

FREQUENCY DOMAIN ROBUST REGULATION OF SIGNALS GENERATED BY AN INFINITE-DIMENSIONAL EXOSYSTEM

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Abstract. This paper deals with frequency domain robust regulation of signals generated by an infinite-dimensional exosystem. The problem is formulated and the stability types are chosen so that one can generalize the existing finite-dimensional theory to more general classes of infinite-dimensional systems and signals. The main results of this article are extensions of the internal model principle, of a necessary and sufficient solvability condition for the robust regulation problem, and of Davison's simple servo compensator for stable plants in the chosen algebraic framework.

Key words. Robust regulation, infinite-dimensional systems, infinite-dimensional exosystems, frequency domain, internal model.

1. Introduction. The topic of this article is robust regulation which is a central problem in the mathematical control theory. The robust regulation problem consists of two parts: robust stabilization, and robust asymptotic tracking and disturbance rejection. Asymptotic tracking and disturbance rejection means that the error between the reference and output signals vanishes as time goes to infinity despite some disturbance signals. The reference and disturbance signals are generated by an exogenous system called the exosystem. Robustness of regulation means that the output of the closed loop asymptotically tracks the reference signals even if the plant is perturbed. Robustness is of fundamental importance since small errors in mathematical models of real world phenomena are unavoidable and stem from various sources such as model simplifications, erroneous parameter estimation etc.

The robust regulation problem of finite-dimensional linear multi-input multi-output (MIMO) systems was solved in the 1970s. Francis, Wonham, and Davison had a central role in this work [5, 6, 7, 9, 10]. Many authors have generalized the finite-dimensional results to infinite-dimensional and non-linear systems in the time domain as well as in the frequency domain since then [1, 13, 15, 17, 30, 32, 35, 36, 37, 41].

Robust regulation of signals generated by an infinite-dimensional exosystem is a challenging problem [15, 39]. For the recent development on this subject, see [14, 18, 31] and the references therein. One can generate very general classes of signals, e.g., general periodic functions, by using infinite-dimensional exosystems. Signals that are generated by an infinite-dimensional exosystem and their frequency domain counterparts are called infinite-dimensional signals in this paper. The main emphasis in the recent research has been on the time domain, while the frequency domain theory has got only a little attention. The purpose of this paper is to develop theory for robust regulation in the frequency domain.

There exist multiple different rings of stable transfer functions each of which has its own special features and is suitable for certain purposes [24]. Thus, it is natural that the robust regulation problem has been studied in many different algebraic frameworks. Rational matrices are suitable for finite-dimensional systems, and robust regulation is well-understood in this framework [11, 37]. The famous Internal Model Principle was presented by Francis and Wonham in [9], roughly speaking, it states that a controller can robustly regulate a reference signal if and only if it contains a

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reduplicated model – an internal model – of the dynamics of the signal. Vidyasagar formulated the frequency domain version of the Internal Model Principle in [37] by using coprime factorizations. The internal model is defined for rational matrices by using the largest invariant factor of the denominator matrix of the signal generator. He was also able to formulate a simple solvability condition for the robust regulation problem by using the internal model and coprime factorizations of the plant.

The signal class chosen for this article is the set of the Laplace transforms of the signals generated by an infinite-dimensional diagonal exosystem in [14]. In order to formulate the internal model principle, one needs a suitable internal model for the chosen signal class. An H^∞ -stable function that corresponds to the largest invariant factor is presented for the chosen signal class. It is an internal model for this signal class and enables generalization of the finite-dimensional theory to infinite-dimensional signals.

Robust regulation of infinite-dimensional signals was first studied by Inoue et al. in [19] for single-input single-output finite-dimensional plants and further developed by Hara et al. [15, 40] for MIMO rational plants. This repetitive control scheme has various practical applications, see the references in [21]. Hara et al. presented a controller for non-strictly proper plants that was capable to robustly regulate L -periodic functions. The controller contained a delay loop of length L which served as an internal model. Hara and his co-authors faced a problem with stabilizability. They found out that a controller cannot contain an internal model of the L -periodic reference signals, i.e. to achieve perfect tracking, and simultaneously have a realization that exponentially stabilizes the closed loop system unless the plant does not vanish at infinity. This is a severe restriction since many of the plants stemming from physical systems are strictly proper, i.e., approach zero at high frequencies. Robust regulation of infinite-dimensional signals was later studied by Ylinen et al. in [41], and they faced similar restrictions.

The problem encountered by Hara et al. and Ylinen et al. indicates that the commonly used stability types such as H^∞ -stability are too strong. In order to overcome the problem, two new rings of stable transfer functions denoted by \mathbf{P} and \mathbf{S} are introduced in this article. They extend H^∞ by allowing the plant transfer function to grow on the imaginary axis. \mathbf{P} -stability allows polynomial growth on the imaginary axis, while \mathbf{S} -stability does not set any limiting growth bound there. In that sense \mathbf{P} - and \mathbf{S} -stability resemble the polynomial and strong stability in the time domain, respectively. Weakening the stability type makes it possible to stabilize the closed loop that contains an internal model of infinite-dimensional signals even if the plant transfer function is strictly proper. The robustness properties of the defined stability rings are not known and are out of the scope of this article. This is why the robustness of regulation is defined assuming that closed loop stability should imply regulation. This is exactly how robustness was defined in [3] and allows one to develop robust regulation theory apart from the topological aspect of the problem.

Infinite-dimensional systems, e.g., systems modelling diffusion or vibration processes or systems with delays, require non-rational transfer function classes. Finite-dimensional robust regulation theory generalizes straightforwardly to certain classes of transfer functions that are suitable for infinite-dimensional systems. In particular, robust regulation of reference and disturbance signals that are Laplace transforms of signals generated by a finite-dimensional exosystem – they are called finite-dimensional signals in this article – have been generalized to certain classes of infinite-dimensional systems [3, 13, 35]. In [13] Hämmäläinen and Pohjolainen generalized Davison's [5]

robust controller for stable rational plants to stable plants in the Callier-Desoer class of transfer functions. This tunable controller has a simple structure and requires only minimal information about the plant. In [35] Rebarber and Weiss extended the Davison's controller to H^∞ -setting.

In this article, the plants are assumed to have right and left coprime factorizations over \mathbf{P} or \mathbf{S} . The assumption allows one to use the stability results presented in [8, 38] and provides a simple framework to study the robust regulation problem. The chosen class of plants is suitable for infinite-dimensional systems and contains the Callier-Desoer class of transfer functions as a proper subset. The adopted algebraic framework allows one to easily generalize the Davison's controller to certain stable plants and infinite-dimensional signals. Combining these controllers appropriately with any stabilizing controller of a given plant yields a new controller design procedure that consists of two main parts: one first constructs an arbitrary stabilizing controller for the given plant and then finds a robustly regulating controller for the numerator matrix of the original plant. This way, one can stabilize the original plant by using any available techniques and separately design a robustly regulating controller for a stable plant, which in general is an easier task than constructing a robustly regulating controller for an unstable plant. A particular class of transfer functions for which the proposed controller design procedure is suitable is the Callier-Desoer class of transfer functions.

On a general level, the frequency domain robust regulation problem is an abstract algebraic problem. This approach allows one to handle both finite- and infinite-dimensional signals as well as plants in various algebraic structures at once. The abstract algebraic approach was used by Nett in [26] where the robust regulation problem was considered in a very general algebraic framework. Nett was able to formulate the internal model principle under very general conditions. The downside of such an approach is that the less one knows about the underlying algebraic structure the less results are obtainable. Generalization of the results of rational matrices to general algebraic structures was also discussed in [37], but only the results concerning stabilization were generalized.

The frequency domain robust regulation problem with infinite-dimensional signals is considered in this article. The reference and disturbance signals have an infinite number of simple poles on the imaginary axis. The signal class requires specific algebraic structures, so one is not able to consider the problem on as general level as in [26]. However, the algebraic structures allow one to consider controller configurations and signal classes in more detail than those in [26]. In addition, the ring of stable transfer functions is not assumed to be a topological ring unlike in [26]. The main contributions of this article are:

- (i) New stability rings \mathbf{P} and \mathbf{S} are presented. They enable one to handle infinite-dimensional reference and disturbance signals.
- (ii) The restriction that the plant cannot be strictly proper that was faced in [15, 40, 41] is removed.
- (iii) It is shown that the internal model principle holds for plants with right and left coprime factorizations over \mathbf{P} and \mathbf{S} .
- (iv) The necessary and sufficient condition presented in [37] for the solvability of the robust regulation problem is generalized for plants with right and left coprime factorizations over \mathbf{P} and \mathbf{S} .
- (v) Davison's robust controller for H_-^∞ -stable transfer functions studied in [5, 13, 35] is generalized to infinite-dimensional signals.

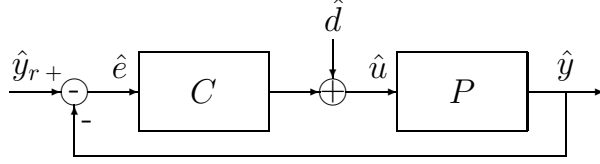


FIG. 2.1. A control configuration.

- (vi) A design procedure for constructing robustly regulating controllers for unstable plants is proposed.

Although, the purpose of this article is not to study the connection between the time domain and the frequency domain robust regulation problems it is interesting to observe the similarities between the two approaches. Infinite-dimensionality of the reference and disturbance signals cause exponential stability to be unachievable in the time domain, see [14, 18, 30]. By weakening the stability type, e.g., considering strong or polynomial stability instead of exponential stability, it is possible to solve the robust regulation problem with infinite-dimensional exosystem in the time domain. The frequency domain equivalence of this observation is shown in this article. Since there are only a few results available for robustness properties of weaker stability types than exponential stability, see [28] and the references therein, the robustness of regulation in [14, 18, 30] was understood in the same sense as in this article. However, an extra assumption related to the smoothness of the signals was required in the time domain.

The paper is organized as follows. Notations and preliminary results are presented in Section 2. The problem is formulated in Section 3. In Section 4, generating elements for the reference and disturbance signals that are crucial for the internal model principle are presented. Section 5 is dedicated for studying the robust regulation problem with **P**-stability. The internal model principle is formulated and shown to hold with **P**-stability in Section 5.1. Then a necessary and sufficient condition for the solvability presented in [11, 37] for rational transfer matrices is shown to hold for the current problem in Section 5.2. Finally, a controller design procedure is proposed in Section 5.3 and Section 5.4. In Section 6, it is seen that the theoretical results with **P**-stability generalize straightforwardly to **S**-stability. In addition, a solvability condition given in terms of blocking zeros that was presented in [12] for finite-dimensional plants is shown to hold in the chosen algebraic setting. Examples that illustrate the theoretical results are provided in Section 7. Finally, Section 8 includes concluding remarks and some directions for future research.

2. Notations and Preliminary Results. Throughout the article \mathbf{R} denotes an integral domain with an identity element and $\mathbf{F}_{\mathbf{R}}$ denotes the field of fractions over \mathbf{R} . The set of all matrices with elements in a set S is denoted by $\mathcal{M}(S)$ and all matrices with n rows and m columns is denoted by $S^{n \times m}$. A matrix $P \in \mathbf{F}_{\mathbf{R}}$ is said to be **R**-stable if $P \in \mathcal{M}(\mathbf{R})$.

Consider the control configuration of Figure 2.1. The closed loop transfer function from (\hat{y}_r, \hat{d}) to (\hat{e}, \hat{u}) is

$$H(P, C) = \begin{bmatrix} (I + PC)^{-1} & -(I + PC)^{-1}P \\ C(I + PC)^{-1} & I - P(I + PC)^{-1}C \end{bmatrix}.$$

A controller $C \in \mathcal{M}(\mathbf{F}_{\mathbf{R}})$ is said to be **R**-stabilizing for $P \in \mathcal{M}(\mathbf{F}_{\mathbf{R}})$ if $\det(I + PC)^{-1} \neq 0$ and $H(P, C) \in \mathcal{M}(\mathbf{R})$.

The coprime factorization approach to stabilization provides a strong and simple framework for studying stability of the closed loop. In particular it allows one to characterize all the stabilizing controllers for a given plant. The stability results of [38, 37] that are relevant here are presented next.

A pair (N, D) (a pair (\tilde{N}, \tilde{D})) of \mathbf{R} -matrices is *right (left) coprime* if there exists such matrices \tilde{Y} and \tilde{X} (Y and X) in $\mathcal{M}(\mathbf{R})$ that

$$\tilde{Y}N + \tilde{X}D = I \quad \left(\tilde{N}Y + \tilde{D}X = I \right).$$

A pair (N_p, D_p) (a pair $(\tilde{N}_p, \tilde{D}_p)$) of \mathbf{R} -matrices is called a *right (left) coprime factorization* of $P \in \mathcal{M}(\mathbf{F}_\mathbf{R})$ if

1. the pair (N_p, D_p) (the pair $(\tilde{N}_p, \tilde{D}_p)$) is right (left) coprime pair of \mathbf{R} -stable matrices,
2. D_p (\tilde{D}_p) is a square matrix such that $\det(D_p) \neq 0$ ($\det(\tilde{D}_p) \neq 0$), and
3. $P = N_p D_p^{-1}$ ($P = \tilde{D}_p^{-1} \tilde{N}_p$).

An \mathbf{R} -stable matrix is called \mathbf{R} -*unimodular* if it has an \mathbf{R} -stable inverse matrix. If P has a right (left) coprime factorization (N_p, D_p) ($(\tilde{N}_p, \tilde{D}_p)$), then (N, D) ((\tilde{N}, \tilde{D})) is another right (left) coprime factorizations of P if and only if $N = N_p U$ and $D = D_p U$ ($\tilde{N} = U \tilde{N}_p$ and $\tilde{D} = U \tilde{D}_p$) for some \mathbf{R} -unimodular matrix U . The next lemma is a version of [38, Lemma 3.1] and [37, Lemma 8.3.2].

LEMMA 2.1. *Let $P \in \mathcal{M}(\mathbf{F}_\mathbf{R})$ be given and have a right coprime factorization (N_p, D_p) (a left coprime factorization $(\tilde{N}_p, \tilde{D}_p)$). The following are equivalent:*

1. *A controller $C \in \mathcal{M}(\mathbf{F}_\mathbf{R})$ stabilizes P .*
2. *There exists a left (right) coprime factorization of C and for any left coprime factorization $(\tilde{N}_c, \tilde{D}_c)$ (right coprime factorization (N_c, D_c)) of C*

$$\tilde{N}_c N_p + \tilde{D}_c D_p \quad \left(\tilde{N}_p N_c + D_p \tilde{D}_c \right)$$

is \mathbf{R} -unimodular.

