Robust Regulation for Exponentially Stable Boundary Control Systems in Hilbert Space

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Abstract

Finite-dimensional robust control of exponentially stable infinite-dimensional linear systems in Hilbert space will be considered. Both distributed and boundary controls are present. Distributed, boundary and measurement disturbances are allowed. It is shown that signals that are linear combinations of sinusoids with polynomial coefficients can be asymptotically tracked and asymptotically rejected, by a finite dimensional, low gain, robust controller, which can easily be tuned experimentally.

I. Introduction

Consider the control system in Fig. 1 where the stable plant P is described by the equations

$$\dot{z}(t) = A_0 z(t) + B_d u_d(t) + E_d w_d(t), \tag{1a}$$

$$S_b z(t) = B_b u_b(t) + E_b w_b(t), \tag{1b}$$

$$y(t) = Cz(t) + D_d u_d(t) + D_b u_b(t) + E_m w_m(t), \tag{1c}$$

for $t \geq 0$ and with the initial condition $z(0) = z_0 \in \mathcal{D}(A_0)$. Equations (1) model a large class of distributed parameter systems, including parabolic and hyperbolic partial differential equations with boundary-distributed control and disturbances [1]. The advantage of the time domain formulation (1) over the frequency domain formulation for example in [2] is that the disturbance signals w_d and w_b and the control signals $u = (u_d, u_b)$ (see Fig. 1) can directly enter the state, instead of entering at the control signal u as in [2]. This is important in many practical situations with spatially distributed sources, e.g., in acoustics where the disturbances are sources of sound inside a spatial domain or at the boundary of the domain, and the control may enter inside the domain or at the boundary using for example smart materials. Note also that in the state space formulation a disturbance at the control signal u can easily be incorporated into w_d and w_b, but the converse is usually not possible.

The control objective is to design a finite-dimensional, low-gain, controller C_ε that stabilizes the closed-loop system and achieves robust regulation, by which we mean that the controller exponentially asymptotically tracks the reference signal r, exponentially asymptotically rejects the disturbance signals w_d, w_b and w_m, and tolerates some perturbations in the plant and the controller. The reference signal r and disturbance signal w are assumed to be of the general form

$$\sum_{k=-n}^{n} a_k(t) e^{i\omega_k t}, \tag{2}$$

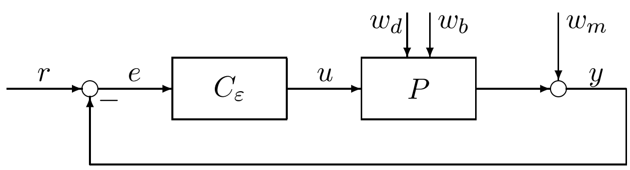

Fig. 1. The closed-loop system with the stable plant P, the reference signal r, the disturbance signals w_d, w_b and w_m, and the controller C_ε.

where the coefficients $a_k(t)$ are polynomials with known degrees but possibly unknown coefficients, and the ω_k are known frequencies. Signals of the form (2) include the most commonly used signals in applications, such as sinusoids, constants and ramps. Periodic signals which have Fourier series can also be approximated by signals of the form (2).

The low gain robust regulator problem for infinite-dimensional systems has been treated by many authors: Pohjolainen in [1], the authors in [3] and [2], Xu and Jerbi [4], Logemann and Townley [5] and Rebarber and Weiss [6]. In [4] the reference and disturbance signals are constants and the plant is described by equations (1) without the boundary term but the output operator C is allowed to be unbounded. In [5] the reference and disturbance signals are constants but the plant is in the class of stabe well-posed regular systems, which is more general than that described by equations (1). In [6] Rebarber and Weiss give a partial generalization of [2]. The reference and disturbance signals are of the form (2) but with constant coefficients, and the plant is in the class of stable well-posed systems, which is larger than the class of regular systems. Due to the general nature of the systems they consider, they achieve tracking only in the sense that the exponentially weighted error signal e is in $L^2[0,\infty)$ (see Fig. 1). This means that the error signal e might not go to zero, but its energy will decay rapidly. Although there might not be much difference in practice, from an engineering point of view it is preferable to have guaranteed rate of decay for the error signal, which exponentially asymptotic tracking provides.

This paper generalizes the robust regulation results for the boundary control system (1), first given by Pohjolainen for constant reference and disturbance signals in [1], to the more general class (2). In [1] Pohjolainen proved robust regulation for the boundary control system (1) (without the feedthrough and output disturbance terms) in Banach space for constant reference and disturbance signals. The authors have earlier given partial extensions to [1]. In [3] the class of reference and disturbance signals was extended to linear combinations of constants and sinusoids, but the class of plants was smaller, consisting of plants described by equations (1) without the boundary term and with the assumption that A_0 is the generator of an analytic semigroup. Using frequency domain methods in [2] the authors further extended the class of reference and disturbance signals to signals of the form (2), but the plant was assumed to be in the CD-algebra, which is not as large as (1). In this paper we show that the finite-dimensional controller proposed by the authors in [2] achieves robust regulation for the larger class of plants (1). A distinguishing feature of the controller is that apart from a small positive tuning parameter ε the controller is completely fixed by the values $P(i\omega_k)$ of the plant transfer matrix at the reference and disturbance signal frequencies. Hence the controller can be tuned with frequency response measurements made from the original stable plant [7].

The structure of the paper is as follows: In Section II-B the boundary controls and disturbances are transformed into the state equation. Because our controls are sufficiently smooth in time, in Section III-A we are able to replace the time derivatives of the controls as state feedback, and are led to an extended system consisting of the plant and the controller. In Section III-B the stability of the extended system is proved by showing that the resolvent of the extended system operator is in H^∞. This leads to a simpler proof than trying to generalize the perturbation theory approach used in [1]. The drawback is that the results are valid only in a Hilbert space. In Section III-C it is shown that the stability of the closed-loop system implies tracking. Finally in Section IV boundary control of a one-dimensional damped wave equation is given as an example to illustrate the theory.

II. Problem Formulation

A. Notation

$\mathcal{L}(X,Y)$ is the set of bounded linear operators from the normed space X to the normed space Y. The domain, range, null space, and spectrum of a linear operator A are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, and $\sigma(A)$, respectively. The C_0-semigroup $T_A(t)$ generated by A is exponentially stable if there are positive constants M and α such that $\|T_A(t)\| \leq Me^{-\alpha t}$ for $t \geq 0$. In this case we also say that A is exponentially stable. A polynomial is stable if all its roots have negative real parts.

B. The Plant

Let the plant be described by the equations (1). The state of the system $z(t) \in X$, the distributed control $u_d(t) \in U_d$, the distributed disturbance $w_d(t) \in W_d$, the boundary control $u_b(t) \in U_b$, the boundary disturbance $w_b(t) \in W_b$, the output disturbance $w_m(t) \in W_m$, the measurement $y(t) \in Y$, where X is a Hilbert space and U_d,

W_d, U_b, W_b, W_m, Y are finite-dimensional spaces, with $\dim U_d = m_d$, $\dim W_d = l_d$, $\dim U_b = m_b$, $\dim W_b = l_b$, $\dim W_m = l_m$, $\dim Y = p$.

