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Project 3

Building a Model for the Solar System Using Ordinary Differential Equations

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Abstract

In this project we developed a model simulating the motion of the planets in the Solar System by using both the Forward Euler and the Velocity Verlet method to integrate Newton's equations of motion. The Velocity Verlet method was found to outperform the Forward Euler method in terms of accuracy and stability, with a small cost of increased computational time due to more floating point operations per iteration. A simulation of Mercury's precessing orbit around the Sun because of correcting terms introduced by GR was run. We got a precession of 42.75 arcsecond per century, opposed by the observed 43, indicating a good correspondence.

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1 Introduction

Throughout history many models of the Solar System have been developed, which first culminated when Newton provided a mathematical framework for predicting the motion of celestial bodies along with a physical explanation with his laws of motion and universal gravitation, before in more recent age, Einstein expanded on the theory of gravity with his theory of general relativity. Applying Newton's laws to the celestial bodies in the Solar System results in a set of coupled ordinary differential equations. Systems made of a number of interacting bodies, usually referred to as the n-body problem, are notorious for making analytic solutions impractical or even impossible, due to their complexity. These systems are therefore predominantly solved by the means of numerical methods. In this project we will develop a model simulating the motion of celestial bodies in the Solar System by using the integration methods Forward Euler and Velocity Verlet to solve the coupled differential equations.

First, we limit ourselves to a hypothetical Solar System with only Earth orbiting around the Sun, known as the two-body problem. In order to compare the numerical methods, we constrain Earth's orbit to be circular. A visual comparison of the solutions generated by the methods will be carried out, as well as a comparison between their conservation properties, specifically how well they conserve energy and angular momentum, and their computational times. Furthermore, the escape velocity of the Earth, which has an analytical solution, will be sought numerically by trial and error (by changing the initial velocity) with the Velocity Verlet method. The effects of letting the classical Newtonian inverse-square gravity creep towards a cube-inverse gravity will also be studied.

Next, we add Jupiter to our hypothetical Solar System with a fixed Sun, thus extending the system to a three-body problem. Here, we will study how Jupiter, having a mass roughly 1000 times smaller than the Sun, alters the Earth's motion. The calculations will also be performed by increasing the mass of Jupiter by a factor 10 and 1000, with the stability of the Verlet solver as the main subject of interest.

Moving on, we seek to make our model more realistic by having a dynamic Sun. In order to do this, we make the center-of-mass position of the three-body system our origin, and give the Sun an initial velocity which makes the total momentum of the system exactly zero. This model will further be extended to include all the planets in the Solar System.

Finally, we will study whether our not numerical solver reproduce the precession of the perihelion of Mercury predicted by Einstein's general theory of relativity.

This project is structured by first presenting a theoretical overview of the problems along with derivations of the aforementioned numerical methods in Section 2, followed by a presentation on how we approach the problems in Section 3. Next, the results of the problems generated by the implementation of our numerical methods are presented in Section 4, before subsequently they and the approach are discussed and concluded upon in Section 5 and Section 6, respectively. Lastly, we outline possible further continuations of the model, based on our results, in Section 7.

2 Theory

2.1 Model of the Solar System

In the classical theory of gravity the force, $\mathbf{F}_{\mathrm{G}}$, between a planet and the Sun is given by Newton's law of universal gravitation

$$\mathbf{F}_{\mathrm{G}} = G\frac{M_{\odot}m}{|\mathbf{r}|^2}\hat{\mathbf{r}}, \tag{2.1}$$

where G is the gravitational constant, $M_{\odot}$ is the mass of the sun, m is the mass of the planet, $\mathbf{r}$ is the distance between the centers of the masses, and $\hat{\mathbf{r}}$ is the unit vector from the planet to the Sun. By assuming that the mass of the Sun is much greater than the planets, a good approximation is to neglect the motion of the Sun. Newton's second law of motion gives the following differential equations for the planet

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{F_x}{m} = a_x,$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{F_y}{m} = a_y,$$
$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = \frac{F_z}{m} = a_z$$

These can be written in a more compact form as

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \mathbf{a} = \frac{\mathbf{F}}{m} = \frac{GM_{\odot}}{|\mathbf{r}|^2}\hat{\mathbf{r}} \tag{2.2}$$

The general method is to rewrite the second order differential equation into two coupled differential equations on the form

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \tag{2.3}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{GM_{\odot}}{r^3}\mathbf{r} \tag{2.4}$$

2.2 The Forward Euler Method

The Forward Euler method is a first order procedure for solving ordinary differential equations with a given initial value. A Taylor expansion of the position results in

$$\mathbf{r}(t+\mathrm{d}t) = \mathbf{r}(t) + \mathbf{v}(t)\cdot\mathrm{d}t + \frac{\mathbf{a}(t)}{2!}\cdot\mathrm{d}t^2 + \mathcal{O}\left(\mathrm{d}t^3\right)$$

The Forward Euler method truncate at first order which give local error of $\mathcal{O}(\mathrm{d}t^2)$, which leads to the equations

$$\mathbf{r}(t+\mathrm{d}t) = \mathbf{r}(t) + \mathbf{v}(t)\cdot\mathrm{d}t,$$
$$\mathbf{v}(t+\mathrm{d}t) = \mathbf{v}(t) + \mathbf{a}(t)\cdot\mathrm{d}t$$

Since the velocity appears in the equation for position does the order of computation matter. By discretizing $t_i = t_0 + i \cdot \mathrm{d}t$, $\mathbf{r}(t_i) = \mathbf{r}_i$, and $\mathbf{v}(t_i) = \mathbf{v}_i$, the Forward Euler method can be written on the form

$$\mathbf{v}_{i+1} = \mathbf{v}_i + \mathbf{a}_i \cdot \mathrm{d}t, \tag{2.5}$$

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \mathbf{v}_i \cdot \mathrm{d}t, \tag{2.6}$$

where $\mathrm{d}t = (t_\mathrm{f} - t_0)/N$ with N as the number of time steps.

2.3 The Velocity Verlet Method

By using the same discretization procedure as in Section 2.2 the Taylor expansion of the position and velocity can be written, respectively, as

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \mathbf{v}_i \cdot \mathrm{d}t + \frac{\mathbf{a}_i}{2!} \cdot \mathrm{d}t^2 + \mathcal{O}\left(\mathrm{d}t^3\right), \tag{2.7}$$

and

$$\mathbf{v}_{i+1} = \mathbf{v}_i + \mathbf{a}_i \cdot \mathrm{d}t + \frac{\mathbf{a}_i'}{2!} \cdot \mathrm{d}t^2 + \mathcal{O}\left(\mathrm{d}t^3\right) \tag{2.8}$$

The quantity $\mathbf{a}_i'$ can be decided by Taylor expanding the acceleration

$$\mathbf{a}_{i+1} = \mathbf{a}_i + \mathbf{a}_i' \cdot \mathrm{d}t + \mathcal{O}\left(\mathrm{d}t^2\right)$$

Here, the term to first order is kept, which means that

$$\mathbf{a}_i' = \frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{\mathrm{d}t} \tag{2.9}$$

Truncating Equation (2.7) and Equation (2.8) followed by inserting Equation (2.9), results in the Velocity Verlet algorithm

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \mathbf{v}_i \cdot \mathrm{d}t + \frac{\mathbf{a}_i}{2} \cdot \mathrm{d}t^2, \tag{2.10}$$

$$\mathbf{v}_{i+1} = \mathbf{v}_i + \frac{\mathrm{d}t}{2}\left(\mathbf{a}_{i+1} + \mathbf{a}_i\right) \tag{2.11}$$

2.4 Scaling the Solar System

Here, we aim to find units more suited for describing the quantities in the Solar System that also make the numerical methods less prone to machine error. For the dimension length we will use Astronomical units (AU), which is defined as the average distance between the Sun and Earth, that is $1\,\mathrm{AU} = 1.5 \cdot 10^{11}$ m. Furthermore, we will use years (yr) instead of seconds (s) for time, since years better match the time evolution of the Solar System. It is also advantageous to express the mass of the celestial bodies in terms of solar masses ($M_\odot$). For circular motion we have that the force must obey the following relation

$$F_\mathrm{G} = \frac{M_\mathrm{Earth} v^2}{r} = G\frac{M_\odot M_\mathrm{Earth}}{r^2}, \tag{2.12}$$

where v is the velocity of Earth. Using the fact that Earth's orbit is almost circular with a radius of 1 AU and period of 1 year, the latter equation can be rewritten as

$$v^2 r = v^2 \cdot 1\,\mathrm{AU} = GM_\odot = 4\pi^2\,\mathrm{AU}^3/\mathrm{yr}^2, \tag{2.13}$$

since, by definition, $v = 2\pi$ AU/yr. Equation (2.2) can thus be written as

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \frac{4\pi^2}{|\mathbf{r}|^2}\hat{\mathbf{r}}\,\mathrm{AU}^3/\mathrm{yr}^2, \tag{2.14}$$

where r and t have the units AU and yr, respectively. This is the equation we will solve numerically.

