StudyThinking

卷二 数学物理方程笔记

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第一章 Fourier Transform

1.1 definition

$$F(\lambda)=\mathcal{F}[f(x)]=\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x$$

$$f(x)=\mathcal{F}^{-1}[f(x)]=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}\,\mathrm{d}\lambda$$

- $\mathcal{F}\left[e^{-a|x|}\right], a>0$

$$\mathcal{F}\left[e^{-a|x|}\right]=\int_{-\infty}^{\infty} e^{-a|x|}e^{-i\lambda x}\,\mathrm{d}x$$

$$=\int_{-\infty}^{0} e^{ax-i\lambda x}\,\mathrm{d}x+\int_{0}^{\infty} e^{-ax-i\lambda x}\,\mathrm{d}x$$

$$=\frac{1}{a-i\lambda}e^{(a-i\lambda)x}|_{-\infty}^{0}-\frac{1}{a+i\lambda}e^{-(a+i\lambda)x}|_{0}^{\infty}$$

$$=\frac{1}{a-i\lambda}+\frac{1}{a+i\lambda}$$

$$=\frac{2a}{a^2+\lambda^2}$$

$*\lim_{t\to\infty} e^{-(\beta+i\lambda)t}=0, \beta>0$

- $\mathcal{F}\left[e^{-ax^2}\right], a>0$

$$\mathcal{F}\left[e^{-ax^2}\right]=\int_{-\infty}^{\infty} e^{-ax^2}e^{-i\lambda x}\,\mathrm{d}x$$

$$=\int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\lambda}{2a}\right)^2-\frac{\lambda^2}{4a}}\,\mathrm{d}x$$

$$=e^{-\frac{\lambda^2}{4a}}\int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\lambda}{2a}\right)^2}\,\mathrm{d}x$$

$$=\sqrt{\frac{\pi}{a}}e^{-\frac{\lambda^2}{4a}}$$

$*$Gaussian integral: $\int_{-\infty}^{\infty} e^{-ax^2}\,\mathrm{d}x=\sqrt{\frac{\pi}{a}}$

- $\mathcal{F}[\cos\eta x^2], \eta>0$

$$\mathcal{F}[\cos\eta x^2]=\int_{-\infty}^{\infty}\cos\eta x^2 e^{-i\lambda x}\,\mathrm{d}x$$

$$=\int_{-\infty}^{\infty}\frac{e^{-i\eta x^2}+e^{i\eta x^2}}{2}e^{-i\lambda x}\,\mathrm{d}x$$

$$=\frac{1}{2}\left(\int_{-\infty}^{\infty} e^{i\eta\left(x-\frac{\lambda}{2\eta}\right)^2-i\frac{\lambda^2}{4\eta}}\,\mathrm{d}x+\int_{-\infty}^{\infty} e^{-i\eta\left(x+\frac{\lambda}{2\eta}\right)^2+i\frac{\lambda^2}{4\eta}}\,\mathrm{d}x\right)$$

$$=\frac{1}{2}e^{-i\frac{\lambda^2}{4\eta}}\sqrt{\frac{\pi}{\eta}}e^{i\frac{\pi}{4}}+\frac{1}{2}e^{i\frac{\lambda^2}{4\eta}}\sqrt{\frac{\pi}{\eta}}e^{-i\frac{\pi}{4}}$$

$$=\frac{1}{2}\sqrt{\frac{\pi}{\eta}}\left[\cos\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)-i\sin\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)+\cos\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)+i\sin\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)\right]$$

$$=\sqrt{\frac{\pi}{\eta}}\cos\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)$$

- $\mathcal{F}[\sin\eta x^2], \eta>0$

$$\mathcal{F}\left[\sin\eta x^2\right] = \sqrt{\frac{\pi}{\eta}}\sin\left(\frac{\lambda^2}{4\eta}+\frac{\pi}{4}\right)$$

$*$Euler's formula: $e^{ix} = \cos x + i\sin x$

$*$Gaussian-like integrals: $\int_{-\infty}^{\infty} e^{\pm i\eta x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{\eta}}e^{\pm i\frac{\pi}{4}}, \eta > 0$

1.2 properties

1.2.1 linear property

$$\mathcal{F}[\alpha f(x) + \beta g(x)] = \alpha\mathcal{F}[f(x)] + \beta\mathcal{F}[g(x)]$$

1.2.2 shift property

$$\mathcal{F}[f(x-b)] = e^{-i\lambda b}\mathcal{F}[f(x)]$$

1.2.3 differentiation property

$$\mathcal{F}[f'(x)] = i\lambda\mathcal{F}[f(x)]$$

$$\mathcal{F}\left[f^{(n)}(x)\right] = (i\lambda)^n\mathcal{F}[f(x)]$$

$$\mathcal{F}[xf(x)] = i\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{F}[f(x)]$$

$$\mathcal{F}[x^n f(x)] = i^n\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{F}[f(x)]$$

