UNIVERSITEIT VAN AMSTERDAM

BEELDBEWERKEN

Image Processing

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1 Exercises

1.1 Analytical Local Structure

For the first part of this exercise the derivatives of (1) are calculated and shown below. Calculated are $f_x, f_{xx}, f_{xy}, f_y and f_{yy}$. These are all partial derivatives and because of that $f_{xy} = 0$.

$$f(x, y) = A * sin(V * x) + B * cos(W * y) \tag{1}$$

$$f_x = V * A * sin(V * x) \tag{2}$$

$$f_{xx} = V^2 * A * sin(V * x) \tag{3}$$

$$f_{xy} = 0 \tag{4}$$

$$f_y = W * B * cos(W * y) \tag{5}$$

$$f_{yy} = W^2 * B * cos(W * y) \tag{6}$$

In part 2 of the assignment a visualisation of the function (1) was made that can be seen in figure 1. To create this image of (1) the meshgrid function was used, which creates 2 2D arrays with the arranged values either increasing per sub-array or withing the sub-array, respectively X and Y from the example.

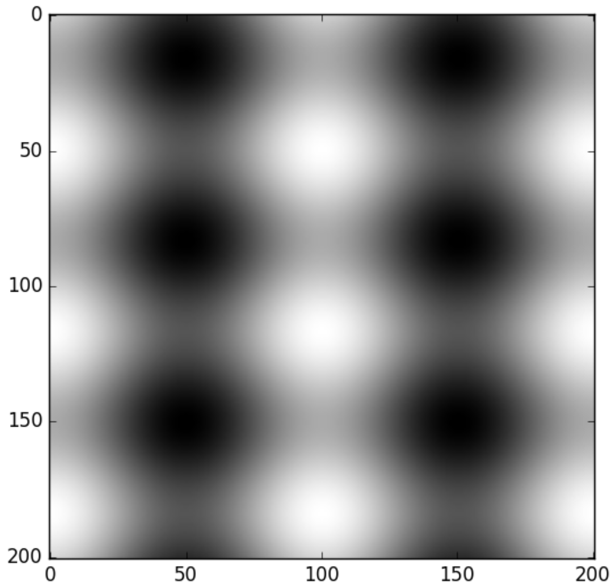

Figure 1: Visualisation of (1)

In the third part of the assignment the visualisations of (2) and (5) were to be made. The results can be seen in figure 2 and 3. To create the images in figure 2 and 3 the same code was used that created the image in figure one, except the formula that was used changed to respectively (2) and (5) instead of (1).

