Lecture 9
Floating Point

School of Computer Science and Engineering
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3. Arithmetic for Computers

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3.5 Floating Point

- **Real numbers**

  $3.14159265\ldots_{ten}$ (pi), $2.71828\ldots_{ten}$ (e),

  0.000000001_{ten} , $0.1_{ten} \times 10^{-8}$ or $1.0_{ten} \times 10^{-9}$,

  $3{,}155{,}760{,}000_{ten}$, $0.00315576 \times 10^{12}$ or 3.15576×10^{9}

- **Scientific notation**
  - $a \times 10^{b}$ (a times ten raised to the power of b)
    - where the exponent b is an integer, and the significand (or mantissa) a is any real number
    - $1.0_{ten} \times 10^{-9}$, 315.576×10^{7}
  - A notation that renders numbers with a single digit to the left of the decimal point (in our text)

- **Normalized number**
  - A number in scientific notation that has no leading 0s (i.e. $1 \le |a| < 10$).

Floating-Point Representation

- **Floating point numbers in binary form**

  $\pm 1.xxxxxxxxx_{\text{two}} \times 2^{yyyy}$

- **Sign**
  - 1 bit
- **Exponent**
  - 8 bits (including the sign of the exponent)
- **Fraction (=significand=mantissa)**
  - 23 bits, fraction
  - sign and magnitude representation

<table>
<tr><td>31</td><td>30</td><td>29</td><td>28</td><td>27</td><td>26</td><td>25</td><td>24</td><td>23</td><td>22</td><td>21</td><td>20</td><td>19</td><td>18</td><td>17</td><td>16</td><td>15</td><td>14</td><td>13</td><td>12</td><td>11</td><td>10</td><td>9</td><td>8</td><td>7</td><td>6</td><td>5</td><td>4</td><td>3</td><td>2</td><td>1</td><td>0</td></tr>
<tr><td>s</td><td colspan="8">exponent</td><td colspan="23">fraction</td></tr>
<tr><td>1 bit</td><td colspan="8">8 bits</td><td colspan="23">23 bits</td></tr>
</table>

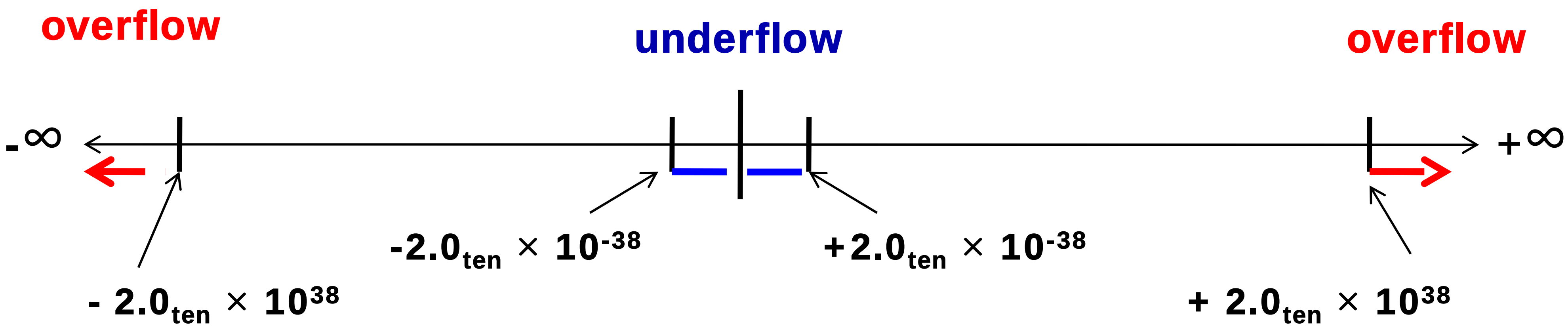
Floating Point Numbers

- **General form of floating-point numbers**

  $(-1)^S \times F \times 2^E$

- **Tradeoff between accuracy and range**
  - Large significand … increased accuracy
  - Large exponent … increased range of numbers
- **Range of floating-point numbers in MIPS**

