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a) Given,

projection of vector y on x is z.

Using the properties of vector projection

$$z = \frac{y \cdot x}{x \cdot x} x$$

then,

$$|z|^2 = \left| \frac{y \cdot x}{x \cdot x} x \right|^2$$

$$= \frac{(y \cdot x)^2}{(x \cdot x)(x \cdot x)} |x|^2$$

$$= \frac{(y \cdot x)^2}{(x \cdot x)\,|x|^2} \cdot |x|^2 = \frac{(y \cdot x)^2}{(x \cdot x)}$$

$$|z|^2 = \left[ \frac{(y \cdot x)^2}{x \cdot x} \right]$$

Now let us compute

$$y^T z = y^T \left[ \frac{(y \cdot x)}{x \cdot x} \cdot x \right] \quad \left[ \because z = \frac{y \cdot x}{x \cdot x} x \right]$$

$$= \frac{y \cdot x}{x \cdot x} (y^T \cdot x)$$ [Also $y^T \cdot x = y \cdot x$ Because vector dot product is commutative]

$$= \frac{y \cdot x}{x \cdot x} (y \cdot x)$$

$$= \frac{(y \cdot x)^2}{x \cdot x}$$

Thus,

we got,

$$|z|^2 = \frac{(y \cdot x)^2}{x \cdot x} \quad \text{and} \quad y^T \cdot z = \frac{(y \cdot x)^2}{x \cdot x}$$

Thus,

$$y^T z = |z|^2$$

$$= .$$

b) Given,

eq'n of the circle is $(x-2)^2 + (y-5)^2 = 9$

$\Rightarrow$ Centre of the circle is $(2, 5)$

Radius of the circle is 3

Because, $(x-h)^2 + (y-k)^2 = r^2$

Let $(x, y)$ be the point of the circle with has farthest distant from origin $(0, 0)$

Then,

$$d = \sqrt{(x-0)^2 + (y-0)^2}$$

$$= \sqrt{x^2 + y^2}$$

$$d^2 = x^2 + y^2$$

Let $f(x, y) = x^2 + y^2$

When, $d^2$ have to be maximized

Let that function be $f(x, y)$.

In Lagrangian method, our constraint is

$$g(x, y) = (x-2)^2 + (y-5)^2 - 9 = 0.$$

Then,

$$L(x, y, \lambda) = f(x,y) - \lambda\,[g(x,y)]$$

$$= x^2 + y^2 - \lambda\left[(x-2)^2 + (y-5)^2 - 9\right]$$

Let us find,

$$\frac{\delta L}{\delta x} = 2x - 2\lambda(x-2)$$

$$\frac{\delta L}{\delta x} = 0$$

$$\Rightarrow \quad 2x = 2\lambda(x-2)$$

$$x = \lambda x - 2\lambda$$

$$x(1-\lambda) = 1-2\lambda$$

$$\boxed{x = \frac{2\lambda}{\lambda - 1}}$$

$$\frac{\delta L}{\delta y} = 2y - 2\lambda(y-5)$$

$$\frac{\delta L}{\delta y} = 0$$

$$\cancel{2}y = \cancel{2}\lambda(y-5)$$

$$y = \lambda y - 5\lambda$$

$$\boxed{y = \frac{5\lambda}{\lambda - 1}}$$

$$\frac{\delta L}{\delta \lambda} = \frac{\delta}{\delta \lambda}\left[x^2 + y^2 - \lambda\left[(x-2)^2 + (y-5)^2 - 9\right]\right]$$

$$= (x-2)^2 + (y-5)^2 - 9$$

Solving,

$$\frac{\delta L}{\delta \lambda} = 0$$

$$(x-2)^2 + (y-5)^2 - 9 = 0$$

$$(x-2)^2 + (y-5)^2 = 9$$

Substituting $x$ and $y$ from equations

we got,

$$\left(\frac{2\lambda}{\lambda-1} - 2\right)^2 + \left(\frac{5\lambda}{\lambda-1} - 5\right)^2 = 9$$

