square root problem on $\mathbb{R}^n$ is the statement that

$$\begin{aligned} &\mathcal{D}(\sqrt{-a \operatorname{div} A\nabla}) = \mathrm{W}^{1,2}(\mathbb{R}^n) \\ &\|\sqrt{-a \operatorname{div} A\nabla} u\| \simeq \|\nabla u\|. \end{aligned} \tag{K1}$$

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This was answered in the positive in 2002 by Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh and Phillipe Tchamitchian in [AHLMcT].

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- A second related question is the following. Suppose that J_t is a family of *closed, densely-defined, Hermitian* forms on $\mathscr{H}$ with domain $\mathcal{W}$ and $L(t)$ the associated self-adjoint operators to J_t with domain $\mathcal{W}$.

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- Counterexample to this second question by McIntosh in 1982 in [Mc82].

Motivations from PDE

For $k = 1, 2$, let $L_k = -\operatorname{div} A_k \nabla$ where $A_k \in \mathrm{L}^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^n))$ non-negative self-adjoint and L_k uniformly elliptic.

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Let u_k be solutions to the wave equation with respect to L_k with the same initial data. That is,

$$\begin{aligned} \partial_t^2 u_k + L_k u_k &= 0 \\ \partial_t u_k\big|_{t=0} &= g \in \mathrm{L}^2(\mathbb{R}^n) \\ u_k(0) &= f \in \mathrm{W}^{1,2}(\mathbb{R}^n). \end{aligned}$$

Suppose there exists a $C > 0$

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Then, whenever $t > 0$, the following estimate holds:

$$\|u_1(t) - u_2(t)\| + \| \int_0^t \nabla(u_1(s) - u_2(s))\ ds\| \\ \leq Ct\|A_1 - A_2\|_\infty(\|\nabla f\| + \|g\|).$$

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See [Aus].

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By solving the Kato square root problem (K1) for *complex* coefficients A, we are able to automatically obtain (P) from (K1).

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Assume $a \in \mathrm{L}^\infty(\mathcal{M})$ and $A = (A_{ij}) \in \mathrm{L}^\infty(\mathcal{M}, \mathcal{L}(\mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M}))$.

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Consider the following second order differential operator $\mathrm{L}_A : \mathcal{D}(\mathrm{L}_A) \subset \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ defined by

$$\mathrm{L}_A u = aS^*ASu = -a\,\mathrm{div}(A_{11}\nabla u) - a\,\mathrm{div}(A_{10}u) + aA_{01}\nabla u + aA_{00}u.$$

The main theorem on manifolds

Theorem (B.-Mc, 2012)

Let $\mathcal{M}$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$. *Suppose the following ellipticity condition holds: there exists* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2$$
$$\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2 \|u\|^2_{\mathrm{W}^{1,2}}$$

for $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{W}^{1,2}(\mathcal{M})$. *Then,* $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{M})$ *and* $\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{M})$.

Lipschitz estimates

Since we allow the coefficients a and A to be *complex*, we obtain the following stability result as a consequence:

Theorem (B.-Mc, 2012)

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with $|\mathrm{Ric}| \leq C$ and $\mathrm{inj}(\mathcal{M}) \geq \kappa > 0$. Suppose that there exist $\kappa_1, \kappa_2 > 0$ such that $\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2$ and $\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2 \|u\|^2_{\mathrm{W}^{1,2}}$ for $v \in \mathrm{L}^2(\mathcal{M})$ and $u \in \mathrm{W}^{1,2}(\mathcal{M})$. Then for every $\eta_i < \kappa_i$, whenever $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, the estimate

$$\|\sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\|u\|_{\mathrm{W}^{1,2}}$$

holds for all $u \in \mathrm{W}^{1,2}(\mathcal{M})$. The implicit constant depends in particular on A, a and η_i.

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For an invertible $A \in \mathrm{L}^\infty(\mathcal{L}(\boldsymbol{\Omega}(\mathcal{M})))$, we consider perturbing D to obtain the operator $\mathrm{D}_A = \mathrm{d} + A^{-1}\mathrm{d}^* A$.

Curvature endomorphism for forms

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$$\mathrm{R}\,\omega = \mathrm{Rm}_{ijkl}\,\theta^i \wedge (\theta^j \llcorner (\theta^k \wedge (\theta^l \llcorner \omega)))$$

for $\omega \in \mathbf{\Omega}_x(\mathcal{M})$.

