Passivity-based control of implicit port-Hamiltonian systems with holonomic constraints

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Passivity-based control of implicit port-Hamiltonian systems

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Abstract

Implicit port-Hamiltonian representations of mechanical systems are considered from a control perspective. Energy shaping is used for the purpose of stabilizing a desired equilibrium. When using implicit models, the problem turns out to be a simple quadratic programming problem (as opposed to the partial differential equations that need to be solved when using explicit representations). The described approach is generalized to address the problem of stabilization of homoclinic orbits thus leading to the formulation of swing-up strategies for underactuated systems.

Keywords: Hamiltonian Dynamics, Holonomic Constraints, Implicit Models, Passivity, Pendulum, Swing-up

1. Introduction

The Hamiltonian formalism is used to describe the dynamics of a wide class of systems including mechanical [1, 3], electrical [4, 5, 6, 7], and thermodynamic [8, 9] ones.

In many cases there are constraints imposed on the system coordinates. These constraints reflect the internal structure of the system, for instance, rigid connections between the system's elements. From the geometrical viewpoint, the action of these constraints results in restricting the system's evolution to a submanifold of the state space.

When the system is subject to the action of external forces it is convenient to consider a pair of (energy-adjoint) port variables (u, y) such that their product is equal to the power supplied into the system. Such model is referred to as a port-Hamiltonian system (see [10] for the original definition as stated with respect to Hamiltonian systems in explicit form).

In general, there are two different approaches to the representation of systems evolving on manifolds: the explicit representation with the dynamics having the

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form of an ordinary differential equation on the manifold and the implicit representation with the dynamics described by a set of differential-algebraic equations usually evolving in a Euclidean space (see, e.g., [11] for a related discussion on constrained Hamiltonian systems). There has been a lot of research on the analysis and control of explicit systems [12, 13]. However, not many results on the control of Hamiltonian systems in implicit formulation have been presented so far. Thus, the primary goal of this contribution is to provide an elaborated approach to the control of Hamiltonian systems in implicit form. The two main issues addressed in this paper are the potential energy shaping for the stabilization of unstable equilibria and the swing-up strategy. The obtained results are illustrated with two examples.

We remark that there is a series of papers presenting a unified approach to the description and analysis of implicit Hamiltonian systems on the base of (generalized) Dirac structures (see, e.g., [14] and references therein). It has been shown that Dirac structures can be used for the analysis of interconnection properties [15] of (implicit) Hamiltonian systems (see also the book [16] for more details). Recently, there has been a paper devoted to the control of (discretized) infinite-dimensional implicit Hamiltonian systems, [17]. However, the authors feel that it is sometimes more advantageous to have a closer look at the object under study. In this sense, the approach presented in this paper allows one to consider the problem at hand at a practical level.

The paper is organized as follows: in Section 2, an implicit representation of port-Hamiltonian systems is presented and a couple of simple models are derived within the described framework. In Section 3, the energy shaping approach is presented in detail and a number of illustrative examples is given. In particular, Subsection 3.5 discusses swing-up strategies for underactuated systems. Finally, Section 4 presents the conclusions and the directions for future research.

2. Implicit port-Hamiltonian systems

2.1. Mechanical systems with holonomic constraints

Consider a controlled mechanical system with the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^{*n} \to \mathbb{R}$. Let there be a number of *holonomic* constraints $c(r) = 0$, $c : \mathbb{R}^n \to \mathbb{R}^k$, restricting the configuration space of the system to an $(n-k)$-dimensional submanifold Γ of the configuration space $\mathbb{R}^n$. Using the Hamiltonian formalism, the dynamics of this system are described by a set of differential-algebraic equations of the form [18, 11]:

$$\begin{bmatrix} \dot{r} \\ \dot{p} \end{bmatrix} = J \left(\nabla H(x) + \begin{bmatrix} \nabla c(r) \\ 0 \end{bmatrix} \lambda \right) + \begin{bmatrix} 0 \\ g(r) \end{bmatrix} u \tag{1a}$$

$$0 = c(r) \tag{1b}$$

$$y = \begin{bmatrix} \nabla_r^\top H(x) & \nabla_p^\top H(x) \end{bmatrix} \begin{bmatrix} 0 \\ g(r) \end{bmatrix} , \tag{1c}$$

where the state is given by $x^\top = \begin{pmatrix} r^\top & p^\top \end{pmatrix}$ with $r \in \mathbb{R}^n$ and $p \in \mathbb{R}^{*n}$ the positions and momenta, respectively,

$$\nabla c(r) = \frac{\partial c^\top}{\partial r}(r)$$

is the transposed Jacobian of the vector-valued function $c(r)$, $\lambda \in \mathbb{R}^{*k}$ is the vector of implicit variables that enforce the holonomic constraints, $(u, y) \in \mathbb{R}^{*m} \times \mathbb{R}^m$ are the conjugated external port variables, and $g(x)$ is a $(n \times m)$-matrix such that $\operatorname{rank} \hat{g}(x) = m$ for all $x \in \mathbb{R}^n \times \mathbb{R}^{*n}$. The $[2n \times 2n]$-matrix J is the structure matrix of the canonical symplectic form.

Here and forth all functions are assumed to be smooth enough and the gradient is assumed to be a column vector.

Equations (1) correspond to a port-Hamiltonian system, [12, 14], with an augmented Hamiltonian function $\widetilde{H}(x) = H(x) + c(r)\lambda$ (see [1, p. 48] for a more general treatment).