We also define $U = U_d \times U_b$, $W = W_d \times W_b \times W_m$ and $u(t) = (u_d(t), u_b(t)) \in U$, $w(t) = (w_d(t), w_b(t), w_m(t)) \in W$.

The system operator $A_0 : X \supset \mathcal{D}(A_0) \to X$ is assumed to be linear and closed, and the boundary operator $S_b : X \supset \mathcal{D}(S_b) \to H$, where H is a Hilbert space, is assumed to satisfy $\mathcal{D}(A_0) \subset \mathcal{D}(S_b)$.

Additionally the control operators $B_d \in \mathcal{L}(U_d, X)$, $B_b \in \mathcal{L}(U_b, H)$, the disturbance operators $E_d \in \mathcal{L}(W_d, X)$, $E_b \in \mathcal{L}(W_b, H)$, $E_m \in \mathcal{L}(W_m, Y)$, the measurement operator $C \in \mathcal{L}(X, Y)$, and the feedthrough operators $D_d \in \mathcal{L}(U_d, Y)$, $D_b \in \mathcal{L}(U_b, Y)$.

We also need the following assumptions

1) The operator $A : X \supset \mathcal{D}(A) \to X$, with $\mathcal{D}(A) = \mathcal{D}(A_0) \cap \mathcal{N}(S_b)$ and $Ax = A_0 x$ for $x \in \mathcal{D}(A)$, is assumed to be the infinitesimal generator of an exponentially stable C_0 semigroup.
2) There are operators $G_u \in \mathcal{L}(U_b, X)$ and $G_w \in \mathcal{L}(W_b, X)$ such that $\mathcal{R}(G_u) \subset \mathcal{D}(A_0)$, $\mathcal{R}(G_w) \subset \mathcal{D}(A_0)$, and $S_b G_u = B_b$, $S_b G_w = E_b$.

We also define $N_u = A_0 G_u \in \mathcal{L}(U_b, X)$ and $N_w = A_0 G_w \in \mathcal{L}(W_b, X)$.

Following [8] we next transform the boundary controls and disturbances into the state equations by defining $x = z - G_u u_b - G_w w_b$. If $z(t) \in \mathcal{D}(A_0)$, then $x(t) \in \mathcal{D}(A_0)$ and $S_b x(t) = 0$. Hence $x(t) \in \mathcal{D}(A)$ and assuming that u_b and w_b are differentiable we get the new state equation

$$\dot{x} = Ax + B_d u_d + N_u u_b - G_u \dot{u}_b + E_d w_d + N_w w_b - G_w \dot{w}_b. \tag{3}$$

It can be shown that if z is a solution of (1), then x is a solution of (3),and conversely [8]. We show later that u and $\dot{u}_b$ are of class C^1. Since A generates a C_0 semigroup and w is smooth, this guarantees that equation (3), and consequently (1), has a unique solution. The output y is given by

$$y = Cx + D_d u_d + (CG_u + D_b) u_b + CG_w w_b + E_m w_m.$$

It is easily seen that the transfer matrices from u to y and from w to y can be written as

$$P_u(s) = C(sI - A)^{-1} B_u(s) + D_u, \qquad P_w(s) = C(sI - A)^{-1} B_w(s) + D_w,$$

where $D_u = \begin{bmatrix} D_d, & CG_u + D_b \end{bmatrix}$, $D_w = \begin{bmatrix} 0, & CG_w, & E_m \end{bmatrix}$, $B_u(s) = \begin{bmatrix} B_d, & N_u - sG_u \end{bmatrix}$ and $B_w(s) = \begin{bmatrix} E_d, & N_w - sG_w, & 0 \end{bmatrix}$. Then the state-equation and the output can be written in the form

$$\dot{x} = Ax + B_u(0)u - G_u \dot{u}_b + B_w(0) w - G_w \dot{w}_b, \tag{4a}$$
$$y = Cx + D_u u + D_w w. \tag{4b}$$

C. Reference and Disturbance Signals

The reference and disturbance signals are assumed to be of the form (2), where the frequencies ω_k are known and the coefficients $a_k(t)$ are polynomials of known degree $m_k - 1$ but possibly unknown coefficients.

D. The Controller

For $\varepsilon > 0$ we define the controller C_ε by

$$C_\varepsilon(s) = \sum_{k=-n}^{n} C_{\varepsilon,k}(s), \tag{5}$$

where

$$C_{\varepsilon,k}(s) = \frac{\varepsilon^{m_k} K_k((s - i\omega_k)/\varepsilon)}{(s - i\omega_k)^{m_k}}, \qquad k = -n, \ldots, n, \tag{6}$$

and

$$K_k(s) = \sum_{l=0}^{m_k - 1} K_{kl} s^l, \qquad K_{kl} \in \mathbb{C}^{m \times p}, \qquad k = -n, \ldots, n, \tag{7}$$

where $m = m_d + m_b$. We assume that the polynomial matrices $K_k(s)$ satisfy the stability conditions

$$\det(s^{m_k} I_p + P_u(i\omega_k) K_k(s)) \quad \text{is stable for } k = -n, \ldots, n. \tag{8}$$

Note that the conditions (8) are robust in the sense that they allow perturbations in the $P_u(i\omega_k)$ and the matrix gains K_{kl} (the coefficients of $K_k(s)$). This is especially important for the values $P_u(i\omega_k)$ since in practice they have to be measured from the plant. Then robustness guarantees that the measurements do not have to be very precise.

The next theorem gives a state-space realization for the controller (5), which is minimal if the stability condition (8) is satisfied.

Theorem 1:

(a) Let $\varepsilon > 0$ and $q_k = pm_k$. The controller $C_{\varepsilon,k}$, defined in (6), has the state-space realization

$$A_{c,k} = \begin{bmatrix} i\omega_k I_p & I_p & 0 & \ldots & 0 \\ 0 & i\omega_k I_p & I_p & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & i\omega_k I_p & I_p \\ 0 & \ldots & 0 & 0 & i\omega_k I_p \end{bmatrix} \in \mathbb{C}^{q_k \times q_k}, \qquad B_{c,k} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_p \end{bmatrix} \in \mathbb{C}^{q_k \times p},$$

$$C_{c,k} = \begin{bmatrix} \varepsilon^{m_k} K_{k0} & \varepsilon^{m_k - 1} K_{k1} & \ldots & \varepsilon K_{k,m_k-1} \end{bmatrix} \in \mathbb{C}^{m \times q_k},$$

The realization is controllable and is observable if $\operatorname{rank} K_{k0} = p$.

