Plasma

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1 Introduction

Plasmas light up when energy from line radiation is in the optical range, sort of like burning metals. However, flames light up because of blackbody radiation, which is different. However, flames can still ionize air molecules up to a certain degree, like plasmas (but weaker). There are a wide range of values that they can take, e.g. $n < 1\text{cm}^{-3}, T \approx 1\text{K}$ to $n \geq 10^{26}\text{cm}^{-3}, T \approx 10^{11}\text{K}$.

Plasma is usually created from gases. It can be produced by

- Electron impact ionization
- Ion impact
- Fast neutral
- X-rays, lasers, etc
- Others, see Dolan 49, 50

It can be destroyed by

- Volume Recombination
  - Radiative recombination Where ions and electrons combine to form a neutral, releasing $E = h\nu$
  - 3-body recombination Two electrons and an ion collide, producing an electron and a neutral

- Surface Recombination Charged particles are attracted to a surface, where they recombine

Example 1.1 (Plasma density established by balance between production & destruction)**.** We look at radiative recombination and electron impact as the destruction and creation methods. At steady-state,

$$n_e n_g \overline{\sigma v_{\text{iz}}} = n_e n_i \overline{\sigma v_{\text{rec}}} = n_e^2 \overline{\sigma v_{\text{rec}}}$$

Then

$$\frac{n_g}{n_i} = \frac{\overline{\sigma v_{\text{rec}}}}{\overline{\sigma v_{\text{iz}}}}$$

where iz stands for ionization. At $T = 100\text{eV}$, the ratio is around 10^{-7}. At $T = 10\text{eV}$, the ratio is 10^{-5}eV.

Example 1.2 (Electron ionization and ion diffusion to walls)**.** In the previous example, both are scaled by volume, which can be cancelled out. In this case, one is governed by volume and the other by area, so we need to keep these values in mind. Creation and loss rate are then

$$R_c = n_e \overline{n_n \sigma v_{\text{iz}}} V$$

$$R_L = D_\perp \frac{dn_i}{dr} A$$

Assuming a torus with inner radius a and outer radius R, we get

$$\overline{n_n} = \frac{2D_\perp}{a^2 \overline{\sigma v_{\text{iz}}}}$$

Plugging in some values,

$$\overline{n_n} \approx \frac{2 \times 1}{1^2 \times 10^{-14}} \approx 10^{14}$$

Note: the n and g subscripts are basically the same. Source: prof

If we want to compare the loss rates, we get the ratio

$$\frac{R_{\text{diff}}}{R_{\text{rec}}} = \frac{D_\perp \frac{n_e}{a} \times 2\pi a \times 2\pi R}{n_e^2 \overline{\sigma v_{\text{rec}}} \times \pi a^2 2\pi R}$$

Using generic values of $T = 10^4\text{eV}, n = 10^{20}\text{m}^{-3}, a = 1\text{m}, D_\perp = 1\text{m}^2\,\text{s}^{-1}$.

1.1 Thermodynamic Equilibrium and the Saha Equation

We can consider the 3 body recombination as the inverse of electron impact ionization. Hence, it is possible to construct this as a thermodynamic equilibrium.

$$R_{\text{iz}} = a(T)e^{-u/kT} n_e n_n$$
$$R_{\text{rec}} = b(T)n_e^2 n_i = b(T)n_e^3$$

Setting both equal gives

$$\frac{n_e^2}{n_n^2} = \frac{c(T)e^{-u/kT}}{n_n}$$

2 Basic Properties of Plasmas

There is charge neutrality, so $n_e \approx n_i$. There is also Debye shielding, where

$$\lambda_D = \left(\frac{\varepsilon_0 kT}{n_e e^2}\right)^2$$

Gyroradius is

$$R = \frac{mv_\perp}{z_e B}$$

and gyrofrequency is

$$\Omega = \frac{z_e B}{m}$$

2.1 Plasma Frequency

To derive this, we assume no B field, no thermal motion, ions fixed, infinite plasma, and one dimensional motion. We start with the momentum, continuity, and Poisson's equations.

