To find the partial fraction decomposition of the function $f(x) = \frac{27}{(x-1)^2(x+2)^3}$, you need to express it in the form of a sum of simpler rational functions. The decomposition will be:

$$\frac{27}{(x-1)^2(x+2)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3}$$

First, multiply both sides by the common denominator $(x-1)^2(x+2)^3$:

$$27 = A(x+2)^3(x-1) + B(x+2)^3 + C(x-1)^2(x+2)^2 + D(x-1)^2(x+2) + E(x-1)^2$$

Next, you need to find the constants A, B, C, D, E by substituting suitable values for x and solving the resulting equations.

Step-by-Step Solution:

1. Substitute $x = 1$ to find B:

$$27 = B(1+2)^3 \implies 27 = 27 \cdot B \implies B = 1$$

2. Substitute $x = -2$ to find E:

$$27 = E(-2-1)^2 \implies 27 = E \cdot 9 \implies E = 3$$

3. Substitute $x = 0$ to find an equation involving A, C, and D:

$$27 = A(-1)(2)^3(-1) + B(2)^3 + C(-1)^2(2)^2 + D(-1)^2(2) + E(-1)^2$$

Using $B = 1$ and $E = 3$:

$$27 = -8A + 8 + 4C + 2D + 3$$

Combine like terms:

$$27 = -8A + 4C + 2D + 11 \implies 16 = -8A + 4C + 2D \quad \text{(Equation 1)}$$

4. Substitute $x = -1$ to find another equation involving A, C, and D:

$$27 = A(-2)(1)^3 + B(1)^3 + C(-2)^2(1) + D(1)(-2)^2 + E(-2)^2$$

Using $B = 1$ and $E = 3$:

$$27 = -2A + 1 + 4C + 4D + 12$$

Combine like terms:

$$27 = -2A + 4C + 4D + 13 \implies 14 = -2A + 4C + 4D \quad \text{(Equation 2)}$$

5. Substitute $x = 2$ to find another equation involving A, C, and D:

$$27 = A(1)(4)^3 + B(4)^3 + C(1)(4)^2 + D(4)(1) + E(1)$$

Using $B = 1$ and $E = 3$:

$$27 = 64A + 64 + 16C + 4D + 3$$

Combine like terms:

$$27 = 64A + 16C + 4D + 67 \implies -40 = 64A + 16C + 4D \quad \text{(Equation 3)}$$

To solve the system of equations:

$$\begin{cases} 16 = -8A + 4C + 2D \quad \text{(Equation 1)} \\ 14 = -2A + 4C + 4D \quad \text{(Equation 2)} \\ -40 = 64A + 16C + 4D \quad \text{(Equation 3)} \end{cases}$$

you can use the elimination method to solve for A, C, and D.
First, you simplify the equations by dividing them:
Equation 1:

$$8 = -4A + 2C + D \quad \text{(Equation 4)}$$

Equation 2:

$$7 = -A + 2C + 2D \quad \text{(Equation 5)}$$

Equation 3:

$$-10 = 16A + 4C + D \quad \text{(Equation 6)}$$

Now, you eliminate D from Equation 4 and Equation 5.
Multiply Equation 4 by 2:

$$16 = -8A + 4C + 2D \quad \text{(Equation 7)}$$

Subtract Equation 5 from Equation 7:

$$16 - 7 = (-8A + 4C + 2D) - (-A + 2C + 2D)$$

$$9 = -8A + 4C + 2D + A - 2C - 2D$$

$$9 = -7A + 2C$$

$$7A = 2C - 9$$

$$C = \frac{7A + 9}{2} \quad \text{(Equation 8)}$$

Next, you substitute C from Equation 8 into Equation 6 to find A.

$$-10 = 16A + 4\left(\frac{7A + 9}{2}\right) + D$$

$$-10 = 16A + 2(7A + 9) + D$$

$$-10 = 16A + 14A + 18 + D$$

$$-10 = 30A + 18 + D$$

$$-28 = 30A + D$$

$$D = -28 - 30A \quad \text{(Equation 9)}$$

Then, you substitute D from Equation 9 into Equation 5 to find A.

$$7 = -A + 2C + 2(-28 - 30A)$$

$$7 = -A + 2C - 56 - 60A$$

$$7 = -61A + 2C - 56$$

$$63 = -61A + 2C$$

Substitute $C = \frac{7A+9}{2}$:

$$63 = -61A + 2\left(\frac{7A+9}{2}\right)$$

$$63 = -61A + 7A + 9$$

$$63 = -54A + 9$$

$$54 = -54A$$

$$A = -1$$

Finally, you use $A = -1$ to find C and D.
Substitute $A = -1$ into Equation 8:

$$C = \frac{7(-1) + 9}{2}$$

$$C = \frac{-7 + 9}{2}$$

$$C = 1$$

Substitute $A = -1$ into Equation 9:

$$D = -28 - 30(-1)$$

$$D = -28 + 30$$

$$D = 2$$

So, the solution to the system of equations is:

$$A = -1, \quad C = 1, \quad D = 2$$

Or you can just use Wolfram Alpha to solve 3 Equation System using following link: https://www.wolframalpha.com/widgets/view.jsp?id=e6bfd79be503e98ee35900cc07b0d5eb

Final Partial Fraction Decomposition:

$$\frac{27}{(x-1)^2(x+2)^3} = \frac{-1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x+2} + \frac{2}{(x+2)^2} + \frac{3}{(x+2)^3}$$