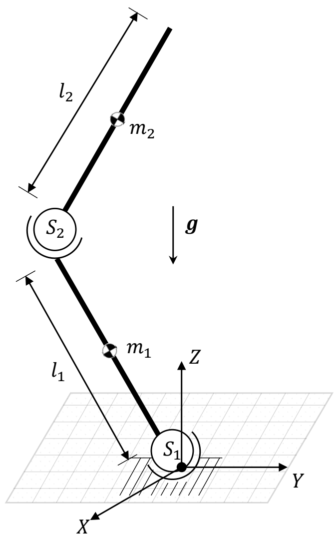
Swing-up controller

AN & HM

This paper is an extension of the results presented by X. Xin and M. Kaneda in [3]. Xin and Kaneda proposed an energy-based swing-up controller for the underactuated planar robot, called Acrobot, by applying the LaSalle's Invariant Theorem.

Similar to the energy-based control approach developed in [3], this paper presents an energy-based control solution for the swing-up problem of the underactuated double spherical pendulum.

1 Double Spherical Pendulum Dynamics



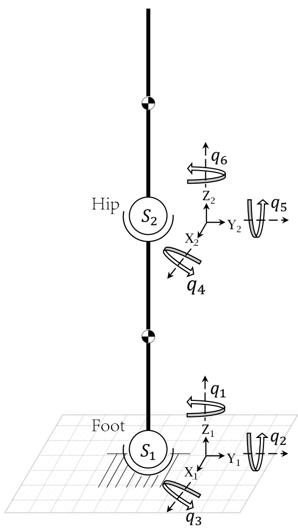


Figure 1: Double Spherical Pendulum

Consider the two-link underactuated 3D open chain-manipulator, called double spherical pendulum (DSP). The analysis is performed considering the system is subjected to gravity with no friction at the joints. The free-body diagram of the DSP is shown in Fig. 1. The open chain-manipulator is composed by two rigid bodies modeled as uniform rods, with two spherical joints: a passive one located at the foot (base) and an active one at the hip. The equation of motion of the DSP is given by

$$\boldsymbol{I}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} - \gamma(\boldsymbol{q}) = \boldsymbol{\tau} \tag{1}$$

expressed in block matrix form gives

$$\begin{bmatrix} \boldsymbol{I}_{11} & \boldsymbol{I}_{12} \\ \boldsymbol{I}_{12}{}^T & \boldsymbol{I}_{22} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_F \\ \ddot{\boldsymbol{q}}_H \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}_F \\ \dot{\boldsymbol{q}}_H \end{bmatrix} - \begin{bmatrix} \gamma_F \\ \gamma_H \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\tau}_H \end{bmatrix} \tag{2}$$

where $\boldsymbol{q}$ is a six dimensional vector called generalized coordinates. $\boldsymbol{I}(\boldsymbol{q}) \in \mathbb{R}^{6\times6}$ denotes the generalized inertia matrix; $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) \in \mathbb{R}^{6\times6}$ is the Coriolis matrix; the vector $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ gives the Coriolis and centrifugal forces. $\gamma(\boldsymbol{q}) \in \mathbb{R}^6$ includes the gravity term which acts at the joints, and $\boldsymbol{\tau} \in \mathbb{R}^6$ with $\boldsymbol{\tau}_H \in \mathbb{R}^3$ being the vector of actuator torques applied at the hip spherical joint.

Thus, we define our non-linear system as

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \ddot{\boldsymbol{q}}(t) \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{I}^{-1}(\boldsymbol{q})\big(- \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \gamma(\boldsymbol{q})\big) + \boldsymbol{I}^{-1}(\boldsymbol{q})\boldsymbol{\tau} \end{bmatrix} \tag{3}$$

with

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{q}(t) & \dot{\boldsymbol{q}}(t) \end{bmatrix} \tag{4}$$

Note that the generalized coordinates and its derivatives $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), \ddot{\boldsymbol{q}}(t))$ are time dependent, but for brevity the t parameter is often suppress in the equations whenever possible.

Proposition 1. *The equation of motion* (1) *satisfies:* $\dot{\boldsymbol{I}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - 2\,\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathbb{R}^{6\times 6}$ *is a skew-symmetric matrix.*

Proof. See Appendix D □

1.1 DSP Energy

The total energy of the DSP is expressed in terms of the generalized coordinates as

$$E(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{I}(\boldsymbol{q})\dot{\boldsymbol{q}} + P(\boldsymbol{q}) \tag{5}$$

where $P(\boldsymbol{q})$ is the potential energy of the DSP and, in terms of the generalized coordinates, is given by [see Appendix B]

$$P(\boldsymbol{q}) = \beta_1 \cos q_2 \cos q_3 + \beta_2 \Big(\cos q_2 \cos(q_3 + q_4) \cos q_5 - \sin q_2 \sin q_5 \Big) \tag{6}$$

where β_1 and β_2 are constants given by

$$\begin{aligned} \beta_1 &= \Big(m_1 \frac{l_1}{2} + m_2\, l_1 \Big)\, g \\ \beta_2 &= m_2 \frac{l_2}{2}\, g \end{aligned}$$

Proposition 2. *The gravity term* $\boldsymbol{\gamma}$ *from the equation of motion* (1) *satisfies*

$$\boldsymbol{\gamma}(\boldsymbol{q}) = -\frac{\partial P(\boldsymbol{q})}{\partial \boldsymbol{q}} \tag{7}$$

Proof. See Appendix C □

Proposition 3. *The DSP system has the following passivity property:*

$$\dot{E}(\dot{\boldsymbol{q}}) = {\dot{\boldsymbol{q}}_H}^T \boldsymbol{\tau}_H \tag{8}$$

Proof. Differentiating the total energy of the DSP (5) yields

$$\dot{E}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{I}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot{\boldsymbol{I}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \dot{P}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{9}$$

Substituting $\boldsymbol{I}(\boldsymbol{q})\ddot{\boldsymbol{q}}$ from Eq. (1) into Eq. (9) yields

$$\dot{E}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{\tau} + \dot{\boldsymbol{q}}^T \boldsymbol{\gamma}(\boldsymbol{q}) + \frac{1}{2}\dot{\boldsymbol{q}}^T \Big(\dot{\boldsymbol{I}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - 2\,\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \Big)\dot{\boldsymbol{q}} + \dot{P}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{10}$$

From Proposition 2, we can define the derivative of the potential energy as

$$\dot{P}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\dot{\boldsymbol{q}}^T \boldsymbol{\gamma}(\boldsymbol{q}) \tag{11}$$

Applying this result along Proposition 1 yields

$$\dot{E}(\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{\tau} = {\dot{\boldsymbol{q}}_H}^T \boldsymbol{\tau}_H \tag{12}$$

□

2 Control Law of the DSP

The passivity property of the system, shown in Eq. (8), suggests to use the total energy $E(\boldsymbol{q}, \dot{\boldsymbol{q}})$ in the controller design. Thus, let $V : \mathbb{R}^{12} \to \mathbb{R}$ be a continuously differentiable function given by

$$V(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{k_E}{2} \left(E(\boldsymbol{q}, \dot{\boldsymbol{q}}) - E_{top}\right)^2 + \frac{k_D}{2}\, \dot{\boldsymbol{q}}_H{}^T \dot{\boldsymbol{q}}_H + \frac{k_P}{2}\, \boldsymbol{q}_H{}^T \boldsymbol{q}_H \tag{13}$$

where $k_P > 0$, $k_E > 0$, $k_D > 0$ are constants; and E_{top} denotes the total energy at the upright position given by

$$E_{top} = \beta_1 + \beta_2 \tag{14}$$

The derivative of function $V(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is given by

$$\dot{V}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = k_E \left(E(\boldsymbol{q}, \dot{\boldsymbol{q}}) - E_{top}\right) \dot{E}(\dot{\boldsymbol{q}}) + k_D\, \dot{\boldsymbol{q}}_H^T\, \ddot{\boldsymbol{q}}_H + k_P\, \dot{\boldsymbol{q}}_H{}^T\, \boldsymbol{q}_H \tag{15}$$

Applying Proposition 3, yields

$$\begin{aligned} \dot{V}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= k_E \left(E(\boldsymbol{q}, \dot{\boldsymbol{q}}) - E_{top}\right) \dot{\boldsymbol{q}}_H{}^T \boldsymbol{\tau}_H + k_D\, \dot{\boldsymbol{q}}_H^T\, \ddot{\boldsymbol{q}}_H + k_P\, \dot{\boldsymbol{q}}_H{}^T\, \boldsymbol{q}_H \\ &= \dot{\boldsymbol{q}}_H{}^T \Big(k_E \left(E(\boldsymbol{q}, \dot{\boldsymbol{q}}) - E_{top}\right) \boldsymbol{\tau}_H + k_D\, \ddot{\boldsymbol{q}}_H + k_P\, \boldsymbol{q}_H \Big) \end{aligned} \tag{16}$$

Proposition 4. *Suppose*

$$k_E \left(E(\boldsymbol{q}, \dot{\boldsymbol{q}}) - E_{top}\right) \boldsymbol{\tau}_H + k_D\, \ddot{\boldsymbol{q}}_H + k_P\, \boldsymbol{q}_H = -k_V\, \dot{\boldsymbol{q}}_H \qquad \forall t \geq 0 \tag{17}$$

where $k_V > 0$ is a constant, then the function $V(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is a decreasing function for all t.

Proof. Substituting Eq. (17) into Eq. (16), yields

$$\dot{V}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -k_V\, \dot{\boldsymbol{q}}_H{}^T \dot{\boldsymbol{q}}_H \leq 0 \qquad \forall t \geq 0 \tag{18}$$

Thus, the function $\dot{V}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is a decreasing function. $\square$

Now, we proceed to define our control law to determine the input $\boldsymbol{\tau}_H$. To this end, from Eq. (1), we find an expression for the hip generalized acceleration

$$\ddot{\boldsymbol{q}}_H = \bar{\boldsymbol{I}}_{22}(\boldsymbol{q})\, \boldsymbol{\tau}_H + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{19}$$

where

$$\bar{\boldsymbol{I}}_{22}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{O}_{3\times3} & \mathbf{1}_{3\times3} \end{bmatrix} \boldsymbol{I}(\boldsymbol{q})^{-1} \begin{bmatrix} \boldsymbol{O}_{3\times3} \\ \mathbf{1}_{3\times3} \end{bmatrix}$$

$$\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} \boldsymbol{O}_{3\times3} & \mathbf{1}_{3\times3} \end{bmatrix} \boldsymbol{I}(\boldsymbol{q})^{-1} \Big(-\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}} + \gamma(\boldsymbol{q}) \Big)$$

Substituting Eq. (19) in Eq. (17) yields

$$\Big(k_E \left(E(\boldsymbol{q}, \dot{\boldsymbol{q}}) - E_{top}\right) \mathbf{1}_{3\times3} + k_D\, \bar{\boldsymbol{I}}_{22}(\boldsymbol{q}) \Big) \boldsymbol{\tau}_H = -k_V\, \dot{\boldsymbol{q}}_H - k_D\, \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - k_P\, \boldsymbol{q}_H \tag{20}$$

Therefore, when

$$\Big(k_E \left(E(\boldsymbol{q}) - E_{top}\right) \mathbf{1}_{3\times3} + k_D\, \bar{\boldsymbol{I}}_{22}(\boldsymbol{q}) \Big) > 0 \quad \forall t \geq 0 \tag{21}$$

we can define $\boldsymbol{\tau}_H$ as

$$\boldsymbol{\tau}_H = \Big(k_E \left(E(\boldsymbol{q}, \dot{\boldsymbol{q}}) - E_{top}\right) \mathbf{1}_{3\times3} + k_D\, \bar{\boldsymbol{I}}_{22}(\boldsymbol{q}) \Big)^{-1} \Big(-k_V\, \dot{\boldsymbol{q}}_H - k_D\, \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - k_P\, \boldsymbol{q}_H \Big) \tag{22}$$

Proposition 5. *Suppose that $k_P > 0$, $k_E > 0$ and $k_V > 0$. Then the existence of the controller $\boldsymbol{\tau}_H$ is guaranteed for the DSP starting from any initial state for all future time if and only if k_D satisfies*

$$k_D > \left\{ \max_{\boldsymbol{q}} \frac{k_E}{\lambda_{min}(\bar{\boldsymbol{I}}_{22}(\boldsymbol{q}))} \left(E_{top} - P(\boldsymbol{q})\right) \right\} \tag{23}$$

Proof.