From the geometrical viewpoint, (1) describe the system evolution on the cotangent bundle of $\mathbb{R}^n$, denoted $T^*\mathbb{R}^n$. The state space manifold $T^*\mathbb{R}^n$ is endowed with the canonical symplectic form $\omega = dr^i \wedge dp_i$ (where Einstein's summation convention is implied). This symplectic form defines a canonical isomorphism between the tangent and cotangent spaces: $\Omega : T(T^*\mathbb{R}^n) \to T^*(T^*\mathbb{R}^n)$ defined by $\Omega(X)(\cdot) = \omega(X, \cdot)$. The vector field $X \in T\left(T^*\mathbb{R}^n\right)$ is written as

$$X = D_H + D_c\lambda + X_g u \tag{2a}$$

$$0 = c \tag{2b}$$

$$y = X_g(H)\ , \tag{2c}$$

where

$$D_H = \Omega^{-1}(dH) = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial r^i} - \frac{\partial H}{\partial r^i}\frac{\partial}{\partial p_i} \tag{3}$$

is the Hamiltonian vector field,

$$D_c\lambda = \Omega^{-1}(dc) = -\frac{\partial c^j}{\partial r^i}\lambda_j \frac{\partial}{\partial p_i} \tag{4}$$

is the vector field of the internal (constraint) forces, and

$$X_g u = g_i^j u_j \frac{\partial}{\partial p_i}$$

is the control vector field. Note that D_c and X_g are the tuples of vector fields: $D_c = \left({D_c}^1, \ldots, {D_c}^k\right)$, resp. $X_g = \left({X_g}^1, \ldots, {X_g}^m\right)$, which are assumed to be linearly independent. Thus, for instance, the application of D_c to a smooth function $f(x)$ yields a vector $D_c f(x) = \left({D_c}^1 f(x), \ldots, {D_c}^k f(x)\right)$.

In the following, D_z will denote the vector field generated by the function $z(x)$, i.e., $D_z = \Omega^{-1}(dz)$. This can be alternatively formulated using Poisson brackets [2]: $D_z = \{\cdot, z\}$.

Equation (1b) constrains the configuration space of (1). We wish to ensure that these constraints are preserved under the system dynamics. To do so we require X to be tangential to Γ, i.e., $X(c^i) = 0$ for any $i = 1, \ldots, k$. This yields the so-called hidden (secondary) constraints,

$$G^i(x) = X(c^i) = \frac{\partial H(x)}{\partial p_j} \frac{\partial c^i(x)}{\partial r^j} = 0 \; . \tag{5}$$

Now, considering $T^*\mathbb{R}^n$ as a state space manifold, we say that (1) evolves on a submanifold $\mathcal{M}_\Gamma \subset T^*\mathbb{R}^n$,

$$\mathcal{M}_\Gamma = \{x = (r, p) \in \mathbb{R}^n \times \mathbb{R}^{*n} | c^i(x) = 0, \, G^i(x) = 0, \, i = 1, \ldots, k\} \; .$$

All results formulated below will hold for $x \in \mathcal{M}_\Gamma$.

Before proceeding to the main part we require the following regularity conditions to hold.

Assumption 1. *The following holds:*

i) The constraints are regular, i.e.,

$$\dim \operatorname{span} \left\{ dc^i(r) \right\}_{r \in \Gamma} = k \; ,$$

where $dc^i(r) \in T_r^*\mathbb{R}^n$ are the differentials of $c^i(r)$ interpreted as the elements of the cotangent vector space $T_r^*\mathbb{R}^n$. Note that the manifold Γ is an integral manifold of the distribution generated by dc_i, i.e.,

$$T\Gamma = \ker \left(\operatorname{span} \left\{ dc^i \right\} \right) \; , \quad i = 1, \ldots, k$$

ii) The initial conditions belong to $\mathcal{M}_\Gamma$, i.e.,

$$x(0) = (r(0), p(0)) \in \mathcal{M}_\Gamma \; .$$

iii) The energy is separable and positive definite w.r.t. p, i.e.,

$$H(x) = P(r) + K(p) \; , \; K(p) = \frac{1}{2} p^\top M^{-1} p \; , \; M > 0 \; ,$$

where P and K are the potential and kinetic energy, respectively.

Assumptions i) and iii) guarantee that $\mathcal{M}_\Gamma$ is a proper subbundle of $T^*\mathbb{R}^n$. Indeed, for any $r \in \Gamma$, the hidden constraints define a linear subspace of codimension k, which is interpreted as the cotangent subspace to Γ at x.

Item i) and strict convexity in iii) ensure that the λ_i exist and are uniquely defined. More precisely, applying the vector field to the hidden constraints yields the condition

$$X(G) = D_H^2(c) + D_c D_H(c) \lambda + X_g D_H(c) u = 0 \; , \tag{6}$$

which implicitly defines λ as a function of x and u. Notice that the $(k\times k)$-matrix defined by

$$D_c D_H(c) = D_c(G) = D_c\left(\frac{\partial H}{\partial p_i}\frac{\partial c}{\partial r^i}\right) = -\frac{\partial^2 H}{\partial p_i \partial p_j}\frac{\partial c^a}{\partial r^i}\frac{\partial c^b}{\partial r^j} \tag{7}$$

is negative definite as follows from Assumptions i) and iii) and hence, invertible. This ensures the well-posedness of the problem.

Assumption ii) guarantees that there are no jumps in the system's trajectories.

Finally, separability of the Hamiltonian in item iii) is, from a computational point of view, one of the main advantages of the implicit modeling framework. Indeed, if the Hamiltonian function can be represented by a sum of two terms, where the first one depends only on p and the second only on q, then it is possible to design symplectic integration schemes by simply composing the elementary integration steps (see, e.g., [19]). Furthermore, the separbility property is necessary for formulating the result of Theorem 5.

Note that the fulfillment of the hidden constraints (5) implies that the Hamiltonian is invariant under the action of the vector field of constraint forces, i.e., $D_c(H) = 0$. This can be easily expressed in terms of Poisson brackets: $D_H(c^i) = \{c^i, H\} = -{D_c}^i(H) = 0$. Alternatively, one can say that the internal forces do not produce work as there is no displacement in the direction of the constraint forces and hence they do not alter the total energy of the system. Furthermore, the vector field D_c is also tangential to the submanifold Γ, i.e.,

$$D_c(c) = 0\ . \tag{8}$$

The following proposition states that the passivity property, which is central for port-Hamiltonian systems, can be readily extended to the case of the constrained dynamics (1).