(b) Let $q = \sum_{k=-n}^{n} q_k$. The controller C_ε, defined in (5), has the state-space realization

$$A_c = \operatorname{diag}(A_{c,-n}, \ldots, A_{c,n}) \in \mathbb{C}^{q \times q}, \quad B_c = \begin{bmatrix} B_{c,-n}^T & \ldots & B_{c,n}^T \end{bmatrix}^T \in \mathbb{C}^{q \times p}, \quad C_c = \begin{bmatrix} C_{c,-n} & \ldots & C_{c,n} \end{bmatrix} \in \mathbb{C}^{m \times q},$$

The realization is controllable. It is observable if $\operatorname{rank} K_{k0} = p$ for each $k = -n, \ldots, n$.

Proof:

(a) It is easily seen that $(sI - A_{c,k})^{-1} B_{c,k} = (s - i\omega_k)^{-m_k} \begin{bmatrix} I_p & (s - i\omega_k) I_p & \ldots & (s - i\omega_k)^{m_k - 1} I_p \end{bmatrix}^T$. Hence $C_{c,k}(sI - A_{c,k})^{-1} B_{c,k} = C_{\varepsilon,k}(s)$, and $(A_{c,k}, B_{c,k}, C_{c,k})$ is a realization of $C_{\varepsilon,k}(s)$. Clearly the matrix $\begin{bmatrix} i\omega_k I - A_{c,k}, & B_{c,k} \end{bmatrix}$ has full rank, which proves the controllability of the realization. To prove observability, we show that the matrix

$$F = \begin{bmatrix} i\omega_k I - A_{c,k} \\ C_{c,k} \end{bmatrix}$$

has full rank, or equivalently, trivial nullspace. Let $x = (x_0, \ldots, x_{m_k - 1}) \in \mathbb{C}^{q_k}$. Then the equation $Fx = 0$ implies that $x_l = 0$ for $l = 1, \ldots, m_k - 1$ and $K_{k0} x_0 = 0$. Clearly the equation $K_{k0} x_0 = 0$ has only the trivial solution iff $\operatorname{rank} K_{k0} = p$. Hence the realization is observable iff $\operatorname{rank} K_{k0} = p$.

(b) Because of the block-diagonal structure of A_c and the fact that $\sigma(A_{c,k}) \cap \sigma(A_{c,l}) = \emptyset$ for $k \neq l$, the claim is easily proved with similar reasoning as in (a).

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Substituting $s = 0$ into condition (8) gives $\det(P_u(i\omega_k) K_k(0)) = \det(P_u(i\omega_k) K_{k0}) \neq 0$, which implies that $\operatorname{rank} K_{k0} = p$ for $k = -n, \ldots, n$. Hence we have the Corollary

Corollary 2: If the controller (5) satisfies stability conditions (8), then the realization given in Theorem 1 is minimal.

III. Main Results

In this section we show that the controller given in Section II-D stabilizes the plant given by equations (4). In Section III-A we give the closed loop equations, in Section III-B we show that the controller stabilizes the plant, and in Section III-C we show that stabilization implies tracking, i.e., if the controller stabilizes the plant then it will also asymptotically track the reference signals and asymptotically reject the disturbance signals.

A. The Closed Loop System

According to Section II-D the controller has the state-space representation

$$\dot{\xi} = A_c \xi + B_c e, \tag{9a}$$
$$u = C_c \xi, \tag{9b}$$

where A_c, B_c, and C_c were defined in Theorem 1, $e = r - y$ is the error signal, and C_c is partitioned as

$$C_c = \begin{bmatrix} C_{c,d} \\ C_{c,b} \end{bmatrix}, \qquad C_{c,d} \in \mathbb{C}^{m_d \times q},\ C_{c,b} \in \mathbb{C}^{m_b \times q}.$$

Define the extended state-space $X_e = X \times \mathbb{C}^q$, and let $x_e(t) = (x(t), \xi(t)) \in X_e$ be the extended state. Using equations (4a) and (9a) we get the state-equation for x_e

$$\begin{aligned} \dot{x}_e &= \begin{bmatrix} Ax + B_u(0)u - G_u\dot{u}_b + B_w(0)w - G_w\dot{w}_b \\ A_c\xi + B_c r - B_c C x - B_c D_u u - B_c D_w w \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ -B_c C & A_c \end{bmatrix} x_e + \begin{bmatrix} B_u(0) \\ -B_c D_u \end{bmatrix} u + \begin{bmatrix} -G_u \\ 0 \end{bmatrix} \dot{u}_b + \begin{bmatrix} B_w(0) \\ -B_c D_w \end{bmatrix} w + \begin{bmatrix} -G_w \\ 0 \end{bmatrix} \dot{w}_b + \begin{bmatrix} 0 \\ B_c \end{bmatrix} r. \end{aligned} \tag{10}$$

When the control loop is closed we have from equation (9b) $u = C_c\xi$ and $u_b = C_{c,b}\xi$ and hence

$$\dot{u}_b = C_{c,b}A_c\xi + C_{c,b}B_c e = \begin{bmatrix} -C_{c,b}B_c C & C_{c,b}(A_c - B_c D_u C_c) \end{bmatrix} x_e - C_{c,b}B_c D_w w + C_{c,b}B_c r \tag{11}$$

Substituting (9b) and (11) into (10), we get the closed loop system

$$\dot{x}_e = A_e x_e + Ew + F\dot{w}_b + Dr, \tag{12}$$

where $\mathcal{D}(A_e) = \mathcal{D}(A) \times \mathbb{C}^q$ and

$$\begin{aligned} A_e &= \begin{bmatrix} A + \tilde{G}_u C_c B_c C & B_u(0)C_c - \tilde{G}_u C_c(A_c - B_c D_u C_c) \\ -B_c C & A_c - B_c D_u C_c \end{bmatrix} \\ E &= \begin{bmatrix} B_w(0) + \tilde{G}_u C_c B_c D_w \\ -B_c D_w \end{bmatrix}, \qquad F = \begin{bmatrix} -G_w \\ 0 \end{bmatrix}, \qquad D = \begin{bmatrix} -\tilde{G}_u C_c B_c \\ B_c \end{bmatrix}, \end{aligned}$$

and $\tilde{G}_u = \begin{bmatrix} 0 & G_u \end{bmatrix}$. We can write A_e in the form $A_e = A_1 + A_2$, where

$$A_1 = \begin{bmatrix} A & 0 \\ 0 & A_c \end{bmatrix}, \qquad A_2 = \begin{bmatrix} \tilde{G}_u C_c B_c C & B_u(0)C_c - \tilde{G}_u C_c(A_c - B_c D_u C_c) \\ -B_c C & -B_c D_u C_c \end{bmatrix}.$$

Clearly A_1 generates a C_0-semigroup and since A_2 is bounded, standard results in perturbation theory show that A_e is also a generator of a C_0-semigroup [8]. Hence, since r and w are smooth, equation (12) has a unique continuously differentiable solution. Then it follows from equation (11) that $\dot{u}_b$ is of class C^1 and from $u = C_c\xi$ that u is also of class C^1. This justifies the claim made in Section II-B about the solution of equation (4a).