3. *There exists such a left coprime factorization $(\tilde{N}_c, \tilde{D}_c)$ (a right coprime factorizations (N_c, D_c)) of C that*

$$\tilde{N}_c N_p + \tilde{D}_c D_p = I \quad \left(\tilde{N}_p N_c + D_p \tilde{D}_c = I \right).$$

The next lemma gives a parametrization of all stabilizing controllers. It appeared in an abstract algebraic setting for the first time in [38, Lemma 3.2] and was later repeated in [37, Theorem 8.3.12].

LEMMA 2.2. *Let $P \in \mathcal{M}(\mathbf{F}_\mathbf{R})$ be given and have a right coprime factorization (N_p, D_p) and a left coprime factorization $(\tilde{N}_p, \tilde{D}_p)$. If $X, Y, \tilde{X}, \tilde{Y} \in \mathcal{M}(\mathbf{R})$ are such that*

$$\tilde{Y}N + \tilde{X}D = I \quad \text{and} \quad \tilde{N}Y + \tilde{D}X = I.$$

then C stabilizes P if and only if there exist $\tilde{R} \in \mathcal{M}(\mathbf{R})$ such that $\det(\tilde{X} - \tilde{R}\tilde{N}_p) \neq 0$ and $(\tilde{Y} + \tilde{R}\tilde{D}_p, \tilde{X} - \tilde{R}\tilde{N}_p)$ is a left coprime factorization of C , or, equivalently,

there exists $R \in \mathcal{M}(\mathbf{R})$ such that $\det(X - N_p R) \neq 0$ and $(Y + D_p R, X - N_p R)$ is a right coprime factorization of C .

A sequence of elements is denoted by $(a_k)_{k \in \mathbb{Z}}$. The set of square summable complex sequences is denoted by ℓ^2 . The real part of a complex number s is denoted by $\Re(s)$. The Hardy space of all analytic complex functions that are bounded in $\mathbb{C}_\beta^+ = \{s \in \mathbb{C} \mid \Re(s) > \beta\}$ is denoted by H_β^∞ . Shorthand notation H^∞ is used for H_0^∞ and the set of all functions that belong to H_β^∞ for some $\beta < 0$ is denoted H_-^∞ . Notation $B_\gamma(\alpha)$ denotes a α -centered complex disc of radius γ .

3. Problem Formulations. In what follows the robust regulation problem is formulated. For that one needs to define classes of reference and disturbance signals and suitable stability types.

3.1. Reference and Disturbance Signals. The classes of reference and disturbance signals are defined next. Let $(a_k)_{k \in \mathbb{Z}} \subset \mathbb{C}^n$ and $(b_k)_{k \in \mathbb{Z}} \subset \mathbb{C}^m$ be fixed sequences of complex vectors that satisfy $\|a_k\| = \|b_k\| = 1$ for all $k \in \mathbb{Z}$. In addition, let $(\omega_k)_{k \in \mathbb{Z}}$ be a sequence of real numbers such that $\omega_{k+1} - \omega_k > 4\gamma$ for a fixed real number $\gamma > 0$. The reference signals are chosen to be of the form

$$(3.1) \quad \hat{y}_r(s) = \sum_{k \in \mathbb{Z}} \frac{\alpha_k}{s - i\omega_k} a_k,$$

where $(\alpha_k)_{k \in \mathbb{Z}} \in \ell^2$. The disturbance signals are of the form

$$(3.2) \quad \hat{d}(s) = \sum_{k \in \mathbb{Z}} \frac{\beta_k}{s - i\omega_k} b_k$$

where $(\beta_k)_{k \in \mathbb{Z}} \in \ell^2$. The set of all the reference signals of the form (3.1) is denoted by $\hat{\mathcal{Y}}(a_k)$ and the set of all disturbance signals of the form (3.2) is denoted by $\hat{\mathcal{D}}(b_k)$.

The chosen classes of reference and disturbance signals contain the Laplace transforms of the time domain reference and disturbance signals generated by the diagonal infinite-dimensional exosystem of [14]. The classes of signals defined above are slightly larger than the classes of Laplace transforms since for the Laplace transforms of the diagonal exosystem one would have absolutely summable $(\alpha_k)_{k \in \mathbb{Z}}$ and $(\beta_k)_{k \in \mathbb{Z}}$.

3.2. Stability Types. It was noted in [41] that need for closed loop H^∞ -stability together with infinite-dimensional reference and disturbance signals restricts the set of plants for which one can solve the robust regulation problem. To overcome this difficulty one should weaken the H^∞ -stability. Since the reference and disturbance signals have unstable poles only on the imaginary axis it is sufficient to relax the boundedness conditions there. This in mind one defines the following two rings of stable transfer functions.

DEFINITION 3.1. A complex function f is said to be **P**-stable if it satisfies the following conditions

1. f is analytic in an open set containing $\overline{\mathbb{C}^+}$,
2. $f \in H_\beta^\infty$ for all $\beta > 0$,
3. there exists constants $M > 0$ and $\alpha > 0$ such that $|f(i\omega)| \leq M(1 + |\omega|)^\alpha$ for all $\omega \in \mathbb{R}$.

The set of all **P**-stable functions is denoted by **P**.

A complex function satisfying the first and the second item above, but not necessarily the third, is said to be **S**-stable and the set of all **S**-stable functions is denoted by **S**.

It is a matter of an elementary exercise to show, that \mathbf{P} and \mathbf{S} together with pointwise sum and multiplication form integral domains with identity elements. Thus, the stability results of [38] can be used as such except for the robustness results which require the ring to be a topological ring. It is also seen immediately that \mathbf{P} is a subring of \mathbf{S} .

The plants and the controllers are assumed to be from $\mathbf{F}_{\mathbf{P}}$ or $\mathbf{F}_{\mathbf{S}}$ depending of the stability type considered at that moment. The sets $\mathbf{F}_{\mathbf{P}}$ and $\mathbf{F}_{\mathbf{S}}$ contain all proper rational functions and all functions in the Callier-Desoer algebra $\hat{\mathcal{B}}(0)$. They also contain all of the functions in \mathbf{F}_{H^∞} that are meromorphic in an open set containing the imaginary axis, but not the \mathbf{F}_{H^∞} that are not meromorphic on the imaginary axis. For example, $\mathbf{F}_{H^\infty} \subset \mathbf{F}_{\mathbf{P}} \subset \mathbf{F}_{\mathbf{S}}$.

3.3. The $(\hat{\mathcal{Y}}, \hat{\mathcal{D}}, \mathbf{R})$ -Robust Regulation Problem. Let $\hat{\mathcal{Y}}$ and $\hat{\mathcal{D}}$ be sets of reference and disturbance signals, respectively. Let $P \in \mathcal{M}(\mathbf{F}_{\mathbf{R}})$ be given. The $(\hat{\mathcal{Y}}, \hat{\mathcal{D}}, \mathbf{R})$ -regulation problem is defined in the following way: Find a controller $C \in \mathcal{M}(\mathbf{F}_{\mathbf{R}})$ such that

1. C is \mathbf{R} -stabilizes P , and
2. for all $\hat{y}_r \in \hat{\mathcal{Y}}$ and $\hat{d} \in \hat{\mathcal{D}}$

$$\hat{e} = \begin{bmatrix} (I + PC)^{-1} & -(I + PC)^{-1}P \end{bmatrix} \begin{bmatrix} \hat{y}_r \\ \hat{d} \end{bmatrix} \in \mathcal{M}(\mathbf{R}).$$

Let $\hat{\mathcal{Y}}$ and $\hat{\mathcal{D}}$ be sets of reference and disturbance signals, respectively. Let $P \in \mathcal{M}(\mathbf{F}_{\mathbf{R}})$ be given. The $(\hat{\mathcal{Y}}, \hat{\mathcal{D}}, \mathbf{R})$ -robust regulation problem is defined in the following way: Find a controller $C \in \mathcal{M}(\mathbf{F}_{\mathbf{R}})$ such that

1. C \mathbf{R} -stabilizes P , and
2. C solves the $(\hat{\mathcal{Y}}, \hat{\mathcal{D}}, \mathbf{R})$ -regulation problem for all the plants it stabilizes.

If $\hat{\mathcal{Y}} = \hat{\mathcal{Y}}(a_k)$ and $\hat{\mathcal{D}} = \hat{\mathcal{D}}(b_k)$, then by referring to \mathbf{R} -regulation or \mathbf{R} -robust regulation problem one means $(\hat{\mathcal{Y}}, \hat{\mathcal{D}}, \mathbf{R})$ -robust regulation or $(\hat{\mathcal{Y}}, \hat{\mathcal{D}}, \mathbf{R})$ -robust regulation problems, respectively. A controller solving the $(\hat{\mathcal{Y}}, \hat{\mathcal{D}}, \mathbf{R})$ -(robust) regulation problem is called $(\hat{\mathcal{Y}}, \hat{\mathcal{D}}, \mathbf{R})$ -(robustly) regulating or shortly (robustly) regulating if there is not risk of confusion.

In what follows the frequency domain version of the internal model principle is considered. Before proceeding one should make precise what is meant by the internal model principle because two different rings of stable transfer functions \mathbf{P} and \mathbf{S} are considered in this article and the internal model principle is dependent of the ring chosen.

DEFINITION 3.2. Consider a controller $C \in \mathcal{M}^{n \times m}(\mathbf{F}_{\mathbf{R}})$. If C has a right coprime factorization (N_c, D_c) , then it is said that it contains $\theta \in \mathbf{R}$ as its internal model if $\theta^{-1}D_c \in \mathcal{M}(\mathbf{R})$.

One can now state the internal model principle. In this article the internal model principle for the $(\hat{\mathcal{Y}}, \hat{\mathcal{D}}, \mathbf{R})$ -robust regulation problem is formulated as follows: Assume that a plant P had both left and a right coprime factorizations. It is said that the internal model principle holds if there exists an element $\theta \in \mathbf{R}$ such that a controller C solves the $(\hat{\mathcal{Y}}, \hat{\mathcal{D}}, \mathbf{R})$ -robust regulation problem if and only if it contains θ as its internal model.

4. Minimal Generating Elements for Signals. If rational transfer functions are considered, it is customary that the reference and disturbance signals are of the form Θv , where Θ is a rational matrix called the signal generator and v is an arbitrary

stable vector. The signal generator Θ has a left coprime factorization (N, D) and the internal model principle can be formulated by using the largest invariant factor θ of D . The element θ has the property that the generator $\theta^{-1}I$ can generate all the signals the generator Θ can. In addition, θ is minimal in the sense that, if $\theta_0^{-1}I$ generates all the reference and disturbance signals, then θ divides θ_0 . The purpose of this section is to find an element θ that can be seen as a largest invariant factor of sets $\hat{\mathcal{Y}}(a_k)$ and $\hat{\mathcal{D}}(b_k)$. Such elements are called minimal generating elements.

DEFINITION 4.1. *If $\theta \in \mathbf{R}$ is such that every $\hat{y}_r \in \hat{\mathcal{Y}}(a_k)$ and $\hat{d} \in \hat{\mathcal{D}}(b_k)$ can be presented in the form $\hat{y}_r = \theta^{-1}\hat{y}_0$ and $\hat{d} = \theta^{-1}\hat{d}_0$, where $\hat{y}_0 \in \mathbf{R}^{n \times 1}$ and $\hat{d}_0 \in \mathbf{R}^{m \times 1}$, then θ is called an **R-generating element**. An **R-generating element** θ is called **minimal** if for any **R-generating element** θ_0 there exists $w_0 \in \mathbf{R}$ such that $\theta_0 = \theta w_0$.*

LEMMA 4.2. *Consider a series $\sum_{k \in \mathbb{Z}} \frac{A_k}{s - i\omega_k}$, where $A_k \in \mathcal{M}(\mathbb{C})$. Let $\omega_k \in \mathbb{R}$ be such that $\omega_{k+1} - \omega_k > 4\gamma$ for a fixed constant $\gamma > 0$. If $\|A_k\| \in \ell^2$, then*

1.

$$\left\| \sum_{k \in \mathbb{Z}} \frac{A_k}{s - i\omega_k} \right\| \leq \sqrt{\sum_{k \in \mathbb{Z}} \|A_k\|^2} \sqrt{\sum_{n=1}^{\infty} \frac{2}{\gamma^2 n^2}} < \infty,$$

2.

$$\left\| \sum_{k \in \mathbb{Z}} \frac{1}{(s - i\omega_k)(s - i\omega_k + \epsilon)} \right\| < \sum_{n=1}^{\infty} \frac{2}{\gamma^2 n^2} < \infty$$

for all $s \in \overline{\mathbb{C}}_+ \cap \bigcup_{k \in \mathbb{Z}} B_\gamma(i\omega_k)$ and $\epsilon > 0$.

Proof. Fix $s \in \mathbb{C}$ and denote the integer for which $\min_{k \in \mathbb{Z}} \{|s - \omega_k|\}$ is attained by l . By assumptions $|s - i\omega_k| > (|k - l| + 1)\gamma$ for all $k \neq l$. If $|s - i\omega_k| > \gamma$, then by Cauchy-Schwarz inequality

$$\begin{aligned} \left\| \sum_{k \in \mathbb{Z}} \frac{A_k}{s - i\omega_k} \right\| &\leq \sum_{k \in \mathbb{Z}} \frac{\|A_k\|}{|s - i\omega_k|} \\ &\leq \sum_{k \in \mathbb{Z}} \frac{\|A_k\|}{\gamma(|k - l| + 1)} \\ &\leq \sqrt{\sum_{k \in \mathbb{Z}} \|A_k\|^2} \sqrt{\sum_{n=1}^{\infty} \frac{2}{\gamma^2 n^2}}. \end{aligned}$$

The second item follows from the first one by noting that $|s - i\omega_k| > |s - i\omega_k + \epsilon|$ in $\overline{\mathbb{C}}_+ \cap \bigcup_{k \in \mathbb{Z}} B_\gamma(i\omega_k)$. \square

A **P-generating element** θ for the signals in $\hat{\mathcal{Y}}(a_k)$ and in $\hat{\mathcal{D}}(b_k)$ is presented in the next theorem. It is needed in the parametrization of robustly regulating controllers and to give solvability conditions for regulation problems.

THEOREM 4.3. *Let $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ be a sequence of strictly positive real numbers. Define*

$$(4.1) \quad \theta(s) = \left(1 + \epsilon_1 \sum_{k \in \mathbb{Z}} \frac{h_k}{s - i\omega_k} \right)^{-1},$$

where the real numbers ω_k satisfy $\omega_{k+1} - \omega_k > 4\gamma$ for some fixed $\gamma > 0$ and for all $k \in \mathbb{Z}$. The constant $\epsilon_1 > 0$ is chosen so that for all $k \in \mathbb{Z}$ one has

$$(4.2) \quad \epsilon_1 \|\Phi_{1k}(s)\Phi_{2k}(s)\| < \frac{1}{2} \text{ for } s \in B_\gamma(i\omega_k) \cap \overline{\mathbb{C}}_+,$$

where $\Phi_{1k} = (1 + \epsilon_1 \frac{h_k}{s - i\omega_k})^{-1}$ and $\Phi_{2k}(s) = \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{h_l}{s - i\omega_l}$.

