Lecture 12: NP-$Complete$ness

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NB: *These notes are from CIS511 at Penn. The course followed Michael Sipser's Introduction to the Theory of Computation (3ed) text.*

Cook-Levin and NP-$Complete$ Languages

From last time, we sketched a proof of the Cook-Levin Theorem - any problem in NP can be converted into an instance of SAT in time and size polynomial in the original input.

Let's show that $3SAT = \{\phi | \phi \ is\ a\ satisfiable\ Boolean\ formula\ in\ 3\text{-}CNF\}$ is NP-$Complete$.

Proof: We will show that SAT reduces to $3SAT$ and $3SAT$ is in NP to show $3SAT$ is NP-$Complete$, because Cook-Levin gives us the reduction from any NP language to SAT, by the transitivity of composition of polynomial reductions.

The reduction is as follows:

Take some instance ϕ of SAT. If we want $y \Leftrightarrow x_1 \wedge x_2$, then we can say $(\neg x_1 \vee \neg x_2 \vee y)$. We don't need to worry about the ORs. Taking the AND of all of these clauses gets us a logically equivalent $\phi' \in 3SAT$.

For the same reason SAT is in NP, $3SAT$ is as well, as it is easily verifiable.

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Karp took Cook's proof and showed a number of problems are NP-$Complete$. Some of these are:

- Independent Set - a subset of vertices, no two of which are adjacent (inputs are $\langle G, k\rangle$, for IS of size k). The reduction from $3SAT$ involves creating triangle for each clause and connecting negations of corresponding variables. Set k to the number of clauses.
- $CLIQUE$ is NP-$Complete$. Take a $\langle G, k\rangle$ instance of IS. Create G^C where two vertices are adjacent in G^C if and only if they are not adjacent in G. Now, the IS in G is exactly a k-clique in G^C.
- $VERTEX\ COVER$ is a vertex set of size k such that every edge is incident to some element in the set. The complement of an independent set is a vertex cover, by definition, so a $|V| - k$ independent set is a k vertex cover.