ExerciseThinking

卷二 数学物理方程习题解

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日期：2025/04/17

目　录

解答多有错误,仅作留档查看

第一章 典型方程与定解问题

1.1 判断方程类型

1.1.1

例：$x^2 u_{xx} - y^2 u_{yy} = 0$.

$$a = x^2, b = 0, c = -y^2$$

$$\Delta = b^2 - ac = x^2 y^2$$

若 x, y 其中一个为 0,则

$$\Delta = 0$$

方程为抛物型.

若 x, y 均不为 0,则

$$\Delta > 0$$

方程为双曲型.

1.1.2

例：$u_{xx} + xy u_{yy} = 0$.

$$a = 1, b = 0, c = xy$$

$$\Delta = b^2 - ac = -xy$$

若 x, y 其中一个为 0,则

$$\Delta = 0$$

方程为抛物型.

若 x, y 均不为 0 且异号,则

$$\Delta > 0$$

方程为双曲型.

若 x, y 均不为 0 且同号,则

$$\Delta < 0$$

方程为椭圆型.

1.2 化下列方程为标准形

1.2.1

例：$u_{xx} + 4u_{xy} + 5u_{yy} + u_x + u_y = 0$.

$$a = 1, b = 2, c = 5$$

$$\Delta = b^2 - ac = -1$$

有

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5 = 0$$

则

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \pm i$$

$$\mathrm{d}y = (2 \pm i)\,\mathrm{d}x$$

$$y = (2 \pm i)x + C$$

则

$$\varphi(x,y) = 2x - y \pm ix = C$$

取 $\xi = x, \eta = 2x - y$,有

$$J = \frac{D(\xi,\eta)}{D(x,y)} = \det(\xi_x, \xi_y; \eta_x, \eta_y) = -1 \neq 0$$

$$u_x = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x} = u_\xi + 2u_\eta$$

$$u_y = u_\xi \frac{\partial \xi}{\partial y} + u_\eta \frac{\partial \eta}{\partial y} = -u_\eta$$

$$u_{xx} = u_{\xi\xi} + 4u_{\eta\xi} + 4u_{\eta\eta}$$

$$u_{yy} = u_{\eta\eta}$$

$$u_{xy} = -u_{\eta\xi} - 2u_{\eta\eta}$$

代入得

$$u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\eta\eta} - 4u_{\xi\eta} - 8u_{\eta\eta} + 5u_{\eta\eta} + u_\xi + 2u_\eta - u_\eta = 0$$

化简得

$$u_{\xi\xi} + u_{\eta\eta} = -u_\xi - u_\eta$$

1.2.2

例: $u_{xx} - 4u_{xy} + u_{yy} = 0$.

$$a = 1, b = -2, c = 1$$

$$\Delta = b^2 - ac = 3$$

有

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 1 = 0$$

则

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2 \pm \sqrt{3}$$

$$\mathrm{d}y = \left(-2 \pm \sqrt{3}\right)\mathrm{d}x$$

$$y = \left(-2 \pm \sqrt{3}\right)x + C$$

则

$$\varphi(x,y) = y + \left(2 \pm \sqrt{3}\right)x = C$$

取 $\xi = y + \left(2 + \sqrt{3}\right)x, \eta = y + \left(2 - \sqrt{3}\right)x$,有

$$J = \frac{D(\xi,\eta)}{D(x,y)} = \det(\xi_x, \xi_y; \eta_x, \eta_y) = 2\sqrt{3} \neq 0$$

$$u_x = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x} = \left(2 + \sqrt{3}\right)u_\xi + \left(2 - \sqrt{3}\right)u_\eta$$

$$u_y = u_\xi \frac{\partial \xi}{\partial y} + u_\eta \frac{\partial \eta}{\partial y} = u_\xi + u_\eta$$

$$u_{xx} = \left(7 + 4\sqrt{3}\right)u_{\xi\xi} + 2u_{\eta\xi} + \left(7 - 4\sqrt{3}\right)u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}$$

$$u_{xy} = \left(2 + \sqrt{3}\right)u_{\xi\xi} + \left(2 - \sqrt{3}\right)u_{\eta\eta} + 4u_{\eta\xi}$$

代入化简得

$$\left(7 + 4\sqrt{3}\right)u_{\xi\xi} + 2u_{\eta\xi} + \left(7 - 4\sqrt{3}\right)u_{\eta\eta} + u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta} - 4\left[\left(2 + \sqrt{3}\right)u_{\xi\xi} + \left(2 - \sqrt{3}\right)u_{\eta\eta} + 4u_{\eta\xi}\right] = 0$$

即

$$u_{\xi\eta} = 0$$

1.3 求四种不同边值条件下对应的固有值问题的解

1.3.1

例：$T'' + \lambda a^2 T = 0, X'' + \lambda X = 0, X(0) = 0, X(l) = 0$.

当 $\lambda < 0$,

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

由 $X(0) = 0, X(l) = 0$ 得

$$c_1 + c_2 = 0, c_1 e^{\sqrt{-\lambda}l} + c_2 e^{-\sqrt{-\lambda}l} = 0$$

解得

$$c_1 = c_2 = 0$$

当 $\lambda = 0$,

$$X(x) = c_1 x + c_2$$

由 $X(0) = 0, X(l) = 0$ 得

$$c_2 = 0, c_1 l + c_2 = 0$$

解得

$$c_1 = c_2 = 0$$

当 $\lambda > 0$,

$$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

由 $X(0) = 0, X(l) = 0$ 得

$$c_1 = 0, c_2 \sin\sqrt{\lambda}l = 0$$

避免平凡解后解得

$$c_1 = 0, c_2 = 1, \lambda_k = \left(\frac{k\pi}{l}\right)^2, k = 1, 2, \ldots$$

将 $\lambda_k = \left(\frac{k\pi}{l}\right)^2$ 代入 $X(x)$ 得

$$X_k(x) = \sin\frac{k\pi}{l}x$$

同样的有

$$T_k''(t) + \left(\frac{k\pi}{l}a\right)^2 T_k(t) = 0$$

由于 $\left(\frac{k\pi}{l}a\right)^2 > 0$,则

$$T_k(x) = A_k \cos\frac{k\pi}{l}at + B_k \sin\frac{k\pi}{l}at$$

$$u_k(x, t) = \left(A_k \cos\frac{k\pi}{l}at + B_k \sin\frac{k\pi}{l}at\right)\sin\frac{k\pi}{l}x$$

由线性叠加原理得

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_k \cos \frac{k\pi}{l} at + B_k \sin \frac{k\pi}{l} at\right) \sin \frac{k\pi}{l} x$$

其中

$$A_k = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{k\pi}{l} x \, \mathrm{d}x$$

$$B_k = \frac{2}{k\pi a} \int_0^l \psi(x) \sin \frac{k\pi}{l} x \, \mathrm{d}x$$

1.3.2

例: $T'' + \lambda a^2 T = 0, X'' + \lambda X = 0, X'(0) = 0, X'(l) = 0$.

当 $\lambda < 0$,

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

$$X'(x) = c_1 \sqrt{-\lambda} e^{\sqrt{-\lambda}x} - c_2 \sqrt{-\lambda} e^{-\sqrt{-\lambda}x}$$

由 $X'(0) = 0, X'(l) = 0$ 得

$$c_1 \sqrt{-\lambda} - c_2 \sqrt{-\lambda} = 0, c_1 \sqrt{-\lambda} e^{\sqrt{-\lambda}l} - c_2 \sqrt{-\lambda} e^{-\sqrt{-\lambda}l} = 0$$

解得

$$c_1 = c_2 = 0$$

当 $\lambda = 0$,

$$X(x) = c_1 x + c_2$$

$$X'(x) = c_1$$

由 $X'(0) = 0, X'(l) = 0$ 得

$$c_1 = 0$$

解得

$$X(x) = 1$$

当 $\lambda > 0$,

$$X(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$X'(x) = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} x + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} x$$

由 $X'(0) = 0, X'(l) = 0$ 得

$$c_2 = 0, -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} l = 0$$

避免平凡解后解得

$$c_1 = 1, c_2 = 0, \lambda_k = \left(\frac{k\pi}{l}\right)^2, k = 1, 2, \ldots$$

