



StudyThinking

卷二 数学物理方程笔记

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第一章 Fourier Transform

1.1 definition

$$F(\lambda) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx$$

$$f(x) = \mathcal{F}^{-1}[F(\lambda)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda$$

• $\mathcal{F}[e^{-a|x|}], a > 0$

$$\begin{aligned}\mathcal{F}[e^{-a|x|}] &= \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\lambda x} dx \\&= \int_{-\infty}^0 e^{ax-i\lambda x} dx + \int_0^{\infty} e^{-ax-i\lambda x} dx \\&= \frac{1}{a-i\lambda} e^{(a-i\lambda)x} \Big|_{-\infty}^0 - \frac{1}{a+i\lambda} e^{-(a+i\lambda)x} \Big|_0^{\infty} \\&= \frac{1}{a-i\lambda} + \frac{1}{a+i\lambda} \\&= \frac{2a}{a^2+\lambda^2}\end{aligned}$$

* $\lim_{t \rightarrow \infty} e^{-(\beta+i\lambda)t} = 0, \beta > 0$

• $\mathcal{F}[e^{-ax^2}], a > 0$

$$\begin{aligned}\mathcal{F}[e^{-ax^2}] &= \int_{-\infty}^{\infty} e^{-ax^2} e^{-i\lambda x} dx \\&= \int_{-\infty}^{\infty} e^{-a(x+\frac{i\lambda}{2a})^2 - \frac{\lambda^2}{4a}} dx \\&= e^{-\frac{\lambda^2}{4a}} \int_{-\infty}^{\infty} e^{-a(x+\frac{i\lambda}{2a})^2} dx \\&= \sqrt{\frac{\pi}{a}} e^{-\frac{\lambda^2}{4a}}\end{aligned}$$

* Gaussian integral: $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$

• $\mathcal{F}[\cos \eta x^2], \eta > 0$

$$\begin{aligned}\mathcal{F}[\cos \eta x^2] &= \int_{-\infty}^{\infty} \cos \eta x^2 e^{-i\lambda x} dx \\&= \int_{-\infty}^{\infty} \frac{e^{-i\eta x^2} + e^{i\eta x^2}}{2} e^{-i\lambda x} dx \\&= \frac{1}{2} \left(\int_{-\infty}^{\infty} e^{i\eta(x-\frac{\lambda}{2\eta})^2 - i\frac{\lambda^2}{4\eta}} dx + \int_{-\infty}^{\infty} e^{-i\eta(x+\frac{\lambda}{2\eta})^2 + i\frac{\lambda^2}{4\eta}} dx \right) \\&= \frac{1}{2} e^{-i\frac{\lambda^2}{4\eta}} \sqrt{\frac{\pi}{\eta}} e^{i\frac{\pi}{4}} + \frac{1}{2} e^{i\frac{\lambda^2}{4\eta}} \sqrt{\frac{\pi}{\eta}} e^{-i\frac{\pi}{4}} \\&= \frac{1}{2} \sqrt{\frac{\pi}{\eta}} \left[\cos\left(\frac{\lambda^2}{4\eta} - \frac{\pi}{4}\right) - i \sin\left(\frac{\lambda^2}{4\eta} - \frac{\pi}{4}\right) + \cos\left(\frac{\lambda^2}{4\eta} - \frac{\pi}{4}\right) + i \sin\left(\frac{\lambda^2}{4\eta} - \frac{\pi}{4}\right) \right] \\&= \sqrt{\frac{\pi}{\eta}} \cos\left(\frac{\lambda^2}{4\eta} - \frac{\pi}{4}\right)\end{aligned}$$

• $\mathcal{F}[\sin \eta x^2], \eta > 0$

$$\mathcal{F}[\sin \eta x^2] = \sqrt{\frac{\pi}{\eta}} \sin\left(\frac{\lambda^2}{4\eta} + \frac{\pi}{4}\right)$$

*Euler's formula: $e^{ix} = \cos x + i \sin x$

*Gaussian-like integrals: $\int_{-\infty}^{\infty} e^{\pm i\eta x^2} dx = \sqrt{\frac{\pi}{\eta}} e^{\pm i\frac{\pi}{4}}, \eta > 0$

1.2 properties

1.2.1 linear property

$$\mathcal{F}[\alpha f(x) + \beta g(x)] = \alpha \mathcal{F}[f(x)] + \beta \mathcal{F}[g(x)]$$

