Assignment 1: Interpolation Methods

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Abstract

The following sections give answers to Assignment 1 in the UiO course MAT4110: Introduction to Numerical Analysis. The codes producing the results discussed in Problem 1 and Problem 3, are available at https://github.com/lasse-steinnes/MAT4110/tree/master/Assignment1.

Problem 1

Let $f(x) = e^x$ for $x \in [0,1]$. Want to find the approximation in $\mathbb{P}_1$ w.r.t. the L2-norm

$$\|g\|_{L^2} = \left(\int_0^1 |g(x)|^2 dx \right)^{\frac{1}{2}}. \tag{1}$$

Hence, want to minimize

$$E =< f-p, f-p >= \|f-p\|_{L^2} = \left(\int_0^1 |f-p|^2 dx \right)^{\frac{1}{2}}, \tag{2}$$

so that

$$E = \|f\|^2 - \sum_{i=0}^{1} \left(2c_i < f, pi > + c_i^2 < p_i, p_i > \right) \tag{3}$$

Choose basis $\mathbb{P}_1 = \{1, x - 1/2\}$ to be used in the interpolation. $\mathbb{P}_1$ has a monic and orthogonal basis, since the inner product $< p_0, p_1 >= 0$. To minimize eq. 1 one has to find the coeffiecients c_i so that

$$\frac{\delta E}{\delta c_i} = -2 < p_i, f > +2c_i < p_i, p_i >= 0, \tag{4}$$

or by rewriting

$$c_i = \frac{< p_i, f >}{< p_i, p_i >}. \tag{5}$$

For $i = 0$ one gets

$$< p_0, f >= \int_0^1 e^x dx = [e^x]_0^1 = e - 1, \tag{6}$$

and since $< p_0, p_0 >= 1$,

$$c_0 = e - 1. \tag{7}$$

For $i = 1$,

$$< p_i, f >= \int_0^1 (x - \frac{1}{2}) e^x dx. \tag{8}$$

Using integration by parts yields

$$< p_i, f >= [xe^x]_0^1 - \int_0^1 e^x dx - \int_0^1 \frac{1}{2} e^x dx \tag{9}$$

$$= e - (e - 1) - \frac{1}{2}(e - 1) = \frac{3 - e}{2}, \tag{10}$$

and

$$< p_1, p_1 >= \int_0^1 (x - \frac{1}{2})^2 dx = \int_0^1 x^2 - x + \frac{1}{4} dx \tag{11}$$

$$= \left[\frac{1}{3}x^3 - \frac{x^2}{2} + \frac{x}{4}\right]_0^1 = \frac{1}{12}. \tag{12}$$

All together

$$c_1 = \frac{12}{2}(3 - e) = 6(3 - e). \tag{13}$$

$$p = c_0 p_0 + c_1 p_1 = (e - 1) + 6(e - 3)(x - \frac{1}{2}) \tag{14}$$

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The interpolation p gives the least squares best approximation of f (Fig. 1).

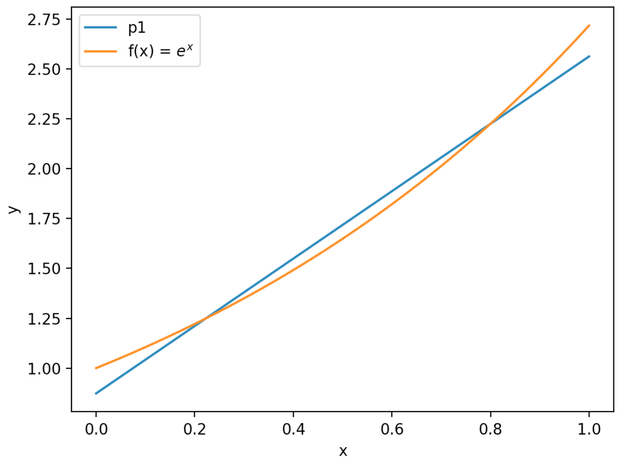

Figure 1: **Least squares best approximation of $f(x) = e^x$:** The $\mathbb{P}_1 = \{1, x - 1/2\}$ approximation and the analytical function is plotted as a function of x.

