Robustness of Controllers for SISO-Plants and Signals Generated by an Infinite-Dimensional Exosystem

Petteri Laakkonen *, and Seppo Pohjolainen †
Department of Mathematics
Tampere University of Technology
P.O. Box 553, FI-33101 Tampere, Finland
* Email: petteri.laakkonen@tut.fi
† Email: seppo.pohjolainen@tut.fi

***Abstract*— Robust regulation of signals generated by infinite-dimensional exosystems have received some attention lately. Robustness have been understood in the sense that strong stability and solvability of a Sylvester equation should imply asymptotic tracking. It is not known what perturbations preserve strong stability or solvability of the Sylvester equation, so it is not entirely clear how robust a robustly regulating controller actually is. The purpose of this article is to give a detailed analysis of the robustness properties of a controller. We give a simple necessary and sufficient condition that guarantees the solvability of the Sylvester equation for single input single output plants. The solvability condition relates smoothness of the reference signals to the solvability of the Sylvester equation and helps us to understand robustness of controllers.**

I. INTRODUCTION

Robust regulation of signals generated by an infinite-dimensional exosystem have received some attention lately [1]–[4]. The purpose in robust regulation is to control a given system in such a manner that the measured information follows a specified reference signal. In addition, the control configuration should work despite some small perturbations of the system. In this article, we choose the reference signals to be generated by the diagonal exosystem

$$\dot{v} = Sv, \qquad v(0) = v_0 \in W, \tag{1a}$$
$$y_r = Fv, \tag{1b}$$

where $F \in \mathcal{B}(W, Y)$, i.e., F is a linear bounded operator from W to Y, and S is the linear diagonal operator

$$S = \sum_{k\in\mathbb{Z}} \mathrm{i}\omega_k \langle \cdot, \phi_k \rangle \phi_k, \tag{2}$$

with the domain

$$\mathcal{D}(S) = \left\{ v \in W \,\middle|\, \sum_{k\in\mathbb{Z}} |\langle v, \phi_k\rangle|^2 < \infty \right\}$$

in a Hilbert-space W with an orthonormal basis $(\phi_k)_{k\in\mathbb{Z}}$. The operator has point spectrum $\mathrm{i}\omega_k$ on the imaginary axis.

With exosystems of the proposed form one can generate the class of almost periodic signals. Particularly, all periodic signals belong to this class of signals. In [3] and [5] the exosystem's state operator contained a finite number of non-trivial Jordan blocks, so that the signals could be polynomially increasing. The theory presented in this paper can be easily extended to such a case, since all the results crucially depends on the decay rate of the plant transfer function at frequencies $\mathrm{i}\omega_k$ as $|\omega_k| \to \infty$ and the smoothness of the reference signals – both are properties that do not depend on the finitely many Jordan blocks. Because of space limitations we only consider the diagonal exosystem. A very general generator that can generate all the bounded periodic signals with one exosystem was considered in [1] and [6], but actual design of robustly regulating controllers is very difficult in this case because of the essential spectrum of the exosystem.

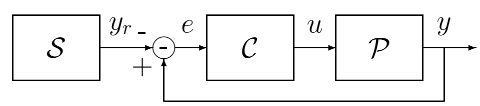

Fig. 1. The control configuration.

The basic setup of the robust regulation problem is depicted in Figure 1. In the figure, $\mathcal{P}$ is a given nominal plant of the form:

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0 \in X, \tag{3a}$$
$$y = Cx + Du. \tag{3b}$$

The state operator A is assumed to be the infinitesimal generator of a C_0-semigroup on a Hilbert space X. The inner product of X is denoted by $\langle \cdot, \cdot \rangle_X$. In addition, $B \in \mathcal{B}(\mathbb{C}, X)$, $C \in \mathcal{B}(X, \mathbb{C})$, and $D \in \mathcal{B}(\mathbb{C}, \mathbb{C})$. We consider only single input single output (SISO) systems in this article.

The controller $\mathcal{C}$ is an error feedback controller:

$$\dot{z} = \mathcal{G}_1 z + \mathcal{G}_2 e, \qquad z(0) = z_0, \tag{4a}$$
$$u = Kz, \tag{4b}$$

where $\mathcal{G}_1$ is the infinitesimal generator of a C_0-semigroup on a Hilbert space Z, $\mathcal{G}_2 \in \mathcal{B}(\mathbb{C}, Z)$, and $K \in \mathcal{B}(Z, \mathbb{C})$. The inner product of Z is denoted by $\langle \cdot, \cdot \rangle_Z$. The input of the controller is the error $e = y_r - y$ between the reference and output signals. Combining the plant and the error feedback controller gives the extended closed loop system

$$\dot{x}_e = A_e x_e + B_e v, \qquad x_e(0) = x_{e0} = \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \in X_e, \tag{5a}$$

$$e = C_e x_e + D_e v, \tag{5b}$$

where $X_e = X \times Z$ with the inner product defined by $\langle (x_1, z_1), (x_2, z_2) \rangle = \langle x_1, x_2 \rangle_X + \langle z_1, z_2 \rangle_Z$. The operators are $C_e = \begin{bmatrix} -C & -DK \end{bmatrix}$, $D_e = F$,

$$A_e = \begin{bmatrix} A & BK \\ -\mathcal{G}_2 C & \mathcal{G}_1 - \mathcal{G}_2 DK \end{bmatrix}, \text{ and } B_e = \begin{bmatrix} 0 \\ \mathcal{G}_2 F \end{bmatrix}. \tag{6}$$

