Force to Torque - Natural Orthogonal Complement Mapping Analysis

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1 Newton-Euler analysis

The analysis is performed base on the *Recursive Newton-Euler algorithm* Algorithm 1 equations. The objective of this document is to find a relation between the reaction force at the foot ($[\boldsymbol{f}_F]_0$) and the 3-dimensional torque at the hip ($\boldsymbol{\tau}_H = \begin{bmatrix} \tau_x & \tau_y & \tau_z \end{bmatrix}^T$).

Algorithm 1 Recursive Newton-Euler (Inverse dynamics)

function IDNE($model$, $\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{q}}$, $\boldsymbol{Q}|_1^N$, $\boldsymbol{\delta}|_1^N$, $\boldsymbol{f}$, $\boldsymbol{n}$)
▷ KINEMATIC COMPUTATIONS
$[\boldsymbol{c}_0]_1 \leftarrow \boldsymbol{0}$ ▷ Base CoM position
$[\boldsymbol{\omega}_0]_1 \leftarrow \boldsymbol{0}$ ▷ Base angular velocity
$[\dot{\boldsymbol{c}}_0]_1 \leftarrow \boldsymbol{0}$ ▷ Base CoM velocity
$[\dot{\boldsymbol{\omega}}_0]_1 \leftarrow \boldsymbol{0}$ ▷ Base angular acceleration
$[\ddot{\boldsymbol{c}}_0]_1 \leftarrow [-\boldsymbol{g}]_1$ ▷ Base CoM acceleration
$[\boldsymbol{\delta}_0]_1 \leftarrow \boldsymbol{0}$
for $i = 1 : N_B$ **do**
$[\boldsymbol{c}_i]_{i+1} \leftarrow \boldsymbol{Q}_i^T[\boldsymbol{c}_{i-1} + \boldsymbol{\delta}_{i-1}]_i + [\boldsymbol{\rho}_i]_{i+1}$
$[\boldsymbol{\omega}_i]_{i+1} \leftarrow \boldsymbol{Q}_i^T[\boldsymbol{\omega}_{i-1} + \dot{\theta}_i \boldsymbol{e}_i]_i$
$[\boldsymbol{u}_{i-1}]_i \leftarrow [\boldsymbol{\omega}_{i-1} \times \boldsymbol{\delta}_{i-1}]_i$
$[\boldsymbol{v}_i]_{i+1} \leftarrow [\boldsymbol{\omega}_i \times \boldsymbol{\rho}_i]_{i+1}$
$[\dot{\boldsymbol{c}}_i]_{i+1} \leftarrow \boldsymbol{Q}_i^T[\dot{\boldsymbol{c}}_{i-1} + \boldsymbol{u}_{i-1}]_i + [\boldsymbol{v}_i]_{i+1}$
$[\dot{\boldsymbol{\omega}}_i]_{i+1} \leftarrow \boldsymbol{Q}_i^T[\dot{\boldsymbol{\omega}}_{i-1} + \boldsymbol{\omega}_{i-1} \times \dot{\theta}_i \boldsymbol{e}_i + \ddot{\theta}_i \boldsymbol{e}_i]_i$
$[\ddot{\boldsymbol{c}}]_{i+1} \leftarrow \boldsymbol{Q}_i^T[\ddot{\boldsymbol{c}}_{i-1} + \dot{\boldsymbol{\omega}}_{i-1} \times \boldsymbol{\delta}_{i-1} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{u}_{i-1}]_i + [\dot{\boldsymbol{\omega}}_i \times \boldsymbol{\rho}_i + \boldsymbol{\omega}_i \times \boldsymbol{v}_i]_{i+1}$
end for
▷ DYNAMIC COMPUTATIONS
$[\boldsymbol{f}_{N_B}^P]_{N_B+1} \leftarrow [m_{N_B} \ddot{\boldsymbol{c}}_{N_B} - \boldsymbol{f}]_{N_B+1}$
$[\boldsymbol{n}_{N_B}^P]_{N_B+1} \leftarrow [\boldsymbol{I}_{N_B} \dot{\boldsymbol{\omega}}_{N_B} + \boldsymbol{\omega}_{N_B} \times \boldsymbol{I}_{N_B} \boldsymbol{\omega}_{N_B} - \boldsymbol{n} + \boldsymbol{\rho}_{N_B} \times \boldsymbol{f}_{N_B}^P]_{N_B+1}$
$[\tau_{N_B}]_{N_B} \leftarrow (\boldsymbol{Q}_{N_B}[\boldsymbol{n}_{N_B}^P]_{N_B+1})_z$
for $i = N_B - 1 : 1$ **do**
$[\boldsymbol{\phi}_{i+1}]_{i+1} \leftarrow \boldsymbol{Q}_{i+1}[\boldsymbol{f}_{i+1}^P]_{i+2}$
$[\boldsymbol{f}_i^P]_{i+1} \leftarrow [m_i \ddot{\boldsymbol{c}}_i + \boldsymbol{\phi}_{i+1}]_{i+1}$
$[\boldsymbol{n}_i^P]_{i+1} \leftarrow [\boldsymbol{I}_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \boldsymbol{I}_i \boldsymbol{\omega}_i + \boldsymbol{\rho}_i \times \boldsymbol{f}_i^P + \boldsymbol{\delta}_i \times \boldsymbol{\phi}_{i+1}]_{i+1} + \boldsymbol{Q}_{i+1}[\boldsymbol{n}_{i+1}^P]_{i+2}$
$[\tau_i]_i \leftarrow (\boldsymbol{Q}_i[\boldsymbol{n}_i^P]_{i+1})_z$
end for
end function

1.1 Dynamic Computations - Inward Recursion

Let

$$\boldsymbol{e} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

This vector denotes the rotation orientation of each generalized coordinate. For the case of the Newton-Euler algorithm and the architecture of the spatial pendulum this orientation is always parallel to the Z axis of the body-coordinate frame in the *six bodies model*.

The *six bodies model* for the spatial pendulum consists in one inertial frame (F_0) and 6 body-coordinate frames

$(F_2 - F_7)$. Now we define the equations for the forces and moments using this model and Algorithm 1

