The world of rough metrics

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Divergence form equations

Let $x \mapsto A(x) = (a_{ij}(x))$ symmetric, measurable matrix function.

Suppose $\kappa > 0$ and $\Lambda < \infty$ such that $x\ \mathcal{L}$ a.e.,

$$\kappa |u|^2_{\mathbb{R}^n} \leq A(x)u \cdot u \leq \Lambda |u|^2_{\mathbb{R}^n}. \tag{1}$$

Then,

$$\mathrm{L}_A = \mathrm{d}^* A \mathrm{d} = -\sum_{i,j=1}^{n} \partial_j (a_{ij}) \partial_i$$

non-negative self-adjoint operator on $\mathrm{L}^2(\mathbb{R}^n, \mathcal{L})$.

- Lax-Milgram theorem circa 1954,
- *a priori estimates* of De Giorgi-Moser-Nash circa 1957:
  if $u \in \mathrm{dom}(\mathrm{L}_A)$ such that $\mathrm{L}_A u = 0 \implies u$ Hölder continuous.

Geometric perspective

Let g be smooth metric tensor on $\mathbb{R}^n \implies x \mapsto B(x)$ such that $\mathrm{g}_x(u,v) = \delta(B(x)u,v) = B(x)u \cdot v$.

If g satisfies: $\exists C \geq 1$ s.t.

$$C^{-1}|u|_\delta \leq |u|_\mathrm{g} \leq C|u|_\delta,$$

then B satisfies (1).

Measure: $\mu_\mathrm{g} = \sqrt{\det B}\mathcal{L}$.

Laplacian: $\Delta_\mathrm{g} = \mathrm{d}_\mathrm{g}^* \mathrm{d}.$

For $u \in \operatorname{dom}(\Delta_{\mathrm{g}})$ and $v \in \mathrm{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$:

$$\begin{aligned}
\langle \Delta_{\mathrm{g}} u, v \rangle_{\mathrm{L}^2(\mathbb{R}^n, \mu_{\mathrm{g}})} &= \int_{\mathbb{R}^n} \mathrm{g}(\mathrm{d}u, \overline{\mathrm{d}v})\ d\mu_{\mathrm{g}} \\
&= \int_{\mathbb{R}^n} (B\mathrm{d}u) \cdot \overline{v}\ (\det B)^{\frac{1}{2}}\ d\mathcal{L} \\
&= \int_{\mathbb{R}^n} ((\det B)^{\frac{1}{2}} B\mathrm{d}u) \cdot \overline{\mathrm{d}v}\ d\mathcal{L} \\
&= \int_{\mathbb{R}^n} \mathrm{d}^{*,\delta}((\det B)^{\frac{1}{2}} B)\mathrm{d}u\ \overline{v} \quad (\det B)^{-\frac{1}{2}}\ d\mu_{\mathrm{g}} \\
&= \left\langle (\det B)^{-\frac{1}{2}} \mathrm{d}^{*,\delta}((\det B)^{\frac{1}{2}} B)\mathrm{d}u, v \right\rangle_{\mathrm{L}^2(\mathbb{R}^n, \mu_{\mathrm{g}})}.
\end{aligned}$$

I.e.

$$\Delta_{\mathrm{g}} = (\det B)^{-\frac{1}{2}} \mathrm{d}^{*,\delta}((\det B)^{\frac{1}{2}} B)\mathrm{d}.$$

Kato's square root problem

$(\mathcal{M}, \mathrm{g})$ Riemannian manifold.

Operator $\mathrm{L}_{B,B_0} := \mathrm{d}_{\mathrm{g}}^* B\mathrm{d} + B_0$, with $\mathbb{C}$-valued coefficients.

