Force to Torque - Articulated Body Mapping Analysis

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1 Articulated Body Algorithm

The analysis is performed base on the *Articulated Body algorithm*, Algorithm 1 equations. The objective of this document is to find a relation between the reaction force at the foot ($[\boldsymbol{f}_F]_0$) and the 3-dimensional torque at the hip ($\boldsymbol{\tau}_H = \begin{bmatrix}\tau_x & \tau_y & \tau_z\end{bmatrix}^T$).

Algorithm 1 Articulated-Body algorithm

function FDaba($model$, $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{q}}$, $\boldsymbol{\tau}$, $\bar{\boldsymbol{f}}_{ext}|_1^{N_B}$, $\bar{\boldsymbol{a}}_g$)
  $\bar{\boldsymbol{v}}_0 \leftarrow \boldsymbol{0}$ ▷ Base twist
  $\bar{\boldsymbol{a}}_0 \leftarrow -\bar{\boldsymbol{a}}_g$ ▷ Base twist derivative
  for $i = 1 : N_B$ **do**
    $[\boldsymbol{X}_J, \bar{\boldsymbol{S}}_i] \leftarrow jcalc\big(jtype(i), q_i, \dot{q}_i\big)$
    $\bar{\boldsymbol{v}}_J = \bar{\boldsymbol{S}}_i \dot{q}_i$
    ${}^i\boldsymbol{X}_{\lambda_i} \leftarrow \boldsymbol{X}_J \boldsymbol{X}_i^{tree}$
    $\bar{\boldsymbol{v}}_i \leftarrow {}^i\boldsymbol{X}_{\lambda_i} \bar{\boldsymbol{v}}_{\lambda_i} + \bar{\boldsymbol{v}}_J$
    $\bar{\boldsymbol{c}}_i \leftarrow \bar{\boldsymbol{v}}_i \times \bar{\boldsymbol{v}}_J$
    $\bar{\boldsymbol{I}}_i^A \leftarrow \bar{\boldsymbol{I}}_i$
    $\bar{\boldsymbol{p}}_i^A \leftarrow \bar{\boldsymbol{v}}_i \times^* \bar{\boldsymbol{I}}_i \bar{\boldsymbol{v}}_i - \bar{\boldsymbol{f}}_i^x$
  end for
  for $i = N_B : 1$ **do**
    $\bar{\boldsymbol{U}}_i \leftarrow \bar{\boldsymbol{I}}_i^A \bar{\boldsymbol{S}}_i$
    $d_i \leftarrow \bar{\boldsymbol{S}}_i^T \bar{\boldsymbol{U}}_i$
    $u_i \leftarrow \tau_i - \bar{\boldsymbol{S}}_i^T \bar{\boldsymbol{p}}_i^A$
    if $\lambda_i \neq 0$ **then**
      $\bar{\boldsymbol{I}}^a \leftarrow \bar{\boldsymbol{I}}_i^A - \bar{\boldsymbol{U}}_i d_i^{-1} \bar{\boldsymbol{U}}_i^T$
      $\bar{\boldsymbol{p}}^a \leftarrow \bar{\boldsymbol{p}}_i^A + \bar{\boldsymbol{I}}^a \bar{\boldsymbol{c}}_i + \bar{\boldsymbol{U}}_i d_i^{-1} u_i$
      $\bar{\boldsymbol{I}}_{\lambda_i}^A \leftarrow \bar{\boldsymbol{I}}_{\lambda_i}^A + {}^{\lambda_i}\boldsymbol{X}_i^* \bar{\boldsymbol{I}}^a \, {}^i\boldsymbol{X}_{\lambda_i}$
      $\bar{\boldsymbol{p}}_{\lambda_i}^A \leftarrow \bar{\boldsymbol{p}}_{\lambda_i}^A + {}^{\lambda_i}\boldsymbol{X}_i^* \bar{\boldsymbol{p}}^a$
    end if
  end for
  for $i = 1 : N_B$ **do**
    $\bar{\boldsymbol{a}}' \leftarrow {}^i\boldsymbol{X}_{\lambda_i} \bar{\boldsymbol{a}}_{\lambda_i} + \bar{\boldsymbol{c}}_i$
    $\ddot{q}_i \leftarrow d_i^{-1}(u_i - \bar{\boldsymbol{U}}_i^T \bar{\boldsymbol{a}}')$
    $\bar{\boldsymbol{a}}_i \leftarrow \bar{\boldsymbol{a}}' + \bar{\boldsymbol{S}}_i \ddot{q}_i$
  end for
end function

