StudyThinking

卷二 数学物理方程笔记

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第一章 Fourier Transform

1.1 definition

$$F(\lambda)=\mathcal{F}[f(x)]=\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x$$

$$f(x)=\mathcal{F}^{-1}[f(x)]=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}\,\mathrm{d}\lambda$$

- $\mathcal{F}\left[e^{-a|x|}\right], a>0$

$$\mathcal{F}\left[e^{-a|x|}\right]=\int_{-\infty}^{\infty} e^{-a|x|}e^{-i\lambda x}\,\mathrm{d}x$$

$$=\int_{-\infty}^{0} e^{ax-i\lambda x}\,\mathrm{d}x+\int_{0}^{\infty} e^{-ax-i\lambda x}\,\mathrm{d}x$$

$$=\frac{1}{a-i\lambda}e^{(a-i\lambda)x}|_{-\infty}^{0}-\frac{1}{a+i\lambda}e^{-(a+i\lambda)x}|_{0}^{\infty}$$

$$=\frac{1}{a-i\lambda}+\frac{1}{a+i\lambda}$$

$$=\frac{2a}{a^2+\lambda^2}$$

$*\lim_{t\to\infty} e^{-(\beta+i\lambda)t}=0, \beta>0$

- $\mathcal{F}\left[e^{-ax^2}\right], a>0$

$$\mathcal{F}\left[e^{-ax^2}\right]=\int_{-\infty}^{\infty} e^{-ax^2}e^{-i\lambda x}\,\mathrm{d}x$$

$$=\int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\lambda}{2a}\right)^2-\frac{\lambda^2}{4a}}\,\mathrm{d}x$$

$$=e^{-\frac{\lambda^2}{4a}}\int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\lambda}{2a}\right)^2}\,\mathrm{d}x$$

$$=\sqrt{\frac{\pi}{a}}e^{-\frac{\lambda^2}{4a}}$$

$*$Gaussian integral: $\int_{-\infty}^{\infty} e^{-ax^2}\,\mathrm{d}x=\sqrt{\frac{\pi}{a}}$

- $\mathcal{F}\left[\cos\eta x^2\right], \eta>0$

$$\mathcal{F}\left[\cos\eta x^2\right]=\int_{-\infty}^{\infty}\cos\eta x^2 e^{-i\lambda x}\,\mathrm{d}x$$

$$=\int_{-\infty}^{\infty}\frac{e^{-i\eta x^2}+e^{i\eta x^2}}{2}e^{-i\lambda x}\,\mathrm{d}x$$

$$=\frac{1}{2}\left(\int_{-\infty}^{\infty} e^{i\eta\left(x-\frac{\lambda}{2\eta}\right)^2-i\frac{\lambda^2}{4\eta}}\,\mathrm{d}x+\int_{-\infty}^{\infty} e^{-i\eta\left(x+\frac{\lambda}{2\eta}\right)^2+i\frac{\lambda^2}{4\eta}}\,\mathrm{d}x\right)$$

$$=\frac{1}{2}e^{-i\frac{\lambda^2}{4\eta}}\sqrt{\frac{\pi}{\eta}}e^{i\frac{\pi}{4}}+\frac{1}{2}e^{i\frac{\lambda^2}{4\eta}}\sqrt{\frac{\pi}{\eta}}e^{-i\frac{\pi}{4}}$$

$$=\frac{1}{2}\sqrt{\frac{\pi}{\eta}}\left[\cos\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)-i\sin\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)+\cos\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)+i\sin\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)\right]$$

$$=\sqrt{\frac{\pi}{\eta}}\cos\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)$$

- $\mathcal{F}\left[\sin\eta x^2\right], \eta>0$

$$\mathcal{F}\left[\sin \eta x^2\right] = \sqrt{\frac{\pi}{\eta}} \sin\left(\frac{\lambda^2}{4\eta} + \frac{\pi}{4}\right)$$

$*$Euler's formula: $e^{ix} = \cos x + i \sin x$

$*$Gaussian-like integrals: $\int_{-\infty}^{\infty} e^{\pm i\eta x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{\eta}} e^{\pm i\frac{\pi}{4}}, \eta > 0$

1.2 properties

1.2.1 linear property

$$\mathcal{F}[\alpha f(x) + \beta g(x)] = \alpha \mathcal{F}[f(x)] + \beta \mathcal{F}[g(x)]$$

1.2.2 shift property

$$\mathcal{F}[f(x-b)] = e^{-i\lambda b} \mathcal{F}[f(x)]$$

1.2.3 differentiation property

$$\mathcal{F}[f'(x)] = i\lambda \mathcal{F}[f(x)]$$

$$\mathcal{F}\left[f^{(n)}(x)\right] = (i\lambda)^n \mathcal{F}[f(x)]$$

$$\mathcal{F}[x f(x)] = i \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{F}[f(x)]$$

$$\mathcal{F}[x^n f(x)] = i^n \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{F}[f(x)]$$

1.2.4 convolution property

$$\mathcal{F}[f(x) * g(x)] = \mathcal{F}[f(x)] \cdot \mathcal{F}[g(x)]$$

$$\mathcal{F}[f(x) \cdot g(x)] = \frac{1}{2\pi} \mathcal{F}[f(x)] * \mathcal{F}[g(x)]$$

第二章 Laplace Transform

2.1 definition

$$F(p) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t$$

- $\mathcal{L}[1]$

$$\mathcal{L}[1] = \int_0^\infty 1\cdot e^{-pt}\,\mathrm{d}t$$
$$= -\tfrac{1}{p}e^{-pt}|_0^\infty$$
$$= \tfrac{1}{p}$$