将 $\lambda_k = \left(\frac{k\pi}{l}\right)^2$ 代入 $X(x)$ 得

$$X_k(x) = \cos \frac{k\pi}{l} x$$

将 $\lambda_k = \left(\frac{k\pi}{l}\right)^2, k = 0, 1, 2, \ldots$ 代入 $T'' + \lambda a^2 T = 0$ 得

$$T_0(x) = A_0 + B_0 t, k = 0$$

$$T_k(x) = A_k \cos \frac{k\pi}{l} at + B_k \sin \frac{k\pi}{l} at, k = 1, 2, \ldots$$

由线性叠加原理得

$$u(x,t) = A_0 + B_0 t + \sum_{k=1}^{\infty} \left(A_k \cos \frac{k\pi}{l} at + B_k \sin \frac{k\pi}{l} at\right) \cos \frac{k\pi}{l} x$$

其中

$$A_0 = \frac{1}{l} \int_0^l \varphi(x)\,\mathrm{d}x$$

$$B_0 = \frac{1}{l} \int_0^l \psi(x)\,\mathrm{d}x$$

$$A_k = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{k\pi}{l} x\,\mathrm{d}x$$

$$B_k = \frac{2}{k\pi a} \int_0^l \psi(x) \cos \frac{k\pi}{l} x\,\mathrm{d}x$$

可以改写为

$$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{k=1}^{\infty} \left(A_k \cos \frac{k\pi}{l} at + B_k \sin \frac{k\pi}{l} at \right) \cos \frac{k\pi}{l} x$$

$$A_k = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{k\pi}{l} x\,\mathrm{d}x, k = 0, 1, 2, \dots$$

$$B_k = \frac{2}{k\pi a} \int_0^l \psi(x) \cos \frac{k\pi}{l} x\,\mathrm{d}x, k = 1, 2, \dots$$

$$B_0 = \frac{2}{l} \int_0^l \psi(x)\,\mathrm{d}x$$

1.3.3

例：$X'' + \lambda X = 0, X'(0) = 0, X(l) = 0.$

当 $\lambda < 0,$

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

$$X'(x) = c_1 \sqrt{-\lambda} e^{\sqrt{-\lambda}x} - c_2 \sqrt{-\lambda} e^{-\sqrt{-\lambda}x}$$

由 $X'(0) = 0, X(l) = 0$ 得

$$c_1 \sqrt{-\lambda} - c_2 \sqrt{-\lambda} = 0, c_1 e^{\sqrt{-\lambda}l} + c_2 e^{-\sqrt{-\lambda}l} = 0$$

解得

$$c_1 = c_2 = 0$$

当 $\lambda = 0,$

$$X(x) = c_1 x + c_2$$

$$X'(x) = c_1$$

由 $X'(0) = 0, X(l) = 0$ 得

$$c_1 = 0, c_1 l + c_2 = 0$$

解得

$$c_1 = c_2 = 0$$

当 $\lambda > 0,$

$$X(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$$

$$X'(x) = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda}x + c_2 \sqrt{\lambda} \cos \sqrt{\lambda}x$$

由 $X'(0) = 0, X(l) = 0$ 得

$$c_2 = 0, c_1 \cos \sqrt{\lambda}l = 0$$

避免平凡解后解得

$$c_1 = 1, c_2 = 0, \lambda_k = \left(\frac{(2k-1)\pi}{2l} \right)^2, k = 1, 2, \dots$$

将 $\lambda_k = \left(\frac{(2k-1)\pi}{2l} \right)^2$ 代入 $X(x)$ 得

$$X_k(x) = \cos \frac{(2k-1)\pi}{2l} x$$

将 $\lambda_k = \left(\frac{k\pi}{l}\right)^2, k = 0, 1, 2, \dots$ 代入 $T'' + \lambda a^2 T = 0$ 得

$$T_k(x) = A_k \cos \frac{(2k-1)\pi}{2l} at + B_k \sin \frac{(2k-1)\pi}{2l} at, k = 1, 2, \dots$$

由线性叠加原理得

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_k \frac{(2k-1)\pi}{2l} at + B_k \sin \frac{(2k-1)\pi}{2l} at\right) \cos \frac{(2k-1)\pi}{2l} x$$

其中

$$A_k = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{(2k-1)\pi}{2l} x \, \mathrm{d}x$$

$$B_k = \frac{4}{(2k-1)\pi a} \int_0^l \psi(x) \cos \frac{(2k-1)\pi}{2l} x \, \mathrm{d}x$$

1.3.4

例：$X'' + \lambda X = 0, X(0) = 0, X'(l) = 0$.

当 $\lambda < 0$,

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

$$X'(x) = c_1 \sqrt{-\lambda} e^{\sqrt{-\lambda}x} - c_2 \sqrt{-\lambda} e^{-\sqrt{-\lambda}x}$$

由 $X(0) = 0, X'(l) = 0$ 得

$$c_1 + c_2 = 0, c_1 \sqrt{-\lambda} e^{\sqrt{-\lambda}l} - c_2 \sqrt{-\lambda} e^{-\sqrt{-\lambda}l} = 0$$

解得

$$c_1 = c_2 = 0$$

当 $\lambda = 0$,

$$X(x) = c_1 x + c_2$$

$$X'(x) = c_1$$

由 $X(0) = 0, X'(l) = 0$ 得

$$c_1 = c_2 = 0$$

当 $\lambda > 0$,

$$X(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$X'(x) = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} x + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} x$$

由 $X(0) = 0, X'(l) = 0$ 得

$$c_1 = 0, c_2 \sqrt{\lambda} \cos \sqrt{\lambda} l = 0$$

避免平凡解后解得

$$c_1 = 0, c_2 = 1, \lambda_k = \left(\frac{(2k-1)\pi}{2l}\right)^2, k = 1, 2, \dots$$

将 $\lambda_k = \left(\frac{(2k-1)\pi}{2l}\right)^2$ 代入 $X(x)$ 得

$$X_k(x) = \sin \frac{(2k-1)\pi}{2l} x$$

将 $\lambda_k = \left(\frac{k\pi}{l}\right)^2, k = 0, 1, 2, \dots$ 代入 $T'' + \lambda a^2 T = 0$ 得

$$T_k(x) = A_k \cos \frac{(2k-1)\pi}{2l} at + B_k \sin \frac{(2k-1)\pi}{2l} at, k = 1, 2, \dots$$

由线性叠加原理得

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_k \frac{(2k-1)\pi}{2l} at + B_k \sin \frac{(2k-1)\pi}{2l} at \right) \sin \frac{(2k-1)\pi}{2l} x$$

其中

$$A_k = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{(2k-1)\pi}{2l} x \, \mathrm{d}x$$

$$B_k = \frac{4}{(2k-1)\pi a} \int_0^l \psi(x) \sin \frac{(2k-1)\pi}{2l} x \, \mathrm{d}x$$

1.4 解下列定解问题

1.4.1

例：$\begin{cases} u_t = a^2 u_{xx} \\ u|_{t=0} = \varphi(x) \\ u_x|_{x=0} = u_x|_{x=l} = 0 \end{cases}$

令 $u(x,t) = X(x)T(t)$

有

$$X'' + \lambda X = 0, X'(0) = X'(l) = 0$$

$$T' + a^2 \lambda T = 0$$

$\lambda < 0$,无非零解

$\lambda = 0$,解得 $X_0 = A_0$

$\lambda > 0$,

$$X = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

由 $X'(0) = X'(l) = 0$

$$\lambda_k = \left(\frac{k\pi}{l} \right)^2$$

则

$$X_k(x) = A_k \cos \frac{k\pi}{l} x$$

$\lambda = 0$,解得 $T_0 = B_0$, $u_0 = X_0 T_0 = A_0 B_0 = C_0$

$\lambda > 0$

$$T_{k'} + \frac{a^2 k^2 \pi^2}{l^2} T_k = 0$$

$$T_k(t) = B_k e^{-\frac{a^2 k^2 \pi^2}{l^2} t}$$

$$u_k = X_k T_k = C_k e^{-\frac{a^2 k^2 \pi^2}{l^2} t} \cos \frac{k\pi}{l} x$$

由叠加原理

$$u(x,t) = C_0 + \sum_{k=1}^{\infty} C_k e^{-\frac{a^2 k^2 \pi^2}{l^2} t} \cos \frac{k\pi}{l} x$$

由 $u(x,0) = \varphi(x) = C_0 + \sum_{k=1}^{\infty} C_k \cos \frac{k\pi}{l} x$

$$C_0 = \frac{1}{l} \int_0^l \varphi(x) \, \mathrm{d}x$$

$$C_k = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{k\pi}{l} x \, \mathrm{d}x$$

