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AKSHAT GUPTA

2021515

ASSIGNMENT 1

Q1) $X \sim$ iid from Exponential (λ)
(for $\lambda = 1, 2, 3, 4$)

$$\Rightarrow f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and $\lambda > 0$

$$\Rightarrow L(\lambda) = \prod_{i=1}^{n} f_\lambda(x_i)$$

$$= \lambda^n \prod_{i=1}^{n} e^{-\lambda x_i}$$

$$= \lambda^n e^{-\lambda \left(\sum_{i=1}^{n} x_i\right)}$$

$$\Rightarrow \log(L(\lambda)) = n \cdot \ln(\lambda) - \lambda\left(\sum_{i=1}^{n} x_i\right)$$

$\Rightarrow$ To get MLE, take derivative of log-likelihood and equate to 0
(because maximising $L(\theta) \equiv$ maximising $\log(L(\theta))$)

$$\Rightarrow \frac{d(\log(L(\lambda)))}{d\lambda} = 0$$

$$\Rightarrow \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \boxed{\lambda = \frac{1}{n} \sum_{i=1}^{n} x_i} \longrightarrow \text{MLE for exponential distribution}$$

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This is also the MoM estimate for λ.

$$\Rightarrow \quad \hat{\lambda}_{MLE} = \hat{\lambda}_{MoM} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\boxed{\log(L(\lambda))_{(\hat{\lambda}_{MLE})} = n \cdot \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}$$

(b) For $\lambda = 1$

$\hat{\lambda}_{MLE} = 1.047623$

For $\lambda = 2$

$\hat{\lambda}_{MLE} = 2.005902$

For $\lambda = 3$

$\hat{\lambda}_{MLE} = 3.057668$

For $\lambda = 4$

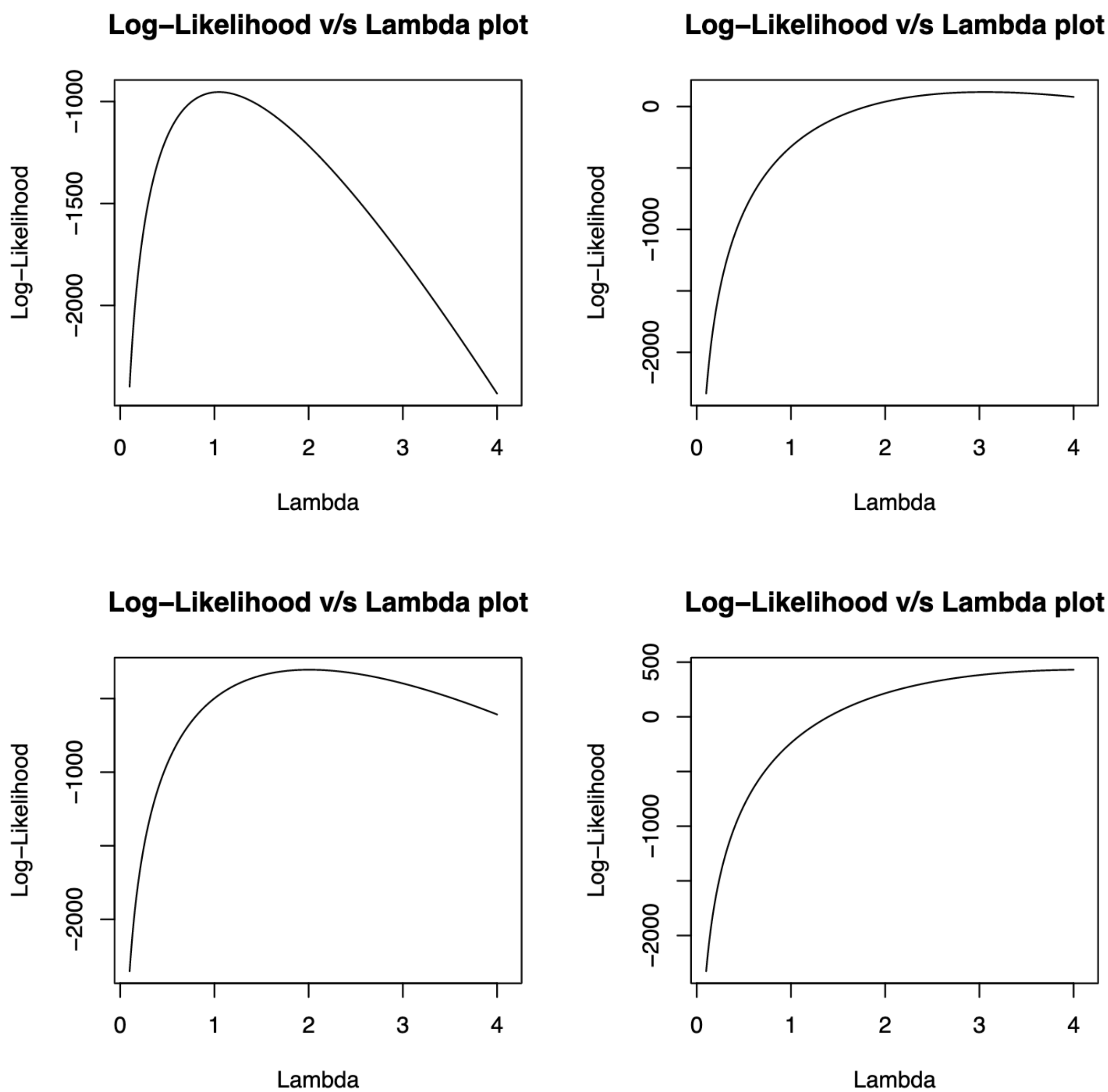
$\hat{\lambda}_{MLE} = 4.186223$

The program defines a -ve log-likelihood function and minimises it using

nlminb (...)

call taking lower value of $\lambda = 0$, and upper = Inf

The graphs obtained for each corresponding value of lambda are as shown below:



For each value of lambda, the likelihood function takes the maximum value at estimated lambda ≈ true lambda.