- $\mathcal{L}[t]$

$$\mathcal{L}[t] = \int_0^\infty te^{-pt}\,\mathrm{d}t$$
$$= -\tfrac{1}{p}te^{-pt}|_0^\infty + \tfrac{1}{p}\int_0^\infty e^{-pt}\,\mathrm{d}t$$
$$= \tfrac{1}{p^2}$$

- $\mathcal{L}[e^{at}]$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at}e^{-pt}\,\mathrm{d}t$$
$$= \int_0^\infty e^{-(p-a)t}\,\mathrm{d}t$$
$$= \tfrac{1}{p-a}$$

- $\mathcal{L}[\cos\omega t]$

$$\mathcal{L}[e^{at}] = \int_0^\infty \cos\omega t e^{-pt}\,\mathrm{d}t$$
$$= -\tfrac{1}{p}\cos\omega t e^{-pt}|_0^\infty - \tfrac{\omega}{p}\int_0^\infty \sin\omega t e^{-pt}\,\mathrm{d}t$$
$$= \tfrac{1}{p} + \tfrac{\omega}{p^2}\sin\omega t e^{-pt}|_0^\infty - \tfrac{\omega^2}{p^2}\mathcal{L}[\cos\omega t]$$
$$= \tfrac{p}{p^2+\omega^2}$$

- $\mathcal{L}[\sin\omega t]$

$$\mathcal{L}[e^{at}] = \int_0^\infty \sin\omega t e^{-pt}\,\mathrm{d}t$$
$$= \tfrac{1}{2i}\int_0^\infty (e^{i\omega t} - e^{-i\omega t})e^{-pt}\,\mathrm{d}t$$
$$= \tfrac{1}{2i}\int_0^\infty \left(e^{-(p-i\omega)t} - e^{-(p+i\omega)t}\right)\mathrm{d}t$$
$$= \tfrac{1}{2i}\left(\tfrac{1}{p-i\omega} - \tfrac{1}{p+i\omega}\right)$$
$$= \tfrac{\omega}{p^2+\omega^2}$$

- $\mathcal{L}[\cosh\omega t]$

$$\mathcal{L}[\cosh\omega t] = \int_0^\infty \cosh\omega t e^{-pt}\,\mathrm{d}t$$
$$= \frac{1}{2}\int_0^\infty \left(e^{-(p-\omega)t} + e^{-(p+\omega)t}\right)\mathrm{d}t$$
$$= \frac{1}{2}\left(\frac{1}{p-\omega} + \frac{1}{p+\omega}\right)$$
$$= \frac{p}{p^2-\omega^2}$$

- $\mathcal{L}[\sinh\omega t]$

$$\mathcal{L}[\sinh\omega t] = \frac{\omega}{p^2-\omega^2}$$

2.2 properties

2.2.1 linear property

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha\mathcal{L}[f(t)] + \beta\mathcal{L}[g(t)]$$

2.2.2 shift property

$$\mathcal{L}[e^{at}f(t)] = F(p-a)$$

2.2.3 differentiation property

$$\mathcal{L}[f'(t)] = p\mathcal{L}[f(t)] - f(0)$$
$$\mathcal{L}\left[f^{(n)}(t)\right] = p^n F(p) - p^{n-1}f(0) - \ldots - pf^{(n-2)}(0) - f^{(n-1)}(0)$$
$$\mathcal{L}[tf(t)] = -\frac{\mathrm{d}}{\mathrm{d}p}F(p)$$
$$\mathcal{L}[t^n f(t)] = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}p^n}F(p)$$

2.2.4 convolve property

$$f(t) * g(t) = \int_0^t f(t)g(t-\tau)\,\mathrm{d}\tau$$
$$\mathcal{L}[f(t) * g(t)] = F(p)\cdot G(p)$$

第三章 Fundamental Equations of Mathematical Physics

3.1 wave equation

$$u_{tt} = a^2 u_{xx}$$

$$u_{tt} = a^2 (u_{xx} + u_{yy})$$

$$u_{tt} = a^2 (u_{xx} + +u_{yy} + u_{zz})$$

3.2 wave equation with a source term

$$u_{tt} = a^2 u_{xx} + f(x, t)$$

$$u_{tt} = a^2 (u_{xx} + u_{yy}) + f(x, y, t)$$

$$u_{tt} = a^2 (u_{xx} + +u_{yy} + u_{zz}) + f(x, y, z, t)$$

3.3 heat equation

$$u_t = a^2 u_{xx}$$

$$u_t = a^2 (u_{xx} + u_{yy})$$

$$u_t = a^2 (u_{xx} + +u_{yy} + u_{zz})$$

3.4 heat equation with a source term

$$u_t = a^2 u_{xx} + f(x, t)$$

$$u_t = a^2 (u_{xx} + u_{yy}) + f(x, y, t)$$

$$u_t = a^2 (u_{xx} + +u_{yy} + u_{zz}) + f(x, y, z, t)$$

3.5 classification of second-order partial differential equations

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = g$$

$*$A, B, C, D, E, F and G are functions of x and y, but not of u.

$$\Delta = B^2 - AC$$

$\Delta > 0$(hyperbolic equation)

$$u_{xy} = [\cdots], u_{xx} - u_{yy} = [\cdots]$$

$\Delta = 0$(parabolic equation)

$$u_{xx} = [\cdots], u_{yy} = [\cdots]$$

$\Delta < 0$(elliptic equation)

$$u_{xx} + u_{yy} = [\cdots]$$

$*$ $[\cdots]$ represents all terms that do not contain second-order partial derivatives.

3.6 simplification of second-order partial differential equations