1.1 Body inertia

$$\bar{\boldsymbol{I}}_i = \begin{bmatrix} \boldsymbol{I}_i + m_i\, CPM(\boldsymbol{\rho}_i) CPM(\boldsymbol{\rho}_i)^T & m_i\, CPM(\boldsymbol{\rho}_i) \\ m_i\, CPM(\boldsymbol{\rho}_i)^T & m_i\, \boldsymbol{1}_{3\times 3} \end{bmatrix}$$

where

$$CPM(\boldsymbol{\rho}) = \begin{bmatrix} 0 & -\rho_z & \rho_y \\ \rho_z & 0 & -\rho_x \\ -\rho_y & \rho_x & 0 \end{bmatrix} \quad \text{given that} \quad \boldsymbol{\rho} = \begin{bmatrix} \rho_x \\ \rho_y \\ \rho_z \end{bmatrix}$$

Note that we are only interested in the bias forces ($\boldsymbol{p}_i^A$) and the wrench f_1, because these variables depend on the torques. We assumed that any other variable not related to the torques can be contain in a single variable, this is the case of the articulated body inertias ($\boldsymbol{I}_i^A$) and twist ($\boldsymbol{v}$).

2 Rigid-Body bias forces analysis

This algorithm is based on a six body scheme, therefore there are 6 body-frames ($F_1 - F_6$) and one inertial frame (F_0).

Let

$$\boldsymbol{S} = \begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0\end{bmatrix}^T \tag{1}$$

$$d_i = \boldsymbol{S}^T \boldsymbol{I}_i^A \boldsymbol{S} \quad i = 1, \cdots, 6 \quad d_i \in \mathbb{R}^1 \tag{2}$$

$$\boldsymbol{b}_i = \frac{1}{d_i} \boldsymbol{I}_i^A \boldsymbol{S} \quad i = 1, \cdots, 6 \quad \boldsymbol{b}_i \in \mathbb{R}^6 \tag{3}$$

$$\boldsymbol{A}_i = \boldsymbol{1}_{6x6} - \boldsymbol{b}_i \boldsymbol{S}^T \quad i = 1, \cdots, 6 \quad \boldsymbol{A}_i \in \mathbb{R}^{6\times6} \tag{4}$$