$$m_e n_e \left(\frac{\partial v}{\partial t} + \vec{v} \cdot \frac{\partial \vec{v}}{\partial x}\right) = -e n_e \vec{E}$$
$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e v) = 0$$

$$\varepsilon_0 \frac{\partial E}{\partial x} = e(n_i - n_e)$$

We use linearization, and assume small perturbations in n_e, v_e, E with a base value (0) and change (1). We use the reference frame where $v_0 = 0$. Dividing, we get for momentum and continuity

$$m \frac{\partial v_1}{\partial t} = -eE_1$$

$$\frac{\partial n_1}{\partial t} + n_0 \frac{\partial v_1}{\partial x} = 0$$

where we ignore second order terms. Poisson's euqation becomes

$$\varepsilon_0 \frac{\partial E_1}{\partial x} = -en_1$$

We assume

$$v_1 = \hat{v}_1 e^{i(kx-\omega t)}$$

and similarly for n_1 and E_1; i.e. a sinusoidal solution. Then plugging into the 3 equations, we get

$$\hat{v}_1 \neq 0 \Rightarrow \omega^2 = \frac{n_0 e^2}{\varepsilon_0 m}$$

Below the plasma frequency, it is opaque. Using $\omega = 2\pi f$, we get $f_p = 9\sqrt{n_e}$. For $n = 10^{20}\text{m}^{-3}$, $f_p = 10^{11}$Hz, on the order of microwaves.

2.2 Propagation of EM waves through a Plasma

Show if $\omega_{\text{EM}} > \omega_{P_e}$, then propagation occurs, vice versa.

Definition 2.1. An EM wave can propagate if the electric field can be felt further on.

Speed of electrons moving due to the field is

$$a = \sqrt{\frac{kT_e}{m_e}}$$

Response time is around $\frac{\lambda_D}{a}$. For $\frac{1}{f} > \Delta t$, we get the equivalent expression in terms of ω as desired.

Recall Debye shielding. We can in fact measure plasma density by finding the cutoff frequency, equal to $9\sqrt{n_e}$.

Example 2.1 (Laser Fusion)**.** $n_e \approx 5 \times 10^{28}\text{m}^{-3}$. To further use laser to penetrate and heat it, the cutoff frequency is around $2 \times 10^{15}\text{s}^{-1}$, which gives a wavelength of $\lambda = 150\text{nm}$. Now CO2 lasers have a wavelength of 10,000 nm, Nd - glass with 640 nm, or KF with 250 nm. Frequency multipliers are needed.

3 Plasma Transport Properties

3.1 1D Fluid Equations for One Species

Continuity:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} n\vec{v} = 0$$

Momentum:

$$mn\frac{\partial \vec{v}}{\partial t} + mn\vec{v}\frac{\partial \vec{v}}{\partial x} = e_n(\vec{E} + \vec{v} \times \vec{B}) - \nabla p + C_{\text{coll}}$$

Energy:

$$p = nkT$$

Note that here $\vec{v}$ is fluid velocity, which is much smaller than particle velocity. For an isothermal plasma, T becomes a constant, and so

$$p = nkT \Rightarrow \nabla p = kT\nabla n = m\frac{kT}{m}\nabla n$$

The isothermal speed of sound is $a = \sqrt{\frac{kT}{m}}$, which gives

$$\nabla p = ma^2\nabla n$$

In an adiabetic case (no energy transfer), $p = Cn^\gamma$, where $\gamma = \frac{C_p}{C_v}$. Then

$$\nabla p = \gamma C n^{\gamma-1}\nabla n = \gamma\frac{p}{n}\nabla n = m\frac{\gamma kT}{m}\nabla n$$

Hence $\nabla p = ma^2\nabla n$ where $a = \sqrt{\frac{\gamma kT}{m}}$.

Recall the 1D continuity Equation. For an infinitesimal volume, net outflow is

$$\frac{\partial nv}{\partial x}\Delta x A = \frac{\partial nv}{\partial x}\Delta x \Delta y \Delta z$$

This is equal to the negative of the change of mass in the volume, $-\frac{\partial n}{\partial t}\Delta x \Delta y \Delta z$.