  $\mathbf{2.0_{ten} \times 10^{-38} \sim 2.0_{ten} \times 10^{38}}$



ANSI / IEEE Std 754-1985

- **IEEE standard for binary floating-point arithmetic**
- **Hidden-bit scheme**

  $(-1)^s \times (1 + \text{fraction}) \times 2^E$

  $= (-1)^s \times (1 + (s1 \times 2^{-1}) + (s2 \times 2^{-2}) + (s3 \times 2^{-3}) + (s4 \times 2^{-4}) + \ldots) \times 2^E$
- **In this book,**
  - significand … represent the 24/53-bit number that is 1 plus the fraction
  - fraction … represent the 23/52-bit number
- **32-bit single format**
  - 1-bit sign, 8-bit exponent, 23-bit fraction
- **64-bit double format**
  - 1-bit sign, 11-bit exponent, 52-bit fraction

Double and Quad Precision

- **Double precision floating-point number**
  - 11 exponent bits
  - 52 fraction bits

$$2.0_{ten} \times 10^{-308} \sim 2.0_{ten} \times 10^{308}$$

<table>
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<tr><td>s</td><td colspan="11">exponent</td><td colspan="20">fraction</td></tr>
<tr><td>1 bit</td><td colspan="11">11 bits</td><td colspan="20">20 bits</td></tr>
<tr><td colspan="32">fraction (continued)</td></tr>
<tr><td colspan="32">32 bits</td></tr>
</table>

- **Quad precision floating-point number**
  - IEEE 754-2008 binary128 standard
  - 15 exponent bits
  - 112 significand bits

IEEE 754 Encoding

Single precision		Double precision		Object represented
Exponent	Fraction	Exponent	Fraction	
0	0	0	0	0.0
0	Nonzero	0	Nonzero	± denormalized number
1~254	Anything	1~2046	Anything	± floating point number
255	0	2047	0	±infinity
255	Nonzero	2047	Nonzero	NAN (Not A Number)

Figure 3.13

Sorting Floating Point Numbers

- Keep encoding that is somewhat compatible with 2's complement
  - e.g., 0.0 in FP is 0 in two's complement
  - Can compare two FP numbers in the same way as comparing 2's complement integers

- Placing the sign in the most significant bit
- Placing exponent before the significand
- But what with the negative exponents ?

Example: 2's complement exponents

- **$1.0_{two} \times 2^{-1}$**

  0 11111111 00000000000000000000000

- **$1.0_{two} \times 2^{+1}$**

  0 00000001 00000000000000000000000

- **$1.0_{two} \times 2^{-1}$ looks like a bigger one than $1.0_{two} \times 2^{+1}$**

  Undesirable !

Biased Notation

- Can reuse integer comparison hardware
  - If the most negative exponent = 00…000
  - and the most positive exponent = 11...111
- $(-1)^{\text{Sign}} \times (1 + \text{Fraction}) \times 2^{(\text{Exponent - } \textbf{Bias})}$
- **Exponent biases in IEEE 754**
  - 127 for single precision

    $-126 \le \text{exponent} \le +127$
  - 1023 for double precision

    $-1022 \le \text{exponent} \le +1023$

Biased Exponent with Bias=127

	How it is interpreted		How it is encoded	
	Decimal Exponent	signed 2's complement	Biased Notation	Decimal Value of Biased Notation
∞, NaN	For infinities		11111111	255
	127	01111111	11111110	254
	...	...	...	...
	2	00000010	10000001	129
Getting closer to zero	1	00000001	10000000	128
	0	00000000	01111111	127
	-1	11111111	01111110	126
	-2	11111110	01111101	125
	...	...	...	...
	-126	10000010	00000001	1
Zero	For Denorms	10000001	00000000	0