Assume: $\kappa_1 \leq B_0(x) \leq \kappa_2$, $B \in \mathrm{L}^\infty(\mathcal{M}; \mathrm{End}(\mathrm{T}^*\mathcal{M}))$ and $x - \mu - \text{a.e.}$,

$$\mathrm{Re}\ \mathrm{g}_x(B(x)u, u) \geq \kappa |u|_{\mathrm{g}_x}^2.$$

On $\mathscr{H} := \mathrm{L}^2(\mathcal{M}) \oplus \left(\mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathcal{M}; \mathrm{T}^*\mathcal{M})\right)$:

$$\Pi_{\mathrm{g}}(B, B_0) = \begin{pmatrix} 0 & S^* X \\ S & 0 \end{pmatrix}, \qquad \Pi_{\mathrm{g}}(B, B_0)^2 = \begin{pmatrix} \mathrm{L}_{B,B_0} & 0 \\ 0 & SS^*X \end{pmatrix}.$$

where $S : \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathcal{M}; \mathrm{T}^*\mathcal{M})$ given by $Su = (u, \mathrm{d}u)$.

N.B. $\Pi_{\mathrm{g}}(B, B_0)$ *not* self-adjoint - only ω-bisectorial for $\omega < \pi/2$.

Key estimate:

$$\|f(\Pi_{\mathrm{g}}(B, B_0))\|_{\mathrm{L}^2 \to \mathrm{L}^2} \lesssim \|f\|_\infty. \tag{2}$$

for f bounded on a sector containing the spectrum of L_{B,B_0}, and holomorphic in the interior of the sector. I.e., $f(\zeta) = e^{-\zeta}$.

Implies: $\mathrm{dom}(\sqrt{\mathrm{L}_{B,B_0}}) = \mathrm{H}^1(\mathcal{M}, \mathrm{g})$.
(Kato square root problem problem c.f [AHL$^+$02, AKM06, BM16])

If h another metric on $\mathcal{M}$ s.t. $\exists C \geq 1$, $C^{-1}|u|_{\mathrm{g}} \leq |u|_{\mathrm{h}} \leq C|u|_{\mathrm{g}}$:

(2) holds for g $\quad\Longleftrightarrow\quad$ (2) holds for h.

Idea: $\Pi_{\mathrm{h}}(B, B_0) = \Pi_{\mathrm{g}}(\tilde{B}, \tilde{B}_0)$.

Why?

$$\|\mathrm{e}^{-t(\Delta_{\mathrm{g}}+I)} - \mathrm{e}^{-t(\Delta_{\mathrm{h}}+I)}\| \lesssim \|\mathrm{g} - \mathrm{h}\|_{\mathrm{L}^\infty}.$$

Fix $\mathcal{M}$ a smooth manifold. I.e. a smooth differentiable structure for $\mathcal{M} \rightsquigarrow$ exterior derivative d.

A measure structure for free:

- $A \subset \mathcal{M}$ *measurable* if for all charts (ψ, U), $\psi(A \cap U) \subset \mathbb{R}^n$ $\mathcal{L}$-measurable.
- $A \subset \mathcal{M}$ *null measure* if for all charts (ψ, U), $\mathcal{L}(\psi(A \cap U)) = 0$.

A measurable $\iff$ μ_{g}-measurable for any smooth g.
A-null measure $\iff$ $\mu_{\mathrm{g}}(A) = 0$.

For any $\mathcal{V} \to \mathcal{M}$ vector bundle, we can talk about $\mathbf{\Gamma}_{\mathrm{R}}(\mathcal{V})$ - measurable sections of $\mathcal{V}$ without a metric on $\mathcal{M}$.

Definition (Rough metric)

Let $\mathrm{g} \in \mathbf{\Gamma}_{\mathrm{R}}(\mathrm{Sym}\ \mathrm{T}^*\mathcal{M} \otimes \mathrm{T}^*\mathcal{M})$ such that: $\forall x \in \mathcal{M}\ \exists (U, \psi)$ chart around x and $\exists C \geq 1$ such that

$$C^{-1} |u|_{(\psi^* \delta)(y)} \leq |u|_{\mathrm{g}(y)} \leq C |u|_{(\psi^* \delta)(y)},$$

$y - \mathrm{a.e.} \in U$ and where δ is the Euclidean metric.

Say g is a *rough metric*.
Chart (U, ψ) satisfies the *local comparability condition*.

- Well-defined induced Radon measure via locally comparable charts:

$$d\mu_{\mathrm{g}}(x) = \sqrt{\det \mathrm{g}(x)}\ d\psi^* \mathcal{L}.$$

- A μ_{g}-measurable $\iff$ A measurable.
- $\mu_{\mathrm{g}}(A) = 0 \iff A$ null-measure.

