L^∞ coefficient operators and non-smooth Riemannian metrics

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Consider the following *uniformly elliptic* second order differential operator $\mathrm{L}_A : \mathcal{D}(\mathrm{L}_A) \subset \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ defined by

$$\mathrm{L}_A u = aS^*ASu = -a\,\mathrm{div}(A_{11}\nabla u) - a\,\mathrm{div}(A_{10}u) + aA_{01}\nabla u + aA_{00}u.$$

where a and $A = (A_{ij})$ are L^∞ multiplication operators.

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That is, that there exist $\kappa_1, \kappa_2 > 0$ such that

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2,\ v \in \mathrm{L}^2$$
$$\mathrm{Re}\,\langle ASu, Su\rangle \geq \kappa_2(\|u\|^2 + \|\nabla u\|^2),\ u \in \mathrm{H}^1$$

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$$\begin{cases} \mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathrm{H}^1(\mathcal{M}) \\ \left\|\sqrt{\mathrm{L}_A}u\right\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{H}^1}\,,\ u \in \mathrm{H}^1(\mathcal{M}) \end{cases}$$

Theorem (B.-Mc [BMc])

Let $(\mathcal{M}, \mathrm{g})$ *be a smooth, complete Riemannian manifold* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$*. Suppose there exist* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\, \langle av, v \rangle \geq \kappa_1 \, \|v\|^2$$
$$\mathrm{Re}\, \langle ASu, Su \rangle \geq \kappa_2 \, \|u\|_{\mathrm{H}^1}^2$$

for $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{H}^1(\mathcal{M})$*. Then,* $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{H}^1(\mathcal{M})$ *and* $\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{H}^1}$ *for all* $u \in \mathrm{H}^1(\mathcal{M})$*.*

Stability

Theorem (B.-Mc [BMc])

Let $(\mathcal{M}, \mathrm{g})$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$. *Suppose that there exist* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle av, v\rangle \geq \kappa_1 \|v\|^2$$
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for $v \in \mathrm{L}^2(\mathcal{M})$ *and* $u \in \mathrm{H}^1(\mathcal{M})$. *Then for every* $\eta_i < \kappa_i$, *whenever* $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, *the estimate*

$$\left\| \sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u \right\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\, \|u\|_{\mathrm{H}^1}$$

holds for all $u \in \mathrm{H}^1(\mathcal{M})$. *The implicit constant depends in particular on* A, a *and* η_i.

History of the problem

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In 1962, Kato showed in [Kato61] that for $0 \leq \alpha < 1/2$ and $A(t)$ *maximal accretive* that

$$\begin{aligned} &\mathcal{D}(A(t)^{\alpha}) = \mathcal{D}(A(t)^{*\alpha}) = \mathcal{D}_{\alpha} = \text{const}, \text{ and} \\ &\|A(t)^{\alpha}u\| \simeq \|A(t)^{*\alpha}u\|, \quad u \in \mathcal{D}_{\alpha}. \end{aligned} \tag{K_α}$$

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In 1982, McIntosh showed that (K2) also did not hold in general in [Mc82].

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Is it then true that

$$\begin{aligned} \mathcal{D}(\sqrt{\mathrm{div}\, A\nabla}) &= \mathrm{H}^1(\mathbb{R}^n) \\ \left\| \sqrt{\mathrm{div}\, A\nabla} u \right\| &\simeq \|\nabla u\| \end{aligned} \tag{K1}$$

This was answered in the positive in 2002 by Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh and Phillipe Tchamitchian in [AHLMcT].

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(H2) Suppose that $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ such that here exist $\kappa_1, \kappa_2 > 0$ satisfying

$$\mathrm{Re}\,\langle B_1 u, u\rangle \geq \kappa_1 \|u\|^2 \quad \text{and} \quad \mathrm{Re}\,\langle B_2 v, v\rangle \geq \kappa_2 \|v\|^2$$

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- (H3) The operators B_1, B_2 satisfy $B_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$.

Let $\Gamma_B^* = B_1 \Gamma^* B_2$, $\Pi_B = \Gamma + \Gamma_B^*$ and $\Pi = \Gamma + \Gamma^*$.

Quadratic estimates and Kato type problems

Proposition

If (H1)-(H3) are satisfied and

$$\int_0^\infty \left\| t\Pi_B(\mathrm{I} + t^2\Pi_B^2)^{-1}u \right\|^2 \ \frac{dt}{t} \simeq \|u\|$$

for $u \in \overline{\mathcal{R}(\Pi_B)}$, *then*

(i) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B^2})$, *and*

(ii) $\|\Gamma u\| + \|\Gamma_B u\| \simeq \|\Pi_B u\| \simeq \|\sqrt{\Pi_B^2} u\|$, *for all* $u \in \mathcal{D}(\Pi_B)$.