Example: Floating-Point Representation

- **Show the IEEE 754 single and double precision representations of -0.75_{ten} .**

[Answer]

- $-0.75_{ten} = -0.11_{two} = -1.1_{two} \times 2^{-1}$
- Single: exponent = -1 + bias = -1 + 127 = 126

  $(-1)^1 \times (1 + .1000...00) \times 2^{(126-127)}$

  = 1 01111110 10000000000000000000000

  = 1011 1111 0100 0000 0000 0000 0000 0000

  = $BF400000_{hex}$
- Double: exponent = -1 + bias = -1 + 1023 = 1022

  $(-1)^1 \times (1 + .1000...00) \times 2^{(1022-1023)}$

  = 1 01111111110 1000000000000000000000000000000000000000...

  = $BFE8000000000000_{hex}$

Example: Converting Binary to Decimal Floating Point

- **What decimal number is represented by**

  1 10000001 01000000000000000000000 ?

[Answer]

$(-1)^S \times (1 + \text{Significand}) \times 2^{(\text{Exponent - Bias})}$

$= (-1)^1 \times (1 + 0.25) \times 2^{(129-127)}$

$= -1 \times 1.25 \times 2^2$

$= -1.25 \times 4$

$= -5.0$

Floating-Point Addition

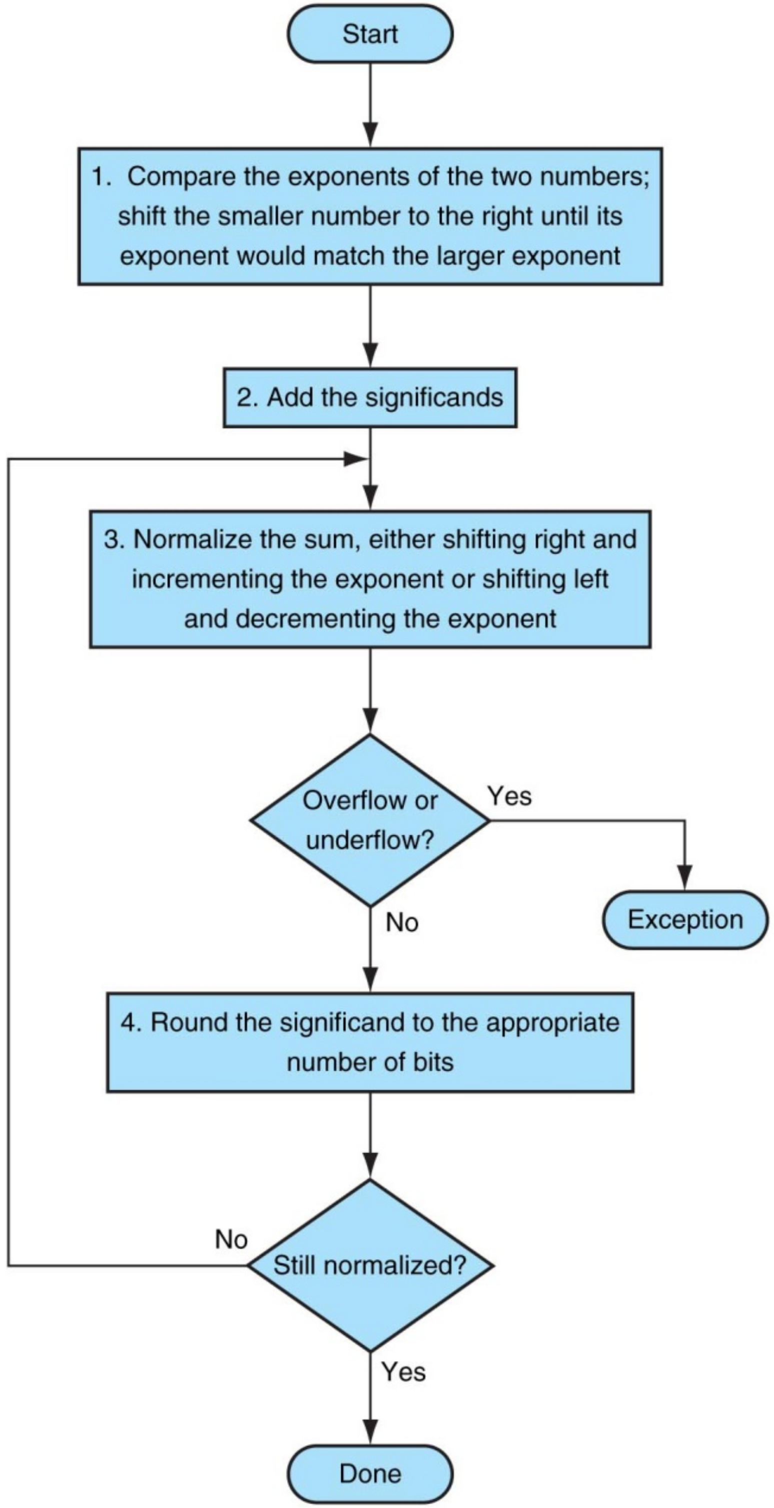

Figure 3.14

Floating Point Addition

- Addition (and subtraction)

$$(\pm F1 \times 2^{E1}) + (\pm F2 \times 2^{E2}) = \pm F3 \times 2^{E3}$$

- Step 0: Restore the hidden bit in F1 and in F2
- Step 1: Align fractions by right shifting F2 by E1 - E2 positions (assuming $E1 \geq E2$) keeping track of (three of) the bits shifted out in G R and S