Proposition 2. *Consider the* restricted state-space $\mathcal{M}_\Gamma$. *System* (1) *is passive whenever $H|_{\mathcal{M}_\Gamma}$, the restriction of H to $\mathcal{M}_\Gamma$, is bounded from below.*

Proof. Taking the derivative of H gives

$$\dot{H} = X(H) = X_g(H)u = g_i^j \frac{\partial H}{\partial p_i} u_j = y^j u_j\ .$$

This equation, together with the lower bound on H, implies passivity. $\square$

Finally, we give a condition for a constrained system to be fully actuated.

Definition 1. We say that (1) is fully actuated whenever

$$\operatorname{span}\left\{\begin{pmatrix}0_n\\ I_n\end{pmatrix}\right\} = \operatorname{span}\{D_c\}\bigcup \operatorname{span}\{X_g\}$$

for all $x \in \mathcal{M}_\Gamma$. We say that (1) is underactuated if it is not fully actuated.

2.2. A simple actuated pendulum

Consider a simple pendulum with mass m_1 held by an ideal massless bar of length l. Let $r^\top = \begin{pmatrix} r^x & r^y \end{pmatrix}$ and $p^\top = \begin{pmatrix} p_x & p_y \end{pmatrix}$ be the position and momenta, respectively. The constraint is given by $c^1(r) = \frac{1}{2}\left(\|r\|^2 - l^2\right) = 0$, while the energy takes the form

$$H(x) = \frac{1}{2m_1}\|p\|^2 + m_1 \bar{g} \cdot r^y$$

with $\bar{g}$ the acceleration due to gravity. Suppose that a torque u_1 is applied to the pendulum axis. The implicit model then takes the form

$$\dot{r} = \frac{1}{m_1} p \tag{9a}$$

$$\dot{p} = -m_1 \bar{g} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} r^x \\ r^y \end{pmatrix} \lambda_1 + \frac{1}{l^2} \begin{pmatrix} r^y \\ -r^x \end{pmatrix} u_1 \tag{9b}$$

$$y^1 = \frac{r_x p^y - r^y p_x}{m_1 l^2} \tag{9c}$$

It is not difficult to verify that Assumption 1 holds, and that the system is fully actuated.

Boundedness of H can be easily established. Given the positive definite form of K, it is only necessary to verify the term $m_1 \bar{g} \cdot r^y$. The term is continuous and restricted to the compact set $\Gamma = \left\{ r \in \mathbb{R}^2 \mid \|r\| = l \right\}$. By the extreme value theorem of Weierstrass, we know that the term is bounded from below and the passivity of the pendulum is confirmed.

2.3. A pendulum on a cart

Consider now an actuated cart with mass m_1, position $r^1 \in \mathbb{R}^2$ and momentum $p_1 \in \mathbb{R}^2$. The cart is constrained to move along the x-axis, which can be expressed as $c^1(r) = 0$ with $c^1(r) = r^{1_y}$. Attached to the cart is a pendulum of length l, mass m_2, position $r_2 \in \mathbb{R}^2$ and momentum $p_2 \in \mathbb{R}^2$. The bond between the cart and the pendulum is expressed as $c^2(r) = 0$ with

$$c^2(r) = \frac{1}{2}\left(\|r^2 - r^1\|^2 - l^2\right) \; .$$

The total energy is given by

$$H(x) = \frac{1}{2m_1}\|p_1\|^2 + \frac{1}{2m_2}\|p_2\|^2 + m_2 \bar{g} \cdot r^{2_y} \; , \tag{10}$$

so the pendulum's equations take the form

$$
\begin{aligned}
\dot{r}^1 &= \frac{1}{m_1} p_1 \\
\dot{r}^2 &= \frac{1}{m_2} p_2 \\
\dot{p} &= -m_2 \bar{g} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & r^{1_x} - r^{2_x} \\ 1 & r^{1_y} - r^{2_y} \\ 0 & r^{2_x} - r^{1_x} \\ 0 & r^{2_y} - r^{1_y} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u \\
y^1 &= \frac{p_{1_x}}{m_1} .
\end{aligned}
\tag{11}
$$

Again, Assumption 1 holds, but the system is underactuated:

$$
k + m = 2 + 1 < 4 = n .
$$

The constraint $\|r^2 - r^1\| = l$ implies that $\|r^{2_y} - r^{1_y}\| \leq l$. Since $r_{1^y} = 0$, we have $\|r^{2_y}\| \leq l$, which defines a compact set on r^{2_y}. Weierstrass Theorem then implies that the restriction of $m_2 \bar{g} \cdot r^{2_y}$ is bounded from below and the pendulum on a cart is passive as well.

The acrobot, the pendubot and many more mechanical systems can be modeled in this framework.

3. Implicit energy shaping

3.1. The matching equations

Definition 2. Let H_d be a smooth mapping from $\mathbb{R}^n \times \mathbb{R}^{*n}$ to $\mathbb{R}$. We say that H_d is an admissible energy (Hamiltonian) function if for any $(r, p) \in \mathcal{M}_\Gamma$ there exist $\mu \in \mathbb{R}^{*k}$ and $\hat{u} \in \mathbb{R}^{*m}$ such that the matching equation

$$
D_{H_\mathrm{d}} - D_H = D_c \mu + X_g \hat{u} \tag{12}
$$

is satisfied.

Remark 3. Recall that $D_H(c^i) = 0$ and $D_c(c^i) = 0$, and note that $X_g(c^i) = 0$ (X_g acts on functions of p and the c^i depend on r alone). Thus,

$$
D_{H_\mathrm{d}}(c^i) = D_H(c^i) + D_c(c^i)\mu + X_g(c^i)\hat{u} = 0 ,
$$

which implies

$$
{D_c}^i(H_\mathrm{d}) = 0 . \tag{13}
$$

By requiring the closed-loop system to be Hamiltonian, we implicitly impose a constraint akin to the hidden constraints. This simplifies the energy balance later required for stability analysis, as the rate of change in H_d remains independent of μ. On the other hand, this substantially reduces the class of stabilizable systems.