Theorem 1 (**Weyl inequalities**). *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ Hermitian matrices, where the eigenvalues of each matrix have the following relations*

$$\lambda_1(\boldsymbol{A}) \leq \cdots \leq \lambda_n(\boldsymbol{A}) \quad ; \quad \lambda_1(\boldsymbol{B}) \leq \cdots \leq \lambda_n(\boldsymbol{B})$$

then the following inequality holds for $k \in [1:n]$

$$\lambda_k(\boldsymbol{A}) + \lambda_1(\boldsymbol{B}) \leq \lambda_k(\boldsymbol{A}+\boldsymbol{B}) \leq \lambda_k(\boldsymbol{A}) + \lambda_n(\boldsymbol{B}) \tag{24}$$

Let the two real symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ be given by

$$\boldsymbol{A} = k_E \left(E(\boldsymbol{q},\dot{\boldsymbol{q}}) - E_{top}\right) \mathbf{1}_{3\times 3} \quad \text{and} \quad \boldsymbol{B} = k_D\, \bar{\boldsymbol{I}}_{22}(\boldsymbol{q})$$

By Theorem 1 with $k = 1$, meaning $\lambda_k = \lambda_1 \equiv \lambda_{min}$, we have

$$\lambda_{min}(\boldsymbol{A}+\boldsymbol{B}) \geq \lambda_{min}(\boldsymbol{A}) + \lambda_{min}(\boldsymbol{B})$$

Assuming $\boldsymbol{A}+\boldsymbol{B} > 0$, it follows $\lambda_{min}(\boldsymbol{A}+\boldsymbol{B}) > 0$, which implies

$$\lambda_{min}(\boldsymbol{A}) + \lambda_{min}(\boldsymbol{B}) > 0$$

and leads to the following inequality

$$\lambda_{min}(\boldsymbol{B}) > -\lambda_{min}(\boldsymbol{A}) \tag{25}$$

By definition, matrix $\boldsymbol{A}$ has three identical eigenvalues, thus $\lambda_{min}(\boldsymbol{A}) = k_E\,(E(\boldsymbol{q},\dot{\boldsymbol{q}}) - E_{top})$, meaning

$$k_D\, \lambda_{min}(\bar{\boldsymbol{I}}_{22}(\boldsymbol{q})) > -k_E\,(E(\boldsymbol{q},\dot{\boldsymbol{q}}) - E_{top}) \tag{26}$$

This inequality will hold for all t if

$$k_D > \max_{\boldsymbol{q}} \left\{ \frac{k_E}{\lambda_{min}(\bar{\boldsymbol{I}}_{22}(\boldsymbol{q}))} (E_{top} - E(\boldsymbol{q},\dot{\boldsymbol{q}})) \right\}$$

By definition, $\min_{(\boldsymbol{q},\dot{\boldsymbol{q}})}\{E(\boldsymbol{q},\dot{\boldsymbol{q}})\} = P(\boldsymbol{q})$. Applying this result in the inequality above, yields

$$k_D > \left\{ \max_{\boldsymbol{q}} \frac{k_E}{\lambda_{min}(\bar{\boldsymbol{I}}_{22}(\boldsymbol{q}))}\,(E_{top} - P(\boldsymbol{q})) \right\} \tag{27}$$

□

3 Stability Analysis

The objective of this paper is to propose a swing-up controller for the DSP using the following LaSalle's theorem:

Theorem 2 (**LaSalle's theorem**). *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(\boldsymbol{x}) \leq 0$ in Ω. Let S be the set of all points in Ω where $\dot{V}(\boldsymbol{x}) = 0$. Let W be the largest invariant set in S. Then every solution starting in Ω approaches W as $t \to \infty$.*

Using the function $V(\boldsymbol{q},\dot{\boldsymbol{q}})$ in (13), we defined the set $\Omega \subset \mathbb{R}^{12}$ as

$$\Omega = \{(\boldsymbol{q},\dot{\boldsymbol{q}}) \mid V(\boldsymbol{q},\dot{\boldsymbol{q}}) \leq c\} \tag{28}$$

where c is a positive constant.

Proposition 6. *Suppose that $k_P > 0$, $k_E > 0$ and $k_V > 0$ hold, and k_P satisfies* (23). *Let function V be defined as* (13). *Then every solution starting in Ω approaches the invariant set W given by*

$$W = \{(\boldsymbol{q},\dot{\boldsymbol{q}}) \mid (\boldsymbol{q}_F,\dot{\boldsymbol{q}}_F) \ \textit{satisfies}\ (35)\ \textit{and}\ \boldsymbol{q}_H(t) \equiv {\boldsymbol{q}_H}^*\} \tag{29}$$

as $t \to \infty$

Proof. Since $\dot{V}(\boldsymbol{q},\dot{\boldsymbol{q}}) \leq 0$ in Ω, this implies the set Ω is a compact invariant set. Let set S be

$$S = \{(\boldsymbol{q},\dot{\boldsymbol{q}}) \mid \dot{V}(\boldsymbol{q},\dot{\boldsymbol{q}}) = 0\} \quad \text{such that} \quad S \subset \Omega \tag{30}$$

It follows that for all $(\boldsymbol{q},\dot{\boldsymbol{q}}) \in S$

$$\dot{V}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -k_V\, \dot{\boldsymbol{q}}_H{}^T \dot{\boldsymbol{q}}_H = 0 \tag{31}$$

which implies

$$\lim_{t\to\infty} \boldsymbol{q}_H(t) = \boldsymbol{q}_H{}^* \tag{32}$$

where $\boldsymbol{q}_H{}^*$ denotes a constant. Substituting $\dot{\boldsymbol{q}}_H = \boldsymbol{0}$ in Eq. (8) yields

$$\dot{E}(\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}_H{}^T \boldsymbol{\tau}_H = 0 \tag{33}$$

meaning

$$\lim_{t\to\infty} E(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t)) = E^* \tag{34}$$

where E^* denotes a constant.

Substituting $\boldsymbol{q}_H(t) \equiv \boldsymbol{q}_H{}^*$ and $E(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t)) \equiv E^*$ into the energy equation (5) yields

$$\dot{\boldsymbol{q}}_F{}^T \boldsymbol{I}_{11}(\boldsymbol{q})\dot{\boldsymbol{q}}_F = 2E^* - 2\Big(\beta_1 \cos q_2 \cos q_3 + \beta_2(\cos q_2 \cos(q_3 + q_4{}^*)\cos q_5{}^* - \sin q_2 \sin q_5{}^*)\Big) \tag{35}$$

Therefore, the largest invariant set W can be expressed as

$$W = \{(\boldsymbol{q},\dot{\boldsymbol{q}}) \mid (\boldsymbol{q}_F,\dot{\boldsymbol{q}}_F) \text{ satisfies (35) and } \boldsymbol{q}_H(t) \equiv \boldsymbol{q}_H{}^*\} \tag{36}$$

By Theorem 2, we ensures that every $(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t))$ starting in Ω approaches to the largest invariant set W as $t \to \infty$. □

Proposition 7. *Consider the DSP* (1) *and the controller* (22). *Suppose that* k_D *satisfies* (23), $k_P > 0$, $k_E > 0$ *and* $k_V > 0$ *hold. Then the invariant set* W *defined in* (29) *satisfies*

$$W = W_{top} \cup \Theta \quad \textit{with} \quad W_{top} \cap \Theta = \emptyset \tag{37}$$

where W_{top} *defined in Eq.* (40) *is the homoclinic orbit Eq.* (39) *with* $\boldsymbol{q}_H = \boldsymbol{0}$ *corresponding to the case* $E^* = E_{top}$, *and* Θ *defined in Eq.* (46) *is the equilibrium set corresponding to the case of* $E^* \neq E_{top}$, *and* E^* *is the convergent value of the energy of the DSP.*

Proof. Evaluating Eq. (17) at $(\boldsymbol{q},\dot{\boldsymbol{q}}) \in W$, meaning substituting $\boldsymbol{q}_H(t) \equiv \boldsymbol{q}_H{}^*$ and $E(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t)) \equiv E^*$ yields

$$k_E\,(E^* - E_{top})\,\boldsymbol{\tau}_H + k_P\,\boldsymbol{q}_H{}^* = \boldsymbol{0} \tag{38}$$

This leads us to address two cases:

Case 1: $E^* = E_{top}$

By inspection of Eq. (38), when $E^* = E_{top}$, implies $\boldsymbol{q}_H{}^* = \boldsymbol{0}$. Applying this result in Eq. (35) yields

$$\dot{\boldsymbol{q}}_F{}^T \boldsymbol{I}_{11}(\boldsymbol{q})\dot{\boldsymbol{q}}_F = 2\,E_{top}(1 - \cos q_2 \cos q_3) \tag{39}$$

Thus, as $t \to \infty$, the closed-loop solution $(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t))$ approaches the following invariant set:

$$W_{top} = \{(\boldsymbol{q},\dot{\boldsymbol{q}}) \mid (\boldsymbol{q}_F,\dot{\boldsymbol{q}}_F) \text{ satisfies Eq. (39) and } \boldsymbol{q}_H = \boldsymbol{0}\} \tag{40}$$

The equilibrium point of the orbit (39) is

$$(\boldsymbol{q}_F,\dot{\boldsymbol{q}}_F) = \left(\begin{bmatrix} q_1^* & 0 & 0 \end{bmatrix}^T, \boldsymbol{0}\right)$$

meaning the motion of the first link satisfies

$$\lim_{t\to\infty} \boldsymbol{q}_F = \begin{bmatrix} q_1^* & 0 & 0 \end{bmatrix}^T \quad ; \quad \lim_{t\to\infty} \dot{\boldsymbol{q}}_F = \boldsymbol{0} \tag{41}$$

This shows that the DSP can enter any arbitrarily small neighborhood of the upright equilibrium under the controller (22).

Case 2: $E^* \neq E_{top}$

By inspection of Eq. (38),when $E^* \neq E_{top}$, implies $\boldsymbol{\tau}_H$ is constant. Therefore Eq. (17) becomes

$$k_E\,(E^* - E_{top})\,\boldsymbol{\tau}_H{}^* + k_P\,\boldsymbol{q}_H{}^* = \mathbf{0} \tag{42}$$

Assumption 1
When $E^* \neq E_{top}$ occurs under the controller (22), then the closed-loop solution $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ approaches an equilibrium point $(\boldsymbol{q}_F{}^*, \boldsymbol{q}_H{}^*, \mathbf{0}, \mathbf{0})$ as $t \to \infty$.