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$$\Gamma_{\mathrm{g}} = \begin{bmatrix} 0 & 0 \\ S & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } B_2 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}.$$

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Then,

$$\Gamma_{\mathrm{g}}^* = \begin{bmatrix} 0 & S^* \\ 0 & 0 \end{bmatrix} \text{ and } \Pi_B^2 = \begin{bmatrix} \mathrm{L}_A & 0 \\ 0 & * \end{bmatrix}$$

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and

$$\Pi_{B,\mathrm{g}}(u,0) = (0, u, \nabla u) \text{ and } \sqrt{\Pi_{B,\mathrm{g}}^2}(u,0) = (\sqrt{\mathrm{L}_A}u, 0).$$

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Proposition

Let $\mathcal{M}$ be a smooth, complete manifold with smooth metric g. Suppose there exist $\eta, \kappa > 0$ such that $|\mathrm{Ric}_{\mathrm{g}}| \leq \eta$ and $\mathrm{inj}(\mathcal{M}, \mathrm{g}) \geq \kappa$. Furthermore, suppose that there exist κ_1, κ_2 such that

$$\begin{aligned} \mathrm{Re} \langle av, v \rangle &\geq \kappa_1 \|v\|^2, \ v \in \mathrm{L}^2(\mathcal{M}, \mathrm{g}) \\ \mathrm{Re} \langle ASu, Su \rangle &\geq \kappa_2 \|u\|_{\mathrm{H}^1}^2, \ u \in \mathrm{H}^1(\mathcal{M}, \mathrm{g}). \end{aligned}$$

Then,

$$\int_0^\infty \left\| t\Pi_B(\mathrm{I} + t^2\Pi_B^2)^{-1} u \right\|^2 \frac{dt}{t} \simeq \|u\|^2$$

for $u \in \overline{\mathcal{R}(\Pi_B)}$.

Rough metrics

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Let $\mathrm{h} \in \mathrm{C}^0(\mathcal{T}^{(2,0)}\mathcal{M})$. Then, define

$$\|\mathrm{h}\|_{\mathrm{op,g}} = \operatorname{esssup}_{\mu_{\mathrm{g}}(x)-\text{a.e.}} \sup_{|u|_{\mathrm{g}}=|v|_{\mathrm{g}}=1} |\mathrm{h}_x(u,v)|,$$

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We say that two C^0 metrics g and $\tilde{\mathrm{g}}$ are δ-close if $\|\mathrm{g}-\tilde{\mathrm{g}}\|_{\mathrm{op},\mathrm{g}} < \delta$ for $\delta > 0$.

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That is, exactly, we can absorb the lack of regularity of g in terms of a symmetric, bounded operator in $\tilde{\mathrm{g}}$. The operator $\|B\| = \|\tilde{\mathrm{g}} - \mathrm{g}\|_{\mathrm{op},\mathrm{g}}$. The change of measures

$$\theta = \frac{d\mu_{\mathrm{g}}}{d\mu_{\tilde{\mathrm{g}}}}$$

is given in terms of $\mathrm{I} + B$.

Stability of function spaces

Proposition

Let g *and* $\tilde{\mathrm{g}}$ *be two* C^0 *metrics on* $\mathcal{M}$*. If there exists* $\delta \in [0,1)$ *such that* $\|\mathrm{g} - \tilde{\mathrm{g}}\|_{\mathrm{op},\mathrm{g}} < \delta$*, then*

(i) *the spaces* $\mathrm{L}^2(\mathcal{M}, \mathrm{g}) = \mathrm{L}^2(\mathcal{M}, \tilde{\mathrm{g}})$ *with*

$$(1-\delta)^{\frac{n}{4}} \|\cdot\|_{2,\mathrm{g}} \leq \|\cdot\|_{2,\tilde{\mathrm{g}}} \leq (1+\delta)^{\frac{n}{4}} \|\cdot\|_{2,\mathrm{g}},$$

(ii) *the Sobolev spaces* $\mathrm{H}^1(\mathcal{M}, \tilde{\mathrm{g}}) = \mathrm{H}^1(\mathcal{M}, \mathrm{g})$ *with*

$$\frac{(1-\delta)^{\frac{n}{4}}}{1+\delta} \|\cdot\|_{\mathrm{H}^1,\mathrm{g}} \leq \|\cdot\|_{\mathrm{H}^1,\tilde{\mathrm{g}}} \leq \frac{(1+\delta)^{\frac{n}{4}}}{1-\delta} \|\cdot\|_{\mathrm{H}^1,\mathrm{g}}.$$

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Proposition

$\Gamma_{\mathrm{g}}^* = \Theta \Gamma_{\tilde{\mathrm{g}}}^* C$ where Θ is a bounded multiplication operator on $\mathrm{L}^2(\mathcal{M})$ and C is a bounded multiplication operator on $\mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$.

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Thus,

$$\Pi_{B,\mathrm{g}} = \Gamma_{\mathrm{g}} + B_1 \Gamma_{\mathrm{g}}^* B_2 = \Gamma_{\tilde{\mathrm{g}}} + B_1 \Theta \Gamma_{\tilde{\mathrm{g}}}^* C B_2.$$

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The idea is to reduce the Kato problem for $\Pi_{B,\mathrm{g}}$ to a Kato problem for $\Pi_{\tilde{B},\tilde{\mathrm{g}}} = \Gamma_{\tilde{\mathrm{g}}} + \tilde{B}_1 \Gamma_{\tilde{\mathrm{g}}}^* \tilde{B}_2$ where $\tilde{B}_1 = B_1 \Theta$ and $\tilde{B}_2 = C B_2$ but now with a smooth metric $\tilde{\mathrm{g}}$.

