Asymptotic behaviour of infinite systems ODEs

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Main Problem

Study asymptotics of infinite systems of the form

$$\dot{x}_k(t) = A_0 x_k(t) + A_1 x_{k-1}(t), \quad k \in \mathbb{Z},\ t \geq 0,$$

where $A_0, A_1 \in \mathbb{C}^{m \times m}$ do not depend on $k \in \mathbb{Z}$.

Problem

We want to study the rates of convergence

$$\sup_{k \in \mathbb{Z}} \|x_k(t) - y_k\|_{\mathbb{C}^m} \to 0, \qquad \textit{as} \quad t \to \infty.$$

A Concrete Model: An Infinite Vehicle Platoon

$$\begin{pmatrix} \dot{y}_k \\ \dot{w}_k \\ \dot{a}_k \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix} \begin{pmatrix} y_k \\ w_k \\ a_k \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{k-1} \\ w_{k-1} \\ a_{k-1} \end{pmatrix}$$

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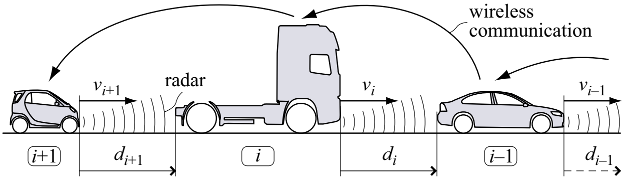

Figure: Source: Ploeg *et. al.*, IEEE, 2011.

A Concrete Model: An Infinite Vehicle Platoon

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$y_k(t) =$ displacement from ideal distance between k and $k-1$

$w_k(t) =$ displacement from ideal velocity of kth vehicle

$a_k(t) =$ acceleration of kth vehicle

Objective: Choose $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ so that $\sup_{k\in\mathbb{Z}} |y_k| \to 0$ as $t \to \infty$.

Infinite Systems of Differential Equations

Our system can be formulated as an abstract Cauchy problem

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \in X$$

on $X = \ell^p(\mathbb{C}^m)$ for $1 \leq p \leq \infty$ by choosing $x(t) = (x_k(t))_{k\in\mathbb{Z}}$ and

$$Ax = (A_0 x_k + A_1 x_{k-1})_{k\in\mathbb{Z}}.$$

i.e.

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ & A_1 & A_0 & & \\ & & A_1 & A_0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

Here $A \in \mathcal{L}(X)$ and our system belongs to the class of "Spatially invariant systems" (Bamieh et. al., Curtain–Iftime–Zwart, and others).

The Platoon Model

$$\begin{pmatrix} \dot{y}_k \\ \dot{w}_k \\ \dot{a}_k \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix}}_{= A_0} \begin{pmatrix} y_k \\ w_k \\ a_k \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{= A_1} \begin{pmatrix} y_{k-1} \\ w_{k-1} \\ a_{k-1} \end{pmatrix}$$

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Challenge: The matrices A_0 and A_1 do not commute.

A More Abstract Class

Assumption

Assume $A_1 \neq 0$, $\sigma(A_0) \subset \mathbb{C}_-$, and there exists $\phi : \mathbb{C} \to \mathbb{C}$ s.t.

$$A_1(\lambda - A_0)^{-1}A_1 = \phi(\lambda)A_1, \qquad \lambda \in \mathbb{C} \setminus \sigma(A_0).$$

$\phi(\cdot)$ is called *characteristic function* of the infinite system.

Lemma

If A_1 is of rank one such that $A_1 = ab^* \in \mathbb{C}^{m\times m}$, then

$$\phi(\lambda) = b^*(\lambda - A_0)^{-1}a.$$

Clearly $\phi(\lambda) = \frac{n(\lambda)}{d(\lambda)}$ where $\deg n(\cdot) \leq \deg d(\cdot)$.

Main Aims and Results

(i) Characterize spectrum of A

(ii) Present conditions for uniform boundedness of $T(t)$

(iii) Study rates of convergence of $\|T(t)x - y\| \to 0$ as $t \to \infty$.

For all these purposes use the characteristic function $\phi(\cdot)$:

$$A_1(\lambda - A_0)^{-1}A_1 = \phi(\lambda)A_1, \qquad \lambda \in \mathbb{C} \setminus \sigma(A_0).$$

Main idea: Existence of $\phi(\cdot)$ compensates for the lack of commutativity of A_0 and A_1.