$$
\begin{aligned}
[\boldsymbol{f}_6]_7 &= m_6 \ddot{\boldsymbol{c}}_6 \\
[\boldsymbol{n}_6]_7 &= \boldsymbol{I}_6 \dot{\boldsymbol{\omega}}_6 + \boldsymbol{\omega}_6 \times \boldsymbol{I}_6 \boldsymbol{\omega}_6 + \boldsymbol{\rho}_6 \times \boldsymbol{f}_6 \\
\tau_6 &= \boldsymbol{e}^T \boldsymbol{Q}_6 \\
[\boldsymbol{f}_5]_6 &= \boldsymbol{Q}_6 [\boldsymbol{f}_6]_7 \\
[\boldsymbol{n}_5]_6 &= \boldsymbol{Q}_6 [\boldsymbol{n}_6]_7 \\
\tau_5 &= \boldsymbol{e}^T \boldsymbol{Q}_5 [\boldsymbol{n}_5]_6 = \boldsymbol{e}^T \boldsymbol{Q}_5 \boldsymbol{Q}_6 [\boldsymbol{n}_6]_7 \\
[\boldsymbol{f}_4]_5 &= \boldsymbol{Q}_5 [\boldsymbol{f}_5]_6 = \boldsymbol{Q}_5 \boldsymbol{Q}_6 [\boldsymbol{f}_6]_7 \\
[\boldsymbol{n}_4]_5 &= \boldsymbol{Q}_5 [\boldsymbol{n}_5]_6 = \boldsymbol{Q}_5 \boldsymbol{Q}_6 [\boldsymbol{n}_6]_7 \\
\tau_4 &= \boldsymbol{e}^T \boldsymbol{Q}_4 [\boldsymbol{n}_4]_5 = \boldsymbol{e}^T \boldsymbol{Q}_4 \boldsymbol{Q}_5 \boldsymbol{Q}_6 [\boldsymbol{n}_6]_7 \\
[\boldsymbol{f}_3]_4 &= m_3 \ddot{\boldsymbol{c}}_3 + \boldsymbol{Q}_4 [\boldsymbol{f}_4]_5 \\
&= m_3 \ddot{\boldsymbol{c}}_3 + \boldsymbol{Q}_4 \boldsymbol{Q}_5 \boldsymbol{Q}_6 [\boldsymbol{f}_6]_7 \\
[\boldsymbol{n}_3]_4 &= \boldsymbol{I}_3 \dot{\boldsymbol{\omega}}_3 + \boldsymbol{\omega}_3 \times \boldsymbol{I}_3 \boldsymbol{\omega}_3 + \boldsymbol{\rho}_3 \times \boldsymbol{f}_3 + \delta_3 \times \boldsymbol{Q}_4 [\boldsymbol{f}_4]_5 + \boldsymbol{Q}_4 [\boldsymbol{n}_4]_5 \\
&= \boldsymbol{I}_3 \dot{\boldsymbol{\omega}}_3 + \boldsymbol{\omega}_3 \times \boldsymbol{I}_3 \boldsymbol{\omega}_3 + \boldsymbol{\rho}_3 \times \boldsymbol{f}_3 + \boldsymbol{\delta}_3 \times \boldsymbol{Q}_4 \boldsymbol{Q}_5 \boldsymbol{Q}_6 [\boldsymbol{f}_6]_7 + \boldsymbol{Q}_4 \boldsymbol{Q}_5 \boldsymbol{Q}_6 [\boldsymbol{n}_6]_7 \\
\tau_3 &= \boldsymbol{e}^T \boldsymbol{Q}_3 [\boldsymbol{n}_3]_4 \\
[\boldsymbol{f}_2]_3 &= \boldsymbol{Q}_3 [\boldsymbol{f}_3]_4 \\
[\boldsymbol{n}_2]_3 &= \boldsymbol{Q}_3 [\boldsymbol{n}_3]_4 \\
\tau_2 &= \boldsymbol{e}^T \boldsymbol{Q}_2 [\boldsymbol{n}_2]_3 = \boldsymbol{e}^T \boldsymbol{Q}_2 \boldsymbol{Q}_3 [\boldsymbol{n}_3]_4 \\
[\boldsymbol{f}_1]_2 &= \boldsymbol{Q}_2 [\boldsymbol{f}_2]_3 = \boldsymbol{Q}_2 \boldsymbol{Q}_3 [\boldsymbol{f}_3]_4 \\
[\boldsymbol{n}_1]_2 &= \boldsymbol{Q}_2 [\boldsymbol{n}_2]_3 = \boldsymbol{Q}_2 \boldsymbol{Q}_3 [\boldsymbol{n}_3]_4 \\
\tau_1 &= \boldsymbol{e}^T \boldsymbol{Q}_1 [\boldsymbol{n}_1]_2 = \boldsymbol{e}^T \boldsymbol{Q}_1 \boldsymbol{Q}_2 \boldsymbol{Q}_3 [\boldsymbol{n}_3]_4
\end{aligned}
$$

Note that $\boldsymbol{I}_3$ and $\boldsymbol{I}_6$ denote 3×3 inertia matrices corresponding to the lower and upper link, respectively, and should not be confused with the generalized inertia matrix $\boldsymbol{I}(\boldsymbol{q})$.From these equations we conclude the system can be reduced from 7 coordinate frames to 3 coordinate frames: inertial frame (F_0) and 2 body-coordinate frames (F_1 and F_2)

$$[\boldsymbol{f}_H]_2 = m_2 \ddot{\boldsymbol{c}}_2 \tag{1}$$
$$[\boldsymbol{n}_H]_2 = \boldsymbol{I}_2 \dot{\boldsymbol{\omega}}_H + \boldsymbol{\omega}_H \times \boldsymbol{I}_2 \boldsymbol{\omega}_H + \boldsymbol{\rho}_2 \times \boldsymbol{f}_H \tag{2}$$

$$[\boldsymbol{f}_F]_1 = m_1 \ddot{\boldsymbol{c}}_{\boldsymbol{1}} + \boldsymbol{R}_2 [\boldsymbol{f}_H]_2 \tag{3}$$
$$[\boldsymbol{n}_F]_1 = \boldsymbol{I}_1 \dot{\boldsymbol{\omega}}_F + \boldsymbol{\omega}_F \times \boldsymbol{I}_1 \boldsymbol{\omega}_F + \boldsymbol{\rho}_1 \times \boldsymbol{f}_F + \boldsymbol{\delta}_1 \times \boldsymbol{R}_2 [\boldsymbol{f}_H]_2 + \boldsymbol{R}_2 [\boldsymbol{n}_H]_2 \tag{4}$$

and the expressions for $\boldsymbol{\tau}_F$ and $\boldsymbol{\tau}_H$ becomes

$$\boldsymbol{\tau}_F = \begin{bmatrix} \tau_{Fx} \\ \tau_{Fy} \\ \tau_{Fz} \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}^T \boldsymbol{R}_1 \\ \boldsymbol{e}^T \boldsymbol{Q}_2 \boldsymbol{Q}_3 \\ \boldsymbol{e}^T \boldsymbol{Q}_3 \end{bmatrix} [\boldsymbol{n}_F]_1 \quad ; \quad \boldsymbol{\tau}_H = \begin{bmatrix} \tau_{Hx} \\ \tau_{Hy} \\ \tau_{Hz} \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}^T \boldsymbol{R}_2 \\ \boldsymbol{e}^T \boldsymbol{Q}_5 \boldsymbol{Q}_6 \\ \boldsymbol{e}^T \boldsymbol{Q}_6 \end{bmatrix} [\boldsymbol{n}_H]_2 \tag{5}$$

where

$$\boldsymbol{R}_1 = \boldsymbol{Q}_1 \boldsymbol{Q}_2 \boldsymbol{Q}_3 \quad ; \quad \boldsymbol{R}_2 = \boldsymbol{Q}_4 \boldsymbol{Q}_5 \boldsymbol{Q}_6 \tag{6}$$

are rotation matrices that establish the following relation between coordinate frames

$$[\boldsymbol{f}_F]_2 = {\boldsymbol{R}_2}^T [\boldsymbol{f}_F]_1 \quad ; \quad [\boldsymbol{f}_F]_1 = {\boldsymbol{R}_1}^T [\boldsymbol{f}_F]_0 \tag{7}$$

Note that the notation $[\bullet]_i$ is omitted in those vectors and matrices expressed in the same frame that the vector on the LHS of the equation. The subscripts H and F denotes *hip* and *foot* respectively, and they are used to identify the vectors related to the first spherical joint at the foot or the second at the hip.