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This can be seen as an extension of Ricci curvature for forms, since $\mathrm{g}(\mathrm{R}\,\omega, \eta) = \mathrm{Ric}(\omega^\flat, \eta^\flat)$ whenever $\omega, \eta \in \mathbf{\Omega}^1_x(\mathcal{M})$ and where $\flat : \mathrm{T}^*\mathcal{M} \to \mathrm{T}\mathcal{M}$ is the flat isomorphism through the metric g.

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The Weitzenböck formula then asserts that $\mathrm{D}^2 = \mathrm{tr}_{12}\,\nabla^2 + \mathrm{R}$.

Theorem (B., 2012)

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold and let $\beta \in \mathbb{C} \setminus \{0\}$. Suppose there exist $\eta, \kappa > 0$ such that $|\mathrm{Ric}| \leq \eta$ and $\mathrm{inj}(\mathcal{M}) \geq \kappa$. Furthermore, suppose there is a $\zeta \in \mathbb{R}$ satisfying $\mathrm{g}(\mathrm{R}\, u, u) \geq \zeta\, |u|^2$, for $u \in \mathbf{\Omega}_x(\mathcal{M})$ and $A \in \mathrm{L}^\infty(\mathcal{L}(\mathbf{\Omega}(\mathcal{M})))$ and $\kappa_1 > 0$ satisfying

$$\mathrm{Re}\, \langle Au, u \rangle \geq \kappa_1 \|u\|^2.$$

Then, $\mathcal{D}(\sqrt{\mathrm{D}_A^2 + |\beta|^2}) = \mathcal{D}(\mathrm{D}_A) = \mathcal{D}(\mathrm{d}) \cap \mathcal{D}(\mathrm{d}^* A)$ and $\|\sqrt{\mathrm{D}_A^2 + |\beta|^2} u\| \simeq \| \mathrm{D}_A\, u\| + \|u\|$.

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Let A_i denote the right-translation of a_i and $A^i = A_i^*$. Let $\mathrm{span}\{A_1, \ldots, A_k\} = \mathcal{A} \subset \mathrm{T}\mathcal{G}$ be the bundle obtained through the right-translation of $\mathfrak{a}$ and $\mathcal{A}^* = \{A^1, \ldots, A^k\}$ the dual of $\mathcal{A}$.

Subelliptic distance

Theorem of Carathéodory-Chow tells us that for any two points $x, y \in \mathcal{G}$, we can find an absolutely continuous curve $\gamma : [0,1] \to \mathcal{G}$ such that

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By also considering ∇ as a closed, densely-defined operator on $\mathrm{L}^2(\mathcal{M})$, we obtain the first-order Sobolev space $\mathrm{W}^{1,2}(\mathcal{G})' = \mathcal{D}(\nabla) = \cap_{i=1}^k \mathcal{D}(A_i)$.

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By also considering ∇ as a closed, densely-defined operator on $\mathrm{L}^2(\mathcal{M})$, we obtain the first-order Sobolev space $\mathrm{W}^{1,2}(\mathcal{G})' = \mathcal{D}(\nabla) = \cap_{i=1}^k \mathcal{D}(A_i)$.

We write the divergence as $\mathrm{div} = -\nabla^*$. Then, the subelliptic Laplacian associated to $\mathcal{A}$ is

$$\Delta = -\mathrm{div}\,\nabla = -\sum_{i=1}^k A_i^2.$$

Nilpotent Lie groups

The Lie group $\mathcal{G}$ is *nilpotent* if the inductively defined sequence $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}],\ \mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}], \ldots$ is eventually zero.

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Theorem (B.-E.-Mc., 2012)

Let $(\mathcal{G}, \mathrm{d}, \mu)$ be a connected, nilpotent Lie group with $\mathfrak{a}$ an algebraic basis, d the associated sub-elliptic distance, and μ the left Haar measure. Suppose that $a, A \in \mathrm{L}^\infty$ and that there exist $\kappa_1, \kappa_2 > 0$ satisfying

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2, \quad \textit{and} \quad \mathrm{Re}\,\langle A\nabla u, \nabla u\rangle \geq \kappa_2 \|\nabla u\|^2.$$

for every $v \in \mathrm{L}^2(\mathcal{G})$ and $u \in \mathrm{W}^{1,2}(\mathcal{G})'$. Then, $\mathcal{D}(\sqrt{-a\,\mathrm{div}\,A\nabla}) = \mathrm{W}^{1,2}(\mathcal{G})'$ and $\|\sqrt{-a\,\mathrm{div}\,A\nabla} u\| \simeq \|\nabla u\|$ for $u \in \mathrm{W}^{1,2}(\mathcal{G})'$.