1. There exists a choice of $\epsilon_1 > 0$ for which (4.2) is satisfied for all $k \in \mathbb{Z}$.
2. The generating element θ is in H^∞ , and it is analytic in an open set containing \mathbb{C}^+ .
3. If $h_k > M(|\omega_k| + 1)^{-\alpha}$ for some $\alpha > 0$ and $M > 0$, then θ is a **P**-generating element.
4. θ is an **S**-generating element.

Proof. One has $\left|1 + \epsilon_1 \frac{h_k}{s - i\omega_k}\right| > 1$ since $\Re\left(\frac{h_k}{s - i\omega_k}\right) > 0$ for all s such that $\Re(s) > 0$. Thus, $\|\Phi_k(s)\| < 1$ in $B_\gamma(i\omega_k) \cap \overline{\mathbb{C}}^+$. Lemma 4.2 implies that there exists $M > 0$ such that $|\Phi_{2k}(s)| < M$ in $B_\gamma(i\omega_k) \cap \overline{\mathbb{C}}^+$ for all $k \in \mathbb{Z}$. It follows that the choice $\epsilon_1 = \frac{1}{2M}$ completes the proof of the first item.

To prove the second item note that $1 + \epsilon_1 \sum_{k \in \mathbb{Z}} \frac{h_k}{s - i\omega_k}$ is analytic everywhere except at $i\omega_k$, $k \in \mathbb{Z}$. It is clear that $\theta(s)$ is analytic in \mathbb{C}^+ provided that it is bounded there. Recall that $\Re\left(\frac{h_k}{s - i\omega_k}\right) > 0$ for all s such that $\Re(s) > 0$, so $\left|1 + \epsilon_1 \sum_{k \in \mathbb{Z}} \frac{h_k}{s - i\omega_k}\right| > 1$. This implies boundedness of θ .

The third item it is proved by showing that $\hat{y}_0 = \theta \hat{y}_r$ is in $\mathbf{P}^{n \times 1}$ for an arbitrary $\hat{y}_r = \sum_{k \in \mathbb{Z}} \frac{\alpha_k}{s - i\omega_k} a_k \in \hat{\mathcal{Y}}(f_k)$. The proof for the disturbance signals is identical.

Denote $U_\beta = \bigcup_{k \in \mathbb{Z}} B_\beta(i\omega_k)$, where $0 < \beta < \gamma$. Lemma 4.2 shows that there exists $M_\beta > 0$ such that $\|\hat{y}_r\| < M_\beta$ for all $s \in \overline{\mathbb{C}}^+ \setminus U_\beta$. Since $\|\theta(s)\| < 1$ in $\overline{\mathbb{C}}^+$, it follows that $\|\hat{y}_0(s)\| < M_\beta$ for all $s \in \overline{\mathbb{C}}^+ \setminus U_\beta$. In particular, this implies that $\hat{y}_0 \in H_\beta^\infty$ for an arbitrary $\beta > 0$.

The proof is completed if one is able to show that \hat{y}_0 is polynomially bounded on the imaginary axis, since in that case \hat{y}_0 is analytic in an open set that contains $i\mathbb{R}$. By the above discussion, the claim follows if one can show that \hat{y}_0 is polynomially bounded in U_γ . To this end fix $k \in \mathbb{Z}$. The decomposition $\theta = \Phi_{1k}(1 + \epsilon_1 \Phi_{1k}(s)\Phi_{2k}(s))^{-1}$ shows that

$$(4.3) \quad \hat{y}_0(s) = \left(\Phi_{1k}(s) \frac{\alpha_k}{s - i\omega_k} a_k + \Phi_{1k}(s) \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{\alpha_l}{s - i\omega_l} a_l \right) (1 + \epsilon_1 \Phi_{1k}(s)\Phi_{2k}(s))^{-1}.$$

One has $|(1 + \epsilon_1 \Phi_{1k}(s)\Phi_{2k}(s))^{-1}| < 2$ for all $s \in B_\gamma(i\omega_k)$ by (4.2). It is easy to see that $\left\| \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{\alpha_l}{s - i\omega_l} a_l \right\| < M_\gamma$ and $|\Phi_{1k}(s)| < 1$ in $B_\gamma(i\omega_k)$. Thus, (4.3) implies that polynomial boundedness of $\Phi_{1k}(s) \frac{\alpha_k}{s - i\omega_k} a_k$ in U_γ is a sufficient condition for \hat{y}_0 to be in $\mathbf{P}^{n \times 1}$. Note that $|\alpha_k| \leq M_2$ for some fixed $M_2 > 0$ independent of $k \in \mathbb{Z}$ and $\|a_k\| = 1$ for all $k \in \mathbb{Z}$. By assumption,

$$\left\| \Phi_{1k}(s) \frac{\alpha_k}{s - i\omega_k} a_k \right\| = \frac{|\alpha_k|}{|s - i\omega_k + \epsilon_1 h_k|} \leq \frac{M_2}{h_k} < \frac{M_2}{M} (1 + |\omega_k|)^\alpha$$

for all $s \in \overline{\mathbb{C}}^+ \cap B_\gamma(i\omega_k)$ which completes the proof of the third item.

It was shown above that $\hat{y}_0 = \theta \hat{y}_r \in H_\beta^\infty$ for all $\beta > 0$, so the fourth item holds if \hat{y}_0 is analytic in an open set containing $i\mathbb{R}$. The claim follows by noting that the zeros of θ blocks the poles of \hat{y}_r . \square

A sequence $(h_k)_{k \in \mathbb{Z}}$ must be fixed for the above generating element. Another generating element is proposed below so that no such choice needs to be made.

THEOREM 4.4. *Define*

$$(4.4) \quad \theta(s) = \left(1 + \sum_{k \in \mathbb{Z}} \frac{\epsilon_2^2}{(s - i\omega_k)(s - i\omega_k + \epsilon_2)} \right)^{-1},$$

where the real numbers ω_k satisfy $\omega_{k+1} - \omega_k > 4\gamma$ for some fixed and for all $k \in \mathbb{Z}$. The constant $\epsilon_2 > 0$ is chosen so that for all $k \in \mathbb{Z}$ one has

$$(4.5) \quad \epsilon_2^2 \left\| \sum_{k \in \mathbb{Z}} \frac{1}{(s - i\omega_k)(s - i\omega_k + \epsilon_2)} \right\| < \frac{1}{2} \text{ for all } s \in \overline{\mathbb{C}} \setminus \bigcup_{k \in \mathbb{Z}} B_\gamma(i\omega_k)$$

and

$$(4.6) \quad \epsilon_2 \|\Phi_{1k}(s)\Phi_{2k}(s)\| < \frac{1}{2} \text{ for all } s \in B_\gamma(i\omega) \cap \overline{\mathbb{C}}^+$$

where $\Phi_{1k}(s) = \left(1 + \frac{\epsilon_2^2}{(s - i\omega_k)(s - i\omega_k + \epsilon_2)} \right)^{-1}$ and $\Phi_{2k}(s) = \sum_{l \in \mathbb{Z}, l \neq k} \frac{\epsilon_2}{(s - i\omega_l)(s - i\omega_l + \epsilon_2)}$.

1. There exists a choice of $\epsilon_2 > 0$ for which (4.5) and (4.6) are satisfied for all $k \in \mathbb{Z}$.
2. The generating element θ is in H^∞ , and it is analytic in an open set containing $\overline{\mathbb{C}}^+$.
3. Elements $\hat{y}_0 = \theta \hat{y}_r$ and $\hat{d}_0 = \theta \hat{d}$ are analytic in an open set containing $\overline{\mathbb{C}}^+$ and belong to H^∞ for every $\hat{y}_r \in \hat{\mathcal{Y}}(a_k)$ and $\hat{d} \in \hat{\mathcal{D}}(b_k)$.
4. θ is a **P**-generating element.
5. θ is an **S**-generating element.

Proof. In order to prove the first item set $z = \frac{s - i\omega_k}{\epsilon_2}$. Now

$$(4.7) \quad \Phi_{1k}(s) = \left(1 + \frac{1}{z(z+1)} \right)^{-1} = \frac{z(z+1)}{z^2 + z + 1}.$$

Denote $U = \bigcup_{k \in \mathbb{Z}} B_\gamma(i\omega_k)$. Lemma 4.2 shows that there exists a constant $M > 0$ such that $\left| \sum_{k \in \mathbb{Z}} \frac{1}{(s - i\omega_k)(s - i\omega_k + \epsilon_2)} \right| < M$. Thus, (4.5) holds if $\epsilon_2 < \frac{1}{\sqrt{2M}}$.

Since the poles of $z^2 + z + 1$ have real part $-\frac{1}{2}$ and $\frac{z(z+1)}{z^2 + z + 1}$ is strictly proper $M_1 = \sup_{z \in \overline{\mathbb{C}}^+} \left\{ \left| \frac{z(z+1)}{z^2 + z + 1} \right| \right\} < \infty$. The function Φ_{1k} is bounded by a constant $M_1 > 0$ independent of k in $B_\gamma(i\omega_k)$ since $\Re(s) \geq 0$ implies $\Re(z) \geq 0$. The existence of $M_2 > 0$ for which $|\Phi_{2k}(s)| < M_2$ for all $s \in B_\gamma(i\omega) \cap \overline{\mathbb{C}}^+$ and $k \in \mathbb{Z}$ can be proved by using Lemma 4.2. Choosing $\epsilon_2 < \min \left\{ \frac{1}{\sqrt{2M}}, \frac{1}{2M_1M_2} \right\}$ completes the proof of the first item.

To show the second item observe that if θ is bounded in $\overline{\mathbb{C}}^+$, then it is analytic in an open set containing $\overline{\mathbb{C}}^+$. Boundedness of θ in $\overline{\mathbb{C}}^+ \setminus U$ follows by (4.5). Note that $\theta = \Phi_{1k} (1 + \epsilon_2 \Phi_{1k} \Phi_{2k})^{-1}$. Boundedness in U follows by (4.6) since Φ_{1k} is bounded in U .

The two last items follow by the third item so only it needs to be shown. The proof of the third item is similar to the proof of the third item of Theorem 4.3. The only

difference is that one only needs to show boundedness of $\hat{y}_0 = \theta \hat{y}$ on the imaginary axis instead of polynomial boundedness. Again one can write

$$(4.8) \quad \hat{y}_0(s) = \left(\Phi_{1k}(s) \frac{\alpha_k}{s - i\omega_k} a_k + \Phi_{1k}(s) \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{\alpha_l}{s - i\omega_l} a_l \right) (1 + \epsilon_2 \Phi_{1k}(s) \Phi_{2k}(s))^{-1}.$$

Denote $z = \frac{s - i\omega_k}{\epsilon_2}$. By (4.7), one has

$$\begin{aligned} \sup_{s \in \mathbb{C}^+} \left\| \Phi_{1k}(s) \frac{\alpha_k}{s - i\omega_k} a_k \right\| &= \sup_{z \in \mathbb{C}^+} \left| \left(1 + \frac{1}{z(z+1)} \right)^{-1} \frac{\alpha_k}{z\epsilon_2} \right| \\ &= \frac{|\alpha_k|}{\epsilon_2} \sup_{z \in \mathbb{C}^+} \frac{z+1}{z^2 + z + 1} < M_3, \end{aligned}$$

where $M_3 > 0$ is a constant independent of k . Since $(1 + \epsilon_2 \Phi_{1k}(s) \Phi_{2k}(s))^{-1}$ and Φ_{1k} are bounded in $\overline{\mathbb{C}^+}$, it follows by (4.8) that \hat{y}_0 is bounded on the imaginary axis. \square

The two generators above are very similar, but in the first one a choice of a sequence must be made. Such a choice naturally affects the class of signals the generator can produce. A natural question arises, whether or not the sets of signals the above signal generators can produce are different. This question is answered by the next theorem.

If θ is an element in \mathbf{R} denote the set of signals θ^{-1} generates in \mathbf{R} by $\theta^{-1}\mathbf{R} = \{\theta^{-1}w_0 \mid w_0 \in \mathbf{R}\}$. Before proceeding to the theorem a simple lemma illustrating the relation between the generators and the sets of the reference signals they generate is given.

LEMMA 4.5. *Let $\theta_1, \theta_2 \in \mathbf{R}$. $\theta_1^{-1}\mathbf{R} \subseteq \theta_2^{-1}\mathbf{R}$ if and only if $\theta_2\theta_1^{-1} \in \mathbf{R}$.*

Proof. If $\theta_1^{-1}\mathbf{R} \subseteq \theta_2^{-1}\mathbf{R}$, then there exists $w_0 \in \mathbf{R}$ such, that

$$\theta_2\theta_1^{-1} = \theta_2(\theta_1^{-1} \cdot 1) = \theta_2(\theta_1^{-1}w_0) = w_0 \in \mathbf{R}.$$

It remains to show the necessity part. Write $\theta_1^{-1}w_0 = \theta_2^{-1}\theta_2\theta_1^{-1}w_0 = \theta_2^{-1}v_0$ for a $w_0 \in \mathbf{R}$, where $v_0 = \theta_2\theta_1^{-1}w_0$. If $\theta_2\theta_1^{-1} \in \mathbf{R}$, then $v_0 \in \mathbf{R}$. Thus, $\theta_1^{-1}\mathbf{R} \subseteq \theta_2^{-1}\mathbf{R}$. \square

THEOREM 4.6. *Fix $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ and denote the generators (4.1) and (4.4) by θ_1 and θ_2 , respectively. The following claims hold:*

1. $\theta_1^{-1}H^\infty \subsetneq \theta_2^{-1}H^\infty$.
2. If $h_k > M(|\omega_k| + 1)^{-\alpha}$ for some $\alpha > 0$ and $M > 0$, then $\theta_1^{-1}\mathbf{P} = \theta_2^{-1}\mathbf{P}$.
3. If there exists no $\alpha > 0$ and $M > 0$ such that $h_k > M(|\omega_k| + 1)^{-\alpha}$, then $\theta_1^{-1}\mathbf{P} \subsetneq \theta_2^{-1}\mathbf{P}$.
4. $\theta_1^{-1}\mathbf{S} = \theta_2^{-1}\mathbf{S}$.
5. θ_2 is a minimal \mathbf{P} -generating element. If $h_k > M(|\omega_k| + 1)^{-\alpha}$ for some $\alpha > 0$ and $M > 0$, then θ_1 is a minimal \mathbf{P} -generating element.
6. θ_1 and θ_2 are minimal \mathbf{S} -generating elements.