1.4.2

例：$\begin{cases} u_{tt} = a^2 u_{xx} \text{ if } 0<x<1 \mid t>0 \\ u|_{t=0} = \begin{cases} x \text{ if } 0 \leq x \leq \frac{1}{2} \\ -x \text{ if } \frac{1}{2} \leq x \leq 1 \end{cases} \\ u_t|_{t=0} = x(1-x) \text{ if } 0 \leq x \leq 1 \\ u|_{x=0} = u|_{x=1} = 0 \text{ if } t \geq 0 \end{cases}$

令 $u(x,t) = X(x)T(t)$

$$X'' + \lambda X = 0$$

$$T'' + a^2 \lambda T = 0$$

$$X(0)T(t) = X(1)T(t) = 0$$

$\lambda \le 0$,只有平凡零解

$\lambda > 0$

$$\lambda_k = (k\pi)^2$$

$$X_k(x) = \sin k\pi x$$

$$T_k(t) = A_k \cos k\pi at + B_k \sin k\pi at$$

$$u(x,t) = \sum_{k=1}^{\infty} (A_k \cos k\pi at + B_k \sin k\pi at) \sin \frac{k\pi}{l}x$$

其中

$$A_k = 2\left(\int_0^{\frac{1}{2}} x \sin k\pi x \,\mathrm{d}x - \int_{\frac{1}{2}}^1 x \sin k\pi x \,\mathrm{d}x\right)$$

$$= \frac{2}{k\pi}\left(-\int_0^{\frac{1}{2}} x \,\mathrm{d}\cos k\pi x + \int_{\frac{1}{2}}^1 x \,\mathrm{d}\cos k\pi x\right)$$

$$= \frac{2}{k\pi}\left[\int_0^{\frac{1}{2}} \cos k\pi x \,\mathrm{d}x - \int_{\frac{1}{2}}^1 \cos k\pi x \,\mathrm{d}x + (-1)^k\right]$$

$$= \frac{2}{k\pi}\left[\frac{4}{k\pi}(-1)^{k+1} + (-1)^k\right]$$

$$= \frac{2}{k\pi}(-1)^k\left(1 - \frac{4}{k\pi}\right)$$

$$B_k = \frac{2}{k\pi a}\int_0^1 x(1-x) \sin k\pi x \,\mathrm{d}x$$

$$= -\frac{2}{k^2\pi^2 a}\int_0^1 x(1-x) \,\mathrm{d}\cos k\pi x$$

$$= \frac{2}{k^2\pi^2 a}\int_0^1 (1-2x) \cos k\pi x \,\mathrm{d}x$$

$$= -\frac{4}{k^3\pi^3 a}\int_0^1 x \,\mathrm{d}\sin k\pi x$$

$$= \frac{4}{k^4\pi^4 a}\left[1 - (-1)^k\right]$$

1.5 相容性条件

1.5.1

例：$\begin{cases} u_{tt} = a^2 u_{xx} \text{ if } 0 < x < l \mid t > 0 \\ u|_{t=0} = \varphi(x). u_t|_{t=0} = \psi(x) \text{ if } 0 \le x \le l \\ u_x|_{x=0} = u_x|_{x=1} = 0 \text{ if } t \ge 0 \end{cases}$

答：进行分离变量后得到固有值问题

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X'(l) = 0 \end{cases}$$

由例题 1.1.2 可知

$$u_0(x,t) = \frac{1}{2}(A_0 + B_0 t)$$

$$u_k(x,t) = \left(A_k \cos \frac{k\pi}{l}at + B_k \sin \frac{k\pi}{l}at\right) \cos \frac{k\pi}{l}x, k = 1,2,...$$

$$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{k=1}^{\infty}\left(A_k \cos \frac{k\pi}{l}at + B_k \sin \frac{k\pi}{l}at\right) \cos \frac{k\pi}{l}x$$

其中

$$A_k = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{k\pi}{l} x \, \mathrm{d}x, k = 0, 1, 2, \ldots$$

$$B_k = \frac{2}{k\pi a} \int_0^l \psi(x) \cos \frac{k\pi}{l} x \, \mathrm{d}x, k = 1, 2, \ldots$$

$$B_0 = \frac{2}{l} \int_0^l \psi(x) \, \mathrm{d}x$$

则解的存在性条件为：

$$\sum_{k=1}^\infty u_k(x,t) = \sum_{k=1}^\infty \left(A_k \cos \frac{k\pi}{l} at + B_k \sin \frac{k\pi}{l} at \right) \cos \frac{k\pi}{l} x$$

$$\sum_{k=1}^\infty (u_k)_t, \sum_{k=1}^\infty (u_k)_{tt}, \sum_{k=1}^\infty (u_k)_x, \sum_{k=1}^\infty (u_k)_{xx}$$

一致收敛，其优级数为

$$\sum_{k=1}^\infty k^m (|A_k| + |B_k|), m = 0, 1, 2$$

若 $\varphi(x) \in C^2, \psi(x) \in C^1, \varphi^{(3)}(x)$ 与 $\psi''(x)$ 分段连续,且 $\varphi(0) = \psi(0) = 0, \varphi''(0) = \varphi''(l) = 0, \psi(0) = \psi(l) = 0$，根据引理 3.1,上述优级数收敛,则例题 3.1 的解可以表示为形如 $u(x,t)$ 的级数.

第二章 分离变量法

2.1 解定解问题

2.1.1

例：$u_{xx} + u_{yy} = 0, 0 < x < a, 0 < y < b$

$u|_{x=0} = u|_{x=a} = 0, 0 \le y \le b$

$u_y|_{y=0} = f(x), u|_{y=b} = 0, 0 \le x \le a$

$$u(x,y) = X(x)Y(y)$$

将其代入分离变量可得固有值问题

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(a) = 0 \end{cases}$$

当 $\lambda \le 0$，无非零解

当 $\lambda > 0$，有

$$\lambda_k = \left(\frac{k\pi}{a}\right)^2, X(x) = \sin\frac{k\pi}{a}x, k = 1, 2, \ldots$$

将 λ_k 代入关于 Y 的方程

$$Y''(y) - \left(\frac{k\pi}{a}\right)^2 Y(y) = 0$$

解得

$$Y(y) = A_k ch\frac{k\pi}{a}y + B_k sh\frac{k\pi}{a}y$$

则

$$u(x,y) = \sum_1^\infty \left(A_k ch\frac{k\pi}{a}y + B_k sh\frac{k\pi}{a}y\right)\sin\frac{k\pi}{a}x$$

由边界条件得

$$f(x) = \sum_1^\infty B_k \frac{k\pi}{a}\sin\frac{k\pi}{a}x$$

$$\sum_1^\infty \left(A_k ch\frac{k\pi b}{a} + B_k sh\frac{k\pi b}{a}\right)\sin\frac{k\pi}{a}x = 0$$

其中

$$A_k = \frac{2}{k\pi}\int_0^a f(x)\sin\frac{k\pi}{a}x\,\mathrm{d}x, k = 1, 2, \ldots$$

相应的

$$B_k = -\frac{2ch\frac{k\pi b}{a}}{k\pi sh\frac{k\pi b}{a}}\int_0^l \varphi(x)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$

2.1.2

例：$u_{xx} + u_{yy} = f(x), 0 < x < a, 0 < y < b$

$u|_{x=0} = A, u|_{x=a} = B, 0 \le y \le b$

$u_y|_{y=0} = g(x), u|_{y=b} = 0, 0 \le x \le a$

$$u(x,y) = v(x,y) + w(x,y)$$

取

$$w(x,y) = \frac{B-A}{a}x + A$$

则有

$$v_{xx} + v_{yy} = f(x,y)$$

$$v|_{x=0} = v_t|_{x=a} = 0$$

$$v_y|_{y=0} = g_1(x), v|_{y=b} = g_2(x)$$

其中

$$f(x,y) = f(x) + \frac{A-B}{a}x - A$$

$$g_1(x) = g(x)$$

$$g_2(x) = \frac{A-B}{a}x - A$$

设 $v(x,y) = \sum_1^\infty v_k(y)\sin\frac{k\pi}{a}x$ 得

$$v_k''(y) - \left(\frac{k\pi}{a}\right)^2 v_{k(y)} = f_k(y)$$

$$v_k'(0) = g_{1k}(x), v_k(b) = g_{2k}(x)$$

解得

$$v_k(y) = A_k ch\frac{k\pi}{a}y + B_k sh\frac{k\pi}{a}y + \frac{a}{k\pi}\int_0^y f_{k(\tau)} sh\frac{k\pi}{a}(y-\tau)\,\mathrm{d}\tau$$

