The world of rough metrics

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4 December 2020

SPP 2026
GEOMETRY AT INFINITY

Universität Potsdam

Divergence form equations

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  if $u \in \mathrm{dom}(\mathrm{L}_A)$ such that $\mathrm{L}_A u = 0 \implies u$ Hölder continuous.

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Laplacian: $\Delta_\mathrm{g} = \mathrm{d}_\mathrm{g}^* \mathrm{d}.$

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I.e.

$$\Delta_{\mathrm{g}} = (\det B)^{-\frac{1}{2}} \mathrm{d}^{*,\delta}((\det B)^{\frac{1}{2}} B)\mathrm{d}.$$

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$$\Pi_{\mathrm{g}}(B, B_0) = \begin{pmatrix} 0 & S^* X \\ S & 0 \end{pmatrix}, \qquad \Pi_{\mathrm{g}}(B, B_0)^2 = \begin{pmatrix} \mathrm{L}_{B,B_0} & 0 \\ 0 & SS^* X \end{pmatrix}.$$

where $S : \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathcal{M}; \mathrm{T}^*\mathcal{M})$ given by $Su = (u, \mathrm{d}u)$.

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N.B. $\Pi_{\mathrm{g}}(B, B_0)$ *not* self-adjoint - only ω-bisectorial for $\omega < \pi/2$.

Key estimate:

$$\|f(\Pi_{\mathrm{g}}(B, B_0))\|_{\mathrm{L}^2 \to \mathrm{L}^2} \lesssim \|f\|_{\infty}. \tag{2}$$

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Implies: $\mathrm{dom}(\sqrt{\mathrm{L}_{B,B_0}}) = \mathrm{H}^1(\mathcal{M}, \mathrm{g})$.
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Why?

$$\|\mathrm{e}^{-t(\Delta_{\mathrm{g}}+I)} - \mathrm{e}^{-t(\Delta_{\mathrm{h}}+I)}\| \lesssim \|\mathrm{g} - \mathrm{h}\|_{\mathrm{L}^\infty}.$$

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For any $\mathcal{V} \to \mathcal{M}$ vector bundle, we can talk about $\mathbf{\Gamma}_{\mathrm{R}}(\mathcal{V})$ - measurable sections of $\mathcal{V}$ without a metric on $\mathcal{M}$.

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- Well-defined induced Radon measure via locally comparable charts:

$$d\mu_{\mathrm{g}}(x) = \sqrt{\det \mathrm{g}(x)}\ d\psi^*\mathcal{L}.$$

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Say g is a *rough metric*.
Chart (U, ψ) satisfies the *local comparability condition*.

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- $\mu_{\mathrm{g}}(A) = 0 \iff A$ null-measure.

Lebesgue and Sobolev Spaces

Tensor bundle: $\mathcal{T}^{(p,q)}\mathcal{M} := (\otimes_{i=0}^{p}\mathrm{T}^*\mathcal{M}) \otimes (\otimes_{i=0}^{q}\mathrm{T}\mathcal{M})$.

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See [Ban16].

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☣ In general $\mathrm{dom}(\Delta_D) \neq \mathrm{H}^2(\mathcal{M})$. ☣

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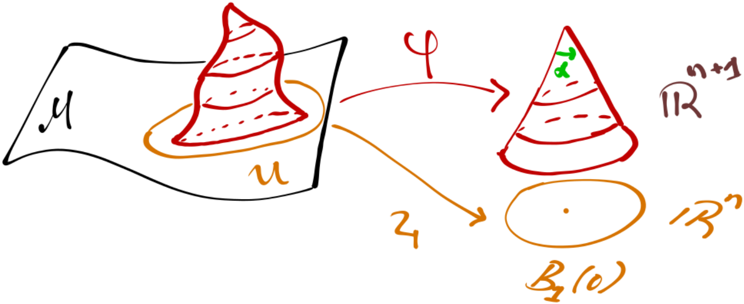