Evaluating the equation of motion of the DSP (1) under Assumption 1 yields

$$-\begin{bmatrix}\boldsymbol{\gamma}_F{}^* \\ \boldsymbol{\gamma}_H{}^*\end{bmatrix} = \begin{bmatrix}\mathbf{0} \\ \boldsymbol{\tau}_H{}^*\end{bmatrix}$$

where $\boldsymbol{\gamma}_F \in \mathbb{R}^3$ and $\boldsymbol{\gamma}_H \in \mathbb{R}^3$ are the elements of the gravity term $\boldsymbol{\gamma}$ of the equation of motion (1). These two vectors are defined in Appendix C as follows

$$\begin{aligned}
\gamma_1{}^* &= 0 \\
\gamma_2{}^* &= \beta_1\ \sin q_2{}^* \cos q_3{}^* + \beta_2 \left(\sin q_2{}^* \cos(q_3{}^* + q_4{}^*) \cos q_5{}^* + \cos q_2{}^* \sin q_5{}^*\right) = 0 \\
\gamma_3{}^* &= \cos q_2{}^* \Big(\beta_1\ \sin q_3{}^* + \beta_2\ \sin(q_3{}^* + q_4{}^*) \cos q_5{}^*\Big) = 0
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
-\gamma_4^* &= -\beta_2\ \cos q_2{}^* \sin(q_3{}^* + q_4{}^*) \cos q_5{}^* = \tau_x^* \\
-\gamma_5^* &= -\beta_2 \left(\cos q_2{}^* \cos(q_3{}^* + q_4{}^*) \sin q_5{}^* + \sin q_2{}^* \cos q_5{}^*\right) = \tau_y^* \\
\gamma_6^* &= 0 = \tau_z^*
\end{aligned} \tag{44}$$

Under the same circumstances, the total energy of the DSP becomes $E^* = P(\boldsymbol{q}^*)$. Substituting these results in Eq. (42), we obtain

$$-k_E\,(P(\boldsymbol{q}^*) - E_{top})\,\boldsymbol{\gamma}_H{}^* + k_P\,\boldsymbol{q}_H{}^* = \mathbf{0} \quad \text{with} \quad P(\boldsymbol{q}^*) \neq E_{top} \tag{45}$$

Then, we can define the equilibrium point set for the closed-loop system as

$$\Theta = \{(\boldsymbol{q}^*, \mathbf{0}) \mid \boldsymbol{q}^* \text{ satisfies (43) and (45)}\} \tag{46}$$

From analyzing these two cases, we conclude the largest invariant set W defined in (29) satisfies

$$W = W_{top} \cup \Theta \quad \text{with} \quad W_{top} \cap \Theta = \emptyset \tag{47}$$

□

4 Equilibrium Point Set Analysis

If one of the equilibrium points in Θ is stable in sense of LaSalle stability, then the control objective of swinging the DSP up to any arbitrarily small neighborhood of the upright equilibrium point cannot be realized. Therefore we use the parameter k_P and Proposition 8 to ensure all the elements of set Θ are unstable.

Proposition 8. *The potential energy of the DSP $P(\boldsymbol{q})$ can be expressed as*

$$P(\boldsymbol{q}) = \pm\Phi(\boldsymbol{q}_H) \quad \textit{when} \quad \boldsymbol{\gamma}_F(\boldsymbol{q}) = \mathbf{0} \tag{48}$$

where

$$\Phi(\boldsymbol{q}_H) = \sqrt{\beta_1{}^2 + \beta_2{}^2 + 2\,\beta_1\beta_2 \cos q_4 \cos q_5} \tag{49}$$

Proof. See Appendix C. □

Proposition 9. *Consider the DSP* (1) *and the controller* (22). *Suppose that* k_D *satisfies* (23), $k_E > 0$, $k_V > 0$ *hold, and* k_P *satisfies*

$$k_P > \frac{2}{\pi}\, k_E\, \min({\beta_1}^2, {\beta_2}^2) \tag{50}$$

Then all the elements in the equilibrium point set Θ *defined in* (46), *are unstable equilibrium points.*

Proof. Assume the equilibrium point set defined in Eq. (46) is composed by three subsets, such that $\Theta = \Theta_0 \cup \Theta_+ \cup \Theta_-$ with the following definitions

$$\begin{aligned} \Theta_0 &= \{(\boldsymbol{q}^*, \mathbf{0}) \mid (\boldsymbol{q}^*, \mathbf{0}) \in \Theta,\ P(\boldsymbol{q}^*) = 0\} \\ \Theta_+ &= \{(\boldsymbol{q}^*, \mathbf{0}) \mid (\boldsymbol{q}^*, \mathbf{0}) \in \Theta,\ P(\boldsymbol{q}^*) > 0\} \\ \Theta_- &= \{(\boldsymbol{q}^*, \mathbf{0}) \mid (\boldsymbol{q}^*, \mathbf{0}) \in \Theta,\ P(\boldsymbol{q}^*) < 0\} \end{aligned} \tag{51}$$

Set Θ_0 Analysis

For all $(\boldsymbol{q}^*, \mathbf{0}) \in \Theta_0$, implies $P(\boldsymbol{q}^*) = \Phi({\boldsymbol{q}_H}^*) = 0$ owing to ${\boldsymbol{\gamma}_F}^* = \mathbf{0}$ when $E^* \neq E_{top}$. Applying this implication in Eq. (49) yields

$$\Phi({\boldsymbol{q}_H}^*) = {\beta_1}^2 + {\beta_2}^2 + 2\,\beta_1\beta_2 \cos {q_4}^* \cos {q_5}^* = 0$$

which implies

$$\beta_1 = \beta_2 \quad \text{and} \quad \cos {q_4}^* \cos {q_5}^* = -1 \tag{52}$$

then, Eq. (45) becomes

$$k_P\, \boldsymbol{q}_H = k_E\, E_{top} {\boldsymbol{\tau}_H}^* = 2\, k_E\, \beta_1\, {\boldsymbol{\tau}_H}^* \tag{53}$$

which implies

$$\begin{aligned} &k_P\, {q_4}^* = 2\, k_E\, {\beta_1}^2 \cos {q_2}^* \sin {q_3}^* \\ &k_P\, {q_5}^* = 2\, k_E\, {\beta_1}^2 \sin {q_2}^* \cos {q_5}^* \quad \text{with} \quad \cos {q_5}^* = \pm 1 \\ &k_P\, {q_6}^* = 0 \end{aligned} \tag{54}$$

For $\cos {q_4}^* \cos {q_5}^* = -1$ to hold, there are two cases:

- Case 1: $\cos {q_4}^* = -1$ and $\cos {q_5}^* = 1$

  In this case we have

  $${q_4}^* = \pm\,\pi, \pm\,3\pi, \cdots \quad \text{and} \quad {q_5}^* = 0, \pm\,2\pi, \cdots$$

  then, this follows

  $$|k_P\, {q_4}^*| \geq k_P\, \pi \quad \text{and} \quad |k_P\, {q_5}^*| \geq 0$$

  applying condition (50) yields

  $$|k_P\, {q_4}^*| \geq k_P\, \pi > 2\, k_E\, {\beta_1}^2 \tag{55}$$

  along to the fact

  $$2\, k_E\, {\beta_1}^2 \geq 2\, k_E\, {\beta_1}^2 |\cos {q_2}^* \sin {q_3}^*| \tag{56}$$

  shows that for this case, Eq. (53) has no solution.

- Case 2: $\cos {q_4}^* = 1$ and $\cos {q_5}^* = -1$

  In this case we have

  $${q_4}^* = 0, \pm\,2\pi, \cdots \quad \text{and} \quad {q_5}^* = \pm\,\pi, \pm\,3\pi, \cdots$$

  similarly to previous case

  $$|k_P\, {q_4}^*| \geq 0 \quad \text{and} \quad |k_P\, {q_5}^*| \geq k_P\, \pi$$

applying condition (50) yields

$$|k_P\, q_5{}^*| \geq k_P\, \pi > 2\, k_E\, \beta_1{}^2 \tag{57}$$

along to the fact

$$2\, k_E\, \beta_1{}^2 \geq 2\, k_E\, \beta_1{}^2 | - \sin q_2{}^*| \tag{58}$$

shows that for this case, Eq. (53) has no solution.

We conclude that when condition (50) holds, Θ_0 *is an empty set.*

Set Θ_+ Analysis

Let express the potential energy of the DSP as in Eq. (6)

$$P(\boldsymbol{q}) = \beta_1 \cos q_2 \cos q_3 + \beta_2 \Big(\cos q_2 \cos(q_3 + q_4) \cos q_5 - \sin q_2 \sin q_5 \Big) \tag{59}$$

For all $(\boldsymbol{q}^*, \mathbf{0}) \in \Theta$, we know $\boldsymbol{\gamma}_F{}^* = \mathbf{0}$, meaning

$$\begin{aligned}
\gamma_1{}^* &= 0 \\
\gamma_2{}^* &= \beta_1\ \sin q_2{}^* \cos q_3{}^* + \beta_2 \left(\sin q_2{}^* \cos(q_3{}^* + q_4{}^*) \cos q_5{}^* + \cos q_2{}^* \sin q_5{}^* \right) = 0 \\
\gamma_3{}^* &= \beta_1\ \sin q_3{}^* + \beta_2\ \sin(q_3{}^* + q_4{}^*) \cos q_5{}^* = 0
\end{aligned} \tag{60}$$

Computing

$$P(\boldsymbol{q}^*)\ \cos q_2{}^* \sin(q_3{}^* + q_4{}^*) - \gamma_2{}^*\ \sin q_2{}^* \sin(q_3{}^* + q_4{}^*) + \gamma_3{}^*\ \cos(q_3{}^* + q_4{}^*) \tag{61}$$

yields

$$P(\boldsymbol{q}^*)\ \cos q_2{}^* \sin(q_3{}^* + q_4{}^*) = \beta_1\ \sin q_4{}^* \tag{62}$$

Similarly, computing

$$\begin{aligned}
P(\boldsymbol{q}^*)\Big(- \cos q_2{}^* \cos(q_3{}^* + q_4{}^*) \sin q_5{}^* - \sin q_2{}^* \cos q_5{}^* \Big) + \gamma_2{}^* \left(\sin q_2{}^* \cos(q_3{}^* + q_4{}^*) \sin q_5{}^* - \cos q_2{}^* \cos q_5{}^* \right) \\
+ \gamma_3{}^* \sin(q_3{}^* + q_4{}^*) \sin q_5{}^*
\end{aligned} \tag{63}$$

yields

$$P(\boldsymbol{q}^*)\Big(\cos q_2{}^* \cos(q_3{}^* + q_4{}^*) \sin q_5{}^* + \sin q_2{}^* \cos q_5{}^* \Big) = \beta_1 \cos q_4{}^* \sin q_5{}^* \tag{64}$$

From Eq. (45) and since $P(\boldsymbol{q}^*) = \Phi(\boldsymbol{q}_H{}^*) > 0$ we get the following expressions

$$\frac{k_P}{\beta_1\, \beta_2}\, q_4{}^* = (\Phi(\boldsymbol{q}_H{}^*) - E_{top}) \frac{k_E}{\Phi(\boldsymbol{q}_H{}^*)}\ \sin q_4{}^* \cos q_5{}^* \tag{65}$$

$$\frac{k_P}{\beta_1\, \beta_2}\, q_5{}^* = (\Phi(\boldsymbol{q}_H{}^*) - E_{top}) \frac{k_E}{\Phi(\boldsymbol{q}_H{}^*)} \cos q_4{}^* \sin q_5{}^* \tag{66}$$

Note that for all $(\boldsymbol{q}^*, \mathbf{0}) \in \Theta$, $\quad E_{top} \neq \Phi(\boldsymbol{q}_H{}^*)$ which implies that $\boldsymbol{q}_H{}^*$ where $q_4{}^* = q_5{}^* = 0$ can not be a solution of the previous equations. Under the assumption, if $q_4{}^* = q_5{}^*$, then $q_4{}^* = q_5{}^* \neq 0$. We rewrite Eq. (65) and Eq. (66) as

$$q_4{}^* \left(\frac{k_P}{\beta_1\, \beta_2} - \nu_1(\boldsymbol{q}_H{}^*) \right) = 0 \quad \text{with} \quad \nu_1(\boldsymbol{q}_H{}^*) = (\Phi(\boldsymbol{q}_H{}^*) - E_{top}) \frac{k_E}{q_4{}^* \Phi(\boldsymbol{q}_H{}^*)}\ \sin q_4{}^* \cos q_5{}^* \tag{67}$$

$$q_5{}^* \left(\frac{k_P}{\beta_1\, \beta_2} - \nu_2(\boldsymbol{q}_H{}^*) \right) = 0 \quad \text{with} \quad \nu_2(\boldsymbol{q}_H{}^*) = (\Phi(\boldsymbol{q}_H{}^*) - E_{top}) \frac{k_E}{q_5{}^* \Phi(\boldsymbol{q}_H{}^*)} \cos q_4{}^* \sin q_5{}^* \tag{68}$$

These two equations will not have solution if and only if

$$k_P > \beta_1\, \beta_2 \max \left\{ \sup_{q_4{}^* = q_5{}^* \neq 0} \nu_1(\boldsymbol{q}_H{}^*), \sup_{q_4{}^* = q_5{}^* \neq 0} \nu_2(\boldsymbol{q}_H{}^*) \right\} \tag{69}$$