Proof. Item three of Theorem 4.4 shows that there exists such a $w_0 \in H^\infty$ that $\theta_2^{-1}w_0 = \epsilon_1 \sum_{k \in \mathbb{Z}} \frac{h_k}{s - i\omega_k}$. Thus, $\theta_2^{-1}(\theta_2 + w_0) = 1 + \epsilon_1 \sum_{k \in \mathbb{Z}} \frac{h_k}{s - i\omega_k} = \theta_1^{-1}$. Theorem 4.4 shows that $\theta_2 \in H^\infty$, so $\theta_2\theta_1^{-1} \in H^\infty$. This implies that $\theta_1^{-1}H^\infty \subseteq \theta_2^{-1}H^\infty$ by Lemma 4.5. Since w_0 and θ_2 are analytic on the imaginary axis, $\theta_2\theta_1^{-1} \in \mathbf{P} \subseteq \mathbf{S}$. Lemma 4.5 shows that $\theta_1^{-1}\mathbf{P} \subseteq \theta_2^{-1}\mathbf{P}$ and $\theta_1^{-1}\mathbf{S} \subseteq \theta_2^{-1}\mathbf{S}$.

It is shown next that the inclusions in the first and the third items are proper, while the sets are equal in the second and the fourth items. Consider $\theta_1\theta_2^{-1}$ and note

that it is uniformly bounded in $\overline{\mathbb{C}^+} \cap \bigcup_{k \in \mathbb{Z}} B_\gamma(i\omega_k)$. Decompose

$$\theta_1 = \Phi_{1k} (1 + \epsilon_1 \Phi_{1k} \Phi_{2k})^{-1}$$

where Φ_{1k} and Φ_{2k} are the one in Theorem 4.3. The choice of ϵ_1 implies that $\left| (1 + \epsilon_1 \Phi_{1k} \Phi_{2k})^{-1} \right| < \frac{1}{2}$ for all $s \in \overline{\mathbb{C}^+} \cap B_\gamma(i\omega_k)$. Thus,

$$(4.9) \quad |\theta_1(s) \theta_2^{-1}(s)| \leq \frac{1}{2} |\Phi_{1k}(s) \theta_2^{-1}(s)| \leq \frac{1}{2} \|\Phi_{1k}(s) \psi_k(s) - \Phi_{1k} \xi_k(s)\|$$

where $\psi_k(s) = \left(1 + \sum_{l \neq k} \frac{\epsilon_2^2}{(s - i\omega_l)(s - i\omega_l + \epsilon_2)} \right)$ and $\xi_k(s) = \frac{\epsilon_2^2}{(s - i\omega_k)(s - i\omega_k + \epsilon_2)}$.

The supremum of $|\Phi_{1k}(s) \xi_k(s)|$ over $\overline{\mathbb{C}^+} \cap B_\gamma(i\omega_k)$ is $\frac{\epsilon_2}{\epsilon_1 h_k}$, which approach infinity as $|k| \rightarrow \infty$. The arguments presented in the proofs of Theorem 4.3 and Theorem 4.4 show that $|\Phi_{1k}(s) \psi_k(s)|$ is bounded by a constant independent of k . Thus, $\theta_1(s) \theta_2^{-1}(s) \notin H^\infty$, and the first item holds by Lemma 4.5.

If there exists no $\alpha > 0$ and $M > 0$ such that $h_k > M(|\omega_k| + 1)^{-\alpha}$, then there exists no $\alpha' > 0$ and $M' > 0$ such that $\frac{1}{h_k} < M'(|\omega_k| + 1)^{\alpha'}$. Since the supremum of $\left| \Phi_{1k} \frac{\epsilon_2^2}{(s - i\omega_k)(s - i\omega_k + \epsilon_2)} \right|$ attained at $i\omega_k$ is $\frac{\epsilon_2}{\epsilon_1 h_k}$ it follows, that $\theta_1(s) \theta_2^{-1}(s) \notin \mathbf{P}$. The third item follows by Lemma 4.5.

To prove the second item assume that $h_k > (|\omega_k| + 1)^{-\alpha}$ for some $\alpha > 0$. The uniform boundedness of θ_2^{-1} in $\overline{\mathbb{C}^+} \setminus \bigcup_{k \in \mathbb{Z}} B_a(i\omega_k)$ for all $a > 0$ follows by Lemma 4.2. Furthermore, $\theta_2^{-1} \theta_1 \in H_\beta^\infty$ for all $\beta > 0$ since $\theta_1 \in H^\infty$. It remains to show that $\theta_2^{-1} \theta_1$ is polynomially bounded in $\bigcup_{k \in \mathbb{Z}} B_\gamma(i\omega_k)$. The first absolute value in (4.9) is uniformly bounded in $\bigcup_{k \in \mathbb{Z}} B_\gamma(i\omega_k)$. The second absolute value is bounded by $(1 + |\omega_k|)^{-\alpha}$ in $B_\gamma(i\omega_k)$ because its supremum in $B_\gamma(i\omega_k)$ is $\frac{\epsilon_2}{\epsilon_1 h_k}$. Since there is a uniform gap between ω_k , polynomial boundedness on the imaginary axis follows. Thus, $\theta_1 \theta_2^{-1} \in \mathbf{P}$. The second item holds by Lemma 4.5.

It was shown above that $\theta_1 \theta_2^{-1} \in H_\beta^\infty$ for all $\beta > 0$. Since the zeros of θ_1 block the poles of θ_2^{-1} one has shown that $\theta_1 \theta_2^{-1} \in \mathbf{S}$ from which the fourth item follows by Lemma 4.5.

It remains to prove the two last items. The proof of the last item is similar to the proof of the fifth item and is therefore skipped. Theorem 4.3 and Theorem 4.4 show that θ_1 and θ_2 are \mathbf{P} -generating elements under the assumptions made, so it remains to show minimality of the generating elements.

To prove the minimality assume that θ_0 is a \mathbf{P} -generating item. In particular, this means that for any $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ there exists $w_0 \in \mathbf{P}^{n \times 1}$ such that $\theta_0^{-1} w_0 = \sum_{k \in \mathbb{Z}} \frac{\epsilon h_k \overline{a_{ki}}}{s - i\omega_k} a_k \in \mathbf{P}$ where $i = 1, 2, \dots, n$ and a_{ki} denotes the i th element of a_k . Thus, the i th element

$$v_i = \sum_{k \in \mathbb{Z}} \frac{\epsilon h_k |a_{ki}|^2}{s - i\omega_k}$$

of $\theta_0^{-1} w_0$ is in \mathbf{P} . Since $\|a_k\| = 1$,

$$v_0 = \sum_{i=1}^n v_i = \sum_{k \in \mathbb{Z}} \frac{\epsilon h_k \sum_{i=1}^n |a_{ki}|^2}{s - i\omega_k} = \sum_{k \in \mathbb{Z}} \frac{\epsilon h_k}{s - i\omega_k} \in \mathbf{P}.$$

This implies that $\theta_0^{-1}(\theta_0 + w_0) = \theta_1^{-1}$ or in other words $\theta_0 = \theta_1(\theta_0 + w_0)$. This shows minimality of θ_1 . Minimality of θ_2 follows by the second item and Lemma 4.5. \square

5. Solving the P-Robust Regulation Problem. Solvability of the \mathbf{P} -robust regulation problem is studied in this section. The generating elements from the previous section have a central role in the solvability conditions as is shown below.

5.1. The Internal Model Principle for the P-Robust Regulation Problem. It is well known that if \mathbf{R} is an arbitrary integral domain with a unit element and $P \in \mathbf{F}_{\mathbf{R}}$ has both coprime factorizations, then a stabilizing controller containing an internal model is also $(\theta^{-1}\mathbf{R}^{n \times 1}, \theta^{-1}\mathbf{R}^{m \times 1}, \mathbf{R})$ -robustly regulating [26, Theorem 4.2.1]. Actually, [26, Theorem 4.2.1] shows internal model principle for a robust regulation problem defined in terms of a topology. However, the proof of the sufficiency part does not contain any topological arguments, so it can be used as such to show sufficiency for the robust regulation problem defined in this article. A controller containing an internal model is robustly regulating since θ in (4.1) is a \mathbf{P} -generating element. Thus, the lemma below follows.

LEMMA 5.1. *Let $P \in \mathcal{M}(\mathbf{F}_{\mathbf{P}})$ have a right and a left coprime factorization. Fix such a sequence $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ that $h_k > M(1 + |\omega_k|)^\alpha$ for some $\alpha > 0$ and $M > 0$. Let θ be the function defined in (4.1). If (N_c, D_c) is a right coprime factorization of a stabilizing controller C and $\theta^{-1}D_c \in \mathcal{M}(\mathbf{P})$, then C solves the \mathbf{P} -robust regulation problem.*

The next theorem shows that the internal model principle is shown to hold for the $(\hat{\mathcal{Y}}(a_k), \{0\}, \mathbf{P})$ -robust regulation problem. The generating element from Theorem 4.3 serves as an internal model.

THEOREM 5.2. *Let P have both left and right coprime factorizations. Fix a sequence $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ that satisfies $h_k > M(1 + |\omega_k|)^\alpha$ for some $\alpha > 0$ and $M > 0$, and let θ be the function in (4.1). If (N_c, D_c) is a right coprime factorization of a stabilizing controller C , then C solves the $(\hat{\mathcal{Y}}(a_k), \{0\}, \mathbf{P})$ -robust regulation problem if and only if $\theta^{-1}D_c \in \mathcal{M}(\mathbf{P})$.*

Proof. The sufficiency part follows by Lemma 5.1, so it remains to show that the internal model is needed for a controller to be robustly regulating. Assume that C solves the $(\hat{\mathcal{Y}}(a_k), \{0\}, \mathbf{P})$ -robust regulation problem. Since all the left coprime factorizations of C are of form $(U\tilde{N}_c, U\tilde{D}_c)$ for some \mathbf{P} -unimodular U , Lemma 2.1 implies that one can assume that $\tilde{N}_p N_c + \tilde{D}_p D_c = I$ without restricting generality. Denote $Q = \frac{\delta}{(s+1)^\alpha} Q_0$, where α and δ are positive constants chosen so that $\|Q(s)\| \|D_c(s)\| < \epsilon$ in $\overline{\mathbb{C}^+}$ for some $\epsilon < 1$ and for an arbitrary but fixed $Q_0 \in \mathcal{M}(\mathbf{P})$, and in addition $\det(\tilde{D}_p + Q) \neq 0$. Such constants exist because $Q_0, D_c \in \mathcal{M}(\mathbf{P})$.

Consider the plant $P' = (\tilde{D}_p - Q)^{-1} \tilde{N}_p$. Since $\|Q(s)D_c(s)\| < \epsilon < 1$

$$\tilde{N}_p N_c + (\tilde{D}_p - Q)D_c = \tilde{N}_p N_c + \tilde{D}_p D_c - QD_c = I - QD_c$$

is unimodular so the controller stabilizes P' by Lemma 2.1. By using the Neumann

series of $(I - QD_c)^{-1}$ and $(I - D_cQ)^{-1}$, see for example [16], one can show that

$$\begin{aligned}
(I + P'C)^{-1} &= D_c(I - QD_c)^{-1}(\tilde{D}_p + Q) \\
&= D_c \left(I + \sum_{i=1}^{\infty} (QD_c)^i \right) (\tilde{D}_p + Q) \\
&= D_c \left(I + \sum_{i=1}^{\infty} (QD_c)^i \right) (\tilde{D}_p + Q) \\
&= \left(I + \sum_{i=1}^{\infty} (D_cQ)^i \right) (D_c\tilde{D}_p + D_cQ) \\
&= (I - D_cQ)^{-1} (D_c\tilde{D}_p + D_cQ).
\end{aligned}$$

The controller is regulating for P' by assumption, so $(I + P'C)^{-1}\hat{y}_r \in \mathcal{M}(\mathbf{P})$ for every $\hat{y}_r \in \hat{\mathcal{Y}}(f_k, a_k)$. Let $\hat{y}_r \in \hat{\mathcal{Y}}(f_k, a_k)$ be arbitrarily chosen. Since $(I - D_cQ)^{-1}$ is \mathbf{P} -unimodular, $(D_c\tilde{D}_p + D_cQ)\hat{y}_r \in \mathcal{M}(\mathbf{P})$. One has $D_c\tilde{D}_p\hat{y}_r \in \mathcal{M}(\mathbf{P})$ since C is regulating for P . Consequently, $D_cQ\hat{y}_r \in \mathcal{M}(\mathbf{P})$.

Recall that $\hat{y}_r = \sum_{k \in \mathbb{Z}} \frac{\alpha_k}{s - i\omega_k} a_k$ for some sequence $(\alpha_k)_{k \in \mathbb{Z}} \in \ell^2$. Lemma 4.2 shows that \hat{y}_r is bounded in every right half plane \mathbb{C}_β^+ with $\beta > 0$. Thus, only the boundary behavior on the imaginary axis can make $D_cQ\hat{y}_r$ unstable. Consequently, $D_cQ\hat{y}_r = \frac{\delta}{(s+1)^{-\alpha}} D_cQ_0\hat{y}_r \in \mathcal{M}(\mathbf{P})$ only if $D_cQ_0\hat{y}_r \in \mathcal{M}(\mathbf{P})$.

Choose $Q_0 = e_i e_j^T$, where e_l is the l th natural basis vector of \mathbb{C}^n and e_j^T denotes the transpose of e_j . In $\hat{y}_r = \sum_{k \in \mathbb{Z}} \frac{\alpha_k}{s - i\omega_k} a_k$ choose $\alpha_k = h_k \frac{|e_j^T a_k|}{e_j^T a_k}$ and denote $x_{jk} = h_k |e_j^T a_k|$. Vary i and j to show that every element of D_c is divisible by $\sum_{k \in \mathbb{Z}} \frac{x_{jk}}{s - i\omega_k}$. Since $h_k = h_k \|a_k\| \leq \sum_{j=1}^n x_{jk}$, it is shown that $\sum_{k \in \mathbb{Z}} \frac{h_k}{s - i\omega_k}$ divides every element of D_c . Thus, $\theta^{-1}D_c = \left(1 + \epsilon_1 \sum_{k \in \mathbb{Z}} \frac{h_k}{s - i\omega_k}\right) D_c \in \mathcal{M}(\mathbf{P})$. \square

It is interesting to note that the internal model is needed even if there is no disturbance signals and the sequence $(a_k)_{k \in \mathbb{Z}}$ is fixed. In the time domain, this corresponds to a situation where there are no disturbances and perturbations are allowed only in the plant parameters but not in the reference operator. Since a stabilizing controller with an internal model solves the \mathbf{P} -robust regulation problem by Lemma 5.1, the above theorem implies the following corollary.