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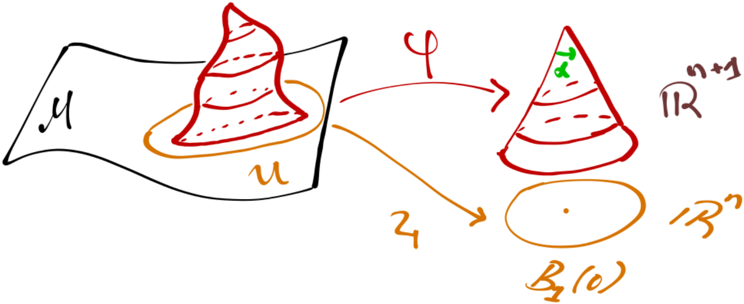

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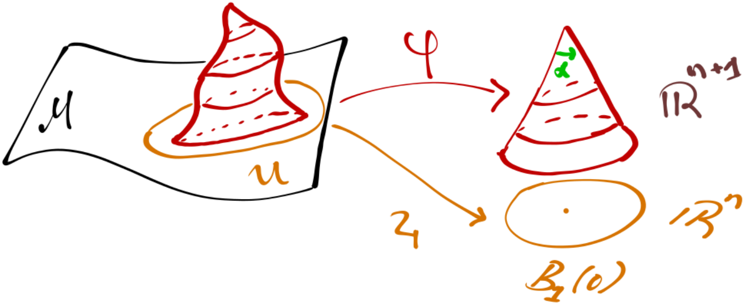

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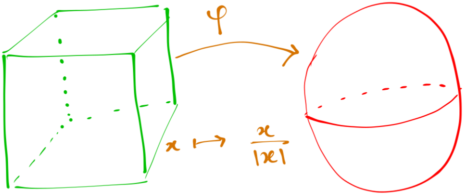

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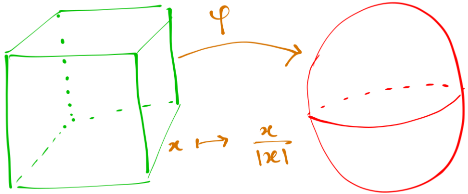

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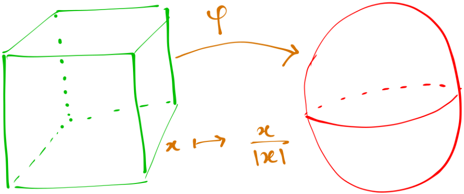

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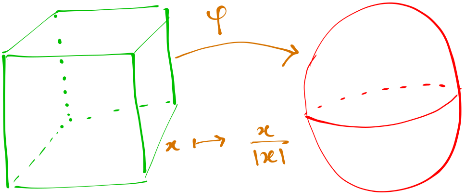

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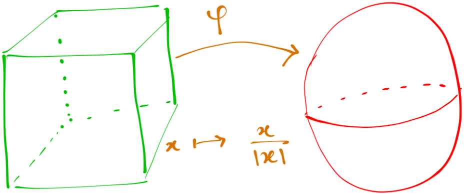

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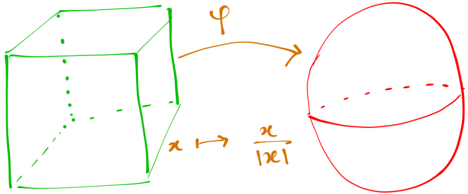

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- Patch (carefully).

Heat equation

Setting so far:

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Question: Synthetic curvature properties in terms of g?

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2. Notions of convergence for $(\mathcal{M}_i, \mathrm{g}_i) \to (\mathcal{M}_\infty, \mathrm{g}_\infty)$?

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Holy grail:

$$\|f(\Pi_{\mathrm{g},B,B_0})\| \lesssim \|f\|_\infty \overset{?}{\Longrightarrow} \text{curvature bound on } (\mathcal{M}, \mathbf{d}_{\mathrm{g}}, \mu_{\mathrm{g}}).$$

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