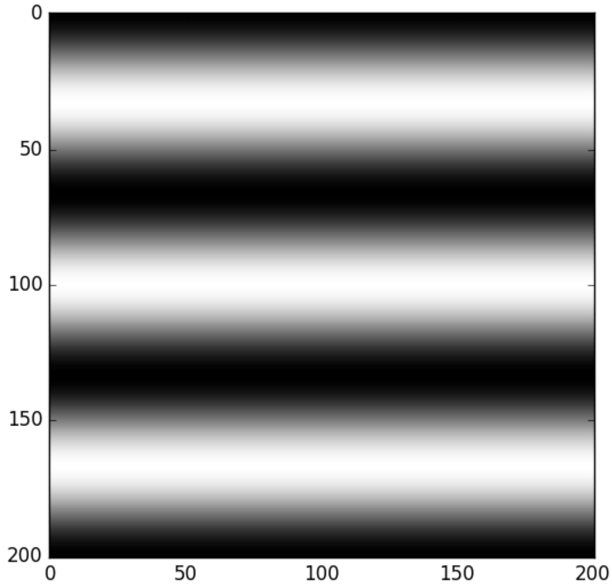

Figure 2: Visualisation of f_x

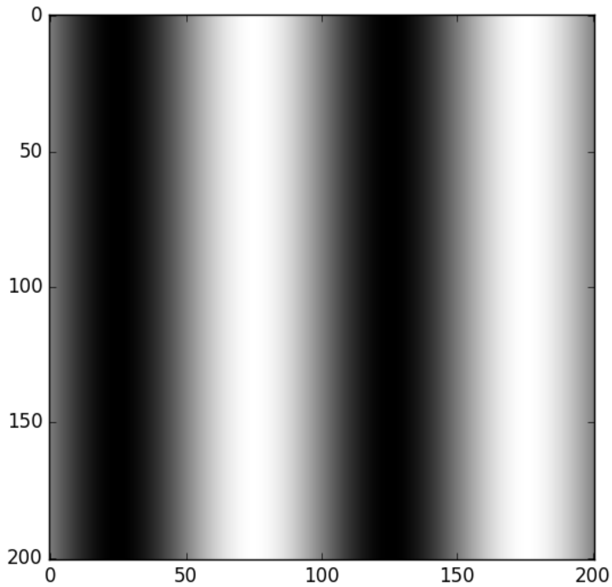

Figure 3: Visualisation of f_y

The final part of this assignment was to create a quiver plot showing the "direction" of the grey values. This means the arrows have to point from the light part of the image to the dark part. This is done by using (2) and (5) in the quiver function together with a meshgrid of 2 arrays consisting of the arranged numbers between -100 and 101 on an interval of 10. The result is shown in figure 4.

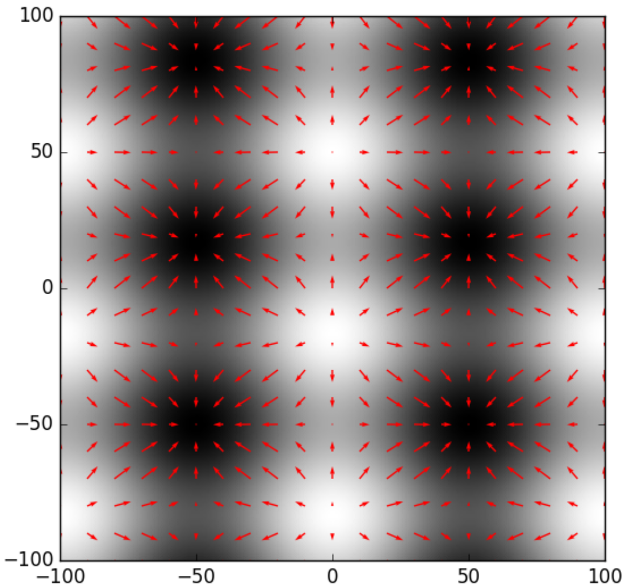

Figure 4: Quiver plot of (1)

1.2 Gaussian Convolution

For the assignment a two dimensional Gaussian kernel was created and an image was convolved with the kernel in the x and y direction.

For the sample size the sigma value s is multiplied by 6 and then incremented by 1, to make it so that the sum of the kernel is more than 0.95. The kernel values are calculated by creating a meshgrid of the x and y values and applying the two dimensional Gaussian function to the values.

