DESCRIPTION OF THE MATHEMATICAL MODELS

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1. Introduction

This document describes the mathematical models in the simulation examples related to the article [1]. The simulation codes can be downloaded using the link below. The considered models are particular cases of the boundary controlled heat equations on one and two-dimensional spatial domains considered in [1, Sec. 5]. The simulation codes utilise the free **RORPack** MATLAB library (link below) for controller construction and simulation of the controlled system. In addition, the one-dimensional example uses the free **Chebfun** MATLAB library (link below) for solving the boundary value problems in construction of the controller parameters.

Download links:

github.com/lassipau/CDC22-Matlab-simulations/	(Simulation codes)
github.com/lassipau/rorpack-matlab/	(**RORPack** library)
www.chebfun.org	(**Chebfun** library)

2. The Heat Equation on a Rectangle

We consider a heat equation on a square $\Omega = [0,1]\times[0,1]$ defined by

$$\begin{aligned}
x_t(\xi,t) &= \Delta x(\xi,t), \quad && x(\xi,0) = x_0(\xi)\\
\frac{\partial x}{\partial n}(\xi,t) &= u(t), && \xi\in\Gamma_1\\
\frac{\partial x}{\partial n}(\xi,t) &= w_{dist}(t), && \xi\in\Gamma_2\\
\frac{\partial x}{\partial n}(\xi,t) &= 0, && \xi\in\Gamma_0\\
y(t) &= \int_{\Gamma_3} x(\xi,t)d\xi.
\end{aligned}$$

Here $\frac{\partial x}{\partial n}$ denotes the outward normal derivative and Γ_0, Γ_1, Γ_2, and Γ_3 are parts of the boundary $\partial\Omega$ of the rectangle defined by (see Figure 1)

$$\begin{aligned}
\Gamma_1 &= \{\, (\xi_1,\xi_2)\in\mathbb{R}^2 \mid 0<\xi_1<1/2,\ \xi_2=0\,\}\\
\Gamma_2 &= \{\, (\xi_1,\xi_2)\in\mathbb{R}^2 \mid \xi_1=0,\ 0<\xi_2<1\,\}\\
\Gamma_0 &= \partial\Omega\setminus(\Gamma_1\cup\Gamma_2)\\
\Gamma_1 &= \{\, (\xi_1,\xi_2)\in\mathbb{R}^2 \mid 0<\xi_1<1,\ \xi_2=1\,\}.
\end{aligned}$$

The system fits in the setting of the example in [1, Sec. 5] with suitable choices of $b,c\in L^2(\partial\Omega;\mathbb{R})$ and $B_d^1(\cdot)\in L^2(\partial\Omega;\mathbb{R}^{1\times n_{d1}})$

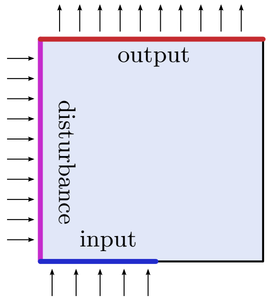

FIGURE 1. Input-output configuration for the 2D heat equation.

In the simulation, the state feedback and output injection operators are use chosen based on LQR/LQG design. Then $K_0 x = \int_\Omega x(\xi) k_0(\xi) d\xi$ for some $k_0 \in L^2(\Omega)$ and the results in [1, Sec. 4] show that for $\omega = \pm\omega_k \in \mathbb{R}$ the values $P_K(i\omega)$ and $P_{KI}(i\omega)$ required in the controller construction could be computed by solving the boundary value problem

$$\begin{aligned} i\omega x_0(\xi) &= \Delta x_0(\xi) + \psi_n(\xi), \\ \frac{\partial x_0}{\partial n}(\xi) &= u_0 + \int_\Omega x_0(\zeta) k_0(\zeta) d\zeta, && \xi \in \Gamma_1 \\ \frac{\partial x_0}{\partial n}(\xi) &= 0, && \xi \in \Gamma_2 \cup \Gamma_0 \\ y_0 &= \int_{\Gamma_3} x_0(\xi) d\xi. \end{aligned}$$

With the choices $u_0 = 1 \in \mathbb{C}$ and $\psi_n = 0 \in L^2(0,1)$ we then have $y_0 = P_K(i\omega)$, and for $u_0 = 0$ and $\psi_n \in L^2(0,1)$ we get $y_0 = P_{KI}(i\omega)\psi_n$. However, based on the results in [1, Sec. 4], the parameter H_K can alternatively be computed as the (approximate) solution of the Sylvester equation

$$G_1 H_K = H_K(A + BK_0) + G_2(C_\Lambda + DK_0) \qquad \text{on} \quad D(A + BK_0),$$

and the simulation code takes advantage of this possibility. In the code H_K is computed using another Finite Difference approximation of the original PDE system. Subsequently, the parameter B_1 can be correspondingly approximated by $B_1 = H_K B + G_2 D$ (where $D = 0$ for our system).

Figure 2 shows an example output of the simulation.