1.2.4 convolution property

$$\mathcal{F}[f(x) * g(x)] = \mathcal{F}[f(x)]\cdot\mathcal{F}[g(x)]$$

$$\mathcal{F}[f(x)\cdot g(x)] = \frac{1}{2\pi}\mathcal{F}[f(x)] * \mathcal{F}[g(x)]$$

1.3 function δ

1.3.1 definition

$$\delta(x-x_0) = \begin{cases}\infty \text{ if } x=x_0\\ 0 \text{ else}\end{cases}$$

$$\int_{-\infty}^{\infty}\delta(x-x_0) = 1$$

1.3.2 properties

$$\delta(x) = \delta(-x)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0) = f(x_0)$$

$$\delta(x) * f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)\,\mathrm{d}\xi = f(x)$$

$$\delta(x-a) * f(x) = \int_{-\infty}^{\infty} f(x-\xi)\delta(\xi-a)\,\mathrm{d}\xi = f(x-a)$$

1.3.3 fourier transform

$$\mathcal{F}[\delta(x-x_0)] = \int_{-\infty}^{\infty}\delta(x-x_0)e^{-i\lambda x}\,\mathrm{d}x = e^{-i\lambda x}$$

when $x_0 = 0$

$$\mathcal{F}[\delta(x)] = 1$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\,\mathrm{d}\lambda$$

since $\delta(x) = \delta(-x)$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}\lambda$$

the fourier transform of 1 can be given

$$\mathcal{F}[1] = 2\pi\delta(\lambda)$$

第二章 Laplace Transform

2.1 definition

$$F(p)=\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-pt}\,\mathrm{d}t$$

- $\mathcal{L}[1]$

$$\mathcal{L}[1]=\int_0^\infty 1\cdot e^{-pt}\,\mathrm{d}t$$
$$=-\tfrac{1}{p}e^{-pt}|_0^\infty$$
$$=\tfrac{1}{p}$$

- $\mathcal{L}[t]$

$$\mathcal{L}[t]=\int_0^\infty te^{-pt}\,\mathrm{d}t$$
$$=-\tfrac{1}{p}te^{-pt}|_0^\infty+\tfrac{1}{p}\int_0^\infty e^{-pt}\,\mathrm{d}t$$
$$=\tfrac{1}{p^2}$$

- $\mathcal{L}[e^{at}]$

$$\mathcal{L}[e^{at}]=\int_0^\infty e^{at}e^{-pt}\,\mathrm{d}t$$
$$=\int_0^\infty e^{-(p-a)t}\,\mathrm{d}t$$
$$=\tfrac{1}{p-a}$$

- $\mathcal{L}[\cos\omega t]$

$$\mathcal{L}[\cos\omega t]=\int_0^\infty \cos\omega te^{-pt}\,\mathrm{d}t$$
$$=-\tfrac{1}{p}\cos\omega te^{-pt}|_0^\infty-\tfrac{\omega}{p}\int_0^\infty \sin\omega te^{-pt}\,\mathrm{d}t$$
$$=\tfrac{1}{p}+\tfrac{\omega}{p^2}\sin\omega te^{-pt}|_0^\infty-\tfrac{\omega^2}{p^2}\mathcal{L}[\cos\omega t]$$
$$=\tfrac{p}{p^2+\omega^2}$$

- $\mathcal{L}[\sin\omega t]$

$$\mathcal{L}[\sin\omega t]=\int_0^\infty \sin\omega te^{-pt}\,\mathrm{d}t$$
$$=\tfrac{1}{2i}\int_0^\infty (e^{i\omega t}-e^{-i\omega t})e^{-pt}\,\mathrm{d}t$$
$$=\tfrac{1}{2i}\int_0^\infty \left(e^{-(p-i\omega)t}-e^{-(p+i\omega)t}\right)\mathrm{d}t$$
$$=\tfrac{1}{2i}\left(\tfrac{1}{p-i\omega}-\tfrac{1}{p+i\omega}\right)$$
$$=\tfrac{\omega}{p^2+\omega^2}$$

- $\mathcal{L}[\cosh\omega t]$

$$\begin{aligned}\mathcal{L}[\cosh \omega t] &= \int_0^\infty \cosh \omega t e^{-pt}\,\mathrm{d}t \\ &= \tfrac{1}{2}\int_0^\infty \left(e^{-(p-\omega)t} + e^{-(p+\omega)t}\right)\mathrm{d}t \\ &= \tfrac{1}{2}\left(\tfrac{1}{p-\omega} + \tfrac{1}{p+\omega}\right) \\ &= \tfrac{p}{p^2-\omega^2}\end{aligned}$$