则

$$v(x,y) = \sum_1^\infty \left[A_k ch\frac{k\pi}{a}y + B_k sh\frac{k\pi}{a}y + \frac{a}{k\pi}\int_0^y f_{k(\tau)} sh\frac{k\pi}{a}(y-\tau)\,\mathrm{d}\tau\right]\sin\frac{k\pi}{a}x$$

整理得

$$u(x,y) = \sum_1^\infty \left[A_k ch\frac{k\pi}{a}y + B_k sh\frac{k\pi}{a}y + \frac{a}{k\pi}\int_0^y f_{k(\tau)} sh\frac{k\pi}{a}(y-\tau)\,\mathrm{d}\tau\right]\sin\frac{k\pi}{a}x$$
$$+ \frac{B-A}{a}x + A$$

其中 A_k, B_k 或许老天知道

2.2 解定解方程

2.2.1

例：
$$\begin{cases} u_t = a^2 u_{xx} & \text{if } 0 < x < l. t > 0 \\ u|_{t=0} = 0 & \text{if } 0 \le x \le l \\ u_x|_{x=0} = \mu_1(x). u_x|_{x=l} = \mu_2(x) & \text{if } t \ge 0 \end{cases}$$

答：令

$$u(x,t) = v(x,t) + w(x,t)$$

将其代入边值条件

$$\begin{cases} \mu_1(t) = u_x|_{x=0} = v_x|_{x=0} + w_x|_{x=0} \\ \mu_2(t) = u_x|_{x=l} = v_x|_{x=l} + w_x|_{x=l} \end{cases}$$

为使 $v_x|_{x=0} = v_x|_{x=l} = 0$,必须有

$$w_x|_{x=0} = \mu_1(t), w_x|_{x=l} = \mu_2(t)$$

取

$$w_x(x,t) = \frac{l-x}{l}\mu_1(t) + \frac{x}{l}\mu_2(t)$$

$$w(x,t) = -\frac{(l-x)^2}{2l}\mu_1(t) + \frac{x^2}{2l}\mu_2(t)$$

此时 $v(x,t)$ 满足以下定解问题

$$\begin{cases} v_t = a^2 v_{xx} + f(x,t) & \text{if } 0 < x < l. t > 0 \\ v|_{t=0} = \varphi(x) & \text{if } 0 \le x \le l \\ u_x|_{x=0} = u_x|_{x=l} = 0 & \text{if } t \ge 0 \end{cases}$$

其中

$$f(x,t) = -\frac{a^2}{l}\mu_1(t) + \frac{a^2}{l}\mu_2(t) + \frac{(l-x)^2}{2l}\mu_1'(t) - \frac{x^2}{2l}\mu_2'(t)$$

$$\varphi(x) = \frac{(l-x)^2}{2l}\mu_1(0) - \frac{x^2}{2l}\mu_2(0)$$

不妨令

$$v(x,t) = \sum_{k=1}^{\infty} v_k(t)\sin\frac{k\pi}{l}x$$

代入定解方程有

$$\sum_{k=1}^{\infty} v_k'(t)\sin\frac{k\pi}{l}x = -a^2\sum_{k=1}^{\infty}\left(\frac{k\pi}{l}\right)^2 v_k(t)\sin\frac{k\pi}{l}x$$
$$+\sum_{k=1}^{\infty} f_k(t)\sin\frac{k\pi}{l}x$$

$$\sum_{k=1}^{\infty} v_k(0)\sin\frac{k\pi}{l}x = +\sum_{k=1}^{\infty}\varphi_k\sin\frac{k\pi}{l}x$$

其中

$$f_k(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$

$$\varphi_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$

解得

$$v_k(t) = \varphi_k e^{-\left(\frac{k\pi a}{l}\right)^2 t} + \int_0^t f_k(\tau)e^{-\left(\frac{k\pi a}{l}\right)^2(t-\tau)}\,\mathrm{d}\tau$$

则

$$v(x,t) = \sum_{k=1}^{\infty}\left[\varphi_k e^{-\left(\frac{k\pi a}{l}\right)^2 t} + \int_0^t f_k(\tau)e^{-\left(\frac{k\pi a}{l}\right)^2(t-\tau)}\,\mathrm{d}\tau\right]\sin\frac{k\pi}{l}x$$

相应的

$$u(x,t) = \sum_{k=1}^{\infty}\left[\varphi_k e^{-\left(\frac{k\pi a}{l}\right)^2 t} + \int_0^t f_k(\tau)e^{-\left(\frac{k\pi a}{l}\right)^2(t-\tau)}\,\mathrm{d}\tau\right]\sin\frac{k\pi}{l}x$$
$$-\frac{(l-x)^2}{2l}\mu_1(t) + \frac{x^2}{2l}\mu_2(t)$$

2.2.2

例：$\begin{cases} u_{tt}=a^2 u_{xx}+bu \text{ if} 0<x<l.t>0 \\ u|_{t=0}=u_t|_{t=0}=0 \text{ if} 0\leq x\leq l \\ u|_{x=0}=\mu_1(x).u|_{x=l}=\mu_2(x) \text{ if } t\geq 0 \end{cases}$.

答：令

$$u(x,t) = v(x,t) + w(x,t)$$

取

$$w(x,t) = \frac{l-x}{l}\mu_1(t) + \frac{x}{l}\mu_2(t)$$

则有

$$v_{tt} = a^2 v_{xx} + bv + f(x,t)$$

$$v|_{t=0} = \varphi(x), v_t|_{t=0} = \psi(x)$$

$$v|_{x=0} = v|_{x=l} = 0$$

其中

$$f(x,t) = \frac{l-x}{l}\mu_1(t) + \frac{x}{l}\mu_2(t) - \frac{l-x}{l}\mu_1''(t) - \frac{x}{l}\mu_2''(t)$$

$$\varphi(x) = -\frac{l-x}{l}\mu_1(0) - \frac{x}{l}\mu_2(0)$$

$$\psi(x) = -\frac{l-x}{l}\mu_1'(0) - \frac{x}{l}\mu_2'(0)$$

先考虑排除 $f(x,t)$ 的齐次方程,令 $v_1(x,t) = X(x)T(t)$ 分离变量得

$$\frac{T''}{a^2 T} = \frac{X''}{x} + \frac{b}{a^2} = -\lambda$$

解得

$$\lambda_k + \frac{b}{a^2} = \left(\frac{k\pi}{l}\right)^2, k = 1, 2, \ldots$$

$$X_k(x) = \sin\frac{k\pi}{l}x$$

$$T_k(t) = A_k\cos\omega_k x + B_k\sin\omega_k x$$

其中

$$\omega_k = \sqrt{\left(\frac{k\pi a}{l}\right)^2 - b}$$

$$A_k = \frac{2}{l}\int_0^l \varphi(x)\sin\omega_k x\,\mathrm{d}x$$

$$B_k = \frac{2}{w_k l}\int_0^l \psi(x)\sin\omega_k x\,\mathrm{d}x$$

$$v_1(x,t) = \sum_{k=1}^{\infty}(A_k\cos\omega_k x + B_k\sin\omega_k x)\sin\frac{k\pi}{l}x$$

由齐次化原理处理 $f(x,t)$ 得

$$v_2(x,t) = \sum_{k=1}^{\infty}\frac{2}{\omega_k l}\int_0^t\int_0^l f(\xi,\tau)\sin\frac{k\pi}{l}\xi\sin\omega_k(t-\tau)\,\mathrm{d}\xi\,\mathrm{d}x\sin\frac{k\pi}{l}x$$

$$v(x,t) = v_1(x,t) + v(x,t)$$

$$= \sum_{k=1}^{\infty}\left(A_k\cos\omega_k x + B_k\sin\omega_k x + \frac{2}{\omega_k l}\int_0^t\int_0^l f(\xi,\tau)\sin\frac{k\pi}{l}\xi\sin\omega_k(t-\tau)\,\mathrm{d}\xi\,\mathrm{d}x\right)\sin\frac{k\pi}{l}x$$

最后

$$u(x,t) = v(x,t) + w(x,t)$$

$$= \sum_{k=1}^{\infty}\left(A_k\cos\omega_k x + B_k\sin\omega_k x + \frac{2}{\omega_k l}\int_0^t\int_0^l f(\xi,\tau)\sin\frac{k\pi}{l}\xi\sin\omega_k(t-\tau)\,\mathrm{d}\xi\,\mathrm{d}x\right)\sin\frac{k\pi}{l}x$$

$$+\frac{l-x}{l}\mu_1(t) + \frac{x}{l}\mu_2(t)$$

第三章 积分变换法

3.1 求下列函数的 Fourier 变换

3.1.1

例：$f(x) = \begin{cases} |x| \text{ if } x \leq a \\ 0 \text{ if } t > 0 \end{cases}$.