Spectrum of the System

Characteristic function $\phi(\cdot)$ determines the spectrum of A:

Theorem

Let $X = \ell^p(\mathbb{C}^m)$ *with* $1 \leq p \leq \infty$. *Then for* $\lambda \in \mathbb{C} \setminus \sigma(A_0)$

$$\lambda \in \sigma(A) \qquad \textit{if and only if} \qquad |\phi(\lambda)| = 1.$$

Moreover, for $\lambda \in \sigma(A) \setminus \sigma(A_0)$ *we have*

- $\mathrm{Ker}(\lambda - A) \neq \{0\}$ *if and only if* $p = \infty$
- $\overline{\mathrm{Ran}(\lambda - A)} = X$ *if and only if* $1 < p < \infty$.

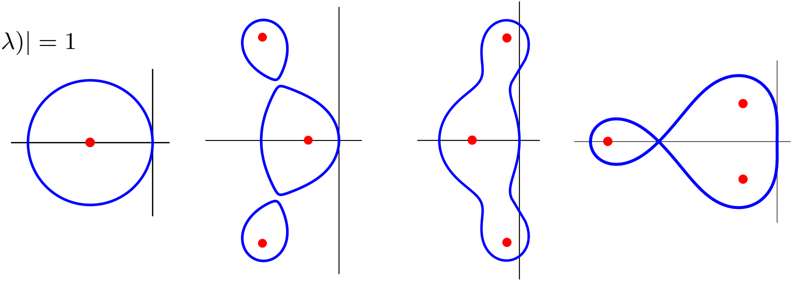
The type of spectrum depends on p, but the location does not.

Spectrum of the Platoon System

$$\begin{pmatrix} \dot{y}_k \\ \dot{w}_k \\ \dot{a}_k \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix} \begin{pmatrix} y_k \\ w_k \\ a_k \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{k-1} \\ w_{k-1} \\ a_{k-1} \end{pmatrix}$$

Characteristic function: $\phi(\lambda) = \dfrac{\alpha_0}{p(\lambda)} = \dfrac{\alpha_0}{\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0}$.

$|\phi(\lambda)| = 1$



Uniform Boundedness of the Semigroup

Uniform Boundedness of the Semigroup

Theorem

Let $1 \leq p \leq \infty$*. If* $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$,

$$\sup_{0<\lambda\leq 1} \frac{\lambda}{1-|\phi(\lambda)|} < \infty \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \sup_{\lambda>0} \frac{\lambda^{n+1}}{n!} \sum_{\ell=1}^{\infty} \left| \frac{d^n}{d\lambda^n} \phi(\lambda)^\ell \right| < \infty,$$

then the semigroup generated by A *is uniformly bounded.*

Proof.

A fairly direct Hille–Yosida approach using a resolvent formula. □

Property: Systems for $m \geq 2$ are typically not contractive. In particular, the platoon system is <u>never</u> contractive.

Sufficient Conditions for Uniform Boundedness

Lemma

Let $\phi(\lambda) = \frac{p(0)}{p(\lambda)}$ *for a polynomial* $p(\lambda)$ *with roots in* $\mathbb{C}_-$ *and* $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$. *If* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is* ***completely monotone****, i.e.,*

$$(-1)^n \frac{d^n}{d\lambda^n} \phi(\lambda) \geq 0, \qquad \lambda > 0$$

then

$$\sup_{n \in \mathbb{N}} \sup_{\lambda > 0} \frac{\lambda^{n+1}}{n!} \sum_{\ell=1}^{\infty} \left| \frac{d^n}{d\lambda^n} \phi(\lambda)^\ell \right| < \infty.$$

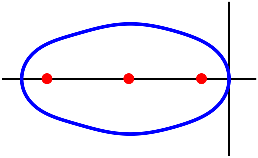
Bernstein: $\phi(\cdot)$ is completely monotone if and only if it is a Laplace transform of a finite Borel measure on $\mathbb{R}_+$.

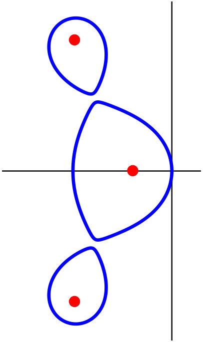
Sufficient Conditions for Uniform Boundedness

Lemma

Uniform boundedness in the following cases:

- $\phi(\lambda) = \dfrac{a_1 \cdots a_q}{(\lambda + a_1) \cdots (\lambda + a_q)}, \quad a_k > 0$
- $\phi(\lambda) = \dfrac{(a^2 + b^2)c}{(\lambda^2 + 2a\lambda + a^2 + b^2)(\lambda + c)}, \; a \geq c$
- $\phi(\lambda) = \phi_1(\lambda)\phi_2(\lambda)$ *where* ϕ_1 *and* ϕ_2 *are completely monotone.*