Setting $u = \hat{u}+\upsilon$ and substituting (12) into (1) gives the new port-Hamiltonian system

$$\begin{bmatrix} \dot{r} \\ \dot{p} \end{bmatrix} = J\left(\nabla H_d(x) + \begin{bmatrix} \nabla c(r) \\ 0 \end{bmatrix} (\lambda - \mu)\right) + \begin{bmatrix} 0 \\ g(r) \end{bmatrix} \upsilon \tag{14a}$$

$$0 = c(r) \tag{14b}$$

$$y_{\mathrm{d}} = \begin{bmatrix} \nabla_r^\top H_d(x) & \nabla_p^\top H_d(x) \end{bmatrix} \begin{bmatrix} 0 \\ g(r) \end{bmatrix} , \tag{14c}$$

with port variables $(\upsilon_j, y_{\mathrm{d}}^j)$. The respective vector field is thus

$$X_d = D_{H_d} + D_c(\lambda - \mu) + X_g \upsilon \,. \tag{15}$$

Since $\lambda - \mu$ is an implicit variable, i.e., it is found as the solution to the auxiliary condition (6), the way it is denoted is immaterial and thus it is possible to rename $(\lambda - \mu)$ to λ without changing the system dynamics. This leads us to the following definition.

Definition 4. Given the vector fields of internal forces $\{{D_c}^i\}$, $i = 1, \ldots, k$, two Hamiltonian functions H_1 and H_2, $H_i : \mathbb{R}^n \times \mathbb{R}^{*n} \to \mathbb{R}$, $i = 1, 2$, are said to be equivalent, $H_1(x) \sim H_2(x)$, if

$$D_{H_1} - D_{H_2} \in \operatorname{span}\left\{{D_c}^i\right\}$$

The equivalence class of H, denoted by $[H]$, is defined as

$$[H] = \{\hat{H} : \mathbb{R}^n \times \mathbb{R}^{*n} \to \mathbb{R} | \hat{H} \sim H\}.$$

We have the following proposition.

Proposition 3. *Let $\hat{H}$ satisfy (12) and (13). Then, any $H_d \in [\hat{H}]$ is an admissible energy function.*

Proof. We need to prove that for any $H_{\mathrm{d}} \in [\hat{H}]$ conditions (12) and (13) hold. An H_{d} can be represented as $H_{\mathrm{d}} = \hat{H} + \kappa_i c^i$, $i = 1, \ldots, k$. If $\hat{H}$ satisfies (12), then H_{d} satisfies (12) as well with $\hat{\mu} = \mu - \kappa$. Furthermore, (13) is satisfied as $D_c(c) = 0$. □

This gives additional freedom for choosing H_{d} in (12). Roughly speaking, this additional freedom 'compensates' for the need to solve (12) in a high-dimensional setting (i.e., higher than in the explicit formulation) using the same number of controls $\hat{u}$.

Equation (13) is analogous to the one formulated for the original system. It ensures that the new Hamiltonian vector field $D_{H_{\mathrm{d}}}$ preserves the holonomic constraints c^i, i.e., $D_{H_{\mathrm{d}}}(c^i) = 0$ and that the constraint forces preserve the new energy, i.e., ${D_c}^i(H_{\mathrm{d}}) = 0$ whenever $x \in \mathcal{M}_\Gamma$. See Sec. 2.1 for more details.

Proposition 4. *If $H_{\mathrm{d}}|_{\mathcal{M}_\Gamma}$ is bounded from below, then the closed-loop* (14) *is passive and the storage function is equal to H_{d}.*

Proof. Direct computation gives

$$X_d(H_\mathrm{d}) = X_g(H_\mathrm{d})v = g_i^j \frac{\partial H_\mathrm{d}}{\partial p_i} v_j = y_\mathrm{d}^j v_j \ .$$

$\square$

3.2. Equilibrium stabilization

Let

$$x^* = \begin{pmatrix} r^* \\ 0 \end{pmatrix} \in \mathcal{M}_\Gamma \tag{16}$$

be a desired equilibrium point. It follows from standard Lyapunov theory that x^* is stabilizable whenever H_d is admissible and x^* is a strict minimum of $H_\mathrm{d}|_{\mathcal{M}_\Gamma}$,

$$\arg\min_{x \in \mathcal{M}_\Gamma} H_\mathrm{d}(x) = x^* \ . \tag{17}$$

The problem is easily solvable in the fully actuated case.

Theorem 5. *Let* (1) *be fully actuated. Any* x^* *satisfying* (16) *is an assignable equilibrium and can be stabilized.*

Proof. Set

$$H_\mathrm{d}(x) = a^\top r + \frac{1}{2}(r - r^*)^\top A(r - r^*) + \frac{1}{2} p^\top M^{-1} p \ , \tag{18}$$

where $A = A^\top \in \mathbb{R}^{n \times n}$ satisfies the linear matrix inequality (LMI)

$$A + \nabla^2 c^i(r^*)\xi_i^* + \left(\nabla c(r^*) \nabla c^\top(r^*)\right)^j \bar{\xi}_j^* > 0 \tag{19}$$

for some scalars ξ_i^* and $\bar{\xi}_j^*$, and with

$$a = -\nabla c^i(r^*)\xi_i^* \ . \tag{20}$$

Since the kinetic energy is left unchanged, we have $D_c(H_\mathrm{d}) = D_c(H)$, so (13) is trivially satisfied. Since

$$D_{H_\mathrm{a}} = H_d - H \in \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ I_n \end{pmatrix} \right\} \ ,$$

equation (12) is solvable on account of full actuation. Thus, the closed-loop is passive with storage function (18). To show stability, it suffices to prove (17).