2.5 Conservation of Energy

The Lagrangian is defined as

$$L = T - V$$

where T is the kinetic energy and V is the potential energy. The kinetic energy and potential energy is given by

$$T = \frac{1}{2}\sum_i m_i v_i^2$$
$$V = \sum_{i\neq j} V(r_{ij}(t))$$

The force is given by the gradient of the potential

$$F_G = -\nabla V = G\frac{M_\odot m}{r^2},$$

which solved for V results in

$$V = -G\frac{M_\odot m}{r}$$

The Lagrangian for the Earth is then given by

$$L = \frac{1}{2}mv^2 + G\frac{M_\odot m}{r} \tag{2.15}$$

The Lagrangian does not have an explicit time dependence, which means that the time derivative is equal to

$$\frac{\partial L}{\partial t} = 0,$$

and thus there is a conserved quantity associated with this independence. In order to find this quantity some formalism must be applied. Assuming that the Euler-Lagrange equations holds, the total time derivative of a generic Lagrangian $L(q(t), \dot{q}(t))$ is given by

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \sum_k \left(\frac{\partial L}{\partial q_k}\frac{\mathrm{d}q_k}{\mathrm{d}t} + \frac{\partial L}{\partial \dot{q}_k}\frac{\mathrm{d}\dot{q}_k}{\mathrm{d}t}\right) + \frac{\partial L}{\partial t}\\ &= \sum_k \left(\frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k\right) + \frac{\partial L}{\partial t}\end{aligned}$$

The second term in the sum can be rewritten as

$$\frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q}$$

This in turn gives

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \sum_k \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) + \frac{\partial L}{\partial t} - \sum_k \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right]}_{=0}\dot{q}_k \\ &= \sum_k \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) + \frac{\partial L}{\partial t},\end{aligned}$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\right) = -\frac{\partial L}{\partial t}$$

The quantity inside the brackets is the Legendre transformation of the Lagrangian, which is the Hamiltonian of the system, so

$$H = \sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L \tag{2.16}$$

This satisfies

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t} = 0 \tag{2.17}$$

For the Earth-Sun system the Hamiltonian is given by

$$\begin{aligned}H &= \frac{\partial L}{\partial v}v - L = mv - \frac{1}{2}mv^2 - G\frac{M_\odot m}{r} \\ &= \frac{1}{2}mv^2 - G\frac{M_\odot m}{r} = T + V,\end{aligned}$$

which is the total energy of the system. Therefore the conserved quantity associated with the explicit time independence of the Lagrangian is the total energy. By Noether's theorem this means that there is a symmetry, and this symmetry is that the Lagrangian is invariant under time translation. Thus is the Hamiltonian equal to

$$H = \frac{1}{2}mv^2 - G\frac{M_\odot m}{r} \tag{2.18}$$

2.6 Conservation of Angular Momentum

The gravitational force that acts between the Sun and the Earth is dependent only on the distance between them, thus the torque is given by

$$\tau = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \Big(f(\mathbf{r})\hat{\mathbf{r}}\Big) = 0$$

The definition of the torque is given by

$$\tau = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t},$$

and it follows that

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = 0 \tag{2.19}$$

Thus the angular momentum is conserved for the sun-earth system, and by Noether's theorem this means that the Lagrangian is invariant under rotations.

2.7 Escape velocity

The escape velocity can be found when the kinetic energy balances out the potential energy that act between the sun and the planet, so the equality is

$$\frac{1}{2}mv^2 = G\frac{M_\odot m}{r}$$

Solving for the velocity gives

$$v = \sqrt{\frac{2GM_\odot}{r}} \tag{2.20}$$

By using that $GM_\odot = 4\pi^2\,\mathrm{AU}^3/\mathrm{yr}^2$ and that the mean distance between the Sun and the Earth is $r = 1\,\mathrm{AU}$, the escape velocity for the Earth is given by

$$v = 2\sqrt{2}\pi\,\mathrm{AU/yr} \tag{2.21}$$

2.8 Closed Orbits

For an object with orbital motion the Lagrangian can be written as

$$\begin{aligned} L &= \frac{1}{2}m\dot{r}^2 + G\frac{Mm}{r} \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}) - V(r) \end{aligned}$$

By applying the Euler-Lagrange equations on this lagrangian, and use that the conjugate momentum is given by $mr^2\dot{\theta} = L$, i.e Angular momentum is conserved, the equations of motion is given by

$$m\frac{d^2r}{dt^2} - \frac{L^2}{mr^3} = -\frac{\partial V}{\partial r}$$

From the definition of Angular momentum this can be rewritten by

$$\frac{d}{dt} = \frac{L^2}{mr^2}\frac{d}{d\theta}$$

into

$$\frac{L^2}{r^2}\frac{d}{d\theta}\left(\frac{L^2}{r^2}\frac{dr}{d\theta}\right) - \frac{L^2}{mr^3} = -\frac{dV}{dr}$$

By introducing a change of variable $u = 1/r$ this equation can be brought into a Binet equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}f(u^{-1}) \equiv F(u)$$

where $f(u^{-1})$ is the radial force. For perfectly circular orbits, the criteria is that the second derivative of the angle is zero. Then for a perfect circle at a radius r_0 this is given by $u_0 = F(u_0)$. So by looking at a small perturbation $\epsilon = u - u_0$ from the perfect circular orbit, and by plugging this expansion into the Binet equation one obtains

$$\frac{d^2\epsilon}{d\theta^2} + \epsilon = \epsilon F'(u_0) + \frac{1}{2}\epsilon^2 F''(u_0) + \cdots$$

For small perturbation all the higher order terms may be negelcted, so to first order

$$\frac{d^2\epsilon}{d\theta^2} + \epsilon = \epsilon F'(u_0)$$

$$\frac{d^2\epsilon}{d\theta^2} = -\epsilon(1 - F'(u_0)) = -a^2\epsilon$$

and since $a^2 = (1 - F'(u_0))$ is just a constant, this equation has solutions

$$\epsilon(\theta) = Acos(a\theta)$$

which mean that a must be a rational number, and it must be positive. By using the definition of $F(u)$ and inserting the condition $F'(u_o) = 0$, one finds that the only $a = 1$ will give a stable orbit. Then

$$\frac{df}{dr} = -2\frac{f}{r}$$

which has solution

$$f(r) = -\frac{k}{r^2} \tag{2.22}$$

so the radial force is attractive and proportional to $1/r^2$. This result is also known as Bertrands theorem, which says that for stable non-circular closed orbits $a = 1$, and for perfectly circular orbits $a = 0$. Bertrands theorem does in fact cover a more general derivation by keeping higher order terms, but that result was neglected here as the main interest was the gravitational force.

2.9 The Three-body Problem

By adding an interaction between the Earth and Jupiter, the Sun-Earth problem is extended to a three body problem. The force added is given by

$$\mathbf{F}_{\mathrm{EJ}} = G\frac{M_{\mathrm{E}}M_{\mathrm{J}}}{r_{\mathrm{EJ}}^2}\hat{\mathbf{r}}$$

where M_{E} and M_{J} is the mass of the Earth and Jupiter, respectively. G is the gravitational constant and r_{EJ} is the distance between them.