答：

$$\mathcal{F}[f(x)] = \int_{-\infty}^{+\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x$$

$$= \int_{-a}^{a} |x|e^{-i\lambda x}\,\mathrm{d}x$$

$$= \int_{0}^{a} xe^{-i\lambda x}\,\mathrm{d}x + \int_{-a}^{0} -xe^{-i\lambda x}\,\mathrm{d}x$$

$$= \int_{0}^{a} x\left(e^{-i\lambda x} + e^{i\lambda x}\right)\,\mathrm{d}x$$

$$= 2\int_{0}^{a} x\cos\lambda x\,\mathrm{d}x$$

$$= \frac{2a\sin\lambda a}{\lambda} + \frac{2(\cos\lambda a - 1)}{\lambda^2}$$

3.1.2

例：$f(x) = \cos\eta x^2$.

答：

$$\mathcal{F}[f(x)] = \int_{-\infty}^{+\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x$$

$$= \int_{-\infty}^{+\infty} \cos\eta x^2 e^{-i\lambda x}\,\mathrm{d}x$$

$$= \frac{1}{2}\left(\int_{-\infty}^{+\infty} e^{-i\lambda x - i\eta x^2}\,\mathrm{d}x + \int_{-\infty}^{+\infty} e^{-i\lambda x + i\eta x^2}\,\mathrm{d}x\right)$$

$$= \frac{1}{2}\left(\int_{-\infty}^{+\infty} e^{-i\eta\left(x + \frac{\lambda}{2\eta}\right)^2 + i\frac{\lambda^2}{4\eta}}\,\mathrm{d}x + \int_{-\infty}^{+\infty} e^{i\eta\left(x - \frac{\lambda}{2\eta}\right)^2 - i\frac{\lambda^2}{4\eta}}\,\mathrm{d}x\right)$$

$$= \frac{1}{2}\left(e^{i\frac{\lambda^2}{4\eta}}\int_{-\infty}^{+\infty} e^{-i\eta x^2}\,\mathrm{d}x + e^{-i\frac{\lambda^2}{4\eta}}\int_{-\infty}^{+\infty} e^{i\eta x^2}\,\mathrm{d}x\right)$$

$$= \frac{1}{2}\left(e^{i\frac{\lambda^2}{4\eta}}\sqrt{\frac{\pi}{i\eta}} + e^{-i\frac{\lambda^2}{4\eta}}\sqrt{-\frac{\pi}{i\eta}}\right)$$

其中

$$\sqrt{\frac{\pi}{i\eta}} = \sqrt{\frac{\pi}{e^{-i\frac{\pi}{2}}\eta}} = \sqrt{\frac{\pi}{\eta}}e^{i\frac{\pi}{4}}$$

$$\sqrt{-\frac{\pi}{i\eta}} = \sqrt{\frac{\pi}{e^{i\frac{\pi}{2}}\eta}} = \sqrt{\frac{\pi}{\eta}}e^{-i\frac{\pi}{4}}$$

$$\mathcal{F}[f(x)] = \frac{1}{2}\sqrt{\frac{\pi}{\eta}}\left(e^{i\frac{\lambda^2}{4\eta} + i\frac{\pi}{4}} + e^{-i\frac{\lambda^2}{4\eta} - i\frac{\pi}{4}}\right)$$

$$\mathcal{F}[f(x)] = \frac{\sqrt{2}}{2}\sqrt{\frac{\pi}{i\eta}}(1 + i)\sin\left(\frac{\lambda^2}{4\eta} + \frac{\pi}{4}\right)$$

3.1.3

例：$f(x) = \sin\eta x^2$.

答：

$$\mathcal{F}[f(x)] = \int_{-\infty}^{+\infty} f(x) e^{-i\lambda x}\, \mathrm{d}x$$

$$= \int_{-\infty}^{+\infty} \sin \eta x^2 e^{-i\lambda x}\, \mathrm{d}x$$

$$= \frac{1}{2i} \left(\int_{-\infty}^{+\infty} e^{-i\lambda x + i\eta x^2}\, \mathrm{d}x - \int_{-\infty}^{+\infty} e^{-i\lambda x - i\eta x^2}\, \mathrm{d}x \right)$$

$$= \frac{\sqrt{2}}{2} \sqrt{\frac{\pi}{i\eta}} (i-1) \sin\left(\frac{\lambda^2}{4\eta} - \frac{\pi}{4} \right)$$

3.2 利用 Fourier 变换的性质求下列函数的 Fourier 变换

3.2.1

例：$f(x) = xe^{-a|x|}$.

答: 令 $g(x) = e^{-a|x|}$，易知 $\mathcal{F}[g(x)] = \frac{2a}{a^2 + \lambda^2}$

$$\mathcal{F}[f(x)] = \mathcal{F}[xg(x)] = i \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{F}[g(x)]$$

$$\mathcal{F}[f(x)] = -\frac{4a\lambda}{(a^2 + \lambda^2)^2}$$

3.2.2

例：$f(x) = e^{-ax^2 + ibx + c}$.

答: 令 $g(x) = e^{-ax^2}$，易知 $\mathcal{F}[g(x)] = \sqrt{\frac{\pi}{a}} e^{-\frac{\lambda^2}{4a}}$

$$\mathcal{F}[f(x)] = e^c \mathcal{F}[e^{ibx} g(x)] = \sqrt{\frac{\pi}{a}} e^{-\frac{(\lambda - b)^2}{4a} + c}$$

3.3 求下列函数的 Fourier 逆变换

3.3.1

例：$F(\lambda) = e^{(-a^2\lambda^2 + ib\lambda + c)t}$.

答:

$$f(x) = \mathcal{F}^{-1}[F(\lambda)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(-a^2\lambda^2 + ib\lambda + c)t} e^{i\lambda x}\, \mathrm{d}\lambda$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-a^2 t \lambda^2 + i(bt+x)\lambda + ct}\, \mathrm{d}\lambda$$

令 $u = \lambda - \frac{i(bt+x)}{2a^2 t}$，则有

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-a^2 t u^2 + \frac{(bt+x)^2}{4a^2 t} + ct}\, \mathrm{d}u$$

$$= \frac{1}{2\pi} e^{\frac{(bt+x)^2}{4a^2 t} + ct} \int_{-\infty}^{+\infty} e^{-a^2 t u^2}\, \mathrm{d}u$$

$$= \frac{1}{2\pi} e^{\frac{(bt+x)^2}{4a^2 t} + ct} \sqrt{\frac{\pi}{a^2 t}}$$

$$f(x) = \frac{1}{\sqrt{4\pi a^2 t}} e^{\frac{(bt+x)^2}{4a^2 t} + ct}$$

3.4 用 Fourier 变换求解定解问题

3.4.1

$$u_t = a^2 u_{xx} + b u_x + cu + f(x,t), -\infty < x < \infty, t > 0$$

例：$u|_{t=0} = 0$.

答：对变量 x 进行 Fourier 变换,$U(\lambda, t) = \mathcal{F}[u(x,t)], F(\lambda, t) = \mathcal{F}[f(x,t)]$

$$U_t = (-a^2\lambda^2 + ib\lambda + c)U + F$$

解得

$$U = (-a^2\lambda^2 + ib\lambda + c) * F$$

$$u(x,t) = \mathscr{F}^{-1}\left[e^{(-a^2\lambda^2 + ib\lambda + c)t} * F\right]$$

$$= \sqrt{\tfrac{\pi}{a^2}t}\, e^{\frac{(bt+x)^2}{4a^2t}+ct} f(x,t)$$

3.5 求下列函数的 Laplace 变换

3.5.1

例：$f(t) = t\cos\omega t$.