Next, we construct the Lagrange function

$$L(x, \xi) = H_\mathrm{d}(x) + c^i(r)\xi_i$$

with Lagrange multipliers ξ_i. The first-order stationarity condition gives

$$a + A(r - r^*) + \nabla c^i(r)\xi_i = 0 \ , \ M^{-1}p = 0 \ , \ c^i(r) = 0 \ ,$$

which are solved by (16) and $\xi_i = \xi_i^*$ if we set a as in (20).

The second-order sufficient condition takes the form [20, p. 68]

$$z^\top \left(A + \nabla^2 c^i(r^*)\xi_i^*\right) z > 0 \tag{21}$$

for all $z \in T_{r^*}\Gamma$, i.e., for all $z \in \mathbb{R}^n$ such that $\nabla(c^i)^\top(r^*)z = 0$, $i = 1, \ldots, k$.

It remains to show that the condition (21) is satisfied whenever (19) holds. LMI (19) can be equivalently written as an inequality involving the quadratic form

$$\langle y, \left(A + \nabla^2 c^i(r^*)\xi_i^* + \left(\nabla c(r^*)\nabla c^\top(r^*)\right)^j \bar{\xi}_j^*\right) y\rangle > 0$$

which must hold for all $y \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Choosing $y \in T_{r^*}$ we recover (21) while the converse, i.e., the existence of $\bar{\xi}^*$ follows from the Finsler theorem [21]. □

The LMI (19) can always be solved by setting $\xi_i = 0$, $\bar{\xi}_j = 0$ and choosing A to be a positive definite matrix. However, the resulting controller can be greatly simplified by exploiting the available degrees of freedom when solving the LMI. In particular, it might be possible to set the matrix A equal to zero (see Sec. 3.3).

Remark 5. The controller obtained from the matching equation (12) with H_d as in (18) provides Lyapunov stability only. As usual, asymptotic stability can then be achieved by adding proper damping.

For underactuated systems, the problem can be solved by searching first a set of $s^i(r)$, $i = 1, \ldots, m+k$, such that

$$\mathrm{span}\{D_s\} = \mathrm{span}\left\{D_c\right\} \bigcup \mathrm{span}\left\{X_g\right\} \ . \tag{22}$$

By setting the desired Hamiltonian as $H_\mathrm{d}(x) = H(x) + f(s(r))$, it is ensured that H_d is assignable for any differentiable $f : \mathbb{R}^{m+k} \to \mathbb{R}$. Then f is chosen such that (17) holds.

The described approach has a number of advantages compared to solving the equilibrium stabilization problem in local coordinates [22, 23, 24]. In particular, one needs to solve a simple quadratic program instead of a partial differential equation. Furthermore, the obtained control is expressed in global coordinates, hence, there are no singularities. Finally, it turns out that an implicit Hamiltonian system is easier to discretize as the Hamiltonian function written in global coordinates is separable. This fact can be used to design an effective integration scheme [19].

3.3. The simple actuated pendulum

Suppose we want to stabilize the point $x^* = \begin{pmatrix} l & 0 & 0 & 0\end{pmatrix}^\top$ (the right horizontal position), which clearly satisfies (16). A solution set for the LMI (19) is

$$A = 0 \ , \ \xi_1^* = \frac{m_1 \bar{g}}{l} \ , \ \bar{\xi}_1^* = 0 \ ,$$

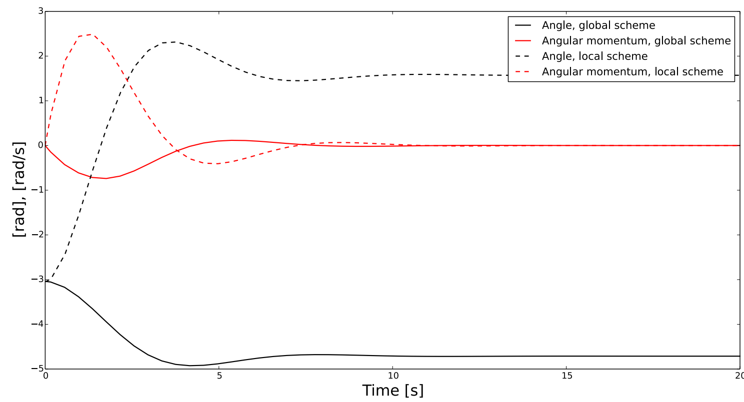

Figure 1: Pendulum trajectories corresponding to the control law obtained using global and local coordinates.

i.e., $\nabla^2 c^1(r^*) = I_2 > 0$. This gives

$$a^\top = -\begin{pmatrix} l & 0 \end{pmatrix} \cdot \frac{m_1 \bar{g}}{l} = -\begin{pmatrix} m_1 \bar{g} & 0 \end{pmatrix}$$

and

$$H_{\mathrm{d}}(x) = -m_1 \bar{g} r^x + \frac{1}{2m_1} \|p\|^2 .$$

$$m_1 \bar{g} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} r^x \\ r^y \end{pmatrix} \mu + \frac{1}{l^2} \begin{pmatrix} r^y \\ -r^x \end{pmatrix} \hat{u} . \tag{23}$$

The solution is simply

$$\mu = -m_1 \bar{g} \frac{r^y + r^x}{l^2} \quad \text{and} \quad \hat{u} = m_1 \bar{g}(r^y - r^x) .$$

A local coordinate chart for Γ is $\theta \mapsto \begin{pmatrix} l \sin\theta & l\cos\theta \end{pmatrix}^\top$ with $\theta \in (-\pi, \pi)$. In local coordinates, the control takes the form

$$\hat{u} = m_1 \bar{g} l (\cos\theta - \sin\theta) . \tag{24}$$

Since it was constructed using global coordinates, the controller does not exhibit undesirable phenomena such as *unwinding*. To illustrate this we consider the problem using local coordinates only. The Hamiltonian takes the form

$$H(\theta, p_\theta) = \frac{1}{2m_1 l^2} p_\theta^2 + m_1 \bar{g} l \cos\theta ,$$

with p_θ the angular momentum. The system equations are

$$\begin{aligned} \dot{\theta} &= \frac{1}{m_1 l^2} p_\theta \\ \dot{p}_\theta &= m_1 \bar{g} l \sin\theta + u_1 . \end{aligned}$$

