Discrete Image Reconstruction using Parallel Beam Geometry

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Abstract

In the context of the computerized axial tomography (CAT or CT) scan, the discrete image reconstruction using parallel beam geometry is a one of the algorithms to reconstruct the cross-sectional images of the given object. Evenly spaced set of X-ray beams are passed through an object and the intensity of the beams at input and output are measured. The aim is to reconstruct an image of the internal structure of an object from the intensity variations arising from beams passing through heterogeneous material. This will be achieved by solving the filtered back projection formula for different readings from different angles.

1 Introduction

A computerized axial tomography (CAT or CT) scan is one of the most important non-invasive medical imaging techniques, which was developed in the early 1970's by Godfrey Hounsfield and Allen Cormack. X-ray CT reconstructs a cross-sectional image by computing the attenuation coefficient distribution of an object from projection data, which records the relative number of photons passing through the object.

2 Mathematical Description

In our study of CT-scan, we will consider a 2D slice of the sample and assume which lies and centered on the XY-plane.

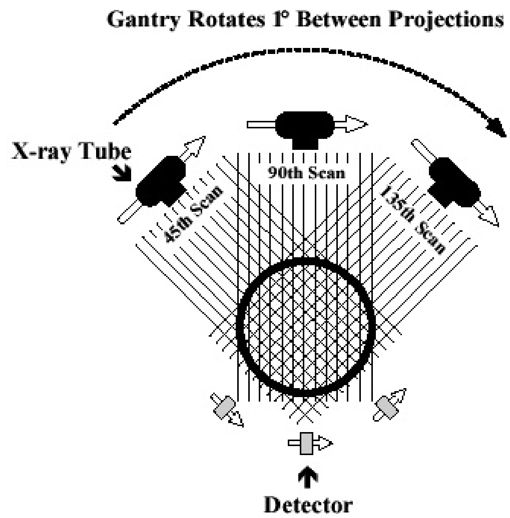

Figure 1: CT Scanner using parallel beam geometry

Mathematical model The absorption coefficient function $f(x,y)$ can be reconstructed using the *modified filtered back-projection formula* [1],

$$f(x,y) \approx \frac{1}{2}\mathscr{B}(\mathscr{F}^{-1}A * \mathscr{R}f)(x,y). \tag{1}$$

where

- Radon transform $\mathscr{R}$ is given by

$$\mathscr{R}f = \int_{l_{t,\theta}} f ds = \int_{s=-\infty}^{\infty} f(tcos(\theta) - ssin(\theta), tsin(\theta) + scos(\theta))ds = ln\left(\frac{I_0}{I_1}\right). \tag{2}$$

  This is the ratio of the intensity at input and output which is the integral of $f(x,y)$ along $l_{t,\theta}$ passing throught point $(t\cos(\theta), t\sin(\theta))$, which forms the input data provided. The line $l_{t,\theta}$ is described as $l_{t,\theta} = \{(t\cos(\theta) - s\sin(\theta), t\sin(\theta) + s\cos(\theta)) : -\infty < s < \infty\}$.

- Back-projection $\mathscr{B}$ of a function $h = h(t,\theta)$ is given by

$$\mathscr{B}h(x_0,y_0) := \frac{1}{\pi}\int_{\theta=0}^{\pi} h(x_0\cos(\theta) + y_0\sin(\theta), \theta)d\theta \tag{3}$$

  Back-projection averages $h(t,\theta)$ over angles θ and when it applys to Radon transformation it gives an average $\mathscr{R}f$ over all lines passing through (x_0,y_0) which approximate $f(x_0,y_0)$.

- $\mathscr{F}^{-1}$ is the inverse Fourier transform.

Proof We know that the Fourier inversion of Fourier transformation of any suitable function $f(x,y)$ is itself, i.e.

$$f(x,y) = \mathscr{F}_2^{-1}\mathscr{F}_2 f(x,y).$$

By applying the definition of 2D inverse Fourier transform,

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathscr{F}_2 f(X,Y) e^{i(xX+yY)} dX dY. \tag{4}$$

Let $X = S\cos(\theta)$ and $Y = S\sin(\theta)$, $0 \leq \theta \leq \pi$ and $S \in \mathbb{R}$ which gives

$$f(x,y) = \frac{1}{4\pi^2}\int_{0}^{\pi}\int_{-\infty}^{\infty} \mathscr{F}_2 f(S\cos(\theta), S\sin(\theta)) e^{iS(x\cos(\theta)+y\sin(\theta))}|S| dS d\theta. \tag{5}$$

Using the fact that for 2D Fourier transformation of f is connected with Radon transformation as following (Central slice theorem), i.e.

$$\mathscr{F}_2 f(S\cos(\theta), S\sin(\theta)) = \mathscr{F}(\mathscr{R}f)(S,\theta) \tag{6}$$

and the definition of Back-Projection, one could arrive at the "*filtered back-projection formula*",

$$f(x,y) = \frac{1}{2}\mathscr{B}\left\{\mathscr{F}^{-1}\left[|S|\mathscr{F}(\mathscr{R}f)(S,\theta)\right]\right\}(x,y). \tag{7}$$

The filtered back-projection formula (7) is the fundamental basis for image reconstruction and $|S|$ works as the filter function in the back projection. However, due to the fact that filter $|S|$ amplifies high frequency noise, in practice, we need replace $|S|$ by low-pass filter A which close to $|S|$ near 0 and vanishes when $|S|$ is large.

$$\begin{aligned} f(x,y) &\approx \frac{1}{2}\mathscr{B}\left\{\mathscr{F}^{-1}\left[A\mathscr{F}(\mathscr{R}f)(S,\theta)\right]\right\}(x,y) \qquad (8)\\ &= \frac{1}{2}\mathscr{B}(\mathscr{F}^{-1}A \ast \mathscr{R}f)(x,y), \qquad (9) \end{aligned}$$

where, $(\ast)$ denotes the convolution of $\mathscr{F}^{-1}A$ and $\mathscr{R}f$.

Here are some of the low-pass filters most commonly used in medical imaging.

1. The Ram-Lak filter:

$$A_1(\omega) = |\omega| \cdot \sqcap_L(\omega) = \begin{cases} |\omega| & \text{if } |\omega| \leq L, \\ 0 & \text{if } |\omega| > L. \end{cases}$$

2. The Shepp-Logan filter:

$$\begin{aligned} A_3(\omega) &= |\omega| \cdot \cos(\pi\omega/(2L)) \cdot \sqcap_L(\omega) \\ &= \begin{cases} |\omega|\cos(\pi\omega/(2L)) & \text{if } |\omega| \leq L, \\ 0 & \text{if } |\omega| > L. \end{cases} \end{aligned}$$

3. The low-pass cosine filter:

$$\begin{aligned} A_3(\omega) &= |\omega| \cdot \cos(\pi\omega/(2L)) \cdot \sqcap_L(\omega) \\ &= \begin{cases} |\omega|\cos(\pi\omega/(2L)) & \text{if } |\omega| \leq L, \\ 0 & \text{if } |\omega| > L. \end{cases} \end{aligned}$$

References

[1] T.G. Feeman. *The Mathematics of Medical Imaging: A Beginners Guide*. Springer Undergraduate Texts in Mathematics and Technology. Springer, 2009.