The output regulation problem is to find a controller (4) such that

1) the state operator A_e generates a strongly stable C_0-semigroup, and
2) the measurement y asymptotically tracks the reference signal y_r for all initial states x_{e0} and v_0, i.e., $e \to 0$ as $t \to \infty$.

The output regulation problem with finite-dimensional exosystems was considered by Byrness et al. [7], and their results were later generalized to infinite-dimensional exosystems [1]–[3]. The solvability of the output regulation problem was shown to be related to the solvability of a certain Sylvester equation. The following result is from [2], where the exosystem (1) was considered.

Theorem 1: If A_e generates a strongly stable C_0-semigroup and there exists an operator $\Sigma \in \mathcal{B}(W, X_e)$ such that it satisfies $\Sigma \mathcal{D}(S) \subseteq \mathcal{D}(A_e)$ and the constrained Sylvester equations

$$\Sigma S - A_e \Sigma = B_e \text{ on } \mathcal{D}(S), \tag{7}$$

$$C_e \Sigma + D_e = 0, \tag{8}$$

then the controller (4) solves the output regulation problem.

The above theorem is a consequence of a very important observation that if (7) is solvable, then the error is

$$e(t) = C_e T_e(t)(x_{e0} - \Sigma v_0) + (C_e \Sigma + D_e) v(t),$$

where T_e is the semigroup generated by A_e. By (8), $(C_e \Sigma + D_e) v(t) = 0$. On the other hand, $C_e T_e(t)(x_{e0} - \Sigma v_0) \to 0$ by strong stability of the closed loop system, so $e(t) \to 0$, as $t \to \infty$.

Theorem 1 shows that if we design a controller so that (7) implies (8), then it solves the output regulation problem for any plant such that the closed loop is strongly stable and (7) is solvable. A controller with this property was called robustly regulating in [1]–[3], and we adopt this characterization of robustness. Thus, *the robust regulation problem* is to find a controller such that

1) it solves the output regulation problem for the nominal plant $\mathcal{P}$, and
2) the controller solves the output regulation problem for all perturbed plants $\mathcal{P}'$ of the form (3) for which the closed loop system remains strongly stable and the Sylvester equation (7) is solvable for the perturbed state and input operators A_e' and B_e' of the extended plant (5).

The internal model structure [1], the $\mathcal{G}$-conditions [2], [3] and the p-copy internal model [3] are necessary and sufficient conditions for a controller to be robustly regulating. They are restatements of the famous Internal Model Principle of finite-dimensional systems presented by Francis and Wonham [8]. We recall the following result from [3].

Theorem 2: If the controller (4) satisfies the $\mathcal{G}$-conditions

$$\mathcal{R}(\mathrm{i}\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \tag{9a}$$

$$\mathcal{N}(\mathcal{G}_2) = \{0\}, \tag{9b}$$

then it solves the robust regulation problem.

It was shown in [2] that the robust regulation problem is solvable for the infinite-dimensional exosystem considered in this article. However, it is not clear how robust a robustly regulating controller actually is. The reason for this is two fold. First, controllers that are robust in the sense of [1]–[3] are robust to perturbations that preserve stability. Robustness properties of stability types that are weaker than exponential stability are largely unknown. Some results on polynomial stability can be found in [9]. Robustness of stability is out of the scope of this article, and is a subject of a future research. Secondly, it is not known what perturbations preserve solvability of the Sylvester equation (7) which was assumed for every perturbed plant.

In this article, we study the robustness of controllers in [2], [4] and [5] in order to clarify robustness properties of controllers and make a detailed robustness analysis on them. The results of this article help us to better understand theoretical limitations of robust control of infinite-dimensional exosystems. It is shown, that the robustness of controllers essentially depends on the robustness of solvability of (7) and the strong stability. We give an explicit solvability condition for (7). It is related to the smoothness of the reference signals and the asymptotic decay rate of the plant transfer function and provides a tool to understand the robustness of controllers. The solvability condition appeared for the first time in [4].

The rest of the paper is organized as follows. The standing assumptions for the plant, the controller and the exosystem are made and discussed in Section II. In Section III, we give a solvability condition for the Sylvester equation (7). The main aim of this article is to analyze the robustness of controllers, which is done in the Section IV. A part of this analysis is based on the solvability condition presented in Section III. Concluding remarks are made in Section V.