Using the following inequality

$$\sup_{q_4 \in (\pi, 2\pi), q_5{}^*} \nu_1(\boldsymbol{q}_H{}^*) \leq \frac{2}{\pi} \min\left(\frac{\beta_2}{\beta_1}, \frac{\beta_1}{\beta_2}\right) \quad \text{and} \quad \sup_{q_5 \in (\pi, 2\pi), q_4{}^*} \nu_2(\boldsymbol{q}_H{}^*) \leq \frac{2}{\pi} \min\left(\frac{\beta_2}{\beta_1}, \frac{\beta_1}{\beta_2}\right) \tag{70}$$

proved in Appendix F, we find inequality (69) can be rewritten as

$$k_P > \beta_1 \, \beta_2 \frac{2}{\pi} \min\left(\frac{\beta_2}{\beta_1}, \frac{\beta_1}{\beta_2}\right) = \frac{2}{\pi} \min({\beta_1}^2, {\beta_2}^2) \tag{71}$$

We conclude that if condition (50) holds, then Eq. (65) and Eq. (66) do not have solution and therefore Θ_+ *is an empty set.*

Set Θ_- Analysis

For all $(\boldsymbol{q}^*, \boldsymbol{0}) \in \Theta_-$, $P(\boldsymbol{q}^*) < 0$, which implies $P(\boldsymbol{q}^*) = -\Phi(\boldsymbol{q}_H{}^*) < E_{top}$. From Eq. (45) we find the following expressions

$$\frac{k_P}{\beta_1 \, \beta_2} \, q_4{}^* = (\Phi(\boldsymbol{q}_H{}^*) + E_{top}) \frac{k_E}{\Phi(\boldsymbol{q}_H{}^*)} \, \sin q_4{}^* \cos q_5{}^* \tag{72}$$

$$\frac{k_P}{\beta_1 \, \beta_2} \, q_5{}^* = (\Phi(\boldsymbol{q}_H{}^*) + E_{top}) \frac{k_E}{\Phi(\boldsymbol{q}_H{}^*)} \cos q_4{}^* \sin q_5{}^* \tag{73}$$

Note that the left-hand side of both equations is linear with respect $q_4{}^*$ and $q_5{}^*$ respectively; and the right-hand side in both equations is a bounded odd periodic function of $q_4{}^*$ and $q_5{}^*$ respectively. This implies this system of simultaneous equations with two unknowns has a finite number of solutions for any $k_P > 0$.

For a given $q_4{}^*$ and $q_5{}^*$, we use the following equations

$$\sin q_2{}^* = \frac{\beta_2}{\Phi} \sin q_5{}^* \tag{74}$$

$$\cos q_2{}^* \sin q_3{}^* = \frac{\beta_2}{\Phi} \sin q_4{}^* \cos q_5{}^* \tag{75}$$

$$\cos q_2{}^* \cos q_3{}^* = -\frac{1}{\Phi} (\beta_1 + \beta_2 \cos q_4{}^* \cos q_5{}^*) \tag{76}$$

to find a unique solution for $q_2{}^*$ and $q_3{}^*$. The detailed procedure used to find these equations can be found in Appendix G. We conclude, Θ_- *has a finite number of equilibrium points.*

Stability of the equilibrium points in Θ_-

By definition the function V is a non-increasing function under the controller (22), therefore if we use as initial condition any equilibrium point plus a perturbation $\boldsymbol{\epsilon}$, the function V will have the following relation

$$V(\boldsymbol{q}_F{}^* + \boldsymbol{\epsilon}, \boldsymbol{q}_H{}^*, \boldsymbol{0}, \boldsymbol{0}) < V(\boldsymbol{q}_F{}^*, \boldsymbol{q}_H{}^*, \boldsymbol{0}, \boldsymbol{0}) \quad \text{with} \quad \boldsymbol{\epsilon} = \epsilon \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \quad \forall \epsilon \notin \{0, \pm 2\pi, \cdots\} \tag{77}$$

meaning the pendulum does not approach the equilibrium point $(\boldsymbol{q}_F{}^*, \boldsymbol{q}_H{}^*, \boldsymbol{0}, \boldsymbol{0})$, but it approaches the invariant set W_{top} or other equilibrium point with a smaller value when evaluated in the $V(\boldsymbol{q}, \dot{\boldsymbol{q}})$ function. Therefore, *all the equilibrium point* $(\boldsymbol{q}_F{}^*, \boldsymbol{q}_H{}^*, \boldsymbol{0}, \boldsymbol{0}) \in \Theta_-$ *are unstable.* □

Appendix A Rotation Matrix Analysis

For this analysis we are using the rotation matrix definition of the Natural Orthogonal methodology [4]. In this methodology the rotation matrices are defines as

$$\boldsymbol{Q}_i = \begin{bmatrix} \cos \bar{q}_i & -\lambda_i \sin \bar{q}_i & \mu_i \sin \bar{q}_i \\ \sin \bar{q}_i & \lambda_i \cos \bar{q}_i & -\mu_i \cos \bar{q}_i \\ 0 & \mu_i & \lambda_i \end{bmatrix} \tag{78}$$

where

$$\lambda_i = \cos \alpha_i$$

$$\mu_i = \sin \alpha_i$$

$$\bar{q}_i = q_i + q_{ini_i}$$

For the DSP under consideration

$$\boldsymbol{q}_{ini} = \begin{bmatrix} \pi & -\frac{\pi}{2} & \pi & 0 & \frac{\pi}{2} & 0 \end{bmatrix}^T \tag{79}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \frac{\pi}{2} & \frac{\pi}{2} & 0 & \frac{\pi}{2} & \frac{\pi}{2} & 0 \end{bmatrix}^T \tag{80}$$

Note that both vectors are constant and they depend on the architecture of the DSP. Thus, the rotation matrix for each generalized coordinate is as follows

$$\lambda_1 = \cos\left(\frac{\pi}{2}\right) = 0 \quad ; \quad \mu_1 = \sin\left(\frac{\pi}{2}\right) = 1 \qquad \lambda_4 = \cos\left(\frac{\pi}{2}\right) = 0 \quad ; \quad \mu_4 = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\boldsymbol{Q}_1 = \begin{bmatrix} -\cos q_1 & 0 & -\sin q_1 \\ -\sin q_1 & 0 & \cos q_1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \boldsymbol{Q}_4 = \begin{bmatrix} \cos q_4 & 0 & \sin q_4 \\ \sin q_4 & 0 & -\cos q_4 \\ 0 & 1 & 0 \end{bmatrix} \tag{81}$$

$$\lambda_2 = \cos\left(\frac{\pi}{2}\right) = 0 \quad ; \quad \mu_2 = \sin\left(\frac{\pi}{2}\right) = 1 \qquad \lambda_5 = \cos\left(\frac{\pi}{2}\right) = 0 \quad ; \quad \mu_5 = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\boldsymbol{Q}_2 = \begin{bmatrix} \sin q_2 & 0 & -\cos q_2 \\ -\cos q_2 & 0 & -\sin q_2 \\ 0 & 1 & 0 \end{bmatrix} \qquad \boldsymbol{Q}_5 = \begin{bmatrix} -\sin q_5 & 0 & \cos q_5 \\ \cos q_5 & 0 & \sin q_5 \\ 0 & 1 & 0 \end{bmatrix} \tag{82}$$

$$\lambda_3 = \cos(0) = 1 \quad ; \quad \mu_3 = \sin(0) = 0 \qquad \lambda_6 = \cos(0) = 1 \quad ; \quad \mu_6 = \sin(0) = 0$$

$$\boldsymbol{Q}_3 = \begin{bmatrix} -\cos q_3 & \sin q_3 & 0 \\ -\sin q_3 & -\cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{Q}_6 = \begin{bmatrix} \cos q_6 & -\sin q_6 & 0 \\ \sin q_6 & \cos q_6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{83}$$

Compute the rotation matrix $\boldsymbol{Q}_1\boldsymbol{Q}_2$

$$\boldsymbol{Q}_1\boldsymbol{Q}_2 = \begin{bmatrix} -\cos q_1 \sin q_2 & -\sin q_1 & \cos q_1 \cos q_2 \\ -\sin q_1 \sin q_2 & \cos q_1 & \sin q_1 \cos q_2 \\ -\cos q_2 & 0 & -\sin q_2 \end{bmatrix} \tag{84}$$

Compute the rotation matrix $\boldsymbol{Q}_2\boldsymbol{Q}_3$

$$\boldsymbol{Q}_2\boldsymbol{Q}_3 = \begin{bmatrix} -\sin q_2 \cos q_3 & \sin q_2 \sin q_3 & -\cos q_2 \\ \cos q_2 \cos q_3 & -\cos q_2 \sin q_3 & -\sin q_2 \\ -\sin q_3 & -\cos q_3 & 0 \end{bmatrix} \tag{85}$$

Using Eq. (84) and the definition for $\boldsymbol{Q}_3$, we find an expression for the rotation matrix from the inertial frame to the body frame of the first link

$$\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3 = \begin{bmatrix} \cos q_1 \sin q_2 \cos q_3 + \sin q_1 \sin q_3 & -\cos q_1 \sin q_2 \sin q_3 + \sin q_1 \cos q_3 & \cos q_1 \cos q_2 \\ \sin q_1 \sin q_2 \cos q_3 - \cos q_1 \sin q_3 & -\sin q_1 \sin q_2 \sin q_3 - \cos q_1 \cos q_3 & \sin q_1 \cos q_2 \\ \cos q_2 \cos q_3 & -\cos q_2 \sin q_3 & -\sin q_2 \end{bmatrix} \tag{86}$$

Compute the rotation matrix $\boldsymbol{Q}_5\boldsymbol{Q}_6$

$$\boldsymbol{Q}_5\boldsymbol{Q}_6 = \begin{bmatrix} -\sin q_5 \cos q_6 & \sin q_5 \sin q_6 & \cos q_5 \\ \cos q_5 \cos q_6 & -\cos q_5 \sin q_6 & \sin q_5 \\ \sin q_6 & \cos q_6 & 0 \end{bmatrix} \tag{87}$$

Using Eq. (87) and the definition for $\boldsymbol{Q}_4$, we find an expression for the rotation matrix from the body frame of the first link to the body frame of the second link

$$\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6 = \begin{bmatrix} -\cos q_4 \sin q_5 \cos q_6 + \sin q_4 \sin q_6 & \cos q_4 \sin q_5 \sin q_6 + \sin q_4 \cos q_6 & \cos q_4 \cos q_5 \\ -\sin q_4 \sin q_5 \cos q_6 - \cos q_4 \sin q_6 & \sin q_4 \sin q_5 \sin q_6 - \cos q_4 \cos q_6 & \sin q_4 \cos q_5 \\ \cos q_5 \cos q_6 & -\cos q_5 \sin q_6 & \sin q_5 \end{bmatrix} \tag{88}$$

