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Calculus

Volume 1

Calculus Volume 1

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The screenshot shows a digital book interface for 'College Success'. The top navigation bar includes a back arrow, the title 'College Success', and the chapter heading '2.2 The Motivated Learner'. On the left, a table of contents lists chapters from Preface to Chapter 10. The main content area displays the 'Resilience and Grit' section. At the top of this section is a search bar labeled 'Search this book' and a 'My highlights' button. Below the search bar is a text block about grit and resilience, followed by a photo of a runner with a prosthetic leg crossing a finish line. A floating callout box contains the word 'resilience' highlighted with a blue border, along with five colored circular icons (yellow, green, blue, purple, pink) and 'Save' and 'Cancel' buttons. A caption below the photo reads: 'Figure 2.3 U.S. Army veteran and captain of the U.S. Invictus team, Will Reynolds, races to the finish line. (Credit: DoD News / Flickr / Attribution 2.0 Generic (CC-BY 2.0))'. A note at the bottom states: 'The study showed that grit and perseverance were better predictors of academic success and achievement than talent or IQ.'

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PREFACE

Welcome to *Calculus Volume 1*, an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

About OpenStax

OpenStax is a nonprofit based at Rice University, and it's our mission to improve student access to education. Our first openly licensed college textbook was published in 2012, and our library has since scaled to over 25 books for college and AP® courses used by hundreds of thousands of students. OpenStax Tutor, our low-cost personalized learning tool, is being used in college courses throughout the country. Through our partnerships with philanthropic foundations and our alliance with other educational resource organizations, OpenStax is breaking down the most common barriers to learning and empowering students and instructors to succeed.

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Format

You can access this textbook for free in web view or PDF through OpenStax.org, and for a low cost in print.

About *Calculus Volume 1*

Calculus is designed for the typical two- or three-semester general calculus course, incorporating innovative features to enhance student learning. The book guides students through the core concepts of calculus and helps them understand how those concepts apply to their lives and the world around them. Due to the comprehensive nature of the material, we are offering the book in three volumes for flexibility and efficiency. Volume 1 covers functions, limits, derivatives, and integration.

Coverage and scope

Our *Calculus Volume 1* textbook adheres to the scope and sequence of most general calculus courses nationwide. We have worked to make calculus interesting and accessible to students while maintaining the mathematical rigor inherent in the subject. With this objective in mind, the content of the three volumes of *Calculus* have been developed and arranged to provide a logical progression from fundamental to more advanced concepts, building upon what students have already learned and emphasizing connections between topics and between theory and applications. The goal of each section is to enable students not just to recognize concepts, but work with them in ways that will be useful in later courses and future careers. The organization and pedagogical features were developed and vetted with feedback from mathematics educators dedicated to the project.

Volume 1

- Chapter 1: Functions and Graphs
- Chapter 2: Limits
- Chapter 3: Derivatives
- Chapter 4: Applications of Derivatives
- Chapter 5: Integration
- Chapter 6: Applications of Integration

Volume 2

- Chapter 1: Integration
- Chapter 2: Applications of Integration
- Chapter 3: Techniques of Integration
- Chapter 4: Introduction to Differential Equations
- Chapter 5: Sequences and Series
- Chapter 6: Power Series
- Chapter 7: Parametric Equations and Polar Coordinates

Volume 3

- Chapter 1: Parametric Equations and Polar Coordinates
- Chapter 2: Vectors in Space
- Chapter 3: Vector-Valued Functions
- Chapter 4: Differentiation of Functions of Several Variables
- Chapter 5: Multiple Integration
- Chapter 6: Vector Calculus
- Chapter 7: Second-Order Differential Equations

Pedagogical foundation

Throughout *Calculus Volume 1* you will find examples and exercises that present classical ideas and techniques as well as modern applications and methods. Derivations and explanations are based on years of classroom experience on the part of long-time calculus professors, striving for a balance of clarity and rigor that has proven successful with their students. Motivational applications cover important topics in probability, biology, ecology, business, and economics, as well as areas of physics, chemistry, engineering, and computer science. **Student Projects** in each chapter give students opportunities to explore interesting sidelights in pure and applied mathematics, from determining a safe distance between the grandstand and the track at a Formula One racetrack, to calculating the center of mass of the Grand Canyon Skywalk or the terminal speed of a skydiver. **Chapter Opening Applications** pose problems that are solved later in the chapter, using the ideas covered in that chapter. Problems include the hydraulic force against the Hoover Dam, and the comparison of relative intensity of two earthquakes. **Definitions, Rules, and Theorems** are highlighted throughout the text, including over 60 **Proofs** of theorems.