Now, we compute each of the bias forces as follows

$$\begin{aligned}
\boldsymbol{p}_6^A &= \boldsymbol{p}_6 \\
\boldsymbol{p}_5^A &= {\boldsymbol{X}_6}^T \Big[\boldsymbol{A}_6 \boldsymbol{p}_6^A + \boldsymbol{I}_6^a \boldsymbol{c}_6\Big] + \boldsymbol{X}_6 \boldsymbol{b}_6 \tau_6 \\
&= \boldsymbol{k}_6 + {\boldsymbol{X}_6}^T \boldsymbol{b}_6\, \tau_6 \\
\boldsymbol{p}_4^A &= {\boldsymbol{X}_5}^T \Big[\boldsymbol{A}_5 \boldsymbol{p}_5^A + \boldsymbol{I}_5^a \boldsymbol{c}_5\Big] + {\boldsymbol{X}_5}^T \boldsymbol{b}_5\, \tau_5 \\
&= {\boldsymbol{X}_5}^T \boldsymbol{I}_5^a \boldsymbol{c}_5 + {\boldsymbol{X}_5}^T \boldsymbol{A}_5 \Big[\boldsymbol{k}_6 + {\boldsymbol{X}_6}^T \boldsymbol{b}_6\, \tau_6\Big] + {\boldsymbol{X}_5}^T \boldsymbol{b}_5\, \tau_5 \\
&= \boldsymbol{k}_5 + {\boldsymbol{X}_5}^T \boldsymbol{A}_5 {\boldsymbol{X}_6}^T \boldsymbol{b}_6\, \tau_6 + {\boldsymbol{X}_5}^T \boldsymbol{b}_5\, \tau_5 \\
\boldsymbol{p}_3^A &= {\boldsymbol{X}_4}^T \Big[\boldsymbol{A}_4 \boldsymbol{p}_4^A + \boldsymbol{I}_4^a \boldsymbol{c}_4\Big] + {\boldsymbol{X}_4}^T \boldsymbol{b}_4\, \tau_4 \\
&= {\boldsymbol{X}_4}^T \boldsymbol{I}_4^a \boldsymbol{c}_4 + {\boldsymbol{X}_4}^T \boldsymbol{A}_4 \Big[\boldsymbol{k}_5 + {\boldsymbol{X}_5}^T \boldsymbol{A}_5 {\boldsymbol{X}_6}^T \boldsymbol{b}_6\, \tau_6 + {\boldsymbol{X}_5}^T \boldsymbol{b}_5\, \tau_5\Big] + {\boldsymbol{X}_4}^T \boldsymbol{b}_4\, \tau_4 \\
&= \boldsymbol{k}_4 + {\boldsymbol{X}_4}^T \boldsymbol{A}_4 {\boldsymbol{X}_5}^T \boldsymbol{A}_5 {\boldsymbol{X}_6}^T \boldsymbol{b}_6\, \tau_6 + {\boldsymbol{X}_4}^T \boldsymbol{A}_4 {\boldsymbol{X}_5}^T \boldsymbol{b}_5\, \tau_5 + {\boldsymbol{X}_4}^T \boldsymbol{b}_4\, \tau_4 \\
\boldsymbol{p}_2^A &= {\boldsymbol{X}_3}^T \Big[\boldsymbol{A}_3 \boldsymbol{p}_3^A + \boldsymbol{I}_3^a \boldsymbol{c}_3\Big] + {\boldsymbol{X}_3}^T \boldsymbol{b}_3\, \tau_3 \\
&= {\boldsymbol{X}_3}^T \boldsymbol{I}_3^a \boldsymbol{c}_3 + {\boldsymbol{X}_3}^T \boldsymbol{A}_3 \Big[\boldsymbol{k}_4 + {\boldsymbol{X}_4}^T \boldsymbol{A}_4 {\boldsymbol{X}_5}^T \boldsymbol{A}_5 {\boldsymbol{X}_6}^T \boldsymbol{b}_6\, \tau_6 + {\boldsymbol{X}_4}^T \boldsymbol{A}_4 {\boldsymbol{X}_5}^T \boldsymbol{b}_5\, \tau_5 + {\boldsymbol{X}_4}^T \boldsymbol{b}_4\, \tau_4\Big] + {\boldsymbol{X}_3}^T \boldsymbol{b}_3\, \tau_3 \\
&= \boldsymbol{k}_3 + {\boldsymbol{X}_3}^T \boldsymbol{A}_3 {\boldsymbol{X}_4}^T \boldsymbol{A}_4 {\boldsymbol{X}_5}^T \boldsymbol{A}_5 {\boldsymbol{X}_6}^T \boldsymbol{b}_6\, \tau_6 + {\boldsymbol{X}_3}^T \boldsymbol{A}_3 {\boldsymbol{X}_4}^T \boldsymbol{A}_4 {\boldsymbol{X}_5}^T \boldsymbol{b}_5\, \tau_5 + {\boldsymbol{X}_3}^T \boldsymbol{A}_3 {\boldsymbol{X}_4}^T \boldsymbol{b}_4\, \tau_4 + {\boldsymbol{X}_3}^T \boldsymbol{b}_3\, \tau_3 \\
\boldsymbol{p}_1^A &= {\boldsymbol{X}_2}^T \Big[\boldsymbol{A}_2 \boldsymbol{p}_2^A + \boldsymbol{I}_2^a \boldsymbol{c}_2\Big] + {\boldsymbol{X}_2}^T \boldsymbol{b}_2\, \tau_2 \\
&= {\boldsymbol{X}_2}^T \boldsymbol{I}_2^a \boldsymbol{c}_2 + {\boldsymbol{X}_2}^T \boldsymbol{A}_2 \Big[\boldsymbol{k}_3 + {\boldsymbol{X}_3}^T \boldsymbol{A}_3 {\boldsymbol{X}_4}^T \boldsymbol{A}_4 {\boldsymbol{X}_5}^T \boldsymbol{A}_5 {\boldsymbol{X}_6}^T \boldsymbol{b}_6\, \tau_6 + {\boldsymbol{X}_3}^T \boldsymbol{A}_3 {\boldsymbol{X}_4}^T \boldsymbol{A}_4 {\boldsymbol{X}_5}^T \boldsymbol{b}_5\, \tau_5 + {\boldsymbol{X}_3}^T \boldsymbol{A}_3 {\boldsymbol{X}_4}^T \boldsymbol{b}_4\, \tau_4 \\
&+ {\boldsymbol{X}_3}^T \boldsymbol{b}_3\, \tau_3\Big] + {\boldsymbol{X}_2}^T \boldsymbol{b}_2\, \tau_2 \\
&= \boldsymbol{k}_2 + {\boldsymbol{X}_2}^T \boldsymbol{A}_2 {\boldsymbol{X}_3}^T \boldsymbol{A}_3 {\boldsymbol{X}_4}^T \boldsymbol{A}_4 {\boldsymbol{X}_5}^T \boldsymbol{A}_5 {\boldsymbol{X}_6}^T \boldsymbol{b}_6\, \tau_6 + {\boldsymbol{X}_2}^T \boldsymbol{A}_2 {\boldsymbol{X}_3}^T \boldsymbol{A}_3 {\boldsymbol{X}_4}^T \boldsymbol{A}_4 {\boldsymbol{X}_5}^T \boldsymbol{b}_5\, \tau_5 + {\boldsymbol{X}_2}^T \boldsymbol{A}_2 {\boldsymbol{X}_3}^T \boldsymbol{A}_3 {\boldsymbol{X}_4}^T \boldsymbol{b}_4\, \tau_4 \\
&+ {\boldsymbol{X}_2}^T \boldsymbol{A}_2 {\boldsymbol{X}_3}^T \boldsymbol{b}_3\, \tau_3 + {\boldsymbol{X}_2}^T \boldsymbol{b}_2\, \tau_2
\end{aligned} \tag{5}$$