Using Eq. (85) and Eq. (88) compute the rotation matrix $\boldsymbol{Q}_2\boldsymbol{Q}_3\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6$

$$\boldsymbol{R} = \boldsymbol{Q}_2\boldsymbol{Q}_3\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6 = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \tag{89}$$

where

$$R_{11} = \sin q_2 \cos(q_3 + q_4) \sin q_5 \cos q_6 - \sin q_2 \sin(q_3 + q_4) \sin q_6 - \cos q_2 \cos q_5 \cos q_6$$

$$R_{12} = -\sin q_2 \cos(q_3 + q_4) \sin q_5 \sin q_6 - \sin q_2 \sin(q_3 + q_4) \cos q_6 + \cos q_2 \cos q_5 \sin q_6$$

$$R_{13} = -\sin q_2 \cos(q_3 + q_4) \cos q_5 - \cos q_2 \sin q_5$$

$$R_{21} = \cos q_2 \sin(q_3 + q_4) \sin q_6 - \cos q_2 \cos(q_3 + q_4) \sin q_5 \cos q_6 - \sin q_2 \cos q_5 \cos q_6$$

$$R_{22} = \cos q_2 \cos(q_3 + q_4) \sin q_5 \sin q_6 + \cos q_2 \sin(q_3 + q_4) \cos q_6 + \sin q_2 \cos q_5 \sin q_6$$

$$R_{23} = \cos q_2 \cos(q_3 + q_4) \cos q_5 - \sin q_2 \sin q_5$$

$$R_{31} = \sin(q_3 + q_4) \sin q_5 \cos q_6 + \cos(q_3 + q_4) \sin q_6$$

$$R_{32} = \cos(q_3 + q_4) \cos q_6 - \sin(q_3 + q_4) \sin q_5 \sin q_6$$

$$R_{33} = -\sin(q_3 + q_4) \cos q_5$$

Using the definition for $\boldsymbol{Q}_3$ and Eq. (88) compute the rotation matrix $\boldsymbol{Q}_3\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6$

$$\boldsymbol{S} = \boldsymbol{Q}_3\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6 = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \tag{90}$$

where

$$S_{11} = \cos(q_3 + q_4) \sin q_5 \cos q_6 - \sin(q_3 + q_4) \sin q_6$$

$$S_{12} = -\sin(q_3 + q_4) \cos q_6 - \cos(q_3 + q_4) \sin q_5 \sin q_6$$

$$S_{13} = -\cos(q_3 + q_4) \cos q_5$$

$$S_{21} = \cos(q_3 + q_4) \sin q_6 + \sin(q_3 + q_4) \sin q_5 \cos q_6$$

$$S_{22} = \cos(q_3 + q_4) \cos q_6 - \sin(q_3 + q_4) \sin q_5 \sin q_6$$

$$S_{23} = -\sin(q_3 + q_4) \cos q_5$$

$$S_{31} = \cos q_5 \cos q_6$$

$$S_{32} = -\cos q_5 \sin q_6$$

$$S_{33} = \sin q_5$$

Using Eq. (86) and Eq. (88), we find the rotation matrix from the inertial frame to the body frame of the second link

$$\boldsymbol{Q} = \boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6 = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \tag{91}$$

where

$$\begin{aligned}
Q_{11} &= (\cos q_1 \sin q_2 \cos q_3 + \sin q_1 \sin q_3)(-\cos q_4 \sin q_5 \cos q_6 + \sin q_4 \sin q_6) \\
&\quad + (-\cos q_1 \sin q_2 \sin q_3 + \sin q_1 \cos q_3)(-\sin q_4 \sin q_5 \cos q_6 - \cos q_4 \sin q_6) + \cos q_1 \cos q_2 \cos q_5 \cos q_6 \\
Q_{12} &= (\cos q_1 \sin q_2 \cos q_3 + \sin q_1 \sin q_3)(\cos q_4 \sin q_5 \sin q_6 + \sin q_4 \cos q_6) \\
&\quad + (-\cos q_1 \sin q_2 \sin q_3 + \sin q_1 \cos q_3)(\sin q_4 \sin q_5 \sin q_6 - \cos q_4 \cos q_6) - \cos q_1 \cos q_2 \cos q_5 \sin q_6 \\
Q_{13} &= (\cos q_1 \sin q_2 \cos q_3 + \sin q_1 \sin q_3) \cos q_4 \cos q_5 + (-\cos q_1 \sin q_2 \sin q_3 + \sin q_1 \cos q_3) \sin q_4 \cos q_5 \\
&\quad + \cos q_1 \cos q_2 \sin q_5 \\
Q_{21} &= (\sin q_1 \sin q_2 \cos q_3 - \cos q_1 \sin q_3)(-\cos q_4 \sin q_5 \cos q_6 + \sin q_4 \sin q_6) \\
&\quad + (-\sin q_1 \sin q_2 \sin q_3 - \cos q_1 \cos q_3)(-\sin q_4 \sin q_5 \cos q_6 - \cos q_4 \sin q_6) + \sin q_1 \cos q_2 \cos q_5 \cos q_6 \\
Q_{22} &= (\sin q_1 \sin q_2 \cos q_3 - \cos q_1 \sin q_3)(\cos q_4 \sin q_5 \sin q_6 + \sin q_4 \cos q_6) \\
&\quad + (-\sin q_1 \sin q_2 \sin q_3 - \cos q_1 \cos q_3)(\sin q_4 \sin q_5 \sin q_6 - \cos q_4 \cos q_6) - \sin q_1 \cos q_2 \cos q_5 \sin q_6 \\
Q_{23} &= (\sin q_1 \sin q_2 \cos q_3 - \cos q_1 \sin q_3) \cos q_4 \cos q_5 + (-\sin q_1 \sin q_2 \sin q_3 - \cos q_1 \cos q_3) \sin q_4 \cos q_5 \\
&\quad + \sin q_1 \cos q_2 \sin q_5 \\
Q_{31} &= \cos q_2 \sin(q_3 + q_4) \sin q_6 - \Big(\sin q_2 \cos q_5 + \cos q_2 \cos(q_3 + q_4) \sin q_5 \Big) \cos q_6 \\
Q_{32} &= \cos q_2 \sin(q_3 + q_4) \cos q_6 + \Big(\sin q_2 \cos q_5 + \cos q_2 \cos(q_3 + q_4) \sin q_5 \Big) \sin q_6 \\
Q_{33} &= -\sin q_2 \sin q_5 + \cos q_2 \cos(q_3 + q_4) \cos q_5
\end{aligned}$$

Appendix B Potential Energy of the DSP

This Appendix has the objective to find an expression for the potential energy of the DSP in terms of the generalized coordinates. The potential energy of the DSP is given by

$$P(\boldsymbol{q}) = -m_1\, \boldsymbol{g}^T \boldsymbol{c}_1 - m_2\, \boldsymbol{g}^T \boldsymbol{c}_2 \tag{92}$$

where

$$\boldsymbol{g} = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T \quad ; \quad g = 9.81$$

and, $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ represent the position of the center of mass (CoM) of the lower and upper link, respectively. Note that both CoM are expressed in the inertial frame.

The expression for $\boldsymbol{c}_1$ in the inertial frame is given by

$$[\boldsymbol{c}_1]_0 = \boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3\, \boldsymbol{\rho}_1 \tag{93}$$

where

$$[\boldsymbol{\rho}_1]_1 = \frac{l_1}{2} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T = \overline{O_1G_1} \qquad l_1 \equiv \text{lower link length}$$

Note that this vector is expressed in the body frame of the first link, F_1.

Substituting Eq. (86) in Eq. (93) yields

$$[\boldsymbol{c}_1]_0 = \frac{l_1}{2} \begin{bmatrix} \cos q_1 \sin q_2 \cos q_3 + \sin q_1 \sin q_3 \\ \sin q_1 \sin q_2 \cos q_3 - \cos q_1 \sin q_3 \\ \cos q_2 \cos q_3 \end{bmatrix} \tag{94}$$

On the other hand, $\boldsymbol{c}_2$ expressed in the inertial frame is given by

$$[\boldsymbol{c}_2]_0 = \boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3\,\boldsymbol{a}_1 + \boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6\,\boldsymbol{\rho}_2 \tag{95}$$

where

$$\begin{aligned}
[\boldsymbol{a}_1]_1 &= l_1 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T = \overline{O_1O_2} \quad ; \quad l_1 \equiv \text{lower link length} \\
[\boldsymbol{\rho}_2]_2 &= \frac{l_2}{2} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T = \overline{O_2G_2} \quad ; \quad l_2 \equiv \text{upper link length}
\end{aligned}$$

Note that both vectors are expressed in the corresponding body frame.

Substituting Eq. (86) and Eq. (88) in Eq. (95) yields

$$\boldsymbol{c}_2 = l_1 \begin{bmatrix} \cos q_1 \sin q_2 \cos q_3 + \sin q_1 \sin q_3 \\ \sin q_1 \sin q_2 \cos q_3 - \cos q_1 \sin q_3 \\ \cos q_2 \cos q_3 \end{bmatrix} + \frac{l_2}{2} \begin{bmatrix} Q_{13} \\ Q_{23} \\ Q_{33} \end{bmatrix} \tag{96}$$

Thus, substituting Eq. (94) and Eq. (96) in Eq. (92), the potential energy expression of the DSP becomes

$$P(\boldsymbol{q}) = \beta_1\,\cos q_2 \cos q_3 + \beta_2 \Big(\cos q_2 \cos(q_3 + q_4) \cos q_5 - \sin q_2 \sin q_5 \Big) \tag{97}$$

where

$$\begin{aligned}
\beta_1 &= \Big(m_1\,\frac{l_1}{2} + m_2\,l_1 \Big)\,g \\
\beta_2 &= m_2\,\frac{l_2}{2}\,g
\end{aligned}$$

Note that q_1 and q_6 don't appear in the final potential energy expression. Meaning *the potential energy doesn't depend on these variables.*

B.1 Partial Derivative of the Potential Energy

Compute the partial derivative of the potential energy of the DSP

$$\begin{aligned}
\frac{\partial P}{\partial q_1} &= 0 \\
\frac{\partial P}{\partial q_2} &= -\beta_1\,\sin q_2 \cos q_3 - \beta_2 \Big(\sin q_2 \cos(q_3 + q_4) \cos q_5 + \cos q_2 \sin q_5 \Big) \\
\frac{\partial P}{\partial q_3} &= -\beta_1\,\cos q_2 \sin q_3 - \beta_2 \Big(\cos q_2 \sin(q_3 + q_4) \cos q_5 \Big) \\
\frac{\partial P}{\partial q_4} &= -\beta_2\,\cos q_2 \sin(q_3 + q_4) \cos q_5 \\
\frac{\partial P}{\partial q_5} &= -\beta_2 \Big(\cos q_2 \cos(q_3 + q_4) \sin q_5 + \sin q_2 \cos q_5 \Big) \\
\frac{\partial P}{\partial q_6} &= 0
\end{aligned} \tag{98}$$

Appendix C Gravity Term Analysis

In this Appendix we find an expression, in term of the generalized coordinates, for the gravity term $\boldsymbol{\gamma} \in \mathbb{R}^6$ of the equation of motion. To this end, we define the gravity term [4] as

$$\boldsymbol{\gamma} = \boldsymbol{T}^T \boldsymbol{w}_G \tag{99}$$

where $\boldsymbol{T} \in \mathbb{R}^{12\times 6}$ represents a partial Jacobian and $\boldsymbol{w}_G \in \mathbb{R}^{12}$ represents the gravitational wrench of the DSP. They are defined as follows

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{t}_{11} & \boldsymbol{t}_{12} & \boldsymbol{t}_{13} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{t}_{21} & \boldsymbol{t}_{22} & \boldsymbol{t}_{23} & \boldsymbol{t}_{24} & \boldsymbol{t}_{25} & \boldsymbol{t}_{26} \end{bmatrix} \tag{100}$$

$$\boldsymbol{w}_G = \begin{bmatrix} \boldsymbol{w}_{G_1} & \boldsymbol{w}_{G_2} \end{bmatrix}^T \tag{101}$$

where

$$\boldsymbol{w}_{G_i} = \begin{bmatrix} \mathbf{0} \\ m_i\, \boldsymbol{g} \end{bmatrix}_i \quad ; \quad [\boldsymbol{g}]_0 = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T \quad ; \quad g = 9.81$$

$\boldsymbol{w}_{G_1}$ and $\boldsymbol{w}_{G_2}$ denotes the gravity wrench of the lower and upper link respectively. Note that the gravity vector, $\boldsymbol{g}$, should be expressed in the corresponding body frame when substitute inside each body wrench expression. Therefore, first we find the expressions for the gravity vector expressed in each of the body frames, F_1 and F_2.

$$[\boldsymbol{g}]_1 = \left(\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3\right)^T \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \tag{102}$$

Substituting Eq. (86) yields

$$[\boldsymbol{g}]_1 = -g \begin{bmatrix} \cos q_2 \cos q_3 \\ -\cos q_2 \sin q_3 \\ -\sin q_2 \end{bmatrix} \tag{103}$$

Similarly, the gravity vector expressed in the second link body frame is given by

$$[\boldsymbol{g}]_2 = (\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6)^T \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \tag{104}$$