When deriving the 1D Momentum Equation, we use $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$. The right hand side is the sum of other forces. Now, consider $G(x,t)$ where $x = x(t)$. Then the total derivative with respect to time is

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial G}{\partial t} + v_x \frac{\partial G}{\partial x}$$

We call the first term the local change, and the second term is the convective change. In 3D, we similarly get

$$\frac{d\vec{G}}{dt} = \frac{\partial \vec{G}}{\partial t} + (\vec{v}\cdot\nabla)\vec{G}$$

3.2 Interspecies Collisions

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{mn}\frac{\partial p}{\partial x} + \frac{C_{\text{coll}}}{mn}$$

We define

$$\frac{C_{\text{coll}}}{mn} \equiv \nu_{a\to b}(v_a - v_b)$$

Where ν is the momentum transfer collision frequency.

Example 3.1 ($E = B = \frac{\partial}{\partial x} = 0$)**.** We have

$$\frac{dv_a}{dt} = \nu_{a\to b}(v_a - v_b)$$

As $t \to \infty, v_a \to v_b$. We also have

$$\nu_{ab} = n_b\overline{\sigma_{ab}\left(\vec{v}_{pa} - \vec{v}_{pb}\right)}$$

Where $\vec{v}_{pa}$ is the individual particle random velocity. Since that of electrons is much greater, we use

$$\nu_{en} \approx n_n \sigma_{en}\overline{c_e}$$

3.3 Diffusion, Mobility and Conductivity

Particles move in a direction opposite to density gradient. This is diffusion if motion is obstructed by collisions. Charged particles move in the direction of $\vec{E}$. Particles ability to move when restricted by collisions is called **mobility**.

Momentum Equation:

We assume steady state, 1D, $\vec{B} = 0$, and a small drift velocity. The

$$v\frac{\partial v}{\partial x} = \frac{eE}{m} - \frac{1}{mn}\frac{dp}{dx} - \nu v$$

The term on the left-hand side is of the order of $\frac{v^2}{L}$, and the middle term on the right-hand side is on the order of $\frac{nkT}{mnL} = \frac{a^2}{L}$. Usually $a^2 >> v^2$, so we can ignore the convective term. Then

$$mnv\nu = -kT\frac{dn}{dx} + neE \Rightarrow nv = -\frac{kT}{m\nu}\frac{dn}{dx}\frac{en}{m\nu}E$$

Its vector form is

$$\vec{\Gamma} = -D\nabla n + n\mu\vec{E}$$

where $D \equiv \frac{kT}{m\nu}$ is the diffusion coefficient, and $\mu = \frac{e}{m\nu}$ is mobility. The Einstein Relation is

$$\frac{D}{\mu} = \frac{kT}{e}$$

If there is no electric field, then

$$\vec{\Gamma} = -D\nabla n$$

which is Fick's Law.

If $\nabla n = 0$, then $\Gamma = n\mu\vec{E} = n\vec{v}_d$.

We now look at the microscopic explanation of diffusion. At x_0, density is n_0. The flux through surface goverened by densities one mean free path away is

$$\Gamma_{\text{forward}} = \frac{1}{4}\left(n_0 - \lambda\frac{dn}{dx}\right)\bar{c}$$

Now

$$\Gamma_{\text{net}} = -\frac{1}{2}\left(\lambda\frac{dn}{dx}\right)\bar{c} = -D\frac{dn}{dx}$$

where

$$D \approx \frac{\overline{c}\lambda}{2} = \frac{\overline{c}^2}{2\nu} = \frac{8kT}{2\pi m\nu} \approx \frac{kT}{m\nu}$$

Electric Current is $\vec{j} = en\vec{v}$ (if $\nabla n = 0$).

$$\vec{j} = en\mu\vec{E} = \frac{e^2 n}{m\nu}\vec{E} = \sigma\vec{E}$$

Electrical Conductivity can hence be written as

$$\sigma \equiv \frac{e^2 n}{m\nu}$$

Example 3.2 (Fluorescent Lamp)**.** For $n_e = 10^{18}$ and $n_{H_2} = 10^{20}$, then $T_e = 10\text{eV}$, so $\overline{c}_e = 2 \times 10^6$. Momentum transfer cross-section is about 10^{-19} (Dolan p51). Now for $r = 1.5\text{cm}$ and $L = 2\text{m}$, at 1 ampere,

$$j = \frac{1}{0.015^2\pi} = 1400\text{A}\,\text{m}^{-2}$$

Then

$$\nu = \sigma_{en}\overline{c}_e n_{H_2} = 2 \times 10^7\text{s}^{-1}$$
$$\mu = \frac{e}{m_e\nu} = 8800\text{C}\,\text{s}\,\text{kg}^{-1}$$
$$\sigma = en_e\mu = 1400\text{C}^2\,\text{s}\,\text{kg}^{-1}\,\text{m}^{-3}$$
$$j = \sigma E \Rightarrow E = 1\text{V}\,\text{m}^{-1}$$

As a check, $v_d = \mu E = 8800 << \overline{c}_e$, and $\lambda_e n = \frac{1}{n_{H_2}\sigma_{en}} = 0.1 << L$.