1.2 Kinematic Computations - Outwards Recursion

Find an expression for $\dot{\boldsymbol{\omega}}_3$ and $\ddot{\boldsymbol{c}}_3$ using the *six bodies model* and Algorithm 1 equations

$$
\begin{aligned}
[\dot{\boldsymbol{\omega}}_1]_2 &= \boldsymbol{Q}_1{}^T[\ddot{q}_1\boldsymbol{e}]_1 \\
[\ddot{\boldsymbol{c}}_1]_2 &= \boldsymbol{Q}_1{}^T[\ddot{\boldsymbol{c}}_0]_1 \quad \text{where} \quad \ddot{\boldsymbol{c}}_0 = \boldsymbol{g}
\end{aligned}
$$

$$
\begin{aligned}
[\dot{\boldsymbol{\omega}}_2]_3 &= \boldsymbol{Q}_2{}^T[\dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times \dot{q}_2\boldsymbol{e} + \ddot{q}_2\boldsymbol{e}]_2 \\
&= (\boldsymbol{Q}_1\boldsymbol{Q}_2)^T[\ddot{q}_1\boldsymbol{e}]_1 + \boldsymbol{Q}_2{}^T[\boldsymbol{\omega}_1 \times \dot{q}_2\boldsymbol{e}]_2 + \boldsymbol{Q}_2{}^T[\ddot{q}_2\boldsymbol{e}]_2 \\
[\ddot{\boldsymbol{c}}_2]_3 &= \boldsymbol{Q}_2{}^T[\ddot{\boldsymbol{c}}_1]_2 = (\boldsymbol{Q}_1\boldsymbol{Q}_2)^T[\ddot{\boldsymbol{c}}_0]_1
\end{aligned}
$$

$$
\begin{aligned}
[\dot{\boldsymbol{\omega}}_3]_4 &= \boldsymbol{Q}_3{}^T[\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times \dot{q}_3\boldsymbol{e} + \ddot{q}_3\boldsymbol{e}]_3 \\
&= \boldsymbol{Q}_3{}^T[\boldsymbol{\omega}_2 \times \dot{q}_3\boldsymbol{e}]_3 + (\boldsymbol{Q}_2\boldsymbol{Q}_3)^T[\boldsymbol{\omega}_1 \times \dot{q}_2\boldsymbol{e}]_2 + (\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3)^T[\ddot{q}_1\boldsymbol{e}]_1 + (\boldsymbol{Q}_2\boldsymbol{Q}_3)^T[\ddot{q}_2\boldsymbol{e}]_2 + \boldsymbol{Q}_3{}^T[\ddot{q}_3\boldsymbol{e}]_3
\end{aligned}
$$

$$
\begin{aligned}
[\ddot{\boldsymbol{c}}_3]_4 &= \boldsymbol{Q}_3{}^T[\ddot{\boldsymbol{c}}_2]_3 + \dot{\boldsymbol{\omega}}_3 \times \boldsymbol{\rho}_3 + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \boldsymbol{\rho}_3) \\
&= (\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3)^T[\ddot{\boldsymbol{c}}_0]_1 + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \boldsymbol{\rho}_3) - \boldsymbol{\rho}_3 \times \dot{\boldsymbol{\omega}}_3
\end{aligned}
$$

$$
\begin{aligned}
[\dot{\boldsymbol{\omega}}_4]_5 &= \boldsymbol{Q}_4{}^T[\dot{\boldsymbol{\omega}}_3 + \boldsymbol{\omega}_3 \times \dot{q}_4\boldsymbol{e} + \ddot{q}_4\boldsymbol{e}]_4 \\
[\ddot{\boldsymbol{c}}_4]_5 &= \boldsymbol{Q}_4{}^T[\ddot{\boldsymbol{c}}_3 + \dot{\boldsymbol{\omega}}_3 \times \boldsymbol{\delta}_3 + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \boldsymbol{\delta}_3)]_4
\end{aligned}
$$

$$
\begin{aligned}
[\dot{\boldsymbol{\omega}}_5]_6 &= \boldsymbol{Q}_5{}^T[\dot{\boldsymbol{\omega}}_4 + \boldsymbol{\omega}_4 \times \dot{q}_5\boldsymbol{e} + \ddot{q}_5\boldsymbol{e}]_5 \\
&= (\boldsymbol{Q}_4\boldsymbol{Q}_5)^T[\dot{\boldsymbol{\omega}}_3]_4 + (\boldsymbol{Q}_4\boldsymbol{Q}_5)^T[\boldsymbol{\omega}_3 \times \dot{q}_4\boldsymbol{e}]_4 + (\boldsymbol{Q}_4\boldsymbol{Q}_5)^T[\ddot{q}_4\boldsymbol{e}]_4 + \boldsymbol{Q}_5{}^T[\boldsymbol{\omega}_4 \times \dot{q}_5\boldsymbol{e}]_5 + \boldsymbol{Q}_5{}^T[\ddot{q}_5\boldsymbol{e}]_5 \\
[\ddot{\boldsymbol{c}}_5]_6 &= (\boldsymbol{Q}_4\boldsymbol{Q}_5)^T[\ddot{\boldsymbol{c}}_3 + \dot{\boldsymbol{\omega}}_3 \times \boldsymbol{\delta}_3 + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \boldsymbol{\delta}_3)]_4
\end{aligned}
$$

$$
\begin{aligned}
[\dot{\boldsymbol{\omega}}_6]_7 &= \boldsymbol{Q}_6{}^T[\dot{\boldsymbol{\omega}}_5 + \boldsymbol{\omega}_5 \times \dot{q}_6\boldsymbol{e} + \ddot{q}_6\boldsymbol{e}]_6 \\
&= (\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6)^T[\dot{\boldsymbol{\omega}}_3]_4 + (\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6)^T[\boldsymbol{\omega}_3 \times \dot{q}_4\boldsymbol{e}]_4 + (\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6)^T[\ddot{q}_4\boldsymbol{e}]_4 + (\boldsymbol{Q}_5\boldsymbol{Q}_6)^T[\boldsymbol{\omega}_4 \times \dot{q}_5\boldsymbol{e}]_5 + (\boldsymbol{Q}_5\boldsymbol{Q}_6)^T[\ddot{q}_5\boldsymbol{e}]_5 \\
&\quad + \boldsymbol{Q}_6{}^T[\boldsymbol{\omega}_5 \times \dot{q}_6\boldsymbol{e}]_6 + \boldsymbol{Q}_6{}^T[\ddot{q}_6\boldsymbol{e}]_6 \\
[\ddot{\boldsymbol{c}}_6]_7 &= (\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6)^T[\ddot{c}_3 + \dot{\boldsymbol{\omega}}_3 \times \boldsymbol{\delta}_3 + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \boldsymbol{\delta}_3)]_4 + \dot{\boldsymbol{\omega}}_6 \times \boldsymbol{\rho}_6 + \boldsymbol{\omega}_6 \times (\boldsymbol{\omega}_6 \times \boldsymbol{\rho}_6)
\end{aligned}
$$