2.10 Relativistic Correction

Adding a relativistic correction to the Newtonian gravitational force gives

$$F_{\mathrm{G}} = \frac{GM_{\odot}M_{\mathrm{Mercury}}}{r^2}\left[1 + \frac{3L^2}{r^2c^2}\right], \tag{2.23}$$

where M_{Mercury} is the mass of Mercury, r is the distance between Mercury and the Sun, $L = |\mathbf{r} \times \mathbf{v}|$ is the magnitude of Mercury's orbital angular momentum per unit mass, and c is the speed of light in vacuum. In astronomical units $c = 63239.7263\,\mathrm{AU/yr}$.

An important test of the general theory of relativity was to compare its prediction for the perihelion precession of Mercury to the observed value. The observed value of the perihelion precession, when all classical effects are subtracted, is 43″ (43 arceseconds) per century. The value of the perihelion angle θ_p can be obtained by

$$\tan\theta_\text{p} = \frac{x_\text{p}}{y_\text{p}}, \tag{2.24}$$

where x_p and y_p is the x and y position, respectively, of Mercury at perihelion, that is, at the point where Mercury is at its closest to the Sun. The speed of Mercury at perihelion is 12.44 AU/yr, and the distance from to the Sun is 0.3075 AU.

3 Method

3.1 Implementing the Forward Euler and Velocity Verlet Methods

Algorithm 1 below shows a possible implementation of the Forward Euler method described in Section 2.2.

Algorithm 1 Forward Eulers Method

function EULER(r, v, a, N)
  for $i = 0$ **to** N **do**
    $v_{i+1} := v_i + a_i * \mathrm{d}t$
    $r_{i+1} := r_i + v_i * \mathrm{d}t$
  end for
end function

Table 3.1 tabulates the number of floating point operations in the Forward Euler method given in Algorithm 1.

Table 3.1: *Floating point operations in the Forward Euler method*

Operation	Floating Point
Memory Reads	$6N$
Memory Writes	$2N$
Additions & Subtractions	$2N$
Multiplications & Divisons	$2N$
Total	$12N$

Algorithm 2 below shows a possible implementation of the Velocity Verlet method described in Section 2.3.

Algorithm 2 Velocity Verlet Method

```
function VERLET(r, v, a, N)
    for i = 0 to N do
        r_{i+1} := r_i + v_i * dt + 1/2 a_i * dt^2
        v_{i+1} := v_i + 1/2 (a_{i+1} + a_i) * dt
    end for
end function
```

Table 3.2 tabulates the number of floating point operations in the Velocity Verlet method given in Algorithm 2.

Table 3.2: *Floating point operations in the Velocity Verlet method*

Operation	**Floating Point**
Memory Reads	$10N$
Memory Writes	$2N$
Additions & Subtractions	$4N$
Multiplications & Divisons	$6N$
Total	$22N$

3.2 Comparison of Earth's Orbit

First, we limit ourselves to a hypothetical Solar System with only Earth orbiting around a static Sun under the assumption that the orbit is co-planar. The coupled differential equations we need to solve is then given by Equation (2.3) and Equation (2.4), only here with Equation (2.14) as our point of departure. We impose initial conditions on Earth that make the co-planar orbit circular in accordance with the discussion in Section 2.4, that is $\mathbf{r}_0(x, y) = (1\,\mathrm{AU}, 0)$ and $\mathbf{v}_0(x, y) = (0, 2\pi\,\mathrm{AU/yr})$. Both the result from the Forward Euler method and the Velocity Verlet method will be presented. We run the simulations with the simulation time $T = 5\,\mathrm{yr}$ and the number of integration points $N = 10\,000$.

3.3 Benchmarking the CPU Times

In order to find the complexity of our numerical methods of choice, we extract the behavior as a function of the number of integration points. In the next sections, we also look at their conservation properties. Here, we look at the computational time of both the Forward Euler and the Velocity Verlet method as a function of the number of integration points. The procedure is to benchmark the performance for an increasing number of integration points. To easier interpret the result, we will use a log-log graph which is obtained by using logarithmic scales on both the horizontal and vertical axes. The observational data will also be linearly fitted.

3.4 Testing Conservation Properties

In order to test that the energy is conserved, a calculation of the total energy as outlined in Equation (2.18) is implemented for the hypothetical Earth-Sun system. The result of the calculation is put into an array of which the maximum value and minimal value is extracted. Then a calculation of the energy fluctuation is done by the following

$$\epsilon = \left| \frac{E_{\text{max}} - E_{\text{min}}}{E_{\text{max}}} \right| \tag{3.1}$$

This calculation will be done for several integration points which will give an array of ϵ's. The fluctuations will be plotted as a function of the number of integration points N as a log-log graph. This procedure will be done with both the Forward Euler method and the Velocity Verlet method, in order to extract which of the methods are favourable.

To test that the total angular momentum is conserved, we use the same rationale as for the energy above. Thus a calculation of the angular momentum fluctuation is done by the following

$$\eta = \left| \frac{L_{\text{max}} - L_{\text{min}}}{L_{\text{max}}} \right| \tag{3.2}$$

This calculation will also be done for several integration points, which will give a array of η's. The results are presented in the same manner as in the previous section, for both the Forward Euler and the Velocity Verlet method.

A plot of the total energy as a function of time will also be generated. The calculation will be done with $N = 10\,000$ integration points and simulation time $T = 5$ yr. This will also be done for both the Forward Euler and the Velocity Verlet method.

3.5 Escape Velocity

To find the numerical escape velocity a simulation of Earths orbit for several initial velocities will be presented in a plot. The result found will then be compared with the analytical result found in Section 2.7

Next, we replace the classic inverse-square gravity

$$F_{\text{G}} = G\frac{M_{\odot} M_{\text{Earth}}}{r}$$

by a gravity that creep towards an inverse-cube gravity

$$F_{\text{G}} = G\frac{M_{\odot} M_{\text{Earth}}}{r^{\beta}}, \tag{3.3}$$

where $\beta \in [2, 3]$. A plot of Earth's orbits when β creep from the value 2 to 3 will be generated. This plot will also be compared with the analytical result found in Section 2.8.

3.6 The Three-body Problem

Here, we add Jupiter to our hypothetical Solar System with a fixed sun, thus extending the system to a three-body problem. This problem will be solved by implementing the interaction as discussed in Section 2.9. The subjects of interest are how Jupiter, having a mass roughly 1000 times smaller than the Sun, alters the Earth's motion and the stability of the Verlet solver for the Earth-Jupiter-Sun interaction. The stability will be studied by the same means as in Section 3.4, by looking at the energy fluctuation of the system. In the study of how Jupiter alter Earth's orbit, we will, in addition to letting Jupiter have its normal mass, increase the mass of Jupiter by a factor 10 and 1000.

3.7 Motion of the Solar System

Next, we aim to make the model more realistic with all the celestial bodies in motion, the Sun included. In order to do this, we make the center-of-mass (COM) position of all the celestial bodies our origin, and give the Sun an initial velocity which makes the total momentum of the system exactly zero. The positional COM, $\mathbf{R}_{\text{COM}}$, is given by

$$\mathbf{R}_{\text{COM}} = \sum_i m_i \mathbf{r}_i$$

To make sure $\mathbf{R}_{\text{COM}} = 0$ with the Sun still close to the origin, we define the initial position of the Sun as

$$\mathbf{r}_\odot = -\frac{\mathbf{R}_{\text{COM}}}{M_\odot} = -\frac{\sum_i m_i \mathbf{r}_i}{M_\odot},$$

where the sum is over all the celestial bodies except the Sun. Similarly, we can ensure that the total momentum of the system is exactly zero by letting the initial velocity of the Sun be given by

$$v_\odot = -\frac{\sum_i m_i \mathbf{v}_i}{M_\odot},$$

where the sum is over all the celestial bodies except the Sun. The interaction between the celestial bodies is found by simply extending the three-body problem to the number of bodies at hand.