Substituting (9b) into (4b), we get the output

$$y = Cx + D_u C_c \xi + D_w w = \begin{bmatrix} C & D_u C_c \end{bmatrix} x_e + D_w w = C_e x_e + D_w w, \tag{13}$$

where $C_e = \begin{bmatrix} C & D_u C_c \end{bmatrix}$.

B. Stability of the Closed Loop System

We prove stability of A_e by using the well-known fact [8] that A_e is stable iff $(sI - A_e)^{-1}$ is analytic and bounded in $\operatorname{Re} s > 0$. Define

$$L(s) = (sI - A_c + B_c P_u(s) C_c)^{-1}. \tag{14}$$

Then $(sI - A_e)^{-1}$ is given by

$$(sI - A_e)^{-1} = \begin{bmatrix} X_{11}(s) & X_{12}(s) \\ X_{21}(s) & X_{22}(s) \end{bmatrix},$$

where

$$X_{11}(s) = -(sI-A)^{-1}B_u(s)C_cL(s)B_cC(sI-A)^{-1} + (sI-A)^{-1},$$
$$X_{12}(s) = (sI-A)^{-1}B_u(s)C_cL(s)(I - B_cC(sI-A)^{-1}\tilde{G}_uC_c) + (sI-A)^{-1}\tilde{G}_uC_c,$$
$$X_{21}(s) = -L(s)B_cC(sI-A)^{-1},$$
$$X_{22}(s) = L(s)(I - B_cC(sI-A)^{-1}\tilde{G}_uC_c).$$

Because A is exponentially stable by assumption, the resolvent $(sI-A)^{-1}$ is analytic and bounded in $\operatorname{Re} s > 0$. Hence if we can show that $L(s)$ and $B_u(s)C_cL(s)$ are analytic and bounded in $\operatorname{Re} s > 0$, then $(sI-A_e)^{-1}$ is also analytic and bounded in $\operatorname{Re} s > 0$ and A_e is exponentially stable. This will be done with the next theorem.

Theorem 3: If the controller satisfies the stability condition (8), then there is an $\varepsilon^* > 0$ such that $L(s)$ defined by equation (14) exists and $L(s)$ and $B_u(s)C_cL(s)$ are analytic and bounded in $\operatorname{Re} s > 0$ for every $\varepsilon \in (0, \varepsilon^*)$.

Proof: Clearly $B_u(s)$ (defined in Section II-B) and $P_u(s)$ are analytic in $\operatorname{Re} s > 0$. Hence the elements of the matrix $sI - A_c + B_cP_u(s)C_c$ are analytic in $\operatorname{Re} s > 0$. Therefore $L(s)$ is analytic in $\operatorname{Re} s > 0$ if it exists for every $\operatorname{Re} s > 0$. Hence we only need to prove that $L(s)$ exists and $L(s)$ and $B_u(s)C_cL(s)$ are bounded in $\operatorname{Re} s > 0$. We assume throughout the proof that $\varepsilon \in (0,1)$. Then it is easily seen that $\|C_c\| \le \varepsilon M_C$ for some constant $M_C > 0$. We also define $d = \min_{j \neq k}|\omega_j - \omega_k|$.

First let k be fixed and define $\mathbb{B}_k = \{\, s \in \mathbb{C} \mid |s - i\omega_k| < r_k \text{ and } \operatorname{Re} s > 0 \,\}$, where $0 < r_k < d/2$. Clearly $B_u(s)$ and $P_u(s)$ are bounded in $\mathbb{B}_k$ and $\|P_u(s)\| \le M_{P,k}$ for all $s \in \mathbb{B}_k$ and some constant $M_{P,k} > 0$. Hence we only need to show that $L(s)$ is bounded in $\mathbb{B}_k$.

Partition A_c, B_c and C_c as $A_c = \operatorname{diag}(A_{c,a}, A_{c,k}, A_{c,b})$, $B_c^T = \begin{bmatrix} B_{c,a}^T & B_{c,k}^T & B_{c,b}^T \end{bmatrix}$ and $C_c = \begin{bmatrix} C_{c,a} & C_{c,k} & C_{c,b} \end{bmatrix}$. Also define $\tilde{A}_{c,k} = A_{c,k} - B_{c,k}P_u(i\omega_k)C_{c,k}$, $\tilde{A}_c = \operatorname{diag}(A_{c,a}, \tilde{A}_{c,k}, A_{c,b})$, $\tilde{B}_c^T = \begin{bmatrix} B_{c,a}^T & 0 & B_{c,b}^T \end{bmatrix}$, $\tilde{C}_c = \begin{bmatrix} 0 & P_u(i\omega_k)C_{c,k} & 0 \end{bmatrix}$, and $A(s) = B_c \begin{bmatrix} P_u(s)C_a & (P_u(s) - P_u(i\omega_k))C_{c,k} & P_u(s)C_b \end{bmatrix}$. Then

$$sI - A_c + B_cP_u(s)C_c = sI - \tilde{A}_c + \tilde{B}_c\tilde{C}_c + A(s).$$

It is shown in the Appendix that $sI - \tilde{A}_c$ is invertible, and that there is an $M_1(\varepsilon) > 0$ such that $\|(sI - \tilde{A}_c)^{-1}\| \le M_1(\varepsilon)$ and that there are constants $M_k > 0$ and $M_{A,c} > 0$ such that $\|C_{c,k}(sI - \tilde{A}_{c,k})^{-1}\| \le M_k$ and $\|(sI - A_{c,a})^{-1}\| \le M_{A,c}$, $\|(sI - A_{c,b})^{-1}\| \le M_{A,c}$. It is easily seen that

$$\|\tilde{C}_c(sI - \tilde{A}_c)^{-1}\| = \|P_u(i\omega_k)C_{c,k}(sI - \tilde{A}_{c,k})^{-1}\| \le M_k\|P_u(i\omega_k)\|.$$

Using the formula $(A + BC)^{-1} = A^{-1} - A^{-1}B(CA^{-1}B + I)^{-1}CA^{-1}$ and the fact that $\tilde{C}_c(sI - \tilde{A}_c)^{-1}\tilde{B}_c = 0$ we see that $(sI - \tilde{A}_c + \tilde{B}_c\tilde{C}_c)^{-1}$ exists and is given by

$$(sI - \tilde{A}_c + \tilde{B}_c\tilde{C}_c)^{-1} = (sI - \tilde{A}_c)^{-1}(I - \tilde{B}_c\tilde{C}_c(sI - \tilde{A}_c)^{-1}), \tag{15}$$

which also shows that

$$\|(sI - \tilde{A}_c + \tilde{B}_c\tilde{C}_c)^{-1}\| \le M_1(\varepsilon)(1 + M_k\|\tilde{B}_c\|\|P_u(i\omega_k)\|). \tag{16}$$