General Lie groups

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$$\operatorname{Re} \langle av, v \rangle \geq \kappa_1 \|v\|^2, \quad \textit{and} \quad \operatorname{Re} \langle ASu, Su \rangle \geq \kappa_2 \|u\|_{\mathrm{W}^{1,2\prime}}$$

for every $v \in \mathrm{L}^2(\mathcal{G})$ *and* $u \in \mathrm{W}^{1,2}(\mathcal{G})'$*. Then,* $\mathcal{D}(\sqrt{aS^*AS}) = \mathrm{W}^{1,2}(\mathcal{G})'$ *with* $\|\sqrt{aS^*ASu}\| \simeq \|u\|_{\mathrm{W}^{1,2\prime}} = \|u\| + \|\nabla u\|$.

Operator theory

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The primary operator we consider is $\Pi_B = \Gamma + B_1 \Gamma^* B_2$.

If the quadratic estimates

$$\int_0^\infty \|t\Pi_B(1 + t^2\Pi_B^2)^{-1}u\|^2 \simeq \|u\| \tag{Q}$$

hold for every $u \in \overline{\mathcal{R}(\Pi_B)}$, then, $\mathscr{H}$ decomposes into the spectral subspaces of Π_B as $\mathscr{H} = \mathcal{N}(\Pi_B) \oplus E_+ \oplus E_-$ and

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The Kato problems are then obtained by letting $\mathscr{H} = \mathrm{L}^2(\mathcal{M}) \oplus (\mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M}))$ and letting

$$\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix},\ \Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix},\ B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

Geometry and harmonic analysis

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One can show that this is *not* artificial - the Kato problem on functions immediately provides a solution to the dual problem on vector fields.

Elements of the proofs

Similar in structure to the proof of [AKMc] which is inspired from the proof in [AHLMcT].

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- A dyadic decomposition of the space
- A notion of averaging (in an integral sense)
- Poincaré inequality - on both functions and vector fields
- Control of ∇^2 in terms of Δ.

The case of non-smooth metrics on manifolds

We let $\mathcal{M}$ be a smooth, complete manifold as before but now let g be a C^0 metric. Let μ_{g} denote the volume measure with respect to g.

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We let $\mathcal{M}$ be a smooth, complete manifold as before but now let g be a C^0 metric. Let μ_{g} denote the volume measure with respect to g.

Let $\mathrm{h} \in \mathrm{C}^0(\mathcal{T}^{(2,0)}\mathcal{M})$. Then, define

$$\|\mathrm{h}\|_{\mathrm{op},\mathrm{g}} = \sup_{x\in\mathcal{M}} \sup_{|u|_{\mathrm{g}}=|v|_{\mathrm{g}}=1} |\mathrm{h}_x(u,v)|.$$

If $\tilde{\mathrm{g}}$ is another C^0 metric satisfying $\|\mathrm{g}-\tilde{\mathrm{g}}\|_{\mathrm{op},\mathrm{g}} \leq \delta < 1$, then $\mathrm{L}^2(\mathcal{M},\mathrm{g}) = \mathrm{L}^2(\mathcal{M},\tilde{\mathrm{g}})$ and $\mathrm{W}^{1,2}(\mathcal{M},\mathrm{g}) = \mathrm{W}^{1,2}(\mathcal{M},\tilde{\mathrm{g}})$ with comparable norms.

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Thus,

$$\Pi_{B,\mathrm{g}} = \Gamma_{\mathrm{g}} + B_1\Gamma_{\mathrm{g}}^* B_2 = \Gamma_{\tilde{\mathrm{g}}} + B_1 C^{-1}\Gamma_{\tilde{\mathrm{g}}}^* C B_2.$$

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This allows us to reduce the study of $\Pi_{B,\mathrm{g}}$ for a C^0 metric g to the study of $\Pi_{\tilde{B},\tilde{\mathrm{g}}} = \Gamma_{\tilde{\mathrm{g}}} + \tilde{B}_1\Gamma_{\tilde{\mathrm{g}}}^*\tilde{B}_2$ where $\tilde{B}_1 = B_1 C^{-1}$ and $\tilde{B}_2 = CB_2$, but now with a smooth metric $\tilde{\mathrm{g}}$.

Connection to geometric flows

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Smooth the metric via Ricci flow in the general case? Regularity of the initial metric?

Application to PDE

In the case we are able to find a suitable C^{∞} metric near the C^{0} one, then we have Lipschitz estimates.

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"Stability" of geometries with Ricci bounds and injectivity radius bounds?

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