To make the simulation even more realistic, initial conditions will be retrieved from the National Aeronautics and Space Administration (NASA) from https://ssd.jpl.nasa.gov/horizons.cgi#top, which can be used to generate ephemerides for the celestial bodies in the Solar System.

3.8 Precession of the Perihelion of Mercury

Finally, we will study whether or not our numerical solver reproduce the precession of the perihelion of Mercury predicted by Einstein's theory of general relativity, as discussed in Section 2.10. A comparison between the perihelion angle obtained by both the classical Newtonian gravity and the relativistic correction, given by Equation (2.23), will be presented.

4 Results

4.1 Implementing the Forward Euler and Velocity Verlet Methods

The program containing the implementation of the Forward Euler and Velocity Verlet methods and accompanying programs that produces all the results presented in this project, can be found at the GitHub repository

https://github.com/nicolossus/FYS3150/tree/master/Project3

4.2 Comparison of Earth's Orbit

Figure 4.1 and Figure 4.2 shows the Earth's orbit around the Sun solved by the Forward Euler method and the Velocity Verlet method, respectively, as outlined in Section 3.2.

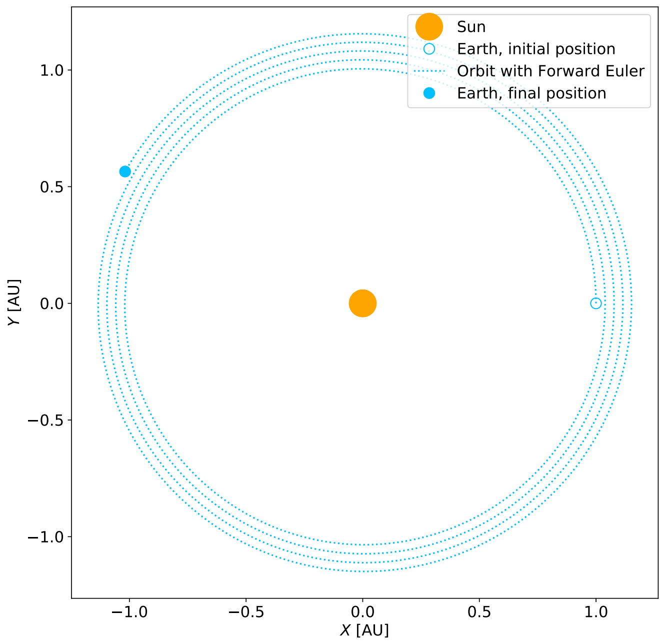


Figure 4.1: *Earth's orbit around the Sun as solved by the Forward Euler method with the simulation time set to* $T = 5\,\mathrm{yr}$ *and the number of integration points set to* $N = 10\,000$. *Here, initial conditions that make the orbit circular have been imposed, namely* $\mathbf{r}_0(x, y) = (1\,\mathrm{AU}, 0)$ *and* $\mathbf{v}_0(x, y) = (0, 2\pi\,\mathrm{AU/yr})$.

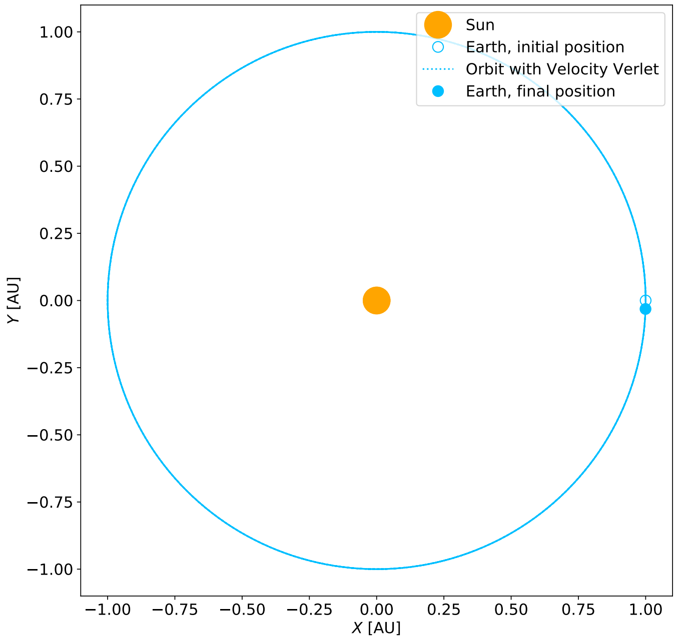


Figure 4.2: *Earth's orbit around the Sun as solved by the Velocity Verlet method with the simulation time set to* $T = 5\,\mathrm{yr}$ *and the number of integration points set to* $N = 10\,000$*. Here, initial conditions that make the orbit circular have been imposed, namely* $\mathbf{r}_0(x, y) = (1\,\mathrm{AU}, 0)$ *and* $\mathbf{v}_0(x, y) = (0, 2\pi\,\mathrm{AU/yr})$*.*

4.3 Benchmarking the CPU Times

Figure 4.3 shows the benchmark of the CPU times of our integration methods, Forward Euler and Velocity Verlet, as outlined in Section 3.3. The benchmark was performed on the same system as in the previous section, only here the number of integration points range from $N = 10^1$ to $N = 10^7$. The observational data have also been linearly fitted.

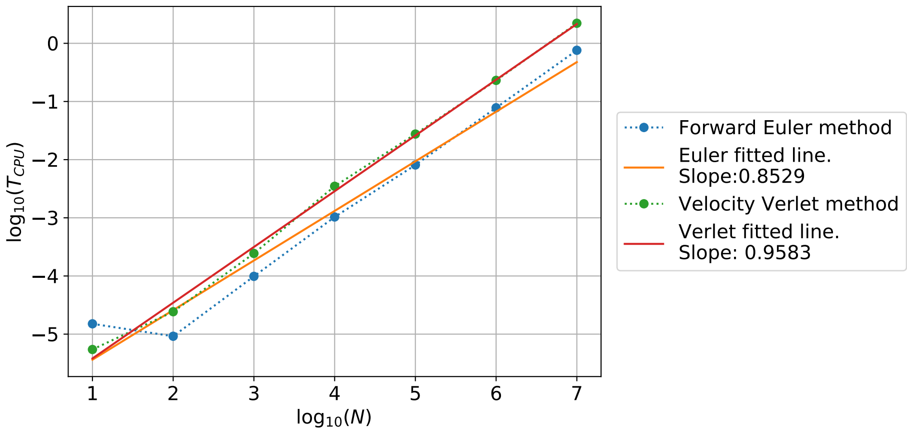


Figure 4.3: *Comparison of CPU times of the Forward Euler method and the Velocity Verlet method. This is a log-log plot of the CPU time T_{CPU} as a function of the number of integration points N. The data have been linearly fitted, and the slope of the fit is stated in the legend.*

4.4 Testing Conservation Properties

Figure 4.4 shows log-log graphs of the energy fluctuations ϵ (given by Equation (3.1)) in the Sun-Earth system generated by using the Forward Euler method and the Velocity Verlet method in accordance with the description in Section 3.4. The fluctuations are plotted as a function of the number of integration points N.