答：

$$\mathscr{L}[f(t)] = \int_0^\infty t\cos\omega t\, e^{-pt}\,\mathrm{d}t$$

$$= \tfrac{1}{p}\int_0^\infty e^{-pt}(\cos\omega t - \omega t\sin\omega t)\,\mathrm{d}t$$

$$= \tfrac{1}{p^2+\omega^2} - \tfrac{\omega}{p}\int_0^\infty t\sin\omega t\, e^{-pt}\,\mathrm{d}t$$

$$= \tfrac{1}{p^2+\omega^2} - \tfrac{\omega}{p^2}\int_0^\infty e^{-pt}(\sin\omega t + \omega t\cos\omega t)\,\mathrm{d}t$$

$$= \tfrac{1}{p^2+\omega^2} - \tfrac{\omega^2}{p^2}\tfrac{1}{p^2+\omega^2} - \tfrac{\omega^2}{p^2}\mathscr{L}[f(t)]$$

$$\mathscr{L}[f(t)] = \tfrac{p^2-\omega^2}{(p^2+\omega^2)^2}$$

例：$f(t) = e^{\omega t}\cos\omega t$.

答：

$$\mathscr{L}\big[e^{\omega t}\cos\omega t\big] = \tfrac{p-\omega}{(p-\omega)^2+\omega^2}$$

3.6 求下列函数的 Laplace 逆变换

3.6.1

例：$F(p) = \dfrac{1}{(p^2+4)^2}$.

答：

$$\mathscr{L}^{-1}[F(p)] = \tfrac{1}{4}\mathscr{L}^{-1}\left[\tfrac{2}{p^2+2^2}\right] * \mathscr{L}^{-1}\left[\tfrac{2}{p^2+2^2}\right]$$

$$= \tfrac{1}{4}\int_0^t \sin 2t\sin 2(t-\tau)\,\mathrm{d}\tau$$

$$= \tfrac{1}{4}\sin 2t\int_0^t \sin 2u\,\mathrm{d}u$$

$$= -\tfrac{1}{8}\sin 2t(\cos 2t - 1)$$

$$= \tfrac{1}{8}\sin 2t - \tfrac{1}{16}\sin 4t$$

3.6.2

例：$F(p) = \dfrac{1}{p^4+5p^2+4}$.

答：

$$\mathscr{L}^{-1}[F(p)] = \tfrac{1}{3}\mathscr{L}^{-1}\left[\tfrac{1}{p^2+1}\right] - \tfrac{1}{6}\mathscr{L}^{-1}\left[\tfrac{2}{p^2+2^2}\right]$$

$$= \tfrac{1}{3}\sin t - \tfrac{1}{6}\sin 2t$$

3.7 用 Laplace 变换求下列常微分方程

3.7.1

例：$y'' + 2ky' + y = e^t, y(0) = 0, y'(0) = 1.$

答:

作 Laplace 变换有：

$$Y = \mathcal{L}[y]$$

$$\mathcal{L}[y'] = p\mathcal{L}[y] - y(0) = pY$$

$$\mathcal{L}[y''] = p\mathcal{L}[y'] - y'(0) = p^2Y - 1$$

则有：

$$p^2Y - 1 + 2kpY + Y = \frac{1}{p-1}$$

$$Y = \frac{p}{(p-1)(p^2+2kp+1)}$$

设

$$Y = \frac{A}{p-1} + \frac{Bp+c}{p^2+2kp+1}$$

解得

$$Y = \frac{1}{2k+2}\left(\frac{1}{p-1} + \frac{1}{p^2+2kp+1} - \frac{p}{p^2+2kp+1}\right)$$

$1°$ $k = -1$

$$Y = \frac{1}{2k+2}\frac{p}{(p-1)^3} = \frac{1}{2k+2}\left(\frac{1}{(p-1)^2} + \frac{1}{(p-1)^3}\right)$$

$$y = \mathcal{L}^{-1}[Y] = \frac{1}{2k+2}\mathcal{L}^{-1}\left[\frac{1}{(p-1)^2} + \frac{1}{(p-1)^3}\right]$$

$$= \frac{1}{2k+2}e^t\left(t^2 + \frac{1}{2}t^3\right)$$

$2°$ $-1 < k < 1$

$$Y = \frac{1}{2k+2}\left[\frac{1}{p-1} + \frac{1+k}{(p+k)^2+\left(\sqrt{1-k^2}\right)^2} - \frac{p+k}{(p+k)^2+\left(\sqrt{1-k^2}\right)^2}\right]$$

$$y = \mathcal{L}^{-1}[Y] = \frac{1}{2k+2}\mathcal{L}^{-1}\left[\frac{1}{p-1} + \frac{1+k}{(p+k)^2+\left(\sqrt{1-k^2}\right)^2} - \frac{p+k}{(p+k)^2+\left(\sqrt{1-k^2}\right)^2}\right]$$

$$= \frac{1}{2k+2}\left[e^t + e^{-kt}\left(\frac{1+k}{\sqrt{1-k^2}}\sin\sqrt{1-k^2}\,t - \cos\sqrt{1-k^2}\,t\right)\right]$$

$2°$ $|k| \geq 1 \setminus k = -1$

$$y = \mathcal{L}^{-1}[Y] = \frac{1}{2k+2}\mathcal{L}^{-1}\left[\frac{1}{p-1} + \frac{1+k}{(p+k)^2+\left(\sqrt{k^2-1}\right)^2} - \frac{p+k}{(p+k)^2+\left(\sqrt{k^2-1}\right)^2}\right]$$

$$= \frac{1}{2k+2}\mathcal{L}^{-1}\left[\frac{1}{p-1} + \frac{1+k}{2\sqrt{k^2-1}}\left(\frac{1}{p+k-\sqrt{k^2-1}} - \frac{1}{p+k+\sqrt{k^2-1}}\right)\right.$$

$$\left. -\frac{1}{2}\left(\frac{1}{p+k-\sqrt{k^2-1}} + \frac{1}{p+k+\sqrt{k^2-1}}\right)\right]$$

$$= \frac{1}{2k+2}\left\{e^t + \frac{k+1}{2\sqrt{k^2-1}}\left[e^{\left(\sqrt{k^2-1}-k\right)t} - e^{\left(-\sqrt{k^2-1}-k\right)t}\right] - \frac{1}{2}\left(e^{\left(\sqrt{k^2-1}-k\right)t} + e^{\left(-\sqrt{k^2-1}-k\right)t}\right)\right\}$$

3.7.2

例：$y'' + 4yt = k\cos\omega t, y(0) = 0, y'(0) = 0.$

答: 对 t 作 Laplace 变换有

$$Y = \mathcal{L}[y]$$

$$\mathcal{L}[y'] = pY$$

$$\mathcal{L}[y'] = p^2 Y$$

$$\mathcal{L}[\cos \omega t] = \frac{p}{p^2 + \omega^2}$$

$$Y = \frac{p}{(p+2^2)(p^2+\omega^2)}$$

$$= \frac{1}{4-\omega^2}\left(\frac{p}{p^2+\omega^2} - \frac{p}{p^2+4}\right)$$

$$y(t) = \frac{1}{4-\omega^2}(\cos \omega t - \cos 2t)$$

3.8 用延拓法求解如下半无界问题

3.8.1

例：$u_t = a^2 u_{xx}, 0 < x < \infty, t > 0$

$u|_{t=0} = \varphi(x), 0 \leq x < \infty$

$u_x|_{x=0} = f_1(t), 0 \leq x < \infty$.

答：

考虑构建一个奇延拓，$u(-x,t) = -u(x,t)$，使其在全空间上定义

同时对 $\varphi(x)$ 进行奇延拓

$$\Phi(x) = \begin{cases} \varphi(x) \text{ if } x \geq 0 \\ -\varphi(-x) \text{ if } x < 0 \end{cases}$$

在 $-\infty < x < \infty$ 上解初值问题

$$u_t = a^2 u_{xx}, u|_{t=0} = \Phi(x)$$

对 $u(x,t), \Phi(x)$ 作关于 x 的 Fourier 变换

$$U(\lambda,t) = \mathscr{F}[u(x,t)], \hat{\Phi}(\lambda) = \mathscr{F}[\Phi(x)]$$

得到

$$U_t = -a^2 \lambda^2 U$$

$$U = \hat{\Phi}(\lambda) e^{-a^2 \lambda^2 t}$$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\Phi}(\lambda) e^{-a^2 \lambda^2 t} e^{i\lambda x} \, d\lambda$$

3.8.2

例：$u_{tt} = a^2 u_{xx} + f(x,t), 0 < x < \infty, t > 0$

$u|_{t=0} = \varphi(x), 0 \leq x < \infty$

$u_t|_{t=0} = \psi(x), 0 \leq x < \infty$

$u_x|_{x=0} = 0, 0 \leq x < \infty$.

