Balancing Control - Angular Momentum

AN & HN

1 Notation

- m is the total mass of the spatial pendulum

$$m = \sum_{i=1}^{2} m_i \tag{1}$$

- $\boldsymbol{c} \in \mathbb{R}^3$ is the center of mass position of the spatial pendulum

$$\boldsymbol{c} = \frac{1}{m} \sum_{i=1}^{2} m_i \boldsymbol{c}_i \tag{2}$$

- $\boldsymbol{h} \in \mathbb{R}^3$ is the linear momentum of the spatial pendulum

$$\boldsymbol{h} = \sum_{i=1}^{2} m_i \boldsymbol{v}_i = m\, \dot{\boldsymbol{c}} \tag{3}$$

- $\boldsymbol{l} \in \mathbb{R}^3$ is the angular momentum of the spatial pendulum at the foot

$$\boldsymbol{l} = \sum_{i=1}^{2} l_i = \sum_{i=1}^{2} (\boldsymbol{c}_i \times m_i\, \dot{\boldsymbol{c}}_i) = \boldsymbol{c} \times \boldsymbol{h} + \boldsymbol{l}_G \tag{4}$$

  where $\boldsymbol{l}_G \in \mathbb{R}^3$ is the angular momentum of the spatial pendulum at its center of mass

- $\boldsymbol{g} \in \mathbb{R}^3$ is the gravity acceleration vector

$$\boldsymbol{g} = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T \quad ; \quad g = 9.81$$

- $\boldsymbol{f}_F \in \mathbb{R}^3$ is the reaction force at the foot

$$\boldsymbol{f}_F = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}^T$$

For a detailed explanation of each equation see Section 5.

1.1 Cross Product Matrix

Let the operator *Cross Product Matrix*, $CPM(\bullet)$, be defined as

$$CPM(\boldsymbol{r}) = \boldsymbol{r}\times = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \quad \text{where} \quad \boldsymbol{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \tag{5}$$

It can be applied in cross product operations as follows

$$\boldsymbol{r} \times \boldsymbol{q} = CPM(\boldsymbol{r})\, \boldsymbol{q}$$

where $\boldsymbol{r}, \boldsymbol{q} \in \mathbb{R}^3$

2 Kinematics and Dynamics

The following are the basic kinematic and dynamic dependencies:

$$\frac{d}{dt}\boldsymbol{h} = \boldsymbol{f}_F + m\,\boldsymbol{g} \tag{6}$$

$$\frac{d}{dt}\boldsymbol{l} = m\,\boldsymbol{c} \times \boldsymbol{g} \tag{7}$$

For a detailed explanation of each equation see Section 6.

3 Direct stabilization of Angular Momentum

The controller will be stabilizing if it enforces the following:

$$\boldsymbol{l} = \boldsymbol{c} \times \boldsymbol{h} + \boldsymbol{l}_G \to \boldsymbol{0} \tag{8}$$

$$\frac{d}{dt}\boldsymbol{l} = m\,\boldsymbol{c} \times \boldsymbol{g} \to \boldsymbol{0} \tag{9}$$

$$\frac{d^2}{dt^2}\boldsymbol{l} = \boldsymbol{h} \times \boldsymbol{g} \to \boldsymbol{0} \tag{10}$$

$$\frac{d^3}{dt^3}\boldsymbol{l} = \boldsymbol{f}_F \times \boldsymbol{g} \to \boldsymbol{0} \tag{11}$$

Applying the $CPM(\bullet)$ operator, the derivatives can be rewritten as

$$\frac{d}{dt}\boldsymbol{l} = -m\,CPM(\boldsymbol{g})\,\boldsymbol{c} \tag{12}$$

$$\frac{d^2}{dt^2}\boldsymbol{l} = -\,CPM(\boldsymbol{g})\,\boldsymbol{h} \tag{13}$$

$$\frac{d^3}{dt^3}\boldsymbol{l} = -\,CPM(\boldsymbol{g})\,\boldsymbol{f}_F \tag{14}$$

where

$$CPM(\boldsymbol{g}) = -g \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -g\,\boldsymbol{K} \quad \text{given that} \quad \boldsymbol{g} = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T$$

