Lecture 52

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1 Recap

Tensors of rank k over a vector space V is a multilinear map

$$T : V^k \to \mathbb{R}$$

Given a basis of V $e_1, \dots, e_n$, we have a basis for V^k composed of $\Phi_I(e_{j1}, \dots, e_{jk})$ which is equal to one if $(j1, \dots, j_k) = I$, where I is an ordered list (that possibly repeats). The dimension is then n^k.

Alternating k tensors are tensors whose sign changes when any two vectors are swapped, i.e.

$$T(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = -T(v_1, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \dots, v_k)$$

They form a linear subspace. Since the sign of permutations are well-defined, we can alternatively define alternating tensors as

$$T^\sigma = \operatorname{sgn}(\sigma)T$$

2 Altnernating tensors

We are looking for a basis for $A^k(V)$, the set of alternating tensors. Consider a set of ordered $k - tuples$. Unlike the ϕ_I as above, we do not need to know the order of the permutation, since it is determined (by definition). Now given a permutation $\bar{I}$ with increasing indices, e.g. $(v_1, v_2, v_3, v_5, \dots)$, we can similarly define a $\phi_{\bar{I}}$, which has value ± 1 on permutations of $\bar{I}$ depending on the sign. We want to say the set of this ϕ forms a basis. It is obvious that they are linearly independent. For an arbitrary tensor f, letting a_I be $f(v_{i1}, \dots, v_{ik})$ with I being increasing, consider $g = \sum_I a_I \phi_I$. For an arbitrary $w \in V^k$, we can write w as a sum of basis vectors. For both f, g, the function is only (possibly) nonzero on basis vectors where there are no repeating indices. For the rest, the fact that f, ϕ are alternating ensures $f(w) = g(w)$. Since w is arbitrary, we can say that ϕ_I spans $A^k(V)$, completing the proof.

Example 2.1. Determinants are alternating tensors.

3 Wedge Product

We want an operation

$$\bigwedge : A^k(V) \times A^l(V) \to A^{k+l}(V)$$

We know we can take the tensor product $w_1 \otimes w_2$, which is alternating except (possibly) for switches between the first k and last l entries. We then desire an operaton to make it alternating.