Then, the system can be reduced to 3 coordinate frames: 2 body-coordinate frames (F_1, F_2) and an inertial frame (F_0), given by

$$
\begin{aligned}
[\dot{\boldsymbol{\omega}}_F]_1 &= \boldsymbol{k}_F + \boldsymbol{A}_F\,\ddot{\boldsymbol{q}}_F &(8)\\
[\ddot{\boldsymbol{c}}_1]_1 &= \boldsymbol{k}_1 - CPM(\boldsymbol{\rho}_1)\dot{\boldsymbol{\omega}}_F &(9)
\end{aligned}
$$

$$
\begin{aligned}
[\dot{\boldsymbol{\omega}}_H]_2 &= \boldsymbol{R}_2{}^T[\dot{\boldsymbol{\omega}}_F]_1 + \boldsymbol{k}_H + \boldsymbol{A}_H\ddot{\boldsymbol{q}}_H &(10)\\
[\ddot{\boldsymbol{c}}_2]_2 &= \boldsymbol{k}_2 + \boldsymbol{R}_2{}^T[\ddot{\boldsymbol{c}}_1 - CPM(\boldsymbol{\delta}_1)\,\dot{\boldsymbol{\omega}}_F]_1 - CPM(\boldsymbol{\rho}_2)\dot{\boldsymbol{\omega}}_H &(11)
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{k}_F(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \boldsymbol{Q}_3{}^T[\boldsymbol{\omega}_2 \times \dot{q}_3\boldsymbol{e}]_3 + (\boldsymbol{Q}_2\boldsymbol{Q}_3)^T[\boldsymbol{\omega}_1 \times \dot{q}_2\boldsymbol{e}]_2 \\
\boldsymbol{k}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{g}) &= \boldsymbol{R}_1{}^T[\ddot{\boldsymbol{c}}_0]_0 + \boldsymbol{\omega}_F \times (\boldsymbol{\omega}_F \times \boldsymbol{\rho}_1) \\
\boldsymbol{k}_H(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= (\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6)^T[\boldsymbol{\omega}_3 \times \dot{q}_4\boldsymbol{e}]_4 + (\boldsymbol{Q}_5\boldsymbol{Q}_6)^T[\boldsymbol{\omega}_4 \times \dot{q}_5\boldsymbol{e}]_5 + \boldsymbol{Q}_6{}^T[\boldsymbol{\omega}_5 \times \dot{q}_6\boldsymbol{e}]_6 \\
\boldsymbol{k}_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \boldsymbol{R}_2{}^T[\boldsymbol{\omega}_F \times (\boldsymbol{\omega}_F \times \boldsymbol{\delta}_1)]_1 + \boldsymbol{\omega}_H \times (\boldsymbol{\omega}_H \times \boldsymbol{\rho}_2) \\
\boldsymbol{A}_F(\boldsymbol{q}) &= \begin{bmatrix} \boldsymbol{R}_1{}^T\boldsymbol{e} & (\boldsymbol{Q}_2\boldsymbol{Q}_3)^T\boldsymbol{e} & \boldsymbol{Q}_3{}^T\boldsymbol{e} \end{bmatrix} \quad ; \quad \boldsymbol{A}_H(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{R}_2{}^T\boldsymbol{e} & (\boldsymbol{Q}_5\boldsymbol{Q}_6)^T & \boldsymbol{Q}_6{}^T\boldsymbol{e} \end{bmatrix} \\
\boldsymbol{\rho}\times = CPM(\boldsymbol{\rho}) &= \begin{bmatrix} 0 & -\rho_z & \rho_y \\ \rho_z & 0 & -\rho_x \\ -\rho_y & \rho_x & 0 \end{bmatrix} \quad \text{given that} \quad \boldsymbol{\rho} = \begin{bmatrix} \rho_x \\ \rho_y \\ \rho_z \end{bmatrix}
\end{aligned}
$$

Applying Eq. (8) in Eq. (9) gives

$$
\begin{aligned}
[\ddot{\boldsymbol{c}}_1]_1 &= \boldsymbol{k}_1 - CPM(\boldsymbol{\rho}_1)[\boldsymbol{k}_F + \boldsymbol{A}_F\,\ddot{\boldsymbol{q}}_F] \\
&= \boldsymbol{k}_3 - CPM(\boldsymbol{\rho}_1)\boldsymbol{A}_F\,\ddot{\boldsymbol{q}}_F &(12)
\end{aligned}
$$

Substituting Eq. (8) in Eq. (10) yields

$$\begin{aligned}[\dot{\boldsymbol{\omega}}_H]_2 &= \boldsymbol{R_2}^T[\boldsymbol{k}_F + \boldsymbol{A}_F\ddot{\boldsymbol{q}}_F]_1 + \boldsymbol{k}_H + \boldsymbol{A}_H\ddot{\boldsymbol{q}}_H \\ &= \boldsymbol{k}_{H2} + \boldsymbol{R_2}^T\boldsymbol{A}_F\ddot{\boldsymbol{q}}_F + \boldsymbol{A}_H\ddot{\boldsymbol{q}}_H \end{aligned} \tag{13}$$

Applying this equation to Eq. (11) gives

$$[\ddot{\boldsymbol{c}}_2]_2 = \boldsymbol{k}_2 + \boldsymbol{R_2}^T[\ddot{\boldsymbol{c}}_1 - CPM(\boldsymbol{\delta}_1)\,\dot{\boldsymbol{\omega}}_F]_1 - CPM(\boldsymbol{\rho}_2)[\boldsymbol{k}_{H2} + \boldsymbol{R_2}^T\boldsymbol{A}_F\ddot{\boldsymbol{q}}_F + \boldsymbol{A}_H\ddot{\boldsymbol{q}}_H] \tag{14}$$

Substituting Eq. (9) and Eq. (8) yields

$$\begin{aligned}[\ddot{\boldsymbol{c}}_2]_2 &= \boldsymbol{k}_2 + \boldsymbol{R_2}^T[\boldsymbol{k}_1 - CPM(\boldsymbol{\rho}_1)\dot{\boldsymbol{\omega}}_F - CPM(\boldsymbol{\delta}_1)\,\dot{\boldsymbol{\omega}}_F]_1 - CPM(\boldsymbol{\rho}_2)[\boldsymbol{k}_{H2} + \boldsymbol{R_2}^T\boldsymbol{A}_F\ddot{\boldsymbol{q}}_F + \boldsymbol{A}_H\ddot{\boldsymbol{q}}_H] \\ &= \boldsymbol{k}_2 + \boldsymbol{R_2}^T[\boldsymbol{k}_1 - CPM(\boldsymbol{\rho}_1 + \boldsymbol{\delta}_1)\,\dot{\boldsymbol{\omega}}_F]_1 - CPM(\boldsymbol{\rho}_2)[\boldsymbol{k}_{H2} + \boldsymbol{R_2}^T\boldsymbol{A}_F\ddot{\boldsymbol{q}}_F + \boldsymbol{A}_H\ddot{\boldsymbol{q}}_H] \\ &= \boldsymbol{k}_2 + \boldsymbol{R_2}^T[\boldsymbol{k}_1 - CPM(\boldsymbol{\rho}_1 + \boldsymbol{\delta}_1)(\boldsymbol{k}_F + \boldsymbol{A}_F\,\ddot{\boldsymbol{q}}_F)]_1 - CPM(\boldsymbol{\rho}_2)[\boldsymbol{k}_{H2} + \boldsymbol{R_2}^T\boldsymbol{A}_F\ddot{\boldsymbol{q}}_F + \boldsymbol{A}_H\ddot{\boldsymbol{q}}_H] \\ &= \boldsymbol{k}_4 - \boldsymbol{R_2}^T CPM(\boldsymbol{\rho}_1 + \boldsymbol{\delta}_1)\boldsymbol{A}_F\,\ddot{\boldsymbol{q}}_F - CPM(\boldsymbol{\rho}_2)[\boldsymbol{R_2}^T\boldsymbol{A}_F\ddot{\boldsymbol{q}}_F + \boldsymbol{A}_H\ddot{\boldsymbol{q}}_H] \\ &= \boldsymbol{k}_4 - \left[\boldsymbol{R_2}^T CPM(\boldsymbol{\rho}_1 + \boldsymbol{\delta}_1) + CPM(\boldsymbol{\rho}_2)\boldsymbol{R_2}^T\right]\boldsymbol{A}_F\,\ddot{\boldsymbol{q}}_F - CPM(\boldsymbol{\rho}_2)\boldsymbol{A}_H\ddot{\boldsymbol{q}}_H \end{aligned} \tag{15}$$

