CV Assignment04

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4.1

For the first part of the task, we take slide 53 of the lecture06 into consideration. From that slide, we know that the eigenvalues correspond to the length of the eigenvectors, and that the eigenvectors are pointing into the direction of the maximum variance and perpendicular to that. As they encode a length neither eigenvalue a, b can be smaller than 0. Further, we know that $\alpha \geq \beta$, so for $\frac{a}{b} = 0$ $a = b = 0$. In that case however $\frac{a}{b} = \frac{0}{0}$, which is not defined therefore $\frac{a}{b} > 0$ for all defined eigenvalues and vectors.

For the second part of the task, we proof that the minimum of the function $f(r) = \frac{(r+1)^2}{r}$ is at $r = 1$ for $r > 0$.

Let's first remodel the function:

$$\begin{aligned} f(r) &= \frac{(r+1)^2}{r} \\ &= \frac{(r^2 + 2r + 1}{r} \\ &= \frac{r^2}{r} + \frac{2r}{r} + \frac{1}{r} \\ &= r + 2 + \frac{1}{r} \\ &= r + \frac{1}{r} + 2 \end{aligned}$$

First, we hypothesize, that from $1 \geq r$ the function is monotonically increasing, from which trivially follows that is limes is its maximum, which is going to be infinity as the range is not limited. So, to proof that the function is monotonically increasing, the criteria is $f(r) < f(r+1)$.

$$
\begin{aligned}
f(r) &< f(r+1) \\
r + \frac{1}{r} + 2 &< (r+1) + \frac{1}{(r+1)} + 2 \\
r + \frac{1}{r} &< (r+1) + \frac{1}{(r+1)} \\
\frac{1}{r} &< 1 + \frac{1}{(r+1)}
\end{aligned}
$$

As $r \geq 1$,because that is the range in which we test the function currently, the maximum on the left side of the inequality is 1. The fraction on the right side however is always greater 0 as $r > 0$, hence the sum is always greater than 1. Therefore the inequality holds true and $f(r)$ is monotonically increasing in the open interval from $r \geq 1$. From this follows, that the minimum value is at its smallest r-value, which is $r = 1$.

Now, we hypothesize, that $f(r)$ in the range $0 < r < 1$, the function is monotonically decreasing. To show this, we check that for every $\epsilon : 0 < \epsilon < 1-r$ $f(r) > f(r+\epsilon)$. As from this follows that its minimum is at the largest r-value, which is $r = 1$.

$$
\begin{aligned}
f(r) &> f(r+\epsilon) \\
r + \frac{1}{r} + 2 &> (r+\epsilon) + \frac{1}{(r+\epsilon)} + 2 \\
r + \frac{1}{r} &> (r+\epsilon) + \frac{1}{(r+\epsilon)} \\
\frac{1}{r} &> \epsilon + \frac{1}{(r+\epsilon)} \\
\frac{1}{r} * (r+\epsilon) &> \epsilon * (r+\epsilon) + 1 \\
\frac{1}{r} * r + \frac{1}{r} * \epsilon &> \epsilon^2 + \epsilon * r + 1 \\
1 + \frac{1}{r} * \epsilon &> \epsilon^2 + \epsilon * r + 1 \\
\frac{1}{r} * \epsilon &> \epsilon^2 + \epsilon * r \\
\frac{1}{r} &> \epsilon + r \\
\frac{1}{r} &> 1 - r + r \\
\frac{1}{r} &> 1
\end{aligned}
$$

As $r < 1$ by defintion of our range, the left side of the equation is always greater

1, hence we know, that the function is monotonically decreasing. Its maximum is at the largest $r-$value, at $r = 1$.

With these two proofs, $f(r)$ is defined over the entire given range and has one minimum at $r = 1$.