Problem 2

When f $\in C^{n+1}([a,b])$ and $\mathbb{P}_n$ interpolates f in $x_0, \ldots x_n \in [a,b]$, then

$$|f - p|_{C([a,b])} \leq \prod_{k=0}^{n} \frac{|x - x_k|}{(n+1)!} \left\| f^{(n+1)} \right\|_{C([a,b])} \tag{15}$$

a)

A maximum estimate of the error (eq. 15) is

$$|f - p|_{C([a,b])} \leq \|f - p\|_{C([a,b]))} = \sup_{x \in [a,b]} |f - p|. \tag{16}$$

Since x $\in [a,b]$, then $|x - x_k| \leq (b - a)$,

$$\|f - p\|_{C([a,b]))} \leq \frac{(b-a)^{n+1}}{(n+1)!} \left\| f^{(n+1)} \right\|_{C([a,b])}. \tag{17}$$

b)

If $[a,b] = [-1,1]$ and the interpolating points are the Chebycheff points on [a,b], then

$$\sup_{x\in[-1,1]} \prod_{k=0}^{n} |x-x_k| = 2^{-n}. \tag{18}$$

On a general interval [a,b], to use the Chebycheff points, let $y \in [a,b]$ and $x \in [-1,1]$ so that

$$g(y) = f(a_0 + a_1 x). \tag{19}$$

Want to determine a_0 and a_1. Knows that $x = -1 \to y = a$, and $x = 1 \to y = b$. Which gives the set of equations

$$\begin{aligned} a &= a_0 + a_1 \cdot (-1), \\ b &= a_0 + a_1 \cdot 1, \\ \Rightarrow a_0 &= \frac{a+b}{2}, \\ \Rightarrow a_1 &= \frac{b-a}{2}. \end{aligned} \tag{20}$$

It follows that

$$y = \frac{a+b}{2} + \frac{b-a}{2}x. \tag{21}$$

Thus using eq. 15 with the Chebycheff points for interpolation

$$\|f-p\|_{C([a,b])} \leq \prod_{k=0}^{n} \frac{1}{2^n(n+1)!} \left\| g^{(n+1)} \right\|_{C([a,b])}. \tag{22}$$

Need to use chain rule $(f \circ y)' = (f' \circ y) \cdot y'$, so that

$$\|f-p\|_{C([a,b])} \leq \prod_{k=0}^{n} \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!} \left\| f^{(n+1)} \right\|_{C([a,b])}. \tag{23}$$

Problem 3

Spline interpolation is an alternative to high order polynomial interpolation. Fix an interval $x \in [a,b]$ and consider the uniform mesh $x_i = a + ih, i = 0, 1, \ldots N$, where $h = (b-a)/N$. In the following problems consider the space of splines

$$S_k^0 = \{s \in C^0([a,b]) : s|_{(x_{i-1},x_i)} \in \mathbb{P}_k \forall i = 1, 2, \ldots, N\} \tag{24}$$

a)

Let $f \in C([a,b])$ and $s \in S_k^0$ the function which in each subinterval $[x_{i-1}, x_i]$ interpolates f at points

$$x_j = x_{i-1} + \frac{j}{k}h, \qquad\qquad j = 0, 1, ..., k. \tag{25}$$

Since s should be equal to f at these k+1 points, this raises k+1 conditions on s. For a polynomial of degree k, there are k+1 degrees of freedom (d.o.f.) to determine. Thus for each d.o.f, there is one condition, and each polynomial $s \in \mathbb{P}_k$ is uniquely determined on $[x_{i-1}, x_i]$, according to the uniqueness of polynomial interpolation (lemma 3.1 in [1]).

b)

Assume $f \in C^{k+1}([a,b])$ and $x \in [x_{i-1}, x_i]$. Let $s = \prod_k^N f$. Thus f is in the space of k+1 continuously derivative functions, so that

$$\left|f(x) - \Pi_k^N f(x)\right| = \frac{\prod_{j=0}^k \left|x - x_j\right|}{(k+1)!}\left|f^{(k+1)}\right|_{C([x_{i-1},x_i])}. \tag{26}$$

Since $x - x_j \leq h$ and $\left|f^{(k+1)}\right| \leq \left\|f^{(k+1)}\right\|$, then

$$\left|f(x) - \Pi_k^N f(x)\right| \leq \frac{h^{k+1}}{(k+1)!}\left\|f^{(k+1)}\right\|_{C([x_{i-1},x_i])} \tag{27}$$

c)

Inserting for $h = (b-a)/N$ in eq. 27, gives an upper error estimate on the interval $x \in [a,b]$,

$$\left|f(x) - \Pi_k^N f(x)\right| \leq \left\|f(x) - \Pi_k^N f(x)\right\|_{C([a,b])} \leq \frac{(b-a)^{k+1}}{N^{k+1}(k+1)!}\left\|f^{(k+1)}\right\|_{C([a,b])} \tag{28}$$

d)

The computational cost $c = k + 1 + (N-1)k = Nk + 1$ is the number of evaluations of f required to find s. Want to find the values of N, k for a fixed c that minimize eq. 28 for

$$\text{(i)} \qquad \sin(x), \qquad x \in [-\pi, \pi] \tag{29}$$

and

$$\text{(ii)} \qquad \frac{1}{1+25x^2}, \qquad x \in [-1,1]. \tag{30}$$

For (i) $|f^{(k+1)}|_{C([-\pi,\pi])} = 1$ and $b-a = 2\pi$, thus

$$\|f(x) - \Pi_k^N\| \leq \frac{(2\pi)^{k+1}}{N^{k+1}(k+1)!} \approx \left[\frac{2\pi e}{N(k+1)}\right]^{k+1} (k+1)^{1/2} = \left[\frac{2\pi k e}{(c-1)(k+1)}\right]^{k+1} (k+1)^{1/2}. \tag{31}$$

In the last two equalities, $N = (c-1)/k$ and Stirling's approximation were used. Stirling's approximation says that $(k+1)! \approx (k+1)^{(k+1)+1/2} e^{-(k+1)}$.

