FU Seminar — Ecker.   07/05/2018.
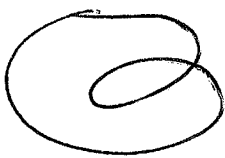
Motivated by Perelman .., by "Adapted".

$\Omega \subset \mathbb{R}^{n+1}$, ~~weird~~ $f : \overline{\Omega} \to \mathbb{R}$. smooth (~~and~~
                                                           (or admissible
                                                            means find
                                                            exp. indus sense)
$\tau > 0$, $\beta : \partial\Omega \to \mathbb{R}$. $\partial\Omega$ "Reasonable".

(later β = M.C. or weak M.C.).

  $(X \text{ Excluded})$.

$$W_\beta(\Omega, f, \tau) := \int \left[\tau |\nabla f|^2 + f - (n+1) \right] u \, d\mu.$$

$$+ 2\tau \int_{\partial\Omega} \beta u \; ; \; u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}},$$

$$M_\beta(\Omega, \tau) := \inf \left\{ W_\beta(\Omega, f, \tau) : \int_\Omega u = 1 \right\}.$$

$$\gamma_\beta(\Omega) := \inf_{\tau > 0} M_\beta(\Omega, \tau).$$

To study $M_\beta(\Omega, I)$ and even existence of $W_\beta(\Omega, f, \tau)$,
one needs some version of log-Sobolev.
+ Jensen + Std Sobolev embedding th$^{\perp}$.   ①

<u>Non-optimal version.</u> If Ω satisfies Gal.-Nirenberg:

$$\left(\int_\Omega |w|^{\frac{n+1}{n}} \, dx\right)^{\frac{n}{n+1}} \leq C_S(\Omega) \cdot \int_\Omega \left(|\nabla w| + |w|\right) dx.$$

$$\forall w \in C^1(\bar\Omega) \quad \left(\text{or } BV(\Omega)\right)$$

then $\forall \varepsilon > 0$, $\forall \varphi \in C^1(\bar\Omega)$ with $\int_\Omega \varphi^2 = 1$. there holds.

$$\int_\Omega \left(\varepsilon |\nabla \varphi|^2 - \varphi^2 \log \varphi^2\right) dx - c(w)\left(1 + \log C_S(\Omega)\right) - \frac{1}{\varepsilon}$$

(proved in Lec 1).

<u>Rmk</u>. Existence of minimizer., upper + lower bound of $M_\beta(\Omega, \tau)$ still to be done.

<u>Fact:</u> Was proven in lecture 1: $M_0(\mathbb{R}^{n+1}, \tau) \geq 0 \ \forall \tau > 0$.

Write $u = q^2 \cdot \dfrac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{n+1}{2}}}$, then $\cancel{W_0(\mathbb{R}^{n+1})}$

$$W_0(\mathbb{R}^{n+1}, f, \tau) = \varepsilon \cdot \mathcal{E}(q, \tau),$$

$$\mathcal{E}(q, \tau) = \int_{\mathbb{R}^{n+1}} \left(4\tau |\nabla q|^2 - q^2 \log q^2\right) \cancel{\cdots} \cdot \underbrace{\frac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{\frac{n+1}{2}}}}_{\text{Gauss meas.}} \cdot dx$$

Th^m:

log-Sob optimal version., Ω open, convex ($\partial\Omega \in C^2$ say).

Then, $\forall q$ with $\int_\Omega q^2 \gamma_{n+1} = 1$,

$$\int_\Omega q^2 \log q^2 \, \gamma_{n+1} + \log\left(\int_\Omega \gamma_{n+1}\right) \leq 2\int_\Omega |\nabla q|^2 \gamma_{n+1}$$

Pf Incomplete to fun, consider. $\mathcal{E}[q] = \underbrace{\int_\Omega \left(2|\nabla q|^2 + q^2 \log q^2\right)\gamma_{n+1}}_{C(q)}$.

$$L = \Delta - x\cdot\nabla, \qquad O-U.$$

$$\int_\Omega w\, Lv\, \gamma_{n+1} = \int_\Omega Lw\, v\cdot\gamma_{n+1} \qquad \left(\begin{array}{l}\text{with Neumann}\\ \text{data}\\ \nabla w\cdot\gamma = \nabla v\cdot\gamma = 0\end{array}\right)$$

$\Rightarrow$ Euler–Lagrange equation for $\mathcal{E}$.

$$\int_\Omega Lw\, \gamma_{n+1} = 0 \quad \text{if} \quad \nabla w\cdot\gamma = 0 \text{ on } \partial\Omega.$$

(constants are in $N(L)$ because γ_{n+1} finite mass).