Assessments that reinforce key concepts

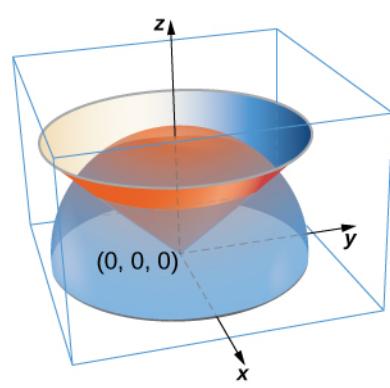
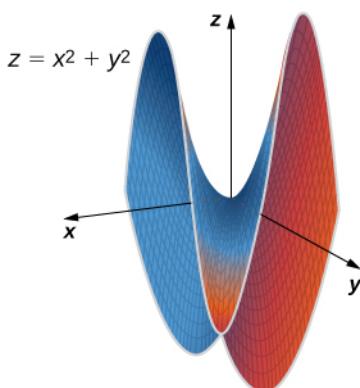
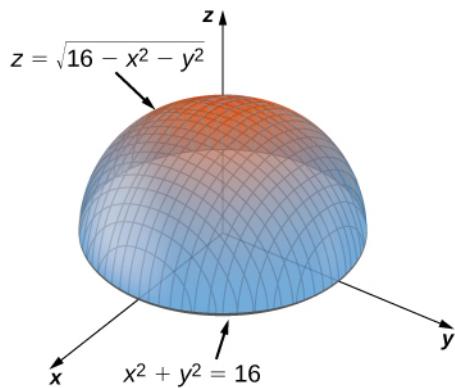
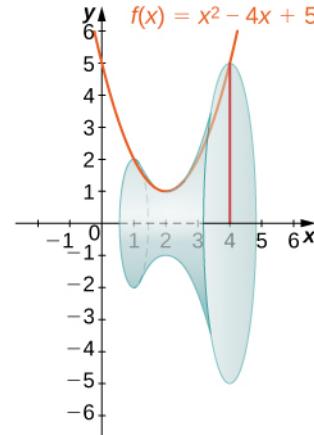
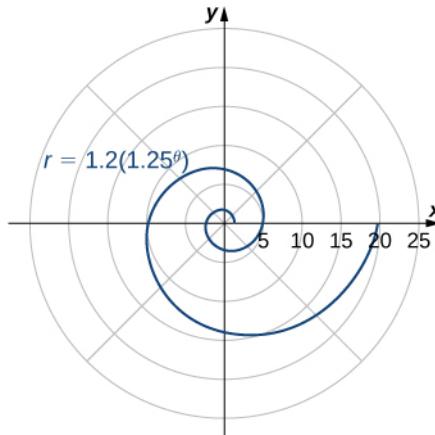
In-chapter **Examples** walk students through problems by posing a question, stepping out a solution, and then asking students to practice the skill with a “Checkpoint” question. The book also includes assessments at the end of each chapter so students can apply what they’ve learned through practice problems. Many exercises are marked with a [T] to indicate they are suitable for solution by technology, including calculators or Computer Algebra Systems (CAS). Answers for selected exercises are available in the **Answer Key** at the back of the book. The book also includes assessments at the end of each chapter so students can apply what they’ve learned through practice problems.

Early or late transcendentals

Calculus Volume 1 is designed to accommodate both Early and Late Transcendental approaches to calculus. Exponential and logarithmic functions are introduced informally in Chapter 1 and presented in more rigorous terms in Chapter 6. Differentiation and integration of these functions is covered in Chapters 3–5 for instructors who want to include them with other types of functions. These discussions, however, are in separate sections that can be skipped for instructors who prefer to wait until the integral definitions are given before teaching the calculus derivations of exponentials and logarithms.

Comprehensive art program

Our art program is designed to enhance students' understanding of concepts through clear and effective illustrations, diagrams, and photographs.