Lebesgue and Sobolev Spaces

Tensor bundle: $\mathcal{T}^{(p,q)}\mathcal{M} := (\otimes_{i=0}^{p} \mathrm{T}^*\mathcal{M}) \otimes (\otimes_{i=0}^{q} \mathrm{T}\mathcal{M})$.
Metric g extends to $\mathcal{T}^{(p,q)}\mathcal{M}$.

$u \in \mathrm{L}^p(\mathcal{T}^{(p,q)}\mathcal{M}, \mathrm{g})$ for $p \in (1, \infty) \iff \int_{\mathcal{M}} |u(x)|_{\mathrm{g}(x)}^p \, d\mu_{\mathrm{g}}(x) < \infty.$

Similarly $u \in \mathrm{L}^\infty(\mathcal{T}^{(p,q)}\mathcal{M}, \mathrm{g})$ if $\exists C < \infty$ such that $|u(x)|_{\mathrm{g}(x)} \leq C$ $x - \mathrm{a.e.}$.

Operator $\mathrm{d}_p = \mathrm{d} : \mathrm{C}^\infty \cap \mathrm{L}^p(\mathcal{M}, \mathrm{g}) \to \mathrm{C}^\infty \cap \mathrm{L}^p(\mathcal{M}; \mathrm{T}^*\mathcal{M}, \mathrm{g})$ *closable* in $\mathrm{L}^p(\mathcal{M}, \mathrm{g})$.

Define:

$$\mathrm{W}^{1,p}(\mathcal{M}, \mathrm{g}) := \mathrm{dom}(\overline{\mathrm{d}_p}), \quad \mathrm{W}_0^{1,p}(\mathcal{M}, \mathrm{g}) := \overline{\mathrm{C}_\mathrm{c}^\infty(\mathcal{M})}^{\|\cdot\|_{\mathrm{W}^{1,p}}}.$$

See [Ban16].

Laplacian

Note: $\mathrm{H}^1(\mathcal{M}, \mathrm{g}) := \mathrm{W}^{1,2}(\mathcal{M}, \mathrm{g})$ is a Hilbert space.

Define: $\mathscr{E}[u, v] := \langle \mathrm{d}u, \mathrm{d}v \rangle_{\mathrm{L}^2(\mathcal{M}, \mathrm{g})}$.

$\mathcal{W} \subset \mathrm{H}^1(\mathcal{M}, \mathrm{g})$ closed subspace such that $\mathrm{H}^1_0(\mathcal{M}, \mathrm{g}) \subset \mathcal{W}$.

$\mathscr{E}_{\mathcal{W}} = \mathscr{E}$ with $\mathrm{dom}(\mathscr{E}_{\mathcal{W}}) = \mathcal{W} \quad \rightsquigarrow \quad \Delta_{\mathrm{g},\mathcal{W}} := \mathrm{d}^{*,\mathrm{g}}_{\mathcal{W}} \mathrm{d}_{\mathcal{W}}$.
Satisfies: $\mathrm{dom}(\sqrt{\Delta_{\mathrm{g},\mathcal{W}}}) = \mathcal{W}$.

$\mathcal{W} = \mathrm{H}^1(\mathcal{M}, \mathrm{g}) \rightsquigarrow \Delta_N$ "Neumann Laplacian",
$\mathcal{W} = \mathrm{H}^1_0(\mathcal{M}, \mathrm{g}) \rightsquigarrow \Delta_D$ "Dirichlet Laplacian".

$\mathcal{M}$ compact $\partial\mathcal{M} = \varnothing$: $\mathrm{W}^{1,p}(\mathcal{M}, \mathrm{g}) = \mathrm{W}^{1,p}_0(\mathcal{M}, \mathrm{g}) = \mathrm{W}^{1,p}(\mathcal{M})$.

$\mathrm{H}^1(\mathcal{M}, \mathrm{g}) = \mathrm{H}^1_0(\mathcal{M}, \mathrm{g}) \iff \Delta_N = \Delta_D$.