~~Assume let~~

$\uparrow$ Let q_0 be the function s.t. $\nabla q_0 \cdot \gamma = 0$, so in particular.

$$\nabla q_0^2 \cdot \gamma = 0.$$

$\left(\circledast\right)$ (This means that $q_0 \in H^1(\Omega^2, \gamma_{n+1})$).

③

Now. let $q^2(t)$, $t>0$, be a solⁿ of

$$\begin{cases} \left(\frac{\partial}{\partial t} - L\right) q^2(t) = 0 & \text{in } \Omega \; t>0. \\ q^2(0) = q_e^2 & \text{in } \Omega. \\ \nabla q^2(t) \cdot \nu = 0 & \text{in } \partial\Omega. \end{cases}$$

We then showed using Bochner $(\partial_t - L) e(q) \leq 0$,

Hence.
$$\partial_t \int_\Omega e(q(t)) \, r_{n+1} \leq -4 \int_\Omega A_{\partial\Omega}\left(\nabla^{\partial\Omega} q_t, \nabla^{\partial\Omega} q_t\right) r_{n+1}$$

tangential ∇.

$$\underline{crude.} \longrightarrow \; \leq 0 \quad \text{if} \quad A_{\partial\Omega} \geq 0.$$

$$\Rightarrow \quad \partial_t \, \mathcal{E}(q(t)) \leq 0 \quad \forall t \geq 0.$$

but. $c := \dfrac{\int q_{t_0}^2 \, r_{n+1}}{\int_\Omega r_{n+1}} = \dfrac{1}{\int r_{n+1}}$. since $\int_\Omega q_0^2 r_{n+1} = 1$.

$\nabla q^2(t) \cdot \nu = 0$ we have. $\partial_t \int_\Omega q^2(t) r_{n+1} = \int_\Omega L q^2(t) r_{n+1}$

$= \int \left(\frac{\partial}{\partial t} - L\right) q^2(t) \, r_{n+1} = 0$ for $1 \in N(L)$.

$\swarrow$ Neumann.

$= \int L q^2(t) \cdot 1 \, r_{n+1}, \quad = 0.$ ④

$\Rightarrow \int (\partial_t - L) q^2(t) r_{n+1} = 0.$

Show: $q^2(t) \to c$ in $L^2(r_{n+1})$, at an exponential
 rate.
 $t \to 0$

$\exists \delta > 0$ dep only in Ω and $\forall w \in C^1(\bar{\Omega})$,

$$\delta \int_\Omega |w - \overline{w}|^2 \, r_{n+1} \leq \int_\Omega |\nabla w|^2 \, r_{n+1}.$$

$$\frac{d}{dt} \int_\Omega |q^2(t) - c|^2 \, r_{n+1} = 2 \int_\Omega |q_{(t)}^2 - c| \frac{\partial q^2}{\partial t} \, r_{n+1}.$$

$$= 2 \int_\Omega |q^2 - c| \, \Delta q^2 \, \partial_{n+1}.$$

$$= -2 \int_\Omega |\nabla q^2|^2 \, r_{n+1}. \quad \text{Since } \nabla c = 0.$$

Poincaré $w \in c^2$.

$$\leq -2, -2\delta \int_\Omega (q^2 - c)^2 \, r_{n+1}.$$

$$\Rightarrow \int_\Omega (q^2(t) - c)^2 \, r_{n+1} \leq e^{-2\delta t} \int_\Omega (q_0^2 - c)^2 \, \partial_{n+1}.$$

We will prove: $\mathcal{E}(q(t)) \leq \mathcal{E}(q_0) < \infty$.

 $\exists c$ indep of t s.t. $\int_\Omega |\nabla q(t)|^2 \, r_{n+1} \leq c.$