The linear relationship in the previous expressions lends itself to linear control. Hence, the closed loop dynamics can be shaped as desired by the choice of constant gains in the closed- loop equation

$$\frac{d^3}{dt^3}\boldsymbol{l} = -\boldsymbol{K}_1 \frac{d^2}{dt^2}\boldsymbol{l} - \boldsymbol{K}_2 \frac{d}{dt}\boldsymbol{l} - \boldsymbol{K}_3\,\boldsymbol{l} \tag{15}$$

in block matrix form becomes

$$\frac{d^3}{dt^3}\begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} = -\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{bmatrix} \frac{d^2}{dt}\begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} - \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_2 \end{bmatrix} \frac{d}{dt}\begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} - \begin{bmatrix} a_3 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & a_3 \end{bmatrix}\begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} \tag{16}$$

Substituting the linear expressions of the derivatives of the angular momentum in the closed-loop equation, yields

$$g\boldsymbol{K}\boldsymbol{f}_F = -\boldsymbol{K}_1\,g\,\boldsymbol{K}\boldsymbol{h} - \boldsymbol{K}_2\,m\,g\,\boldsymbol{K}\boldsymbol{c} - \boldsymbol{K}_3\,\boldsymbol{l} \tag{17}$$

Note that

$$\boldsymbol{K}\boldsymbol{r} = \begin{bmatrix} -r_y & r_x & 0 \end{bmatrix}^T \quad \text{where} \quad \boldsymbol{r} \in \mathbb{R}^3$$

meaning

$$f_x = -a_1\, h_x - m\, a_2\, c_x - \frac{1}{g} a_3\, l_y \tag{18}$$

$$f_y = -a_1\, h_y - m\, a_2\, c_y + \frac{1}{g} a_3\, l_x \tag{19}$$

$$0 = -a_3\, l_z \tag{20}$$

Reducing the expression to two equations becomes

$$\boldsymbol{f}_{Fxy} = -\tilde{\boldsymbol{K}}_1 \boldsymbol{h}_{xy} - m\, \tilde{\boldsymbol{K}}_2 \boldsymbol{c}_{xy} - \frac{1}{g} \tilde{\boldsymbol{K}}_3 \boldsymbol{l}_{xy} \tag{21}$$

where

$$\tilde{\boldsymbol{K}}_1 = \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix} \quad ; \quad \tilde{\boldsymbol{K}}_2 = \begin{bmatrix} a_2 & 0 \\ 0 & a_2 \end{bmatrix} \quad ; \quad \tilde{\boldsymbol{K}}_3 = \begin{bmatrix} 0 & a_3 \\ -a_3 & 0 \end{bmatrix}$$

4 Potential Energy

$$V_i = -\sum_{i=1}^{2} m_i \boldsymbol{g}^T\, \boldsymbol{c}_i \tag{22}$$

$$\frac{d}{dt} V = -\sum_{i=1}^{2} m_i \boldsymbol{g}^T \dot{\boldsymbol{c}}_i = -\boldsymbol{g}^T m\, \dot{\boldsymbol{c}} = -\boldsymbol{g}^T \boldsymbol{h} \tag{23}$$

$$\begin{aligned} \frac{d^2}{dt^2} V &= -\boldsymbol{g}^T m\, \ddot{\boldsymbol{c}} \\ &= -\boldsymbol{g}^T (\boldsymbol{f}_F + m\boldsymbol{g}) \\ &= g\,(f_z - m\, g) \end{aligned} \tag{24}$$

5 Appendix: Pendulum's Properties

5.1 Pendulum's Center of Mass

The pendulum can be seen as a system made up of a collection of particles, where each particle is represented by a link. The *ith* link has a mass m_i which is concentrated at its center of mass G_i. Therefore, the position vector of the pendulum's center of mass, denoted as $\boldsymbol{c}$, is found by the expression, see [1]:

$$\boldsymbol{c} = \frac{1}{m} \sum_{i=1}^{2} m_i \boldsymbol{c}_i \tag{25}$$

where m is the total mass of the pendulum given by $m = \sum_{i=1}^{2} m_i$.

5.2 Pendulum's Momentum

Before computing the linear momentum of the pendulum, it is important to consider the following relation between the position vectors of the centers of mass of each link, $\boldsymbol{c}_i$, and the position vector of the center of mass of the pendulum $\boldsymbol{c}$.