答：

考虑构建一个偶延拓，$u(-x,t) = u(x,t)$，使其在全空间上定义 同时对 $\varphi(x), \psi(x)$ 进行偶延拓

$$\Phi(x) = \begin{cases} \varphi(x) \text{ if } x \geq 0 \\ \varphi(-x) \text{ if } x < 0 \end{cases}$$

$$\Psi(x) = \begin{cases} \psi(x) \text{ if } x \geq 0 \\ \psi(-x) \text{ if } x < 0 \end{cases}$$

于是在 $-\infty < x < +\infty$ 上，方程是一个有源波动方程，用达朗贝尔公式得

$$u(x,t) = \frac{1}{2}\varphi(x+at) + \frac{1}{2}\varphi(x-at) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\, d\xi + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t+\tau)} f(\xi,\tau)\, d\xi\, d\tau$$

第四章 波动方程

4.1 通解法

4.1.1

例：$\begin{cases} 3u_{xx}+10u_{xy}+3u_{yy}=0 \text{ if}-\infty<x<+\infty \wedge y>0 \\ u|_{y=0}=\varphi(x) \text{ if } -\infty<x<+\infty \\ u_y|(y=0)=\psi(x) \text{ if } -\infty<x<+\infty \end{cases}$

答：

$$u_{xx} + \tfrac{10}{3}u_{xy} + u_{yy} = 0$$

形如

$$u_{xx} - (A + B)u_{xy} + ABu_{yy} = 0$$

则

$$A = -3, B = -\tfrac{1}{3}$$

取

$$\xi = y - 3x, \eta = y - \tfrac{1}{3}x$$

$$u_x = -3\tfrac{\partial u}{\partial \xi} - \tfrac{1}{3}\tfrac{\partial u}{\partial \eta}$$

$$u_y = \tfrac{\partial u}{\partial \xi} + \tfrac{\partial u}{\partial \eta}$$

$$u_{xy} = -3\tfrac{\partial^2 u}{\partial \xi^2} - \tfrac{1}{3}\tfrac{\partial^2 u}{\partial \eta^2} - \tfrac{10}{3}\tfrac{\partial^2 u}{\partial \xi \partial \eta}$$

$$u_{xx} = 9\tfrac{\partial^2 u}{\partial \xi^2} + \tfrac{1}{9}\tfrac{\partial^2 u}{\partial \eta^2} + 2\tfrac{\partial^2 u}{\partial \xi \partial \eta}$$

$$u_{yy} = \tfrac{\partial^2 u}{\partial \xi^2} + \tfrac{\partial^2 u}{\partial \eta^2} + 2\tfrac{\partial^2 u}{\partial \xi \partial \eta}$$

代入化简得

$$\tfrac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

$$u(x, y) = F(\xi) + G(\eta) = F(y - 3x) + G\left(y - \tfrac{1}{3}x\right)$$

$$u(x, 0) = F(-3x) + G\left(-\tfrac{1}{3}x\right) = \varphi(x)$$

$$u_y(x, y) = F'(-3x) + G'\left(-\tfrac{1}{3}x\right) = \psi(x)$$

对两边从 0 到 x 积分

$$\int_0^x \psi(\xi)\,\mathrm{d}\xi = -\tfrac{1}{3}F(-3x)\big|_0^x - 3G\left(-\tfrac{1}{3}x\right)\big|_0^x$$

$$u(x, y) = \tfrac{9}{8}\varphi\left(x - \tfrac{1}{3}y\right) - \tfrac{1}{8}\varphi(x - 3y) + \tfrac{3}{8}\int_0^{x-\frac{1}{3}y} \psi(\xi)\,\mathrm{d}\xi - \tfrac{3}{8}\int_0^{x-3y} \psi(\xi)\,\mathrm{d}\xi$$

4.1.2

$$u_{tt} = a^2 u_{xx}$$

$$u|_{t=0} = 0$$

例：$u_t|_{t=0} = 1$.

答：

达朗贝尔公式

$$u(x,t) = \tfrac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \tfrac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,\mathrm{d}\xi$$

$$= \tfrac{1}{2a}\int_{x-at}^{x+at}\mathrm{d}\xi$$

$$= t$$

4.1.3

例: 在半平面 $\{(x,t)|-\infty < x < +\infty, t > 0\}$,求弦振动方程 $u_{tt} = u_{xx}$, $M(2,5)$ 的依赖区间,它是否落在 $(1,0)$ 的影响区域内?

答:

作特征线

$$x + t = 2 + 5 = 7, x - t = 2 - 5 = -3$$

$M(2,5)$ 的依赖区间

$$[-3, 7]$$

作特征线

$$x + t = 1, x - t = 1$$

当 $t = 5$ 时

$$x_1 = -4, x_2 = 6$$

$[-3,6]$ 落在影响区域内, $(6,7]$ 不在影响区域内.

$$u_{tt} = a^2 \Delta u, (x,y) \in R^2, t > 0$$

$$u|_{t=0} = 3x + 2y$$

例: $u_t|_{t=0} = 0$.

答:

取

$$\xi = x + r\cos\theta, \eta = y + r\sin\theta$$

在极坐标下,有

$$u(x,y,t) = \tfrac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{2\pi}\mathrm{d}\theta\int_0^{at}\frac{\varphi(\xi,\eta)}{\sqrt{(at)^2 - r^2}}r\,\mathrm{d}r$$

$$+ \tfrac{1}{2\pi a}\int_0^{2\pi}\mathrm{d}\theta\int_0^{at}\frac{\psi(\xi,\eta)}{\sqrt{(at)^2 - r^2}}r\,\mathrm{d}r$$

$$u(x,y,t) = \tfrac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{r(3x+2y)+3r^2\cos\theta+2r^2\sin\theta}{\sqrt{(at)^2 - r^2}}\,\mathrm{d}\theta\,\mathrm{d}r$$

$$= \tfrac{3x+2y}{2a}\frac{\partial}{\partial t}\int_0^{at}\frac{\mathrm{d}r^2}{\sqrt{(at)^2 - r^2}}$$

$$= 3x + 2y$$

4.1.4

例: 在 $t = 0$ 平面上以 $(0,0)$ 为圆心, 1 为半径的圆域内,给出充分光滑的函数 φ, ψ,试问能否决定初值问题的解在 $(x,y,t) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ 的值?

$$u_{tt} = a^2 \Delta u,$$

$$u|_{t=0} = \varphi(x,y), u_t|_{t=0} = \psi(x,y)$$

答:

点 $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ 在 $t = \frac{1}{2}$ 时的传播半径为 $R = at = \frac{a}{2}$

该点距离 $(0,0)$ $r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

影响区域 $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}\right)^2 \leq \left(\frac{a}{2}\right)^2$

当 $a \geq 4$ 时, φ, ψ 一定落在区域内,可确定 $u\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ 的值,反之不能确认.

4.1.5

例: 证明方程
$\frac{\partial}{\partial x}\left[\left(1 - \frac{x}{h}\right)^2 \frac{\partial u}{\partial x}\right] = \frac{1}{a^2}\left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial t^2}$
的通解为 $u = \frac{1}{h-x}[F(x - at) + G(x + at)]$

答:

取

$$v(x,t) = (h - x)u(x,t)$$

则

$$\frac{\partial}{\partial x}v = -u + (h - x)\frac{\partial}{\partial x}u$$

$$\frac{\partial^2}{\partial x^2}v = -\frac{\partial}{\partial x}u - \frac{\partial}{\partial x}u + (h - x)\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial}{\partial x}\left[\left(1 - \frac{x}{h}\right)^2 \frac{\partial u}{\partial x}\right] = -\frac{2}{h}\left(1 - \frac{x}{h}\right)\frac{\partial u}{\partial x} + \left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial x^2} = \frac{h-x}{h^2}\frac{\partial^2 v}{\partial x^2}$$

$$\frac{1}{a}\left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial t^2} = \frac{1}{a^2}\frac{h-x}{h^2}\frac{\partial^2 v}{\partial t^2}$$

则原方程可以化为

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2}$$

解得

$$v = F(x - at) + G(x + at)$$

$$u = \frac{1}{h-x}[F(x - at) + G(x + at)]$$

$F(x), G(x)$ 由初值条件 φ, ψ 确定.

4.2 求初值问题

4.2.1

$$u_{tt} = a^2 u_{xx} + x^2 - a^2 t^2, -\infty < x < +\infty, t > 0$$

例: $u|_{t=0} = 0, u_t|_{t=0} = 0$.