Substituting Eq. (91) yields

$$[\boldsymbol{g}]_2 = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{12} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} = -g \begin{bmatrix} Q_{31} \\ Q_{32} \\ Q_{33} \end{bmatrix} \tag{105}$$

where

$$\begin{aligned}
Q_{31} &= \cos q_2 \sin(q_3+q_4) \sin q_6 - \Big(\sin q_2 \cos q_5 + \cos q_2 \cos(q_3+q_4) \sin q_5\Big) \cos q_6 \\
Q_{32} &= \cos q_2 \sin(q_3+q_4) \cos q_6 + \Big(\sin q_2 \cos q_5 + \cos q_2 \cos(q_3+q_4) \sin q_5\Big) \sin q_6 \\
Q_{33} &= -\sin q_2 \sin q_5 + \cos q_2 \cos(q_3+q_4) \cos q_5
\end{aligned}$$

Then, Eq. (99) written in matrix form becomes

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{bmatrix} = \begin{bmatrix} \boldsymbol{t}_{11}{}^T & \boldsymbol{t}_{21}{}^T \\ \boldsymbol{t}_{12}{}^T & \boldsymbol{t}_{22}{}^T \\ \boldsymbol{t}_{13}{}^T & \boldsymbol{t}_{23}{}^T \\ \mathbf{0}^T & \boldsymbol{t}_{24}{}^T \\ \mathbf{0}^T & \boldsymbol{t}_{25}{}^T \\ \mathbf{0}^T & \boldsymbol{t}_{26}{}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_{G_1} \\ \boldsymbol{w}_{G_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{t}_{11}{}^T\boldsymbol{w}_{G_1} + \boldsymbol{t}_{21}{}^T\boldsymbol{w}_{G_2} \\ \boldsymbol{t}_{12}{}^T\boldsymbol{w}_{G_1} + \boldsymbol{t}_{22}{}^T\boldsymbol{w}_{G_2} \\ \boldsymbol{t}_{13}{}^T\boldsymbol{w}_{G_1} + \boldsymbol{t}_{23}{}^T\boldsymbol{w}_{G_2} \\ \boldsymbol{t}_{24}{}^T\boldsymbol{w}_{G_2} \\ \boldsymbol{t}_{25}{}^T\boldsymbol{w}_{G_2} \\ \boldsymbol{t}_{26}{}^T\boldsymbol{w}_{G_2} \end{bmatrix} \tag{106}$$

Thus, we conclude

$$\boldsymbol{\gamma}_F = \begin{bmatrix} \boldsymbol{t}_{11}{}^T\boldsymbol{w}_{G_1} + \boldsymbol{t}_{21}{}^T\boldsymbol{w}_{G_2} \\ \boldsymbol{t}_{12}{}^T\boldsymbol{w}_{G_1} + \boldsymbol{t}_{22}{}^T\boldsymbol{w}_{G_2} \\ \boldsymbol{t}_{13}{}^T\boldsymbol{w}_{G_1} + \boldsymbol{t}_{23}{}^T\boldsymbol{w}_{G_2} \end{bmatrix} \quad ; \quad \boldsymbol{\gamma}_H = \begin{bmatrix} \boldsymbol{t}_{24}{}^T\boldsymbol{w}_{G_2} \\ \boldsymbol{t}_{25}{}^T\boldsymbol{w}_{G_2} \\ \boldsymbol{t}_{26}{}^T\boldsymbol{w}_{G_2} \end{bmatrix} \tag{107}$$

The definition for each element, $\boldsymbol{t}_{ij} \in \mathbb{R}^6$ in the partial Jacobian, $\boldsymbol{T}$ is given by

$$\boldsymbol{t}_{ij} = \begin{bmatrix} \boldsymbol{e}_j \\ \boldsymbol{e}_j \times \boldsymbol{r}_{ik} \end{bmatrix} \tag{108}$$

where

$$\boldsymbol{r}_{ik} = \overline{O_k G_i} \tag{109}$$

Note that all the elements of $\boldsymbol{t}_{ij}$ are expressed in the *ith* coordinate frame.

C.1 Analysis of $\boldsymbol{\gamma}_F$

Define the elements of the partial Jacobian $\boldsymbol{T}$ corresponding to $\boldsymbol{\gamma}_F$

$$\begin{aligned}
\boldsymbol{t}_{11} &= \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_1 \times \boldsymbol{r}_{11} \end{bmatrix}_1 \quad ; \quad \boldsymbol{t}_{12} = \begin{bmatrix} \boldsymbol{e}_2 \\ \boldsymbol{e}_2 \times \boldsymbol{r}_{11} \end{bmatrix}_1 \quad ; \quad \boldsymbol{t}_{13} = \begin{bmatrix} \boldsymbol{e}_3 \\ \boldsymbol{e}_3 \times \boldsymbol{r}_{11} \end{bmatrix}_1 \\
\boldsymbol{t}_{21} &= \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_1 \times \boldsymbol{r}_{21} \end{bmatrix}_2 \quad ; \quad \boldsymbol{t}_{22} = \begin{bmatrix} \boldsymbol{e}_2 \\ \boldsymbol{e}_2 \times \boldsymbol{r}_{21} \end{bmatrix}_2 \quad ; \quad \boldsymbol{t}_{23} = \begin{bmatrix} \boldsymbol{e}_3 \\ \boldsymbol{e}_3 \times \boldsymbol{r}_{21} \end{bmatrix}_2
\end{aligned} \tag{110}$$

Let find an expression for each vector $\boldsymbol{e}_j$ expressed in the corresponding coordinate frame.

$$\begin{aligned}
&[\boldsymbol{e}_1]_1 = (\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3)^T\boldsymbol{e} \quad ; \quad [\boldsymbol{e}_2]_1 = (\boldsymbol{Q}_2\boldsymbol{Q}_3)^T\boldsymbol{e} \quad ; \quad [\boldsymbol{e}_3]_1 = \boldsymbol{Q}_3{}^T \\
&[\boldsymbol{e}_1]_2 = (\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6)^T\boldsymbol{e} \quad ; \quad [\boldsymbol{e}_2]_2 = (\boldsymbol{Q}_2\boldsymbol{Q}_3\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6)^T\boldsymbol{e} \quad ; \quad [\boldsymbol{e}_3]_2 = (\boldsymbol{Q}_3\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6)^T\boldsymbol{e}
\end{aligned} \tag{111}$$

where

$$\boldsymbol{e} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

Substituting Eq. (86), Eq. (85) and the definition of $\boldsymbol{Q}_3$ in Eq. (111) yields

$$[\boldsymbol{e}_1]_1 = \begin{bmatrix} \cos q_2 \cos q_3 \\ -\cos q_2 \sin q_3 \\ -\sin q_2 \end{bmatrix} \quad ; \quad [\boldsymbol{e}_2]_1 = \begin{bmatrix} -\sin q_3 \\ -\cos q_3 \\ 0 \end{bmatrix} \quad ; \quad [\boldsymbol{e}_3]_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{112}$$

Substituting Eq. (91), Eq. (89) and Eq. (90) in Eq. (111) yields

$$[\boldsymbol{e}_1]_2 = \begin{bmatrix} Q_{31} \\ Q_{32} \\ Q_{33} \end{bmatrix} \quad ; \quad [\boldsymbol{e}_2]_2 = \begin{bmatrix} R_{31} \\ R_{32} \\ R_{33} \end{bmatrix} \quad ; \quad [\boldsymbol{e}_3]_2 = \begin{bmatrix} \cos q_5 \cos q_6 \\ -\cos q_5 \sin q_6 \\ \sin q_5 \end{bmatrix} \tag{113}$$

where

$$\begin{aligned}
Q_{31} &= \cos q_2 \sin(q_3+q_4)\sin q_6 - \Big(\sin q_2 \cos q_5 + \cos q_2 \cos(q_3+q_4)\sin q_5\Big)\cos q_6 \\
Q_{32} &= \cos q_2 \sin(q_3+q_4)\cos q_6 + \Big(\sin q_2 \cos q_5 + \cos q_2 \cos(q_3+q_4)\sin q_5\Big)\sin q_6 \\
Q_{33} &= -\sin q_2 \sin q_5 + \cos q_2 \cos(q_3+q_4)\cos q_5
\end{aligned}$$

$$\begin{aligned}
R_{31} &= \sin(q_3+q_4)\sin q_5 \cos q_6 + \cos(q_3+q_4)\sin q_6 \\
R_{32} &= \cos(q_3+q_4)\cos q_6 - \sin(q_3+q_4)\sin q_5 \sin q_6 \\
R_{33} &= -\sin(q_3+q_4)\cos q_5
\end{aligned}$$

Define the vectors $\boldsymbol{r}_{ik}$

$$[\boldsymbol{r}_{11}]_1 = \boldsymbol{\rho}_1 \quad ; \quad [\boldsymbol{r}_{21}]_2 = [\boldsymbol{a}_1]_2 + \boldsymbol{\rho}_2 \tag{114}$$

where

$$[\boldsymbol{\rho}_1]_1 = \begin{bmatrix} \frac{l_1}{2} & 0 & 0 \end{bmatrix}^T \quad ; \quad [\boldsymbol{\rho}_2]_2 = \begin{bmatrix} 0 & 0 & \frac{l_2}{2} \end{bmatrix}^T \quad ; \quad [\boldsymbol{a}_1]_1 = \begin{bmatrix} l_1 & 0 & 0 \end{bmatrix}^T$$

Note that $\boldsymbol{\rho}_2$ and $\boldsymbol{a}_1$ should be expressed in F_2. Therefore, we find an expression for $[\boldsymbol{a}_1]_2$

$$[\boldsymbol{a}_1]_2 = (\boldsymbol{Q}_4\boldsymbol{Q}_5\boldsymbol{Q}_6)^T[\boldsymbol{a}_1]_1 \tag{115}$$

Substituting Eq. (88) yields

$$[\boldsymbol{a}_1]_2 = l_1 \begin{bmatrix} -\cos q_4 \sin q_5 \cos q_6 + \sin q_4 \sin q_6 \\ \cos q_4 \sin q_5 \sin q_6 + \sin q_4 \cos q_6 \\ \cos q_4 \cos q_5 \end{bmatrix} \tag{116}$$

By straightforward computation, the elements of $\boldsymbol{\gamma}_F$ become

$$\begin{aligned}\gamma_1 &= (\boldsymbol{e}_1 \times \boldsymbol{r}_{11})^T\, m_1\, [\boldsymbol{g}]_1 + (\boldsymbol{e}_1 \times \boldsymbol{r}_{21})^T\, m_2\, [\boldsymbol{g}]_2 \\ &= 0 \end{aligned} \tag{117}$$

$$\begin{aligned}\gamma_2 &= (\boldsymbol{e}_2 \times \boldsymbol{r}_{12})^T\, m_1\, [\boldsymbol{g}]_1 + (\boldsymbol{e}_2 \times \boldsymbol{r}_{22})^T\, m_2\, [\boldsymbol{g}]_2 \\ &= \beta_1\, \sin q_2 \cos q_3 + \beta_2\, \Big(\sin q_2 \cos(q_3 + q_4) \cos q_5 + \cos q_2 \sin q_5 \Big) \end{aligned} \tag{118}$$

$$\begin{aligned}\gamma_3 &= (\boldsymbol{e}_3 \times \boldsymbol{r}_{13})^T\, m_1\, [\boldsymbol{g}]_1 + (\boldsymbol{e}_3 \times \boldsymbol{r}_{23})^T\, m_2\, [\boldsymbol{g}]_2 \\ &= \beta_1\, \cos q_2 \sin q_3 + \beta_2\, \cos q_2 \sin(q_3 + q_4) \cos q_5 \end{aligned} \tag{119}$$

C.2 Analysis of $\boldsymbol{\gamma}_H$

Define the elements of the partial Jacobian $\boldsymbol{T}$ corresponding to $\boldsymbol{\gamma}_H$

$$\boldsymbol{t}_{24} = \begin{bmatrix} \boldsymbol{e}_4 \\ \boldsymbol{e}_4 \times \boldsymbol{r}_{22} \end{bmatrix}_2 \quad ; \quad \boldsymbol{t}_{25} = \begin{bmatrix} \boldsymbol{e}_5 \\ \boldsymbol{e}_5 \times \boldsymbol{r}_{22} \end{bmatrix}_2 \quad ; \quad \boldsymbol{t}_{26} = \begin{bmatrix} \boldsymbol{e}_6 \\ \boldsymbol{e}_6 \times \boldsymbol{r}_{22} \end{bmatrix}_2 \tag{120}$$