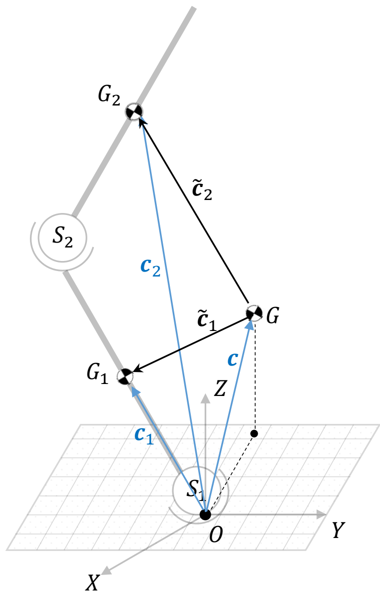

Figure 1: Link's Center of Mass

Further, define the position vector $\boldsymbol{c}_i = \overrightarrow{OG_i}$, $\boldsymbol{c} = \overrightarrow{OG}$ and $\tilde{\boldsymbol{c}}_i = \overrightarrow{GG_i}$, where the relationship is, see Fig. 1.

$$\boldsymbol{c}_i = \boldsymbol{c} + \tilde{\boldsymbol{c}}_i \tag{26}$$

Hence

$$\sum_{i=1}^{2} m_i \tilde{\boldsymbol{c}}_i = \sum_{i=1}^{2} m_i (\boldsymbol{c}_i - \boldsymbol{c}) = \sum_{i=1}^{2} (m_i \boldsymbol{c}_i - m_i \boldsymbol{c}) = \sum_{i=1}^{2} m_i \boldsymbol{c}_i - m\boldsymbol{c} \tag{27}$$

From Eq. (25), the right hand side of the above equation vanishes

$$\sum_{i=1}^{2} m_i \tilde{\boldsymbol{c}}_i = 0 \tag{28}$$

This is an important relationship that will be used in future computations.

The centroidal momentum of the multi rigid-body system is defined as the sum of the linear momenta of each rigid-bodies that compose it

$$\boldsymbol{h} = \sum_{i=1}^{2} \boldsymbol{h}_i = \sum_{i=1}^{2} m_i \dot{\boldsymbol{c}}_i \tag{29}$$

where $\dot{\boldsymbol{c}}_i$ can be expressed as the time derivative of Eq. (26) and denotes the velocity of the center of mass of the pendulum G

$$\dot{\boldsymbol{c}}_i = \dot{\boldsymbol{c}} + \dot{\tilde{\boldsymbol{c}}}_i \tag{30}$$

Substituting the above expression into the momentum equation, Eq. (29), yields

$$\boldsymbol{h} = \sum_{i=1}^{2} m_i (\dot{\boldsymbol{c}} + \dot{\tilde{\boldsymbol{c}}}_i) = \sum_{i=1}^{2} m_i \dot{\boldsymbol{c}} + \sum_{i=1}^{2} m_i \dot{\tilde{\boldsymbol{c}}}_i = \sum_{i=1}^{2} m_i \dot{\boldsymbol{c}} + \frac{D}{Dt}\Big(\sum_{i=1}^{2} m_i \tilde{\boldsymbol{c}}_i\Big) \tag{31}$$

Applying Eq. (28), the derivative of the right hand side vanishes. Hence, the centroidal momentum of the pendulum is

$$\boldsymbol{h} = m\dot{\boldsymbol{c}} \tag{32}$$

5.3 Pendulum's Angular Momentum

The angular momentum of the pendulum with respect to the Foot (point O - origin of inertia frame) is defined as the sum of the angular momenta of each link about the same point. It is given by the following expression

$$\boldsymbol{l} = \sum_{i=1}^{2} \boldsymbol{l}_i = \sum_{i=1}^{2} (\boldsymbol{c}_i \times m_i \dot{\boldsymbol{c}}_i) \tag{33}$$

Substituting Eq. (26) and Eq. (30) into Eq. (33) gives

$$\begin{aligned} &\sum_{i=1}^{2} [(\boldsymbol{c} + \tilde{\boldsymbol{c}}_i) \times m_i(\dot{\boldsymbol{c}} + \dot{\tilde{\boldsymbol{c}}}_i)] = \sum_{i=1}^{2} [\boldsymbol{c} \times m_i(\dot{\boldsymbol{c}} + \dot{\tilde{\boldsymbol{c}}}_i) + \tilde{\boldsymbol{c}}_i \times m_i(\dot{\boldsymbol{c}} + \dot{\tilde{\boldsymbol{c}}}_i)] \\ &= \sum_{i=1}^{2} (\boldsymbol{c} \times m_i \dot{\boldsymbol{c}}) + \sum_{i=1}^{2} (\boldsymbol{c} \times m_i \dot{\tilde{\boldsymbol{c}}}_i) + \sum_{i=1}^{2} (\tilde{\boldsymbol{c}}_i \times m_i \dot{\boldsymbol{c}}) + \sum_{i=1}^{2} (\tilde{\boldsymbol{c}}_i \times m_i \dot{\tilde{\boldsymbol{c}}}_i) \end{aligned} \tag{34}$$