$$\left(\frac{2}{\lambda-1}\right)^2 + \left(\frac{5}{\lambda-1}\right)^2 = 9$$

$$4+25 = 9(\lambda-1)^2$$

$$9\lambda^2 - 18\lambda + 9 = 29$$

$$9(\lambda-1)^2 = 29$$

$$\lambda - 1 = \pm\frac{\sqrt{29}}{3}$$

$$\lambda = \frac{\pm\sqrt{29}}{3} + 1$$

$$\boxed{\lambda = 1 \pm \frac{\sqrt{29}}{3}}$$

Now,

$$x = \frac{2\lambda}{\lambda-1} = \frac{2\left(1 \pm \frac{\sqrt{29}}{3}\right)}{\pm\frac{\sqrt{29}}{3}}$$

$$= \pm\frac{6}{\sqrt{29}} + 2$$

$$= 3.114 \text{ and } 0.885$$

$$y = \frac{5\lambda}{\lambda - 1}$$

$$= \frac{5\left(1 \pm \frac{\sqrt{29}}{3}\right)}{\pm\frac{\sqrt{29}}{3}} = \pm\frac{15}{\sqrt{29}} + 5$$

$$= 7.785 \ \& \ 2.214$$

$$(x, y) = (\underset{x_1}{3.114}, \underset{y_1}{7.785}) \text{ and } (\underset{x_2}{0.885}, \underset{y_2}{2.214})$$

$$\text{Distance} = \sqrt{x_1^2 + y_1^2}$$

$$= \sqrt{(3.114)^2 + (7.785)^2}$$

$$= 8.384$$

$$\sqrt{x_2^2 + y_2^2} = \sqrt{(0.885)^2 + (2.214)^2}$$

$$= 2.383$$

Therefore,

Farthest point from, origin is

$(3.114, 7.785)$

c) Given,

$$D = \begin{bmatrix} 1 & -1 & 8 \\ 4 & 2 & 1 \\ 0 & 1 & 5 \\ 5 & -2 & -5 \\ -2 & 0 & -7 \\ 3 & 5 & 3 \end{bmatrix}$$

Points are

$A_1 = (1, -1, 8)$

$A_2 = (4, 2, 1)$

$A_3 = (0, 1, 5)$

$A_4 = (5, -2, -5)$

$A_5 = (-2, 0, -7)$

$A_6 = (3, 5, 3)$

Also,

Given orthogonal vectors are,

$$x = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

From this,

$$x \cdot x = (2)^2 + (1)^2 + (2)^2 = 9$$

$$y \cdot y = (-1)^2 + (0)^2 + (1)^2 = 2$$

$$\text{Projection of } A_1 = \frac{A_1 \cdot x}{x \cdot x} x + \frac{A_1 \cdot y}{y \cdot y} y$$

$$= \frac{1(2) + 1(-1) + 8(2)}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \frac{1(-1) + 0(-1) + 8(1)}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\frac{17}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \frac{7}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/18 \\ 17/9 \\ 137/18 \end{bmatrix}$$

Projection of $A_2 = \frac{A_2 \cdot x}{x \cdot x} x + \frac{A_2 \cdot y}{y \cdot y} \cdot y$

$$= \frac{4(2)+2(1)+1(2)}{9}\begin{bmatrix}2\\1\\2\end{bmatrix} + \frac{4(-1)+2(0)+1(1)}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$= \frac{12}{9}\begin{bmatrix}2\\1\\2\end{bmatrix} + \left(\frac{-3}{2}\right)\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}25/6\\4/3\\7/6\end{bmatrix}$$

