Lecture 8

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1 Differentiation for Functions of Many Variables

We can rewrite $f : U \to \mathbb{R}^n$ as

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

The derivative of a function f at $x_0 \in U$ can be thought of as a matrix or a linear map. This means

$$f(x) = f(x_0) + g(h) + R(h), R(h) = o(h)$$

where g is linear and R is a correction term that tends to 0 faster than h. Now this linear map can be represented by a unique matrix (up to basis).

Now define

$$Df(x_0)_{ij} = \frac{\partial f_i}{\partial x_j}(x_0)$$

We can prove that this matrix is the desired linear map. For simplicity, assume $f : \mathbb{R}^m \mapsto \mathbb{R}$. Then using that fact that $\frac{\partial f}{\partial x_i} = Df \cdot e_i$, we see that Df is the derivative. We can generalise this by applying the theorem on each row of f, ie each dimension of its range.

Lemma 1.1.

$$f'(a;u) = Df(a) \cdot u$$

where u is a unit vector.

Proof. We assumed differentiability for the LHS to make sense. Then letting B be its derivative,

$$\lim_{t \to 0} \frac{f(a+tu) - f(a) - Btu}{|tu|} = 0$$
$$\lim_{t \to 0} \frac{f(a+tu) - f(a)}{t} - Bu = 0$$

□

Lemma 1.2. *If f is differentiable at a, then*

$$Df(a) = \begin{pmatrix} D_1 f(a) & D_2 f(a) & \dots & D_m f(a) \end{pmatrix}$$

Proof. Note that $Df(a)e_i = f'(a;e_i)$. Then letting $Df(a)_j = \lambda_j$, note that

$$D_j f(a) = f'(a;e_j) = Df(a)e_j = \lambda_j$$

□