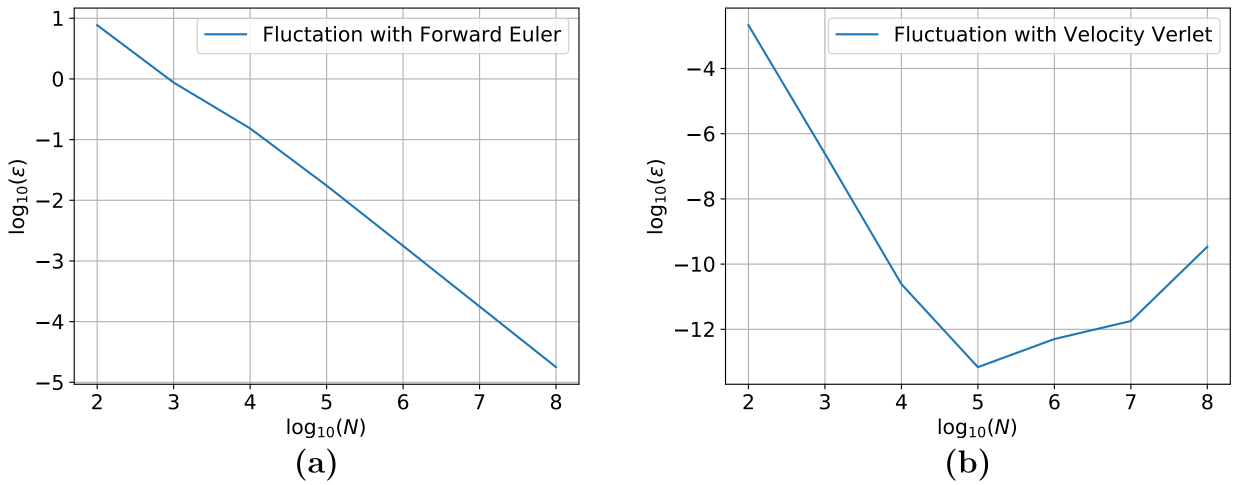
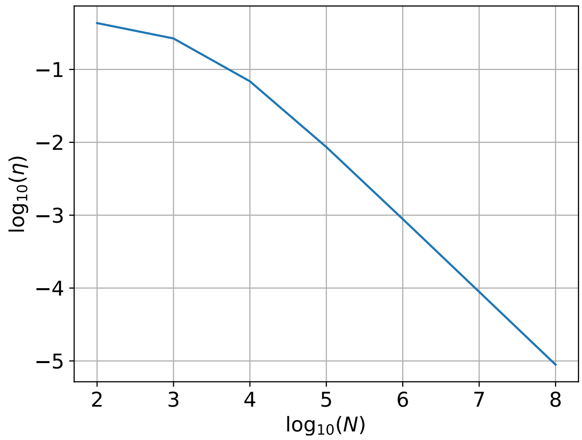


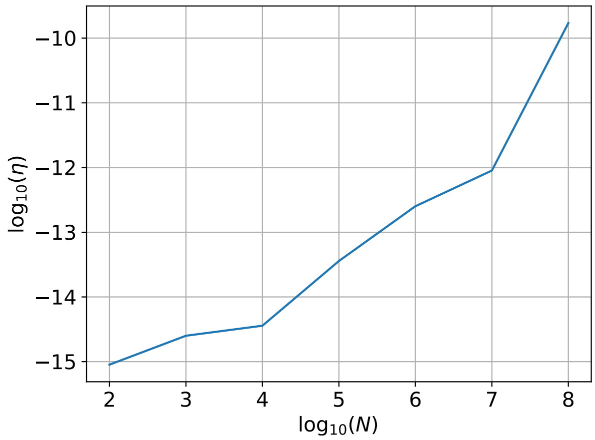
Figure 4.4: *Comparison of conservation properties of **(a)** the Forward Euler method and **(b)** the Velocity Verlet method. These are a log-log graphs of the energy fluctuation ϵ plotted as a function of the number of integration points N.*

Figure 4.5 shows log-log graphs of the angular momentum fluctuations η (given by Equation (3.2)) in the Sun-Earth system generated by using the Forward Euler method and the

Velocity Verlet method in accordance with the description in Section 3.4. The fluctuations are plotted as a function of the number of integration points N.



(a)



(b)

Figure 4.5: *Comparison of conservation properties of* ***(a)*** *the Forward Euler method and* ***(b)*** *the Velocity Verlet method. These are a log-log graphs of the angular momentum fluctuation η plotted as a function of the number of integration points N.*

In Figure 4.6 is a plot of the total energy in the Sun-Earth system is generated as a function of time. The plot shows a comparison of how the total energy changes with time when using the Forward Euler and the Velocity Verlet method. The plot is generated by simulating the motion for $T = 5$ years and with $N = 10\,000$ integration points.

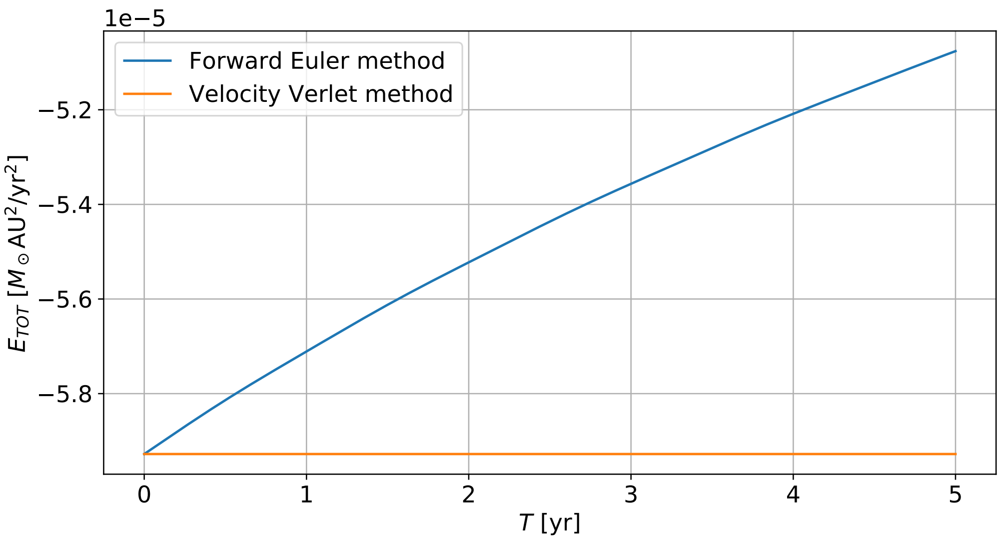


Figure 4.6: *The total energy of the Earth-Sun system as a function of time. The system is simulated for $T = 5$ years, with the motion solved by both the Forward Euler and the Velocity Verlet method with $N = 10\,000$ integration points.*

4.5 Escape velocity

Figure 4.7 shows the orbit of the Earth for different initial velocities (stated in the plot legend), and for which the Earth begins to escape the Sun's clutches.

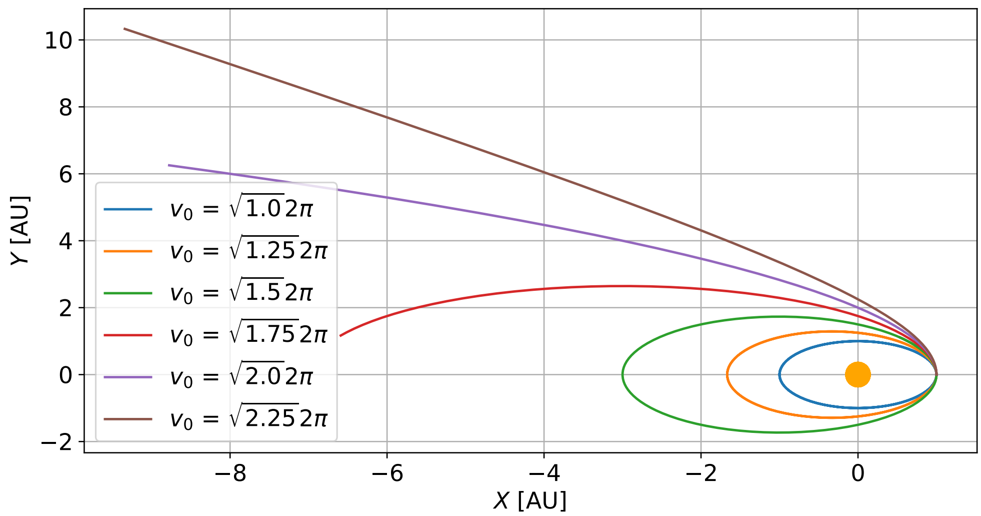
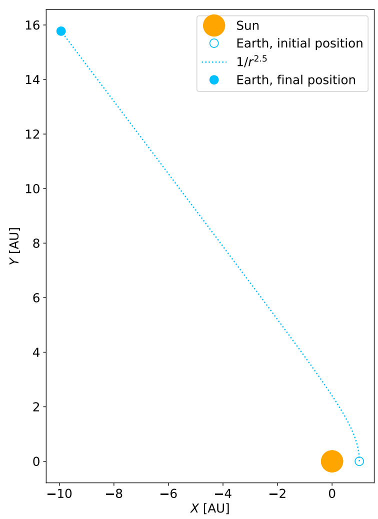
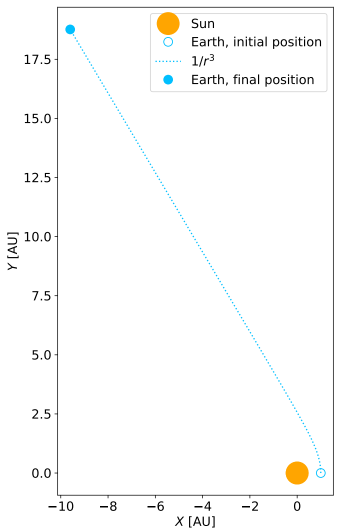