- $\mathcal{L}[\sinh \omega t]$

$$\mathcal{L}[\sinh \omega t] = \tfrac{\omega}{p^2-\omega^2}$$

2.2 properties

2.2.1 linear property

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha\mathcal{L}[f(t)] + \beta\mathcal{L}[g(t)]$$

2.2.2 shift property

$$\mathcal{L}[e^{at} f(t)] = F(p-a)$$

2.2.3 differentiation property

$$\mathcal{L}[f'(t)] = p\mathcal{L}[f(t)] - f(0)$$

$$\mathcal{L}\left[f^{(n)}(t)\right] = p^n F(p) - p^{n-1} f(0) - \ldots - pf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathcal{L}[tf(t)] = -\tfrac{\mathrm{d}}{\mathrm{d}p}F(p)$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \tfrac{\mathrm{d}^n}{\mathrm{d}p^n}F(p)$$

2.2.4 convolve property

$$f(t) * g(t) = \int_0^t f(t)g(t-\tau)\,\mathrm{d}\tau$$

$$\mathcal{L}[f(t) * g(t)] = F(p)\cdot G(p)$$

第三章 Fundamental Equations

3.1 wave equation

$$u_{tt} = a^2 u_{xx}$$

$$u_{tt} = a^2 \left(u_{xx} + u_{yy}\right)$$

$$u_{tt} = a^2 \left(u_{xx} + +u_{yy} + u_{zz}\right)$$

3.2 wave equation with a source term

$$u_{tt} = a^2 u_{xx} + f(x,t)$$

$$u_{tt} = a^2 \left(u_{xx} + u_{yy}\right) + f(x,y,t)$$

$$u_{tt} = a^2 \left(u_{xx} + +u_{yy} + u_{zz}\right) + f(x,y,z,t)$$

3.3 heat equation

$$u_t = a^2 u_{xx}$$

$$u_t = a^2 \left(u_{xx} + u_{yy}\right)$$

$$u_t = a^2 \left(u_{xx} + +u_{yy} + u_{zz}\right)$$

3.4 heat equation with a source term

$$u_t = a^2 u_{xx} + f(x,t)$$

$$u_t = a^2 \left(u_{xx} + u_{yy}\right) + f(x,y,t)$$

$$u_t = a^2 \left(u_{xx} + +u_{yy} + u_{zz}\right) + f(x,y,z,t)$$

3.5 classification of second-order partial differential equations

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

$*$A, B, C, D, E, F and G are functions of x and y, but not of u.

$$\Delta = B^2 - AC$$

$\Delta > 0$(hyperbolic equation)

$$u_{xy} = [\cdots], u_{xx} - u_{yy} = [\cdots]$$

$\Delta = 0$(parabolic equation)

$$u_{xx}=[\cdots], u_{yy}=[\cdots]$$

$\Delta<0$(elliptic equation)

$$u_{xx}+u_{yy}=[\cdots]$$

$*$ $[\cdots]$ represents all terms that do not contain second-order partial derivatives.

3.6 simplification of second-order partial differential equations

$$Au_{xx}+2Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G$$

characteristic equation:

$$A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-2B\frac{\mathrm{d}y}{\mathrm{d}x}+C=0$$

characteristic lines are determined by the solutions of the characteristic equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{B\pm\sqrt{B^2-AC}}{A}$$

$$\begin{vmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y}\\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y}\end{vmatrix}\neq 0$$

$$\xi(x,y)=c_1,\eta(x,y)=c_2$$

$(1)\Delta=B^2-AC>0$

e.g. $u_{xx}-4u_{xy}+u_{yy}=0$

$$A=1,B=-2,C=1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-2\pm\sqrt{3}$$

$$y+\left(2\pm\sqrt{3}\right)x=c$$

$$\xi(x,y)=y+\left(2+\sqrt{3}\right)x,\eta(x,y)=y+\left(2-\sqrt{3}\right)x$$

$(2)\Delta=0$

Characteristic lines $\xi(x,y)=c_1$ are still governed by the characteristic equation, while $\eta(x,y)$ can be any function independent of $\xi(x,y)$, provided that the Jacobian determinant is not equal to zero.