Therefore

$$L(s) = (sI - \tilde{A}_c + \tilde{B}_c\tilde{C}_c)^{-1}(I + A(s)(sI - \tilde{A}_c + \tilde{B}_c\tilde{C}_c)^{-1})^{-1},$$

provided that $\|A(s)(sI - \tilde{A}_c + \tilde{B}_c\tilde{C}_c)^{-1}\| < 1$. From equation (15) we get

$$\|A(s)(sI - \tilde{A}_c + \tilde{B}_c\tilde{C}_c)^{-1}\| \le \|A(s)(sI - \tilde{A}_c)^{-1}\|(1 + M_k\|\tilde{B}_c\|\|P_u(i\omega_k)\|), \tag{17}$$

where

$$\begin{aligned} \|A(s)(sI - \tilde{A}_c)^{-1}\| &\le \|B_c\|(\|P_u(s)C_{c,a}(sI - A_{c,a})^{-1}\| + \|(P_u(s) - P_u(i\omega_k))C_{c,k}(sI - \tilde{A}_{c,k})^{-1}\| \\ &\quad + \|P_u(s)C_{c,b}(sI - A_{c,b})^{-1}\|) \\ &\le 2\varepsilon M_{P,k}M_CM_{A,c}\|B_c\| + M_k\|B_c\|\|P_u(s) - P_u(i\omega_k)\|. \end{aligned} \tag{18}$$

Because P_u is continuous at $i\omega_k$, $\|P_u(s) - P_u(i\omega_k)\|$ can be made arbitrarily small by choosing r_k sufficiently small. Now it follows from inequalities (17) and (18) that $\|A(s)(sI - \tilde{A}_c + \tilde{B}_c\tilde{C}_c)^{-1}\| < 1$ when ε and r_k are sufficiently small. Hence $(I + A(s)(sI - \tilde{A}_c + \tilde{B}_c\tilde{C}_c)^{-1})^{-1}$ exists and is bounded in $\mathbb{B}_k$, which combined with (16) shows that $L(s)$ is bounded for $s \in \mathbb{B}_k$. Because k was arbitrary, $L(s)$ is bounded in $\cup_{k=-n}^{n}\mathbb{B}_k$.

Next let $r = \min_{-n \le k \le n} r_k$ and assume that $s \in \mathbb{B}_c = \{\, s \in \mathbb{C} \mid \operatorname{Re} s > 0 \text{ and } |s - i\omega_k| \ge r,\ k = -n, \dots, n \,\}$ Then $sI - A_c$ is invertible and

$$L(s) = (sI - A_c)^{-1}(I + B_c P_u(s) C_c (sI - A_c)^{-1})^{-1}.$$

Clearly there is a constant $M_r > 0$ such that $\|(sI - A_{c,k})^{-1}\| \le M_r / |s - i\omega_k|$ for $k = -n, \dots, n$ and thus $\|(sI - A_c)^{-1}\|$ is bounded for $s \in \mathbb{B}_c$. From the inequalities $\|B_u(s)\| \le \|B_d\| + \|N_u\| + |s|\|G_u\|$ and $\|C_{c,k}\| \le \varepsilon M_C$ we get

$$\|B_u(s) C_c (sI - A_c)^{-1}\| \le \sum_{k=-n}^{n} \|B_u(s)\| \|C_{c,k}(sI - A_{c,k})^{-1}\| \le \sum_{k=-n}^{n} \|B_u(s)\| \frac{\varepsilon M_C M_r}{|s - i\omega_k|} \le \varepsilon M_B, \tag{19}$$

for some constant $M_B > 0$. Since $\|(sI - A)^{-1}\|$ is bounded in $\mathbb{B}_c$ the estimate (19) gives

$$\|B_c P_u(s) C_c (sI - A_c)^{-1}\| \le \|C\| \|(sI - A)^{-1}\| \|B_u(s) C_c (sI - A_c)^{-1}\| + \|D_u\| \|C_c\| \|(sI - A_c)^{-1}\| \le \varepsilon M$$

for some constant $M > 0$. Clearly $\varepsilon M < 1$ when ε is sufficiently small, and hence $(I + B_c P_u(s) C_c (sI - A_c)^{-1})^{-1}$, and consequently $L(s)$, exists and is bounded in $\mathbb{B}_c$. Finally it follows from (19) that $B_u(s) C_c L(s)$ is bounded in $\mathbb{B}_c$. ■

C. Stability Implies Tracking

In this Section we show that stabilization implies asymptotic tracking and disturbance rejection, i.e., if the controller stabilizes the plant, then it will also asymptotically track reference signals of the form (2) and asymptotically reject disturbance signals of the same form.

Theorem 4: Assume that the extended system operator A_e is stable and let the reference and disturbance signals be

$$r(t) = \sum_{k=-n}^{n} \sum_{j=0}^{m_k - 1} r_{kj} t^j e^{i\omega_k t}, \tag{20a}$$

$$w(t) = \sum_{k=-n}^{n} \sum_{j=0}^{m_k - 1} w_{kj} t^j e^{i\omega_k t}, \tag{20b}$$

where $w_{kj} = (w_{dkj}, w_{bkj}, w_{mkj})$. Then the error signal $e(t) = r(t) - y(t)$ satisfies $\lim_{t \to \infty} e(t) = 0$ exponentially fast.

Proof: From equation (20b) we get

$$w_b(t) = \sum_{k=-n}^{n} \sum_{j=0}^{m_k - 1} w_{bkj} t^j e^{i\omega_k t}, \qquad \dot{w}_b(t) = \sum_{k=-n}^{n} \sum_{j=0}^{m_k - 1} \tilde{w}_{kj} t^j e^{i\omega_k t},$$

where $\tilde{w}_{kj} = i\omega_k w_{bkj} + (j+1) w_{bk,j+1}$ and we define $w_{bkm_k} = 0$. Letting $h_{kj} = D r_{kj} + E w_{kj} + F \tilde{w}_{kj}$ the solution of the closed loop state equation (12) can be written as

$$\begin{aligned} x_e(t) &= T_{A_e}(t) x_e(0) + \int_0^t T_{A_e}(t - \tau)(D r(\tau) + E w(\tau) + F \dot{w}_b(\tau)) d\tau \\ &= T_{A_e}(t) x_e(0) + \sum_{k=-n}^{n} \sum_{j=0}^{m_k - 1} e^{i\omega_k t} \int_0^t \tau^j T_{A_e - i\omega_k I}(t - \tau) h_{kj}\, d\tau \\ &= T_{A_e}(t) x_{e0} - \sum_{k=-n}^{n} \sum_{j=0}^{m_k - 1} \sum_{l=0}^{j} \frac{j!}{(j-l)!} t^{j-l} (A_e - i\omega_k I)^{-(l+1)} h_{kj} e^{i\omega_k t}, \end{aligned}$$

where

$$x_{e0} = x_e(0) + \sum_{k=-n}^{n} \sum_{j=0}^{m_k - 1} j!\, (A_e - i\omega_k I)^{-(j+1)} h_{kj}.$$

Now the error signal can be written as

$$e(t) = r(t) - y(t) = -C_e T_{A_e}(t) x_{e0} + \sum_{k=-n}^{n} \sum_{j=0}^{m_k-1} e_{kj}(t) e^{i\omega_k t},$$

where

$$e_{kj}(t) = [C_e(A_e - i\omega_k I)^{-1} h_{kj} + r_{kj} - D_w w_{kj}] t^j + \sum_{l=1}^{j} \frac{j!}{(j-l)!} t^{j-l} C_e (A_e - i\omega_k I)^{-(l+1)} h_{kj}.$$

Since C_e is bounded and $T_{A_e}(t)$ is stable, we have $\lim_{t\to\infty} C_e T_{A_e}(t) x_{e0} = 0$. Hence, to prove tracking it is sufficient to show that $e_{kj}(t) = 0$ for $t \geq 0$.