Figure 4.7: *Orbit of the Earth for different initial velocities (stated in the legend). The plot shows for which initial velocities the Earth has its escape velocity. The simulation time was set to $T = 3$ years with $N = 10\,000$ integration points.*

Figure 4.8 shows the orbit of the Earth when β in Equation (3.3) creep from the value 2 to 3, as discussed in Section 3.5. The value of β is stated in the plot legends.



(a)



(b)

Figure 4.8: *Earth's orbit when using Equation (3.3) as the gravitational force. The value of β is specified in each plot.*

4.6 The Three-body Problem

Figure 4.9 shows log-log graphs of the energy fluctuation ϵ (given by Equation (3.1)) in the Earth-Jupiter-Sun system generated by using the Velocity Verlet method. The fluctuation is plotted as a function of the number of integration points N.

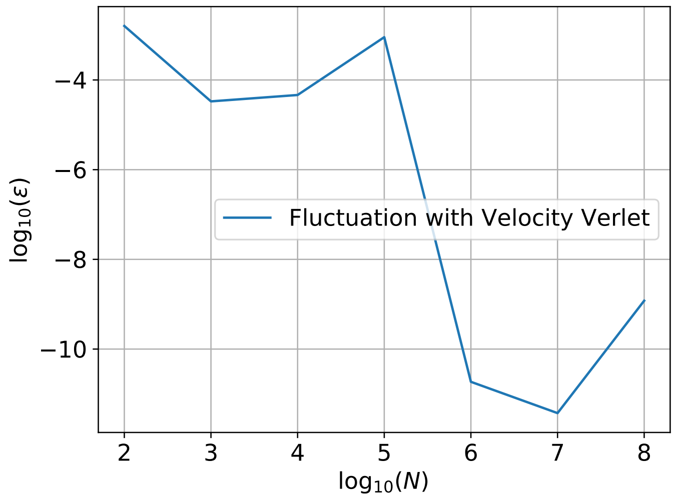


Figure 4.9: *Stability of the Verlet solver for Earth-Jupiter-Sun system, here visualized as a log-log graph of the energy fluctuation ϵ as a function of the number of integration points N with the simulation set to $T = 22$ yr.*

Figure 4.10 shows the motion of the Earth-Jupiter-Sun system with the simulation time set to $T = 22$ years and the number of integration points $N = 1\,000\,000$. In this figure Jupiter has its natural mass.

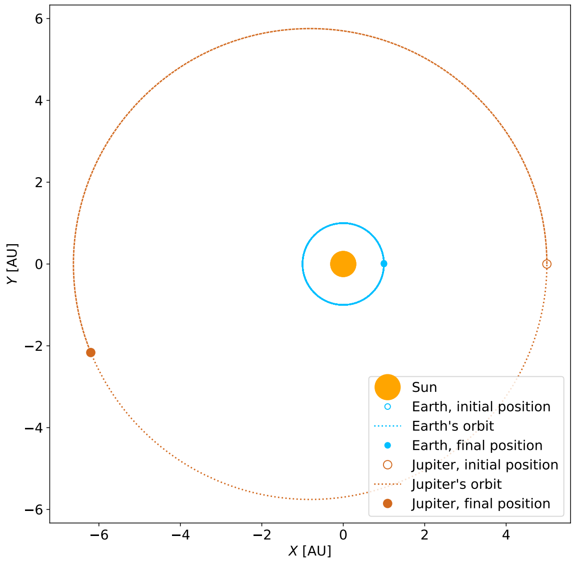


Figure 4.10: *Motion of the Earth-Jupiter-Sun system simulated with the simulation time set to $T = 22$ years and the number of integration points set to $N = 1\,000\,000$. In this particular system Jupiter has its natural mass (roughly 1000 times smaller than the Sun).*

Figure 4.11 and Figure 4.12 shows the motion of the Earth-Jupiter-Sun system with the simulation time set to $T = 22$ years and the number of integration points $N = 1\,000\,000$. In these figures we have increased Jupiter's mass by a factor of 10 and 1000, respectively.

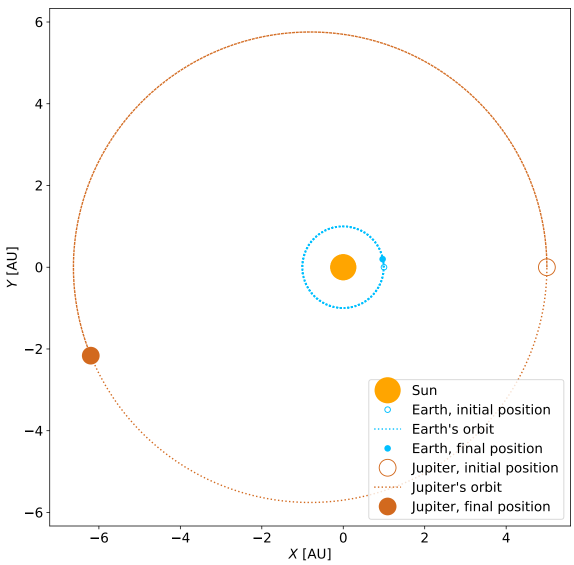


Figure 4.11: *Motion of the Earth-Jupiter-Sun system simulated with the simulation time set to $T = 22$ years and the number of integration points set to $N = 1\,000\,000$. In this particular system Jupiter has its natural mass increased by a factor 10.*

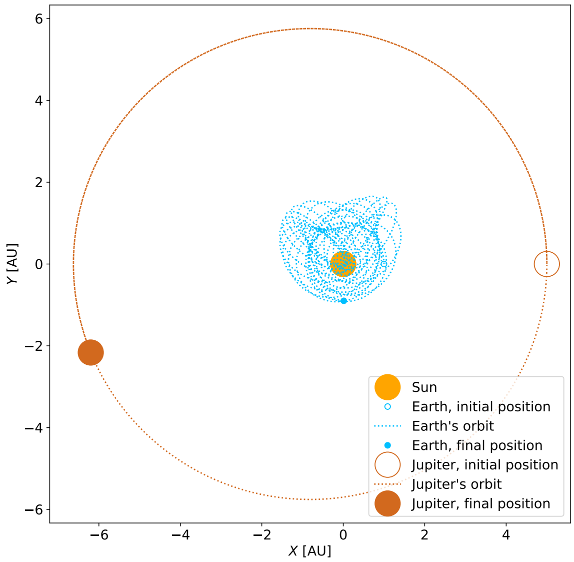


Figure 4.12: *Motion of the Earth-Jupiter-Sun system simulated with the simulation time set to $T = 22$ years and the number of integration points set to $N = 1\,000\,000$. In this particular system Jupiter has its natural mass increased by a factor 1000, making the mass almost the same as the Sun's.*

4.7 Motion of the Solar System

We here seek to make our model more realistic, by including the motion of the Sun as outlined in Section 3.7. Figure 4.13 shows the motion of the Earth-Jupiter-Sun system, with the Sun being dynamic. Initial positions and velocities of the planets have been retrieved from NASA, and the initial position and velocity of the Sun have been calculated from these.