II. Assumptions

In this article, the smoothness is controlled by restricting the set of allowed reference operators. It is shown later in this article that the smoothness of the signals is important for the solvability of the Sylvester equation (7). We define

$$\mathscr{Y}(f_k) = \left\{ F \in \mathcal{B}(W, \mathbb{C}) \;\middle|\; \sum_{k \in \mathbb{Z}} |f_k|^{-2} |F \phi_k|^2 < \infty \right\}, \tag{10}$$

where $(f_k)_{k\in\mathbb{Z}}$ is a fixed bounded sequence of strictly positive real numbers. In addition, we assume that there is a uniform gap between the poles of the exosystem. It is a simplifying assumption that guarantees that the exosystem does not have essential spectrum. This is important since every robustly regulating controller contains a copy of the exosystem and essential spectrum would cause problems with stabilizability.

Assumption 1: The exosystem (1) is assumed to satisfy the following conditions:

1) The constants $\omega_k \in \mathbb{R}$ in (2) are in increasing order and there exists a positive constant $\gamma > 0$ such that $\omega_k - \omega_{k-1} > 4\gamma$ for all $k \in \mathbb{Z}$.
2) the reference operator $F \in \mathscr{Y}(f_k)$.

In many physically important models we can assume that the plant is exponentially stabilizable and detectable. Since it is assumed in the existing literature giving actual controllers for infinite-dimensional exosystems, see [1], [2] and [5], we take it as a standing assumption. The invertibility of the plant transfer function on the spectra of the exosystem is a necessary condition for the solvability of the Sylvester equation [3], [7]. In order to receive a simple solvability condition, we assume that the spectrum of A and S are disjoint. In what follows, we denote the resolvent set of an operator A by $\rho(A)$.

Assumption 2: The plant (3) is assumed to satisfy the following conditions:

1) The pair (A, B) is exponentially stabilizable,
2) the pair (A, C) is exponentially detectable,
3) the plant transfer function $\mathcal{P}(s)$ is right invertible at $\mathrm{i}\omega_k$ for all $k \in \mathbb{Z}$, and
4) $\mathrm{i}\omega_k \in \rho(A)$.

The $\mathcal{G}$-conditions (9) are necessary for a controller to be robustly regulating if the spectrums of A_e and S are disjoint [3]. In this article we assume the spectrums to be disjoint, so we can assume the $\mathcal{G}$-conditions without restricting generality. Furthermore, we assume that the spectra of $\mathcal{G}_1$ does not have accumulation points at $\mathrm{i}\omega_k$ and that the assosiated eigenspace is finite-dimensional. These assumptions imply that the controller does not have essential spectrum at $\mathrm{i}\omega_k$. This is a sensible assumption, since if the controller would have essential spectrum at $\mathrm{i}\omega_k$ it would be extremely difficult to stabilize the closed loop, see Section 4.6 of [1].

Assumption 3: The controller (4) is assumed to satisfy the following conditions:

1) $\mathrm{i}\omega_k$ is an isolated spectrum point of $\mathcal{G}_1$ for all $k \in \mathbb{Z}$,
2) if γ_k is a simple closed curve that separates $\mathrm{i}\omega_k$ from the rest of the spectrum and $P_k = -\frac{1}{2\pi\mathrm{i}}\int_{\gamma_k} R(s, \mathcal{G}_1)\, ds$, then $Z_k = P_k Z$ is finite-dimensional, and
3) the controller satisfies the $\mathcal{G}$-conditions (9).

Remark 1: If 1) of Assumption 3 holds, then P_k in 2) is a projection and Z is a direct sum of Z_k and $Z_k' = (I - P_k)Z$ by [10, III-§4].

In addition, we assume that the spectrum of the exosystem and the closed loop system are disjoint. This is not a severe restriction, since we are the ones who design the controller and we can construct a controller that satisfies this assumption if the assumptions that were made above hold.

Assumption 4: The closed loop (5) is assumed to satisfy $\mathrm{i}\omega_k \in \rho(A_e)$.

III. Solvability of the Sylvester Equation

In this section, the solvability of the Sylvester equation (7) is studied. The main result is a condition for the solvability that provides insight into what perturbations are allowed to the plant parameters. We recall the following result of [2].

Lemma 1: If $\mathrm{i}\omega_k \in \rho(A_e)$ for all $k \in \mathbb{Z}$, then a unique bounded solution to (7) exists in W if and only if

$$\sup_{\|x_e\|\le 1} \sum_{k\in\mathbb{Z}} |\langle R(\mathrm{i}\omega_k, A_e)\, B_e \phi_k, x_e\rangle|^2 < \infty. \qquad (11)$$

If the unique solution exists, it is

$$\Sigma = \sum_{k\in\mathbb{Z}} \langle \cdot, \phi_k\rangle R(\mathrm{i}\omega_k, A_e)\, B_e \phi_k.$$

The sum condition (11) relates the operator F to the closed loop system consisting of the plant and the controller. Thus, it gives a relation between the solvability and the smoothness of the reference signals. This connection is clarified later in this article.