Note $\boldsymbol{X}_i$ is a 6×6 matrix which denotes the transformation matrices from frame F_i to F_{i+1}. Similarly ${\boldsymbol{X}_i}^T$ transforms from frame F_{i+1} to F_i

3 Reaction force analysis

3.1 Twist derivative

The first generalized acceleration is given by

$$\ddot{q}_1 = \frac{1}{d_1}\Big[\tau_1 - \boldsymbol{S}^T \boldsymbol{p}_1^A - \boldsymbol{S}^T \boldsymbol{I}_1^A \Big(\boldsymbol{X}_1 \boldsymbol{a}_0 + \boldsymbol{c}_1\Big)\Big] \tag{6}$$

Applying it in the equation of the first twist derivative yields

$$\begin{aligned}
\boldsymbol{a}_1 &= \boldsymbol{X}_1\boldsymbol{a}_0 + \boldsymbol{c}_1 + \boldsymbol{S}\ddot{q}_1 \\
&= \boldsymbol{X}_1\boldsymbol{a}_0 + \boldsymbol{c}_1 + \boldsymbol{S}\frac{1}{d_1}\Big[\tau_1 - \boldsymbol{S}^T\boldsymbol{p}_1^A - \boldsymbol{S}^T\boldsymbol{I}_1^A\Big(\boldsymbol{X}_1\boldsymbol{a}_0 + \boldsymbol{c}_1\Big)\Big] \\
&= \Big[\boldsymbol{1}_{6\times 6} - \frac{1}{d_1}\boldsymbol{S}\boldsymbol{S}^T\boldsymbol{I}_1^A\Big]\Big[\boldsymbol{X}_1\boldsymbol{a}_0 + \boldsymbol{c}_1\Big] + \boldsymbol{S}\frac{1}{d_1}\Big[\tau_1 - \boldsymbol{S}^T\boldsymbol{p}_1^A\Big] \\
&= \boldsymbol{k}_1 + \boldsymbol{S}\frac{1}{d_1}\Big[\tau_1 - \boldsymbol{S}^T\boldsymbol{p}_1^A\Big] \qquad (7)
\end{aligned}$$

3.2 Wrench

The first wrench expressed in body-frame F_1 is given by

$$\begin{aligned}
\boldsymbol{f}_1 &= \boldsymbol{I}_1^A\boldsymbol{a}_1 + \boldsymbol{p}_1^A \\
&= \boldsymbol{I}_1^A\Big(\boldsymbol{k}_1 + \boldsymbol{S}\frac{1}{d_1}\Big[\tau_1 - \boldsymbol{S}^T\boldsymbol{p}_1^A\Big]\Big) + \boldsymbol{p}_1^A \\
&= \boldsymbol{I}_1^A\boldsymbol{k}_1 + \boldsymbol{b}_1\,\tau_1 + \boldsymbol{A}_1\boldsymbol{p}_1^A \qquad (8)
\end{aligned}$$