A reasonable target Hamiltonian is

$$H_{\mathrm{d}}(\theta, p_\theta) = \frac{1}{2m_1 l^2} p_\theta^2 + \frac{m_1 \bar{g} l}{2} \left(\theta - \frac{\pi}{2}\right)^2 ,$$

and it is achieved by the control

$$\hat{u} = m_1 \bar{g} l \left(\frac{\pi}{2} - \theta - \sin\theta\right) \tag{25}$$

The control laws (24) and (25) were simulated for a pendulum with parameters satisfying

$$m_1 \bar{g} l = 1 \quad \text{and} \quad m_1 l^2 = 1$$

and with initial condition $\theta(0) = 0.1 - \pi$, $p_\theta(0) = 0$. A damping term $v^1 = -y^1$ was included in order to achieve asymptotic stability. The results are shown in Fig. 1. Both controllers steer the system to the desired angle $\theta = \frac{1}{2}\pi = -\frac{3}{2}\pi$, but, with the local controller, the pendulum makes an unnecessarily long excursion with large overshoots in angular position and momentum.

3.4. The pendulum on a cart

It can be verified that

$$s^1(r) = r^{1_y} , \quad s^2(r) = \frac{1}{2}\|r^2 - r^1\|^2 \quad \text{and} \quad s^3(r) = r^{1_x}$$

satisfy (22). Given the constraints $c^1(r) = c^2(r) = 0$, it is clear that the only Hamiltonian functions of interest are of the form

$$\begin{aligned} H_{\mathrm{d}}(x) &= H(x) + f(s^3(r)) \\ &= \frac{1}{2m_1}\|p_1\|^2 + \frac{1}{2m_2}\|p_2\|^2 + m_2 \bar{g} \cdot r^{2_y} + f(r^{1_x}) , \end{aligned}$$

and that the stabilizable equilibria have the structure

$$x^* = \begin{pmatrix} r_* \\ 0 \end{pmatrix} ,$$

where $r^* \in \left\{ \begin{pmatrix} r_*^{1_x} & 0 & r_*^{1_x} & -l \end{pmatrix}^\top \mid r_*^{1_x} \in \mathbb{R} \right\}$.

Condition (17) is indeed satisfied with the choice $f(r^{1_x}) = \frac{1}{2}(r^{1_x} - r_*^{1_x})^2$. It can be readily checked that the condition (19) is satisfied as well. The matching equation is then

$$\begin{pmatrix} r_*^{1_x} - r^{1_x} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & r^{1_x} - r^{2_x} \\ 1 & r^{1_y} - r^{2_y} \\ 0 & r^{2_x} - r^{1_x} \\ 0 & r^{2_y} - r^{1_y} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \hat{u} . \tag{26}$$

The solution is simply $\mu_1 = \mu_2 = 0$ and $\hat{u} = r_*^{1_x} - r^{1_x}$.

3.5. Swing-up strategies

In its most general form, a passivity-based controller acts on a port-Hamiltonian system with damping and produces another port-Hamiltonian system, but with different structure and different dynamics. For fully actuated systems, it is possible to stabilize an unstable equilibrium by shaping the potential energy alone. However, for the under-actuated case, shaping the potential energy is not sufficient for stabilizing a desired equilibrium point: the kinetic energy has to be modified as well, which in turn requires solving a PDE (see, e.g., [24]). This approach is too complex to be of practical value or to be generalized (also, it requires a previous partial linearization). Such complexity is the price that has to be paid if one insists on having a closed-loop Hamiltonian system.

An alternative approach does not require the closed-loop system to have a Hamiltonian form. It consists in defining a generalized energy function and to drive an underactuated mechanical system about an open-loop unstable equilibrium by forcing its trajectories to converge to an energy level-set that includes such equilibrium. That is to say, it is proposed to stabilize a homoclinic orbit that contains the desired equilibrium, but the equilibrium itself is not stabilized. Once the system state is sufficiently close to the unstable equilibrium, the controller can be switched to a linear law that stabilizes it [26, 27].

The first step consists in defining a more general class of energy functions as described below.

Definition 6. Let $V : \mathbb{R}^n \times \mathbb{R}^{*n} \to \mathbb{R}$ be a smooth function. We say that V is a *(generalized) admissible energy function* if there exist a pair $(\hat{u}, \hat{\mu})$ satisfying the constraint

$$D_H^2(c) + D_c D_H(c)\hat{\mu} + X_g D_H(c)\hat{u} = 0 \tag{27}$$

and such that V is invariant with respect to the resulting vector field X_V:

$$X_V - D_H = D_c\hat{\mu} + X_g\hat{u}, \tag{28}$$

i.e., $X_V(V) = 0$.

Remark 7. Definition 6 generalizes the notion of an admissible energy function given in Definition 2, insofar as the closed-loop system X_V does not need to have the Hamiltonian form (14). On the other hand, the conditions ${D_c}^i(V)$ might not hold, so one has to take the specific value of μ into consideration when establishing an energy balance for V. Such value is computed from (27), which was obtained by differentiating $c(r)$ twice. Confront Remark 3.

For example, the function $V = (H - H_0)^2/2$, with H_0 a constant, is always admissible.

Setting $H_0 = H(x^*)$ serves as a starting point for the proposed swing-up strategy. By adding proper damping, V can be forced to converge to its minimum value, i.e., $H \to H_0$. However, this does not necessarily imply that x will converge to x^*, since there might be orbits contained in the level-set $\{x \in \mathcal{M}_\Gamma \mid H(x) = H_0\}$ which do not necessarily include x^*. To remedy this, the level-set can be further restricted (see Remark 9 below).