- Step 2: Add the resulting F2 to F1 to form F3
- Step 3: Normalize F3 (so it is in the form 1.XXXXX …)
  - If F1 and F2 have the same sign $\rightarrow$ F3 $\in [1,4)$ $\rightarrow$ 1 bit right shift F3 and increment E3 (check for overflow)
  - If F1 and F2 have different signs $\rightarrow$ F3 may require *many* left shifts each time decrementing E3 (check for underflow)
- Step 4: Round F3 and possibly normalize F3 again
- Step 5: Rehide the most significant bit of F3 before storing the result

Example: Floating Point Addition

- Add

$$(0.5 = 1.0000 \times 2^{-1}) + (-0.4375 = -1.1100 \times 2^{-2})$$

  - Step 0:
  - Step 1:
  - Step 2:
  - Step 3:
  - Step 4:
  - Step 5:

Example: Floating Point Addition

- Add

$$(0.5 = 1.0000 \times 2^{-1}) + (-0.4375 = -1.1100 \times 2^{-2})$$

- Step 0: Hidden bits restored in the representation above
- Step 1: Shift significand with the smaller exponent (1.1100) right until its exponent matches the larger exponent (so once)
- Step 2: Add significands

  $$1.0000 + (-0.111) = 1.0000 – 0.111 = 0.001$$
- Step 3: Normalize the sum, checking for exponent over/underflow

  $$0.001 \text{ x } 2^{-1} = 0.010 \text{ x } 2^{-2} = .. = 1.000 \text{ x } 2^{-4}$$
- Step 4: The sum is already rounded, so we're done
- Step 5: Rehide the hidden bit before storing