Recall the expression for $\boldsymbol{f}_F$, Eq. (3)

$$\begin{aligned}[\boldsymbol{f}_F]_1 &= m_1\ddot{\boldsymbol{c}}_1 + \boldsymbol{R}_2[\boldsymbol{f}_H]_2 \\ &= m_1\ddot{\boldsymbol{c}}_1 + m_2\boldsymbol{R}_2[\ddot{\boldsymbol{c}}_2]_2 \end{aligned} \tag{16}$$

Expressed at the inertia frame becomes

$$[\boldsymbol{f}_F]_0 = m_1\boldsymbol{R}_1[\ddot{\boldsymbol{c}}_1]_1 + m_2\boldsymbol{R}_1\boldsymbol{R}_2[\ddot{\boldsymbol{c}}_2]_2 \tag{17}$$

Substituting Eq. (12) yields

$$\begin{aligned}[\boldsymbol{f}_F]_0 &= m_1\boldsymbol{R}_1[\boldsymbol{k}_3 - CPM(\boldsymbol{\rho}_1)\boldsymbol{A}_F\,\ddot{\boldsymbol{q}}_F]_1 + m_2\boldsymbol{R}_1\boldsymbol{R}_2[\boldsymbol{k}_4 - \left[\boldsymbol{R_2}^T CPM(\boldsymbol{\rho}_1 + \boldsymbol{\delta}_1) + CPM(\boldsymbol{\rho}_2)\boldsymbol{R_2}^T\right]\boldsymbol{A}_F\,\ddot{\boldsymbol{q}}_F \\ &\quad - CPM(\boldsymbol{\rho}_2)\boldsymbol{A}_H\ddot{\boldsymbol{q}}_H]_2 \\ &= \boldsymbol{k}_5 + \boldsymbol{F}\boldsymbol{A}_F\,\ddot{\boldsymbol{q}}_F + \boldsymbol{G}\boldsymbol{A}_H\ddot{\boldsymbol{q}}_H \end{aligned} \tag{18}$$

where

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{q}) &= -\left[m_1\boldsymbol{R}_1 CPM(\boldsymbol{\rho}_1) + m_2\boldsymbol{R}_1 CPM(\boldsymbol{\rho}_1 + \boldsymbol{\delta}_1) + m_2\boldsymbol{R}_1\boldsymbol{R}_2 CPM(\boldsymbol{\rho}_2)\boldsymbol{R_2}^T\right] \\ \boldsymbol{G}(\boldsymbol{q}) &= -m_2\boldsymbol{R}_1\boldsymbol{R}_2 CPM(\boldsymbol{\rho}_2)\end{aligned}$$

In the next section we find an expression for $\ddot{\boldsymbol{q}}_F$ and $\ddot{\boldsymbol{q}}_H$ in terms of $\boldsymbol{\tau}_H$

1.3 Equation of motion of the spatial pendulum

The equation of motion of the spatial pendulum is

$$\boldsymbol{I}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \gamma(\boldsymbol{q}) = \boldsymbol{\tau} \tag{19}$$

$$\begin{bmatrix}\boldsymbol{I}_{11} & \boldsymbol{I}_{12} \\ \boldsymbol{I}_{21} & \boldsymbol{I}_{22}\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{q}}_F \\ \ddot{\boldsymbol{q}}_H\end{bmatrix} + \begin{bmatrix}\boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{q}}_F \\ \dot{\boldsymbol{q}}_H\end{bmatrix} - \begin{bmatrix}\gamma_F \\ \gamma_H\end{bmatrix} = \begin{bmatrix}\boldsymbol{\tau}_F \\ \boldsymbol{\tau}_H\end{bmatrix} \tag{20}$$

where $\boldsymbol{I}(\boldsymbol{q})$ is the 6×6 generalized inertia matrix and $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Coriolis coefficient matrix. We are looking for an expression for $\ddot{\boldsymbol{q}}_F$ without considering gravity and only one active joint at the hip. Thus, the equation of motion becomes

$$\begin{bmatrix}\boldsymbol{I}_{11} & \boldsymbol{I}_{12} \\ \boldsymbol{I}_{21} & \boldsymbol{I}_{22}\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{q}}_F \\ \ddot{\boldsymbol{q}}_H\end{bmatrix} + \begin{bmatrix}\boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{q}}_F \\ \dot{\boldsymbol{q}}_H\end{bmatrix} = \begin{bmatrix}\boldsymbol{0} \\ \boldsymbol{\tau}_H\end{bmatrix} \tag{21}$$

So

$$\ddot{\boldsymbol{q}}_F = \bar{\boldsymbol{I}}_{12}\boldsymbol{\tau}_H + \boldsymbol{k}_{MF} \tag{22}$$

$$\ddot{\boldsymbol{q}}_H = \bar{\boldsymbol{I}}_{22}\boldsymbol{\tau}_H + \boldsymbol{k}_{MH} \tag{23}$$

where

$$\bar{\boldsymbol{I}}_{12}(\boldsymbol{q}) = \begin{bmatrix}\boldsymbol{1} & \boldsymbol{0}\end{bmatrix} \boldsymbol{I}(\boldsymbol{q})^{-1} \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{1}\end{bmatrix} \quad ; \quad \boldsymbol{k}_{MF}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\begin{bmatrix}\boldsymbol{1} & \boldsymbol{0}\end{bmatrix} \boldsymbol{I}(\boldsymbol{q})^{-1}\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$$

$$\bar{\boldsymbol{I}}_{22}(\boldsymbol{q}) = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{1}\end{bmatrix} \boldsymbol{I}(\boldsymbol{q})^{-1} \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{1}\end{bmatrix} \quad ; \quad \boldsymbol{k}_{MH}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\begin{bmatrix}\boldsymbol{0} & \boldsymbol{1}\end{bmatrix} \boldsymbol{I}(\boldsymbol{q})^{-1}\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$$

Substituting the result in Eq. (19) yields

$$\begin{aligned}[\boldsymbol{f}_F]_0 &= \boldsymbol{k}_5 + \boldsymbol{F}\boldsymbol{A}_F[\bar{\boldsymbol{I}}_{12}\boldsymbol{\tau}_H + \boldsymbol{k}_{MF}] + \boldsymbol{G}\boldsymbol{A}_H[\bar{\boldsymbol{I}}_{22}\boldsymbol{\tau}_H + \boldsymbol{k}_{MH}]\\ &= \boldsymbol{b} + \left[\boldsymbol{F}\boldsymbol{A}_F\bar{\boldsymbol{I}}_{12} + \boldsymbol{G}\boldsymbol{A}_H\bar{\boldsymbol{I}}_{22}\right]\boldsymbol{\tau}_H\\ &= \boldsymbol{b} + \boldsymbol{M}\boldsymbol{\tau}_H \end{aligned} \tag{24}$$