Similarly to the procedure for $\boldsymbol{\gamma}_F$, find an expression for each vector $\boldsymbol{e}_j$ expressed in the upper link body frame.

$$[\boldsymbol{e}_4]_2 = (\boldsymbol{Q}_4 \boldsymbol{Q}_5 \boldsymbol{Q}_6)^T \boldsymbol{e} \quad ; \quad [\boldsymbol{e}_5]_2 = (\boldsymbol{Q}_5 \boldsymbol{Q}_6)^T \boldsymbol{e} \quad ; \quad [\boldsymbol{e}_6]_2 = \boldsymbol{Q}_6{}^T \boldsymbol{e} \tag{121}$$

where

$$\boldsymbol{e} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

Substituting Eq. (88), Eq. (87) and the definition of $\boldsymbol{Q}_6$ yields

$$[\boldsymbol{e}_4]_2 = \begin{bmatrix} \cos q_5 \cos q_6 \\ -\cos q_5 \sin q_6 \\ \sin q_5 \end{bmatrix} \quad ; \quad [\boldsymbol{e}_5]_2 = \begin{bmatrix} \sin q_6 \\ \cos q_6 \\ 0 \end{bmatrix} \quad ; \quad [\boldsymbol{e}_6]_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{122}$$

Define the vectors $\boldsymbol{r}_{ik}$ as

$$\boldsymbol{r}_{22} = \boldsymbol{\rho}_2 \tag{123}$$

where

$$\boldsymbol{\rho}_2 = \frac{l_2}{2} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

Thus, $\boldsymbol{\gamma}_H$ elements are given by

$$\begin{aligned}\gamma_4 &= (\boldsymbol{e}_4 \times \boldsymbol{r}_{24})^T m_2\, [\boldsymbol{g}]_2 \\ &= \beta_2\, \cos q_2 \sin(q_3 + q_4) \cos q_5 \end{aligned} \tag{124}$$

$$\begin{aligned}\gamma_5 &= (\boldsymbol{e}_5 \times \boldsymbol{r}_{25})^T\, m_2\, [\boldsymbol{g}]_2 \\ &= \beta_2\, \Big(\cos q_2 \cos(q_3 + q_4) \sin q_5 + \sin q_2 \cos q_5 \Big) \end{aligned} \tag{125}$$

$$\begin{aligned}\gamma_6 &= (\boldsymbol{e}_6 \times \boldsymbol{r}_{26})^T m_2\, [\boldsymbol{g}]_2 \\ &= 0 \end{aligned} \tag{126}$$

C.3 Summary

Expressions for the gravity term vector, $\boldsymbol{\gamma}$, of the equation of motion

$$\begin{aligned}\gamma_1 &= 0 \\ \gamma_2 &= \beta_1\, \sin q_2 \cos q_3 + \beta_2\, \Big(\sin q_2 \cos(q_3 + q_4) \cos q_5 + \cos q_2 \sin q_5 \Big) \\ \gamma_3 &= \beta_1\, \cos q_2 \sin q_3 + \beta_2\, \cos q_2 \sin(q_3 + q_4) \cos q_5 \\ \gamma_4 &= \beta_2\, \cos q_2 \sin(q_3 + q_4) \cos q_5 \\ \gamma_5 &= \beta_2\, \Big(\cos q_2 \cos(q_3 + q_4) \sin q_5 + \sin q_2 \cos q_5 \Big) \\ \gamma_6 &= 0 \end{aligned}$$

These results along with results from Appendix B show that

$$\gamma = -\frac{\partial P}{\partial \boldsymbol{q}} \tag{127}$$

Note that the variables q_1 and q_6 don't appear in $\boldsymbol{\gamma}$, therefore we conclude *the gravity term of the equation of motion of the spatial pendulum doesn't depend on them.*

Appendix D Structural property of the DSP

In this Appendix we prove

$$\dot{\boldsymbol{I}} - 2\,\boldsymbol{C} \equiv \text{ skew symmetric matrix} \tag{128}$$

where $\dot{\boldsymbol{I}}$ denotes the derivative of the generalized inertia matrix and $\boldsymbol{C}$ denotes the Coriolis coefficient matrix. From [4], we can define the Coriolis coefficient matrix as

$$\boldsymbol{C} = \boldsymbol{T}^T \boldsymbol{M} \dot{\boldsymbol{T}} + \boldsymbol{T}^T \boldsymbol{W} \boldsymbol{M} \boldsymbol{T} \tag{129}$$

and the generalized inertia matrix as

$$\boldsymbol{I} = \boldsymbol{T}^T \boldsymbol{M} \boldsymbol{T} \tag{130}$$

which implies the derivative of the generalized inertia matrix is given by

$$\dot{\boldsymbol{I}} = \boldsymbol{T}^T \boldsymbol{M} \dot{\boldsymbol{T}} + \dot{\boldsymbol{T}}^T \boldsymbol{M} \boldsymbol{T} + \boldsymbol{T}^T \boldsymbol{W} \boldsymbol{M} \boldsymbol{T} - \boldsymbol{T}^T \boldsymbol{M} \boldsymbol{W} \boldsymbol{T} \tag{131}$$

where $\boldsymbol{T}$ represents a partial Jacobian, $\dot{\boldsymbol{T}}$ represents the rate of change of the partial Jacobian, $\boldsymbol{M}$ represents the manipulator mass and $\boldsymbol{W}$ represents the manipulator angular velocity, each of them is given by

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{O}_{6\times 6} \\ \boldsymbol{O}_{6\times 6} & \boldsymbol{M}_2 \end{bmatrix} \quad \text{where} \quad \boldsymbol{M}_i = \begin{bmatrix} \boldsymbol{I}_i & \boldsymbol{O}_{3\times 3} \\ \boldsymbol{O}_{3\times 3} & m_i\,\boldsymbol{1}_{3\times 3} \end{bmatrix} \tag{132}$$

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_1 & \boldsymbol{O}_{6\times 6} \\ \boldsymbol{O}_{6\times 6} & \boldsymbol{W}_2 \end{bmatrix} \quad \text{where} \quad \boldsymbol{W}_i = \begin{bmatrix} \boldsymbol{\Omega}_i & \boldsymbol{O}_{3\times 3} \\ \boldsymbol{O}_{3\times 3} & \boldsymbol{O}_{3\times 3} \end{bmatrix} \tag{133}$$

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{t}_{11} & \boldsymbol{t}_{12} & \boldsymbol{t}_{13} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{t}_{21} & \boldsymbol{t}_{22} & \boldsymbol{t}_{23} & \boldsymbol{t}_{24} & \boldsymbol{t}_{25} & \boldsymbol{t}_{26} \end{bmatrix} \tag{134}$$

$$\dot{\boldsymbol{T}} = \begin{bmatrix} \dot{\boldsymbol{t}}_{11} & \dot{\boldsymbol{t}}_{12} & \dot{\boldsymbol{t}}_{13} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \dot{\boldsymbol{t}}_{21} & \dot{\boldsymbol{t}}_{22} & \dot{\boldsymbol{t}}_{23} & \dot{\boldsymbol{t}}_{24} & \dot{\boldsymbol{t}}_{25} & \dot{\boldsymbol{t}}_{26} \end{bmatrix} \tag{135}$$

Note that $\boldsymbol{\Omega}_i$ is a skew symmetric matrix given by

$$\boldsymbol{\Omega}_i = CPM(\boldsymbol{\omega}_i) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad \text{given that} \quad \boldsymbol{\omega}_i = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

The partial Jacobian, $\boldsymbol{T}$ was defined in Appendix C.

On the other hand, the definition for each element in the rate of change of the partial Jacobian, $\dot{\boldsymbol{T}}$, is given by

$$\dot{\boldsymbol{t}}_{ij} = \begin{bmatrix} \boldsymbol{\omega}_j \times \boldsymbol{e}_j \\ (\boldsymbol{\omega}_j \times \boldsymbol{e}_j) \times \boldsymbol{r}_{ik} + \boldsymbol{e}_j \times \dot{\boldsymbol{r}}_{ik} \end{bmatrix} \tag{136}$$

where

$$\dot{\boldsymbol{r}}_{ik} = \boldsymbol{\omega}_k \times \boldsymbol{a}_k + \cdots + \boldsymbol{\omega}_{i-1} \times \boldsymbol{a}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{\rho}_i \tag{137}$$

Now, let

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{T}^T \boldsymbol{M} \dot{\boldsymbol{T}} \\ \boldsymbol{B} &= \boldsymbol{T}^T \boldsymbol{W} \boldsymbol{M} \boldsymbol{T} \\ \boldsymbol{D} &= \boldsymbol{T}^T \boldsymbol{M} \boldsymbol{W} \boldsymbol{T} \end{aligned}$$

such that

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$$
$$\dot{\boldsymbol{I}} = \boldsymbol{A} + \boldsymbol{A}^T + \boldsymbol{B} - \boldsymbol{D}$$

Thus we are interested in

$$\dot{\boldsymbol{I}} - 2\,\boldsymbol{C} = -\boldsymbol{A} + \boldsymbol{A}^T - \boldsymbol{B} - \boldsymbol{D} = \boldsymbol{S} \equiv \text{ skew symmetric matrix} \tag{138}$$

Using a Matlab script with symbolic variables, we compute these matrices $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{D}$. Then, we verify

$$s_{ii} = -b_{ii} - d_{ii} = 0 \quad \text{for} \quad i = 1, \cdots, 6$$

and

$$s_{ij} + s_{ji} = [-a_{ij} + a_{ji} - b_{ij} - d_{ij}] + [-a_{ji} + a_{ij} - b_{ji} - d_{ji}] = 0 \quad \text{for} \quad i, j = 1, \cdots 6$$

which implies

$$s_{ii} = 0 \quad \text{and} \quad s_{ij} = -s_{ji}$$

therefore matrix $\boldsymbol{S}$ is skew symmetric matrix and it has the property

$$\boldsymbol{z}^T \Big(\dot{\boldsymbol{I}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - 2\,\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \Big) \boldsymbol{z} = 0 \quad \forall \boldsymbol{z}$$

Appendix E Potential energy function

In this Appendix we show that under certain conditions the potential energy function, Eq. (6) can only depend on the generalized coordinates at the hip, $\boldsymbol{q}_H$. First, assume $\boldsymbol{\gamma}_F = \boldsymbol{0}$, meaning [Appendix C]

$$\gamma_2 = \beta_1\, \sin q_2 \cos q_3 + \beta_2 \Big(\sin q_2 \cos(q_3 + q_4) \cos q_5 + \cos q_2 \sin q_5 \Big) = 0$$
$$\gamma_3 = \cos q_2 \Big(\beta_1\, \sin q_3 + \beta_2\, \sin(q_3 + q_4) \cos q_5 \Big) = 0$$

which implies, in this case,

$$\gamma_3 = \beta_1\, \sin q_3 + \beta_2\, \sin(q_3 + q_4) \cos q_5 = 0$$

Applying these equations the square of the potential energy can be rewritten as

$$P^2 = P^2 + {\gamma_2}^2 + {\gamma_3}^2 \tag{139}$$

where

$$\begin{aligned} P^2 &= {\beta_1}^2 \cos^2 q_3 \cos^2 q_2 + 2\,\beta_1 \beta_2 \Big(-\sin q_2 \cos q_2 \cos q_3 \sin q_5 + \cos^2 q_2 \cos q_3 \cos(q_3 + q_4) \cos q_5 \Big) + {\beta_2}^2 \Big(\sin^2 q_2 \sin^2 q_5 \\ &\quad - 2\, \sin q_2 \cos q_2 \cos(q_3 + q_4) \sin q_5 \cos q_5 + \cos^2 q_2 \cos^2(q_3 + q_4) \cos^2 q_5 \Big) \end{aligned}$$

$$\begin{aligned} {\gamma_2}^2 &= {\beta_1}^2 \sin^2 q_2 \cos^2 q_3 + 2\,\beta_1 \beta_2 \Big(\sin^2 q_2 \cos q_3 \cos(q_3 + q_4) \cos q_5 + \sin q_2 \cos q_2 \cos q_3 \sin q_5 \Big) + {\beta_2}^2 \Big(\cos^2 q_2 \sin^2 q_5 \\ &\quad + 2\, \sin q_2 \cos q_2 \cos(q_3 + q_4) \sin q_5 \cos q_5 + \sin^2 q_2 \cos^2(q_3 + q_4) \cos^2 q_5 \Big) \end{aligned}$$

$${\gamma_3}^2 = {\beta_1}^2 \sin^2 q_3 + 2\,\beta_1 \beta_2 \sin q_3 \sin(q_3 + q_4) \cos q_5 + {\beta_2}^2 \sin^2(q_3 + q_4) \cos^2 q_5$$