Before proceeding, we write $R(s, A_e)\, B_e$ in terms of the plant and the controller transfer functions. In $\rho(A)$,

$$\begin{aligned} sI - A_e = &\begin{bmatrix} I & 0 \\ -\mathcal{G}_2 C R(s, A) & I \end{bmatrix} \\ &\times \begin{bmatrix} sI - A & 0 \\ 0 & sI - \mathcal{G}_1 + \mathcal{G}_2 \mathcal{P}(s) K \end{bmatrix} \\ &\times \begin{bmatrix} I & -R(s, A)\, BK \\ 0 & I \end{bmatrix}. \end{aligned}$$

Denote $W(s) = (sI - \mathcal{G}_1 + \mathcal{G}_2 \mathcal{P}(s) K)^{-1}$. By using the above equation, we can calculate

$$R(s, A_e) = \begin{bmatrix} Q_{11}(s) & Q_{12}(s) \\ Q_{21}(s) & Q_{22}(s) \end{bmatrix}, \qquad (12)$$

where

$$\begin{aligned} Q_{11}(s) &= R(s, A)\,(I - BKW(s)\mathcal{G}_2 C R(s, A)), \\ Q_{12}(s) &= R(s, A)\, BKW(s), \\ Q_{21}(s) &= -W(s)\mathcal{G}_2 C R(s, A), \text{ and} \\ Q_{22}(s) &= W(s). \end{aligned}$$

The Woodbury matrix identity to shows that

$$\begin{aligned} W(s) = R(s, \mathcal{G}_1) - &R(s, \mathcal{G}_1)\, \mathcal{G}_2 \\ &\times (I + \mathcal{P}(s)\mathcal{C}(s))^{-1}\, \mathcal{P}(s) K R(s, \mathcal{G}_1), \end{aligned}$$

where

$$\mathcal{P}(s) = C R(s, A)\, B + D \quad \text{and} \quad \mathcal{C}(s) = K R(s, \mathcal{G}_1)\, \mathcal{G}_2.$$

It follows that

$$W(s)\mathcal{G}_2 = R(s, \mathcal{G}_1)\, \mathcal{G}_2\, (I + \mathcal{P}(s)\mathcal{C}(s))^{-1}.$$

Substitute this into (12) to get

$$R(s, A_e)\, B_e = \begin{bmatrix} R(s, A)\, B\mathcal{C}(s)(I + \mathcal{P}(s)\mathcal{C}(s))^{-1} \\ R(s, \mathcal{G}_1)\, \mathcal{G}_2\, (I + \mathcal{P}(s)\mathcal{C}(s))^{-1} \end{bmatrix} F. \quad (13)$$

We can now state the main result of this section.

Theorem 3: Let the assumptions of Section II hold. Write $K = \langle \cdot, k_0 \rangle$ and $\mathcal{G}_2 y = y g_2$ where $k_0, g_2 \in Z$. The Sylvester equation (7) is solvable if and only if

$$\sup_{\|z\|_Z \le 1} \sum_{k \in \mathbb{Z}} \left| \frac{\langle (\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2, z \rangle_Z}{\langle (\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z} \right|^2 \left| \frac{F\phi_k}{\mathcal{P}(\mathrm{i}\omega_k)} \right|^2 < \infty, \tag{14}$$

where ν_k is the order of the pole of $R(s, \mathcal{G}_1)$ at $\mathrm{i}\omega_k$.

Proof: Substituting (13) into (11) and using $\langle (x_1, z_1), (x_2, z_2) \rangle = \langle x_1, x_2 \rangle_X + \langle z_1, z_2 \rangle_Z$ we see that (11) is equivalent to

$$\sup_{\|x\|_X \le 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \lim_{s \to \mathrm{i}\omega_k} W_1(s) F\phi_k, x \right\rangle_X \right|^2 < \infty, \tag{15a}$$

$$\sup_{\|z\|_Z \le 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \lim_{s \to \mathrm{i}\omega_k} W_2(s) F\phi_k, z \right\rangle_Z \right|^2 < \infty, \tag{15b}$$

where $W_1(s) = R(s, A) B \mathcal{C}(s)(I + \mathcal{P}(s)\mathcal{C}(s))^{-1}$ and $W_2(s) = R(s, \mathcal{G}_1) \mathcal{G}_2 (I + \mathcal{P}(s)\mathcal{C}(s))^{-1}$. The limits above exist because $i\omega_k \in \rho(A_e)$.

The next step is to calculate the limits. Fix $k \in \mathbb{Z}$. By [3, Lemma 6.4.], there exists exactly one eigenvector z_1 at $\mathrm{i}\omega_k$. Since Z_k in the second item of Assumption 3 is finite-dimensional, it follows by [10, III-§5] that $Z_k = \operatorname{span}\{z_1, \ldots, z_{\nu_k}\}$ where z_i, $i = 1, \ldots, \nu_k$, are the generalized eigenvectors. Furthermore, in a neighborhood of $\mathrm{i}\omega_k$ the resolvent has the Laurent expansion $R(s, \mathcal{G}_1) = \sum_{i=1}^{\nu_k} \frac{1}{(s - \mathrm{i}\omega_k)^i} (\mathcal{G}_1 - \mathrm{i}\omega_k I)^{i-1} P_k + \sum_{i=0}^{\infty} (s - \mathrm{i}\omega_k)^i R_i$ where the matrices R_i are of no interest.