2 Summary

2.1 Reduced Newton-Euler equations

$$[\boldsymbol{f}_H]_2 = m_2\ddot{\boldsymbol{c}}_2 \tag{25}$$

$$[\boldsymbol{n}_H]_2 = \boldsymbol{I}_2\dot{\boldsymbol{\omega}}_H + \boldsymbol{\omega}_H \times \boldsymbol{I}_2\boldsymbol{\omega}_H + \boldsymbol{\rho}_2 \times \boldsymbol{f}_H \tag{26}$$

$$[\boldsymbol{f}_F]_1 = m_1\ddot{\boldsymbol{c}}_1 + \boldsymbol{R}_2[\boldsymbol{f}_H]_2 \tag{27}$$

$$[\boldsymbol{n}_F]_1 = \boldsymbol{I}_1\dot{\boldsymbol{\omega}}_F + \boldsymbol{\omega}_F \times \boldsymbol{I}_1\boldsymbol{\omega}_F + \boldsymbol{\rho}_1 \times \boldsymbol{f}_F + \boldsymbol{\delta}_1 \times \boldsymbol{R}_2[\boldsymbol{f}_H]_2 + \boldsymbol{R}_2[\boldsymbol{n}_H]_2 \tag{28}$$

$$[\dot{\boldsymbol{\omega}}_F]_1 = \boldsymbol{k}_F + \boldsymbol{A}_F\,\ddot{\boldsymbol{q}}_F \tag{29}$$

$$[\ddot{\boldsymbol{c}}_1]_1 = \boldsymbol{k}_1 - CPM(\boldsymbol{\rho}_1)\dot{\boldsymbol{\omega}}_F \tag{30}$$

$$[\dot{\boldsymbol{\omega}}_H]_2 = \boldsymbol{R}_2{}^T[\dot{\boldsymbol{\omega}}_F]_1 + \boldsymbol{k}_H + \boldsymbol{A}_H\ddot{\boldsymbol{q}}_H \tag{31}$$

$$[\ddot{\boldsymbol{c}}_2]_2 = \boldsymbol{k}_2 + \boldsymbol{R}_2{}^T[\ddot{\boldsymbol{c}}_1 - CPM(\boldsymbol{\delta}_1)\,\dot{\boldsymbol{\omega}}_F]_1 - CPM(\boldsymbol{\rho}_2)\dot{\boldsymbol{\omega}}_H \tag{32}$$

where

$$\begin{aligned}\boldsymbol{k}_F(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \boldsymbol{Q}_3{}^T[\boldsymbol{\omega}_2 \times \dot{q}_3\boldsymbol{e}]_3 + (\boldsymbol{Q}_2\boldsymbol{Q}_3)^T[\boldsymbol{\omega}_1 \times \dot{q}_2\boldsymbol{e}]_2\\ \boldsymbol{k}_1(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{g}) &= \boldsymbol{R}_1{}^T[\ddot{\boldsymbol{c}}_0]_0 + \boldsymbol{\omega}_F \times (\boldsymbol{\omega}_F \times \boldsymbol{\rho}_1)\\ \boldsymbol{k}_H(\boldsymbol{q},\dot{\boldsymbol{q}}) &= (\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6)^T[\boldsymbol{\omega}_3 \times \dot{q}_4\boldsymbol{e}]_4 + (\boldsymbol{Q}_5\boldsymbol{Q}_6)^T[\boldsymbol{\omega}_4 \times \dot{q}_5\boldsymbol{e}]_5 + \boldsymbol{Q}_6{}^T[\boldsymbol{\omega}_5 \times \dot{q}_6\boldsymbol{e}]_6\\ \boldsymbol{k}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \boldsymbol{R}_2{}^T[\boldsymbol{\omega}_F \times (\boldsymbol{\omega}_F \times \boldsymbol{\delta}_1)]_1 + \boldsymbol{\omega}_H \times (\boldsymbol{\omega}_H \times \boldsymbol{\rho}_2)\\ \boldsymbol{A}_F(\boldsymbol{q}) &= \begin{bmatrix}\boldsymbol{R}_1{}^T\boldsymbol{e} & (\boldsymbol{Q}_2\boldsymbol{Q}_3)^T\boldsymbol{e} & \boldsymbol{Q}_3{}^T\boldsymbol{e}\end{bmatrix} \quad ; \quad \boldsymbol{A}_H(\boldsymbol{q}) = \begin{bmatrix}\boldsymbol{R}_2{}^T\boldsymbol{e} & (\boldsymbol{Q}_5\boldsymbol{Q}_6)^T & \boldsymbol{Q}_6{}^T\boldsymbol{e}\end{bmatrix}\end{aligned}$$

and

$$\boldsymbol{\tau}_H = \begin{bmatrix}\tau_{Hx}\\ \tau_{Hy}\\ \tau_{Hz}\end{bmatrix} = \begin{bmatrix}\boldsymbol{e}^T\boldsymbol{R}_2\\ \boldsymbol{e}^T\boldsymbol{Q}_5\boldsymbol{Q}_6\\ \boldsymbol{e}^T\boldsymbol{Q}_6\end{bmatrix}[\boldsymbol{n}_H]_2 \tag{33}$$

2.2 Mapping

$$\begin{aligned}\boldsymbol{f}_F &= \boldsymbol{b} + \left[\boldsymbol{F}\boldsymbol{A}_F\bar{\boldsymbol{I}}_{12} + \boldsymbol{G}\boldsymbol{A}_H\bar{\boldsymbol{I}}_{22}\right]\boldsymbol{\tau}_H\\ &= \boldsymbol{b} + \boldsymbol{M}\boldsymbol{\tau}_H\end{aligned} \tag{34}$$

where

$$\boldsymbol{b} = \boldsymbol{f}_0$$

meaning $\boldsymbol{b}$ is the reaction force vector at the foot when $\boldsymbol{\tau}_H = \boldsymbol{0}$. The expression for matrices $\boldsymbol{F}, \boldsymbol{A}_F, \boldsymbol{G}$ and $\boldsymbol{A}_H$ is given in next section.

3 Coefficients analysis

$$\boldsymbol{A}_F(\boldsymbol{q}_F) = \begin{bmatrix} \boldsymbol{R}_1{}^T\boldsymbol{e} & (\boldsymbol{Q}_2\boldsymbol{Q}_3)^T\boldsymbol{e} & \boldsymbol{Q}_3{}^T\boldsymbol{e} \end{bmatrix} = \begin{bmatrix} \cos\bar{q}_3\sin\bar{q}_2 & \sin\bar{q}_3 & 0 \\ -\sin\bar{q}_2\sin\bar{q}_3 & \cos\bar{q}_3 & 0 \\ -\cos\bar{q}_2 & 0 & 1 \end{bmatrix} \tag{35}$$

$$\boldsymbol{A}_H(\boldsymbol{q}_H) = \begin{bmatrix} \boldsymbol{R}_2{}^T\boldsymbol{e} & (\boldsymbol{Q}_5\boldsymbol{Q}_6)^T & \boldsymbol{Q}_6{}^T\boldsymbol{e} \end{bmatrix} = \begin{bmatrix} \cos\bar{q}_6\sin\bar{q}_5 & \sin\bar{q}_6 & 0 \\ -\sin\bar{q}_5\sin\bar{q}_6 & \cos\bar{q}_6 & 0 \\ -\cos\bar{q}_5 & 0 & 1 \end{bmatrix} \tag{36}$$

