Section - 2

Examples:

1 Sol: Given,

$$Y_i \sim N(\mu_i, \sigma^2)$$

Probability density fn (PDF) of a normal distribution

with mean μ_i and variance σ^2 is

$$f(y_i; \mu_i, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - \mu_i)^2}{2\sigma^2}\right)$$

$$(y_i - \mu_i)^2 = y_i^2 - 2\mu_i y_i + \mu_i^2$$

$$-\frac{(y_i^2 - 2\mu_i y_i + \mu_i^2)}{2\sigma^2} = \frac{-y_i^2}{2\sigma^2} + \frac{2\mu_i y_i}{2\sigma^2} + \frac{\mu_i^2}{2\sigma^2}$$

$$= \frac{-y_i^2}{2\sigma^2} + \frac{\mu_i y_i}{\sigma^2} - \frac{\mu_i^2}{2\sigma^2}$$

General form of exp family distribution:

$$f(y_i; \theta_i) = a(\theta_i)\, b(y_i) \exp(K(y_i) Q(\theta_i))$$

Rewriting our expression:

$$f(y_i; \mu_i, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-y_i^2}{2\sigma^2} + \frac{\mu_i y_i}{\sigma^2} - \frac{\mu_i^2}{2\sigma^2}\right)$$

Comparing with exp family form:

Natural parameter : $Q(\theta_i) = \frac{\mu_i}{\sigma^2}$

$k(y_i) = y_i$

Since, the natural parameter of an exp family form is the quantity that appears as the coefficient of y_i, we can conclude that:

μ_i is the natural parameter.

2. Sol: Given,

$$Y_i \sim \text{Binom}(n_i, \pi_i)$$

For $Y_i \sim \text{Binom}(n_i, \pi_i)$ PMF is

$$P(y_i; n_i, \pi_i) = \binom{n_i}{y_i} \pi_i^{y_i} (1-\pi_i)^{n_i - y_i}$$

Taking natural logarithm:

$$\ln P(y_i; n_i, \pi_i) = \ln \binom{n_i}{y_i} + y_i \ln \pi_i + (n_i - y_i)\ln(1-\pi_i)$$

$$= \ln \binom{n_i}{y_i} + y_i \ln \pi_i + n_i \ln(1-\pi_i) - y_i \ln(1-\pi_i)$$

$$= \ln \binom{n_i}{y_i} + y_i \ln \frac{\pi_i}{1-\pi_i} + n_i \ln (1-\pi_i)$$

It can be expressed as:

$$P(y_i) = \binom{n_i}{y_i} \exp\left(y_i \ln\left(\frac{\pi_i}{1-\pi_i}\right) + n_i \ln(1-\pi_i)\right)$$

By Comparing with exp family form

$$\exp\left(k(y_i) Q(\theta_i)\right)$$

Natural Parameter $\theta_i = \ln\left(\frac{\pi_i}{1-\pi_i}\right)$

Example 3: $Y_i \sim \text{Poisson}(\mu_i)$

The probability Mass Fn is:

$$P(y_i ; \mu_i) = \frac{e^{-\mu_i} \mu_i^{y_i}}{y_i!}, \quad y_i = 0, 1, 2 \dots$$

taking natural log:

$$\ln p(y_i ; \mu_i) = -\mu_i + y_i \ln \mu_i - \ln y_i!$$

Comparing with exp family form:

$$\exp\left(K(y_i) Q(\theta_i)\right)$$

Natural parameter $Q(\theta_i) = \ln \mu_i$

Example 4: Given,

$$(Y_1, \dots Y_k) \sim \text{Multinomial}(N, (\pi_1, \pi_2 \dots \pi_k))$$

Probability Mass Function for a multinomial dist is

$$P(y_1, y_2 \dots y_k : N, \pi_1, \pi_2 \dots \pi_k)$$

$$= \frac{N!}{y_1!\, y_2! \dots y_k!} \pi_1^{y_1} \pi_2^{y_2} \dots \pi_k^{y_k}$$

where,

$$Y_1 + Y_2 + \dots + Y_k = N$$

Taking Natural logarithm:

$$\ln p(y_1, y_2 \dots y_k) = \ln\left(\frac{N!}{y_1!\, y_2! \dots y_k!}\right) + \sum_{i=1}^{K} y_i \ln \pi_i$$

Expanding the terms,

$$\ln p(y_1, y_2 \dots y_k) = \ln N! - \cancel{\ln} \sum_{i=1}^{K} \ln y_i! + \sum_{i=1}^{K} y_i \ln \pi_i$$

General form of an exponential family is.

$$P(y_1 \cdots y_k) = a(\theta)\, b(y) \exp\left(\sum_{j=1}^{m} k_j(y)\, Q_j(\theta)\right)$$

To compare;

first term $\ln N! - \sum_{i=1}^{k} \ln y_i!$ is independent

of π_i, so it belongs to base measure $b(y)$

Summation;

$$\sum_{i=1}^{k} y_i \ln \pi_i \rightarrow \text{contains parameters.}$$

Since,

$$\sum_{i=1}^{k} \pi_i = 1$$

$$\pi_k = 1 - \sum_{i=1}^{k-1} \pi_i$$

Now ~~that~~,

$$\ln \pi_i = \ln\left(\frac{\pi_i}{\pi_k}\right) + \ln \pi_k \quad \text{for } i = 1, 2 \cdots k-1$$

Now, substituting this in

$$\sum_{i=1}^{k} y_i \ln \pi_i = \sum_{i=1}^{k-1} y_i \left(\ln \frac{\pi_i}{\pi_k} + \ln \pi_k\right) + y_k \ln \pi_k.$$

Expanding,

$$= \sum_{i=1}^{k-1} y_i \ln \frac{\pi_i}{\pi_k} + \sum_{i=1}^{k-1} y_i \ln \pi_k + y_k \ln \pi_k$$

Since,

$\sum_{i=1}^{k} y_i = N$, we substitute $y_k = N - \sum_{i=1}^{k-1} y_i$

$$= \sum_{i=1}^{k-1} y_i \ln \frac{\pi_i}{\pi_k} + \left(N - \sum_{i=1}^{k-1} y_i\right) \ln \pi_k$$

$$= \sum_{i=1}^{k-1} y_i \ln \frac{\pi_i}{\pi_k} + N \ln \pi_k - \sum_{i=1}^{k-1} y_i \ln \pi_k$$

$$= \sum_{i=1}^{k-1} y_i \ln \frac{\pi_i}{\pi_k} + N \ln \pi_k - \ln \pi_k \sum_{i=1}^{k-1} y_i$$

Since, $\sum_{i=1}^{k-1} y_i + y_k = N$, we substitue $y_k = N - \sum_{i=1}^{k-1} y_i$

So;

$$= \sum_{i=1}^{k-1} y_i \ln \frac{\pi_i}{\pi_k} + (N-N) \ln \pi_k$$

$$= \sum_{i=1}^{k-1} y_i \ln \frac{\pi_i}{\pi_k}$$

$\therefore$ $\boxed{Q_i(\theta) = \ln \frac{\pi_i}{\pi_k}}$ $\rightarrow$ Required natural Parameters