Assumption 6. *There exists a function $z : r \mapsto z(r) \in \mathbb{R}^m$ such that*

$$\frac{\partial}{\partial r^i} z^j(r) = g_i^j(r) \ .$$

Notice that this assumption is equivalent to $D_H(z) = y$. In other words, we assume that the time-integral of the passive output is a function of the positions. This assumption is satisfied by most mechanical systems. Typically, the passive output y is a function of velocities, so $z(r)$ is a simple function of the positions. For the pendulum (9), we have

$$z(r) = \arctan\left(\frac{r^y}{r^x}\right) \ .$$

Indeed,

$$X(z) = D_H(z) = \frac{1}{1 + \left(\frac{r^y}{r^x}\right)^2} \left(\frac{p_y r^x - p_x r^y}{m_1 (r^x)^2}\right) = \\ \frac{1}{(r^x)^2 + (r^y)^2} \left(\frac{p_y r^x - p_x r^y}{m_1}\right) = y^1 \ .$$

For the pendulum on a cart we have $z = r^{1_x}$ and $X(z) = \frac{p_{1_x}}{m_1} = y^1$.

Theorem 7. *Let the system* (1) *be passive and let the following conditions be satisfied:*

1. *$m \leq n - k$,*
2. *The columns of g and ∇c are linearly independent.*

Then the following statements hold true.

i) For any $f : \mathbb{R}^m \to \mathbb{R}$, the function

$$V(x) = \frac{1}{2} k_H \left(H(x) - H_0\right)^2 + f(z) + \frac{1}{2} k_v \|y\|^2 \tag{29}$$

is admissible, provided k_v is large enough.

ii) Set $u = \hat{u} + v$. The system is passive with storage function V, input v and output

$$y_\mathrm{d} = y \cdot \big[\big(k_H (H - H_0) I_m + k_v X_g^2(H)\big) - \\ k_v D_c X_g(H) (D_c D_H(c))^{-1} X_g D_H(c)\big]$$

Proof. Direct computations give

$$X_V(V) = k_H (H - H_0) X_\mathrm{d}(H) + \nabla^\top f(z) X_\mathrm{d}(z) + k_v y X_\mathrm{d}(y) \ .$$

Condition (28) implies

$$\begin{aligned} X_V(H) &= y\hat{u} \\ X_V(z) &= D_H(z) = y^\top \\ X_V(y) &= D_H X_g(H) + D_c X_g(H)\hat{\mu} + X_g^2(H)\hat{u}\ . \end{aligned}$$

Thus, according to Definition 6, we need to show the existence of $\hat{u}$ and $\hat{\mu}$ such that

$$y \cdot \left[\left(k_H(H - H_0)I_m + k_v X_g^2(H) \right) \hat{u} + k_v D_c X_g(H)\hat{\mu} + \nabla f(z) + k_v D_H X_g(H) \right] = 0$$

and (27) hold. Written in matrix notation, a sufficient condition is

$$\begin{pmatrix} k_H(H - H_0)I_m + k_v X_g^2(H) & k_v D_c X_g(H) \\ X_g D_H(c) & D_c D_H(c) \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{\mu} \end{pmatrix} = - \begin{pmatrix} k_v D_H X_g(H) + \nabla f(z) \\ D_H^2(c) \end{pmatrix}\ . \quad (30)$$

Notice that

$$X_g^2(H) = X_g \left(g_i^a \frac{\partial H}{\partial p^i} \right) = \frac{\partial^2 H}{\partial p^i \partial p^j} g_i^a g_j^b \quad (31)$$

defines a positive definite matrix and

$$D_c X_g(H) = D_c \left(g_i^a \frac{\partial H}{\partial p_i} \right) = -\frac{\partial^2 H}{\partial p^i \partial p^j} g_i^a \frac{\partial c^b}{\partial r^j} \quad (32)$$

$$X_g D_H(c) = X_g \left(\frac{\partial H}{\partial p_i} \frac{\partial c^a}{\partial r^i} \right) = \frac{\partial^2 H}{\partial p^i \partial p^j} \frac{\partial c^a}{\partial r^i} g_j^b\ . \quad (33)$$

The matrix $D_c D_H(c) = -D_\lambda^2(H)$ is negative definite, hence invertible. Now, by Schur's argument, the matrix in front of $\hat{u}$ and $\hat{\mu}$ in (30) is invertible if, and only if,

$$k_H(H - H_0)I_m + k_v X_g^2(H) - k_v D_c X_g(H)(D_c D_H(c))^{-1} X_g D_H(c)$$

is invertible. We wish to show that the last two summands form a positive definite matrix, i.e.,

$$X_g^2(H) - D_c X_g(H) \left(D_\lambda^2(H) \right)^{-1} X_g D_c(H) > 0. \quad (34)$$

This would imply that one can choose k_v large enough to ensure that the whole expression is positive definite (note that H is bounded).

Taking into account that $D_c X_g(H) = g^T \cdot \nabla_p^2 H \cdot \nabla c$, (34) can be rewritten as

$$g^T \left[\nabla_p^2 H - \nabla_p^2 H \cdot \nabla c \left(\nabla c^T \cdot \nabla_p^2 H \cdot \nabla c \right)^{-1} \nabla c^T \cdot \nabla_p^2 H \right] g > 0. \quad (35)$$

Let us denote $U = (\nabla_p^2 H)^{\frac{1}{2}} g$ and $V = (\nabla_p^2 H)^{\frac{1}{2}} \nabla c$. Both U and V have full column rank, $\mathrm{rank}(U) = m$, $\mathrm{rank}(V) = k$. Now (35) can be rewritten as:

$$U^T \left(I_{[n\times n]} - V(V^T V)^{-1} V^T \right) U > 0 \tag{36}$$

The term $(V^T V)^{-1} V^T$ is the left (Moore-Penrose) pseudo-inverse of V, denoted V^+. Furthermore, $(I - VV^+)$ is the orthogonal projector onto the kernel of V^T (see [28] for details).