It should be noted that the second term vanishes, once the result of Eq. (28) is applied. This is clear after rewriting this term as shown below

$$\sum_{i=1}^{2} (\boldsymbol{c} \times m_i \dot{\tilde{\boldsymbol{c}}}_i) = \boldsymbol{c} \times \frac{D}{dt} \sum_{i=1}^{2} (m_i \tilde{\boldsymbol{c}}_i)$$

In a similar way, we can apply Eq. (28) to the effect that

$$\sum_{i=1}^{2} (\tilde{\boldsymbol{c}}_i \times m_i \dot{\boldsymbol{c}}) = -\dot{\boldsymbol{c}} \times \sum_{i=1}^{2} (m_i \tilde{\boldsymbol{c}}_i)$$

Thus, the expression for the angular momentum of the pendulum about point O is finally given by

$$\boldsymbol{l} = \sum_{i=1}^{2} (\boldsymbol{c} \times m_i \dot{\boldsymbol{c}}) + \sum_{i=1}^{2} (\tilde{\boldsymbol{c}}_i \times m_i \dot{\tilde{\boldsymbol{c}}}_i) \tag{35}$$

From [2], the angular momentum about the center of mass of the pendulum, point G, is given by

$$\boldsymbol{l}_G = \sum_{i=1}^{2} (\tilde{\boldsymbol{c}}_i \times m_i \dot{\boldsymbol{c}}_i) \tag{36}$$

Substituting Eq. (30) in the previous equation, it can be rewritten as follows

$$\begin{aligned} \boldsymbol{l}_G &= \sum_{i=1}^{2} (\tilde{\boldsymbol{c}}_i \times m_i(\dot{\boldsymbol{c}} + \dot{\tilde{\boldsymbol{c}}}_i)) \\ &= \sum_{i=1}^{2} (\tilde{\boldsymbol{c}}_i \times m_i \dot{\boldsymbol{c}}) + \sum_{i=1}^{2} (\tilde{\boldsymbol{c}}_i \times m_i \dot{\tilde{\boldsymbol{c}}}_i) \end{aligned}$$

Note that the first term vanishes by the result of Eq. (28), yielding

$$\boldsymbol{l}_G = \sum_{i=1}^{2} (\tilde{\boldsymbol{c}}_i \times m_i \dot{\tilde{\boldsymbol{c}}}_i) \tag{37}$$

Substituting this result in Eq. (35), yields

$$\begin{aligned} \boldsymbol{l} &= \sum_{i=1}^{2} (\boldsymbol{c} \times m_i \dot{\boldsymbol{c}}) + \sum_{i=1}^{2} (\tilde{\boldsymbol{c}}_i \times m_i \dot{\tilde{\boldsymbol{c}}}_i) \\ &= \boldsymbol{c} \times m\dot{\boldsymbol{c}} + \boldsymbol{l}_G \end{aligned}$$

Thus, the final expression for the angular momentum of the pendulum about point O is given by

$$\boldsymbol{l} = \boldsymbol{c} \times \boldsymbol{h} + \boldsymbol{l}_G \tag{38}$$

6 Appendix: Kinematics and Dynamics of the Spatial Pendulum

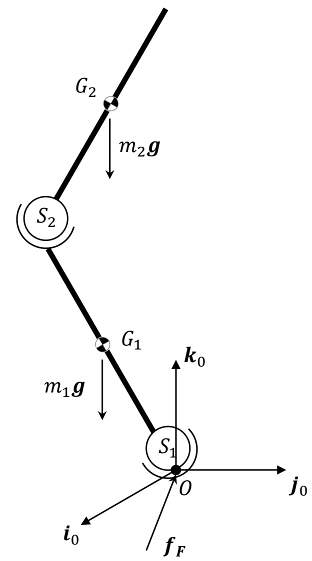

Figure 2: DSP's free-body diagram

6.1 The Newton Equation of the DSP

The Second Newton Law states that the time derivative (in an inertial system of coordinates) of the total momentum of a system of particles equals the sum of all forces acting on this system. From Fig. 2, the Newton equation of the system can be expressed as follows