Additional resources

Student and instructor resources

We've compiled additional resources for both students and instructors, including Getting Started Guides, an instructor solution manual, and PowerPoint slides. Instructor resources require a verified instructor account, which can be requested on your OpenStax.org log-in. Take advantage of these resources to supplement your OpenStax book.

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1 | FUNCTIONS AND GRAPHS



Figure 1.1 A portion of the San Andreas Fault in California. Major faults like this are the sites of most of the strongest earthquakes ever recorded. (credit: modification of work by Robb Hannawacker, NPS)

Chapter Outline

- [**1.1** Review of Functions](#)
- [**1.2** Basic Classes of Functions](#)
- [**1.3** Trigonometric Functions](#)
- [**1.4** Inverse Functions](#)
- [**1.5** Exponential and Logarithmic Functions](#)

Introduction

In the past few years, major earthquakes have occurred in several countries around the world. In January 2010, an earthquake of magnitude 7.3 hit Haiti. A magnitude 9 earthquake shook northeastern Japan in March 2011. In April 2014, an 8.2-magnitude earthquake struck off the coast of northern Chile. What do these numbers mean? In particular, how does a magnitude 9 earthquake compare with an earthquake of magnitude 8.2? Or 7.3? Later in this chapter, we show how logarithmic functions are used to compare the relative intensity of two earthquakes based on the magnitude of each earthquake (see [Example 1.39](#)).

Calculus is the mathematics that describes changes in functions. In this chapter, we review all the functions necessary to study calculus. We define polynomial, rational, trigonometric, exponential, and logarithmic functions. We review how to evaluate these functions, and we show the properties of their graphs. We provide examples of equations with terms involving these functions and illustrate the algebraic techniques necessary to solve them. In short, this chapter provides the foundation for the material to come. It is essential to be familiar and comfortable with these ideas before proceeding to the formal introduction of calculus in the next chapter.

1.1 | Review of Functions

Learning Objectives

- 1.1.1 Use functional notation to evaluate a function.
- 1.1.2 Determine the domain and range of a function.
- 1.1.3 Draw the graph of a function.
- 1.1.4 Find the zeros of a function.
- 1.1.5 Recognize a function from a table of values.
- 1.1.6 Make new functions from two or more given functions.
- 1.1.7 Describe the symmetry properties of a function.

In this section, we provide a formal definition of a function and examine several ways in which functions are represented—namely, through tables, formulas, and graphs. We study formal notation and terms related to functions. We also define composition of functions and symmetry properties. Most of this material will be a review for you, but it serves as a handy reference to remind you of some of the algebraic techniques useful for working with functions.

Functions

Given two sets A and B , a set with elements that are ordered pairs (x, y) , where x is an element of A and y is an element of B , is a relation from A to B . A relation from A to B defines a relationship between those two sets. A function is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the *input*; the element of the second set is called the *output*. Functions are used all the time in mathematics to describe relationships between two sets. For any function, when we know the input, the output is determined, so we say that the output is a function of the input. For example, the area of a square is determined by its side length, so we say that the area (the output) is a function of its side length (the input). The velocity of a ball thrown in the air can be described as a function of the amount of time the ball is in the air. The cost of mailing a package is a function of the weight of the package. Since functions have so many uses, it is important to have precise definitions and terminology to study them.

Definition

A **function** f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the **domain** of the function. The set of outputs is called the **range** of the function.

For example, consider the function f , where the domain is the set of all real numbers and the rule is to square the input. Then, the input $x = 3$ is assigned to the output $3^2 = 9$. Since every nonnegative real number has a real-value square root, every nonnegative number is an element of the range of this function. Since there is no real number with a square that is negative, the negative real numbers are not elements of the range. We conclude that the range is the set of nonnegative real numbers.

For a general function f with domain D , we often use x to denote the input and y to denote the output associated with x . When doing so, we refer to x as the **independent variable** and y as the **dependent variable**, because it depends on x . Using function notation, we write $y = f(x)$, and we read this equation as “ y equals f of x .” For the squaring function described earlier, we write $f(x) = x^2$.

The concept of a function can be visualized using [Figure 1.2](#), [Figure 1.3](#), and [Figure 1.4](#).



Figure 1.2 A function can be visualized as an input/output device.