1.2.2 shift property

$$\mathcal{F}[f(x - b)] = e^{-i\lambda b} \mathcal{F}[f(x)]$$

1.2.3 differentiation property

$$\mathcal{F}[f'(x)] = i\lambda \mathcal{F}[f(x)]$$

$$\mathcal{F}[f^{(n)}(x)] = (i\lambda)^n \mathcal{F}[f(x)]$$

$$\mathcal{F}[xf(x)] = i \frac{d}{d\lambda} \mathcal{F}[f(x)]$$

$$\mathcal{F}[x^n f(x)] = i^n \frac{d}{d\lambda} \mathcal{F}[f(x)]$$

1.2.4 convolution property

$$\mathcal{F}[f(x) * g(x)] = \mathcal{F}[f(x)] \cdot \mathcal{F}[g(x)]$$

$$\mathcal{F}[f(x) \cdot g(x)] = \frac{1}{2\pi} \mathcal{F}[f(x)] * \mathcal{F}[g(x)]$$

第二章 Laplace Transform

2.1 definition

$$F(p) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-pt} dt$$

- $\mathcal{L}[1]$

$$\begin{aligned}\mathcal{L}[1] &= \int_0^\infty 1 \cdot e^{-pt} dt \\ &= -\frac{1}{p}e^{-pt} \Big|_0^\infty \\ &= \frac{1}{p}\end{aligned}$$

- $\mathcal{L}[t]$

$$\begin{aligned}\mathcal{L}[t] &= \int_0^\infty te^{-pt} dt \\ &= -\frac{1}{p}te^{-pt} \Big|_0^\infty + \frac{1}{p} \int_0^\infty e^{-pt} dt \\ &= \frac{1}{p^2}\end{aligned}$$

- $\mathcal{L}[e^{at}]$

$$\begin{aligned}\mathcal{L}[e^{at}] &= \int_0^\infty e^{at}e^{-pt} dt \\ &= \int_0^\infty e^{-(p-a)t} dt \\ &= \frac{1}{p-a}\end{aligned}$$

- $\mathcal{L}[\cos \omega t]$

$$\begin{aligned}\mathcal{L}[\cos \omega t] &= \int_0^\infty \cos \omega t e^{-pt} dt \\ &= -\frac{1}{p} \cos \omega t e^{-pt} \Big|_0^\infty - \frac{\omega}{p} \int_0^\infty \sin \omega t e^{-pt} dt \\ &= \frac{1}{p} + \frac{\omega}{p^2} \sin \omega t e^{-pt} \Big|_0^\infty - \frac{\omega^2}{p^2} \mathcal{L}[\cos \omega t] \\ &= \frac{p}{p^2 + \omega^2}\end{aligned}$$

- $\mathcal{L}[\sin \omega t]$

$$\begin{aligned}\mathcal{L}[\sin \omega t] &= \int_0^\infty \sin \omega t e^{-pt} dt \\ &= \frac{1}{2i} \int_0^\infty (e^{i\omega t} - e^{-i\omega t}) e^{-pt} dt \\ &= \frac{1}{2i} \int_0^\infty (e^{-(p-i\omega)t} - e^{-(p+i\omega)t}) dt \\ &= \frac{1}{2i} \left(\frac{1}{p-i\omega} - \frac{1}{p+i\omega} \right) \\ &= \frac{\omega}{p^2 + \omega^2}\end{aligned}$$

- $\mathcal{L}[\cosh \omega t]$

$$\begin{aligned}
\mathcal{L}[\cosh \omega t] &= \int_0^\infty \cosh \omega t e^{-pt} dt \\
&= \frac{1}{2} \int_0^\infty (e^{-(p-\omega)t} + e^{-(p+\omega)t}) dt \\
&= \frac{1}{2} \left(\frac{1}{p-\omega} + \frac{1}{p+\omega} \right) \\
&= \frac{p}{p^2 - \omega^2}
\end{aligned}$$

• $\mathcal{L}[\sinh \omega t]$

$$\mathcal{L}[\sinh \omega t] = \frac{\omega}{p^2 - \omega^2}$$

2.2 properties

2.2.1 linear property

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$$

2.2.2 shift property

$$\mathcal{L}[e^{at} f(t)] = F(p - a)$$

2.2.3 differentiation property

$$\mathcal{L}[f'(t)] = p \mathcal{L}[f(t)] - f(0)$$

$$\mathcal{L}[f^{(n)}(t)] = p^n F(p) - p^{n-1} f(0) - \dots - p f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathcal{L}[t f(t)] = -\frac{d}{dp} F(p)$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{dp^n} F(p)$$