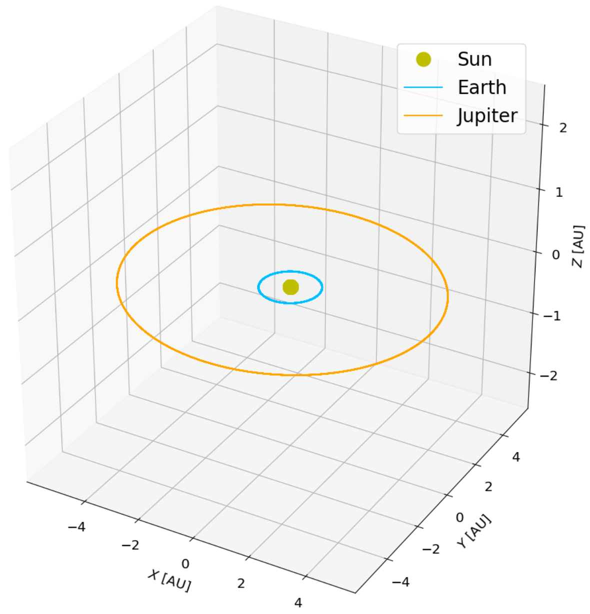


Figure 4.13: *Motion of the Earth-Jupiter-Sun system, with the Sun also in motion.*

Figure 4.14 shows the orbits of the inner planets of the Solar System, namely Mercury, Venus, Earth, and Mars.

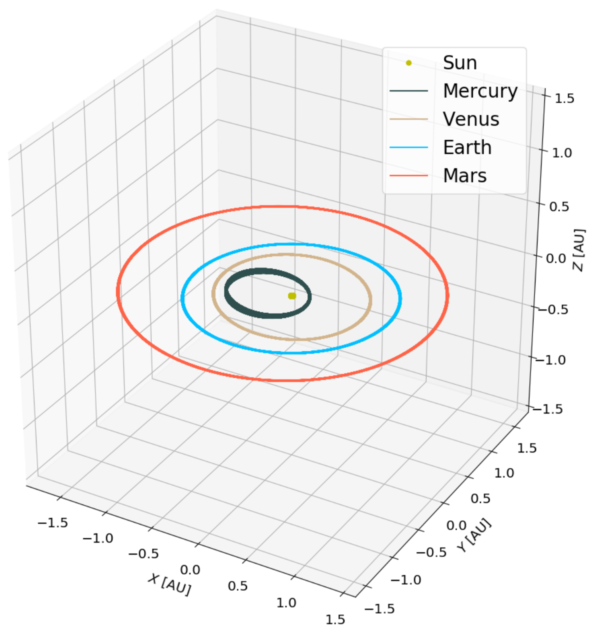


Figure 4.14: *Motion of the inner planets in the Solar System, with the planets stated in the legend.*

Figure 4.15 shows the orbits of the inner planets of the Solar System, namely Jupiter, Saturn, Uranus, and Neptune.

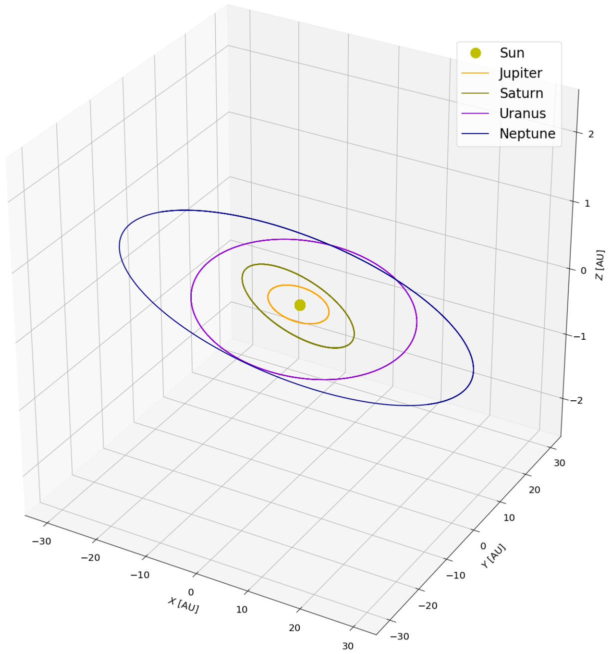


Figure 4.15: *Motion of the outer planets in the Solar System, with the planets stated in the legend.*

Figure 4.16 shows the full model of all the planets in the Solar System, with Pluto included for historical reasons.

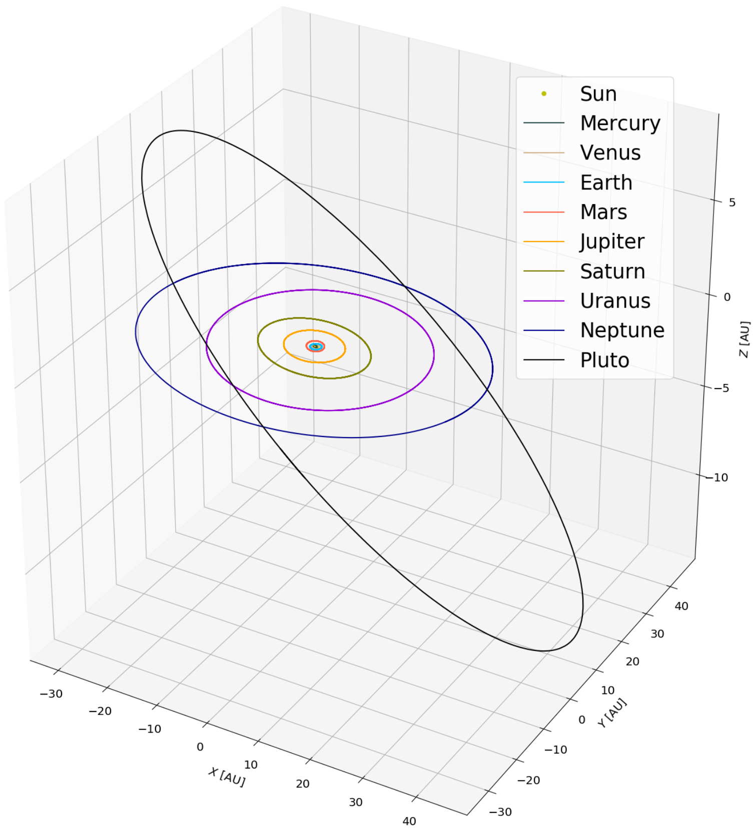


Figure 4.16: *Full model of all the planets in the Solar System, with Pluto included for historical reasons.*

4.8 Precession of the Perihelion of Mercury

Figure 4.17 shows the precession of the perihelion of Mercury with both Newton's and Einstein's gravity. This simulation was run over one century, with Mercury as the only planet in orbit around the Sun, starting with Mercury at perihelion on the x-axis. That is, the initial conditions, as discussed in Section 2.10, were $\mathbf{r}_0(x, y, z) = (0.3075\,\mathrm{AU}, 0, 0)$ and $\mathbf{v}_0(x, y, z) = (0, 12.44\,\mathrm{AU/yr}, 0)$. The simulation used $N = 100\,000\,000$ as the number of integration points.