$$\boldsymbol{F}\boldsymbol{A}_F = -\Big[m_1\boldsymbol{R}_1\, CPM(\boldsymbol{\rho}_1) - m_2\boldsymbol{R}_1\, CPM(\boldsymbol{\rho}_1+\boldsymbol{\delta}_1) - m_2\boldsymbol{R}_1\boldsymbol{R}_2\, CPM(\boldsymbol{\rho}_2)\boldsymbol{R}_2{}^T\Big]\boldsymbol{A}_F = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \tag{37}$$

$$\begin{aligned}
a_{11} &= \beta_1\Big[\cos\bar{q}_1\sin\bar{q}_3 - \sin\bar{q}_1\cos\bar{q}_2\cos\bar{q}_3\Big] + \beta_2\Big[\cos\bar{q}_1\sin(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5 - \sin\bar{q}_1\cos\bar{q}_2\cos(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5\Big] \\
a_{12} &= -\cos\bar{q}_1\Big[\beta_1\cos\bar{q}_3\sin\bar{q}_2 + \beta_2\Big(\cos\bar{q}_2\cos\bar{q}_5 + \sin\bar{q}_2\cos(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5\Big)\Big] \\
a_{13} &= \beta_1\Big[\sin\bar{q}_1\cos\bar{q}_3 - \cos\bar{q}_1\cos\bar{q}_2\sin\bar{q}_3\Big] + \beta_2\Big[\sin\bar{q}_1\cos(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5 - \cos\bar{q}_1\cos\bar{q}_2\sin(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5\Big] \\
a_{21} &= \beta_1\Big[\sin\bar{q}_1\sin\bar{q}_3 + \cos\bar{q}_1\cos\bar{q}_2\cos\bar{q}_3\Big] + \beta_2\Big[\sin\bar{q}_1\sin(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5 + \cos\bar{q}_1\cos\bar{q}_2\cos(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5 - \cos\bar{q}_1\sin\bar{q}_2\cos\bar{q}_5\Big] \\
a_{22} &= -\sin\bar{q}_1\Big[\beta_1\sin\bar{q}_2\cos\bar{q}_3 + \beta_2\Big(\cos\bar{q}_2\cos\bar{q}_5 + \sin\bar{q}_2\cos(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5\Big)\Big] \\
a_{23} &= -\beta_1\Big[\cos\bar{q}_1\cos\bar{q}_3 + \sin\bar{q}_1\cos\bar{q}_2\sin\bar{q}_3\Big] - \beta_2\Big[\cos\bar{q}_1\cos(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5 + \sin\bar{q}_1\cos\bar{q}_2\sin(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5\Big] \\
a_{31} &= 0 \\
a_{32} &= \beta_1\cos\bar{q}_2\cos\bar{q}_3 + \beta_2\Big[\cos\bar{q}_2\cos(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5 - \sin\bar{q}_2\cos\bar{q}_5\Big] \\
a_{33} &= -\sin\bar{q}_2\Big[\beta_1\sin\bar{q}_3 + \beta_2\sin(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5\Big]
\end{aligned}$$

where

$$\begin{aligned}
\beta_1 &= m_1\,\frac{l_1}{2} + m_2\, l_1 \\
\beta_2 &= m_2\,\frac{l_2}{2}
\end{aligned}$$

$$\boldsymbol{G}\boldsymbol{A}_H = -\Big[m_2\,\boldsymbol{R}_1\boldsymbol{R}_2\, CPM(\boldsymbol{\rho}_2)\Big]\boldsymbol{A}_H = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & 0 \end{bmatrix} \tag{38}$$

$$\begin{aligned}
b_{11} &= \beta_2\sin\bar{q}_5\Big[\sin\bar{q}_1\cos(\bar{q}_3+\bar{q}_4) - \cos\bar{q}_1\cos\bar{q}_2\sin(\bar{q}_3+\bar{q}_4)\Big] \\
b_{12} &= \beta_2\Big[\cos\bar{q}_1\sin\bar{q}_2\sin\bar{q}_5 + \sin\bar{q}_1\sin(\bar{q}_3+\bar{q}_4)\cos\bar{q}_5 + \cos\bar{q}_1\cos\bar{q}_2\cos(\bar{q}_3+\bar{q}_4)\cos\bar{q}_5\Big] \\
b_{13} &= 0 \\
b_{21} &= -\beta_2\sin\bar{q}_5\Big[\cos\bar{q}_1\cos(\bar{q}_3+\bar{q}_4) + \sin\bar{q}_1\cos\bar{q}_2\sin(\bar{q}_3+\bar{q}_4)\Big] \\
b_{22} &= \beta_2\Big[\sin\bar{q}_1\sin\bar{q}_2\sin\bar{q}_5 - \cos\bar{q}_1\sin(\bar{q}_3+\bar{q}_4)\cos\bar{q}_5 + \sin\bar{q}_1\cos\bar{q}_2\cos(\bar{q}_3+\bar{q}_4)\cos\bar{q}_5\Big] \\
b_{23} &= 0 \\
b_{31} &= -\beta_2\sin\bar{q}_2\sin(\bar{q}_3+\bar{q}_4)\sin\bar{q}_5 \\
b_{32} &= \beta_2\Big[\sin\bar{q}_2\cos(\bar{q}_3+\bar{q}_4)\cos\bar{q}_5 - \cos\bar{q}_2\sin\bar{q}_5\Big] \\
b_{33} &= 0
\end{aligned}$$

In this section the computations of each rotation matrix ($\boldsymbol{Q}_1 - \boldsymbol{Q}_6$) were omitted, but they are available in the Appendix of this document.

4 Appendix: Rotation matrices

For this analysis we are using the rotation matrix definition of the Natural Orthogonal methodology. In this methodology the rotation matrices are defines as

$$\boldsymbol{Q}_i = \begin{bmatrix} \cos\bar{q}_i & -\lambda_i \sin\bar{q}_i & \mu_i \sin\bar{q}_i \\ \sin\bar{q}_i & \lambda_i \cos\bar{q}_i & -\mu_i \cos\bar{q}_i \\ 0 & \mu_i & \lambda_i \end{bmatrix} \tag{39}$$

where

$$\lambda_i = \cos\alpha_i$$

$$\mu_i = \sin\alpha_i$$

$$\bar{q}_i = q_i + q_{ini_i}$$

For the spatial pendulum under consideration

$$\boldsymbol{q}_{ini} = \begin{bmatrix} \pi & -\frac{\pi}{2} & \pi & 0 & \frac{\pi}{2} & 0 \end{bmatrix}^T \tag{40}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \frac{\pi}{2} & \frac{\pi}{2} & 0 & \frac{\pi}{2} & \frac{\pi}{2} & 0 \end{bmatrix}^T \tag{41}$$