Stability Properties (without rates)

Theorem

Let $X = \ell^p(\mathbb{C}^m)$*. Assume* $\phi(\lambda) = \frac{p(0)}{p(\lambda)}$ *is completely monotone and* $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$*.*

- *If* $1 < p < \infty$*, then* $T(t)$ *is strongly stable, i.e.,* $T(t)x \to 0$
- *If* $p = 1$*, then* $T(t)x \to 0$ *as* $t \to \infty$ *for all* $x \in \overline{\mathrm{Ran}(A)} \neq X$*.*
- *If* $p = \infty$ *and* $x = x_0 + x_1 \in \overline{\mathrm{Ran}(A)} \oplus \mathrm{Ker}(A) \neq X$*, then*

$$T(t)x \to x_1 \qquad \text{as} \qquad t \to \infty.$$

Rates of Convergence

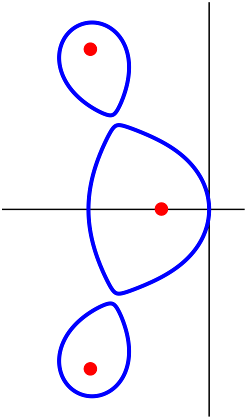
Next aim: Find rates of convergence for

$$\|T(t)x\| \to 0 \qquad \text{as} \quad t \to \infty$$

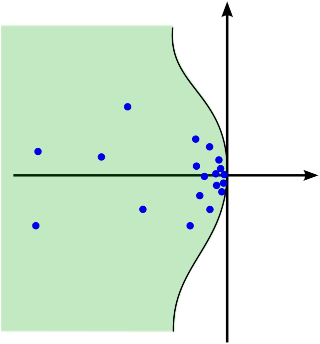
and

$$\|T(t)x - y\| \to 0 \qquad \text{as} \quad t \to \infty$$

under the assumption $\sigma(A) \cap i\mathbb{R} = \{0\}$.



Rates for $T(t)$ via $R(is, A) = (is - A)^{-1}$



Theorem (Martinez '11)

If $\|R(is,A)\| \le M_R|s|^{-\alpha}$, *then*

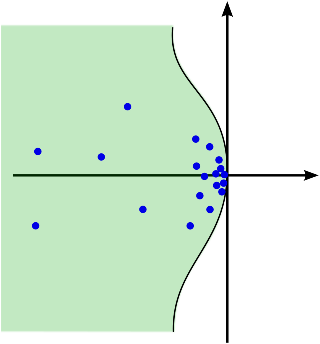
$$\|T(t)x - x_1\| = O\left(\frac{\log t}{t}\right)^{1/\alpha}$$

whenever

$$x = x_0 + x_1 \in \mathrm{Ran}(A) \oplus \mathrm{Ker}(A).$$

- On Hilbert, $\log t$ can be omitted [Batty, Chill & Tomilov '16]. (preprint '13).

Further Properties of the Class



Perturbation theory on Hilbert space X:

Theorem (LP 2014)

Assume $\|R(is,A)\| \le M_R |s|^{-\alpha}$, $0 \in \sigma_c(A)$.

Also $A+BC$ *stable if for* $\beta + \gamma \ge \alpha$

$$\|(-A^{-1})^\beta B\|, \quad \|(-A^{-*})^\gamma C^*\|$$

are small, B, C *Hilbert–Schmidt.*

- Dual to the perturbation theory of polynomial stability [LP 2012].

Convergence Rates via Resolvent Estimates

Theorem

Let $1 \leq p \leq \infty$. *If* $0 \in \sigma(A) \subset \mathbb{C}_- \cup \{0\}$, *then for* $0 < s \leq 1$

$$\|R(is, A)\| \asymp \frac{1}{|1 - |\phi(is)||} = O(|s|^{-n_\phi})$$

for an even integer $n_\phi \geq 2$. *Thus if* $T(t)$ *is uniformly bounded and* $x = x_0 + x_1 \in \operatorname{Ran}(A) \oplus \operatorname{Ker}(A)$, *then*

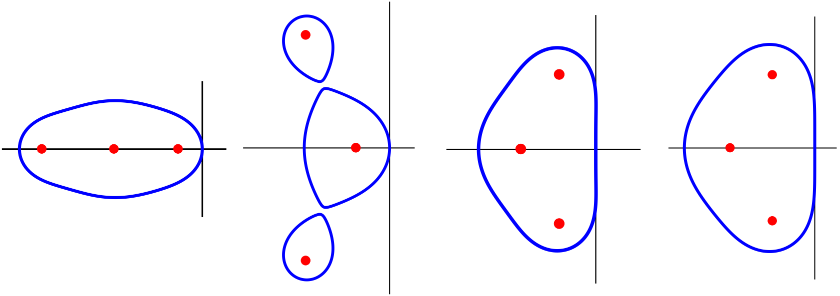
$$\|T(t)x - x_1\| = O\left(\left(\frac{\log t}{t}\right)^{1/n_\phi}\right)$$

Batty, Chill & Tomilov '16: Logarithm can be omitted if $p = 2$.