Expressed in the inertia frame

$$[\boldsymbol{f}_1]_0 = {\boldsymbol{X}_1}^T\boldsymbol{I}_1^A\boldsymbol{k}_1 + {\boldsymbol{X}_1}^T\boldsymbol{b}_1\,\tau_1 + {\boldsymbol{X}_1}^T\boldsymbol{A}_1\boldsymbol{p}_1^A \qquad (9)$$

Substituting Eq. (5) in the previous equation yields

$$\begin{aligned}
[\boldsymbol{f}_1]_0 &= {\boldsymbol{X}_1}^T\boldsymbol{I}_1^A\boldsymbol{k}_1 + {\boldsymbol{X}_1}^T\boldsymbol{b}_1\,\tau_1 + {\boldsymbol{X}_1}^T\boldsymbol{A}_1\Big[\boldsymbol{k}_2 + {\boldsymbol{X}_2}^T\boldsymbol{A}_2{\boldsymbol{X}_3}^T\boldsymbol{A}_3{\boldsymbol{X}_4}^T\boldsymbol{A}_4{\boldsymbol{X}_5}^T\boldsymbol{A}_5{\boldsymbol{X}_6}^T\boldsymbol{b}_6\,\tau_6 \\
&\quad + {\boldsymbol{X}_2}^T\boldsymbol{A}_2{\boldsymbol{X}_3}^T\boldsymbol{A}_3{\boldsymbol{X}_4}^T\boldsymbol{A}_4{\boldsymbol{X}_5}^T\boldsymbol{b}_5\,\tau_5 + {\boldsymbol{X}_2}^T\boldsymbol{A}_2{\boldsymbol{X}_3}^T\boldsymbol{A}_3{\boldsymbol{X}_4}^T\boldsymbol{b}_4\,\tau_4 + {\boldsymbol{X}_2}^T\boldsymbol{A}_2{\boldsymbol{X}_3}^T\boldsymbol{b}_3\,\tau_3 + {\boldsymbol{X}_2}^T\boldsymbol{b}_2\,\tau_2\Big] \\
&= \boldsymbol{k}_0 + {\boldsymbol{X}_1}^T\boldsymbol{A}_1{\boldsymbol{X}_2}^T\boldsymbol{A}_2{\boldsymbol{X}_3}^T\boldsymbol{A}_3{\boldsymbol{X}_4}^T\boldsymbol{A}_4{\boldsymbol{X}_5}^T\boldsymbol{A}_5{\boldsymbol{X}_6}^T\boldsymbol{b}_6\,\tau_6 + {\boldsymbol{X}_1}^T\boldsymbol{A}_1{\boldsymbol{X}_2}^T\boldsymbol{A}_2{\boldsymbol{X}_3}^T\boldsymbol{A}_3{\boldsymbol{X}_4}^T\boldsymbol{A}_4{\boldsymbol{X}_5}^T\boldsymbol{b}_5\,\tau_5 \\
&\quad + {\boldsymbol{X}_1}^T\boldsymbol{A}_1{\boldsymbol{X}_2}^T\boldsymbol{A}_2{\boldsymbol{X}_3}^T\boldsymbol{A}_3{\boldsymbol{X}_4}^T\boldsymbol{b}_4\,\tau_4 + {\boldsymbol{X}_1}^T\boldsymbol{A}_1{\boldsymbol{X}_2}^T\boldsymbol{A}_2{\boldsymbol{X}_3}^T\boldsymbol{b}_3\,\tau_3 + {\boldsymbol{X}_1}^T\boldsymbol{A}_1{\boldsymbol{X}_2}^T\boldsymbol{b}_2\,\tau_2 + {\boldsymbol{X}_1}^T\boldsymbol{b}_1\,\tau_1 \\
&= \boldsymbol{k}_0 + {\boldsymbol{X}_1}^T\boldsymbol{b}_1\,\tau_1 + {\boldsymbol{X}_1}^T\boldsymbol{A}_1{\boldsymbol{X}_2}^T\boldsymbol{b}_2\,\tau_2 + {\boldsymbol{X}_1}^T\boldsymbol{A}_1{\boldsymbol{X}_2}^T\boldsymbol{A}_2{\boldsymbol{X}_3}^T\boldsymbol{b}_3\,\tau_3 + {\boldsymbol{X}_1}^T\boldsymbol{A}_1{\boldsymbol{X}_2}^T\boldsymbol{A}_2{\boldsymbol{X}_3}^T\boldsymbol{A}_3\Big[{\boldsymbol{X}_4}^T\boldsymbol{b}_4\,\tau_4 \\
&\quad + {\boldsymbol{X}_4}^T\boldsymbol{A}_4{\boldsymbol{X}_5}^T\boldsymbol{b}_5\,\tau_5 + {\boldsymbol{X}_4}^T\boldsymbol{A}_4{\boldsymbol{X}_5}^T\boldsymbol{A}_5{\boldsymbol{X}_6}^T\boldsymbol{b}_6\,\tau_6\Big] \qquad (10)
\end{aligned}$$

