Lecture 6

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January 18, 2024

1 Wave Impedance

Recall

$$\tilde{V}(d) = V_0^+ e^{j\beta d}[1 + \Gamma e^{-j2\beta d}]$$
$$\tilde{I}(d) = \frac{V_0^+}{Z_0} e^{j\beta d}[1 - \Gamma e^{-j2\beta d}]$$

where the reflection coefficient is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

If we look at the standing waves, or the (time) maximum of $|\tilde{V}(d)|, |\tilde{I}(d)|$, they have a wavelength half of that of the voltage/current. Voltage and current are out of phase by π.

Definition 1.1 (Standing Wave Ratio)**.** The standing wave ratio is

$$S = \frac{|\tilde{V}|_{\text{max}}}{|\tilde{V}|_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

For a matched line, $S = 1 = 0\text{dB}$. For an open circuit, $|\Gamma| = 1, S = \infty = -\infty\text{dB}$.

Recall the reactance is positive for inductors and negative for capacitors. Then

$$\Gamma = \frac{jX - Z_0}{jX + Z_0} \Rightarrow |\Gamma| = 1$$

Definition 1.2 (Wave Impedance)**.**

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = Z_0 \frac{1+\Gamma_d}{1-\Gamma_d}$$

Definition 1.3 (Input Impedance)**.**

$$Z_i = Z(d=l) = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

Example 1.1. For $v_g = 10\text{V}, R_g = 50\Omega, l = 2.25\lambda, Z_0 = 50\Omega, v_p = 3 \times 10^8 \text{m s}^{-1}, f = 1\text{GHz}, Z_L = 100 + j75\Omega$, we get
wavelength: $\lambda = \frac{v_p}{f} = 30\text{cm}$
length: $l = 2.25\lambda = 67.5\text{cm}$
Reflection coefficient: $\Gamma = \frac{100+j75-50}{100+j75+50} = 0.537e^{j29.7^\circ}$
To solve for input impedance, first note

$$\beta l = \frac{2\pi}{\lambda} \times 2.25\lambda = 4.5\pi$$

Then

$$Z_i = Z_0 \times \frac{Z_L + jZ_0 \tan(4.5\pi)}{Z_0 + jZ_L \tan(4.5\pi)} = Z_0 \times \frac{Z_L + jZ_0\infty}{Z_0 + jZ_L\infty} = \frac{Z_0^2}{Z_L}$$

Plugging the numbers in, this gives $Z_i = 16 - j12\Omega$. Note that the addition of the transmission line turns the imaginary component of the load from positive to negative. In other words, the load goes from a resistor and inductor in series to a resistor and capacitor in series. This equivalent circuit is V_g, Z_0, Z_i in series. This makes it trivial to find V_i, which is

$$V_i = 10 \times \frac{16 - j12}{66 - j12} = 298e^{-j26.6^\circ}\text{V}$$

But then this is equal to $V_0^+ e^{j\beta l}(1 + \Gamma e^{-j2\beta l})$. Plugging $\beta l = 4.5\pi$, we get

$$V_i = V_0^+ j(1 - \Gamma)$$

which gives V_0^+.