COROLLARY 5.3. *Let P have both left and right coprime factorizations. Fix a sequence $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ that satisfies $h_k > M(1 + |\omega_k|)^\alpha$ for some $\alpha > 0$ and $M > 0$, and let θ be the function in (4.1). A stabilizing controller C with a right coprime factorization (N_c, D_c) solves the \mathbf{P} -robust regulation problem if and only if $\theta^{-1}D_c \in \mathcal{M}(\mathbf{P})$.*

It follows directly from Lemma 5.1 that a controller containing an internal model of form (4.1) is $(\{0\}, \hat{\mathcal{D}}(f_k, b_k), \mathbf{P})$ -robustly regulating. It is shown in the next example that if only the disturbance signals contain unstable dynamics a controller can be robustly regulating even if it does not contain a full internal model.

EXAMPLE 5.4. *Consider the $(\{0\}, \left\{\begin{bmatrix} 1 & 0 \\ s & 0 \end{bmatrix}^T, \mathbf{P}\right\})$ -robust regulation problem. Let the plant P and the controller C be*

$$P = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} \frac{s+1}{s} & 0 \\ 0 & 1 \end{bmatrix}.$$

The plant admits a right coprime factorization (P, I) which is also a left coprime factorization. A right and a left coprime factorization of C is given by (I, C^{-1}) because C^{-1} is \mathbf{P} -stable and $P + C^{-1} = I$. This also shows that C stabilizes P . All plants that C stabilizes are of the form

$$P' = (P + C^{-1}R)(I - IR)^{-1},$$

provided that $\det(I - IR) \neq 0$. Since $P'(I + CP') = (P + C^{-1}R)C^{-1}$ and $C^{-1} \begin{bmatrix} 1 & 0 \\ s & 0 \end{bmatrix}^T$ is stable the controller is robustly regulating. However, $\frac{1}{s}C^{-1}$ is not stable, so the controller contains only a partial internal model.

The above example shows that the internal model principle does not necessarily hold for $(\{0\}, \hat{\mathcal{D}}(f_k, b_k), \mathbf{P})$ -robust regulation problem. This is an interesting observation and might be of use if the disturbance signals are appropriately structured and contain an infinite number of unstable poles that do not appear in the reference signals. Such a situation might happen if the disturbance signals affect only one input component of the system and the reference signals have only a finite number of unstable poles. Further study on this subject is out of the scope of this article, but it is an interesting subject for further research.

5.2. A Necessary and Sufficient Condition for Solvability. The solvability condition presented in the next theorem is a version of the one given in [11, 37] for rational matrices. Due to the differences of the algebraic structures the proofs of [11, 37] cannot be used.

THEOREM 5.5. *Let P have right and left coprime factorizations. Fix a sequence $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ such that $h_k > M(1 + |\omega_k|)^\alpha$ for some $\alpha > 0$ and $M > 0$. Let θ be the function in (4.1). If (N_p, D_p) is a right coprime factorization of P , then the \mathbf{P} -robust regulation problem is solvable if and only if $(\theta I, N_p)$ is left coprime.*

Proof. The sufficiency part of the theorem is shown first. Assume that $(\theta I, N_p)$ is left coprime. Now there exist \mathbf{P} -stable matrices V and W such that $\theta V + N_p W = I$. Since (N_p, D_p) is right coprime, there exists stable matrices \tilde{X} and \tilde{Y} such that $\tilde{Y}N_p + \tilde{X}D_p = I$. Let $(\tilde{N}_p, \tilde{D}_p)$ be a left coprime factorization of P which exists by the assumption. There exist stable matrices X and Y such that $\tilde{N}_p Y + \tilde{D}_p X = I$. One can assume that $\det(X) \neq 0$ without restricting generality.

Choose $R = (W + \theta R_0)X$, where R_0 is chosen so that $\det(X - N_p R) \neq 0$. The controller $C = (Y + D_p R)(X - N_p R)^{-1}$ stabilizes P by Lemma 2.2. Since

$$X - N_p R = X - N_p (W + \theta R_0)X = (I - N_p W + \theta R_0)X = \theta(V + R_0)X$$

Theorem 5.2 shows that C is robustly regulating.

In order to show the necessity part, let C be a robustly regulating controller. Lemma 2.1 shows that C possesses a left coprime factorization $(\tilde{N}_c, \tilde{D}_c)$ for which $\tilde{N}_c N_p + \tilde{D}_c D_p = I$ since it stabilizes P . There exists a left coprime factorization $(\tilde{N}_p, \tilde{D}_p)$ of P by assumption, and Lemma 2.1 shows that there exists a right coprime factorization (N_c, D_c) of C such that $\tilde{N}_p N_c + \tilde{D}_p D_c = I$. Theorem 5.2 implies that

$$V := \theta^{-1}(I - N_p N_c) = \theta^{-1}(I + PC)^{-1} = \theta^{-1}D_c \tilde{D}_p \in \mathcal{M}(\mathbf{P}).$$

Thus $\theta V + N_p N_c = I$, which completes the proof. \square

The above theorem gives a necessary and sufficient condition in terms of coprimeness of the numerator of a left coprime factorization and the generating element. Since (P, I) is a right coprime factorization of a stable plant P the above theorem implies the following corollary.

COROLLARY 5.6. *Let P be a \mathbf{P} -stable matrix. Fix such a sequence $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ that such that $h_k > M(1 + |\omega_k|)^\alpha$ for some $\alpha > 0$ and $M > 0$. Let θ be the function in (4.1). Let θ be the one in (4.1). The \mathbf{P} -robust regulation problem is solvable if and only if $(\theta I, P)$ is left coprime.*

REMARK 5.7. *Results of this section do not use any properties of the ring \mathbf{P} , but only the internal model principle. Thus the results can be generalized as long as the internal model is required for the robustness of regulation under stability.*

5.3. Controller Design for H_-^∞ -stable Plants. The main result of this section is the following theorem giving a necessary and sufficient condition for the robust regulation problem to be solvable for a given H_-^∞ -stable plant.

THEOREM 5.8. *If $P(s)$ H_-^∞ -stable, then there exists a controller solving the \mathbf{P} -robust regulation problem if and only if there exists a right inverse $P^r(i\omega_k)$ of P at $i\omega_k$ for every $k \in \mathbb{Z}$ and constants $M, \alpha > 0$ such that $\|P^r(i\omega_k)\| < M(1 + |\omega_k|)^\alpha$ for all $k \in \mathbb{Z}$.*

The proof of the above theorem is divided into necessity and sufficiency parts given by Theorem 5.9 and Theorem 5.11, respectively. The necessity part holds for all plants with a right coprime factorization, while the sufficiency part is proved only for H_-^∞ -stable plants.

THEOREM 5.9. *Consider $P \in \mathbf{F}_{\mathbf{P}}^{n \times m}$. If the \mathbf{P} -robust regulation problem is solvable, then P is right invertible at $i\omega_k$ and there exist right inverses and constants $\alpha, M > 0$ such that $\|P^r(i\omega_k)\| < M(1 + |\omega_k|)^\alpha$ for all $k \in \mathbb{Z}$.*

Proof. Let C solve the \mathbf{P} -robust regulation problem. Since (P, I) is a left coprime factorization of P and C is a stabilizing controller, there exists a right coprime factorization of (N_c, D_c) of C that satisfies $PN_c + D_c = I$.

It follows from Theorem 5.2 that $D_c(i\omega_k) = 0$, so $P(i\omega_k)N_c(i\omega_k) = I$. Thus, $N_c(i\omega_k)$ is a right inverse of $P(i\omega_k)$. Since $N_c \in \mathcal{M}(\mathbf{P})$ one has $\|N_c(i\omega_k)\| < M(1 + |\omega_k|)^\alpha$ for some $M > 0$ and $\alpha > 0$ and for all $k \in \mathbb{Z}$. \square

It is shown next that for H_-^∞ -stable plants the robust regulation problem is solvable if the transfer function vanish at $i\omega_k$ only polynomially fast as $|k| \rightarrow \infty$. To this end, a controller solving the robust regulation problem is presented. The controller to be presented is a generalization of those presented in [13, 35]. The controller considered is of the form

$$(5.1) \quad C_\epsilon(s) = \epsilon \left(C_0(s) + \sum_{k \in \mathbb{Z}} \frac{K_k}{s - i\omega_k} \right),$$

where C_0 is H_-^∞ -stable and $(\|K_k\|)_{k \in \mathbb{Z}} \in \ell^2$. The only difference to the controller of [35] is, that the sum here is infinite. Some assumptions on the choice of the design parameters K_k are needed.

ASSUMPTION 5.10. *Denote $G_k = P(i\omega_k)K_k$.*

1. G_k is invertible.
2. $\|G_k^{-1}\| < M(1 + |\omega_k|)^\alpha$ for some $\alpha > 0$ and $M > 0$.
3. There exists $M > 0$ such that $\left\| \left(I + \frac{G_k}{z} \right)^{-1} \right\| < M$ in for all $z \in \overline{\mathbb{C}^+}$ and $k \in \mathbb{Z}$.

4. There exists $M > 0$ such that $\left\| (I + zG_k^{-1})^{-1} \right\| < M$ holds for all $z \in \overline{\mathbb{C}^+}$ and $k \in \mathbb{Z}$.

The first assumption is crucial for the controller to contain an internal model. The second assumption is needed for the controller to be \mathbf{P} -stabilizing. Since $K_k G_k^{-1}$ is a right inverse of $P(i\omega_k)$ the second item restricts the convergence speed of the plant. The last two are technical assumptions related to the stability. The last assumption may be weakened, since only polynomial bound is required in the proof of Lemma 5.14.

THEOREM 5.11. *If $P(s) \in \mathcal{M}(H_-^\infty)$ and there exist a right inverse $P^r(i\omega_k)$ of P at $i\omega_k$ for all $k \in \mathbb{Z}$ and constants $M, \alpha > 0$ such that $\|P^r(i\omega_k)\| < M(1 + |\omega_k|)^\alpha$ for all $k \in \mathbb{Z}$, then there exists a constant $\epsilon^* > 0$ and a choice of parameters K_k in (5.1) such that the controller solves the \mathbf{P} -robust regulation problem for all $\epsilon \in (0, \epsilon^*]$.*

The proof of the above theorem uses a series of lemmas and is postponed later. A technical lemma is given first. Lemma 5.13 and Lemma 5.14 show that if Assumption 5.10 holds, then the proposed controller is stabilizing. Lemma 5.15 shows that the controller is regulating. Finally, Lemma 5.16 shows that a choice of parameters satisfying Assumption 5.10 can be made.

LEMMA 5.12. *Let $P(s) \in \mathcal{M}(H_\beta^\infty)$ for some $\beta \in \mathbb{R}$. Then for all $\epsilon > 0$ there exist a real number $M > 0$ such that $\left\| \frac{P(s) - P(s_0)}{s - s_0} \right\| < M$ for all $s, s_0 \in \mathbb{C}_{\beta+\epsilon}^+$.*

Proof. Fix $s_0 \in \mathbb{C}_{\beta+\epsilon}^+$ and choose $\delta < \min\{\frac{\epsilon}{3}, 1\}$. By assumption, there exists $M_0 > 0$ such that $\|P(s)\| < M_0$ for all $s \in \mathbb{C}_\beta^+$. Let γ be the simple positively oriented circular path around s_0 with radius 2δ . By the Cauchy's differentiation formula,

$$\begin{aligned} \|P^{(n)}(s_0)\| &= \left\| \frac{n!}{2\pi i} \oint_\gamma \frac{P(s)}{(s - s_0)^{n+1}} ds \right\| \\ &\leq \frac{n!}{2\pi} \oint_\gamma \frac{\|P(s)\|}{|s - s_0|^{n+1}} ds \\ &\leq \frac{n!}{2\pi} \oint_\gamma \frac{M_0}{(2\delta)^{n+1}} ds = \frac{M_0 n!}{(2\delta)^n}. \end{aligned}$$

Let $s \in B_\delta(s_0)$ be arbitrary. By using the Taylor series of $P(s)$ at s_0 , one gets

$$\begin{aligned} \left\| \frac{P(s) - P(s_0)}{s - s_0} \right\| &= \left\| \frac{\sum_{n=0}^{\infty} \frac{1}{n!} P^{(n)}(s_0) (s - s_0)^n - P(s_0)}{s - s_0} \right\| \\ &= \left\| \sum_{n=1}^{\infty} \frac{1}{n!} P^{(n)}(s_0) (s - s_0)^{n-1} \right\| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \|P^{(n)}(s_0)\| |s - s_0|^{n-1} \\ &< \sum_{n=1}^{\infty} \frac{1}{n!} \frac{M_0 n!}{(2\delta)^n} \delta^{n-1} \\ &= \frac{M_0}{\delta} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{2M_0}{\delta}. \end{aligned}$$

Thus, $\left\| \frac{P(s) - P(s_0)}{s - s_0} \right\| < \frac{2M_0}{\delta}$ for all $s \in B_\delta(s_0)$. It is easy to see that the same upper

bound holds also for all $s \in \mathbb{C}_{\beta+\epsilon}^+ \setminus B_\delta(s_0)$. The claim follows since the limit does not depend on s_0 . \square

LEMMA 5.13. *Let $P \in \mathcal{M}(H_-^\infty)$. If the design parameters K_k in (5.1) satisfy Assumption 5.10, then there exists a constant $\epsilon^* > 0$ such that $(I + PC)^{-1} \in \mathcal{M}(\mathbf{P} \cap H^\infty)$ for all $\epsilon \in (0, \epsilon^*]$.*

Proof. Denote $U = \overline{\mathbb{C}^+} \cap \bigcup_{k \in \mathbb{Z}} B_\gamma(i\omega_k)$. Lemma 4.2 and uniform boundedness of $P(s)$ imply that $P(s) \sum_{k \in \mathbb{Z}} \frac{K_k}{s - i\omega_k}$ is uniformly bounded in $\overline{\mathbb{C}^+} \setminus U$. Thus, $P(s)C_\epsilon(s)$ is uniformly bounded in $\overline{\mathbb{C}^+} \setminus U$. Choose $M_1 > 0$ such that $\left\| P(s) \sum_{k \in \mathbb{Z}} \frac{K_k}{s - i\omega_k} \right\| \leq M_1$ for all $s \in \overline{\mathbb{C}^+} \setminus U$, and set $\epsilon_1 = \frac{1}{2M_1}$. Use the Neumann series, see for example [16, Corollary 5.6.16], to show that

$$\left\| (I + P(s)C_\epsilon(s))^{-1} \right\| = \left\| \sum_{i=0}^{\infty} (-P(s)C_\epsilon(s))^i \right\| \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

for all $\epsilon \in (0, \epsilon_1]$.