☣ In general $\mathrm{dom}(\Delta_D) \neq \mathrm{H}^2(\mathcal{M})$. ☣

Examples

1. $\mathcal{M}$ any smooth manifold, g a C^∞ complete Riemannian metric, $\exists \eta \in \mathbb{R}$ s.t. $\mathrm{Ric}(\mathrm{g}) \geq \eta \mathrm{g}$. Then,
$$\Delta_N = \Delta_D = \mathrm{d}_2^{*,\mathrm{g}} \overline{\mathrm{d}_2} = -\overline{\mathrm{tr}\, \nabla^{\mathrm{T}^*\mathcal{M}} \mathrm{d}_2} = -\overline{\mathrm{tr}\, \nabla^{\mathrm{T}\mathcal{M}} \nabla} := \Delta,$$
and $\mathrm{dom}(\Delta) = \mathrm{H}^2_0(\mathcal{M}, \mathrm{g}) = \mathrm{H}^2(\mathcal{M}, \mathrm{g}) = \left\{u \in \mathrm{L}^2(\mathcal{M}) : u \in \mathrm{H}^1(\mathcal{M}, \mathrm{g}), \nabla^{\mathrm{g}} \mathrm{d}u \in \mathrm{L}^2(\mathcal{M}; \mathcal{T}^{(2,0)}\mathcal{M})\right\}$.

2. $\mathcal{M}$ any smooth manifold, g a C^0 Riemannian metric $\implies$ g rough metric.

3. $\mathcal{M} = \Omega \subset \mathbb{R}^n$, bounded smooth domain, $\mathrm{g} = \delta$. $\Delta_D = -\sum_{j=1}^n \partial_j^2$ with Dirichlet BCs, $\Delta_N = -\sum_{j=1}^n \partial_j^2$ with Neumann BCs.

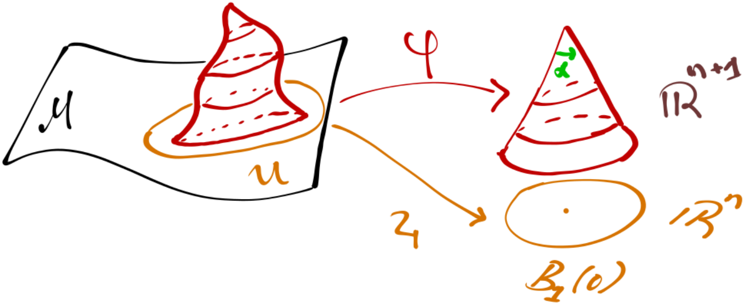
4. $f : \mathbb{R}^n \to \mathbb{R}$ Lipschitz. Pullback metric $f^* \delta_{n+1}(u, v) = \mathrm{d}fu \cdot \mathrm{d}fv$ rough metric on $\mathbb{R}^n$.

5. $\mathcal{M}$, $\mathcal{N}$ smooth manifolds, $\varphi : \mathcal{M} \to \mathcal{N}$ Lipeomorphism. If h a C^0 Riemannian metric on $\mathcal{N}$, $\varphi^* \mathrm{h}$ rough metric on $\mathcal{M}$.

6. $\mathcal{M}$ smooth, (ψ, U) chart such that $\psi(U) = B_1(0)$. For $\alpha \in (0, \pi]$, let

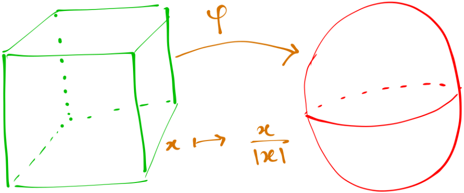
$$\varphi(x) = \left(\psi(x), \cot\left(\frac{\alpha}{2}\right)(1 - |\psi(x)|_{\mathbb{R}^n})\right).$$

Suppose $g|_{\mathcal{M}\setminus U} \in C^\infty$ and $g|_U = \psi^* \delta_{n+1}$.



Then, g is a rough metric and $g = dr^2 + \sin^2(\alpha) r^2 dy^2$ in polar coordinates around x.

7. $\square^n = \partial[-1,1]^{n+1}$ Euclidean cube.
$\varphi : \square^n \to \mathrm{S}^n \subset \mathbb{R}^{n+1}$, radial projection $\varphi(x) := \frac{x}{|x|}$.