  0 01111011 00000000000000000000000

Floating-Point Adder

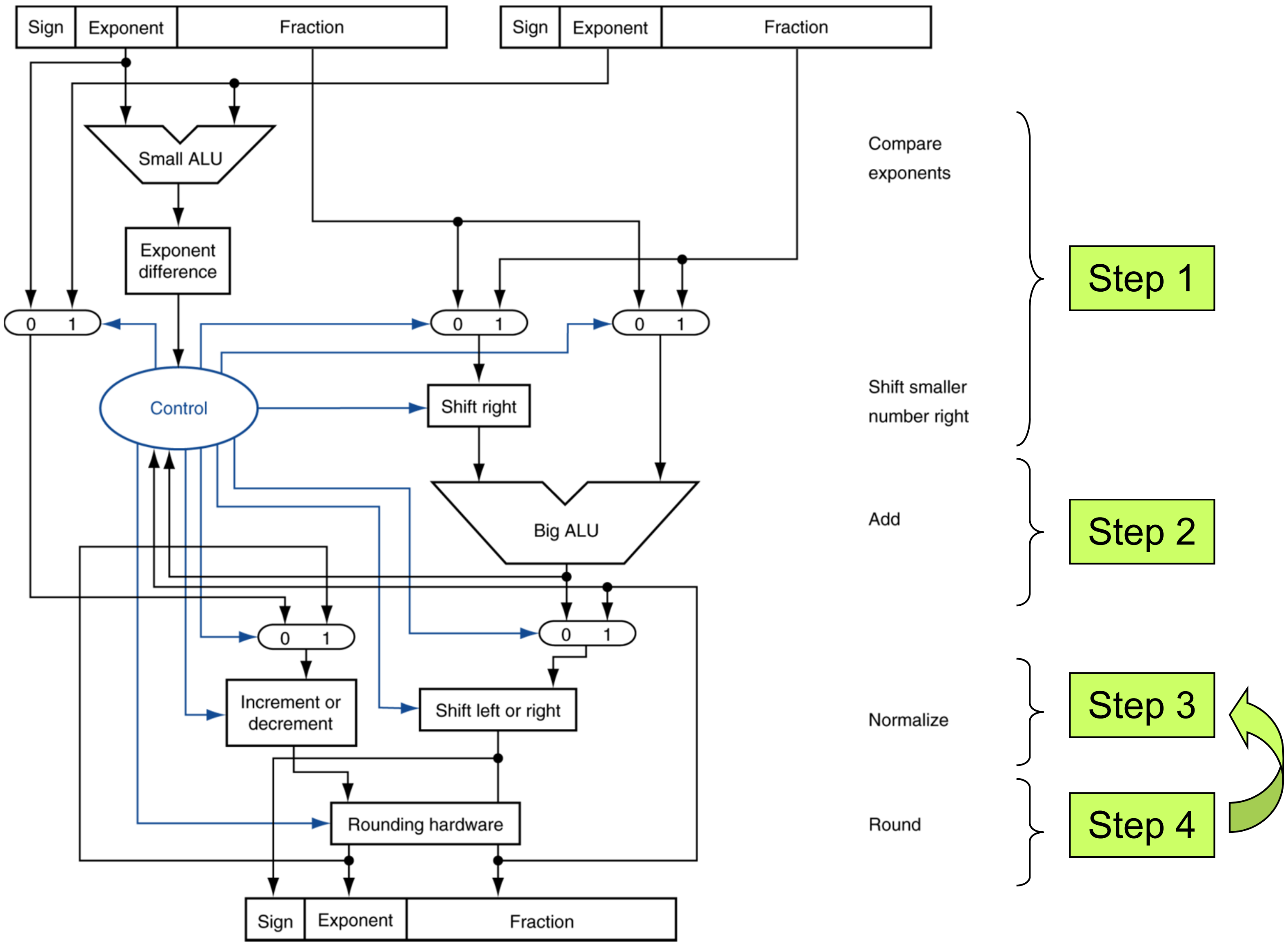

Figure 3.15

Floating-Point Multiplication

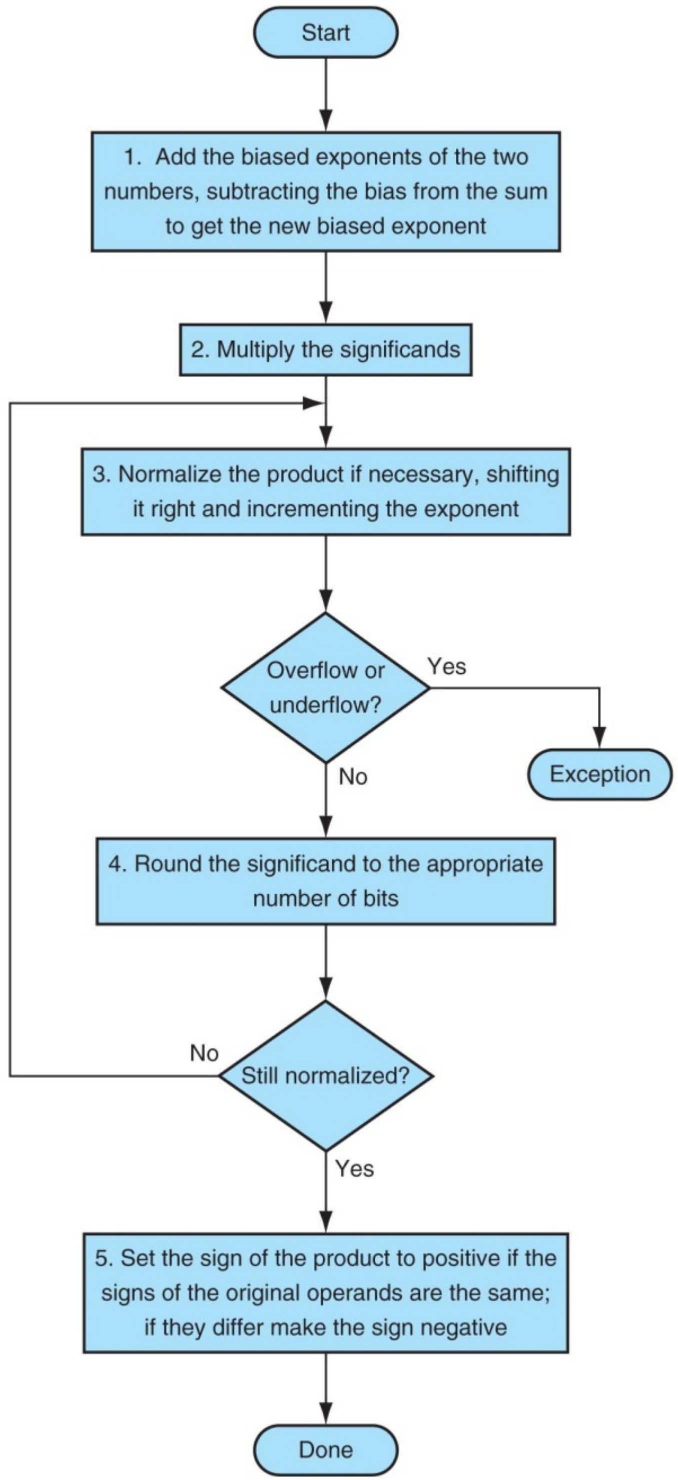

Figure 3.16

Floating Point Multiplication

- Multiplication

$$(\pm F1 \times 2^{E1}) \text{ x } (\pm F2 \times 2^{E2}) = \pm F3 \times 2^{E3}$$

- Step 0: Restore the hidden bit in F1 and in F2
- Step 1: Add the two (biased) exponents and subtract the bias from the sum, so E1 + E2 – 127 = E3

  also determine the sign of the product (which depends on the sign of the operands (most significant bits))
- Step 2: Multiply F1 by F2 to form a double precision F3
- Step 3: Normalize F3 (so it is in the form 1.XXXXX …)
  - Since F1 and F2 come in normalized $\rightarrow$ F3 $\in [1,4)$ $\rightarrow$ 1 bit right shift F3 and increment E3
  - Check for overflow/underflow
- Step 4: Round F3 and possibly normalize F3 again
- Step 5: Rehide the most significant bit of F3 before storing the result

