On the Structure of Robust Controllers for Infinite-Dimensional Systems

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***Abstract*— Starting from a very general formulation of the Internal Model Principle it is shown that a robust controller tracking/rejecting signals generated by an infinite-dimensional exosystem can be decomposed into a servocompensator and a stabilizing controller. The servocompensator contains an internal model of the exosystem generating the reference and disturbance signals and the stabilizing controller stabilizes the infinite-dimensional closed-loop system.**

***Index Terms*— Infinite-Dimensional Systems, Robust Regulation, Internal Model Principle**

I. Introduction

One of the main results of classical control theory of finite-dimensional linear systems is the Internal Model Principle (IMP) due to Francis and Wonham [1], and Davison [2], [3]. This principle asserts that any error feedback controller which achieves closed loop stability also achieves robust output regulation if and only if the controller contains a suitably duplicated model of the dynamic structure of the exosystem which generates the reference and disturbance signals which the controller is required to track/reject.

The approach of Francis and Wonham is based on geometric theory. Davison's approach is non-geometric and leads to a remarkably simple result showing that a robust controller can be divided into two parts: a servocompensator and a stabilizing controller. The servocompensator contains an internal model of the dynamics of the reference and disturbance signals in the form of a p-copy of the exosystem, where p is the dimension of the output space. The role of the stabilizing controller is to stabilize the extended system consisting of the servocompensator and the plant.

In this paper we use a new characterization of IMP based on the Internal Model Structure (IMS) of Immonen [4]. The IMS has been shown [5] to be equivalent to so called $\mathcal{G}$-conditions (Definition 3 in this paper). Using the $\mathcal{G}$-conditions we show that if the reference and disturbance signals are generated by a finite-dimensional exosystem, then the controller can be decomposed into a servocompensator and a stabilizing controller generalizing Davison's result to infinite-dimensional plants. This also gives a new proof for the finite-dimensional case.

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II. Preliminaries

A. The Exosystem

The reference and disturbance signals are assumed to be generated by the exosystem

$$\dot{v} = Sv, \qquad v(0) \in W, \tag{1}$$

where $S : \mathcal{D}(S) \subset W \to W$ is a generator of C_0-semigroup $T_S(t)$ on a Hilbert-space W. Then the solution of (1) is given by $v(t) = T_S(t)v(0)$ for $t \geq 0$. We assume that the spectrum of S is a pure point spectrum $\sigma_p(S) = \{\, \mathrm{i}\omega_k \mid k \in J \subset \mathbb{Z} \,\}$ with $\omega_k \neq \omega_l$ for $k \neq l$. If the index set J is infinite we also demand that $\inf_{k\neq l}|\omega_k - \omega_l| > 0$.

B. The Plant

The plant P is described by the equations

$$\dot{x} = Ax + Bu + F_s v, \qquad x(0) \in X \tag{2a}$$

$$y = Cx + Du + F_m v, \tag{2b}$$

where the state $x(t) \in X$, the input $u(t) \in U$, the output $y(t) \in Y$ and the signal $v(t) \in W$ is a solution of (1). The spaces X, U and Y are Hilbert spaces. The system operator $A : \mathcal{D}(A) \subset X \to X$ is the generator of a C_0-semigroup $T_A(t)$, all the other operators are bounded: $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$, $D \in \mathcal{L}(U,Y)$, $F_s \in \mathcal{L}(W,X)$, and $F_m \in \mathcal{L}(W,Y)$. We also assume that $\sigma_p(S) \subset \rho(A)$ and that the transfer function of the plant $P(s) = C(sI - A)^{-1}B + D \in \mathcal{L}(U,Y)$ is boundedly invertible for $s \in \sigma_P(S)$.

The reference signal $r : [0,\infty) \to Y$ is given by $r = F_r v$ where $F_r \in \mathcal{L}(W,Y)$. Combining the plant equations (2) and the tracking error $e = y - r = y - F_r v$ we can write the plant equations into the more convenient form

$$\dot{x} = Ax + Bu + Ev, \qquad x(0) \in X \tag{3a}$$

$$e = Cx + Du + Fv, \tag{3b}$$

where $E = F_s$ and $F = F_m - F_r$.

C. The Controller

The controller is defined by the equations

$$\dot{z} = \mathcal{G}_1 z + \mathcal{G}_2 e, \qquad z(0) \in Z \tag{4a}$$

$$u = Kz, \tag{4b}$$

where $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a C_0-semigroup on the Hilbert space Z, $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$ and $K \in \mathcal{L}(Z,U)$.