Note that both vectors are constant and they depend on the architecture of the spatial pendulum. Thus, the rotation matrix for each generalized coordinate is as follows

$$\lambda_1 = \cos\left(\frac{\pi}{2}\right) = 0 \quad ; \quad \mu_1 = \sin\left(\frac{\pi}{2}\right) = 1 \qquad \lambda_4 = \cos\left(\frac{\pi}{2}\right) = 0 \quad ; \quad \mu_4 = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\boldsymbol{Q}_1 = \begin{bmatrix} \cos\bar{q}_1 & 0 & \sin\bar{q}_1 \\ \sin\bar{q}_1 & 0 & -\cos\bar{q}_1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \boldsymbol{Q}_4 = \begin{bmatrix} \cos\bar{q}_4 & 0 & \sin\bar{q}_4 \\ \sin\bar{q}_4 & 0 & -\cos\bar{q}_4 \\ 0 & 1 & 0 \end{bmatrix} \tag{42}$$

$$\lambda_2 = \cos\left(\frac{\pi}{2}\right) = 0 \quad ; \quad \mu_2 = \sin\left(\frac{\pi}{2}\right) = 1 \qquad \lambda_5 = \cos\left(\frac{\pi}{2}\right) = 0 \quad ; \quad \mu_5 = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\boldsymbol{Q}_2 = \begin{bmatrix} \cos\bar{q}_2 & 0 & \sin\bar{q}_2 \\ \sin\bar{q}_2 & 0 & -\cos\bar{q}_2 \\ 0 & 1 & 0 \end{bmatrix} \qquad \boldsymbol{Q}_5 = \begin{bmatrix} \cos\bar{q}_5 & 0 & \sin\bar{q}_5 \\ \sin\bar{q}_5 & 0 & -\cos\bar{q}_5 \\ 0 & 1 & 0 \end{bmatrix} \tag{43}$$

$$\lambda_3 = \cos(0) = 1 \quad ; \quad \mu_3 = \sin(0) = 0 \qquad \lambda_6 = \cos(0) = 1 \quad ; \quad \mu_6 = \sin(0) = 0$$

$$\boldsymbol{Q}_3 = \begin{bmatrix} \cos\bar{q}_3 & -\sin\bar{q}_3 & 0 \\ \sin\bar{q}_3 & \cos\bar{q}_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{Q}_6 = \begin{bmatrix} \cos\bar{q}_6 & -\sin\bar{q}_6 & 0 \\ \sin\bar{q}_6 & \cos\bar{q}_6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{44}$$

Compute the rotation matrix $\boldsymbol{Q}_1\boldsymbol{Q}_2$

$$\begin{aligned} \boldsymbol{Q}_1\boldsymbol{Q}_2 &= \begin{bmatrix} \cos\bar{q}_1 & 0 & \sin\bar{q}_1 \\ \sin\bar{q}_1 & 0 & -\cos\bar{q}_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\bar{q}_2 & 0 & \sin\bar{q}_2 \\ \sin\bar{q}_2 & 0 & -\cos\bar{q}_2 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\bar{q}_1 \cos\bar{q}_2 & \sin\bar{q}_1 & \cos\bar{q}_1 \sin\bar{q}_2 \\ \sin\bar{q}_1 \cos\bar{q}_2 & -\cos\bar{q}_1 & \sin\bar{q}_1 \sin\bar{q}_2 \\ \sin\bar{q}_2 & 0 & -\cos\bar{q}_2 \end{bmatrix} \end{aligned} \tag{45}$$

Compute the rotation matrix $\boldsymbol{Q}_2\boldsymbol{Q}_3$

$$\begin{aligned} \boldsymbol{Q}_2\boldsymbol{Q}_3 &= \begin{bmatrix} \cos\bar{q}_2 & 0 & \sin\bar{q}_2 \\ \sin\bar{q}_2 & 0 & -\cos\bar{q}_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\bar{q}_3 & -\sin\bar{q}_3 & 0 \\ \sin\bar{q}_3 & \cos\bar{q}_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\bar{q}_2 \cos\bar{q}_3 & -\cos\bar{q}_2 \sin\bar{q}_3 & \sin\bar{q}_2 \\ \sin\bar{q}_2 \cos\bar{q}_3 & -\sin\bar{q}_2 \sin\bar{q}_3 & -\cos\bar{q}_2 \\ \sin\bar{q}_3 & \cos\bar{q}_3 & 0 \end{bmatrix} \end{aligned} \tag{46}$$

Using Eq. (45) and the definition for $\boldsymbol{Q}_3$, we find an expression for the rotation matrix from the inertial frame to the body frame of the first link

$$\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3 = \boldsymbol{Q}_1\boldsymbol{Q}_2 \begin{bmatrix} \cos\bar{q}_3 & -\sin\bar{q}_3 & 0 \\ \sin\bar{q}_3 & \cos\bar{q}_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\bar{q}_1\cos\bar{q}_2\cos\bar{q}_3 + \sin\bar{q}_1\sin\bar{q}_3 & -\cos\bar{q}_1\cos\bar{q}_2\sin\bar{q}_3 + \sin\bar{q}_1\cos\bar{q}_3 & \cos\bar{q}_1\sin\bar{q}_2 \\ \sin\bar{q}_1\cos\bar{q}_2\cos\bar{q}_3 - \cos\bar{q}_1\sin\bar{q}_3 & -\sin\bar{q}_1\cos\bar{q}_2\sin\bar{q}_3 - \cos\bar{q}_1\cos\bar{q}_3 & \sin\bar{q}_1\sin\bar{q}_2 \\ \sin\bar{q}_2\cos\bar{q}_3 & -\sin\bar{q}_2\sin\bar{q}_3 & -\cos\bar{q}_2 \end{bmatrix} \tag{47}$$

Compute the rotation matrix $\boldsymbol{Q}_5\boldsymbol{Q}_6$

$$\boldsymbol{Q}_5\boldsymbol{Q}_6 = \begin{bmatrix} \cos\bar{q}_5 & 0 & \sin\bar{q}_5 \\ \sin\bar{q}_5 & 0 & -\cos\bar{q}_5 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\bar{q}_6 & -\sin\bar{q}_6 & 0 \\ \sin\bar{q}_6 & \cos\bar{q}_6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\bar{q}_5\cos\bar{q}_6 & -\cos\bar{q}_5\sin\bar{q}_6 & \sin\bar{q}_5 \\ \sin\bar{q}_5\cos\bar{q}_6 & -\sin\bar{q}_5\sin\bar{q}_6 & -\cos\bar{q}_5 \\ \sin\bar{q}_6 & \cos\bar{q}_6 & 0 \end{bmatrix} \tag{48}$$

Using Eq. (48) and the definition for $\boldsymbol{Q}_4$, we find an expression for the rotation matrix from the body frame of the first link to the body frame of the second link

$$\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6 = \begin{bmatrix} \cos\bar{q}_4 & 0 & \sin\bar{q}_4 \\ \sin\bar{q}_4 & 0 & -\cos\bar{q}_4 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{Q}_5\boldsymbol{Q}_6$$

$$= \begin{bmatrix} \cos\bar{q}_4\cos\bar{q}_5\cos\bar{q}_6 + \sin\bar{q}_4\sin\bar{q}_6 & -\cos\bar{q}_4\cos\bar{q}_5\sin\bar{q}_6 + \sin\bar{q}_4\cos\bar{q}_6 & \cos\bar{q}_4\sin\bar{q}_5 \\ \sin\bar{q}_4\cos\bar{q}_5\cos\bar{q}_6 - \cos\bar{q}_4\sin\bar{q}_6 & -\sin\bar{q}_4\cos\bar{q}_5\sin\bar{q}_6 - \cos\bar{q}_4\cos\bar{q}_6 & \sin\bar{q}_4\sin\bar{q}_5 \\ \sin\bar{q}_5\cos\bar{q}_6 & -\sin\bar{q}_5\sin\bar{q}_6 & -\cos\bar{q}_5 \end{bmatrix} \tag{49}$$