(Note that $x_1 = 0$ if $1 \leq p < \infty$)

Decay Rates for the Platoon System

For the platoon system, the possible exponents n_ϕ are 2, 4 and 6.



Corresponding rates are $\left(\frac{\log t}{t}\right)^{\frac{1}{2}}$, $\left(\frac{\log t}{t}\right)^{\frac{1}{4}}$ and $\left(\frac{\log t}{t}\right)^{\frac{1}{6}}$ (though the uniform boundedness condition is not valid in the last two cases).

Characterizing Initial States Leading to Convergence

Problem

Characterize elements $x \in \mathrm{Ran}(A)$ *and* $x \in \overline{\mathrm{Ran}(A)}$.

Motivation: Initial states $x \in \mathrm{Ran}(A) \oplus \mathrm{Ker}(A)$ lead to convergence

$$\|T(t)x\| = O\left(\frac{\log t}{t}\right)^{1/n_\phi}.$$

In the cases $X = \ell^1(\mathbb{C}^m)$ and $X = \ell^\infty(\mathbb{C}^m)$

$$t \mapsto T(t)x$$

converges to some $y \in X$ if and only if $x \in \overline{\mathrm{Ran}(A)} \oplus \mathrm{Ker}(A)$.

Theorem

Let $X = \ell^\infty(\mathbb{C}^m)$, $0 \in \sigma(A) \subset \mathbb{C}_- \cup \{0\}$, $T(t)$ *is bdd,* $\phi'(0) \neq 0$.

$x \in \overline{\mathrm{Ran}(A)} \oplus \mathrm{Ker}(A)$ *if and only if there exists* $y_0 \in \mathrm{Ran}(A_1)$

$$\sup_{k\in\mathbb{Z}} \left\| \frac{1}{n} \sum_{j=1}^{n} \phi(0)^{j-k} A_1 A_0^{-1} x_{k-j} - y_0 \right\|_{\mathbb{C}^m} \to 0, \quad n \to \infty, \tag{1}$$

Theorem

Let $X = \ell^\infty(\mathbb{C}^m)$, $0 \in \sigma(A) \subset \mathbb{C}_- \cup \{0\}$, $T(t)$ *is bdd,* $\phi'(0) \neq 0$.

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$$\sup_{k\in\mathbb{Z}} \left\| \frac{1}{n} \sum_{j=1}^{n} \phi(0)^{j-k} A_1 A_0^{-1} x_{k-j} - y_0 \right\|_{\mathbb{C}^m} \to 0, \quad n \to \infty, \tag{1}$$

Moreover, if the decay in (1) *is like* $O(n^{-1})$ *as* $n \to \infty$ *then* $x = x_0 + x_1 \in \mathrm{Ran}(A) \oplus \mathrm{Ker}(A)$ *and*

$$\|T(t)x - x_1\| = O\left(\left(\frac{\log t}{t}\right)^{1/n_\phi}\right), \quad t \to \infty.$$

Quantified Decay for the Platoon System

$$\begin{pmatrix}\dot y_k\\ \dot w_k\\ \dot a_k\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\zeta^3 & -3\zeta^2 & -3\zeta\end{pmatrix}\begin{pmatrix}y_k\\ w_k\\ a_k\end{pmatrix} + \begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}y_{k-1}\\ w_{k-1}\\ a_{k-1}\end{pmatrix}$$

Theorem

Let $X = \ell^\infty(\mathbb{C}^3)$. *If there exists* $c \in \mathbb{R}$

$$\sup_{k\in\mathbb{Z}} \left| c - \frac{1}{n}\sum_{j=1}^{n} y_{k-j}(0)\right| = O\left(\frac{1}{n}\right) \quad \textit{as} \quad n \to \infty,$$

then $\|T(t)x - x_1\| = O(\sqrt{\log t/t})$ *where* $x_1 = ((c, -\zeta c/3, 0)^T)_{k\in\mathbb{Z}}$

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Paunonen, L. and Seifert, D. *Asymptotic behaviour of coupled systems in discrete and continuous time*, Journal of Dynamics and Differential Equations, online, 2016.

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Preprints available on Arxiv.

Conclusions

Thank You!