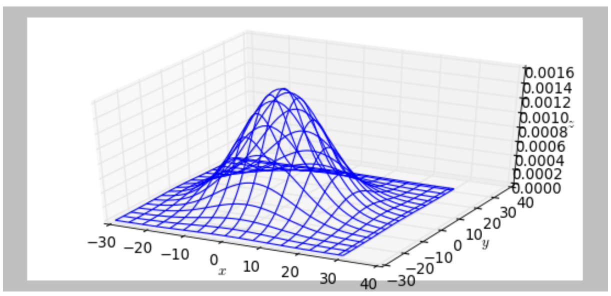

Figure 5: 3D visualization of Gaussian kernel with $\sigma = 10$

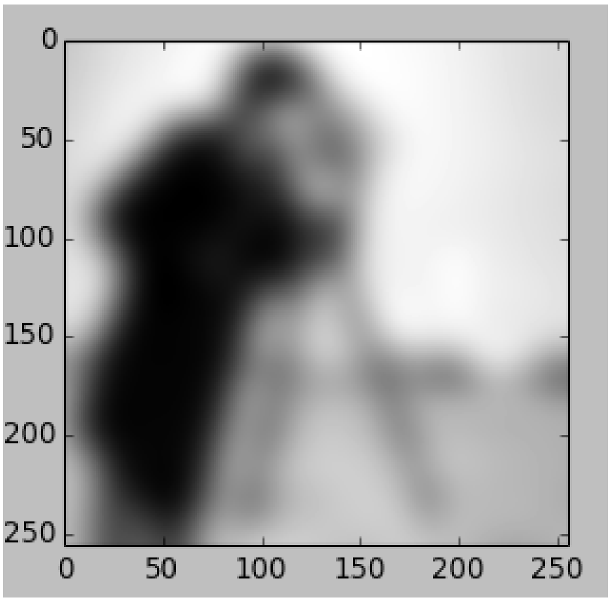

Figure 6: Camaraman.png convolved with Gaussian function $\sigma = 10$

Testing

The following table and graph shows the time needed for convolving an image related to the σ value.

Table 1: Relationship between σ and time

σ	1	2	3	5	7	9	11	15	19
time(s)	0.258	0.274	0.330	0.442	0.580	0.789	0.861	1.624	3.162

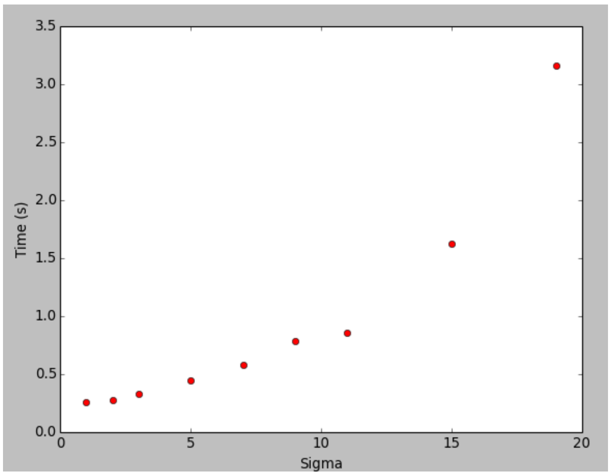

Figure 7: Relationship between σ value and time

From figure 7 it is visible that the order of computational complexity of the two dimensional Gaussian convolution is $O(n^2)$.

1.3 Seperable Gaussian Convolution

The two dimensional Gaussian function $G^\sigma(x,y)$ can be rewritten as two one dimensional Gaussian functions.

$$G^\sigma(x,y) = g^\sigma(x)g^\sigma(y)$$

In the function above the g^σ is the one dimensional Gaussian function. To convolve an image in both the x and y direction the two one dimensional kernels are calculated. Then the original image is convolved in the x direction with the $g^\sigma(x)$ kernel and the with the $g^\sigma(y)$ kernel in the y direction.

Testing

The following table and graph show the relationship between time and σ value for the one dimensional Gaussian convolutions.

Table 2: Relationship between σ and time

σ	1	2	3	5	7	9	11	15	19
time(s)	0.296	0.249	0.331	0.305	0.261	0.322	0.261	0.269	0.260

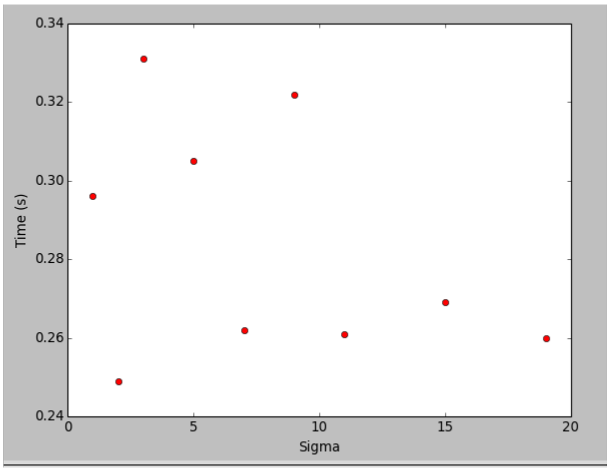

Figure 8: Relationship between σ value and time

From figure 8 it is noticeable that with the values it is not easy to determine the order of complexity. The suspected order should be $O(n)$. Better testing is required to give a definite answer to the question.