Projection of $A_3 = \frac{A_3 \cdot x}{x \cdot x} x + \frac{A_3 \cdot y}{y \cdot y} y$

$$= \frac{0(2)+1(1)+5(2)}{9}\begin{bmatrix}2\\1\\2\end{bmatrix} + \frac{0(-1)+1(0)+5(1)}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$= \frac{11}{9}\begin{bmatrix}2\\1\\2\end{bmatrix} + 5/2\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-1/18\\11/9\\89/18\end{bmatrix}$$

Projection of $A_4 = \frac{A_4 \cdot x}{x \cdot x} x + \frac{A_4 \cdot y}{y \cdot y} y$

$$= \frac{5(2)+(-2)(1)+(-5)(2)}{9}\begin{bmatrix}2\\1\\2\end{bmatrix} + \frac{5(-1)+(-2)(0)+5(1)}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}41/9\\-2/9\\-49/9\end{bmatrix}$$

Projection of $A_5 = \frac{-2(2)+0(1)+(-7)(2)}{9}\begin{bmatrix}2\\1\\2\end{bmatrix} + \frac{-2(-1)+0+(-7)}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$$= \begin{bmatrix}-3/2\\-2\\-13/2\end{bmatrix}$$

Projection of $A_6 = \frac{A_5 \cdot x}{x \cdot x} x + \frac{A_5 \cdot y}{y \cdot y} y$

$$= \frac{3(2)+5(1)+3(2)}{9}\begin{bmatrix}2\\1\\2\end{bmatrix} + \frac{3(-1)+5(0)+3(1)}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}34/9\\17/9\\34/9\end{bmatrix}$$

Mean Square Error (for six point)

$$= \frac{1}{6}\sum_{i=1}^{6} |A_i - \text{proj}(A_i)|^2$$

$$= \frac{1}{6}\Big[|A_1 - \text{proj}(A_1)|^2 + |A_2 - \text{proj}(A_2)|^2 + |A_3 - \text{proj}(A_3)|^2 + |A_4 - \text{proj}(A_4)|^2 + |A_5 - \text{proj}(A_5)|^2 + |A_6 - \text{proj}(A_6)|^2\Big]$$

$$= \frac{1}{6}\Big[|(1,-1,8)-(5/18, 17/9, 137/18)|^2 + |(4,2,1)-(\tfrac{25}{6}, \tfrac{4}{3}, \tfrac{7}{6})|^2 + |(0,1,5) - (-1/18, 11/9, 89/18)|^2 + |(5,-2,-5) - (4/9, \tfrac{-2}{9}, \tfrac{-49}{9})|^2 + |(-2,0,-7) - (-3/2, -2, -13/2)|^2 + |(3,5,3) - (34/9, 17/9, 34/9)|^2\Big]$$

$$= \frac{1}{6}[9.388 + 0.5 + 0.055 + 3.55 + 16.5 + 10.88]$$

$$= \frac{1}{6}(40.886) = 6.814.$$

d) Given,

$$A = \begin{bmatrix} 2 & 0 & -4 & 3 \\ -5 & 1 & 8 & 0 \\ 3 & -3 & 0 & 2 \\ 5 & 1 & 2 & 1 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} -0.4286 & -0.0026 & -0.1304 & 0.8939 & -0.0814 \\ 0.8338 & -0.1394 & -0.3332 & 0.3387 & -0.2441 \\ -0.2242 & -0.4303 & -0.7934 & -0.1709 & 0.3255 \\ -0.1063 & -0.8429 & 0.3315 & -0.0522 & -0.4061 \\ 0.2438 & -0.2914 & 0.3728 & 0.2332 & 0.8138 \end{bmatrix}$$

and

$$D = \mathrm{Diag}([11.1826, 6.2933, 3.6019, 2.7147])$$

$$= \begin{bmatrix} 11.1826 & 0 & 0 & 0 \\ 0 & 6.2933 & 0 & 0 \\ 0 & 0 & 3.6019 & 0 \\ 0 & 0 & 0 & 2.7147 \end{bmatrix}$$