By straightforward computation, we find

$$P = \pm \Phi(\boldsymbol{q}_H) \quad \text{when} \quad \boldsymbol{\gamma}_F = \boldsymbol{0} \tag{140}$$

where

$$\Phi(\boldsymbol{q}_H) = \sqrt{{\beta_1}^2 + {\beta_2}^2 + 2\,\beta_1 \beta_2 \cos q_4 \cos q_5} \tag{141}$$

Appendix F Proof of inequality (70)

Note that $\nu_1(q_4{}^*, q_5{}^*)$ is an even function, therefore we just need to consider the case $q_4{}^* \geq 0$. Considering $\sin q_4{}^* < 0$ for $q_4 \in ((2n-1)\pi, 2n\pi)$ for $n \in \mathbb{Z}^+$ and $\Phi(q_4{}^*, q_5{}^*) \leq E_{top}$, we can see

$$\sup_{q_4{}^*, q_5{}^*} \nu_1(q_4{}^*, q_5{}^*) = \sup_{n, q_5{}^*} \left\{ \sup_{q_4 \in ((2n-1)\pi, 2n\pi), q_5{}^*} \nu_1(q_4{}^*, q_5{}^*) \right\} = \sup_{q_4 \in (\pi, 2\pi), q_5{}^*} \nu_1(q_4{}^*, q_5{}^*)$$

where the last inequality holds because $\sup_{q_4 \in ((2n-1)\pi, 2n\pi), q_5{}^*} \nu_1(q_4{}^*, q_5{}^*)$ is a strictly decreasing function. Similarly, we can see

$$\sup_{q_5{}^*, q_4{}^*} \nu_2(q_4{}^*, q_5{}^*) = \sup_{n, q_4{}^*} \left\{ \sup_{q_5 \in ((2n-1)\pi, 2n\pi), q_4{}^*} \nu_2(q_4{}^*, q_5{}^*) \right\} = \sup_{q_5 \in (\pi, 2\pi), q_4{}^*} \nu_2(q_4{}^*, q_5{}^*)$$

Now, we analyze function $\nu_1(q_4{}^*, q_5{}^*)$. Let rewrite

$$\Phi^2(q_4{}^*, q_5{}^*) = {\beta_1}^2 + {\beta_2}^2 + 2\, \beta_1\, \beta_2 \cos q_4{}^* \cos q_5{}^*$$

as

$$\begin{aligned} \Phi^2(q_4{}^*, q_5{}^*) &= {\beta_1}^2 \sin^2 q_4{}^* (\cos^2 q_5{}^* + \sin^2 {q_5}^2) + {\beta_1}^2 \cos^2 q_4{}^* + {\beta_2}^2 + 2\, \beta_1\, \beta_2 \cos q_4{}^* \cos q_5{}^* \\ &= {\beta_1}^2 \sin^2 q_4{}^* \cos^2 q_5{}^* + {\beta_1}^2 \sin^2 q_4{}^* \sin^2 q_5{}^* + {\beta_1}^2 \cos^2 q_4{}^* + {\beta_2}^2 + 2\, \beta_1\, \beta_2 \cos q_4{}^* \cos q_5{}^* \end{aligned} \tag{142}$$

which implies

$$\Phi^2(q_4{}^*, q_5{}^*) \geq {\beta_1}^2 \sin^2 q_4{}^* \cos^2 q_5{}^* \quad \Rightarrow \quad |\Phi(q_4{}^*, q_5{}^*)| \geq |\beta_1\ \sin q_4{}^* \cos q_5{}^*| \quad \Rightarrow \quad \frac{1}{\beta_1} \geq \left| \frac{\sin q_4{}^* \cos q_5{}^*}{\Phi} \right| \tag{143}$$

Similarly, we can find

$$\frac{1}{\beta_2} \geq \left| \frac{\sin q_4{}^* \cos q_5{}^*}{\Phi} \right| \tag{144}$$

Thus, we conclude

$$\left| \frac{\sin q_4{}^* \cos q_5{}^*}{\Phi} \right| \leq \min\left(\frac{1}{\beta_1}, \frac{1}{\beta_2} \right) \tag{145}$$

which implies

$$\sup_{q_4 \in (\pi, 2\pi), q_5{}^*} \nu_1(q_4{}^*, q_5{}^*) = \sup_{q_4 \in (\pi, 2\pi), q_5{}^*} \frac{(\Phi - E_{top})}{q_4{}^*} \frac{\sin q_4{}^* \cos q_5{}^*}{\Phi} \leq \min\left(\frac{1}{\beta_1}, \frac{1}{\beta_2} \right) \sup_{q_4 \in (\pi, 2\pi), q_5{}^*} \frac{E_{top} - \Phi}{\pi}$$

Using $\Phi \geq |\beta_1 - \beta_2|$, we obtain

$$\sup_{q_4 \in (\pi, 2\pi), q_5{}^*} \nu_1(q_4{}^*, q_5{}^*) \leq \min\left(\frac{1}{\beta_1}, \frac{1}{\beta_2} \right) \frac{E_{top} - |\beta_1 - \beta_2|}{\pi}$$

If $\beta_1 \leq \beta_2$

$$\sup_{q_4 \in (\pi, 2\pi), q_5{}^*} \nu_1(q_4{}^*, q_5{}^*) \leq \min\left(\frac{1}{\beta_1}, \frac{1}{\beta_2} \right) \frac{E_{top} - |\beta_2 - \beta_1|}{\pi} = \frac{2}{\pi} \frac{\beta_1}{\beta_2}$$

If $\beta_2 > \beta_1$

$$\sup_{q_4 \in (\pi, 2\pi), q_5{}^*} \nu_1(q_4{}^*, q_5{}^*) \leq \min\left(\frac{1}{\beta_1}, \frac{1}{\beta_2} \right) \frac{E_{top} - |\beta_1 - \beta_2|}{\pi} \leq \frac{2}{\pi} \frac{\beta_2}{\beta_1}$$

which implies

$$\sup_{q_4 \in (\pi, 2\pi), q_5{}^*} \nu_1(q_4{}^*, q_5{}^*) \leq \frac{2}{\pi} \min\left(\frac{\beta_2}{\beta_1}, \frac{\beta_1}{\beta_2} \right) \tag{146}$$

Following a similar procedure to analyze $\nu_2(q_4{}^*, q_5{}^*)$ we find

$$\sup_{q_5 \in (\pi, 2\pi), q_4{}^*} \nu_2(q_4{}^*, q_5{}^*) \leq \frac{2}{\pi} \min\left(\frac{\beta_2}{\beta_1}, \frac{\beta_1}{\beta_2} \right) \tag{147}$$

Appendix G Unique solutions in Θ_-

In this Appendix we show the procedure to find the three equations that show there is a unique value for q_2 and q_3 given q_4 and q_5. To this end, for $q_4, q_5 \in \Theta_-$, we have

$$\begin{aligned} P(\boldsymbol{q}) &= \beta_1 \cos q_2{}^* \cos q_3{}^* + \beta_2 \cos q_2{}^*(\cos q_3{}^* \cos q_4{}^* - \sin q_3{}^* \sin q_4{}^*) \cos q_5{}^* - \beta_2 \sin q_2{}^* \sin q_5{}^* \\ \gamma_2{}^* &= \beta_1 \sin q_2{}^* \cos q_3{}^* + \beta_2 \sin q_2{}^*(\cos q_3{}^* \cos q_4{}^* - \sin q_3{}^* \sin q_4{}^*) \cos q_5{}^* + \beta_2 \cos q_2{}^* \sin q_5{}^* = 0 \\ \gamma_3{}^* &= \beta_1 \sin q_3{}^* + \beta_2 (\sin q_3{}^* \cos q_4{}^* + \cos q_3{}^* \sin q_4{}^*) \cos q_5{}^* = 0 \end{aligned}$$

Rewrite the three equations as follows

$$-\Phi(q_4{}^*, q_5{}^*) = (\beta_1 + \beta_2 \cos q_4{}^* \cos q_5{}^*) \cos q_2{}^* \cos q_3{}^* - \beta_2 \cos q_2{}^* \sin q_3{}^* \sin q_4{}^* \cos q_5{}^* - \beta_2 \sin q_2{}^* \sin q_5{}^* \quad (148)$$

$$0 = (\beta_1 + \beta_2 \cos q_4{}^* \cos q_5{}^*) \sin q_2{}^* \cos q_3{}^* - \beta_2 \sin q_2{}^* \sin q_3{}^* \sin q_4{}^* \cos q_5{}^* + \beta_2 \cos q_2{}^* \sin q_5{}^* \quad (149)$$

$$0 = (\beta_1 + \beta_2 \cos q_4{}^* \cos q_5{}^*) \sin q_3{}^* + \beta_2 \cos q_3{}^* \sin q_4{}^* \cos q_5{}^* \quad (150)$$

Multiplying both sides of Eq. (149) by $\dfrac{\cos q_2{}^*}{\sin q_2{}^*}$ gives

$$(\beta_1 + \beta_2 \cos q_4{}^* \cos q_5{}^*) \cos q_2{}^* \cos q_3{}^* = \beta_2 \left(\cos q_2{}^* \sin q_3{}^* \sin q_4{}^* \cos q_5{}^* - \frac{\cos^2 q_2{}^* \sin q_5{}^*}{\sin q_2{}^*} \right) \quad (151)$$

Substituting it in Eq. (148) yields

$$\sin q_2{}^* = \frac{\beta_2}{\Phi} \sin q_5{}^* \quad (152)$$

Applying Eq. (152) in (149) gives

$$\cos q_2{}^* = \frac{1}{\Phi} \Big(-(\beta_1 + \beta_2 \cos q_4{}^* \cos q_5{}^*) \cos q_3{}^* + \beta_2 \sin q_3{}^* \sin q_4{}^* \cos q_5{}^* \Big) \quad (153)$$

Multiplying both sides of Eq. (150) by $\dfrac{\cos q_3{}^*}{\sin q_3{}^*}$ gives

$$-(\beta_1 + \beta_2 \cos q_4{}^* \cos q_5{}^*) \cos q_3{}^* = \beta_2 \frac{\cos^2 q_3{}^*}{\sin q_3{}^*} \sin q_4{}^* \cos q_5{}^* \quad (154)$$

Substituting this result in Eq. (153) yields

$$\cos q_2{}^* \sin q_3{}^* = \frac{\beta_2}{\Phi} \sin q_4{}^* \cos q_5{}^* \quad (155)$$

Applying Eq. (155) in Eq. (150) gives

$$\cos q_2{}^* \cos q_3{}^* = -\frac{1}{\Phi} (\beta_1 + \beta_2 \cos q_4{}^* \cos q_5{}^*) \quad (156)$$

References

[1] I. Fantoni, R. Lozano and M. Spong, ”Energy based control of the Pendubot”, *IEEE Transactions on Automatic Control*, vol. 45, no. 4, pp. 725-729, 2000.

[2] O. Kolesnichenko and A. Shiriaev, ”Partial stabilization of underactuated EulerLagrange systems via a class of feedback transformations”, *Systems & Control Letters*, vol. 45, no. 2, pp. 121-132, 2002.

[3] X. Xin and M. Kaneda, ”Analysis of the energy-based swing-up control of the Acrobot”, *International Journal of Robust and Nonlinear Control*, vol. 17, no. 16, pp. 1503-1524, 2007.

[4] J. Angeles, *Fundamentals of robotic mechanical systems*, 4th ed. New York: Springer, 2004.