It is easy to see that $Z = \operatorname{span}\{z_{\nu_k}\} \oplus \mathcal{R}(\mathrm{i}\omega_k I - \mathcal{G}_1)$. For (9a) to hold it is necessary that $g_2 = \alpha z_{\nu_k} + z$, where $z \in \mathcal{R}(\mathrm{i}\omega_k I - \mathcal{G}_1)$ and $\alpha \neq 0$. It follows, that $(\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2 = \alpha z_1$. By [3, Lemma 6.4.], $K(\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2 = \alpha K z_1 \neq 0$.

Since $\mathcal{P}(\mathrm{i}\omega_k) \neq 0$ by Assumption 2, the above arguments show that

$$\lim_{s \to \mathrm{i}\omega_k} W_1(s) = \frac{R(\mathrm{i}\omega_k, A) B}{\mathcal{P}(\mathrm{i}\omega_k)} \tag{16}$$

and

$$\lim_{s \to \mathrm{i}\omega_k} W_2(s) = \frac{(\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2}{\langle (\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z \, \mathcal{P}(\mathrm{i}\omega_k)}. \tag{17}$$

Since $R(\mathrm{i}\omega_k, A)$ exists for all $k \in \mathbb{Z}$, it is seen by (16) and (17) that (15) can be written as

$$\sup_{\|x\|_X \le 1} \sum_{k \in \mathbb{Z}} \left| \left\langle \frac{R(\mathrm{i}\omega_k, A) B}{\mathcal{P}(\mathrm{i}\omega_k)} F\phi_k, x \right\rangle_X \right|^2 < \infty, \tag{18a}$$

$$\sup_{\|z\|_Z \le 1} \sum_{k \in \mathbb{Z}} \left| \frac{\langle (\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2, z \rangle_Z}{\langle (\mathcal{G}_1 - \mathrm{i}\omega_k I)^{\nu_k - 1} P_k g_2, k_0 \rangle_Z} \right|^2 \left| \frac{F\phi_k}{\mathcal{P}(\mathrm{i}\omega_k)} \right|^2 < \infty. \tag{18b}$$

The condition (18b) is (14), so the claim follows if (18a) holds whenever (18b) holds. Substituting $z = k_0$ gives a lower bound $\sum_{k \in \mathbb{Z}} \left| \frac{F\phi_k}{\mathcal{P}(\mathrm{i}\omega_k)} \right|^2$ for the supremum (18b). By the Cauchy-Schwarz inequality, $\sum_{k \in \mathbb{Z}} \left| \frac{F\phi_k}{\mathcal{P}(\mathrm{i}\omega_k)} \right|^2 < \infty$ is a sufficient condition for (18a) to hold. ■

Output regulation of the reference signals generated by (1) with a feedforward controller was studied in [11]. With feedforward controllers the solvability condition (14) was reduced to $\sum_{k \in \mathbb{Z}} |F\phi_k|^2 |\mathcal{P}(\mathrm{i}\omega_k)|^{-2} < \infty$. The reason for this is that here we need to include the exosystem into the feedback controller which is unnecessary with feedforward controllers.

IV. Robustness Analysis

Controllers satisfying Assumption 3 exist [4], [9]. Thus, it makes sense to ask what kind of perturbations a controller solving the robust regulation problem tolerates. In this section, we give an answer to this question.

We allow additive bounded perturbations in the plant parameters A, B, C, and D, and in the reference operator F. We don't allow perturbations in the controller, and we assume that perturbed plants satisfy assumptions of Section II. In addition, we assume that the closed loop system is strongly stable, and that the Sylvester equation (7) is solvable. In other words, we have defined the set $\mathscr{V}$ of perturbed plants given by the following definition.

Definition 1: Let an error feedback controller (4) be given and fix S to be the operator in (2). Fix a strictly positive sequence of real numbers $(f_k)_{k \in \mathbb{Z}} \in \ell^\infty$. The class of perturbed plants and reference operators $\mathscr{V}$ is defined to be the set of all 5-tuples (A', B', C', D', F') for which

1) $F' \in \mathscr{Y}(f_k)$,
2) (A', B') is exponentially stabilizable,
3) (A', C') is exponentially detectable,
4) $\mathcal{P}'(\mathrm{i}\omega_k)$ is invertible for all $k \in \mathbb{Z}$,
5) $i\omega_k \in \rho(A')$,
6) $\mathrm{i}\omega_k \in \rho(A'_e)$,
7) A'_e generates a strongly stable C_0-semigroup,
8) the Sylvester equation (7) is solvable, and
9) the additive perturbation $\triangle A = A' - A$ is bounded.

Here A'_e is the perturbed state operator of the extended plant in (6) and $\mathcal{P}'(s)$ is the plant transfer function of the extended plant.