$\mathbf{d}_{\square^n}(x,y) \simeq \mathbf{d}_{\mathrm{S}^n}(\varphi(x),\varphi(y)) \rightsquigarrow \varphi^{-1} : \mathrm{S}^n \to \square^n$ Lipeomorphism.

$(\mathrm{S}^n, (\varphi^{-1})^*(\delta|_{\square^n}))$ isometric to $\square^n \subset \mathbb{R}^{n+1}$.

$\exists B \in \mathbf{\Gamma}_{\mathrm{R}}(\mathrm{Sym\ End}\ (\mathrm{T}^*\mathrm{S}^n))$ such that
$(\varphi^{-1})^*(\delta|_{\square^n})_x(u,v) = \mathrm{g}_{\mathrm{S}^n\,x}(Bu,v)$, x a.e.

$$\Delta_{\square^n} = \mathrm{d}^*_{\square^n}\mathrm{d}_{\square^n} = \varphi^*(\det B)^{-\frac{1}{2}}\mathrm{d}^{*,\mathrm{S}^n}((\det B)^{\frac{1}{2}}B)\mathrm{d}^{\mathrm{S}^n}(\varphi^{-1})^*.$$

Weyl Asymptotics

Theorem (B.-Nursultanov-Rowlett 2018 [BNR20])

$\mathcal{M}$ compact with smooth boundary $\partial\mathcal{M}$. $\mathcal{W} \subset \mathrm{H}^1(\mathcal{M}, \mathrm{g})$ closed subspace, $\mathrm{H}^1_0(\mathcal{M}, \mathrm{g}) \subset \mathcal{W}$. Then,

(i) *$\Delta_{\mathrm{g},\mathcal{W}}$ has discrete non-negative spectrum with finite dimensional eigenspaces, and*

(ii) *Letting $N(\lambda, \Delta_{\mathrm{g},\mathcal{W}})$ be the number of eigenvalues $\leq \lambda$,*

$$\lim_{\lambda\to\infty} \frac{N(\lambda, \Delta_{\mathrm{g},\mathcal{W}})}{\lambda^{\frac{n}{2}}} = \frac{\omega_n}{(2\pi)^n}\, \mu_{\mathrm{g}}(\mathcal{M}).$$

- $\mathrm{dom}(\Delta_{\mathrm{g},\mathcal{W}}) \subset \mathrm{dom}(\sqrt{\Delta_{\mathrm{g},\mathcal{W}}}) = \mathcal{W} \subset \mathrm{H}^1(\mathcal{M},\mathrm{g}) = \mathrm{H}^1(\mathcal{M})$.
- $\Delta_{\mathrm{g},\mathcal{W}}$ self-adjoint $\implies$ $(\imath + \Delta_{\mathrm{g},\mathcal{W}})^{-1} : \mathrm{L}^2(\mathcal{M}) \to \mathrm{dom}(\Delta_{\mathrm{g},\mathcal{W}}) \subset \mathrm{H}^1(\mathcal{M}) \overset{\text{compact}}{\hookrightarrow} \mathrm{L}^2(\mathcal{M})$.
- Lack of distance: how to do domain monotonicity?
- Cover $\mathcal{M}$ *almost-everywhere* by *mutually disjoint* Lipschitz domains in locally comparable charts (ψ, U).
- $\exists B_\psi \in \mathrm{L}^\infty(\mathrm{Sym\ End}(\mathrm{T}^*U))$ s.t. for all $v \in \mathrm{C}^\infty_{\mathrm{c}}(\mathring{U})$

$$\langle \Delta_{\mathrm{g},\mathcal{W}} u, v \rangle_{\mathrm{L}^2(\mathcal{M},\mathrm{g})} = \left\langle B_\psi \mathrm{d}^{\mathbb{R}^n} \psi^* u, \mathrm{d}^{\mathbb{R}^n} v^* \right\rangle_{\mathrm{L}^2(U;\sqrt{\det B_\psi}\ d\mathcal{L})}.$$

- Results of Birman-Solomjak [BS72] yield asymptotics in (ψ, U) for Dirichlet and Neumann problems of induced operator in $\varphi(U)$.
- Patch (carefully).