$$\frac{d}{dt}\boldsymbol{h} = \sum_{i=1}^{2} \boldsymbol{f}_i$$

$$\frac{d}{dt}\sum_{i=1}^{2} m_i \dot{\boldsymbol{c}}_i = \boldsymbol{f}_F + \sum_{i=1}^{2} m_i \boldsymbol{g} \tag{39}$$

which, by virtue of the definition of the centroid, Eq. (25), simplifies to

$$\boldsymbol{f}_F + m\boldsymbol{g} = m\boldsymbol{a} = m\ddot{\boldsymbol{c}} \tag{40}$$

This result implies the center of mass of the pendulum accelerates under the influence of the net external force ($\sum_{i=1}^{2} \boldsymbol{f}_i$) just as a particle accelerates under the influence of the net force acting on it. (Second Newton Law for composites systems).

To see how the components of the reaction force are constrained, Eq. (40) can be expressed in vector form as follows

$$\begin{bmatrix} f_x \\ f_y \\ f_z - mg \end{bmatrix} = \begin{bmatrix} m\ddot{c}_x \\ m\ddot{c}_y \\ m\ddot{c}_z \end{bmatrix} \tag{41}$$

6.2 The Euler Equation of the DSP

The Second Newton Law also states that the time derivative of the total angular momentum of a system of particles with respect to a fixed point in an inertial coordinate system equals to the sum of all external moments and moments due to external forces acting around this fixed point. Thus, from Fig. 2 the Euler equation about point O, can be expressed as follows

$$\frac{d}{dt}\boldsymbol{l} = \sum_{j=1}^{n} \boldsymbol{n}_j^{ext} + \sum_{i=1}^{m} \boldsymbol{n}_i^{F} \tag{42}$$

where $\boldsymbol{n}_j^{ext}$ denotes the jth external moment, while $\boldsymbol{n}_i^F$ denotes the moment due to the external force $\boldsymbol{f}_i$.

Consult Fig. 2 that there are no external moments acting on the pendulum and the only external force is the gravity. Hence, the Euler's equation about O of the pendulum becomes

$$\frac{d}{dt}\boldsymbol{l} = \sum_{i=1}^{2} \boldsymbol{n}_i^F = \sum_{i=1}^{2} (\boldsymbol{c}_i \times m_i \boldsymbol{g}) \tag{43}$$

This equation implies that, for this particular case, the time derivative of the pendulum's angular momentum is equal to the sum of the gravitational torque acting on all links. Now, if we substitute Eq. (33) in this equation, the left hand side can be expressed as

$$\frac{d}{dt}\boldsymbol{l} = \sum_{i=1}^{2} \frac{d}{dt}(\boldsymbol{c}_i \times m_i \boldsymbol{v}_i) = \sum_{i=1}^{2} (\boldsymbol{c}_i \times m_i \boldsymbol{a}_i) \tag{44}$$

because

$$\frac{d}{dt}(\boldsymbol{c}_i \times m_i \boldsymbol{v}_i) = \dot{\boldsymbol{c}}_i \times m_i \dot{\boldsymbol{c}}_i + \boldsymbol{c}_i \times m_i \ddot{\boldsymbol{c}}_i = \boldsymbol{c}_i \times m_i \boldsymbol{a}_i \tag{45}$$

By applying Eq. (25), the right hand side of Eq. (43) can be simplified as follows

$$\begin{aligned} \sum_{i=1}^{2} \boldsymbol{n}_i^F &= \sum_{i=1}^{2} \boldsymbol{c}_i \times m_i \boldsymbol{g} \\ &= \sum_{i=1}^{2} (m_i \boldsymbol{c}_i \times \boldsymbol{g}) \\ &= m\boldsymbol{c} \times \boldsymbol{g} \end{aligned} \tag{46}$$

so, from Eq. (44) and Eq.(46), the Euler equation about O of the pendulum is given by

$$\frac{d}{dt}\boldsymbol{l} = \sum_{i=1}^{2} (\boldsymbol{c}_i \times m_i \boldsymbol{a}_i) = m\boldsymbol{c} \times \boldsymbol{g} \tag{47}$$

References

[1] M. Ardema, *Newton-Euler dynamics.* New York: Springer, 2006.

[2] J. Peraire, and S. Widnall, *Lecture L11 - Conservation Laws for Systems of Particles.*, 2008