0.1 Trigonometric Fourier series

We use symbolic MATLAB to find Fourier series representations and to plot them. We first evaluate the cofficients symbolically and then form partial sums of series.

(1) The Fourier cofficients are given by the formulas ($\omega = \frac{2\pi}{T}$):

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt,$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(k\omega t)dt,$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(k\omega t)dt.$$

(2) The Fourier series is then given by the formula:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos(k\omega t) + b_k\sin(k\omega t)).$$

(3) Then we plot partial sums of Fourier series.

The following MATLAB commands are needed. To evaluate $\int_0^2 t^2 dt$, use commands:
» syms t
» $int(t\hat{}2, 0, 2)$

Example Find the Fourier series representation of the function

$$f(t) = \ t, \quad -\pi < t < \pi \quad \text{period} \quad 2\pi.$$

In the following we find the Fourier series and plot the 6th partial sum to 1st figure. To 2nd figure we plot the 6th partial sum of Fourier series and the function to the same figure (to see how accurate is the approximation). To 3rd figure we plot the (sine) spectrum (ω-domain representation, to understand the physics of this periodic signal).

```
» syms t  k
» T=2*pi;
» w=2*pi /T;
» f=t;
» a0=2/T*int(f,-pi,pi)
» ak=2/T*int(f*cos(k*w*t),t,-pi,pi)
» bk=2/T*int(f*sin(k*w*t),t,-pi,pi)
```

The Fourier cofficients are now found. We want to plot the Fourier series. For example when $k = 1:6$, which means that $k = 1, 2, 3, ..., 6$ and $t = -5 : 0.005 : 5$, which means that $t \in [-5, 5]$ (increment is 0.005). We use the command *subs*, because k and t are symbols. To plot the (sine) spectrum, instead of the command *plot*, we use the command *stem* in order to see peaks in 3rd figure.

```
» k=1:6;
» ak=subs(ak);
» bk=subs(bk);
» f_f=0.5*a0+sum(ak.*cos(k*w*t))+sum(bk.*sin(k*w*t))
» t=-5:0.005:5;
» f_f=subs(f_f);
» figure(1)
» plot(t,f_f)
» f=subs(f);
» figure(2)
» plot(t,f_f,t,f)
» legend('f_f','f')
» figure(3)
» stem(k,bk)
» axis([0 10 -3 3])
```

MATLAB assignments

1. **Square**

   (a) Find the Fourier series representation of the function and plot the 6th partial sum

$$f(t) = \begin{cases} 0, & -\pi < t < -\frac{\pi}{2} \\ 1, & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < t < \pi. \end{cases} \quad \text{period} \quad 2\pi$$

   (b) Plot the 6th partial sum and the function to the same figure.

   (c) Plot the (cosine) spectrum.

2. **Triangle**

   (a) Find the Fourier series representation of the function and plot the 6th partial sum

$$f(t) = \begin{cases} t+1, & -\pi < t < 0 \quad \text{period} \quad 2\pi \\ -t+1, & 0 < t < \pi. \end{cases}$$

   (b) Plot the 6th partial sum and the function to the same figure.

   (c) Plot the (cosine) spectrum.

3. Study the Fourier series applet http://www.falstad.com/fourier/

   Check your solutions in the problems 1 and 2.

0.2 Complex Fourier series

By the following MATLAB example we can illustrate the spectrum of a periodic function. We use complex Fourier series. By finding the Fourier cofficients we can plot the amplitude and phase spectra.

Example

Consider the following periodic signal.

$$f(t) = \begin{cases} 50t, & 0 < t < 0.02 \quad \text{period} \quad 0.1 \\ 0, & 0.02 < t < 0.1 \end{cases}$$

The complex Fourier series is defined by the formula:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{i2\pi kt}{T}}$$

where the complex Fourier cofficiets can be evaluated by the formula:

$$c_k = \frac{1}{T}\int_0^T f(t) e^{-\frac{i2\pi kt}{T}} dt$$

Then the amplitude spectrum is the set of numbers $|c_k|$ and the phase spectrum consists of phase angles of c_k. Use the following commands:

» syms t k
» $T = 0.1;$
» $w = 2 * pi/T;$
» $f = 50 * t$
» $k = -5 : 5;$
» $ck = 1/T * int(f * exp(-i * k * w * t), t, 0, 0.02)$
» $c0 = 1/T * int(f, t, 0, 0.02)$

The cofficients are now found. We want to plot these cofficients for example when $k = -5 : 5$. We use the command *subs*, because k is a symbol.

» $ck = subs(ck);$
» $stem(k, abs(ck))$
» hold on
» $k = 0;$
» $c0 = subs(c0);$
» $stem(k, c0)$

Instead of the command *plot*, we use the command *stem* in order to get peaks in the figure. Then we can plot the phase spectrum:

» figure(2)
» $k = -5 : 5;$
» stem(k,angle(ck))

Example

Use the following in the exercise

» syms t k
» $T = 50;$
» $w = 2 * pi/T;$
» $f = heaviside(t + 0.5) - heaviside(t - 0.5)$
» $k = -100 : 100;$
» $ck = 1/T * int(f * exp(-i * k * w * t), t, -T/2, T/2)$
» $c0 = 1/T * int(f, t, -T/2, T/2)$
» $ck = subs(ck);$
» $stem(k, abs(ck))$
» hold on
» $k = 0;$
» $c0 = subs(c0);$
» $stem(k, c0)$

MATLAB assignments:

1. Plot the amplitude and phase spectrum of the function:

$$f(t) = \begin{cases} 1, & -0.5 < t < 0.5 \quad \text{period} \quad 5 \\ 0, & \text{otherwise} \end{cases}$$

   Let first $k = -5 : 5$. Then $k = -10 : 10$.

2. Consider the function defined in the problem 1. By this program you can illustrate how the Fouries series cofficients approaches the Fourier transform as $T \to \infty$. First let $T = 20$ and $k = -20 : 20$. Then $T = 50$ and $k = -100 : 100$. See what happens. Do this only for the amplitude spectrum.

0.3 Fourier transform

By the following MATLAB examples we find the Fourier transform of given functions and plot their amplitude and phase spectra.

Example 1. Find the Fourier transform of the signal $f(t) = 2e^{-3t}u(t)$. Plot the amplitude and phase spectrum.

```
» syms t w
» f=2*exp(-3*t)*heaviside(t);
» F=fourier(f)
» w=-10:0.01:10;
» F=subs(F);
» figure(1)
» plot(w,abs(F))
» figure(2)
» plot(w,angle(F))
```

Example 2. Find the Fourier transform of the signal

$$f(t) = \begin{cases} 1, & -0.5 \le t < 0.5 \\ 0, & \text{otherwise.} \end{cases}$$

Plot the amplitude and phase spectrum.

```
» syms t w
» f=heaviside(t+0.5)-heaviside(t-0.5);
» F=fourier(f)
» F = simplify(F)
» w=(-10:0.01:10)+eps;
» F=subs(F);
» figure(1)
» plot(w,abs(F))
» figure(2)
» plot(w,angle(F))
```

MATLAB assignment

1. Find the Fourier transform and plot the amplitude and phase spectrum of

$$f(t) = \begin{cases} t+1, & -1 \le t < 0 \\ -t+1, & 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$$