Heat equation

Setting so far: $\mathcal{M}$ manifold, d from differentiable structure, g rough metric $\rightsquigarrow \mu_{\mathrm{g}}$ and L^p, $\mathrm{W}^{1,p}$. *No distance*.

$\mathcal{W} \subset \mathrm{H}^1(\mathcal{M}, \mathrm{g})$ closed subspace, $\mathrm{H}^1_0(\mathcal{M}, \mathrm{g}) \subset \mathcal{W}$, and $\mathrm{C}^\infty(\mathcal{M}) \cap \mathcal{W}$ dense in $\mathcal{W}$.

$\mathrm{d}_{\mathcal{W}} = \mathrm{d}$ with $\mathrm{dom}(\mathrm{d}_{\mathcal{W}}) = \mathcal{W} \quad \rightsquigarrow$ Laplacian: $\Delta_{\mathrm{g},\mathcal{W}} := \mathrm{d}_{\mathcal{W}}^{*,\mathrm{g}} \mathrm{d}_{\mathcal{W}}$.

$u \in \mathrm{C}^1\left((0,\infty), \mathrm{dom}(\Delta_{\mathrm{g},\mathcal{W}})\right)$ solution to the $\Delta_{\mathrm{g},\mathcal{W}}$-heat equation with initial condition $u_0 \in \mathrm{L}^2(\mathcal{M}, \mathrm{g})$ if:

(i) $\partial_t u(\cdot, t) = \Delta_{\mathrm{g},\mathcal{W}} u(\cdot, t) \quad \forall t \in (0, \infty)$
(ii) $\lim_{t \to 0} u(\cdot, t) = u_0$ in $\mathrm{L}^2(\mathcal{M}, \mathrm{g})$.

Borel functional calculus $\rightsquigarrow$ every such solution u uniquely given by:

$$u(\cdot, t) = \mathrm{e}^{-t\Delta_{\mathrm{g},\mathcal{W}}} u_0.$$

Heat kernels

$(t,x,y) \mapsto \rho_t^{g,\mathcal{W}}(x,y) : (0,\infty) \times \mathcal{M} \times \mathcal{M}$ separably measurable, almost-everywhere symmetric in (x,y) is a heat kernel if:

(i) $\lim_{t\to 0} \rho_t^{g,cW}(\cdot,y) = \delta_y$ (delta mass at y),

(ii) if u solution to the heat equation with initial data $u_0 \in L^2(\mathcal{M},g)$,

$$u(t,x) = \int_{\mathcal{M}} \rho_t^{g,\mathcal{W}}(x,y) u_0(y)\, d\mu_g(y).$$

Idea:

1. Show that for $\mathrm{a.e.} - x \in \mathcal{M}$, $\exists C_t < \infty$ s.t. for $v \in L^2(\mathcal{M},g)$,
$$|(e^{-t\Delta_{g,\mathcal{W}}} v)(x)| \le C_t \|v\|_{L^2}. \tag{3}$$
2. Implies $(v \mapsto e^{-t\Delta_{g,\mathcal{W}}} v)(x) \in L^2(\mathcal{M},g)^*$. Riesz Representation theorem: $\exists a_{t,x} \in L^2(\mathcal{M},g)$ such that $(e^{-t\Delta_{g,\mathcal{W}}} v)(x) = \langle a_{t,x}, v\rangle_{L^2}$.
3. Write $\rho_t^{g,\mathcal{W}}(x,y) := \left\langle a_{\frac{t}{2},x}, a_{\frac{t}{2},y} \right\rangle_{L^2}$.
4. Beurling-Deny condition $\|\sqrt{\Delta_{g,\mathcal{W}}}|u|\|_{L^2} \le \|\sqrt{\Delta_{g,\mathcal{W}}} u\|_{L^2} \implies \rho_t^{g,\mathcal{W}} \ge 0$.

$\mathcal{M}$ compact boundaryless $\implies$ $\mathcal{W} = \mathrm{H}^1_0(\mathcal{M}, \mathrm{g}) = \mathrm{H}^1(\mathcal{M}, \mathrm{g}) \rightsquigarrow$ unique Δ_{g}.

