McIntoshery in Geometry

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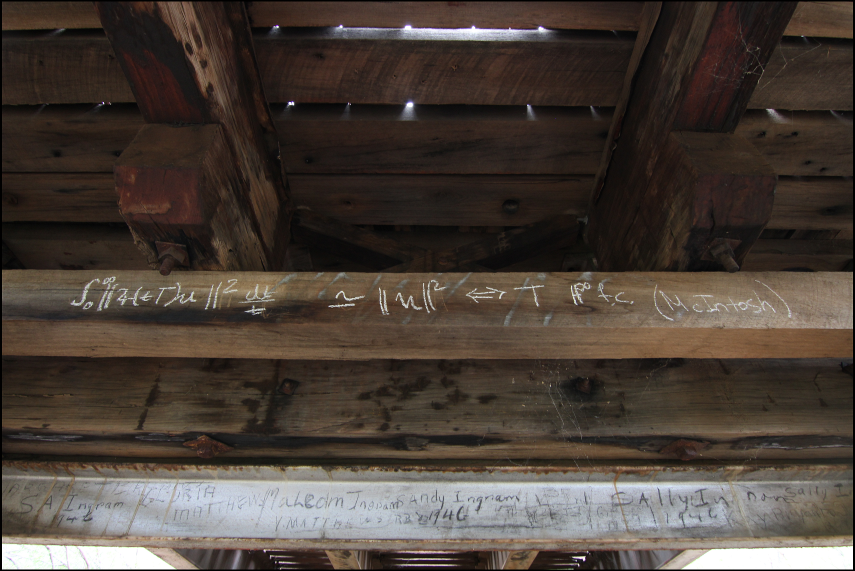
Message from Andreas Rosén



$\int_0^\infty \|tH(tT)u\|^2 \frac{dt}{t} \simeq \|u\|^2 \Leftrightarrow T$ H^∞ f.c. (McIntosh)
SA Ingram
MATTHEW
Malcolm Ingram
Sandy Ingram
1946
Sally In

Part 1 - Spectral flows and harmonic analysis

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- $\star$ Connections to the study of particle physics.
- $\star$ Due to Atiyah and Singer in 1969 in [AS69] on the index theory of skew-adjoint Fredholm operators using topological language.

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Obtain Spin bundle $\not{\Delta}\,\mathcal{M} = \mathrm{P}_{\mathsf{Spin}}(\mathcal{M}) \times_\eta \not{\Delta}\,\mathbb{R}^n$, and Atiyah-Singer Dirac operator $\not{\mathrm{D}}_{\mathrm{g}} : \mathrm{C}^\infty(\not{\Delta}\,\mathcal{M}) \to \mathrm{C}^\infty(\not{\Delta}\,\mathcal{M})$.

An example of a perturbation theorem

Another metric h, the distance to g:

$$\rho_M(\mathrm{g}, \mathrm{h}) = \log\left(\inf\left\{ C \geq 1 : C^{-1}|u|_{\mathrm{g}} \leq |u|_{\mathrm{h}} \leq C|u|_{\mathrm{g}}, u \in \mathrm{T}_x M\right\}\right).$$

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Continuity - requires $\rho_M(\mathrm{g},\mathrm{h}) \leq \varepsilon$ and $\|\nabla^{\mathrm{g}}\mathrm{h}\| \leq \varepsilon$.

Metric perturbations

Theorem (B.-McIntosh-Rosén (2016) in [BMR18])

Suppose $(\mathcal{M}, \mathrm{g})$ *be without boundary and there is a* $\kappa > 0$ *with* $\mathrm{inj}(\mathcal{M}, \mathrm{g}) > \kappa$. *Fix* $C > 0$ *and let* h *be a* $\mathrm{C}^{0,1}$ *metric with* $\rho_M(\mathrm{g}, \mathrm{h}) \leq 1$ *and* $|\nabla^{\mathrm{g}}\mathrm{h}| \leq C$.

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where the implicit constant depends on C, $\dim \mathcal{M}$ *and* C_R *and* κ.

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$$|\nabla^{\mathrm{g}} \mathrm{g}_\varepsilon(0)| = 1.$$

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$\mathrm{D}_{\mathcal{B}} = \mathrm{D}$ with domain

$$\mathcal{D}(\mathrm{D}_{\mathcal{B}}) = \{u \in \mathcal{D}(\mathrm{D}_{\max}) : \mathscr{R}\, u \in \mathcal{B}\},$$

where $\mathscr{R}$ is the trace map.

Boundary value perturbations

Theorem (B.-Rosén (2017))

Let $(\mathcal{M}, \mathrm{g})$ *have* $\Sigma \neq \varnothing$ *with a precompact neighbourhood* Z *of* Σ *and a* $\kappa > 0$ *with* $\mathrm{inj}(\mathcal{M} \setminus Z, \mathrm{g}) > \kappa$.

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Here, $\|\mathcal{B}\|_{\mathrm{Lip}} = \sup_{x\in\Sigma}\left(|\pi_{\mathcal{E}}(x)| + |\nabla\pi_{\mathcal{E}}(x)|\right)$.

Theorem (Cont...)