Example: Floating Point Multiplication

- Multiply

$$(0.5 = 1.0000 \times 2^{-1}) \text{ x } (-0.4375 = -1.1100 \times 2^{-2})$$

  - Step 0:
  - Step 1:
  - Step 2:
  - Step 3:
  - Step 4:
  - Step 5:

Example: Floating Point Multiplication

- Multiply

$$(0.5 = 1.0000 \times 2^{-1}) \text{ x } (-0.4375 = -1.1100 \times 2^{-2})$$

- Step 0: Hidden bits restored in the representation above
- Step 1: Add the exponents (not in bias would be -1 + (-2) = -3 and in bias would be (-1+127) + (-2+127) – 127 = (-1 -2) + (127+127-127) = -3 + 127 = 124
- Step 2: Multiply the significands
  1.0000 x 1.110 = 1.110000
- Step 3: Normalized the product, checking for exp over/underflow
  1.110000 x 2^{-3} is already normalized
- Step 4: The product is already rounded, so we're done
- Step 5: Rehide the hidden bit before storing
  1 01111100 11000000000000000000000

Floating-Point Instructions

Category	Instruction	Example	Meaning	Comments
Arithmetic	FP add single	add.s $f2,$f4,$f6	$f2 = $f4 + $f6	FP add (single precision)
	FP subtract single	sub.s $f2,$f4,$f6	$f2 = $f4 - $f6	FP sub (single precision)
	FP multiply single	mul.s $f2,$f4,$f6	$f2 = $f4 × $f6	FP multiply (single precision)
	FP divide single	div.s $f2,$f4,$f6	$f2 = $f4 / $f6	FP divide (single precision)
	FP add double	add.d $f2,$f4,$f6	$f2 = $f4 + $f6	FP add (double precision)
	FP subtract double	sub.d $f2,$f4,$f6	$f2 = $f4 - $f6	FP sub (double precision)
	FP multiply double	mul.d $f2,$f4,$f6	$f2 = $f4 × $f6	FP multiply (double precision)
	FP divide double	div.d $f2,$f4,$f6	$f2 = $f4 / $f6	FP divide (double precision)
Data transfer	load word copr. 1	lwc1 $f1,100($s2)	$f1 = Memory[$s2 + 100]	32-bit data to FP register
	store word copr. 1	swc1 $f1,100($s2)	Memory[$s2 + 100] = $f1	32-bit data to memory
Condi-tional branch	branch on FP true	bc1t 25	if (cond == 1) go to PC + 4 + 100	PC-relative branch if FP cond.
	branch on FP false	bc1f 25	if (cond == 0) go to PC + 4 + 100	PC-relative branch if not cond.
	FP compare single (eq,ne,lt,le,gt,ge)	c.lt.s $f2,$f4	if ($f2 < $f4) cond = 1; else cond = 0	FP compare less than single precision
	FP compare double (eq,ne,lt,le,gt,ge)	c.lt.d $f2,$f4	if ($f2 < $f4) cond = 1; else cond = 0	FP compare less than double precision