D. The Closed-Loop System

Let $X_e = X \times Z$ be the extended state-space, consisting of the plant and controller states, and let $x_e(t) = (x(t), z(t)) \in X_e$ be the extended state. Combining the equations (2) and (4) we get the closed-loop system

$$\dot{x}_e = A_e x_e + B_e v, \qquad x_e(0) \in X_e \tag{5a}$$
$$e = C_e x_e + D_e v, \tag{5b}$$

where $C_e = [C \ \ DK] \in \mathcal{L}(X_e, Y)$, $D_e = F \in \mathcal{L}(W, Y)$, and $A_e : \mathcal{D}(A_e) = \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1) \subset X_e \to X_e$ and $B_e \in \mathcal{L}(W, X_e)$ are given by

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix}, \qquad B_e = \begin{bmatrix} E \\ \mathcal{G}_2 F \end{bmatrix}.$$

E. The Output Regulation Problem

Definition 1: The Output Regulation Problem (ORP) is as follows: Design a controller (4) such that

1) The closed-loop system operator A_e generates a stable C_0-semigroup.
2) For all initial states $x_e(0) \in X_e$ and $v(0) \in W$ we have $\lim_{t\to\infty} e(t) = 0$.

In The Robust Output Regulation Problem (RORP) it is required in addition that the controller (4) solves the ORP even if in (3) the operators A, B, C, D, E, and F are perturbed to $A + \Delta_A$, $B + \Delta_B$, $C + \Delta_C$, $D + \Delta_D$, $E + \Delta_E$ and $F + \Delta_F$, respectively, in such a way that the closed-loop system remains stable.

III. Previous Results

The authors have previously proved the following theorem.

Theorem 1: If A_e generates a stable C_0-semigroup and there exists an operator $H_{\text{ss}} \in \mathcal{L}(W, X_e)$ which satisfies $H_{\text{ss}}\mathcal{D}(S) \subset \mathcal{D}(A_e)$ and the constrained Sylvester equations

$$H_{\text{ss}} S - A_e H_{\text{ss}} = B_e, \ \text{ on } \mathcal{D}(S) \tag{6a}$$
$$C_e H_{\text{ss}} + D_e = 0, \tag{6b}$$

then the controller (4) solves the ORP.

Proof: See [6]. ■

If S is finite-dimensional, then exponential stability of A_e is sufficient to guarantee that (6a) has a solution. For an infinite-dimensional S necessary and sufficient conditions for the solvability of (6a) are given in [6].

A solution of (6a) does not necessarily satisfy (6b). For this we need the following definition [4].

Definition 2: The controller $(\mathcal{G}_1, \mathcal{G}_2)$ has *Internal Model Structure* (IMS) if for every $\Gamma \in \mathcal{L}(W, Z)$ and $\Delta \in \mathcal{L}(W, Y)$ with $\Gamma\mathcal{D}(S) \subset \mathcal{D}(\mathcal{G}_1)$ the following implication holds:

$$\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 \Delta \quad \text{on } \mathcal{D}(S) \implies \Delta = 0. \tag{7}$$

It is shown in [5], [6] that the controller has IMS if and only if (6a) implies (6b). In particular, if the plant parameters are perturbed in such a way that the closed-loop system remains stable and the perturbed Sylvester equation (6a) has a solution, then the perturbed Sylvester equation (6b) is also satisfied. Therefore we have the following Theorem.

Theorem 2: Assume that the controller (4) has IMS. If A_e generates a stable C_0-semigroup and there exists an operator $H_{\text{ss}} \in \mathcal{L}(W, X_e)$ satisfying $H_{\text{ss}}\mathcal{D}(S) \subset \mathcal{D}(A_e)$ and the Sylvester equation (6a), then the controller solves the RORP.

The definition of IMS given in Definition 2 is not convenient to apply in practice. A more easily checkable condition is given by the following definition.

Definition 3: The controller $(\mathcal{G}_1, \mathcal{G}_2)$ satisfies the $\mathcal{G}$-conditions if

$$\mathcal{N}(\mathcal{G}_2) = \{0\}, \tag{8a}$$
$$\mathcal{R}(\mathcal{G}_2) \cap \mathcal{R}(sI - \mathcal{G}_1) = \{0\}, \qquad \text{for all } s \in \sigma_p(S). \tag{8b}$$

It has been proved by Paunonen and Pohjolainen [5] that the controller has IMS if and only if it satisfies the $\mathcal{G}$-conditions.