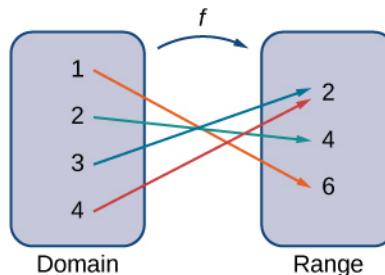


Figure 1.3 A function maps every element in the domain to exactly one element in the range. Although each input can be sent to only one output, two different inputs can be sent to the same output.

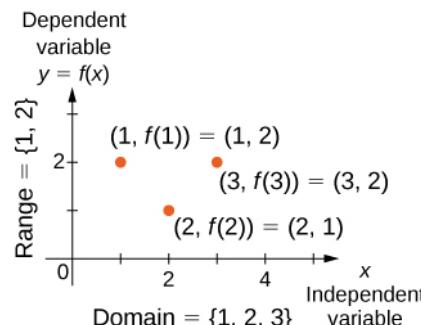


Figure 1.4 In this case, a graph of a function f has a domain of $\{1, 2, 3\}$ and a range of $\{1, 2\}$. The independent variable is x and the dependent variable is y .



Visit this [applet link](http://www.openstax.org/l/grapherrors) (<http://www.openstax.org/l/grapherrors>) to see more about graphs of functions.

We can also visualize a function by plotting points (x, y) in the coordinate plane where $y = f(x)$. The **graph of a function** is the set of all these points. For example, consider the function f , where the domain is the set $D = \{1, 2, 3\}$ and the rule is $f(x) = 3 - x$. In **Figure 1.5**, we plot a graph of this function.

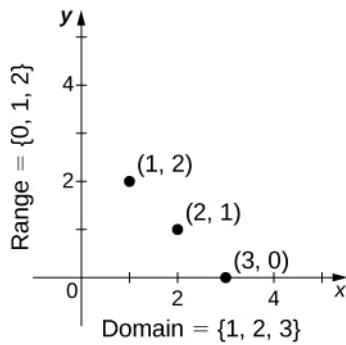


Figure 1.5 Here we see a graph of the function f with domain $\{1, 2, 3\}$ and rule $f(x) = 3 - x$. The graph consists of the points $(x, f(x))$ for all x in the domain.

Every function has a domain. However, sometimes a function is described by an equation, as in $f(x) = x^2$, with no specific domain given. In this case, the domain is taken to be the set of all real numbers x for which $f(x)$ is a real number. For example, since any real number can be squared, if no other domain is specified, we consider the domain of $f(x) = x^2$ to be the set of all real numbers. On the other hand, the square root function $f(x) = \sqrt{x}$ only gives a real output if x is nonnegative. Therefore, the domain of the function $f(x) = \sqrt{x}$ is the set of nonnegative real numbers, sometimes called the *natural domain*.

For the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$, the domains are sets with an infinite number of elements. Clearly we cannot list all these elements. When describing a set with an infinite number of elements, it is often helpful to use set-builder or interval notation. When using set-builder notation to describe a subset of all real numbers, denoted \mathbb{R} , we write

$$\{x|x \text{ has some property}\}.$$

We read this as the set of real numbers x such that x has some property. For example, if we were interested in the set of real numbers that are greater than one but less than five, we could denote this set using set-builder notation by writing

$$\{x|1 < x < 5\}.$$

A set such as this, which contains all numbers greater than a and less than b , can also be denoted using the interval notation (a, b) . Therefore,

$$(1, 5) = \{x|1 < x < 5\}.$$

The numbers 1 and 5 are called the *endpoints* of this set. If we want to consider the set that includes the endpoints, we would denote this set by writing

$$[1, 5] = \{x|1 \leq x \leq 5\}.$$

We can use similar notation if we want to include one of the endpoints, but not the other. To denote the set of nonnegative real numbers, we would use the set-builder notation

$$\{x|0 \leq x\}.$$

The smallest number in this set is zero, but this set does not have a largest number. Using interval notation, we would use the symbol ∞ , which refers to positive infinity, and we would write the set as

$$[0, \infty) = \{x|0 \leq x\}.$$

It is important to note that ∞ is not a real number. It is used symbolically here to indicate that this set includes all real numbers greater than or equal to zero. Similarly, if we wanted to describe the set of all nonpositive numbers, we could write

$$(-\infty, 0] = \{x|x \leq 0\}.$$

Here, the notation $-\infty$ refers to negative infinity, and it indicates that we are including all numbers less than or equal to zero, no matter how small. The set

$$(-\infty, \infty) = \{x|x \text{ is any real number}\}$$

refers to the set of all real numbers.