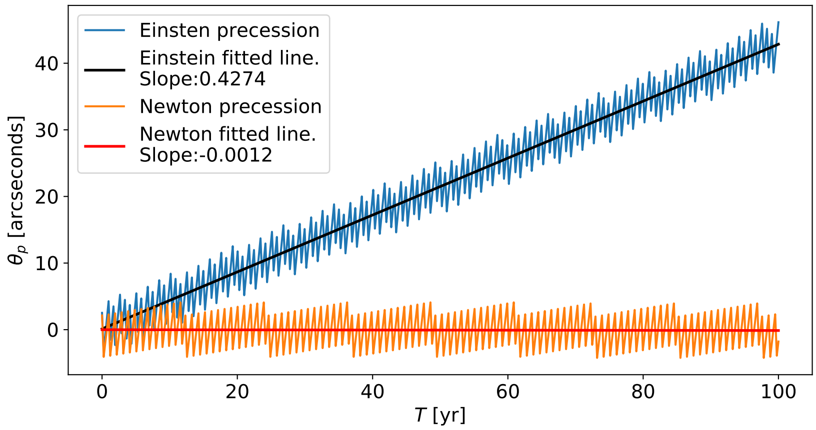


Figure 4.17: *Precession of the perihelion of Mercury obtained by both the classical Newtonian gravity and the relativistic correction provided by Einstein. The data have been linearly fitted, and the slope of the fit is stated in the legend.*

5 Discussion

5.1 Implementing the Forward Euler and Velocity Verlet Methods

The code used in this project (see GitHub) was written with object oriented code. The "write once, run many times" point of object orientation was central in developing the code. However, the final code is not as general as we first intended it to be. Many modules became very specialized to the specific model in this project, and not many-body problems in general. However, certain parts of the code are indeed reusable for other many-body problems, in particular the implementations of the Forward Euler and the Velocity Verlet method. With more time and programming experience with object orientation, more of these specific modules would surely been transformed to more general ones.

5.2 Comparison of Earth's Orbit

In this simulation we imposed initial conditions on Earth that makes the orbit circular and co-planar with the static Sun, and simulated the motion for 5 years. We therefore expect a constant circular orbit and that the final position of the Earth is the same as the initial position. In Figure 4.1, which shows the orbit of the Earth solved by the Forward Euler method, we see that the circular orbit spiral out and that the final position of Earth is almost opposite of the initial position. In contrast, is the orbit of the Earth solved by the Velocity Verlet method, shown in Figure 4.2, constant circular as it overlap with itself over the time evolution and the final position is almost the same as the initial position. Increasing the number of integration points would have increased the accuracy of both methods, but it is still clear-cut that the Velocity Verlet method vastly outperform the Forward Euler method in terms of accuracy.

5.3 Benchmarking the CPU Times

Table 3.1 and Table 3.2 shows that the number of floating point operations with N integration points are $12N$ for the Euler Forward method and $22N$ for the Velocity Verlet method. From this, we expect the methods to have the same complexity, with the Velocity Verlet method a bit slower than the Forward Euler method due to more floating point operations. The benchmark result shown in Figure 4.3 is in accordance with this expectation. The result shows that both have a complexity close to N^1 with approximately $N^{0.95}$ and $N^{0.85}$ for the Forward Euler and the Velocity Verlet method, respectively. The Forward Euler method has an irregularity for the first integration point with $N = 10$, but follows the expected linear trend from the second integration point and forward. The benchmark could have been carried out in a more robust manner by running the simulations for each number of integration points multiple times, and then average the CPU times. However, the behaviour in terms of performance of the numerical methods is clearly shown, and is enough for the qualitative analysis we aim for.

5.4 Testing Conservation Properties

From Section 2.5 we expect that the total Energy for the Earth-Sun system is conserved as a function of time. In Figure 4.4 we calculated the energy fluctuation as a function of integration points. We see that the fluctuation is large for the Forward Euler method, but that the fluctuation decreases as the number of integration points grow, but it is still not sufficient to conserve the total energy. Verlet's method on the other hand has a small fluctuation, with a minimum value of the order 10^{-12} for $N = 10^5$, but we see that as the number of integration points increases after that the fluctuation also increases. It is likely that loss of numerical precision when we calculate the fluctuation is the reason for this increase. Also in Figure 4.6 a plot of the total energy as a function of time in the Earth-Sun sytem was presented, and we see that the Energy is not conserved in the Forward Euler method, but Verlet's method conserves energy.

In Figure 4.5 angular momentum fluctuation as a function of the number of integration points for the Earth-Sun system was presented, and we see that the Forward Euler method again has a large fluctuation, but the fluctuation in Verlet's method is much smaller, but also here we see that the fluctuation increases as the number of integration points increases. Even though it increases the fluctuation is of the order 10^{-10} for $N = 10^8$, which is negligible, and therefore the Verlet method conserves angular momentum.

We implemented unit tests in order to check that if the two methods conserves energy and angular momentum, and there we also found that Forward Euler does not conserve energy, but Verlet's method does. These tests can be found in test.cpp in the GitHub repository (run "make test.x").

5.5 Escape Velocity and Closed Orbits

In Figure 4.7 we generated a plot of Earths orbit around the sun for several different initial velocitys. From Section 2.7 we expect that the Escape velocity of the Earth is equal to $2\sqrt{2}$, and we see in the plot that our numerical result is in exact correspondence with the analytical result.

From Section 2.8 we found that the only stable closed orbits for our potential is when $\beta = 2$, and from Figure 4.8 where we generated two plots with $\beta = 2.5$ and $\beta = 3$ we see that Earth escapes the gravitational field of the Sun, which is in accordance to the analytical result.

5.6 The Three-body Problem

For the three-body problem including Earth, Jupiter and the stationary sun, Figure 4.9 shows that the energy was well conserved during a time period of 22 years (approximately twice of Jupiter's period). The smallest fluctuation in the total energy of approximately $10^{-9}\%$ occurred for $N = 10^7$. The stability of the three-body problem, looking at conservation of energy and angular momentum, was also confirmed by the unit test (test.cpp in the GitHub repository).

For Jupiter's natural mass, Figure 4.10 shows that the trajectory of Earth is visibly unchanged, and the start and end position of Earth is approximately unchanged.

For ten times the natural mass of Jupiter, Figure 4.11 shows little perturbation of Earth's orbit as well. However, the start and endpoints are somewhat displaced, indicating that the heavier Jupiter affects the Earth noticeably.

For 1000 times normal mass, Jupiter weights almost as much as the Sun. Likewise, Earth's orbit is heavily altered, exhibiting chaotically precessing, elliptical orbits.

5.7 Motion of the Solar System

Comparing Figure 4.10 and Figure 4.13, we see that the having the Sun moving makes little visible difference to the dynamics of the system. This is of course because of the Sun's big mass compared to Earth and Jupiter, causing the Sun's trajectory very small, and its center very close to COM.

Looking at the extended Solar System, the inner planets Figure 4.14, the outer planets Figure 4.15, and finally, all the planets Figure 4.16, our modeling reproduces a well-behaved simulation of closed, elliptical orbits, true to our Solar System.

5.8 Precession of the Perihelion of Mercury

As seen from the linear fit of relativistic correction in Figure 4.17, the precession of the perihelion of Mercury was found to be $42.74''$. This is in very good correspondance with the observed value of $43''$. Einstein was right all along.

6 Conclusion

In this project we developed a model simulating the motion of the planets in the Solar System by using both the Forward Euler and the Velocity Verlet method to integrate Newton's equations of motion. It was found that the Velocity Verlet conserves both energy and angular momentum very well, whereas the Forward Euler method does not. The Velocity Verlet methods conservation properties, in terms of energy and angular momentum conservation, are crucial in a model of the Solar System, and many-body interactions in general. In addition were we able to make good simulations for even a relatively low number of integration points with the Verlet solver. Both methods are also relatively fast, in terms of computational speed, even for an increasing number of integration points. The Forward Euler method is a bit faster than the Velocity Verlet method, due to that the latter has more floating point operations per iteration. However, this difference is nearly negligible. Implementing the Velocity Verlet integration method for solving coupled ordinary differential equations is therefore by far the best option, compared to the Forward Euler integration method, for making a fairly realistic model of the Solar System.

7 Future Work

The final code that we generated is not as general as wanted, since many of the modules were made for specific problems and not an overall solver for many body systems. There are certain parts that are reusable, in particular the implementations of the Forward Euler and the Velocity Verlet method. In the next two projects we want to generate a more general code, specifically since we have ideas about choosing the molecular dynamics project. Using more class orientation to generalize and c++ templates to accept different data structures(e.i. a molecule class for molecular dynamics) will be priorities. To improve simulation time, one could introduce cutoff-functions, neglecting planets and bodies that are too small and too far away to contribute significantly to the dynamics of the system.