Ellipticity

Let us define ellipticity of L_A with respect to a metric h by

$$\begin{aligned} \mathrm{Re}\,\langle av, v\rangle_{\mathrm{h}} &\geq \kappa_{1,\mathrm{h}}\,\|v\|_{2,\mathrm{h}}^2\,,\ v \in \mathrm{L}^2(\mathcal{M}) \\ \mathrm{Re}\,\langle ASu, Su\rangle_{\mathrm{h}} &\geq \kappa_{2,\mathrm{h}}\,\|u\|_{\mathrm{H}^1,\mathrm{h}}^2\,,\ u \in \mathrm{H}^1(\mathcal{M}). \end{aligned} \tag{E}$$

Ellipticity (cont.)

Then the loss in the ellipticity constants by transferring from one operator to another is as follows.

Proposition

If $\|g - \tilde{g}\|_{op,g} \leq \delta < 1$, then assuming (E) with respect to g implies (E) with respect to $\tilde{g}$ with an appropriate change of the coefficients a and A with ellipticity constants

$$\kappa_{1,\tilde{g}} = \frac{\kappa_{1,g}}{(1+\delta)^{\frac{n}{2}}} \quad \textit{and} \quad \kappa_{2,\tilde{g}} = \kappa_{2,g} \frac{(1-\delta)}{(1+\delta)^{\frac{n}{2}}}.$$

Theorem (Reduction of the rough problem to the smooth)

Let $\mathcal{M}$ be a smooth manifold with a C^0 metric g and suppose that the ellipticity condition (E) is satisfied for $(\mathcal{M}, \mathrm{g})$. Suppose further that there exists a smooth, complete metric $\tilde{\mathrm{g}}$ satisfying:

(i) *there exists $\eta > 0$ such that $|\mathrm{Ric}_{\tilde{\mathrm{g}}}|_{\tilde{\mathrm{g}}} \leq \eta$,*

(ii) *there exists $\kappa > 0$ such that $\mathrm{inj}(\mathcal{M}, \tilde{\mathrm{g}}) \geq \kappa$,*

(iii) $\|\mathrm{g} - \tilde{\mathrm{g}}\|_{\mathrm{op},\mathrm{g}} < 1$,

Then, the quadratic estimate

$$\int_0^\infty \left\| t\Pi_{B,\mathrm{g}}(\mathrm{I} + t^2\Pi_{B,\mathrm{g}}^2)^{-1}u \right\|_{\mathrm{g}}^2 \, \frac{dt}{t} \simeq \|u\|_{2,\mathrm{g}}^2$$

holds for all $u \in \overline{\mathcal{R}(\Pi_{B,\mathrm{g}})}$.

Compact manifolds

Given a C^0 metric g, we can always find a C^∞ metric $\tilde{\mathrm{g}}$ that is as close as we would like in the $\|\cdot\|_{\mathrm{op,s,g}}$ norm by pasting together Euclidean metrics via a partition of unity.

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Further if we assume that $\mathcal{M}$ is compact, then automatically $|\mathrm{Ric}_{\tilde{\mathrm{g}}}| \leq C_{\tilde{\mathrm{g}}}$ and $\mathrm{inj}(\mathcal{M}, \tilde{\mathrm{g}}) \geq \kappa_{\tilde{\mathrm{g}}} > 0$.

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Theorem

Let $\mathcal{M}$ be a smooth, compact Riemannian manifold and let g be a C^0 metric on $\mathcal{M}$. Then, the quadratic estimate

$$\int_0^\infty \left\| t\Pi_{B,\mathrm{g}}(\mathrm{I} + t\Pi_{B,\mathrm{g}}^2)^{-1} u \right\|^2 \frac{dt}{t} \simeq \|u\|^2$$

is satisfied for all $u \in \overline{\mathcal{R}(\Pi_{B,\mathrm{g}})}$.

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- Intuition is that the smoothing properties of geometric flows could be utilised to find good close smooth metrics.
- The non-uniqueness of existence of solutions to geometric flows are not an issue since we only require *one* good metric near the initial one.
- A good place to start may be the mean curvature flow since such flows have been studied with rough initial data.

The noncompact case

- Although we can obtain arbitrarily close smooth metrics via a partition of unity in the noncompact setting, the desired geometric properties are not automatic.
- Intuition is that the smoothing properties of geometric flows could be utilised to find good close smooth metrics.
- The non-uniqueness of existence of solutions to geometric flows are not an issue since we only require *one* good metric near the initial one.
- A good place to start may be the mean curvature flow since such flows have been studied with rough initial data.
- The first task is to understand the backward behaviour of this flow and the relationship it has to $\|\cdot\|_{\mathrm{op,g}}$.

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