Let k and j be fixed and define $x_q = (z_q, \xi_q)$ by

$$x_q = (A_e - i\omega_k I)^{-q} h_{kj}, \qquad q = 1, \dots, m_k.$$

Then we have to show that

$$C_e x_1 + r_{kj} - D_w w_{kj} = 0 \tag{21}$$

and $C_e x_q = 0$ for $q = 2,\dots,m_k$. In the following we partition ξ_q as $\xi_q = (\xi_{q,-n}, \dots, \xi_{qn})$ where $\xi_{qk} = (\xi_{qk1}, \dots, \xi_{qkm_k})$. For $q = 1$

$$(A_e - i\omega_k I) x_1 = h_{kj} = \begin{bmatrix} -\tilde{G}_u C_c B_c (r_{kj} - D_w w_{kj}) + B_w(0) w_{kj} - G_w \tilde{w}_{kj} \\ B_c(r_{kj} - D_w w_{kj}) \end{bmatrix} \tag{22}$$

The second component of equation (22) gives

$$-B_c C z_1 + (A_c - i\omega_k I - B_c D_u C_c)\xi_1 = B_c(r_{kj} - D_w w_{kj}),$$

and using $C_e = \begin{bmatrix} C & D_u C_c \end{bmatrix}$ we get

$$-B_c(C_e x_1 + r_{kj} - D_w w_{kj}) + (A_c - i\omega_k I)\xi_1 = 0. \tag{23}$$

Using the block structure of B_c, A_c and ξ_1, the kth block of (23) gives

$$\begin{aligned} &-B_{ck}(C_e x_1 + r_{kj} - D_w w_{kj}) + (A_{ck} - i\omega_k I)\xi_{1k} = 0 \\ \iff &-\begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_e x_1 + r_{kj} - D_w w_{kj} \end{bmatrix} + \begin{bmatrix} \xi_{1k2} \\ \vdots \\ \xi_{1km_k} \\ 0 \end{bmatrix} = 0. \end{aligned}$$

Hence (21) holds and additionally $\xi_{1kl} = 0$ for $l = 2,\dots,m_k$. Next we show by induction that for $1 \leq q < m_k$ we have

$$\xi_{qkl} = 0 \qquad \text{for } l = q+1, \dots, m_k. \tag{24}$$

We have already shown that (24) holds for $q = 1$. Now assume that (24) holds for some $q \geq 1$. From the definition of x_q we get

$$\begin{aligned} x_{q+1} = (A_e - i\omega_k I)^{-1} x_q \implies &-B_{ck} C_e x_{q+1} + (A_{ck} - i\omega_k I)\xi_{q+1,k} = \xi_{qk} \\ \implies &-\begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_e x_{q+1} \end{bmatrix} + \begin{bmatrix} \xi_{q+1,k2} \\ \vdots \\ \xi_{q+1,km_k} \\ 0 \end{bmatrix} = \begin{bmatrix} \xi_{qk1} \\ \vdots \\ \xi_{qk,m_k-1} \\ \xi_{qkm_k} \end{bmatrix}. \end{aligned} \tag{25}$$

Hence $\xi_{q+1,kl} = \xi_{qk,l-1}$ for $l = 2,\dots,m_k$, which together with (24) implies that $\xi_{q+1,kl} = 0$ for $l = q+2,\dots,m_k$. This concludes the induction step. In particular, we get from equation (25) that $C_e x_{q+1} = -\xi_{qkm_k} = 0$ for $q = 1,\dots,m_k - 1$, which completes the proof. ■

Tracking is robust with respect to perturbations in the plant and the controller provided that the structure of the controller remains the same, the assumptions made about the plant continue to hold, and the closed loop system remains stable. This can be seen as follows: If the closed loop system is stable we still have $\lim_{t\to\infty} C_e T_{A_e}(t) x_{e0} = 0$, and because of the structure of the controller and the operators A_e and C_e, equations (21) and (25) still hold for the perturbed system.

D. Tuning the Controller

It follows from the stability conditions (8) that apart from the parameter ε, the controller is completely determined by the values $P_u(i\omega_k)$ for $k=-n,\ldots,n$. Since an input $u(t)=u_k e^{i\omega_k t}$ will asymptotically result in the output $y(t)=P_u(i\omega_k)u_k e^{i\omega t}$, the values $P_u(i\omega_k)$ can be obtained by using frequency response measurements as described in [7].

IV. An Example

Consider the damped wave equation

$$p_{tt}(\eta,t)+\beta p_t(\eta,t)+\alpha p(\eta,t)=\gamma p_{\eta\eta}(\eta,t), \qquad \eta\in(a,b),\ t>0, \quad \alpha>0,\beta,\gamma>0, \tag{26a}$$

with boundary control and disturbance

$$p_\eta(a,t)=u_1(t)+w_{b1}(t), \quad p_\eta(b,t)=u_2(t)+w_{b2}(t), \tag{26b}$$

and point observation and disturbance

$$y_1(t)=p(\eta_1,t)+w_{m1}(t), \quad y_2(t)=p(\eta_2,t)+w_{m2}(t), \quad \eta_1,\eta_2\in(a,b),\ \eta_1\neq\eta_2, \tag{26c}$$

and with zero initial conditions. Next we formulate the problem in the form (1). Define $z(t)=(z_1(t),z_2(t))=(p(\cdot,t),p_t(\cdot,t))$ and the operator $A_1=\frac{d^2}{d\eta^2}-\alpha I$ with domain $\mathcal{D}(A_1)=H^2(a,b)$. Let the state space be $X=H^1_{\alpha\gamma}(a,b)\times L^2(a,b)$, where $H^1_{\alpha\gamma}(a,b)$ is the Sobolev space $H^1(a,b)$ with the inner product $\langle z,v\rangle_{H^1_{\alpha\gamma}}=\alpha\langle z,v\rangle_{L^2}+\gamma\langle z',v'\rangle_{L^2}$. Then the inner product in X is given by $\langle (z_1,z_2),(v_1,v_2)\rangle_X=\langle z_1,v_1\rangle_{H^1_{\alpha\gamma}}+\langle z_2,v_2\rangle_{L^2}$. Now problem (26) can be written in the form (1) where

$$A_0=\begin{bmatrix}0 & I\\ A_1 & -\beta I\end{bmatrix}, \qquad \mathcal{D}(A_0)=\mathcal{D}(A_1)\times H^1_{\alpha\gamma}(a,b),$$