IV. Main Results

The following Theorem has been proved in [5] under the assumption $\sigma_p(S) \cap \sigma(A_e) = \emptyset$. We give here a proof with the less restrictive assumptions $\sigma_p(S) \cap \sigma_p(A_e) = \emptyset$ and (9).

Theorem 3: If $\sigma_p(S) \cap \sigma_p(A_e) = \emptyset$, then the following are equivalent for $s \in \sigma_p(S)$.

1) The operators $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the $\mathcal{G}$-conditions (8) and the inclusion
$$\{0\} \times \mathcal{R}(\mathcal{G}_2) \subset \mathcal{R}(sI - A_e). \tag{9}$$
2) The restriction $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)} : \mathcal{N}(sI - \mathcal{G}_1) \to Y$ is a bijection.

Proof: Throughout the proof let $s \in \sigma_p(S)$. Then $s \notin \sigma_p(A_e)$ and the operator $sI - A_e$ is injective.

1) $\implies$ 2).
First we show that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is an injection. Suppose that $z \in \mathcal{N}(sI - \mathcal{G}_1)$ satisfies $P(s)Kz = 0$. Let $x = R(s; A)BKz \in \mathcal{D}(A)$. Then

$$\begin{aligned}(sI - A_e)\begin{bmatrix} x \\ z \end{bmatrix} &= \begin{bmatrix} sI - A & -BK \\ -\mathcal{G}_2 C & sI - \mathcal{G}_1 - \mathcal{G}_2 DK \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} \\ &= \begin{bmatrix} BKz - BKz \\ -\mathcal{G}_2 P(s)Kz + (sI - \mathcal{G}_1)z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.\end{aligned} \tag{10}$$

Since $sI - A_e$ is injective we have $z = 0$. Therefore $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is an injection.

Next we show that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is a surjection. Let $y \in Y$ be arbitrary. Since $(0, -\mathcal{G}_2 y) \in \{0\} \times \mathcal{R}(\mathcal{G}_2)$, it follows from (9) that there is a $(x, z) \in \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1)$ such that

$$\begin{aligned}\begin{bmatrix} 0 \\ -\mathcal{G}_2 y \end{bmatrix} &= (sI - A_e)\begin{bmatrix} x \\ z \end{bmatrix} \\ &= \begin{bmatrix} (sI - A)x - BKz \\ -\mathcal{G}_2(Cx + DKz) + (sI - \mathcal{G}_1)z \end{bmatrix}.\end{aligned}$$

We get from the first equation $x = R(s;A)BKz$ and substituting this into the second equation gives

$$-\mathcal{G}_2 y = -\mathcal{G}_2 P(s)Kz + (sI - \mathcal{G}_1)z \iff$$
$$\mathcal{G}_2(P(s)Kz - y) = (sI - \mathcal{G}_1)z.$$

Since $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the condition (8b) we must have $(sI-\mathcal{G}_1)z = 0$ and $\mathcal{G}_2(P(s)Kz-y) = 0$, and furthermore $P(s)Kz - y = 0$, since $\mathcal{G}_2$ satisfies (8a). Hence $z \in \mathcal{N}(sI - \mathcal{G}_1)$ and $y = P(s)Kz$, so $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is a surjection.

2) $\Longrightarrow$ 1).

First we prove (8a). Let $y \in \mathcal{N}(\mathcal{G}_2)$. Since $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is a surjection, there is a $z \in \mathcal{N}(sI - \mathcal{G}_1)$ such that $y = P(s)Kz$. Then $\mathcal{G}_2 P(s)Kz = 0$. Choosing $x = R(s;A)BKz \in \mathcal{D}(A)$ we get as in (10) that $(sI - A_e)(x, z) = (0, 0)$. Since $sI - A_e$ is injective we have $z = 0$ and therefore $y = 0$.

Next we prove (8b). Let $v \in \mathcal{R}(\mathcal{G}_2) \cap \mathcal{R}(sI-\mathcal{G}_1)$. Then there are $y \in Y$ and $z \in \mathcal{D}(\mathcal{G}_1)$ such that $v = \mathcal{G}_2 y = (sI-\mathcal{G}_1)z$. First we show that there is a $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that $v = \mathcal{G}_2 P(s)Kz_1 = (sI - \mathcal{G}_1)z_1$. Since $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is a surjection, there is a $z_0 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that $P(s)Kz_0 = y - P(s)Kz$. Then $y = P(s)K(z + z_0)$ and since $(sI - \mathcal{G}_1)z = (sI - \mathcal{G}_1)(z + z_0)$, we can choose $z_1 = z + z_0$. Now choosing $x_1 = R(s;A)BKz_1 \in \mathcal{D}(A)$ we get as in (10) that $(sI - A_e)(x_1, z_1) = (0, 0)$. Since $sI - A_e$ is injective we have $z_1 = 0$ and therefore $v = 0$.