We do not consider robustness of the properties 1) and 9). Disregarding robustness of 1) is justified, for example, if the reference operator is fixed. Under the property 9) it makes sense to speak about small perturbations, see Theorem 4.

A. Robustness of Properties from 2) to 6)

The following theorem shows, that a robustly regulating controller is robust to small additive perturbations of the plant operators that do not destroy properties from 7) to 9) of the above definition.

Theorem 4: Assume that $(A, B, C, D, F) \in \mathscr{V}$. There exists a constant $\epsilon > 0$ such that $(A', B', C', D', F) \in \mathscr{V}$ if $\|A - A'\|, \|B - B'\|, \|C - C'\|, |D - D'| < \epsilon$ and the properties from 7) to 9) of Definition 1 hold.

Proof: We first note that exponential stability is robust to small additive perturbations, i.e., there exists a constant $\epsilon > 0$ such that any if A generates an exponentially stable C_0-semigroup and $\|A - A'\| < \epsilon$, then the C_0-semigroup generated by A' is exponentially stable [12]. It follows, that the properties 2) and 3) of Definition 1 are robust to small additive perturbations of A, B, C and D.

Since A is exponentially stabilizable and B is a finite-rank operator, we can present A in the form

$$A = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix},$$

where A_+ is an operator in a finite-dimensional space and A_- generates an exponentially stable C_0-semigroup in a subspace of X [12]. The spectrum of an operator generating an exponentially stable semigroup is in some left half plane $\{s \in \mathbb{C} \,|\, \Re(s) < \beta < 0\}$. Since exponential stability is robust to small additive perturbations, and the spectrum of a finite-dimensional controller is continuous, the property 5) of Definition 1 is robust to small additive perturbations.

We next show that the property 6) is robust to small additive perturbations that preserve strong stability. Note that the operator

$$Q = \begin{bmatrix} 0 & BK \\ -\mathcal{G}_2 C & -\mathcal{G}_2 DK \end{bmatrix} \tag{19}$$

is compact as a finite-rank operator. Since (A, B) is exponentially stabilizable, A does not have essential spectrum in the right half-plane $\mathbb{C}_{-\beta} = \{s \in \mathbb{C} \,|\, \Re(s) > -\beta\}$ for some sufficiently small $\beta > 0$. Since A and $\mathcal{G}_1$ do not have essential spectrum in $\mathbb{C}_{-\beta}$, neither does

$$A_e = \begin{bmatrix} A & 0 \\ 0 & \mathcal{G}_1 \end{bmatrix} + \begin{bmatrix} 0 & BK \\ -\mathcal{G}_2 C & -\mathcal{G}_2 DK \end{bmatrix}$$

by [10, Theorem IV.5.35]. Since A_e can be strongly stable only if it does not have spectrum points on the imaginary axis, we have that $\mathrm{i}\omega_k \in \rho(A_e)$ if A_e generates a strongly stable C_0-semigroup. Furthermore, any perturbations of B, C, and D do not affect the essential spectrum, because the perturbed Q of (19) would remain compact. Since $\mathcal{G}_1$ is not perturbed and A remains exponentially stabilizable under small additive perturbations, we have that the property 6) is preserved under small perturbations that preserve property 7).

It remains to show that the property 4) is robust to small additive perturbations preserving the 7) and 8). Since $\mathrm{i}\omega_k \in \rho(A)$ by assumption, we see that (18a) can hold only if $\mathcal{P}(\mathrm{i}\omega_k) \neq 0$. Thus, the property 4) is actually a necessary condition for 8) to hold if $\mathrm{i}\omega_k \in \rho(A)$. The claim follows since 5) is robust to small additive perturbations. ■

B. Robustness of Properties 7) and 8)

Theorem 4 shows that robustness of a controller is essentially dependent on robustness of the properties 7) and 8) of Definition 1. Since we did not use stability of the closed loop in the proof of Theorem 3 we see that the property 8) does not imply the property 7). The next example shows, that there exists a controller that strongly stabilizes a plant satisfying Assumption 3 that does not satisfy the property 8). Thus, both of the properties are needed.

Example 1: We first recall the observer based controller presented in [5], see also [4]. In (4), we choose $Z = X \times W$,

$$\mathcal{G}_1 = \begin{bmatrix} A + BK_1 + L(C + DK_1) & (B + LD)K_2 \\ 0 & S \end{bmatrix},$$

$$\mathcal{G}_2 = \begin{bmatrix} -L \\ G_2 \end{bmatrix}, \text{ and } K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}.$$

The operator L is chosen so that $A + LC$ is exponentially stable. In addition, $G_2 u = u g_2$, where g_2 is chosen so, that $\langle g_2, \phi_k \rangle \neq 0$ for all $k \in \mathbb{Z}$. Let $A + BK_{11}$ be exponentially stable. By [5, Lemma 14], the unique solution Σ_0 to the Sylvester equation

$$S\Sigma_0 = \Sigma_0(A + BK_{11}) + G_2(C + DK_{11})$$

exits. Setting $K_1 = K_{11} + K_2\Sigma_0$, where K_2 is chosen so that $S + (\Sigma_0 B + G_2 D)K_2$ is strongly stable and $\sigma(S) \cap \sigma((\Sigma_0 B + G_2 D)K_2) = \emptyset$, yields a strongly stabilizing controller that satisfies the $\mathcal{G}$-conditions [5, Theorems 12 and 13].