$B_d=E_d=D_d=D_b=0$, $B_b=E_b=E_m=I$ and $u(t)=(u_1(t),u_2(t))$, $w_b(t)=(w_{b1}(t),w_{b2}(t))$, $w_m(t)=(w_{m1}(t),w_{m2}(t))$. The boundary operator $S_b:X\supset\mathcal{D}(S_b)\to\mathbb{C}^2$ is given by $S_b z=(z_1'(a),z_1'(b))$ with $\mathcal{D}(S_b)=H^2(a,b)\times L^2(a,b)$, and the observation operator $C\in\mathcal{L}(X,\mathbb{C}^2)$ is given by $Cz=(z_1(\eta_1),z_1(\eta_2))$.

Now the system operator is $Az=A_0 z$ for $z\in\mathcal{D}(A)=\mathcal{D}(A_0)\cap\mathcal{N}(S_b)=\{\,(z_1,z_2)\in\mathcal{D}(A_0)\mid z_1'(a)=z_1'(b)=0\,\}$. It is easily seen that $G_u=G_w=G:\mathbb{C}^2\to X$ is given by $G(c_1,c_2)=(c_1g_1+c_2g_2,0)$, where $g_1(\eta)=(\eta-b)^2/(2(a-b))$ and $g_2(\eta)=(\eta-a)^2/(2(b-a))$. Then $N_u=N_w=N=A_0G:\mathbb{C}^2\to X$ is given by $N(c_1,c_2)=(0,c_1A_1g_1+c_2A_1g_2)$.

Making the change of variables $x=z-Gu_b-Gw_b$ we get the state and output equations

$$\begin{aligned}\dot{x}&=Ax+Nu-G\dot{u}+Nw_b-G\dot{w}_b,\\ y&=Cx+CGu+Cgw_b+w_m.\end{aligned}$$

It is easily seen that the transfer matrix is given by

$$P(s)=\frac{1}{k(s)\sinh k(s)(b-a)}\begin{bmatrix}-\cosh k(s)(b-\eta_1) & \cosh k(s)(\eta_1-a)\\ -\cosh k(s)(b-\eta_2) & \cosh k(s)(\eta_2-a)\end{bmatrix},$$

where $k(s)=\sqrt{(s^2+\beta s+\alpha)/\gamma}$. Since $\det P(i\omega)=\sinh k(i\omega)(\eta_1-\eta_2)[k(i\omega)^2\sinh k(i\omega)(b-a)]^{-1}\neq 0$ for all $\omega\in\mathbb{R}$, we can track signals of every frequency. Let the referenece and disturbance signals be

$$r(t)=\begin{bmatrix}1\\ \sin 6t\end{bmatrix}, \qquad w_b(t)=\begin{bmatrix}0.5+0.5\sin 6t\\ 0.5+0.5\sin 6t\end{bmatrix}, \qquad w_m(t)=\begin{bmatrix}0.1-0.2\sin 6t\\ 0.3+0.1\sin 6t\end{bmatrix},$$

and let $a=0$, $b=1$, $\alpha=\gamma=1$, and $\beta=2$, $\eta_1=0.2$, and $\eta_2=0.8$. Choose the matrix gains as

$$K_k=P(i\omega_k)^*\big(P(i\omega_k)P(i\omega_k)^*\big)^{-1/2}, \qquad k=-1,0,1,$$

and $\varepsilon=0.52$. Fig. 2 shows the resulting error signals as a function of time.

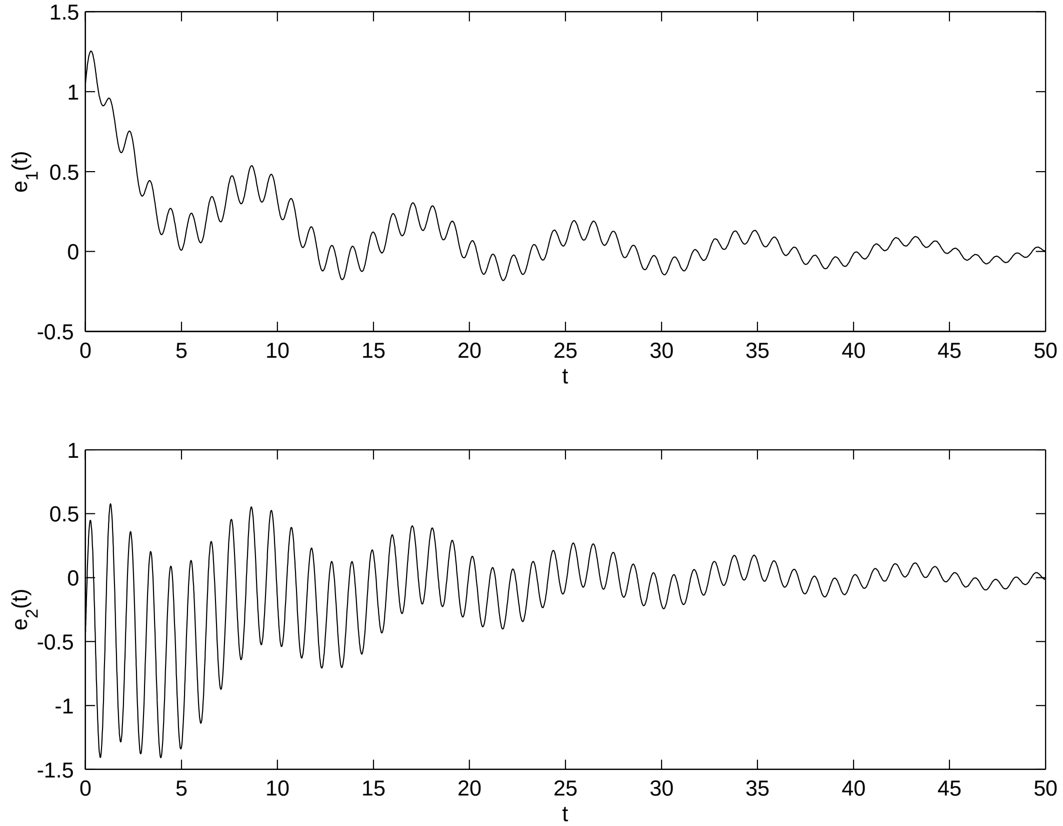

Fig. 2. The error signals e_1 and e_2 as a function of time.