2.2.4 convolve property

$$f(t) * g(t) = \int_0^t f(\tau) g(t - \tau) d\tau$$

$$\mathcal{L}[f(t) * g(t)] = F(p) \cdot G(p)$$

第三章 Fundamental Equations of Mathematical Physics

3.1 wave equation

$$u_{tt} = a^2 u_{xx}$$
$$u_{tt} = a^2 (u_{xx} + u_{yy})$$
$$u_{tt} = a^2 (u_{xx} + u_{yy} + u_{zz})$$

3.2 wave equation with a source term

$$u_{tt} = a^2 u_{xx} + f(x, t)$$
$$u_{tt} = a^2 (u_{xx} + u_{yy}) + f(x, y, t)$$
$$u_{tt} = a^2 (u_{xx} + u_{yy} + u_{zz}) + f(x, y, z, t)$$

3.3 heat equation

$$u_t = a^2 u_{xx}$$
$$u_t = a^2 (u_{xx} + u_{yy})$$
$$u_t = a^2 (u_{xx} + u_{yy} + u_{zz})$$

3.4 heat equation with a source term

$$u_t = a^2 u_{xx} + f(x, t)$$
$$u_t = a^2 (u_{xx} + u_{yy}) + f(x, y, t)$$
$$u_t = a^2 (u_{xx} + u_{yy} + u_{zz}) + f(x, y, z, t)$$

3.5 classification of second-order partial differential equations

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

*A, B, C, D, E, F and G are functions of x and y, but not of u.

$$\Delta = B^2 - AC$$

$\Delta > 0$ (hyperbolic equation)

$$u_{xy} = [\cdot \cdot \cdot], u_{xx} - u_{yy} = [\cdot \cdot \cdot]$$

$\Delta = 0$ (parabolic equation)

$$u_{xx} = [\cdot \cdot \cdot], u_{yy} = [\cdot \cdot \cdot]$$

$\Delta < 0$ (elliptic equation)

$$u_{xx} + u_{yy} = [\cdot \cdot \cdot]$$

* $[\cdot \cdot \cdot]$ represents all terms that do not contain second-order partial derivatives.

3.6 simplification of second-order partial differential equations

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

characteristic equation:

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0$$

characteristic lines are determined by the solutions of the characteristic equation:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$
$$\begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} \neq 0$$

$$\xi(x, y) = c_1, \eta(x, y) = c_2$$

$$(1) \Delta = B^2 - AC > 0$$

$$\text{e.g. } u_{xx} - 4u_{xy} + u_{yy} = 0$$

$$A = 1, B = -2, C = 1$$

$$\frac{dy}{dx} = -2 \pm \sqrt{3}$$

$$y + (2 \pm \sqrt{3})x = c$$

$$\xi(x, y) = y + (2 + \sqrt{3})x, \eta(x, y) = y + (2 - \sqrt{3})x$$

$$(2) \Delta = 0$$

Characteristic lines $\xi(x, y) = c_1$ are still governed by the characteristic equation, while $\eta(x, y)$ can be any function independent of $\xi(x, y)$, provided that the Jacobian determinant is not equal to zero.

$$(3) \Delta < 0$$

$$\text{e.g. } u_{xx} + 4u_{xy} + 5u_{yy} + u_x + u_y = 0$$

$$A = 1, B = 2, C = 5$$

$$\frac{dy}{dx} = 2 \pm i$$

$$2x - y \pm ix = c$$

$$\xi(x, y) = x, \eta(x, y) = 2x - y$$

3.7 constant-coefficient equation

when $A, B, C \in \mathbb{R}$

$$(1) \Delta > 0$$

$$\xi = y - \frac{B + \sqrt{B^2 - AC}}{A}x, \eta = y - \frac{B - \sqrt{B^2 - AC}}{A}x$$

$$(2) \Delta = 0$$

$$\xi = y - \frac{B}{A}x, \eta = y$$

$$(3) \Delta < 0$$

$$\xi = y - \frac{B}{A}x, \eta = \frac{\sqrt{AC - B^2}}{A}x$$

$$\text{e.g. } u_{xx} - (A + B)u_{xy} + ABu_{yy} = 0$$

$$\xi = y + Ax, \eta = y + Bx$$

$$u_{xx} - (A + B)u_{xy} + ABu_{yy} = -(A - B)^2 u_{\xi\eta}$$

第四章 Method of separation of variables

4.1 string vibration equation

$$u_{tt} = a^2 u_{xx} (0 < x < l, t > 0)$$

$$u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x) (0 \leq x \leq l)$$

$$u|_{x=0} = 0, u|_{x=l} = 0 (0 \leq x \leq l)$$