Given a matrix A, its singular value decomposition is

$$A = L \Delta R^T$$

Here,

$$\Delta = D = \mathrm{Diag}([11.1826, 6.2933, 3.6019, 2.7147])$$

then,

in order to obtain $R^T$ the SVD eq will be

$$\boxed{R^T = \Delta^{-1} L^T A}$$

Inverse of diagnol matrix is,

$$\Delta^{-1} = D^{-1} = \mathrm{Diag}\left(\left[\frac{1}{11.1826}, \frac{1}{6.2933}, \frac{1}{3.6019}, \frac{1}{2.7147}\right]\right)$$

$$L^T = \begin{bmatrix} -0.4286 & 0.8338 & -0.2242 & -0.1063 & 0.2488 \\ -0.0026 & -0.01394 & -0.4303 & -0.8429 & -0.2914 \\ -0.1034 & -0.3332 & -0.7934 & 0.3315 & 0.3728 \\ 0.8939 & 0.3387 & -0.1709 & -0.0522 & 0.2332 \\ -0.0814 & -0.2441 & 0.3255 & -0.4069 & 0.8138 \end{bmatrix}$$

$$L^T A = \begin{bmatrix} -0.4286 & 0.8338 & -0.2242 & -0.1063 & 0.2488 \\ -0.0026 & -0.01394 & -0.4303 & -0.8429 & -0.2914 \\ -0.1034 & -0.3332 & -0.7934 & 0.3315 & 0.3728 \\ 0.8939 & 0.3387 & -0.1709 & -0.0522 & 0.2332 \\ -0.0814 & -0.2441 & 0.3255 & -0.4069 & 0.8138 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & -4 & 3 \\ -5 & 1 & 8 & 0 \\ 3 & -3 & 0 & 2 \\ 5 & 4 & 2 & 1 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

$$L^T A = \begin{bmatrix} -6.2303 & 1.8877 & 8.9036 & -1.8405 \\ -4.8136 & -0.2742 & -3.6648 & -1.7113 \\ 0.7365 & 3.1241 & -0.4706 & -1.5655 \\ -0.6794 & 1.2656 & -0.2708 & 2.2877 \\ -0.0003 & 0.0001 & 0.0004 & -0.0001 \end{bmatrix}$$

$$\Delta^{-1} = \begin{bmatrix} \frac{1}{11.1826} & 0 & 0 & 0 \\ 0 & \frac{1}{6.2933} & 0 & 0 \\ 0 & 0 & \frac{1}{3.6019} & 0 \\ 0 & 0 & 0 & \frac{1}{2.7147} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Delta^{-1} = \begin{bmatrix} 0.0894 & 0 & 0 & 0 \\ 0 & 0.1588 & 0 & 0 \\ 0 & 0 & 0.2776 & 0 \\ 0 & 0 & 0 & 0.3683 \end{bmatrix}$$

$$R^{T} = \begin{bmatrix} -6.2303 & 1.8877 & 8.9036 & -1.8405 \\ -4.8136 & -0.2742 & -3.6648 & -1.7113 \\ 0.7365 & 3.1241 & -0.4706 & -1.5655 \\ -0.6794 & 1.2656 & -0.2708 & 2.2877 \\ -0.0003 & 0.0001 & 0.0004 & -0.001 \end{bmatrix} \cdot \begin{bmatrix} 0.0894 & 0 & 0 & 0 \\ 0 & 0.1588 & 0 & 0 \\ 0 & 0 & 0.2776 & 0 \\ 0 & 0 & 0 & 0.3683 \end{bmatrix}$$

$$R^{T} = \begin{bmatrix} -0.5569 & 0.2997 & 2.4716 & -0.6778 \\ -0.4303 & -0.0435 & 9.0173 & -0.6302 \\ 0.0658 & 0.4961 & -0.1306 & -0.5765 \\ -0.0607 & 0.2009 & -0.0751 & 0.8425 \\ -0.00002 & 0.0005 & -0.00011 & -0.0003 \end{bmatrix}$$