1.4 Gaussian derivatives

The Gaussian derivatives of an image can be calculated by convolving an image by the derivative of the Gaussian function. The derivatives of a two dimensional Gaussian function can also be seperated in two one dimensional Gaussian functions. The following equations will analitically show that the derivatives are seperable in two functions.

$$G^\sigma(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

$$\frac{\delta G^\sigma}{\delta x} = -\frac{x}{2\pi\sigma^4} e^{-\frac{(x^2+y^2)}{2\sigma^2}} \qquad \frac{\delta G^\sigma}{\delta y} = -\frac{y}{2\pi\sigma^4} e^{-\frac{(x^2+y^2)}{2\sigma^2}} \tag{7}$$

$$\frac{\delta^2 G^\sigma}{\delta^2 x} = \left(-1 + \frac{x^2}{\sigma^2}\right) \frac{e^{-\frac{(x^2+y^2)}{2\sigma^2}}}{2\pi\sigma^4} \qquad \frac{\delta^2 G^\sigma}{\delta^2 y} = \left(-1 + \frac{y^2}{\sigma^2}\right) \frac{e^{-\frac{(x^2+y^2)}{2\sigma^2}}}{2\pi\sigma^4} \tag{8}$$

$$\frac{\delta^2 G^\sigma}{\delta xy} = \frac{xy}{2\pi\sigma^6} e^{-\frac{(x^2+y^2)}{2\sigma^2}} \tag{9}$$

The equations in (7) are the first x and y derivatives of the two dimensional Gaussian function. The equations in (8) are the second x and y derivatives of the two dimensional Gaussian function. Equation (9) represents the derivatives in x and then y and vice versa of the two dimensional Gaussian function ($\frac{\delta^2 G^s}{\delta xy} = \frac{\delta^2 G^s}{\delta yx}$). The following equations represent the one dimensional Gaussian function and its derivatives.

$$g^\sigma(i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{i^2}{2\sigma^2}}$$

$$\frac{\delta g^\sigma(i)}{\delta i} = -\frac{i}{\sigma^3\sqrt{2\pi}} e^{-\frac{i^2}{2\sigma^2}} \qquad \frac{\delta^2 g^\sigma(i)}{\delta^2 i} = -\frac{\sigma^2 - i^2}{\sigma^5\sqrt{2\pi}} e^{-\frac{i^2}{2\sigma^2}} \tag{10}$$

The equations in (10) are the first and second derivatives of the one dimensional Gaussian function.

$$\begin{aligned} \frac{\delta(g^\sigma(x)g^\sigma(y))}{\delta x} &= \frac{\delta g^\sigma(x)}{\delta x} g^\sigma(y) = -\frac{x}{\sigma^3\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \\ &= -\frac{x}{2\pi\sigma^4} e^{-\frac{(x^2+y^2)}{2\sigma^2}} = \frac{\delta G^\sigma(x,y)}{\delta x} \end{aligned} \tag{11}$$

From equation (11) it can be determined that the two dimensional Gaussian derivative in the x direction is separable in two one dimensional functions. The first derivative of the one dimensional Gaussian function is applied to x and multiplied by the value of the one dimensional Gaussian function of y.

$$\begin{aligned} \frac{\delta(g^\sigma(x)g^\sigma(y))}{\delta y} &= \frac{\delta g^\sigma(y)}{\delta y} g^\sigma(x) = -\frac{y}{\sigma^3\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \\ &= -\frac{y}{2\pi\sigma^4} e^{-\frac{(x^2+y^2)}{2\sigma^2}} = \frac{\delta G^\sigma(x,y)}{\delta y} \end{aligned} \tag{12}$$

From equation (12) it can be determined that the two dimensional Gaussian derivative in the y direction is separable in two one dimensional functions.

$$\begin{aligned}\frac{\delta(g^\sigma(x)g^\sigma(y))^2}{\delta^2 x} &= \frac{\delta g^\sigma(x)^2}{\delta^2 x}g^\sigma(y) = -\frac{\sigma^2 - x^2}{\sigma^5\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{y^2}{2\sigma^2}} \\ &= \left(-1 + \frac{x^2}{\sigma^2}\right)\frac{e^{-\frac{(x^2+y^2)}{2\sigma^2}}}{2\pi\sigma^4} = \frac{\delta^2 G^\sigma}{\delta^2 x}\end{aligned} \tag{13}$$