An appropriate operator K_2 was given in [5]. Note that here we do not need the extra assumption of [5] concerning the asymptotic behavior of the plant transfer function that was needed in order to show the solvability of the Sylvester equation (7). Set

$$\mu_k = \mathrm{i}\omega_k - \frac{|F\phi_k|}{a_k} |\langle g_2, \phi_k \rangle|, \tag{20}$$

where $(a_k)_{k\in\mathbb{Z}}$ is an increasing sequence of positive real numbers such that

$$\sum_{k\in\mathbb{Z}} \left| \frac{\mu_k - \mathrm{i}\omega_k}{\langle B_0, \phi_k \rangle} \right|^2 = \sum_{k\in\mathbb{Z}} \frac{|F\phi_k|^2}{a_k^2} |P_K(\mathrm{i}\omega_k)|^2 < \infty, \tag{21}$$

where $\mathcal{P}_K(s) = (C + DK_{11})(sI - A + BK_{11})^{-1}B + D$. By the arguments in [4, Section 3.3.3], choosing $K_2 = \langle \cdot, h \rangle$, where

$$h = \sum_{k\in\mathbb{Z}} h_k \phi_k, \qquad \overline{h}_k = \frac{\mu_k - \mathrm{i}\omega_k}{\langle B_0, \phi_k \rangle} \prod_{\substack{l\in\mathbb{Z} \\ l\neq k}} \frac{\mathrm{i}\omega_k - \mu_l}{\mathrm{i}\omega_k - \mathrm{i}\omega_l}, \tag{22}$$

yields an operator such that $S + B_0 K_2$ is strongly stable and has point spectrum at μ_k, $k \in \mathbb{Z}$.

Assume, that $\frac{|\mathcal{P}(\mathrm{i}\omega_k)|}{|k|} \to 0$, as $|k| \to \infty$. We see that there exists an operator F such that (14) does not hold. For such F the Sylvester equation (7) is not solvable. On the other hand, the sequence $(a_k)_{k\in\mathbb{Z}}$ can be chosen so that (21) holds regardless of the sequence $\left(\frac{|F\phi_k|}{\mathcal{P}_K(\mathrm{i}\omega_k)}\right)_{k\in\mathbb{Z}}$. Thus, there exists extended plants that are strongly stable, but for which the Sylvester equation (7) is not solvable.

Next we consider the robustness of the solvability of the Sylvester equation (7). Theorem 3 shows that it is solely dependent of the asymptotic decay rate of the plant transfer function and smoothness of the reference signals. Thus, a perturbation can make the Sylvester equation unsolvable only if it makes the plant transfer function to decay faster. The theorem shows that there is a trade-off between the generality of the reference signal class and the robustness of the solvability, since by choosing $(f_k)_{k\in\mathbb{Z}}$ in (10) that decays fast restricts the class of all possible reference signals, but allows a wider class of perturbations that make the plant transfer function to decay faster. This is illustrated by the following example. It also shows that an arbitrarily small additive perturbation may make the Sylvester equation unsolvable. Thus, the robustness properties of the solvability of the Sylvester equation are generally weak.

Fig. 2. A heated bar with non-collocated heater and sensor.

Example 2: The heated bar of Figure 2 have one heater u and one measurement y, and there is some uncertainty in the length of the heated interval ($-\frac{1}{3} < \delta < \frac{1}{3}$). We model heat transfer of the bar by (3), where

$$\begin{aligned}
&Ax(z,t) = \frac{d^2x}{dz^2}(z,t) - x(z,t), \text{with}\\
&\mathcal{D}\left(A\right) = \left\{ h \in \mathbf{L}_2(0,1) \,\middle|\, h, \frac{dh}{dz} \text{ absolutely continuous,}\right.\\
&\qquad\qquad \left. \frac{d^2h}{dz^2} \in \mathbf{L}_2(0,1), \frac{dh}{dz}(0) = \frac{dh}{dz}(1) = 0 \right\},\\
&(Bu)(z,t) = \mathbf{1}_{[\frac{2}{3}+\delta,1]}(z)u(t),\\
&y(t) = Cx(z,t) = \int_0^1 \mathbf{1}_{[0,\frac{1}{3}]}(z)x(z,t)dz
\end{aligned}$$

with the state space $X = \mathbf{L}_2(0,1)$. Function $\mathbf{1}_{[a,b]}(z)$ is the indicator function of the interval $[a,b] \subseteq [0,1]$.