Q2.

Given that the data follows Normal Distribution, the MLE is given by:

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For $X \sim$ iid from $N(\mu, \sigma^2)$,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\Rightarrow L(\mu, \sigma^2) = \prod_{i=1}^{n} f(x_i)$$

$$= \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left(-\tfrac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right)$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\mu)^2\right)$$

$$\Rightarrow \log(L(\mu,\sigma^2)) = \ln(L(\mu,\sigma^2))$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$\Rightarrow$ For MLE, take derivative w.r.t $(\mu, \sigma^2) = 0$

$$\Rightarrow \frac{d}{d\mu}(\log(L(\mu,\sigma^2))) = 0$$

$$\& \frac{d}{d\sigma^2}(\log(L(\mu,\sigma^2))) = 0$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\& \frac{1}{2\sigma^2}\left(\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 - n\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\& \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 - n = 0$$

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$$\Rightarrow \quad \frac{1}{n}\sum_{i=1}^{n} x_i = \mu$$

$$\& \quad \frac{1}{n}\sum_{i=1}^{n} (x_i-\mu)^2 = \sigma^2$$

$\Rightarrow$ MLE for (μ, σ^2) is :

$$\left(\hat{\mu}_{MLE}, \hat{\sigma}^2_{MLE}\right) = \left(\bar{x}, s^2\right)$$

where $\bar{x}$ = Sample Mean

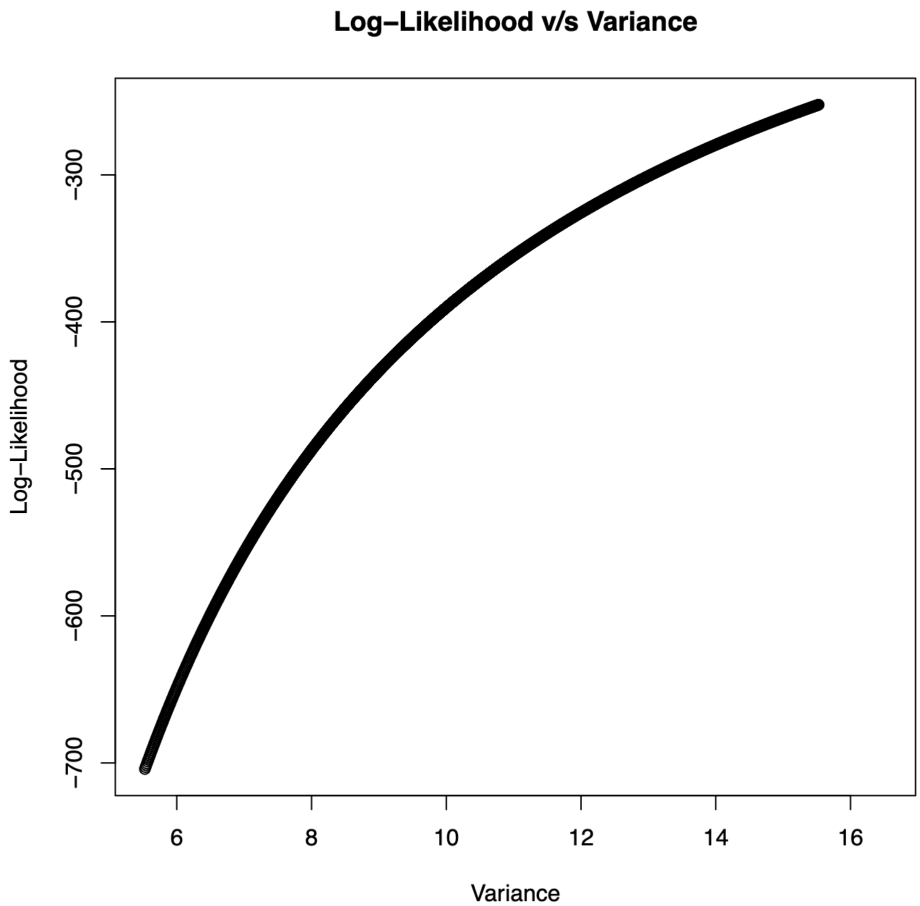
s^2 = Sample Variance

The ML estimates obtained for (μ,σ) are:

$\mu = 4000.044$

$\sigma = 15.52671$

Now assuming the mean (μ) to be known, the likelihood function was plotted against the variance. The graph obtained is as shown below:



This shows that the likelihood function takes the maximum value at $\sigma \approx 15$, which validates the value obtained as the MLE.

$\exp(-\mu)$ is equivalent to defining an exponential distribution for $\lambda = 1$.

$\Rightarrow$ The MLE obtained for $\exp(-\mu)$ was:

$\lambda = 0.001005684$

(which is approximately 0)