Currents tend to be carried by electrons, because $m_e << m_i$, which all else more or less the same.

$$j_e = \frac{e^2 n_e}{m_e\nu_e}\vec{E}$$

and similarly for j_i. Now we assume $\sigma_{en} \approx \sigma_{in}$ (we don't have data on the latter).

We see that most of Ohmic heating goes to the electrons.

$$P_{\text{ohmic}} = \vec{j} \cdot \vec{E} = \sigma\vec{E} \cdot E = \frac{e^2 n}{m\nu}E^2$$

Generally, $m_e\nu_e << m_i\nu_i$, hence most of the power goes to the electrons.

Example 3.3 (Fluorescent Tube (low density))**.** $T_e \approx 50,000\text{K}, T_i \approx 500\text{K}$.

Example 3.4 (Lightning (high density))**.** $T_e \to T_i$

Example 3.5 (Fusion)**.** There is a low density but a large τ_E. For JET, T_e and T_i are within a factor of 2, with the former being greater.

3.4 Ambipolar Diffusion (Flow to a Surface)

If electrons flow to a surface faster, an electric field is produced, which attracts ions. This is where "ambipolar" comes from. They may recombine on such surfaces. In steady state, $\Gamma_e = \Gamma_i = \Gamma$. Then

$$\Gamma_e = -D_e \nabla n_e - n_e \mu_e \vec{E}$$

$$\Gamma_i = -D_i \nabla_i + n_i \mu_i \vec{E}$$

where $\vec{E}$ is the field due to charge separation, i.e. the ambipolar electric field. The difference in signs come from the signs of the charges.

Set $\Gamma_e = \Gamma_i$, use $n_e = n_i, \Delta n_e = \Delta n_i$. Then

$$\Gamma = -\left(\frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}\right)\frac{dn}{dx} = -D_A \frac{dn}{dx}$$

where D_A is the ambipolar diffusion coefficient. Usually $|\mu_e| >> |\mu_i|$, so

$$D_a \approx \left(1 + \frac{D_e \mu_i}{\mu_e D_i}\right) D_i$$

we can use the Einstein relation which gives

$$D_A \approx \left(1 + \frac{T_e}{T_i}\right) D_i$$

Example 3.6. Fusion: $T_e \approx T_i \Rightarrow D_A \approx 2D_i$.
Neon tube: $T_e \approx 100 T_i \Rightarrow D_A \approx 100 D_i$
Note that we always have $D_i < D_A < D_e$.

Example 3.7 (Radial Losses)**.** Hydrogen has a density of 10^{22} and electrons 10^{18}. $T_e = 10\text{eV}, T_i = 0.1\text{eV}, \sigma = 10^{-19}$. Then the mean free path is 10^{-3}. The temperatures give mean speeds of $\bar{c}_e = 2 \times 10^6, \bar{c}_i = 3500$.

$$\nu_{en} = \sigma \bar{c}_e n_{H_2} = 2 \times 10^9$$

$$D_e = \frac{kT_e}{m\nu_{en}} 880$$
$$\nu_{in} = \sigma \bar{c}_i n_{H_2} = 3.5 \times 10^6$$
$$D_i = \frac{kT_i}{m\nu_{in}} = 1.4$$

Thus
$$D_A = D_i \left(1 + \frac{T_e}{T_i}\right) = 140$$

3.5 Diffusion in a Magnetic Field

Recall the equation for Γ. If we turn the electric field off, using $\vec{E} = \vec{v} \times \vec{B}$ and plugging in $\Gamma = nv$, we get

$$\Gamma = -\frac{kT}{m\nu} \nabla n + \frac{e}{m\nu} \Gamma \times \vec{B}$$

Choose $\vec{B} = (0, 0, B)$, and $\frac{dn}{dz} = \frac{dn}{dy} = 0$.
Splitting the equation into its components,

$$\Gamma_x = -D\frac{dn}{dx} + \frac{e}{m\nu}\Gamma_y B$$
$$\Gamma_y = -\frac{e}{m\nu}\Gamma x B$$

