Tutorial 7

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February 29, 2024

1 Wedge Products

Theorem 1.1. *Let V be a vector space, with alternating tensors $f \in A^k(V), g \in A^l(V)$. We define an element $f \wedge g \in A^{k+l}(V)$ such that there is*

- *Associativity: $f \wedge (g \wedge h) = (f \wedge g) \wedge h$*
- *Homogeneity: $(cf) \wedge g = c(f \wedge g) = f \wedge (cg)$*
- *Distributivity: If f, g have the same order, then $(f + g) \wedge h = f \wedge h + g \wedge h, h \wedge (f + g) = h \wedge f + h \wedge g$*
- *Anticommutativity: If f, g re of order k, l respectively, then $g \wedge f = (-1)^{kl} f \wedge g$*
- *Given a basis a_i for V, let ϕ_i be dual bases of V^*, ψ_I be corresponding elementary alternating tensors. If I is an ascending k tuple,*

$$\psi_I = \bigwedge_j \phi_{ij}$$

Proof. Let $F \in \mathcal{L}^k(V)$. Define the linear operation

$$AF = \sum_\sigma (\mathrm{sgn}\sigma) F^\sigma$$

If F is alternating, then the signs cancel out, and obviously $AF = (k!)F$. Note also that

$$\begin{aligned}
(AF)^\tau &= \sum_\sigma (\text{sgn}\sigma)(F^\sigma)^\tau \\
&= \sum_\sigma (\text{sgn}\sigma) F^\tau \circ \sigma \\
&= (\text{sgn}\tau) \sum_\sigma (\text{sgn}\tau)(\text{sgn}\sigma) F^{\tau\circ\sigma} \\
&= (\text{sgn}\tau) \sum_\sigma (\text{sgn}\tau \circ \sigma) F^{\tau\circ\sigma} \\
&= (\text{sgn}\tau) AF
\end{aligned}$$

Then the output of A is alternating. Now define

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g) \in A^{k+l}(V)$$

Since A is linear, homogeneity and distributivity are trivial. For anticommutativity,

$$\begin{aligned}
A(F \otimes G) &= \sum_\sigma (\text{sgn}\sigma)(F \otimes G)^\sigma \\
&= \sum_\sigma (\text{sgn}\sigma)((G \otimes F)^\pi)^\sigma \\
&= (\text{sgn}\pi) \sum_\sigma (\text{sgn}\sigma \circ \pi)(G \times F)^{\sigma\circ\pi} \\
&= (\text{sgn}\pi) A(G \otimes F)
\end{aligned}$$

Now to permutate $F \otimes G$ to $G \otimes F$, we need to move each of the last l entries k steps back, so kl elementary operations are needed. This gives us the sign of π, so

$$A(F \otimes G) = (-1)^{kl} A(G \otimes F)$$

which completes the proof.
Associativity is a bit more involved. First we want to show that $AF = 0 \Rightarrow A(F \otimes G) = 0$. This is trivial, since

$$(\text{sgn}\sigma) F(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) G(v_\sigma(k+1), \ldots, v_{\sigma(k+l)}) = (\text{sgn}\sigma) \times 0 = 0$$

If h has order m, f has order k. We want to show the following.

$$\begin{aligned} AF \wedge h &= \frac{1}{m} A(F \otimes h) \\ \frac{1}{k!m!} A(AF \otimes h) &= \frac{1}{m} A(F \otimes h) \\ A(AF \otimes h - k!F \otimes h) &= 0 \\ A((AF - k!F) \otimes h) &= 0 \end{aligned}$$

Then the first line is true if $A(AF - k!F) = 0$, from the previous identity. But since AF is alternating, and A is linear, $A(AF) = k!AF, A(k!F) = k!AF$. Now writing $F = f \otimes g$,

$$\begin{aligned} f \wedge g &= \frac{1}{k!l!} AF \\ (f \wedge g) \wedge h &= \frac{1}{k!l!} AF \wedge h \\ &= \frac{1}{k!l!m!} A(F \otimes h) \\ &= \frac{1}{k!l!m!} A(f \otimes g \otimes h) \end{aligned}$$

Doing the same but in reverse order, let $G = g \otimes h$, and

$$\begin{aligned} g \wedge h &= \frac{1}{l!m!} AG \\ f \wedge (g \wedge h) &= \frac{1}{l!m!} f \wedge AG \\ &= \frac{(-1)^{k(l+m)}}{l!m!} AG \wedge f \\ &= \frac{(-1)^{k(l+m)}}{k!l!m!} A(g \otimes h \otimes f) \\ &= \frac{(-1)^{k(l+m)}(-1)^{k(l+m)}}{k!l!m!} A(f \otimes g \otimes h) \\ &= \frac{1}{k!l!m!} A(f \otimes g \otimes h) \\ &= (f \wedge g) \wedge h \end{aligned}$$

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