In the case of Runge's equation (ii), then $|f^{(k+1)}|_{C([-1,1])} = 5^{k+1}(k+1)!$ and $b-a = 2$. It follows that

$$\|f(x) - \Pi_k^N\| \leq \frac{2^{k+1}}{N^{k+1}(k+1)!} 5^{k+1}(k+1)! = \left[\frac{10}{N}\right]^{k+1} = \left[\frac{10k}{c-1}\right]^{k+1}. \tag{32}$$

In the last equality, $N = (c-1)/k$ was inserted into the equation. Thus, for a fixed computational cost c, the upper error estimates only depends on k.

For $c \to \infty$ (asymptotic behaviour) it is clear that the error for $\sin(x)$ decrease for all k (Fig. 2). A requirement for a decreasing error for $\sin(x)$ (eq. 31) is $N(k+1) > 2\pi e$ since $(k+1)^{1/2}$ increase slowly. However with Runge's function, the error increases for $k > 10$ (Fig.2). From eq. 32 a requirement for the error to decrease with increasing k is $N > 10$.

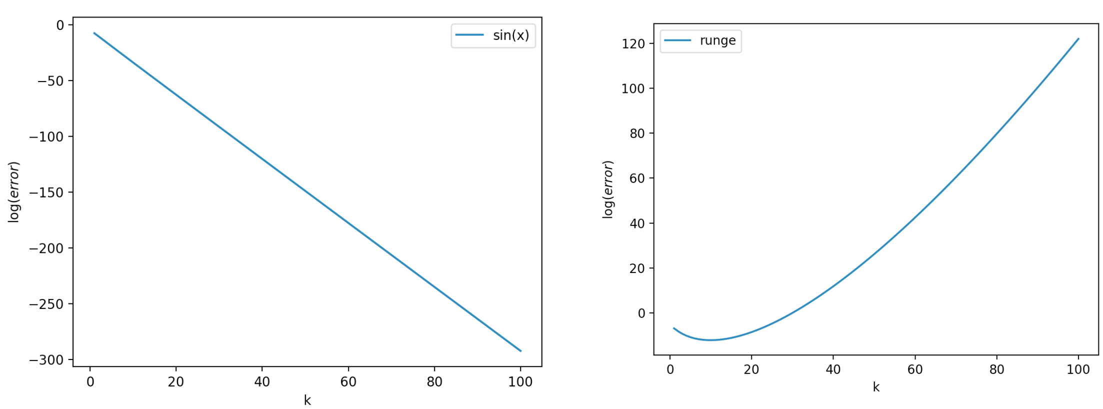

Figure 2: **Error estimates of spline interpolation**. **Left:** $\sin(x)$ **Right:** Runge's function. See eq. 31 and eq. 32 for error estimates. Plotted with c = 300 (asymptotic behaviour).

e)

The function `piecewisepol(f,a,b,N,k)` interpolates the function f on the interval $x \in [a,b]$, divided into N sub-intervals with splines of degree k and smoothness 0 (Fig 3 and 4).

For $\sin x$ a low number of sub-intervals N ($N = 1$), and high k ($k = 8$) better describes the oscillatory behaviour of this function, rather than $(N,k) = (8,1)$ (Fig. 3).

However, in the case of Runge's function, which has an increasing Lebesque constant with k and a quickly growing k-th order derivative, $(N,k) = (8,1)$ performs a lot better than $(N,k) = (1,8)$. So a low order polynomial interpolating spline $s \in \mathbb{P}_k$ would be preferable for Runge with N large (optimally N > 10 as decribed above).