Then, we have the perturbation estimate

$$\left\| \frac{\not{\mathrm{D}}_{\mathcal{B}}}{\sqrt{1 + \not{\mathrm{D}}_{\mathcal{B}}^2}} - \frac{\not{\mathrm{D}}_{\tilde{\mathcal{B}}}}{\sqrt{1 + \not{\mathrm{D}}_{\tilde{\mathcal{B}}}^2}} \right\|_{\mathrm{L}^2 \to \mathrm{L}^2} \lesssim \hat{\delta}_\infty(\mathcal{B}, \tilde{\mathcal{B}}),$$

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$$\not{\Delta}_{\mathrm{B},x}\,\Sigma = \left\{ (\psi, \vec{\mathrm{n}}(x) \cdot \mathrm{B}(x)\psi) : \psi \in \not{\Delta}^+_x\,\Sigma \right\}$$

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$\not{\mathrm{D}}$-elliptic and self-adjoint whenever $\mathrm{B}(x)^{*} = \mathrm{B}(x)$.

Part 2 - Boundary value problems for first-order elliptic operators

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- $\star$ Boundary conditions - ellipticity, self-adjointness, regularity of eigensections?

Motivation

$\mathcal{M}$ Riemannian Spin manifold, smooth compact boundary Σ,

$$\mathcal{W} := \mathrm{T}\mathcal{M} \otimes \not{\Delta}\,\mathcal{M} \cong \not{\Delta}\,\mathcal{M} \overset{\perp}{\oplus} \not{\Delta}_{\frac{3}{2}}\,\mathcal{M}.$$

Induced Dirac operator

$$\not{D}_{\mathcal{W}} = \begin{pmatrix} \not{D} & \text{💩}^* \\ \text{💩} & \not{D}_{\mathrm{RS}} \end{pmatrix}.$$

Natural questions:

- ⋆ Boundary conditions - ellipticity, self-adjointness, regularity of eigensections?
- ⋆ Fredholmness and index theorems?

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Rarita-Schwinger $\not{D}_{\mathrm{RS}}$ does not give rise to a symmetric A_{RS}.

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(A7) A and $\tilde{\mathrm{A}}$ are adapted differential operators on $\mathcal{E}_\Sigma = \mathcal{E}|_{\partial M}$ and $\mathcal{F}_\Sigma = \mathcal{F}|_{\partial M}$ respectively to the operators D and D^* respectively.

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$$\|R_t u\|_{\mathrm{L}^2(\Sigma)} \lesssim t\|\mathrm{A}u\|_{\mathrm{L}^2(\Sigma)} + \|u\|_{\mathrm{L}^2(\Sigma)}.$$

Bi-sectorial boundary defining operators

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$$\|u\|^2_{\mathrm{H}^1_{\mathrm{D}}} := \|\eta u\|^2_{\mathrm{H}^1} + \|\mathrm{D}u\|^2 + \|u\|,$$

where η is a compactly supported cutoff near the boundary.

Lemma

For $\theta \in (\omega_r, \pi/2)$ fixed, there exists an inner product $\langle \cdot, \cdot \rangle_{N,\theta}$ such that $|A_r|$ is m-θ-accretive and for which the estimate

$$\|(\partial_t + \mathrm{A})u\|^2_{\mathrm{L}^2(Z_{[0,\infty)})} \simeq \|u'\|^2_{\mathrm{L}^2(Z_{[0,\infty)})} + \|Au\|^2_{\mathrm{L}^2(Z_{[0,\infty)})} - \mathrm{Re}\,\langle |A_r|\,\mathrm{sgn}(A_r)\ u_0, u_0\rangle_{N,\theta} - r\|u_0\|^2_{N,\theta},$$

holds for $u \in \mathrm{C}^\infty_{\mathrm{c}}(Z_{[0,\infty)}; E)$ where $u_0 = u|_\Sigma$.

For any equivalent norm $\| \cdot \|_\star \simeq \| \cdot \|_{\mathrm{L}^2(\mathcal{E}_\Sigma)}$,

$$
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(i) Via Similarity Theorem due to Callier–Grabowski–Le Merdy (c.f. [Haa06] by Haase), there exists $\langle\cdot,\cdot\rangle_\theta$ so that $|\mathrm{A}_r|$ is m-θ-accretive.

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$$\begin{aligned}\big\langle u', |\mathrm{A}_r|\,\mathrm{sgn}(A_r)u\big\rangle_{N,\theta} - \big\langle |\mathrm{A}_r|\,\mathrm{sgn}(A_r)u', u\big\rangle_{N,\theta} \\ = a(\mathrm{sgn}(\mathrm{A}_r)u', u)^{\mathrm{conj}} - a(u, \mathrm{sgn}(\mathrm{A}_r)u') \in \mathrm{Im}\,\mathbb{R}\end{aligned}$$

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Higher regularity

Banach-valued Cauchy problem: $f \in \mathrm{L}^2(Z_{[0,\rho]}, E)$,

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Key estimate:

$$\int_0^\rho \|\partial_t W(t; f)\|^2_{\mathrm{L}^2(\mathcal{E}_\Sigma)}\, dt + \int_0^\rho \||A_r|\, W(t; f)\|^2_{\mathrm{L}^2(\mathcal{E}_\Sigma)} \lesssim \int_0^\rho \|f(t)\|^2_{\mathrm{L}^2(\mathcal{E}_\Sigma)}.$$

Theorem

The following holds:

$$\mathcal{D}(\mathrm{D}_{\mathsf{max}}) \cap \mathrm{H}^{\mathrm{k}+1}_{\mathrm{loc}}(\mathcal{E}) = \{u \in \mathcal{D}(\mathrm{D}_{\mathsf{max}}) : Du \in \mathrm{H}^{\mathrm{k}}_{\mathrm{loc}}(\mathcal{F}) \text{ and } \chi^+(A_r)(u|_\Sigma) \in \mathrm{H}^{\mathrm{k}+\frac{1}{2}}(\mathcal{E}_\Sigma)\}.$$

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