$$\begin{aligned}\frac{\delta(g^\sigma(x)g^\sigma(y))^2}{\delta^2 y} &= \frac{\delta g^\sigma(y)^2}{\delta^2 y}g^\sigma(x) = -\frac{\sigma^2 - y^2}{\sigma^5\sqrt{2\pi}}e^{-\frac{y^2}{2\sigma^2}}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}} \\ &= \left(-1 + \frac{y^2}{\sigma^2}\right)\frac{e^{-\frac{(x^2+y^2)}{2\sigma^2}}}{2\pi\sigma^4} = \frac{\delta^2 G^\sigma}{\delta^2 y}\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\delta(g^\sigma(x)g^\sigma(y))^2}{\delta xy} &= \frac{\delta g^\sigma(x)}{\delta x}\frac{\delta g^\sigma(y)}{\delta y} = -\frac{x}{\sigma^3\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}} - \frac{y}{\sigma^3\sqrt{2\pi}}e^{-\frac{y^2}{2\sigma^2}} \\ &= \frac{xy}{2\pi\sigma^6}e^{-\frac{(x^2+y^2)}{2\sigma^2}} = \frac{\delta^2 G^\sigma}{\delta xy}\end{aligned} \tag{15}$$

Equations (13), (14), (15) separates the two dimensional second derivatives into one dimensional Gaussian (derivative) functions.

Implementation

For the assignment a function gD was created that has parameters $F, s, iorder$ and $jorder$ which represent the image, sigma, x-derivative, y-derivative repectively. First the x-derivative and y-derivative values are calculated (corresponding to $iorder$ and $jorder$). Then the image is convolved with the x-kernel in the x-direction and then in the y-direction with the y-kernel (A and B respectively).

Results

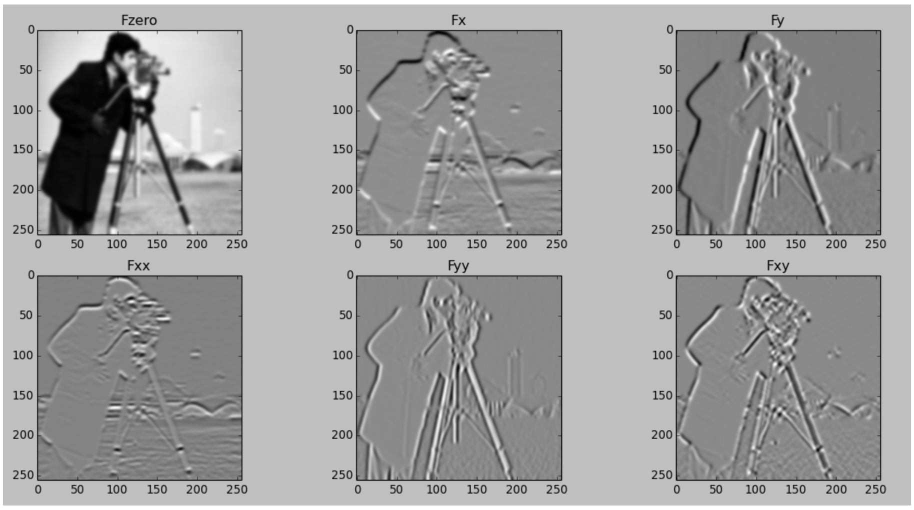

Figure 9: The Gaussian derivatives

1.5 Comparison of Theory and Practice

An analytical and experimental comparison will be made with the first order derivatives in x and y of the function $f(x, y) = Asin(Vx) + Bcos(Wy)$ from the very first assignment.