答:

由达朗贝尔公式有

$$u(x,t) = \frac{1}{2}[\varphi(x + at) + \varphi(x - at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,\mathrm{d}\xi$$

$$+ \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi, \tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

取 $\frac{\partial}{\partial t}P(x,t,\tau)|_{t-\tau=0} = x^2 - a^2 t^2$ 有

$$\begin{cases} P_{tt} = a^2 P_{xx} \\ P|_{t-\tau=0} = 0 \\ P_t|_{t-\tau=0} = x^2 - a^2 t^2 \end{cases}$$

$$P(x, t, \tau) = \frac{1}{2a} \int_{x-a(t-\tau)}^{x+a(t-\tau)} (\xi^2 - a^2 t^2) \, d\xi$$

$$= x^2(t - \tau) + \frac{a^2(t-\tau)^3}{3} - a^2 t^2(t - \tau)$$

再对 $P(x, t, \tau)$ 积分

$$u(x, t) = \int_0^t P(x, t, \tau) \, d\tau$$

$$= \frac{1}{2}x^2 t^2 - \frac{5}{12}a^2 t^4$$

第五章 椭圆型方程

5.1 导出散度公式和 Green 公式

取 $\boldsymbol{A}(x,y) = M\boldsymbol{i} + N\boldsymbol{j}, \boldsymbol{n} = \cos\alpha\boldsymbol{i} + \cos\beta\boldsymbol{j}$

$$\iint_D \frac{\partial M}{\partial x}\,\mathrm{d}\sigma = \int_C M\boldsymbol{i} \cdot \boldsymbol{n}\,\mathrm{d}l$$

$$\iint_D \frac{\partial N}{\partial x}\,\mathrm{d}\sigma = \int_C N\boldsymbol{j} \cdot \boldsymbol{n}\,\mathrm{d}l$$

则

$$\iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial x}\right)\mathrm{d}\sigma = \int_C (M\boldsymbol{i} \cdot \boldsymbol{n} + N\boldsymbol{j} \cdot \boldsymbol{n})\,\mathrm{d}l$$

即

$$\iint_D \nabla \cdot \boldsymbol{A}\,\mathrm{d}\sigma = \int_C \boldsymbol{A} \cdot \boldsymbol{n}\,\mathrm{d}l$$

取

$$\boldsymbol{A} = \nabla \cdot u$$

$$\nabla(\nabla \cdot u) = \Delta u$$

$$\nabla u \cdot \boldsymbol{n} = \frac{\partial u}{\partial n}$$

代入化简得

$$\iint_D \Delta u\,\mathrm{d}\sigma = \int_C \frac{\partial u}{\partial n}\,\mathrm{d}l$$

在上式中取 $M = u\frac{\partial v}{\partial x}, N = u\frac{\partial v}{\partial y}$

$$\iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial x}\right)\mathrm{d}\sigma$$

$$= \iint_D (uv_{xx} + uv_{yy})\,\mathrm{d}\sigma + \iint_D (u_x v_x + u_y v_y)\,\mathrm{d}\sigma$$

$$= \iint_D (u\Delta v)\,\mathrm{d}\sigma + \iint_D (\nabla u \cdot \nabla v)\,\mathrm{d}\sigma$$

$$\int_C (M\boldsymbol{i} \cdot \boldsymbol{n} + N\boldsymbol{j} \cdot \boldsymbol{n})\,\mathrm{d}l$$

$$= \int_C (uv_x\boldsymbol{i} + uv_y\boldsymbol{j}) \cdot \boldsymbol{n}\,\mathrm{d}l$$

$$= \int_C u\frac{\partial v}{\partial n}\,\mathrm{d}l$$

即

$$\iint_D (u\Delta v)\,\mathrm{d}\sigma + \iint_D (\nabla u \cdot \nabla v)\,\mathrm{d}\sigma = \int_C u\frac{\partial v}{\partial n}\,\mathrm{d}l$$

交换 u,v

$$\iint_D (v\Delta u)\,\mathrm{d}\sigma + \iint_D (\nabla v \cdot \nabla u)\,\mathrm{d}\sigma = \int_C v\frac{\partial u}{\partial n}\,\mathrm{d}l$$

两式相减有

$$\iint_D (u\Delta v - v\Delta u)\,\mathrm{d}\sigma = \int_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}l$$

5.2 Green 函数

5.2.1

例：设二维函数 $u(x,y)$ 在 ∂D 为边界的区域 D 内调和,且 $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$,证明

$$\int_C \frac{\partial u}{\partial n}\,\mathrm{d}l = 0$$

答:

在已导出的散度定理中，

$$\iint_D \Delta u \, d\sigma = \int_C \frac{\partial u}{\partial n} \, dl$$

有

$$\Delta u = 0$$

即

$$\int_C \frac{\partial u}{\partial n} \, dl = 0$$

5.2.2

例：三维 Poisson 方程的 Dirichlet 问题,适用 Green 函数表示该问题的解

$$\Delta u = -f(M), M \in \Omega$$
$$u|_{\partial\Omega} = 0 \qquad \qquad .$$

答：

在 Ω 区域中

$$\Delta_M G(M, M') = -\delta(M - M'), M, M' \in \Omega$$

其中 $\delta(M - M')$ 是三维 Dirac delta 函数

在 $\partial\Omega$ 边界上

$$G(M, M') = 0, M \in \partial\Omega$$

则

$$u(M) = \int_\Omega G(M, M') f(M') \, dV(M')$$

5.2.3

例：在平面 $-\infty < x < +\infty, y > 0$ 上求 Green 函数 $G(M, M_0)$

$$-\Delta G = \delta(x - x_0, y - y_0),$$
$$G|_{\partial\Omega} = 0$$

答：

在整个平面上 Laplace 方程的 Green 函数为：

$$G_0(M, M_0) = -\frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

由于边界条件要求 $G(x, 0) = 0$,可以通过镜像法构造一个辅助源点 $M_0' = (x_0, -y_0)$ 并定义

$$G(M, M_0) = G_0(M, M_0) - G_0(M, M_0')$$
$$G_0(M, M_0') = -\frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y + y_0)^2}$$

易得 $G(M, M_0)$ 在区域上调和,且在边界上等于 0

$$G(M, M_0) = -\frac{1}{2\pi} \ln \frac{\sqrt{(x-x_0)^2 + (y-y_0)^2}}{\sqrt{(x-x_0)^2 + (y+y_0)^2}}$$

5.2.4

例：在平面 $-\infty < x < +\infty, y > 0$ 上求解

$$\Delta u = -f(M), -\infty < x < +\infty, y > 0$$
$$u|_{y=0} = \varphi(x)$$

答：

做 Laplace 变换

$$U(\lambda, y) = \mathscr{F}[u(x,y)]$$

$$F(\lambda, y) = \mathscr{F}[f(x,y)]$$

$$\Phi(\lambda) = \mathscr{F}[\varphi(x)]$$

则

$$U_{yy} - \lambda^2 U = -F$$

$$U(\lambda, y) = \Phi(\lambda)e^{-|\lambda|y} + \int_0^y G(\lambda, y-\xi)F(\lambda, \xi)\,\mathrm{d}\xi$$

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\Phi(\lambda)e^{-|\lambda|y} + \int_0^y G(\lambda, y-\xi)F(\lambda, \xi)\,\mathrm{d}\xi\right]e^{i\lambda x}\,\mathrm{d}\lambda$$

5.2.5

例：在圆域 $x^2 + y^2 < R^2$ 上求解

$\Delta u = -f(M), r < R$

$u|_{r=R} = \varphi$

答：

由达朗贝尔公式有

$$u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi$$

$$+ \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

取 $\frac{\partial}{\partial t}P(x,t,\tau)|_{t-\tau=0} = x^2 - a^2t^2$ 有

$$\begin{cases} P_{tt}=a^2 P_{xx} \\ P|_{t-\tau=0}=0 \\ P_t|_{t-\tau=0}=x^2-a^2t^2 \end{cases}$$

$$P(x,t,\tau) = \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)}(\xi^2 - a^2t^2)\,\mathrm{d}\xi$$

$$= x^2(t-\tau) + \frac{a^2(t-\tau)^3}{3} - a^2t^2(t-\tau)$$

再对 $P(x,t,\tau)$ 积分

$$u(x,t) = \int_0^t P(x,t,\tau)\,\mathrm{d}\tau$$

$$= \frac{1}{2}x^2t^2 - \frac{5}{12}a^2t^4$$