Fix h smooth auxiliary metric. $\exists B \in \mathrm{L}^\infty(\mathcal{M}; \mathrm{Sym\ End}(\mathrm{T}^*\mathcal{M}), \mathrm{h})$ s.t. $\mathrm{g}(u, v) = \mathrm{h}(Bu, v)$. Then,

$$\Delta_{\mathrm{g}} = -\theta^{-1} \mathrm{d}^{*,\mathrm{h}} (B\theta) \overline{\mathrm{d}}.$$

$\exists \eta \in \mathbb{R}$ s.t. $\mathrm{Ric}(\mathrm{h}) \geq \eta \mathrm{h}$. Saloff-Coste in [SC92] $\implies$ parabolic Harnack estimates for $u \geq 0$ satisfying

$$\partial_t u = -\theta^{-1} \mathrm{d}^{*,\mathrm{h}} (B\theta) \overline{\mathrm{d}} u = \Delta_{\mathrm{g}} u.$$

Implies (3), and $\exists \alpha > 0$ s.t.

$$(t, x, y) \mapsto \rho^{\mathrm{g}}_t(x, y) \in \mathrm{C}^\omega((0, \infty); \mathrm{C}^\alpha(\mathcal{M} \times \mathcal{M})).$$

See [Ban17].

Example: $\mathrm{g} = \varphi^* \mathrm{h}$, h smooth, $\varphi : \mathcal{M} \to \mathcal{M}$ Lipeomorphism.

$$(x, y) \to \rho^{\mathrm{g}}_t(x, y) = \rho^{\mathrm{h}}_t(\varphi(x), \varphi(y)) \in \mathrm{C}^{0,1}(\mathcal{M} \times \mathcal{M}).$$

Theorem (B.-Bryan 2019 [BB20])

For $\mathcal{M}$ *smooth manifold,* g *rough metric,* $\mathcal{W}$ *subspace as before, there exists a unique heat kernel* $\rho_t^{\mathrm{g},\mathcal{W}}$ *satisfying:*

(i) $\rho_t^{\mathrm{g},\mathcal{W}} > 0$ *for* $t > 0$,

(ii) $\forall K \Subset \mathcal{M}$, $\forall 0 < t_1 < t_2$, $\exists \alpha(K, t_1, t_2)$ *such that*

$$(t, x, y) \mapsto \rho_t^{\mathrm{g},\mathcal{W}} \mathrm{C}^\omega \left([t_1, t_2]; \mathrm{C}^{\alpha(K,t_1,t_2)}(K \times K) \right).$$

1. Cover $\mathcal{M}$ by locally comparable charts. Inside (U, ψ): $\exists B_\psi \in \mathrm{L}^\infty(U; \mathrm{Sym\ End}(\mathrm{T}^*\mathcal{M}))$ such that $\forall v \in \mathrm{C}^\infty_{\mathrm{c}}(U)$,
$$\langle \Delta_{\mathrm{g},\mathcal{W}} u, v \rangle_{\mathrm{L}^2(\mathcal{M},\mathrm{g})} = \left\langle B_\psi \mathrm{d}^{\mathbb{R}^n}(\psi^* u), \mathrm{d}^{\mathbb{R}^n}(\psi^* v) \right\rangle_{\mathrm{L}^2(\psi(U), \sqrt{\det B_\psi}\ \mathcal{L})}.$$

2. Parabolic Harnack estimates in *weighted* Sobolev spaces $\mathrm{H}^1(\psi(U), \sqrt{\det B_\psi}\ \mathcal{L}) \implies$ (3) and regularity.

Varadhan asymptotics

Important paper: [Nor97] by Norris. Setting: $\mathcal{M}$ Lipschitz.

Abstract methods for existence of $\rho_t^{\mathrm{g},\mathcal{W}}$ when $\mathcal{W} = \mathrm{H}_0^1(\mathcal{M}, \mathrm{g})$ or $\mathrm{H}^1(\mathcal{M}, \mathrm{g})$.