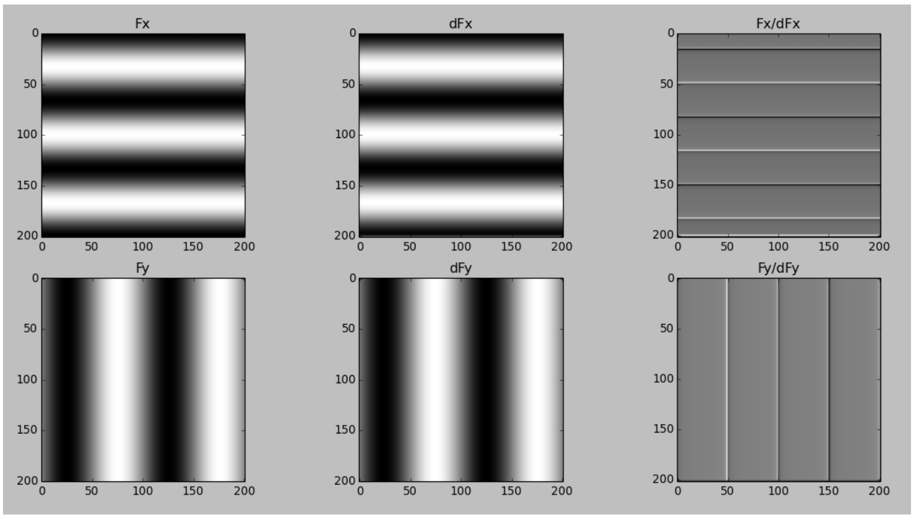

Figure 10: Comparison of the first derivatives

When comparing the two derivatives in the two most right images at points of inflexion the grey values are not constant.

1.6 Canny Edge

Method

As stated in the assignment, finding edges in images is often an important first step in image processing applications. One of the most used edge detectors is the Canny edge detector. The Canny edge detector uses Gaussian derivatives to find edges in images.
To find an edge in the image we look for 2 things:

- $\sqrt{(}G^s(x)^2 + G^s(y)^2)$ has to be bigger than threshold t
- The pixel we are checking has to be a zero crossing, meaning in a neighbourhood of 3 pixels the checked pixel has to lie between 2 pixels where one of the pixels has a $f_{ww} > 0$ and the other pixel has a $f_{ww} < 0$.

If both of the conditions are met, we can use the gradient of the pixel to give it a color, otherwise the pixel stays black.

Implementation

To implement the algorithm described above we use the gD function implemented in the assignment above, Gaussian Derivatives. This function is used to create convolutions of the derivatives of the Gaussian 2D function, f_x, f_{xx}, f_{xy}, f_y and f_{yy} with s = 2.0 becuase that gave us the best result.
The next step is creating a new image, consisting of zero's. We then start looping over all the pixels in the image. The first check to be made is if $\sqrt{(}G^s(x)^2 + G^s(y)^2)$ is bigger than our threshold $t = 0.02$. If this is the case, we can do our zero crossing check. This means that we make a circle around the checked pixel, checking opposite pixels. Each pixel a is put in the formula show at (16). If one pixel is bigger than zero and the other is smaller than zero, or the other way around, an edge has been found and the gradient of the pixel is filled in as colour.

$$f_{ww} = f_x(a)^2 f_{xx}(a) + 2 f_x(a) f_y(a) f_{xy}(a) + f_y(a)^2 f_{yy}(a) \tag{16}$$

Results

The results of the edge detection can be seen in figure 11 below. For this image we used s = 2.0, because a lower s gave too much edges and a higher s too few.

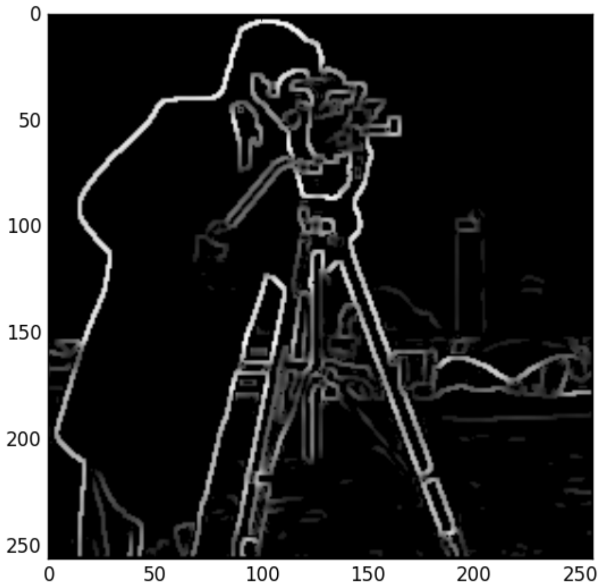

Figure 11: Edges of our cameraman with s = 2