Finally we prove (9). Let $(0, v) \in \{0\} \times \mathcal{R}(\mathcal{G}_2)$ be arbitrary. Then there is an $y \in Y$ such that $v = \mathcal{G}_2 y$. Since $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is a surjection, there is a $z \in \mathcal{N}(sI - \mathcal{G}_1)$ such that $P(s)Kz = -y$. Let $x = R(s;A)BKz \in \mathcal{D}(A)$. Then

$$\begin{aligned}(sI - A_e)\begin{bmatrix}x\\z\end{bmatrix} &= \begin{bmatrix}(sI-A)x - BKz\\ -\mathcal{G}_2(Cx + DKz) + (sI - \mathcal{G}_1)z\end{bmatrix}\\ &= \begin{bmatrix}BKz - BKz\\ -\mathcal{G}_2(P(s)Kz)\end{bmatrix}\\ &= \begin{bmatrix}0\\ -\mathcal{G}_2(-y)\end{bmatrix} = \begin{bmatrix}0\\v\end{bmatrix}.\end{aligned}$$

Therefore $(0, v) \in \mathcal{R}(sI - A_e)$ and the proof is complete. ∎

It follows from Theorem 3 that under the stated conditions $\mathcal{N}(sI - \mathcal{G}_1)$ is isomorphic to Y, and since Y is nontrivial, we must have $\mathcal{N}(sI - \mathcal{G}_1) \neq \{0\}$ and hence $\sigma_p(S) \subset \sigma_p(\mathcal{G}_1)$. Therefore the controller must contain a copy of the dynamics of the exosystem S and thus satisfies the Internal Model Principle. In particular, if $\dim Y = p$, then $\dim \mathcal{N}(sI-\mathcal{G}_1) = p$ and every eigenvalue of S must have multiplicity at least p as an eigenvalue of $\mathcal{G}_1$. Hence $\mathcal{G}_1$ contains a p-copy of S. Note that this also holds for infinite-dimensional exosystems.

In the following we assume that $\dim Y = p$ and the eigenvalues $\mathrm{i}\omega_k$ of the exosystem are simple. The next theorem shows that with a suitable decomposition of the controller state space Z the operator $\mathcal{G}_1$ can be written in a lower triangular form.

Theorem 4: Assume that the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the $\mathcal{G}$-conditions (8) and the equation (9). The space Z can be decomposed into a direct sum $Z = Z_1 \dot{+} Z_2$ so that $\mathcal{G}_1$ can be represented by a lower triangular matrix

$$\begin{bmatrix}G_1 & 0\\ R_1 & R_2\end{bmatrix},$$

where $\sigma_p(S) \subset \sigma_p(G_1)$ and each eigenvalue $\mathrm{i}\omega_k \in \sigma_p(G_1)$ has multiplicity p.

Proof: It follows from Theorem 3 that $\mathcal{N}(sI - \mathcal{G}_1)$ is isomorphic to Y for $s \in \sigma_p(S)$. Hence s must be an eigenvalue of $\mathcal{G}_1$ with multiplicity p and the eigenspaces $\mathcal{N}(sI - \mathcal{G}_1)$ have dimension p. Since the eigenspaces are finite-dimensional, $-\mathrm{i}\omega_k$ is an eigenvalue of ${\mathcal{G}_1}^*$ with multiplicity p. Write Z as a direct sum $Z = \overline{Z}_1 \dot{+} Z_2$ where

$$Z_1 = \sum_{k=-\infty}^{\infty} \mathcal{N}(-\mathrm{i}\omega_k I - {\mathcal{G}_1}^*) \tag{11}$$

and Z_2 is any complementary subspace of $\overline{Z}_1$.

Next we show that $\overline{Z}_1$ is an invariant subspace of ${\mathcal{G}_1}^*$. Let $T_1(\cdot)$ be the semigroup generated by ${\mathcal{G}_1}^*$ and let $t > 0$ be fixed. The subspaces $\mathcal{N}(-\mathrm{i}\omega_k I - {\mathcal{G}_1}^*) \subset \mathcal{D}({\mathcal{G}_1}^*)$ are closed invariant subspaces of ${\mathcal{G}_1}^*$, so it follows from Lemma 2.5.4 of [7] that they are also $T_1(t)$-invariant. Now let $\varepsilon > 0$ and $x \in \overline{Z}_1$ be arbitrary. Then there is an $y \in Z_1$ such that $\|x - y\| < \varepsilon/\|T_1(t)\|$ and y has the representation