Some functions are defined using different equations for different parts of their domain. These types of functions are known as *piecewise-defined functions*. For example, suppose we want to define a function f with a domain that is the set of all real numbers such that $f(x) = 3x + 1$ for $x \geq 2$ and $f(x) = x^2$ for $x < 2$. We denote this function by writing

$$f(x) = \begin{cases} 3x + 1 & x \geq 2 \\ x^2 & x < 2 \end{cases}.$$

When evaluating this function for an input x , the equation to use depends on whether $x \geq 2$ or $x < 2$. For example, since $5 > 2$, we use the fact that $f(x) = 3x + 1$ for $x \geq 2$ and see that $f(5) = 3(5) + 1 = 16$. On the other hand, for $x = -1$, we use the fact that $f(x) = x^2$ for $x < 2$ and see that $f(-1) = 1$.

Example 1.1

Evaluating Functions

For the function $f(x) = 3x^2 + 2x - 1$, evaluate

- a. $f(-2)$
- b. $f(\sqrt{2})$
- c. $f(a + h)$

Solution

Substitute the given value for x in the formula for $f(x)$.

$$\begin{aligned} \text{a. } f(-2) &= 3(-2)^2 + 2(-2) - 1 = 12 - 4 - 1 = 7 \\ \text{b. } f(\sqrt{2}) &= 3(\sqrt{2})^2 + 2\sqrt{2} - 1 = 6 + 2\sqrt{2} - 1 = 5 + 2\sqrt{2} \\ \text{c. } f(a + h) &= 3(a + h)^2 + 2(a + h) - 1 = 3(a^2 + 2ah + h^2) + 2a + 2h - 1 \\ &= 3a^2 + 6ah + 3h^2 + 2a + 2h - 1 \end{aligned}$$



- 1.1** For $f(x) = x^2 - 3x + 5$, evaluate $f(1)$ and $f(a + h)$.

Example 1.2

Finding Domain and Range

For each of the following functions, determine the i. domain and ii. range.

a. $f(x) = (x - 4)^2 + 5$

b. $f(x) = \sqrt{3x + 2} - 1$

c. $f(x) = \frac{3}{x - 2}$

Solution

a. Consider $f(x) = (x - 4)^2 + 5$.

i. Since $f(x) = (x - 4)^2 + 5$ is a real number for any real number x , the domain of f is the interval $(-\infty, \infty)$.

ii. Since $(x - 4)^2 \geq 0$, we know $f(x) = (x - 4)^2 + 5 \geq 5$. Therefore, the range must be a subset of $\{y|y \geq 5\}$. To show that every element in this set is in the range, we need to show that for a given y in that set, there is a real number x such that $f(x) = (x - 4)^2 + 5 = y$. Solving this equation for x , we see that we need x such that

$$(x - 4)^2 = y - 5.$$

This equation is satisfied as long as there exists a real number x such that

$$x - 4 = \pm \sqrt{y - 5}.$$

Since $y \geq 5$, the square root is well-defined. We conclude that for $x = 4 \pm \sqrt{y - 5}$, $f(x) = y$, and therefore the range is $\{y|y \geq 5\}$.

b. Consider $f(x) = \sqrt{3x + 2} - 1$.

i. To find the domain of f , we need the expression $3x + 2 \geq 0$. Solving this inequality, we conclude that the domain is $\{x|x \geq -2/3\}$.

ii. To find the range of f , we note that since $\sqrt{3x + 2} \geq 0$, $f(x) = \sqrt{3x + 2} - 1 \geq -1$. Therefore, the range of f must be a subset of the set $\{y|y \geq -1\}$. To show that every element in this set is in the range of f , we need to show that for all y in this set, there exists a real number x in the domain such that $f(x) = y$. Let $y \geq -1$. Then, $f(x) = y$ if and only if

$$\sqrt{3x + 2} - 1 = y.$$

Solving this equation for x , we see that x must solve the equation

$$\sqrt{3x + 2} = y + 1.$$

Since $y \geq -1$, such an x could exist. Squaring both sides of this equation, we have $3x + 2 = (y + 1)^2$.

Therefore, we need

$$3x = (y + 1)^2 - 2,$$