Decompose $(I + PC_\epsilon)^{-1} = \Phi_{1k} (I + \epsilon \Phi_{2k} \Phi_{1k}^{-1})^{-1}$, where

$$(5.2) \quad \Phi_{1k}(s) = \left(I + \epsilon \frac{P(i\omega_k)K_k}{s - i\omega_k} \right)^{-1}$$

and

$$(5.3) \quad \Phi_{2k}(s) = \frac{P(s) - P(i\omega_k)}{s - i\omega_k} K_k + P(s)C_0(s) + \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{P(s)K_l}{s - i\omega_l}.$$

Assumption 5.10 implies that there exists a constant $M_2 > 0$ such that $\|\Phi_{1k}(s)\| < M_2$ since the real part of $z = \frac{s - i\omega_k}{\epsilon}$ is positive whenever $s \in \overline{\mathbb{C}^+}$. Lemma 5.12 and Lemma 4.2 show that there exists a constant $M_3 > 0$ such that $\|\Phi_{2k}(s)\| < M_3$ for all $s \in B_\gamma(i\omega_k)$. Note that M_2 and M_3 are independent of k . Choose $\epsilon_2 = \frac{1}{2M_2M_3}$.

It follows, that $\left\| (I + P(s)C_\epsilon(s))^{-1} \right\| \leq 2M_2$ for all $\epsilon \in (0, \epsilon_2]$ and $s \in B_\gamma(i\omega_k)$. Since M_2 does not depend on $k \in \mathbb{Z}$ the claim follows by choosing $\epsilon^* = \min\{\epsilon_1, \epsilon_2\}$. \square

LEMMA 5.14. *If the design parameters in K_k in (5.1) satisfy Assumption 5.10, then there exists a constant $\epsilon^* > 0$ such that $C_\epsilon(I + PC_\epsilon)^{-1} \in \mathcal{M}(\mathbf{P})$ for all $\epsilon \in (0, \epsilon^*]$.*

Proof. Lemma 5.13 shows that there exists a choice of $\epsilon_1 > 0$ such that $(I + PC_\epsilon)^{-1}$ is \mathbf{P} -stable for all $\epsilon \in (0, \epsilon_1]$. Lemma 4.2 implies that C_ϵ is uniformly bounded in $\overline{\mathbb{C}^+} \setminus \bigcup_{k \in \mathbb{Z}} B_\delta(i\omega_k)$ for all $\delta < \gamma$. Thus, $C_\epsilon(I + PC_\epsilon)^{-1} \in H_\beta^\infty$ for all $\beta > 0$, and polynomial boundedness needs to be shown only in $\overline{\mathbb{C}^+} \cap \bigcup_{k \in \mathbb{Z}} B_\gamma(i\omega_k)$.

Write

$$C_\epsilon(s)(I + P(s)C_\epsilon(s))^{-1} = H_1(s) + H_2(s),$$

where

$$H_1(s) = \frac{\epsilon K_k}{s - i\omega_k} (I + P(s)C_\epsilon(s))^{-1}$$

and

$$H_2(s) = \left(\epsilon C_0(s) + \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{\epsilon K_l}{s - i\omega_l} \right) (I + P(s)C_\epsilon(s))^{-1}.$$

Lemma 4.2 and Lemma 5.13 show that the absolute value of $H_2(s)$ is bounded in $B_\gamma(i\omega_k)$ by a bound independent of k . Thus, it is sufficient to show polynomial boundedness of $|H_1(s)|$.

To analyze the behavior of $|H_1(s)|$ in $B_\gamma(i\omega_k)$ recall Φ_{1k} and Φ_{2k} defined by (5.2) and (5.3) and decompose $(I + PC_\epsilon)^{-1} = \Phi_{1k} (I + \epsilon\Phi_{2k}\Phi_{1k})^{-1}$. Now

$$H_1(s) = \frac{\epsilon K_k}{s - i\omega_k} \Phi_{1k} (I + \epsilon\Phi_{2k}\Phi_{1k})^{-1}.$$

There exist constants $M > 0$ and $\epsilon_0 > 0$ such that $\|(I + \epsilon\Phi_{2k}\Phi_{1k})^{-1}\| < M$ in $B_\gamma(i\omega_k)$ for all $k \in \mathbb{Z}$ and $\epsilon \in (0, \epsilon_0]$ by Assumption 5.10. Thus, it is sufficient to show polynomial boundedness of $\left| \frac{\epsilon K_k}{s - i\omega_k} \Phi_{1k}(s) \right|$. Write $z = \frac{s - i\omega_k}{\epsilon}$. By Assumption 5.10,

$$\left\| \frac{\epsilon K_k}{s - i\omega_k} \Phi_{1k}(s) \right\| = \left\| \frac{K_k}{z} \left(I + \frac{G_k}{z} \right)^{-1} \right\| = \left\| G_k^{-1} (I + zG_k^{-1})^{-1} \right\| < M(1 + |\omega_k|)^\alpha$$

for some $M > 0$ and $\alpha > 0$ independent of $k \in \mathbb{Z}$. Choice $\epsilon^* = \min\{\epsilon_0, \epsilon_1\}$ completes the proof. \square

LEMMA 5.15. *Let θ be the generating element defined in (4.1). If C_ϵ in (5.1) \mathbf{P} -stabilizes $P \in \mathcal{M}(H^\infty)$, then $\theta^{-1}(I + PC_\epsilon)^{-1} \in \mathcal{M}(\mathbf{P})$.*

Proof. Similar steps as in the proof of Lemma 5.14 show the claim. Details are left to reader. \square

LEMMA 5.16. *Let $P \in \mathcal{M}(H^\infty)$ be right invertible at $i\omega_k$. If the right inverses $P^r(i\omega_k)$ can be chosen so that $\|P^r(i\omega_k)\| < M(1 + |\omega_k|)^\alpha$ for some $\alpha > 0$ and $M > 0$, then there exists such a choice of design parameters K_k that Assumption 5.10 holds.*

Proof. Choose $K_k = g_k P^r(i\omega_k)$. Since $\|P^r(i\omega_k)\| < M(1 + |\omega_k|)^\alpha$ there exists a sequence $(g_k)_{k \in \mathbb{Z}} \in \ell^2$ of positive real numbers such that $g_k > \frac{1}{M_0}(1 + |\omega_k|)^{-\alpha_0}$ and $(\|K_k\|)_{k \in \mathbb{Z}} \in \ell^2$. The first and the second item of Assumption 5.10 hold since $G_k = P(i\omega_k)K_k = g_k I$ and $\|K_k G_k^{-1}\| = \|P^r(i\omega_k)\|$. Since $(I + z^{-1}G_k)^{-1} = z(z + g_k)^{-1}I$ and $(I + zG_k^{-1})^{-1} = (1 + zg_k^{-1})^{-1}I$, it is easy to see that for all $z \in \mathbb{C}^+$ $\|(I + z^{-1}G_k)^{-1}\| < 1$ and $\|(I + zG_k^{-1})^{-1}\| < 1$. Thus, items three and four of Assumption 5.10 hold. \square

Proof of Theorem 5.11 Lemmas 5.13, Lemma 5.14 and Lemma 5.16 show that there exists a choice of $\epsilon^* > 0$ and design parameters K_k in (5.1) such that $(I + PC_\epsilon)^{-1}$ and $C_\epsilon(I + PC_\epsilon)^{-1}$ are \mathbf{P} -stable for all $\epsilon \in (0, \epsilon^*]$. Since P is \mathbf{P} -stable, it follows that C_ϵ is a \mathbf{P} -stabilizing controller for all $\epsilon \in (0, \epsilon^*]$.

Fix an arbitrary $\epsilon \in (0, \epsilon^*]$. Since (P, I) is a left coprime factorization of P , there exists a right coprime factorization (N_c, D_c) of C_ϵ that satisfies $PN_c + D_c = I$. By Lemma 5.15,

$$\theta^{-1}(I + PC_\epsilon)^{-1} = \theta^{-1}(D_c D_c^{-1} + PN_c D_c^{-1})^{-1} = \theta^{-1}D_c \in \mathcal{M}(\mathbf{P}).$$

Theorem 5.2 implies that the controller C_ϵ is robustly regulating. The claim follows since $\epsilon \in (0, \epsilon^*]$ was chosen arbitrarily. \square

COROLLARY 5.17. *Fix a sequence $(h_k)_{k \in \mathbb{Z}}$ of real numbers that satisfy $h_k > M(1 + |\omega_k|)^\alpha$ for some fixed $\alpha > 0$ and $M > 0$. Let θ be the function defined in (4.1). If $P \in H^\infty$, then $(\theta I, P)$ is left coprime if and only if P has a right inverse $P^r(i\omega_k)$ at $i\omega_k$ for all $k \in \mathbb{Z}$ such that $\|P^r(i\omega_k)\| < M(|\omega_k| + 1)^\alpha$ for some fixed positive constants M and α and for all $k \in \mathbb{Z}$.*

Proof. The claim follows by Theorem 5.5, Theorem 5.9, and Theorem 5.11. \square

At the end of this section the results of this article are compared to those of [41]. A controller of form (5.1) was discussed also in [41] by Ylinen et al. with H^∞ -stability. They were not able to show whether or not the controller works when H^∞ -stability of the closed loop was required. The next example illustrates, that the answer is negative. Only SISO-systems are considered and the generating element is chosen to be $\theta(s) = 1 - \epsilon^{-s}$ in the example. It is straightforward to see that one can use similar justification for MIMO-systems and the signal generators considered in [41].

EXAMPLE 5.18. *Consider the reference signals of form $\hat{y}_r(s) = \frac{\phi(s)}{\theta(s)}$, where $\phi(s) \in H^\infty$ and $\theta(s) = 1 - \epsilon^{-s}$. Since the generating element has zeros at $2\pi ki$ for all $k \in \mathbb{Z}$, set $\omega_k = 2\pi k$. Let $P(s)$ be a SISO-plant such that $|P(i\omega_k)| > M$ for some $M > 0$ and for all $k \in \mathbb{Z}$.*

Choose a controller of form (5.1), where $C_0(s) = 0$. Write K_k in the form $K_k = \frac{g_k}{P(i\omega_k)}$. Since the controller should be defined at $s = 1$, say, necessarily $K_k \rightarrow 0$ as $|k| \rightarrow \infty$. It follows, that $g_k \rightarrow 0$ as $|k| \rightarrow \infty$.

The decomposition of $(I + P(s)C_\epsilon(s))^{-1}$ from the proof of Lemma 5.13 gives

$$\frac{1}{\theta(s)} (1 + P(s)C_\epsilon(s))^{-1} = \Phi_{1k}(s) (I + \epsilon\Phi_{2k}(s)\Phi_{1k}^{-1}(s))^{-1}.$$

One can choose sufficiently small $\epsilon > 0$, so that $\epsilon\Phi_{2k}(s)\Phi_{1k}^{-1}(s) < \frac{1}{2}$ near $i\omega_k$. It follows, that

$$\lim_{s \rightarrow i\omega_k} \left| (1 + P(s)C(s))^{-1} \times \frac{1}{\theta(s)} \right| \geq \frac{1}{2} \lim_{s \rightarrow i\omega_k} \left| \frac{s - i\omega_k}{(s - i\omega_k + \epsilon g_k)\theta(s)} \right| = \frac{1}{2\epsilon|g_k|}.$$

The last term approaches infinity as $|k| \rightarrow \infty$, so there exists no such $\epsilon^ > 0$ that $\frac{1}{\theta(s)}(1 + P(s)C_\epsilon(s))^{-1} \in H^\infty$ for all $\epsilon \in (0, \epsilon^*]$. \blacksquare*

The reason Ylinen et al. were able to make a controller of the form

$$(5.4) \quad C_\epsilon = \sum_{k \in \mathbb{Z}} \frac{\epsilon^2 K_k}{(s - i\omega_k)(s - i\omega_k + \epsilon)}$$

to work in H^∞ -setting but not a controller of the form (5.1) is explained by the first item of Theorem 4.6. It shows that if H^∞ stability is considered, the controller (5.4) that contains the generating element (4.4) as an internal model can regulate a wider class of reference signals than the controller (5.1) which contains the generating element (4.1) as its internal model.

The fifth item of Theorem 4.6 shows that if \mathbf{P} -stability is considered instead of H^∞ -stability, it does not matter which one of the generating elements one chooses. This explains why the controller (5.1) can solve \mathbf{P} -robust regulation problem. It also indicates that the controller (5.4) should work too.

5.4. Controller Design for Unstable Plants. Theorem 5.5 only states a condition for the solvability, but does not give an actual controller for unstable plants. However, an idea how to find a robustly regulating controller can be found in the proof of the theorem. In what follows this idea is made precise and a procedure to find a robustly regulating controller is given. For a plant with a right coprime factorization (N_p, D_p) the proposed controller consists of a stabilizing controller C_s and a controller C_i containing the internal model and has the structure depicted in Figure 5.1.

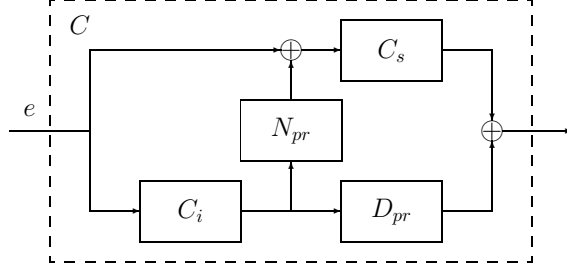


FIG. 5.1. Structure of the designed robustly regulating controller.

Assume that P has both right and left coprime factorizations. Let a right coprime factorization of P be (N_p, D_p) . A robustly regulating controller can be found by using the following procedure:

1. Find a stabilizing controller C_s for P .
2. Find a robustly regulating controller C_i for N_p . Denote

$$\tilde{D}_i = (I + N_p C_i)^{-1} \text{ and } \tilde{N}_i = C_i \tilde{D}_i.$$

3. A robustly regulating controller is given by

$$C = C_s \tilde{D}_i^{-1} + D_p C_i = \begin{bmatrix} C_s & I \end{bmatrix} \begin{bmatrix} I & N_p \\ 0 & D_p \end{bmatrix} \begin{bmatrix} I \\ C_i \end{bmatrix}.$$

The factorization of C_i in the second step is right coprime and satisfies $N_p \tilde{N}_i + \tilde{D}_i = I$. In order to show that C is robustly regulating let (N_c, D_c) be a right coprime factorization of C_s . Such a coprime factorizations exists by Lemma 2.1. Now,

$$C = (N_c + D_p R) (D_c - N_p R)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix},$$

where $R = \tilde{N}_i D_c$. The controller C stabilizes P by Lemma 2.2. In addition,

$$(N_c + D_p R, D_c - N_p R)$$

is a right coprime factorization of C . Since $\theta^{-1} \tilde{D}_i \in \mathcal{M}(\mathbf{P})$ by Theorem 5.2 and

$$D_c - N_p R = \left(I - N_p \tilde{N}_i \right) D_c = \tilde{D}_i D_c,$$

Theorem 5.2 implies that C is robustly regulating.