Matrix (36) is non singular if and only if the rank of the projection of U onto the kernel of $V^\top$ is equal to m, which, in turn, implies that $\mathrm{rank}(U^T V^\perp) = m$, where $V^\perp$ is a matrix, whose rows are orthogonal to the columns of V, i.e., $\mathcal{R}(V^\perp) = \ker(V)$. Returning to the initial notation we write $V^\perp = (\nabla_p^2 H)^{-\frac{1}{2}} \nabla c^\perp$, $\mathrm{rank}(\nabla c^\perp) = n - k$. Now, we have

$$\mathrm{rank}(U^T V^\perp) = \mathrm{rank}(g \nabla c^\perp) = m ,$$

which holds by the linear independence of g and ∇c.

To prove passivity, it suffices to compute $X_V(V)$ with $u = \hat{u} + v$. We have

$$\begin{aligned} X_V(V) = y\big[&\left(k_H(H - H_0) + k_v X_g^2(H)\right)(\hat{u} + v) \\ &+ k_v D_c X_g(H)(\hat{\mu} + \nu) + \nabla f(z) + k_v D_H X_g(H)\big] \end{aligned}$$

or equivalently,

$$X_V(V) = y\big[\left(k_H(H - H_0) + k_v X_g^2(H)\right) v + k_v D_c X_g(H)\nu \big] , \tag{37}$$

where ν is such that

$$D_H^2(c) + D_c D_H(c)(\hat{\mu} + \nu) + X_g D_H(c)(\hat{u} + v) = 0 ,$$

i.e., such that

$$D_c D_H(c)\nu + X_g D_H(c) v = 0 .$$

Solving explicitly for ν and substituting back in (37) gives

$$\begin{aligned} X_V(V) = y\, \big[&\left(k_H(H - H_0) + k_v X_g^2(H)\right) - \\ &k_v D_c X_g(H)(D_c D_H(c))^{-1} X_g D_H(c) \big]\, v . \end{aligned}$$

This proves item ii). $\square$

Remark 8. Theorem 7 does not impose in principle any restriction on the degree of under-actuation, but we note that, for specific problems, the detectability of the resulting output still needs to be verified on a case-by-case basis. The test might fail for overly complex systems or for systems with high degrees of under-actuation.

Remark 9. Note that the function $f(z)$ appearing in (29) serves as an additional "degree of freedom" which is used to further restrict the level-set as was mentioned above.

In the single-input case, the control $\hat{u}$ can be computed using Cramer's rule,

$$\hat{u} = \frac{\begin{vmatrix} -k_v D_H X_g(H) - \nabla f(z) & k_v D_c X_g(H) \\ -D_H^2(c)) & D_c D_H(c) \end{vmatrix}}{\begin{vmatrix} k_H(H-H_0)I_m + k_v X_g^2(H) & k_v D_c X_g(H) \\ X_g D_H(c) & D_c D_H(c) \end{vmatrix}} \tag{38}$$

(the extension to the multi-input case is straightforward). Some of the terms are computed in (31), (32), (33) and (7). For completeness, we compute the remaining ones:

$$\begin{aligned} D_H X_g(H) &= \frac{\partial H}{\partial p_j}\frac{\partial H}{\partial p_i}\frac{\partial g_i^a}{\partial r^j} - \frac{\partial^2 H}{\partial p_j \partial p_i} g_i^a \frac{\partial H}{\partial r^j} \\ D_H^2(c) &= \frac{\partial^2 c^a}{\partial r^j \partial r^i}\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial p_i} - \frac{\partial^2 H}{\partial p_j \partial p_i}\frac{\partial H}{\partial r^j}\frac{\partial c^a}{\partial r^i} \ . \end{aligned}$$

Also, notice that $y_{\mathrm{d}} \cdot y \geq 0$, so we can set $v = -k_y y$ with $k_y > 0$. The control is now completely specified and ensures that $X(V) \leq 0$.

3.6. The pendulum on a cart (cont.)

Take $f(z) = \frac{1}{2}(r^{1_x})^2$, so that the cart converges to $r^{1_x} = 0$. The desired energy level is $H_0 = m_2 \bar{g} l$ which, in view of the restriction $r^{1_x} = 0$, corresponds to the homoclinic orbit that passes through the desired equilibrium

$$x^* = \begin{pmatrix} 0 & 0 & 0 & l \end{pmatrix}^\top \ ,$$

so that almost every trajectory converges to x^*. For completeness, we compute the terms in (38):

$$\begin{aligned} \nabla f &= r^{1_x} \\ D_H X_g(H) &= 0 \\ D_c X_g(H) &= \begin{pmatrix} 0 & \frac{r^{2_x} - r^{1_x}}{m_1} \end{pmatrix} \\ X_g D_H(c) &= -(D_c X_g(H))^\top \\ D_H^2(c) &= \begin{pmatrix} 0 & \left\| \frac{p_2}{m_2} - \frac{p_1}{m_1} \right\| + \bar{g}(r^{1_y} - r^{2_y}) \end{pmatrix}^\top \\ D_g^2(H) &= \frac{1}{m_1} \\ D_c D_H(c) &= \begin{pmatrix} \frac{1}{m_1} & \frac{r^{1_y} - r^{2_y}}{m_1} \\ \frac{r^{1_y} - r^{2_y}}{m_1} & l^2 \frac{m_1 + m_2}{m_1 m_2} \end{pmatrix} \ , \end{aligned}$$

so that the controller is completely specified.

4. Conclusions

In global coordinates, the defining functions (the Hamiltonian and the constraints) of many mechanical systems of interest are quadratic and convex. This representation proves to be useful in an energy shaping scenario, where the control problem turns out to be a simple quadratic programming problem instead of the usual problem of finding the solution of a partial differential equation. Another advantage of computing the closed-loop energy (or Lyapunov) function is that the resulting controller does not exhibit undesired phenomena such as winding.

It is worth noting, however, that once the closed-loop Hamiltonian has been obtained, computing the control is simpler in local coordinates. Thus, the results of [11] can be used in a mixed approach in which H_d or V are computed in global coordinates and the actual control is computed using an explicit representation.

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