Lecture 18

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1 Wave Equation: Plane Waves

Recall in simple media,

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho_v}{\varepsilon_0} \\
\vec{\nabla} \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{H} &= \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}
\end{aligned}
$$

Consider a pair of parallel planes along z. Drawing a closed loop (clockwise) from z to $z + \Delta z$ along each plate, Faraday's Law gives

$$
\begin{aligned}
V_{\text{emf}} &= \oint_C \vec{E} \cdot d\vec{l} \\
&= -\frac{v(z,t)}{h} + 0 + \frac{v(z+\Delta z, t)}{h} \times h + 0 \\
&= \frac{\partial v(z,t)}{\partial z}
\end{aligned}
$$

On the other side, we get

$$
\begin{aligned}
V_{\mathrm{emf}} &= -\frac{\partial}{\partial t}\int \mu\vec{H}\cdot d\vec{s} \\
&= -\frac{\partial}{\partial t}\frac{\mu i}{w}\Delta z \times h \\
&= \frac{\mu h}{w}\Delta z\left(-\frac{\partial i}{\partial t}\right)
\end{aligned}
$$

Therefore

$$\frac{\partial v}{\partial z} = -L'\frac{\partial i}{\partial t}$$

Likewise, using Ampere's Law,

$$
\begin{aligned}
\oint \vec{H}\cdot d\vec{l} &= \int \sigma\vec{E}\cdot d\vec{s} + \frac{d}{dt}\int \varepsilon\vec{E}\cdot d\vec{s} \\
\ldots &= \ldots \\
\frac{\partial i}{\partial z} &= -G'v - C'\frac{\partial v}{\partial t}
\end{aligned}
$$

2 Phasor Form of Maxwell's Equations

Assume (using Einstein summation)

$$\vec{E} = E_i\cos(\omega t + \phi_i)\hat{e}_i$$

Then

$$\vec{E} = \Re\left\{\left[E_i e^{j\phi_i}\hat{e}_i\right]e^{j\omega t}\right\}$$

and the same applies to $\vec{H}$. Then

$$
\begin{aligned}
\vec{\nabla}\times\tilde{\vec{E}} &= -j\omega\mu\tilde{\vec{H}} \\
\vec{\nabla}\times\tilde{\vec{H}} &= \sigma\vec{E} + j\omega\varepsilon\tilde{\vec{E}} \\
&= j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\tilde{\vec{E}}
\end{aligned}
$$

where we define

Definition 2.1 (Complex Permittivity)**.**

$$\varepsilon_c = \varepsilon + \frac{\sigma}{j\omega}$$

Note: For $\varepsilon >> \frac{\sigma}{\omega}, \varepsilon_c \approx \varepsilon$, and the conduction current is much less than the displacement current, so the medium behaves as a good dielectric. Conversely, $\varepsilon_c \approx \frac{\sigma}{j\omega}$, and the medium behaves as a good conductor.

Example 2.1. Seawater has $\varepsilon_r = 81$ AND $\sigma \approx 4$. Then for $f = 1$kHz,

$$\omega\varepsilon = 2 \times 10^3 \pi \times \frac{81 \times 10^{-9}}{36\pi} \approx 4 \times 10^{-6} << 4 = \sigma$$

so we can approximate it as a conductor. For even higher frequencies, e.g. $f = 100$MHz, then $\omega\varepsilon \approx 10^{-1}$.

Recall that in phasor form, the curl of $\vec{E}$ is proportional to $\vec{H}$. Then combining both phasor equations,

$$\begin{aligned}
\vec{\nabla} \times \vec{\nabla} \times \tilde{\vec{E}} &= -j\omega\mu\vec{\nabla} \times \tilde{\vec{H}} \\
\vec{\nabla}(\vec{\nabla} \cdot \tilde{\vec{E}}) - \nabla^2 \tilde{\vec{E}} &= -j\omega\mu \times j\omega\varepsilon_c \tilde{\vec{E}} \\
-\nabla^2 \tilde{\vec{E}} &= \omega^2 \varepsilon_c \mu \tilde{\vec{E}} \\
(\nabla^2 + \omega^2 \varepsilon_c \mu)\tilde{\vec{E}} &= 0
\end{aligned}$$

Doing the same for $\vec{H}$, we get also

$$(\nabla^2 + \omega^2 \varepsilon_c \mu)\tilde{\vec{H}} = 0$$

This is called the wave equation, or Helmholtz equation.

Definition 2.2 (Complex Wavenumber)**.**

$$k_c = \omega\sqrt{\varepsilon_c \mu} = \beta - j\alpha$$

where α is the attenuation constant, and β the phase constant.

For a medium with $\sigma = 0$, we get a real wavenumber $k_c \in \mathbb{R}$.

Assume $\tilde{\vec{E}} = E_x(z)\hat{x}$. The Laplacian then becomes a second derivative, and the Helmholtz equation becomes

$$\frac{d^2\tilde{E}_x}{z^2} + k^2\tilde{E}_x = 0$$

We have solved this previously, and we know

$$\tilde{E}_x = E_x^+ e^{-jkz} + E_x^- e^{jkz}$$

Substituting into the curl of $\vec{E}$,

$$\begin{aligned}\vec{\nabla} \times \tilde{\vec{E}} &= \hat{z}\frac{\partial}{\partial z} \times \tilde{E}_x \\ &= \left((-jk)E_x^+ e^{-jkz} + jkE_x^- e^{jkz}\right)\hat{y}\end{aligned}$$

Solving for $\tilde{\vec{H}}$, considering only the positive wave,

$$\tilde{\vec{H}}^+ = \frac{E_x^+}{\frac{\omega\mu}{k}} e^{-jkz}\hat{y}$$

For a wave that propogates along z, we have an $\vec{E}$ field that oscillates along x, and a $\vec{H}$ field along y, which is what we see on every E&M textbook. They are related by

Definition 2.3 (Intrinsic Wave Impedance)**.**

$$\frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\varepsilon\mu}} = \sqrt{\frac{\mu}{\varepsilon}}$$