All transfer functions have both coprime factorizations in the well known Callier-Desoer class of transfer functions that was introduced in [2]. One can use standard techniques to design a stabilizing controller for a plant in the Callier-Desoer class of transfer functions. In addition, the numerator matrix of a right (or left) coprime factorization is always in H_-^∞ , see [4, Lemma A.7.47], so a robustly regulating controller for the numerator matrix is readily provided by (5.1). Thus, one can combine the above controller design procedure with a controller of form (5.1) which provides a simple way to design a robustly regulating controller.

6. Solving the S-Robust Regulation Problem. Theorem 4.6 shows that the generators in (4.1) and (4.4) are minimal \mathbf{S} -generating elements. Thus, it is not a surprise that a robustly regulating controller contains them as an internal model. The proofs of Lemma 5.1 and Theorem 5.2 apply to the \mathbf{S} -robust regulation problem without changes so one has the following theorem showing that the internal model principle holds for the \mathbf{S} -robust regulation problem. Note that in the theorem extra conditions on the sequence $(h_k)_{k \in \mathbb{Z}}$ are not needed unlike with \mathbf{P} -stability. This is because the \mathbf{S} -stability does not restrict the divergence rate of the closed loop transfer function on the imaginary axis.

THEOREM 6.1. *Consider a plant $P \in \mathcal{M}(\mathbf{F}_\mathbf{S})$ with right and left coprime factorizations, and let the function θ be the one in (4.1) with an arbitrary sequence $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ of positive real numbers. If C is a stabilizing controller with a right coprime factorization (N_c, D_c) , then C solves the $(\hat{\mathcal{Y}}(a_k), \{0\}, \mathbf{S})$ -robust regulation problem if and only if $\theta^{-1}D_c \in \mathcal{M}(\mathbf{S})$.*

COROLLARY 6.2. *Consider a plant $P \in \mathcal{M}(\mathbf{F}_\mathbf{S})$ with right and left coprime factorizations, and let the function θ be the one in (4.1) with an arbitrary sequence $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ of positive real numbers. If C is a stabilizing controller with a right coprime factorization (N_c, D_c) , then C solves the \mathbf{S} -robust regulation problem if and only if $\theta^{-1}D_c \in \mathcal{M}(\mathbf{S})$.*

If θ is the function in (4.1) then $\theta^{-1} \in H_\beta^\infty$ for all $\beta > 0$. Consequently, $\theta^{-1}D \in \mathcal{M}(H_\beta^\infty)$ for all $\beta > 0$, which implies that a transfer function contains θ as its internal model if $\theta^{-1}D \in \mathcal{M}(\mathbf{S})$ is analytic on the imaginary axis. But this is to say that D should block the poles of θ or in other words $D(i\omega_k) = 0$ for all $k \in \mathbb{Z}$. This shows the following corollary.

COROLLARY 6.3. *Consider a plant $P \in \mathcal{M}(\mathbf{F}_\mathbf{S})$ with right and left coprime factorizations, and let the function θ be the one in (4.1) with an arbitrary sequence $(h_k)_{k \in \mathbb{Z}} \in \ell^2$ of positive real numbers. If C is a stabilizing controller with a right coprime factorization (N_c, D_c) , then C solves the \mathbf{S} -robust regulation problem if and only if $D_c(i\omega_k) = 0$ for all $k \in \mathbb{Z}$.*

The previous corollary states a simple condition for a stabilizing controller to be robustly regulating. This allows one to formulate a result corresponding to the time domain blocking zero condition of [12] for robustly regulating controllers.

THEOREM 6.4. *Consider a plant $P \in \mathcal{M}(\mathbf{F}_\mathbf{S})$, and assume that it has right and left coprime factorizations. An \mathbf{S} -stabilizing controller C solves the \mathbf{S} -robust regulation problem if and only if*

$$(6.1) \quad [(I + P'(i\omega_k)C(i\omega_k))^{-1} - (I + P'(i\omega_k)C(i\omega_k))^{-1}P'(i\omega_k)] = 0$$

for all P' which C stabilizes.

Proof. Sufficiency is shown first. Let θ be the function defined in (4.1). Since C stabilizes P' and $\theta^{-1} \in H_\beta^\infty$ for all $\beta > 0$, condition (6.1) implies that

$$\theta^{-1} [(I + P'C)^{-1} - (I + P'C)^{-1}P'] \in \mathcal{M}(\mathbf{P}).$$

Since θ is a \mathbf{S} -generating function, it follows that C is regulating. Thus, C is robustly regulating.

In order to show necessity, let $(\tilde{N}_p, \tilde{D}_p)$ be a left coprime factorization of a plant P' that C stabilizes. Lemma 2.1 implies that there exists a left coprime factorization

(N_c, D_c) of C such that $\tilde{N}_p N_c + \tilde{D}_p D_c = I$. A direct calculation shows that

$$\begin{bmatrix} (I + P'C)^{-1} & -(I + P'C)^{-1} P' \end{bmatrix} = \begin{bmatrix} D_c \tilde{D}_p & -D_c \tilde{N}_p \end{bmatrix}.$$

Corollary 6.3 implies that (6.1) holds. \square

The internal model principle holds by Theorem 6.1, so the solvability condition of Theorem 5.5 holds for **S**-robust regulation problem by Remark 5.7.

THEOREM 6.5. *Let P have right and left coprime factorizations. Fix a sequence $(h_k)_{k \in \mathbb{Z}} \in \ell^2$, and let θ be the function in (4.1). If (N_p, D_p) is a right coprime factorization of P , then the **S**-robust regulation problem is solvable if and only if $(\theta I, N_p)$ is left coprime.*

The controller design of Section 5.3 can be carried out with the **S**-stability. In fact, the proofs are easier, because no attention to the asymptotic behavior on the imaginary axis needs to be paid as long as the closed loop system remains analytic on the imaginary axis. This is why the condition that a plant $P(s) \in \mathcal{M}(H_-^\infty)$ has at most polynomial decay rate on the imaginary axis is not needed.

THEOREM 6.6. *Let the function θ be the one in (4.1). If $P \in \mathcal{M}(H_-^\infty)$ then $(\theta I, P)$ is left coprime if and only if $P(i\omega_k)$ is right invertible for all $k \in \mathbb{Z}$.*

Since the controller design of Section 5.4 for unstable plants is applicable the above theorem gives an easy characterization for the solvability of the robust regulation problem with **S**-stability for all plants in the Callier-Desoer -algebra.

7. Examples. The first example illustrates how selecting a weaker stability type affects the solvability of the robust regulation problem. The two transfer functions considered are from [22] and [23] and stem from a heat equation modelling a one dimensional heated metal bar with two heaters and two sensors.

EXAMPLE 7.1. *Consider the plant transfer functions*

$$P_1(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \text{ and } P_2(s) = \begin{bmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{bmatrix}$$

where

$$\begin{aligned}
g_{11}(s) &= \frac{4 \sinh\left(\frac{1}{4}\sqrt{s+1}\right) \left(\sinh\left(\frac{3}{4}\sqrt{s+1}\right) - \sinh\left(\frac{1}{2}\sqrt{s+1}\right)\right)}{(s+1)^{\frac{3}{2}} \sinh(\sqrt{s+1})}, \\
g_{12}(s) &= \frac{4 \sinh^2\left(\frac{1}{4}\sqrt{s+1}\right)}{(s+1)^{\frac{3}{2}} \sinh(\sqrt{s+1})}, \\
g_{21}(s) &= \frac{\left\{ \begin{array}{l} 2 \cosh(\sqrt{s+1}) - 4 \cosh\left(\frac{3}{4}\sqrt{s+1}\right) \\ + 2 \cosh\left(\frac{1}{2}\sqrt{s+1}\right) \\ + 4 \cosh\left(\frac{1}{4}\sqrt{s+1}\right) - 4 \end{array} \right\}}{(s+1)^{\frac{3}{2}} \sinh(\sqrt{s+1})}, \\
g_{22}(s) &= g_{11}(s), \\
p_{11}(s) &= \frac{1}{s+1} + 7 \frac{\left\{ \begin{array}{l} -\cosh(\sqrt{s+1}) + \cosh\left(\frac{6}{7}\sqrt{s+1}\right) \\ -\cosh\left(\frac{2}{7}\sqrt{s+1}\right) \\ +\cosh\left(\frac{1}{7}\sqrt{s+1}\right) \end{array} \right\}}{2(s+1)^{3/2} \sinh(\sqrt{s+1})}, \\
p_{12}(s) &= \frac{7}{(s+1)^{3/2}} \frac{\sinh\left(\frac{1}{7}\sqrt{s+1}\right) \sinh\left(\frac{3}{7}\sqrt{s+1}\right)}{\sinh(\sqrt{s+1})}, \\
p_{21}(s) &= 7 \frac{\left\{ \begin{array}{l} -\cosh\left(\frac{6}{7}\sqrt{s+1}\right) - 2 \cosh\left(\frac{5}{7}\sqrt{s+1}\right) \\ +\cosh\left(\frac{4}{7}\sqrt{s+1}\right) \\ + 2 \cosh\left(\frac{1}{7}\sqrt{s+1}\right) - 4 \end{array} \right\}}{2(s+1)^{3/2} \sinh(\sqrt{s+1})}, \\
p_{22}(s) &= \frac{7 \sinh\left(\frac{1}{7}\sqrt{s+1}\right) \left(\sinh\left(\frac{5}{7}\sqrt{s+1}\right) - \sinh\left(\frac{4}{7}\sqrt{s+1}\right)\right)}{(s+1)^{3/2} \sinh(\sqrt{s+1})}.
\end{aligned}$$

To simplify considerations note that the transfer functions can be put into form

$$P_i(s) = V_i(s) \Lambda_i(s) U_i(s),$$

where $U_i(s)$ and $V_i(s)$ are invertible for any $s \in \overline{\mathbb{C}^+}$ and $\|U_i(s)\|$, $\|U_i^{-1}(s)\|$, $\|V_i(s)\|$ and $\|V_i^{-1}(s)\|$ are uniformly bounded in $\overline{\mathbb{C}^+} \cap \{s \in \mathbb{C} \mid |s| > \rho\}$ for some $\rho > 0$ [22]. The matrix $\Lambda_i(s) = \text{diag}(q_{i1}(s), q_{i2}(s))$ is a diagonal matrix with diagonal elements

$$q_{11}(s) = q_{12}(s) = \frac{1}{(s+1)\sqrt{s+1}}$$

for $i = 1$, and for $i = 2$

$$q_{21}(s) = \frac{1}{(s+1)\sqrt{s+1}} \quad \text{and} \quad q_{22}(s) = \frac{1}{(s+1)\sqrt{s+1}} e^{-\frac{4}{7}\sqrt{s+1}}.$$

Note that $P_i(s) \in H_-^\infty$ for $i = 1, 2$, and $P_i(i\omega_k)$ is invertible for all $k \in \mathbb{Z}$ and $i = 1, 2$. Since $q_{11}(s)$ and $q_{12}(s)$ are polynomially bounded on the imaginary axis Theorem 5.5 implies that the **P**-robust regulation problem is solvable for $P_1(s)$. Since $q_{22}(s)$ decays exponentially on the imaginary axis Theorem 5.5 implies that the **P**-robust regulation problem is not solvable for $P_2(s)$. However, Theorem 6.5 shows that **S**-robust regulation problem is solvable for both of the plants. ■

In the next example, the controller design procedure of Section 5.4 is used for a transfer function stemming from a time-delay system. Note that the transfer function is initially unstable and is in the Callier-Desoer class of transfer functions.

EXAMPLE 7.2. Consider the transfer function

$$P(s) = \begin{bmatrix} 0 & \frac{1}{s} & \frac{1}{s-2} \\ \frac{1}{s+1} & \frac{1}{s} + \frac{1-\epsilon^{-s}}{s(s+1)} & 0 \end{bmatrix}$$

stemming from a delay system. A right coprime factorization of $P(s)$ is given first. To this end the transfer function is presented as a sum of a rational transfer function containing the unstable poles of the plant and a stable transfer function.

$$P(s) = \begin{bmatrix} 0 & \frac{1}{s} & \frac{1}{s-2} \\ 0 & \frac{1}{s} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{s+1} & \frac{1-\epsilon^{-s}}{s(s+1)} & 0 \end{bmatrix} = P_r(s) + P_s(s).$$

A right coprime factorization of the rational part P_r is (N_r, D_r) where

$$N_r(s) = \begin{bmatrix} 0 & \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & \frac{1}{s+1} & 0 \end{bmatrix} \text{ and } D_r(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{s}{s+1} & 0 \\ 0 & 0 & \frac{s-2}{s+1} \end{bmatrix}.$$

Set $X = I$ and

$$Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}.$$

Now, $YN_r + XD_r = I$. Choose $D_p = D_r$ and

$$N_p = N_r + P_s D_r = \begin{bmatrix} 0 & \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1-\epsilon^{-s}}{(s+1)^2} & 0 \end{bmatrix}.$$

The pair (N_p, D_p) is a right coprime factorization since

$$N_p D_p^{-1} = N_r D_r^{-1} + P_s = P_r + P_s = P$$

and

$$(7.1) \quad YN_p + (X - YP_s)D_p = YN_r + YP_s D_r + XD_r - YP_s D_r = I.$$

Set $N_0 = Y$ and

$$D_0 = X - YP_s = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{s+1} & 1 - \frac{1-\epsilon^{-s}}{s(s+1)} & 0 \\ \frac{3}{s+1} & \frac{3(1-\epsilon^{-s})}{s(s+1)} & 1 \end{bmatrix}.$$

The equation (7.1) shows that $C_0 = D_0^{-1}N_0$ is a stabilizing controller for P .

Since

$$N_p(s) \begin{bmatrix} 0 & s+1 \\ 0 & 0 \\ s+1 & 0 \end{bmatrix} = N_p(s)N_p^r(s) = I,$$

a right inverse of N_p at $i\omega_k$ is $N_p^r(i\omega_k)$. Theorem 5.5 shows that there exists a controller solving the \mathbf{P} -robust regulation problem for N_p since $\|N_p(i\omega_k)\| \leq (1 + |\omega_k|)$.

Recall that $\gamma > 0$ is any number less than one quarter of the minimum distance between the poles $i\omega_k$. Choose $\epsilon^* = \frac{1}{60\gamma}$. A robustly regulating controller for N_p is

$$C_{i\epsilon}(s) = \sum_{k \in \mathbb{Z}} \frac{\epsilon N_p^r(i\omega_k)}{(s - i\omega_k)(1 + |\omega_k|)^2} = \begin{bmatrix} 0 & c_\epsilon(s) \\ 0 & 0 \\ c_\epsilon(s) & 0 \end{bmatrix},$$

where $c_\epsilon(s) = \sum_{k \in \mathbb{Z}} \frac{\epsilon(1+i\omega_k)}{(s-i\omega_k)(1+|\omega_k|)^2}$ and $\epsilon \in (0, \epsilon^*]$.