$(3)\Delta<0$

e.g.$u_{xx}+4u_{xy}+5u_{yy}+u_x+u_y=0$

$$A=1,B=2,C=5$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2\pm i$$

$$2x-y\pm ix=c$$

$$\xi(x,y)=x,\eta(x,y)=2x-y$$

3.7 constant-coefficient equation

when $A,B,C\in\mathbb{R}$

(1)$\Delta>0$

$$\xi=y-\frac{B+\sqrt{B^2-AC}}{A}x,\eta=y-\frac{B-\sqrt{B^2-AC}}{A}x$$

(2)$\Delta=0$

$$\xi=y-\frac{B}{A}x,\eta=y$$

(3)$\Delta<0$

$$\xi=y-\frac{B}{A}x,\eta=\frac{\sqrt{AC-B^2}}{A}x$$

e.g.$u_{xx}-(A+B)u_{xy}+ABu_{yy}=0$

$$\xi=y+Ax,\eta=y+Bx$$

$$u_{xx}-(A+B)u_{xy}+ABu_{yy}=-(A-B)^2u_{\xi\eta}$$

第四章 Separation of Variables

4.1 string vibration equation

$$u_{tt} = a^2 u_{xx} (0 < x < l, t > 0)$$

$$u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x) (0 \le x \le l)$$

$$u|_{x=0} = 0, u|_{x=l} = 0 (t > 0)$$

assume the equation has a solution in the form of separated variables：

$$u(x,t) = X(x)T(t)$$

substitute into the equation:

$$X(x)T''(t) = a^2 X''(x)T(t)$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda$$

obtain ordinary differential equations for the spatial function and the temporal function

$$X''(x) + \lambda X(x) = 0$$

$$T''(t) + \lambda a^2 T(t) = 0$$

from the boundary conditions $X(0) = X(l) = 0$:

$(1)\lambda \le 0$

only the trivial solution

$(2)\lambda > 0$

$$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

since $X(0) = X(l) = 0$

$$c_1 = 0$$

$$c_2 = 1, \sqrt{\lambda} = \frac{k\pi}{l}, k = 1, 2, 3, ...$$

$$X(x) = \sin\frac{k\pi}{l}x$$

similarly:

$$T_k(t) = A_k \cos\frac{k\pi a}{l}t + B_k \sin\frac{k\pi a}{l}t$$

by the principle of superposition:

$$u(x,t)=\sum_{k=1}^{\infty}\left(A_k\cos\frac{k\pi a}{l}t+B_k\sin\frac{k\pi a}{l}t\right)\sin\frac{k\pi}{l}x$$
$$u_t(x,t)=\sum_{k=1}^{\infty}\left(-A_k\frac{k\pi a}{l}\sin\frac{k\pi a}{l}t+B_k\frac{k\pi a}{l}\cos\frac{k\pi a}{l}t\right)\sin\frac{k\pi}{l}x$$

since $u|_{t=0}=\varphi(x), u_t|_{t=0}=\psi(x)$

$$\varphi(x)=\sum_{k=1}^{\infty}A_k\sin\frac{k\pi}{l}x$$
$$\psi(x)=\sum_{k=1}^{\infty}A_k\frac{k\pi a}{l}\sin\frac{k\pi}{l}x$$
$$A_k=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$
$$A_k=\frac{2}{k\pi a}\int_0^l\psi(x)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$

conditions for the application of the method of separation of variables:

(1)The general equation must be linear.

(2)The general equation must be homogeneous.

(3)The boundary conditions must be homogeneous.

4.2 heat conduction equation

$$u_t=a^2u_{xx}(0<x<l,t>0)$$
$$u|_{t=0}=\varphi(x)(0\le x\le l)$$
$$u_x|_{x=0}=0,u_x|_{x=l}=0(t>0)$$

apply separation of variables:

$$X''(x)+\lambda X(x)=0,X'(0)=X'(l)=0$$
$$T'(t)+\lambda a^2T(t)=0$$

(1)$\lambda<0$

only the trival solution

(2)$\lambda=0$

$$X_0=1,\lambda_0=0$$

(2)$\lambda>0$

$$X_k=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$$
$$X_k'=-c_1\sqrt{\lambda}\sin\sqrt{\lambda}x+c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

since $X'(0)=X'(l)=0$

$$c_1=1,c_2=0$$

$$X_k = \cos\frac{k\pi}{l}x, \lambda_k = \left(\frac{k\pi}{l}\right)^2, k = 1, 2, ...$$

similarly:

$$T_0 = A_0, T_k = A_k e^{-\left(\frac{k\pi a}{l}\right)^2 x}$$

by the principle of superposition:

$$u(x,t) = A_0 + \sum_{k=1}^{\infty} A_k e^{-\left(\frac{k\pi a}{l}\right)^2 x} \cos\frac{k\pi}{l}x$$

since $u|_{t=0} = \varphi(x)$

$$A_0 = \frac{1}{l}\int_0^l \varphi(x)\,\mathrm{d}x$$
$$A_k = \frac{2}{l}\int_0^l \varphi(x)\cos\frac{k\pi}{l}x\,\mathrm{d}x$$