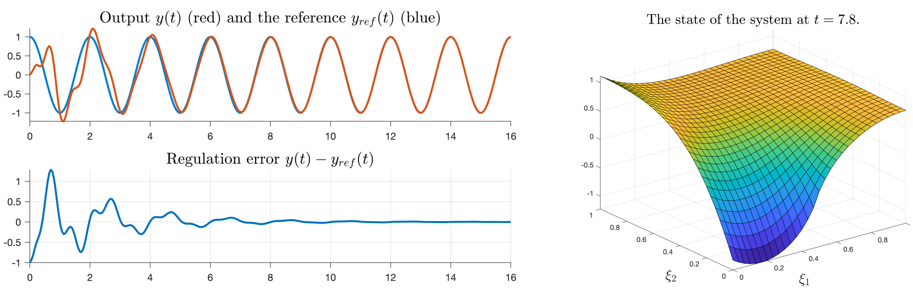

FIGURE 2. Simulation of the 2D heat equation. Output $y(t)$ and tracking error (left) and the temperature profile at $t = 7.8s$ (right).

3. The One-Dimensional Heat Equation

Consider a one-dimensional heat equation on $\Omega = [0,1]$ with non-collocated boundary inputs and outputs,

$$\begin{aligned} x_t(\xi,t) &= (a(\xi)x_\xi)_\xi(\xi,t) + b_d(\xi)w_d^0(t) \\ -x_\xi(0,t) &= u(t) + w_d^1(t), \quad x_\xi(1,t) = w_d^2(t) \\ y(t) &= x(1,t), \qquad x(\xi,0) = x_0(\xi), \end{aligned}$$

where the spatially varying heat conductivity satisfies $a(\cdot) \in C([0,1])$, $a(\xi) > 0$ for all $\xi \in [0,1]$, and $w_d^0(t)$, $w_d^1(t)$, and $w_d^2(t)$ are external disturbance signals. It is well-known that this PDE defines a regular linear system on $X = L^2(0,1)$. The system is unstable due to the eigenvalue at $0 \in \mathbb{C}$.

For stabilizing the system (1) with state feedback and output injection we can choose, e.g., $K_0 = -\langle \cdot, \mathbf{1}\rangle_{L^2}$ and $L = -\mathbf{1}(\cdot) \in L^2(0,1)$, where $\mathbf{1}(\xi) = 1$ for all $\xi \in [0,1]$. Since $K_0 x = -\int_0^1 x(\xi)d\xi$, the results in [1, Sec. 4] show that for $\omega = \pm\omega_k \in \mathbb{R}$ the values $P_K(i\omega)$ and $P_{KI}(i\omega)$ required in the controller construction can be computed by solving the boundary value problem

$$i\omega x(\xi) = (a(\xi)x_\xi)_\xi(\xi) + \psi_n(\xi) \tag{2a}$$

$$-x_\xi(0) = u_0 - \int_0^1 x(\xi)d\xi, \quad x_\xi(1) = 0 \tag{2b}$$

$$y_0 = x(1). \tag{2c}$$

With the choices $u_0 = 1 \in \mathbb{C}$ and $\psi_n = 0 \in L^2(0,1)$ we then have $y_0 = P_K(i\omega)$, and for $u_0 = 0$ and $\psi_n \in L^2(0,1)$ we get $y_0 = P_{KI}(i\omega)\psi_n$. Based on the results in [1, Sec. 4], the simulation code solves the the boundary value problem numerically using the **Chebfun** library and the solution is computed for a finite number of functions ψ_n taken from any orthonormal basis of $L^2(0,1)$ (by default $\{1\} \cup \{\sqrt{2}\cos(2\pi n\cdot)\}_{n=1}^\infty \cup \{\sqrt{2}\sin(2\pi n\cdot)\}_{n=1}^\infty$).

Figure ?? shows an example output of the simulation.

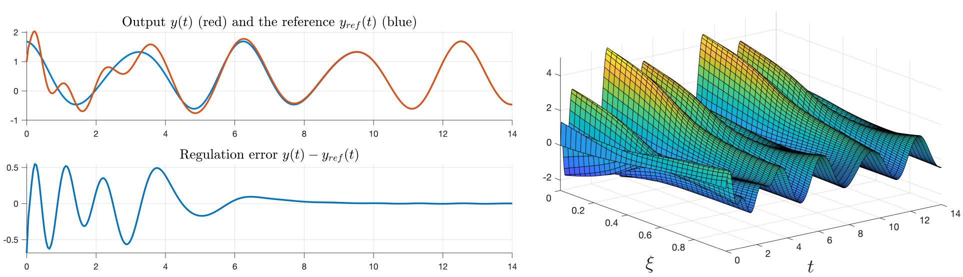

Figure 3. Simulation of the 1D heat equation. Output $y(t)$ and tracking error (left) and the controlled temperature profile (right).

References

[1] Lassi Paunonen and Jukka-Pekka Humaloja. On robust regulation of PDEs: from abstract methods to PDE controllers. In *Proceedings of the 61st IEEE Conference on Decision and Control*, Cancún, Mexico, December 6–9, 2022 (submitted). Preprint available at arxiv.org

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