# Lecture 42

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## 1 Manifolds

A $k$ dimensional manifold in $\mathbb{R}^k$ is a set $M^k \in \mathbb{R}^n$ with the property that $\forall p \in M^k \exists \alpha : U \to \mathbb{R}^n$ wwith $\alpha(U) = V \subseteq M^k$ with open $U \subseteq \mathbb{R}^k, V \subseteq M^k$ satisfying

1. $\alpha \in C^r$
2. $\alpha$ is one-to-one
3. $\alpha^{-1}$ is continuous
4. The rank of $D\alpha$ is $k$

We call $\alpha$ a coordinate chart.

**Example 1.1.** Consider the map $\pi$ from the surface of a sphere to the $\mathbb{R}^2$ plane by means of a line connecting the north pole to said point and finding its intersection with the $\mathbb{R}^2$ plane. This is *not* a coordinate chart, but $\pi^{-1}$ is. Then $\mathbb{S}^2$ is a manifold without boundary.

*Note: If you think $\alpha$ preserves distance, you need to seek medical help.*

**Example 1.2** (Ellipsoid)**.** Is $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\} \subset \mathbb{R}^3$ an open manifold? It is, if you use tangent planes.

**Example 1.3** (Parabaloid)**.** Is $\{z = 5x^2 + 7y^2\} \subset \mathbb{R}^3$ a manifold? It is. Define

$$\alpha(x, y) = (x, y, 5x^2 + 7y^2)$$

It is obvious that $\alpha$ is continuous. Its inverse is also continuous (preimage of open set is open). Finally,

$$D\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 10x & 14y \end{pmatrix}$$

which has rank 2, since its (only) two columns are linearly independent.

Consider any $C^r$ function $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^k$ is open. Similar to the example above, the graph of $f$ is a $k$ dimensional manifold in $\mathbb{R}^{k+1}$.

Now we see why every condition is needed. The rank being $k$ means that there are $k$ "directions" locally, and that they are all smooth. If not, the image of $U \subseteq \mathbb{R}^k$ under $\alpha$ has less than $k$ dimensions, so it doesn't make sense to call it a $k$ dimensional manifold. Without $C^r$, a cone can be a manifold, which is weird because of its vertex.