V. Conclusions

A robust finite-dimensional low-gain controller for state space boundary control systems with boundary and distributed control and disturbances and measurement disturbances is considered. The controller (5), originally given in [2], has been shown to achieve robust regulation for signals of the form (2) and for the plant (1), which is more general than the plant in [2]. Because our controls and disturbances are sufficiently smooth in time, boundary controls and disturbances can be trasformed into the state equation. The stability of the extended system, consisting of the plant and the controller, is proved by showing that the resolvent of the extended system operator is in H^∞. Regulation is then shown to be a consequence of stabilization. The only information required of the plant is the values of the transfer matrix at the reference- and disturbance signal frequencies, which can be measured from the open-loop plant with frequency response measurements. A one-dimensional damped wave equation is given as an example to illustrate the theory. Future work includes enlarging the class of signals to arbitrary periodic functions and, since the tracking proof in Section III-C is valid also for strongly stable systems, to extend the stability proof to strongly stable systems.

Appendix

For the proofs we need the following definitions: $U_k(s) = (s+1)^{-m_k} Q_k(s)$ and

$$U_{\varepsilon,k}(s) = U_k((s - i\omega_k)/\varepsilon) = \frac{\varepsilon^{m_k} Q_k((s-i\omega_k)/\varepsilon)}{(s - i\omega_k + \varepsilon)^{m_k}}. \tag{27}$$

Let $s \in \mathbb{B}_k$ and $\varepsilon \in (0,1)$. Partition $sI - \tilde{A}_{c,k} = sI - A_{c,k} + B_{c,k} P_u(i\omega_k) C_{c,k}$ as

$$sI - \tilde{A}_{c,k} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} (s-i\omega_k)I_p & -I_p & \dots & 0 \\ 0 & (s-i\omega_k)I_p & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & \dots & (s-i\omega_k)I_p \end{bmatrix},$$

$B_1 = \begin{bmatrix} 0 & \dots & -I_p \end{bmatrix}^T$, $C_1 = \begin{bmatrix} \varepsilon^{m_k} P_u(i\omega_k) K_{k0} & \dots & \varepsilon^2 P_u(i\omega_k) K_{k,m_k-2} \end{bmatrix}$ and $D_1 = \varepsilon P_u(i\omega_k) K_{k,m_k-1} + (s - i\omega_k)I_p$. Clearly A_1 is invertible and $\det(sI - \tilde{A}_{c,k}) = \det A_1 \det(D_1 - C_1 A_1^{-1} B_1)$. Let $A_1^{-1} B_1 = X$. Partitioning X in the obvious way and solving the equation $A_1 X = B_1$ we get $X_j = -(s - i\omega_k)^{-(m_k-j-1)} I_p$ for $0 \le j \le m_k - 2$. Hence

$$C_1 A_1^{-1} B_1 = C_1 X = \sum_{j=0}^{m_k-2} C_{1j} X_j = -(s - i\omega_k)^{-(m_k-1)} \sum_{j=0}^{m_k-2} \varepsilon^{m_k-j} P_u(i\omega_k) K_{kj} (s - i\omega_k)^j,$$

and so $D_1 - C_1 A_1^{-1} B_1 = (s - i\omega_k)^{-(m_k-1)} \varepsilon^{m_k} Q_k((s - i\omega_k)/\varepsilon)$. Because $\det A_1 = (s - i\omega_k)^{p(m_k-1)}$ we get $\det(sI - \tilde{A}_{c,k}) = \varepsilon^{pm_k} \det Q_k((s - i\omega_k)/\varepsilon)$. Now it follows from the stability condition (8) that $\det(sI - \tilde{A}_{c,k})$ is bounded away from zero for $s \in \mathbb{B}_k$. Hence $sI - \tilde{A}_{c,k}$ is invertible and there is an $\tilde{M}_\varepsilon > 0$ such that $\|(sI - \tilde{A}_{c,k})^{-1}\| \le \tilde{M}_\varepsilon$ for $s \in \mathbb{B}_k$.

Let $l \ne k$. It follows from the inequality $|s - i\omega_l| \ge d/2$ that $\|(sI - A_{c,l})^{-1}\|$ is bounded in $\mathbb{B}_k$ and thus $\|(sI - A_{c,a})^{-1}\|$ and $\|(sI - A_{c,b})^{-1}\|$ are bounded by some constant $M_{A,c} > 0$. Hence $\|(sI - \tilde{A}_c)^{-1}\| \le \max\{\tilde{M}_\varepsilon, M_{A,c}\} = M_1(\varepsilon)$.

Next we show that $\|C_{c,k}(sI - \tilde{A}_{c,k})^{-1}\|$ is bounded. Let $C_{c,k}(sI - \tilde{A}_{c,k})^{-1} = Y = \begin{bmatrix} Y_0 & \cdots & Y_{m_k-1} \end{bmatrix}$. Then Y satisfies the equations

$$\begin{aligned} (s - i\omega_k) Y_0 + \varepsilon^{m_k} Y_{m_k-1} P_u(i\omega_k) K_{k0} &= \varepsilon^{m_k} K_{k0}, \\ -Y_{j-1} + (s - i\omega_k) Y_j + \varepsilon^{m_k-j} Y_{m_k-1} P_u(i\omega_k) K_{kj} &= \varepsilon^{m_k-j} K_{kj}, \quad \text{for } 1 \le j \le m_k - 2, \\ -Y_{m_k-2} + Y_{m_k-1}(s - i\omega_k + \varepsilon P_u(i\omega_k) K_{k,m_k-1}) &= \varepsilon K_{k,m_k-1}, \end{aligned}$$

which have the solution

$$Y_{m_k-1} = \sum_{l=0}^{m_k-1} \frac{\varepsilon^{m_k-l}(s - i\omega_k)^l}{(s - i\omega_k + \varepsilon)^{m_k}} K_{kl} U_{\varepsilon,k}(s)^{-1},$$

and for $j = 0, \dots, m_k - 2$

$$Y_j = Y_{m_k-1} \Big[(s - i\omega_k)^{m_k-j-1} + \sum_{l=j+1}^{m_k-1} \varepsilon^{m_k-l} (s - i\omega_k)^{l-j-1} P_u(i\omega_k) K_{kl} \Big] - \sum_{l=j+1}^{m_k-1} \varepsilon^{m_k-l} (s - i\omega_k)^{l-j-1} K_{kl}.$$

Because $\varepsilon/|s - i\omega_k + \varepsilon| \le 1$ and $|s - i\omega_k|/|s - i\omega_k + \varepsilon| \le 1$ for $\operatorname{Re} s > 0$, we have $\|Y_{m_k-1}\| \le \sum_{l=0}^{m_k-1} \|K_{kl}\| \|U_k^{-1}\|_\infty$, and so $\|Y_{m_k-1}\|$ is bounded. Clearly each $\|Y_j\|$ is also bounded, and thus there is a constant $M_k > 0$ such that $\|Y\| \le M_k$.

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