$$y = \sum_{k=-\infty}^{\infty} y_k,$$

where $y_k \in \mathcal{N}(-\mathrm{i}\omega_k I - {\mathcal{G}_1}^*)$ and the series converges unconditionally. Now the boundedness of $T_1(t)$ implies that

$$T_1(t)y = \sum_{k=-\infty}^{\infty} T_1(t)y_k,$$

and the series converges unconditionally. Since $T_1(t)y_k \in \mathcal{N}(-\mathrm{i}\omega_k I - {\mathcal{G}_1}^*)$ we have $T_1(t)y \in Z_1$ and because $\|T_1(t)x - T_1(t)y\| \leq \|T_1(t)\|\|x-y\| < \varepsilon$, we have $T_1(t)x \in \overline{Z}_1$. Therefore $\overline{Z}_1$ is $T_1(t)$-invariant, and by Lemma 2.5.3 of [7] it is also ${\mathcal{G}_1}^*$-invariant.

Since $\overline{Z}_1$ is ${\mathcal{G}_1}^*$-invariant, the operator ${\mathcal{G}_1}^*$ can be represented as

$${\mathcal{G}_1}^* = \begin{bmatrix}G_{11} & G_{12}\\ 0 & G_{22}\end{bmatrix},$$

where

$$G_{11} = {\mathcal{G}_1}^*|_{\overline{Z}_1}, G_{12} = P_1{\mathcal{G}_1}^*|_{Z_2}, G_{22} = P_2{\mathcal{G}_1}^*|_{Z_2},$$

and P_1 and P_2 are projections onto $\overline{Z}_1$ and Z_2. respectively. Clearly

$${\mathcal{G}_1}^*|_{\mathcal{N}(-\mathrm{i}\omega_k I - {\mathcal{G}_1}^*)} = -\mathrm{i}\omega_k I,$$

where I is a $p \times p$ identity matrix, so $-\mathrm{i}\omega_k$ is an eigenvalue of G_{11} with multiplicity p. Therefore

$$\mathcal{G}_1 = \begin{bmatrix} G_{11}^* & 0 \\ G_{12}^* & G_{22}^* \end{bmatrix},$$

which is of the claimed form. Moreover, $\mathrm{i}\omega_k$ is an eigenvalue of G_{11}^* with multiplicity p. ∎

Using the decomposition of Z in Theorem 4 we can write $\mathcal{G}_2$ in the form

$$\mathcal{G}_2 = \begin{bmatrix} G_2 \\ R_3 \end{bmatrix},$$

where $G_2 = P_1\mathcal{G}_2$ and $R_3 = P_2\mathcal{G}_2$ and $K = [K_1\ K_2]$ where $K_1 = KP_1$ and $K_2 = KP_2$. Then the controller parameters take the form

$$\mathcal{G}_1 = \begin{bmatrix} G_1 & 0 \\ R_1 & R_2 \end{bmatrix}, \quad \mathcal{G}_2 = \begin{bmatrix} G_2 \\ R_3 \end{bmatrix}, \quad K = [K_1\ K_2],$$

which is the same controller stucture as given by Davison in [3]. The parameters (G_1, G_2) define the servocompensator on the state space Z_1

$$\dot{z}_1 = G_1 z_1 + G_2 e, \tag{12a}$$

which contains an internal model of the exosystem S. The parameters $(R_1, R_2, R_3, K_1, K_2)$ define the stabilizing controller on the state space Z_2

$$\dot{z}_2 = R_1 z_1 + R_2 z_2 + R_3 e \tag{12b}$$

$$u = K_1 z_1 + K_2 z_2. \tag{12c}$$

The purpose of the stabilizing controller is to stabilize the system consisting of the plant and the servocompensator.

This decomposition of the controller into two parts is very useful for design purposes, since it allows us to design the servocompensator and the stabilizing controller independently of one another.

V. Conclusion

In this paper we have given a new proof which shows that a robust controller can be decomposed into a servocompensator and a stabilizing controller. The servocompensator contains a p times duplicated internal model of the exosystem generating the reference and disturbance signals, where p is the dimension of the output space. The purpose of the stabilizing controller is to stabilize the system consisting of the plant and the servocompensator.

We assumed that the exosystem and the controller are finite-dimensional and that the exosystem has simple eigenvalues. Further work includes extending the results to infinite-dimensional exosystems with possibly non-simple eigenvalues. In this case we need to allow the controller also to be infinite-dimensional.

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