4 Mapping

In the case of the spatial pendulum

$$\boldsymbol{\tau}_F = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T \quad \Rightarrow \quad \tau_1 = \tau_2 = \tau_3 = 0 \qquad (11)$$

thus

$$[\boldsymbol{f}_1]_0 = \boldsymbol{k}_0 + {\boldsymbol{X}_1}^T\boldsymbol{A}_1{\boldsymbol{X}_2}^T\boldsymbol{A}_2{\boldsymbol{X}_3}^T\boldsymbol{A}_3\Big[{\boldsymbol{X}_4}^T\boldsymbol{b}_4\,\tau_4 + {\boldsymbol{X}_4}^T\boldsymbol{A}_4{\boldsymbol{X}_5}^T\boldsymbol{b}_5\,\tau_5 + {\boldsymbol{X}_4}^T\boldsymbol{A}_4{\boldsymbol{X}_5}^T\boldsymbol{A}_5{\boldsymbol{X}_6}^T\boldsymbol{b}_6\,\tau_6\Big] \qquad (12)$$

Note that $\boldsymbol{f}_1$ denotes the first wrench, a 6×1 vector. Thus the previous equation can be express in block form as follows

$$\begin{bmatrix}\boldsymbol{n}_F \\ \boldsymbol{f}_F\end{bmatrix}_0 = \begin{bmatrix}\boldsymbol{k}_{0n} \\ \boldsymbol{k}_{0f}\end{bmatrix} + \begin{bmatrix}\boldsymbol{D}_n \\ \boldsymbol{D}_f\end{bmatrix}\boldsymbol{\tau}_{\boldsymbol{H}} \quad \text{given that} \quad \boldsymbol{\tau}_H = \begin{bmatrix}\tau_4 & \tau_5 & \tau_6\end{bmatrix}^T \qquad (13)$$

where $\boldsymbol{D}_n,\ \boldsymbol{D}_f \in \mathbb{R}^{3\times 3}$ and $\boldsymbol{n}_F,\ \boldsymbol{f}_F,\ \boldsymbol{k}_{0n},\ \boldsymbol{k}_{0f} \in \mathbb{R}^3$. Therefore, the reaction force at the foot is given by

$$\boldsymbol{f}_F = \boldsymbol{k}_{0f} + \boldsymbol{D}_f\boldsymbol{\tau}_H \qquad (14)$$

where

$$\boldsymbol{D}_f = \begin{bmatrix}\boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3\end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{d}_1 \end{bmatrix} = \boldsymbol{X}_1{}^T \boldsymbol{A}_1 \boldsymbol{X}_2{}^T \boldsymbol{A}_2 \boldsymbol{X}_3{}^T \boldsymbol{A}_3 \boldsymbol{X}_4{}^T \boldsymbol{b}_4$$

$$\begin{bmatrix} \boldsymbol{e}_2 \\ \boldsymbol{d}_2 \end{bmatrix} = \boldsymbol{X}_1{}^T \boldsymbol{A}_1 \boldsymbol{X}_2{}^T \boldsymbol{A}_2 \boldsymbol{X}_3{}^T \boldsymbol{A}_3 \boldsymbol{X}_4{}^T \boldsymbol{A}_4 \boldsymbol{X}_5{}^T \boldsymbol{b}_5$$

$$\begin{bmatrix} \boldsymbol{e}_3 \\ \boldsymbol{d}_3 \end{bmatrix} = \boldsymbol{X}_1{}^T \boldsymbol{A}_1 \boldsymbol{X}_2{}^T \boldsymbol{A}_2 \boldsymbol{X}_3{}^T \boldsymbol{A}_3 \boldsymbol{X}_4{}^T \boldsymbol{A}_4 \boldsymbol{X}_5{}^T \boldsymbol{A}_5 \boldsymbol{X}_6{}^T \boldsymbol{b}_6$$

and

$$\boldsymbol{k}_{0f} = \boldsymbol{f}_0$$

meaning $\boldsymbol{k}_{0f}$ is the reaction force vector at the foot when $\boldsymbol{\tau}_H = 0$.