Lecture 4

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1 Continued...

Let $f : X \mapsto Y$. Note that for $B, Y - B \in Y$, their preimages $f^{-1}B$ and $f^{-1}(Y - B)$ are disjoint, but their union is X. (prove it yourself!)

Proposition 1.1. *$f : X \mapsto Y$ is continuous iff the preimage of any closed set in Y is closed in X.*

The proof follows from the statement above.

Example 1.1. Let $X \subset \mathbb{R}^2$ be the union of the closed disk with radius 1 around the origin, and the point $(4, 0)$. Let

$$f(x) = \begin{cases} 1 & x \in \text{ closed disk} \\ 0 & x = (4, 0) \end{cases}$$

Then the preimage of any closed set is closed, so $f(x)$ is continuous.

2 Compact Sets

Let (X, d) be a metric space. We take $A \subseteq X$.

Definition 2.1 (Open Cover)**.** An open cover of A is a collection of open sets $\mathcal{U}_i, i \in I$ such that $A \subseteq \bigcup_{i \in I} \mathcal{U}_i$.

Remark. Any open set A has a cover. We can take $\mathcal{U}_1 = X$, or take $\mathcal{U}_i$ to be any open ball around a given point in A, and have a $\mathcal{U}_i$ for every point in A.

Definition 2.2 (Compact)**.** A set is compact iff any open cover has a finite subcover. In other words, there exists a finite number of $\mathcal{U}_i$ (that partially forms said open cover) such that their union is also an open cover.

2.1 Properties of Compact Sets

Proposition 2.1. *Any compact set is closed and bounded.*

Proof. Let A be said compact set.
Boundedness: Define $A_i = \mathcal{U}(0; i)$. Then the union of A_i for $i \in \mathbb{N}$ is obviously an open cover. Then there is a subset of A_is that is also an open cover. Hence A is bounded.
Closedness: since A is bounded, there is a point b in $X - A$. Then define B_i to be the closed ball containing b with radius i, and C_i be its complement. Define a sequence of B_i with $i = \frac{1}{n}, n \in \mathbb{N}$. The the intersection of all such B_i is closed and contains only b, and the union of all such C_i is open and contains all points except for b. Then the union of C_i is an open cover of A. With there being a finite subcover, we know there is a finite n such that $C_{i(n)}$ covers A, so B_i is a closed set containing a. Recall the definition of B_i. Then we have an open $B_i' = \mathcal{U}\left(b, \frac{1}{n}\right)$ that contains b. Since b is arbitrary, the complement of A is open, so A is closed. $\square$

Proposition 2.2. *If A is compact, then $f(A)$ is also compact for any continuous f.*

Proof. Let B_i form an open cover of $f(A)$. Define $A_i = f^{-1}(B_i)$. Then A_i form an open cover of A. Since A is compact, we have a finite collection of $A_i, i \in I'$ that covers A. Then the collection of B_i for $i \in I'$ is a finite subcover of B. $\square$