One now has the right coprime factorization (N_p, D_p) of P , the stabilizing controller C_0 for P and the robustly regulating controller C_i for N_p , so the controller design of Section 5.4 is now applicable. A robustly regulating controller for P is

$$C_\epsilon(s) = [C_0(s) \quad I] \begin{bmatrix} I & N_p(s) \\ 0 & D_p(s) \end{bmatrix} \begin{bmatrix} I \\ C_{i\epsilon}(s) \end{bmatrix} = \begin{bmatrix} 0 & c_\epsilon(s) \\ 0 & \frac{s(1+s+c_\epsilon(s))}{s(s+1)+\epsilon^{-s}-1} \\ 3+c_\epsilon(s) & \frac{-3s(1+s+c_\epsilon(s))}{s(s+1)+\epsilon^{-s}-1} \end{bmatrix},$$

where $c_\epsilon(s) = \sum_{k \in \mathbb{Z}} \frac{\epsilon(1+i\omega_k)}{(s-i\omega_k)(1+|\omega_k|)^2}$ and ϵ is any strictly positive real number less than or equal to $\frac{1}{60\gamma}$. Note that the tuning parameter $\epsilon > 0$ enables online tuning of the controller if necessary. ■

8. Concluding Discussion and Directions for Future Research. The frequency domain robust regulation problem with an infinite-dimensional exosystem was studied in this paper. One was able to formulate the internal model principle and to give a necessary and sufficient condition for solvability by considering **P**-stability and **S**-stability instead of the more commonly used stability types such as H^∞ -stability that are too strong for infinite-dimensional signals. The chosen stability types weakened the boundedness requirement of H^∞ -stable functions on the imaginary axis. This allowed one to achieve robust perfect tracking of infinite-dimensional signals even with strictly proper plant transfer functions. The robustness was understood in the sense that closed loop stability implies regulation, which made topological considerations unnecessary.

There are many applications where the reference signals are generated by an infinite-dimensional exosystem, see the references in [21]. The theoretical results of this paper show that such signals can be tracked perfectly with infinite-dimensional controllers. However, controllers in the engineering applications are often finite-dimensional. Basically, this means that the high frequency components of the reference signals are not controlled, thus leading to non-perfect tracking. The controllers proposed in this article serve as accurate models to be approximated by finite-dimensional controllers in the engineering applications. Robustness is also encountered in the nature [20]. Such systems might contain infinite-dimensional feedback controllers. The results of this article show fundamental limitations for robustness of such control structures.

The formulation of the robust regulation problem used ideas that are familiar from the time domain theory, i.e., the definition of robustness, the stability types, and the classes of the reference and disturbance signals, see [14, 18, 31] and the references therein. Despite the similarities, it is not easy to find out in what sense the time domain and frequency domain approaches correspond to each other. For example, an extra assumption on the smoothness of the reference and disturbance signals in [14, 18, 31] is not needed in this article. A further study on the different stability

types and realization theory is needed in order to understand the connection between the two formulations of the seemingly same problem better.

One of the most important parts of the future work is to provide realization theory for controllers in the frequency domain. This would clarify the connection between time domain and frequency domain robust regulation and make the theory presented here more applicable. A specific research question is: Under which conditions a realization of a frequency domain controller solving the robust regulation problem solves the corresponding time domain robust output regulation problem. A part of this work is to find out in what sense the realizations of **P**-stabilizing or **S**-stabilizing controllers stabilize the extended system in the time domain. For **P**-stability this problem was solved in [29], and it was shown that the output of a realization of a **P**-stable system is bounded for all suitably smooth inputs. This results is analogous to the well known result that H^∞ -stability corresponds to bounded-input bounded-output stability in the time domain, see for example [4]. It is an open question when does **P**-stability of a plant transfer function imply polynomial stability in the time domain. The connection between the strong and **S**-stability is more fuzzy. Strong stability allows accumulation points of point spectra on the imaginary axis, but **S**-stability does not allow accumulation of poles there. Even if the current definition of **S**-stability is sufficient to tackle the problem considered in this article, one might want to revise the definition to allow more complicated signal generators or plants and to find out stronger connections between the strong stability in the time domain and the **S**-stability in the frequency domain.

With the type of robustness considered in this paper, the perturbations tolerated by a robustly regulating controller are just the perturbations preserving closed loop stability. Thus, it is of importance to study the robustness of stability. To be able to use stability theory of topological rings [38], one needs to define a topology in **P** and **S**. It would be better if one is able to find a metric that induces a suitable topology, since it would enable quantitative measurement of robustness and of deeper understanding of robustness of stability. The authors believe that **P**-stability would have relatively good robustness properties on the basis of the recent time domain results on robustness of polynomial stability [28].

One can easily define stability types that are stronger than **S**-stability but weaker than **P**-stability by restricting the boundary behavior of stable plants on the imaginary axis differently, e.g., one can allow exponential growth rate. This would be advantageous since by doing so one would improve the robustness properties of **S**-stability and at the same time be able to robustly regulate plants for which **P**-robust regulation problem is not solvable. An example of a plant for which such an approach would be ideal is the plant transfer function $P_2(s)$ of Example 7.1.

The time domain theory of robust regulation in [18, 27, 31] is suitable for more general signal classes than those considered in this paper. The theory there allows the state operator of the exosystem to be a block diagonal operator with an infinite number of non trivial Jordan blocks on its diagonal. The signals generated by such exosystems correspond to signals with higher order poles in the frequency domain. One possible direction for future research is to generalize the results in this article for such signals. The generalization for signals with a finite number of unstable higher order poles – not necessarily on the imaginary axis – and an infinite number of simple poles on the imaginary axis is believed to be straightforward. One just adds a finite-dimensional controller regulating the dynamics corresponding to the higher order poles, see for example [4, 13, 37], to the controller for infinite-number of simple

poles presented in this article. The theoretical results concerning the solvability of the problem might generalize straightforwardly even for generators with an infinite number of higher order poles. In this case the internal model would contain the higher order poles of the reference signals.

It was assumed that the plant have both left and right coprime factorizations over the ring \mathbf{P} or \mathbf{S} . This is restricting since the algebraic properties of the rings are largely unknown. In particular, the existence of coprime factorizations is not known. In addition, even if coprime factorizations exist, they are hard to find in practice which restricts the applicability the theory developed. There are two things to do to overcome this restriction. First, one should learn more about the algebraic properties of the rings. Then the results of [33, 37] can be used in order to see how restrictive it is to assume existence of both coprime factorizations from the theoretical point of view. Secondly, one can use the stabilization theory based on general factorization developed by Mori [25] and Quadrat [34] to develop robust regulation theory without using coprime factorizations. This is a promising direction of research since the problem formulation of this article would allow the use of the algebraic results presented in [25, 34]. The results obtained would make the theory more applicable and general since no coprime factorizations are needed. There is no need for restricting oneself to a certain ring. Thus, theory can be developed for different rings at once.

REFERENCES

- [1] MAHABALESHWAR K. P. BHAT, *Regulator Theory for Evolution Systems*, PhD thesis, University of Toronto, Toronto, Canada, 1976.
- [2] FRANK M. CALLIER AND CHARLES A. DESOER, *An algebra of transfer functions for distributed linear time-invariant systems*, IEEE Transaction on Circuits and Systems, 25 (1978), pp. 651–662.
- [3] F. M. CALLIER AND C. A. DESOER, *Stabilization, tracking and disturbance rejection in multivariable convolution systems*, Annales de la Société Scientifique de Bruxelles, 94 (1980), pp. 7–51.
- [4] RUTH F. CURTAIN AND HANS J. ZWART, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.
- [5] EDWARD J. DAVISON, *Multivariable tuning regulators: The feedforward and robust control of a general servomechanism problem*, IEEE Transactions on Automatic Control, 21 (1976), pp. 35–47.
- [6] ———, *The robust control of a servomechanism problem for linear time-invariant multivariable systems*, IEEE Transactions on Automatic Control, 21 (1976), pp. 25–34.
- [7] E. J. DAVISON AND A. GOLDENBERG, *Robust control of a general servomechanism problem: The servo compensator*, Automatica, 11 (1975), pp. 461–471.
- [8] C. A. DESOER, RUEY-WEN LIU, JOHN MURRAY, AND RICHARD SAEKS, *Feedback system design: The fractional representation approach to analysis and synthesis*, IEEE Transactions on Automatic Control, 25 (1980), pp. 399–412.
- [9] B.A. FRANCIS AND W.M. WONHAM, *The internal model principle for linear multivariable regulators*, Applied Mathematics & Optimization, 2 (1975), pp. 170–194.
- [10] BRUCE A. FRANCIS, *The linear multivariable regulator problem*, SIAM Journal on Control and Optimization, 15 (1977), pp. 486–505.
- [11] BRUCE A. FRANCIS AND M. VIDYASAGAR, *Algebraic and topological aspects of the regulator problem for lumped linear systems*, Automatica, 19 (1983), pp. 87–90.
- [12] B. A. FRANCIS AND W. M. WONHAM, *The role of transmission zeros in linear multivariable regulators*, International Journal of Control, 22 (1975), pp. 657–681.
- [13] TIMO HÄMÄLÄINEN AND SEPPÖ POHJOLAINEN, *A finite-dimensional robust controller for systems in the CD-algebra*, IEEE Transactions on Automatic Control, 45 (2000), pp. 421–431.
- [14] ———, *Robust regulation of distributed parameter systems with infinite-dimensional exosystems*, SIAM Journal on Control and Optimization, 48 (2010), pp. 4846–4873.
- [15] SHINJI HARA, YUTAKA YAMAMOTO, TOHRU OMATA, AND MICHIO NAKANO, *Repetitive control system: A new type servo system for periodic exogenous signals*, IEEE Transactions on

- Automatic Control, 33 (1988), pp. 659–668.
- [16] ROGER A. HORN AND CHARLES R. JOHNSON, *Matrix Analysis*, Cambridge University Press, 1999.
 - [17] J. HUANG, *Nonlinear Output Regulation: Theory and Applications*, SIAM, 2004.
 - [18] E. IMMONEN, *State Space Output Regulation Theory for Infinite-Dimensional Linear Systems and Bounded Uniformly Continuous Exogenous Signals*, PhD thesis, Tampere University of Technology, Tampere, Finland, 2006.
 - [19] T. INOUE, M. NAKANO, AND S. IWAI, *High accuracy control of servomechanism for repeated contouring*, in 10th Annual Symposium on Incremental Motion Control Systems and Devices, 1981, pp. 285–292.
 - [20] HIROAKI KITANO, *Biological robustness*, Nature, 5 (2004).
 - [21] Z. XIANYI L. CUIYAN, Z. DONGCHUN, *Theory and applications of the repetitive control*, in SICE Annual Conference, Hokkaido Institute of Technology, Sapporo, Japan, August 2004.
 - [22] P. LAAKKONEN AND S. POHJOLAINEN, *Structure at infinity for infinite-dimensional systems*, International Journal of Control, 84 (2011), pp. 702–715.
 - [23] ———, *Solvability of the output regulation problem with a feedforward controller*, in Proceedings of 12th European Control Conference (ECC) 2013, Zürich, Switzerland, July 17–19 2013.
 - [24] HARTMUT LOGEMANN, *Stabilization and regulation of infinite-dimensional systems using coprime factorizations*, in Analysis and Optimization of Systems: State and Frequency Domain Approaches for Infinite-Dimensional Systems, R.F. Curtain, A. Bensoussan, and J.L. Lions, eds., vol. 185 of Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 1993, pp. 102–139.
 - [25] KAZUYOSHI MORI, *Elementary proof of controller parametrization without coprime factorizability*, IEEE Transactions on Automatic Control, 49 (2004), pp. 589–592.
 - [26] CARL N. NETT, *The fractional representation approach to robust linear feedback design: a self-contained exposition*, master’s thesis, Rensselaer Polytechnic Institute, Troy, New York, USA, 1984.
 - [27] L. PAUNONEN, *Output Regulation Theory for Linear Systems with Infinite-Dimensional and Periodic Exosystems*, PhD thesis, Tampere University of Technology, Tampere, Finland, 2011.
 - [28] ———, *Robustness of polynomial stability with respect to unbounded perturbations*, Systems & Control Letters, 62 (2013), pp. 331–337.
 - [29] L. PAUNONEN AND P. LAAKKONEN, *On polynomial stability for linear systems*, in The 21st International Symposium on Mathematical Theory of Networks and Systems (MTNS 2014), Groningen, The Netherlands, July 7–11 2014.
 - [30] L. PAUNONEN AND S. POHJOLAINEN, *Internal model theory for distributed parameter systems*, SIAM Journal of Control and Optimization, 48 (2010), pp. 4753–4775.
 - [31] ———, *Robust controller design for infinite-dimensional exosystems*, International Journal of Robust and Nonlinear Control, (2012), pp. 702–715.
 - [32] SEPPO A. POHJOLAINEN, *Robust multivariable PI-controller for infinite dimensional systems*, IEEE Transactions on Automatic Control, 27 (1982), pp. 17–30.
 - [33] A. QUADRAT, *The fractional representation approach to synthesis problems: An algebraic analysis viewpoint part II: Internal stabilization*, SIAM Journal on Control and Optimization, 42 (2003), pp. 300–320.
 - [34] ———, *On a generalization of the Youla-Kučera parametrization. Part II: the lattice approach to MIMO systems*, Mathematics of Control, Signals, and Systems, 18 (2006), pp. 199–235.
 - [35] RICHARD REBARBER AND GEORGE WEISS, *Internal model based tracking and disturbance rejection for stable well-posed systems*, Automatica, 39 (2003), pp. 1555–1569.
 - [36] HIROYUKI UKAI AND TETSUO IWAZUMI, *General servomechanism problems for distributed parameter systems*, International Journal of Control, 42 (1985), pp. 1195–1212.
 - [37] M. VIDYASAGAR, *Control System Synthesis: A Factorization Approach*, MIT Press, 1985.
 - [38] MATHUKUMALLI VIDYASAGAR, HANS SCHNEIDER, AND BRUCE A. FRANCIS, *Algebraic and topological aspects of feedback stabilization*, IEEE Transactions on Automatic Control, 27 (1982), pp. 880–894.
 - [39] GEORGE WEISS AND MARTIN HÄFELE, *Repetitive control of MIMO systems using h^∞ design*, Automatica, 35 (1999), pp. 1185–1199.
 - [40] YUTAKA YAMAMOTO AND SHINJI HARA, *Internal and external stability and robust stability condition for a class of infinite-dimensional systems*, Automatica, 28 (1992), pp. 81–93.
 - [41] L. YLINEN, T. HÄMÄLÄINEN, AND S. POHJOLAINEN, *Robust regulation of stable systems in the H^∞ -algebra*, International Journal of Control, 79 (2006), pp. 24–35.