The transfer function of the system is

$$\mathcal{P}(s) = \frac{\sinh\left(\frac{1}{3}\sqrt{s+1}\right)\sinh\left(\left(\frac{1}{3}-\delta\right)\sqrt{s+1}\right)}{\sqrt{s+1}(s+1)\sinh\left(\sqrt{s+1}\right)}.$$

At high frequencies $\mathcal{P}(s)$ behaves approximately as $\frac{1}{(s+1)\sqrt{s+1}}\mathrm{e}^{-\left(\frac{1}{3}+\delta\right)\sqrt{s+1}}$. It is seen, that an arbitrarily small perturbation in the length of the heated interval changes the rate of convergence. The change in the decay rate between the perturbed and the original plant transfer functions is $\mathrm{e}^{-\delta\sqrt{s+1}}$. Thus, even a small perturbation changes the decay rate by an exponential degree.

Choose $\omega_k = k$, $k \in \mathbb{Z}$, and consider the Sylvester equation (7). If we choose $f_k = \left|\frac{1}{(k+1)^2\sqrt{k+1}}\mathrm{e}^{-\left(\frac{1}{3}\right)\sqrt{k+1}}\right|$ in (10), then the Sylvester equation is solvable for every $F \in \mathscr{Y}(f_k)$ by Theorem 3. However, any perturbation such that $\delta > 0$ makes the Sylvester equation unsolvable. Thus, we may want to add some extra smoothness to the reference signals by choosing $f_k = \left|\frac{1}{(k+1)^2\sqrt{k+1}}\mathrm{e}^{-\left(\frac{1}{3}+\epsilon\right)\sqrt{k+1}}\right|$, where $\frac{1}{3} \geq \epsilon > 0$. The Sylvester equation is now solvable whenever $\delta \leq \epsilon$, so the solvability is robust to small perturbations in the location of the heater.

Robustness of strong stability is not studied in this article. We just note that an arbitrarily small additive perturbation can make a strongly stable system unstable [1, Example 6.5], so strong stability has weak robustness properties. However, polynomial stability has relatively good robustness properties [9], and it is achievable if the transfer function has a polynomial decay rate [13].

V. Conclusions

We studied robustness of robustly regulating controllers. We showed, that they are robust to perturbations that preserve solvability of the Sylvester equation (7) and strong stability. Both of these properties can be destroyed by an arbitrarily small additive perturbation, so in this sense controllers have only weak robustness properties. Robustness of solvability was shown to be fully dependent on the changes in the decay rate of the plant transfer function.

References

[1] E. Immonen, "State space output regulation theory for infinite-dimensional linear systems and bounded uniformly continuous exogenous signals," Ph.D. dissertation, Tampere University of Technology, Tampere, Finland, 2006.

[2] T. Hämäläinen and S. Pohjolainen, "Robust regulation of distributed parameter systems with infinite-dimensional exosystems," *SIAM Journal on Control and Optimization*, vol. 48, no. 8, pp. 4846–4873, 2010.

[3] L. Paunonen and S. Pohjolainen, "Internal model theory for distributed parameter systems," *SIAM Journal of Control and Optimization*, vol. 48, no. 7, pp. 4753–4775, 2010.

[4] P. Laakkonen, "Robust regulation for infinite-dimensional systems and signals in the frequency domain," Ph.D. dissertation, Tampere University of Technology, Tampere, Finland, 2013. [Online]. Available: http://urn.fi/URN:ISBN:978-952-15-3178-1

[5] L. Paunonen and S. Pohjolainen, "Robust controller design for infinite-dimensional exosystems," *International Journal of Robust and Nonlinear Control*, no. 4, pp. 702–715, 2012. [Online]. Available: dx.doi.org/10.1002/rnc.2920

[6] E. Immonen, "On the internal model structure for infinite-dimensional systems: Two common controller types and repetitive control," *SIAM Journal of Control and Optimization*, vol. 45, no. 6, pp. 2065–2093, 2007.

[7] C. Byrnes, I. Laukǒ, D. Gilliam, and V. Shubov, "Output regulation for linear distributed parameter systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 12, pp. 2236–2252, 2000.

[8] B. Francis and W. Wonham, "The internal model principle for linear multivariable regulators," *Applied Mathematics & Optimization*, vol. 2, no. 2, pp. 170–194, 1975.

[9] L. Paunonen, "Robustness of polynomial stability with respect to unbounded perturbations," *Systems & Control Letters*, vol. 62, pp. 331–337, 2013.

[10] T. Kato, *Perturbation Theory for Linear Operators*. Springer-Verlag, New York, 1966.

[11] P. Laakkonen and S. Pohjolainen, "Solvability of the output regulation problem with a feedforward controller," in *Proceedings of 12th European Control Conference (ECC) 2013*, Zürich, Switzerland, July 17-19 2013.

[12] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag, New York, 1995.

[13] L. Paunonen and S. Pohjolainen, "Output regulation for general infinite-dimensional exosystems," in *Proceedings of the 7th IFAC Symposium on Robust Control Design, Rocond 2012*, Aalborg, Denmark, June 20-22 2012, pp. 162–167.