Figure 3.17

3.10 Concluding Remarks

Core MIPS	Name	Integer	Fl. pt.	Arithmetic core + MIPS-32	Name	Integer	Fl. pt.
add	add	0.0%	0.0%	FP add double	add.d	0.0%	10.6%
add immediate	addi	0.0%	0.0%	FP subtract double	sub.d	0.0%	4.9%
add unsigned	addu	5.2%	3.5%	FP multiply double	mul.d	0.0%	15.0%
add immediate unsigned	addiu	9.0%	7.2%	FP divide double	div.d	0.0%	0.2%
subtract unsigned	subu	2.2%	0.6%	FP add single	add.s	0.0%	1.5%
AND	AND	0.2%	0.1%	FP subtract single	sub.s	0.0%	1.8%
AND immediate	ANDi	0.7%	0.2%	FP multiply single	mul.s	0.0%	2.4%
OR	OR	4.0%	1.2%	FP divide single	div.s	0.0%	0.2%
OR immediate	ORi	1.0%	0.2%	load word to FP double	l.d	0.0%	17.5%
NOR	NOR	0.4%	0.2%	store word to FP double	s.d	0.0%	4.9%
shift left logical	sll	4.4%	1.9%	load word to FP single	l.s	0.0%	4.2%
shift right logical	srl	1.1%	0.5%	store word to FP single	s.s	0.0%	1.1%
load upper immediate	lui	3.3%	0.5%	branch on floating-point true	bc1t	0.0%	0.2%
load word	lw	18.6%	5.8%	branch on floating-point false	bc1f	0.0%	0.2%
store word	sw	7.6%	2.0%	floating-point compare double	c.x.d	0.0%	0.6%
load byte	lbu	3.7%	0.1%	multiply	mul	0.0%	0.2%
store byte	sb	0.6%	0.0%	shift right arithmetic	sra	0.5%	0.3%
branch on equal (zero)	beq	8.6%	2.2%	load half	lhu	1.3%	0.0%
branch on not equal (zero)	bne	8.4%	1.4%	store half	sh	0.1%	0.0%
jump and link	jal	0.7%	0.2%				
jump register	jr	1.1%	0.2%				
set less than	slt	9.9%	2.3%				
set less than immediate	slti	3.1%	0.3%				
set less than unsigned	sltu	3.4%	0.8%				
set less than imm. uns.	sltiu	1.1%	0.1%				

Figure 3.28

Supplement

Accurate Arithmetic

- **Guard bit**
  - Used to provide one fraction bit when shifting left to normalize a result
  - e.g., when normalizing fraction after division or subtraction
- **Round bit**
  - Used to improve rounding accuracy
- **Sticky bit**
  - Used to support Round to nearest even
  - $0.50...00_{ten}$ vs. $0.50...01_{ten}$
  - Is set to a 1 whenever a 1 bit shifts (right) through it
  - e.g., when aligning fraction during addition/subtraction

F = 1 . xxxxxxxxxxxxxxxxxxxxxxx G R S

Example: Rounding with Guard Digits

- Add $2.56_{ten} \times 10^0$ to $2.34_{ten} \times 10^2$.
- Significant digit = 3 decimal digits
- **Round to the nearest number with and without guard and round digits.**

[Answer]

1) Without guard and round digits

$$2.34 \times 10^2 + 0.02 \times 10^2 = 2.36 \times 10^2$$

2) With guard and round digits

2.56×10^0 -> 0.0256×10^2 (guard = 5, round = 6)

$$2.3400 \times 10^2 + 0.0256 \times 10^2 = 2.3656 \times 10^2$$

=> rounded to 2.37×10^2

Elaboration

- **Rounding modes in IEEE 754**
  1) Always round up
     - Toward +∞
  2) Always round down (toward -∞)
     - Toward -∞
  3) Truncate
     - Toward 0
     - Round down if positive, up if negative
  4) Round to nearest even
     - When the Guard || Round || Sticky are 100
     - Always creates a 0 in the least significant bit of fraction

Examples

	7.3	7.5	8.5	7.7	-7.3	-7.5	-8.5	-7.7
Round up	8	8	9	8	-7	-7	-8	-7
Round down	7	7	8	7	-8	-8	-9	-8
Truncate	7	7	8	7	-7	-7	-8	-7
Round to nearest even	7	8	8	8	-7	-8	-8	-8

	+0001.01	-0001.01	+0101.10	+0100.10	-0011.10
Round up	+0010	-0001	+0110	+0101	-0011
Round down	+0001	-0010	+0101	+0100	-0100
Truncate	+0001	-0001	+0101	+0100	-0011
Round to nearest even	+0001	-0010	+0110	+0100	-0100