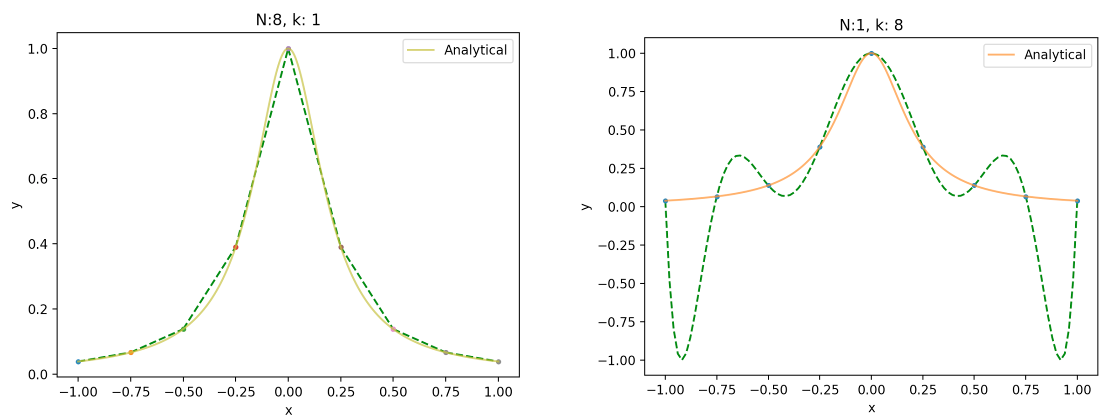

Figure 3: **Interpolation of Runge's function**. **Left:** Spline of degree 1 (k) for 8 sub-intervals (N). **Right:** Spline of degree 8 (k) for 1 sub-interval (N).

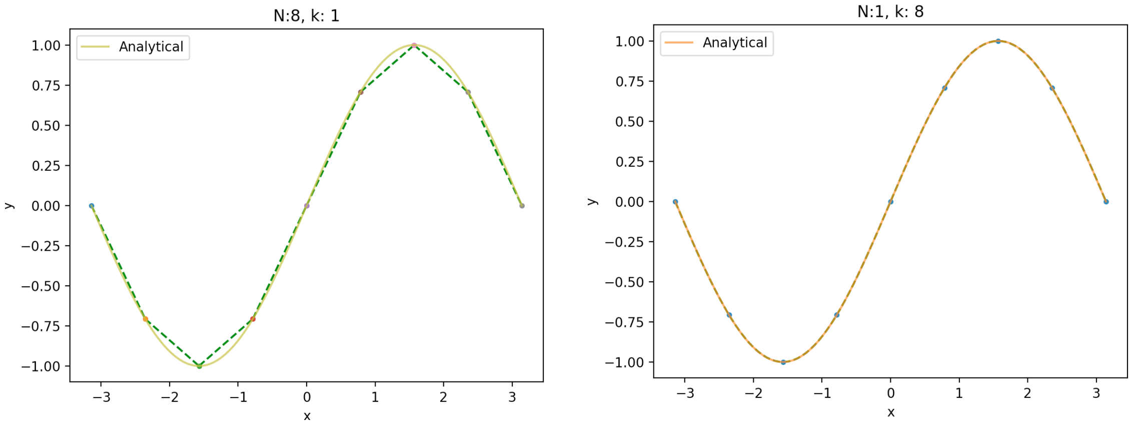

Figure 4: **Interpolation of** $\sin \mathbf{x}$. **Left:** Spline of degree 1 (k) for 8 sub-intervals (N). **Right:** Spline of degree 8 (k) for 1 sub-interval (N).

References

[1] A.C. Faul. *A Concise Introduction to Numerical Analysis*. CRC Press, 2016.

I Appendix

PS: The graphs became very nice with a better (N,k) fit (Fig. 5).

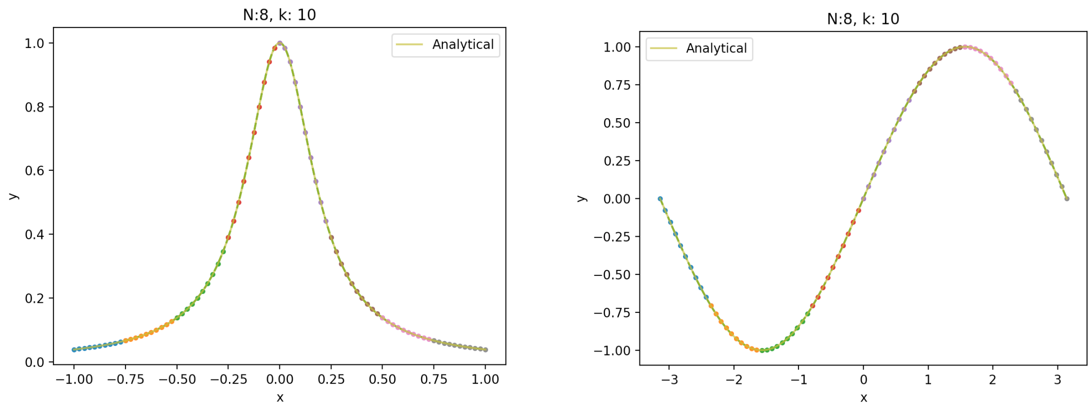

Figure 5: **Interpolation with better (N,k) fit**. **Left:** Runge with splines of degree 10 (k) for 8 sub-intervals (N). **Right:** $\sin x$ with splines of degree 10 (k) for 8 sub-interval (N).