Project 1

a) 3. This means that $y_i \sim N(X_{i,*}\vec{B}, \sigma^2)$

Show that $\mathbb{E}(\hat{\beta}) = \vec{\beta}$

We know that we can express $\hat{\beta} = (X^T X)^{-1} X^T Y$

$\Rightarrow \langle (X^T X)^{-1} X^T Y \rangle$ factor $(X^T X)^{-1} X^T$ is a matrix of constants, thus we can write

$\langle (X^T X)^{-1} X^T Y \rangle = (XX^T)^{-1} X^T \langle Y \rangle$

again, we write $Y = X\vec{\beta} + \varepsilon$, and $\langle X\vec{\beta} + \varepsilon \rangle$

$= X \langle \vec{\beta} \rangle + \langle \varepsilon \rangle$

$\Rightarrow (XX^T)^{-1} X^T (X \langle \vec{B} \rangle + \langle \varepsilon \rangle)$

$= \underbrace{(XX^T)^{-1} X^T X}_{\mathbb{1}} \beta + (XX^T)^{-1} X^T \underbrace{\langle \varepsilon \rangle}_{0} = \mathbb{1} \beta = \beta$

This result means that the estimator of the regression is unbiased.

a.3) For this derivation, we use the following

$\mathrm{Var}(A B) = A \,\mathrm{Var}(B) A^T$

A: not stockastic matrix; B: stockastic matrix

Now $\Rightarrow \mathrm{Var}(\hat{\beta}) = \mathrm{Var}((X^T X)^{-1} X^T Y)$

$(X^T X)^{-1} X^T$ is non stochastic matrix, while Y is a stochastic

$\Rightarrow \mathrm{Var}(\hat{\beta}) = (X^T X)^{-1} X^T \mathrm{Var}(Y) [(X^T X)^{-1} X^T]^T$

We also use the following relations:

$(AB)^T = B^T A^T$ and $(A^{-1})^T = (A^T)^{-1}$

Then $[(X^T X)^{-1} X^T]^T = (X^T)^T ((X^T X)^{-1})^T = X (X^T X)^{-1}$

$\Rightarrow \mathrm{Var}(\hat{\beta}) = (X^T X)^{-1} X^T (\sigma^2 \mathbb{1}) X (X^T X)^{-1}$

$= \sigma^2 \cancel{(X^T X)^{-1}} \cancel{X^T X} (X^T X)^{-1} = \sigma^2 (X^T X)^{-1}$