Distance:

$$\mathbf{d}_{\mathrm{g}}(x, y) := \sup\left\{ f(x) - f(y) : f \in \mathrm{C}^{0,1}(\mathcal{M}),\ |\mathrm{d}f(x)|_{\mathrm{g}} \leq 1 \ x \text{ a.e.} \right\}.$$

Important theorem:

$$\lim_{t \to 0} 4t \log \rho_t^{\mathrm{g},\mathcal{W}}(x, y) = -\mathbf{d}_{\mathrm{g}}^2(x, y).$$

$\rightsquigarrow$ $(\mathcal{M}, \mathbf{d}_{\mathrm{g}}, \mu_{\mathrm{g}})$ measure metric space, infinitesimally Hilbertian.

Question: Synthetic curvature properties in terms of g?

Strum's example of non-uniqueness

Let $\mathcal{M} = \mathbb{R}^n$. In [Stu97] by Sturm, shows $\exists A \in \mathrm{L}^\infty(\mathbb{R}^n; \mathrm{Sym\ Mat}(n))$ s.t. for x a.e.

$$\frac{1}{2}|u|^2_{\mathbb{R}^n} \leq A(x)u \cdot u < |u|^2_{\mathbb{R}^n}$$

- $\mathrm{g}(u, v) := A(x)u \cdot v$ rough metric on $\mathbb{R}^n$,
- $d\mu_{\mathrm{g}}(x) = \sqrt{\det A(x)}\, d\mathcal{L} < d\mathcal{L}$.

But

$$\mathbf{d}_{\mathrm{g}}(x, y) = |x - y| \quad \Longrightarrow \quad \mathcal{H}^{\mathbf{d}_{\mathrm{g}}} \neq \mu_{\mathrm{g}}.$$

Cannot happen for $A \in \mathrm{C}^0 \cap \mathrm{L}^\infty(\mathbb{R}^n; \mathrm{Sym\ Mat}(n))$.

Future outlook

Current works in "progress":

- Study of Weyl asymptotics on $\mathcal{M}$ with boundary, rough metric, for certain *Robin* boundary conditions (with Medet Nursultanov and Julie Rowlett).
- $(\mathcal{M}, \mathrm{g})$ automatically RCD when $\mathcal{M}$ compact. Synthetic bound in terms of g? (with Chiara Rigoni).

Questions:

1. $(\mathcal{M}, \mathrm{g}) \rightsquigarrow (\mathcal{M}, \mathbf{d}_{\mathrm{g}}, \mu_{\mathrm{g}})$. Synthetic curvature properties?
2. Notions of convergence for $(\mathcal{M}_i, \mathrm{g}_i) \to (\mathcal{M}_\infty, \mathrm{g}_\infty)$?

3. Given g, h, suppose $\exists C(\mathrm{g},\mathrm{h}) \geq 1$ such that for x a.e.,

$$C(\mathrm{g},\mathrm{h})^{-1}|u|_{\mathrm{g}(x)} \leq |u|_{\mathrm{h}(x)} \leq C(\mathrm{g},\mathrm{h})|u|_{\mathrm{g}(x)}.$$

Recall $Su = (u, \mathrm{d}u)$ and

$$\Pi_{\mathrm{g}}(B,B_0) = \begin{pmatrix} 0 & S^*X \\ S & 0 \end{pmatrix}, \quad \Pi_{\mathrm{g}}(B,B_0)^2 = \begin{pmatrix} \mathrm{L}_{B,B_0} & 0 \\ 0 & SS^*X \end{pmatrix}.$$

$\Pi_{\mathrm{g}}(B,B_0)$ first-order factorisation of $\mathrm{L}_{B,B_0} = \mathrm{d}_2^{*,\mathrm{g}} B \overline{\mathrm{d}_2}$.

$$\|f(\Pi_{\mathrm{g}}(B,B_0))\| \lesssim \|f\|_\infty \iff \|f(\Pi_{\mathrm{h}}(B,B_0))\| \lesssim \|f\|_\infty.$$

Holy grail:

$$\boxed{\|f(\Pi_{\mathrm{g},B,B_0})\| \lesssim \|f\|_\infty \overset{?}{\Longrightarrow} \text{curvature bound on } (\mathcal{M}, \mathbf{d}_{\mathrm{g}}, \mu_{\mathrm{g}}).}$$

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