Tutorial 1

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1 Problem Solving Flowchart

1. Write out the Lagrangian
2. Apply the Euler-Lagrange equation
3. Solve the differential equation, maybe with approximations

Example 1.1 (Brachistochrone Problem)**.** Find the shape of a track that can get a bead from one point to another (lower) point as fast as possible. In other words, solve for the path that minimises time. First we write out the expression for time.

$$\begin{aligned} T &= \int \frac{ds}{v} \\ &= \int \frac{ds}{\sqrt{2gy}} \\ &= \int \sqrt{\frac{dx^2 + dy^2}{2gy}} \\ &= \int \sqrt{\frac{1 + y'^2}{2gy}} dx \end{aligned}$$

Before applying the Euler-Lagrange equation, note the deus ex machina identity

$$f - y' \frac{\partial f}{\partial y'} = C$$

that applies for functions where $\frac{\partial f}{\partial x} = 0$. Then rewriting the integral as $T = \int f dx$,

$$\begin{aligned}
\frac{\partial f}{\partial y'} &= \frac{y'}{\sqrt{2gy(1+y'^2)}} \\
f - y'\frac{\partial f}{\partial y'} &= \frac{1+y'^2}{\sqrt{2gy(1+y'^2)}} - \frac{y'^2}{\sqrt{2gy(1+y'^2)}} \\
\frac{1}{\sqrt{2gy(1+y'^2)}} &= C \\
(1+y'^2)y &= \frac{1}{2gC^2} \\
&= k^2
\end{aligned}$$

where the new constant k is as defined. It is painfully obvious that the solution is

$$\begin{cases} x &= \frac{1}{2}k^2(\theta - \sin\theta) \\ y &= \frac{1}{2}k^2(1 - \cos\theta) \end{cases}$$

which happens to be a cycloid.