1.

Consider a 1D Ising chain of length L with periodic boundary conditions (ring). The length L is measured in units of the lattice spacing and is a dimensionless integer. Thus there are in all L different spins, numbered from 0 to $L-1$, the periodic boundary condition makes spin L equivalent to spin 0. The Hamiltonian is

$$H = -J \sum_{i=0}^{L-1} \sigma_i \sigma_{i+1} \tag{1}$$

where $J > 0$.

a) Find the average value $\langle m \rangle$ of the magnetization per site

$$m = \frac{1}{L} \sum_{j=0}^{L-1} \sigma_j \tag{2}$$

as a function of temperature T/J.

b) Calculate using the transfer matrix approach the correlation function $C(r)$ for two spins separated by a distance r

$$C(r) = \langle \sigma_0 \sigma_r \rangle - \langle \sigma_0 \rangle \langle \sigma_r \rangle. \tag{3}$$

Give both the exact expression for finite L and the limiting result for $L \to \infty$. What is $C(r)$ for $T = 0$?

2.

a) Make a Monte Carlo (MC) program for the ferromagnetic Ising model that uses the **Wolff algorithm**, i.e. the single cluster version of the Swendsen-Wang algorithm. It should work both for the 1D chain with periodic boundary conditions as well as for the 2D square lattice with periodic boundary conditions in both directions.

b) Check that your program reproduces the exact results you obtained for the 1D chain in problem 1. To be concrete use $L = 16$ and make a plot of $C(r)$ where you plot C vs. r for the MC simulation on top of the exact result from 1b) for two values of the temperature, $T/J = 0.5$ and $T/J = 1$.

Now consider the Ising model on a two dimensional $L \times L$ square lattice with periodic boundary conditions in both directions. L is measured in units of the lattice spacing.

c) Use your MC program to make a plot of the average magnetization per site $\langle m \rangle$ vs. T/J for $L = 16$. What is the value of $\langle m \rangle$ as $T/J \to 0$? And what is it when $T/J \to \infty$?

d) Use your MC program to make a plot of the average magnetization squared per site $\langle m^2 \rangle$ vs. T/J for $L = 16$. What is the value of $\langle m^2 \rangle$ as $T/J \to 0$? And what is it when $T/J \to \infty$?

A common way of finding the critical temperature of a phase transition is to consider a dimensionless ratio of moments of the magnetization, for instance

$$\Gamma = \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2} \tag{4}$$

and to plot Γ vs. T/J for different system sizes L.

e) Show analytically using finite size scaling arguments that the curves of Γ vs. T/J for different values of L cross at the phase transition temperature. Assume that $\Gamma = \Gamma(t, L^{-1})$ i.e. that Γ is a function of the reduced temperature $t \equiv (T - T_c)/T_c$ and the inverse (linear) system size L^{-1} only.

f) Use your MC program to estimate the critical temperature T_c for the 2D square lattice Ising model by plotting Γ vs. T/J for three different system sizes $L = \{8, 16, 32\}$. Compare with the exact result $T_c/J = \frac{2}{\ln 1+\sqrt{2}} \approx 2.2691853$. Do the three curves cross at a single point? If not, what can the reason be?