Putting the second into the first,

$$\Gamma_x = -\frac{D\nu^2}{\Omega^2 + \nu^2}\frac{dn}{dx} = -D_\perp \frac{dn}{dx}$$

where $D_\perp$ is the magnetic fross-field diffusion coefficient, and $\Omega = \frac{eB}{m}$ is the cyclotron frequency. Note that if there are only electrons, we get $\nu = 0$ which gives $D_\perp = 0$. This makes sense, since collisions don't matter if we consider the centre of mass.

Example 3.8. $n = 10^{20}, T = 10\text{keV}, B = 10\text{T}$ Then $\nu_{ei} = 3 \times 10^3, \Omega_e = 1.2 \times 10^{-12}, D_{e\perp} = 3 \times 10^{-18} D_e = 2 \times 10^{-6}$

We will get to show that $D_{i\perp} = D_{e\perp}$ in assignment 5.

Example 3.9 (Classical Confinement time)**.**

$$\tau_f \approx \frac{\overline{n}V}{\Gamma_{\text{loss}}} = \frac{\overline{n} \times 2\pi R \times \pi a^2}{D_\perp(n/a) \times 2\pi R \times 2\pi a} \approx \frac{a^2}{D_\perp}$$

With $a = 1\text{m}, D_\perp = 2 \times 10^{-6} \Rightarrow \tau_p \approx 5 \times 10^5\text{s}$. However, $D_\perp \approx 1$, and is roughly $\frac{1}{B}$.

Bohm Diffusion:

$$D_B = 60\frac{T}{B}$$

where T is in keV and B in T.

$$D_{\perp\text{neoclassical}} \propto \frac{1}{B^2_{\text{poloidal}}}$$

Without B, $D \propto \frac{1}{\nu}$, and with it, $D_\perp \propto \nu$.

3.6 Ohmic Heating a Fusion Plasma

$$\begin{aligned} P_\Omega &= \vec{j} \cdot \vec{E} \\ &= \frac{j^2}{\sigma} \\ &= j^2\frac{m\nu_{ei}}{e^2 n} \\ &= j^2\frac{m_e 10^{-15} n_e T^{-3/2}}{e^2 n_e} \end{aligned}$$

where n_e cancels out. Now at 10 keV, $\sigma_{\text{plasma}} \approx \frac{1}{20}\sigma_{\text{copper}}$.

3.7 Heat Conduction in a Plasma

We define the thermal flux to be

$$Q = K\frac{dT}{dx}$$

where K is the thermal flux. Recall that in assignment 2, we wanted to show that

$$Q = \frac{1}{4} n\bar{c} \times 2kT \neq \frac{3}{2} kT$$

This is because for lower energy particles, only those closer to the surface could pass through, but more higher energy particles could pass through (larger volume). Net flux is $2k(T_1 - T_2)\frac{1}{4}n\tau$. Using the first order Taylor approximation,

$$Q_{\text{net}} = -n\tau\lambda k dT/dx$$

hence $K \approx n\tau\lambda k$. Now

$$\lambda \approx \frac{\bar{c}}{\nu} \Rightarrow K \approx \frac{nk\tau^2}{\nu} \approx \frac{nk}{\nu} \times \frac{kT}{m} = nkD$$

Then for electrons,

$$K = nkD$$

and

$$K \propto T^{5/2}$$

3.8 Cross-Field Heat Conduction

$$K_\perp \approx K \frac{\nu^2}{\Omega^2}$$

where collisionality ν is for **all** particles.

For electrons, momentum transfer collision frequency are approximately the same, with

$$\nu_{ei}^{\text{mom}} \approx \nu_{ee}^{\text{mom}} \approx \nu_{ee}^{E}$$

For ions,

$$\nu_{ii} = \sqrt{\frac{m_e}{2m_i}} \nu_{ei}$$

where the factor of 2 selectively appears in some texts.

$$K_i = n_i k \left(\frac{kT_i}{\nu_{ii} m_i} \right) \neq n_i k D_i$$

$$K_{i\perp} = K_i \frac{\nu_{ii}^2}{\Omega_i^2} \neq n_i k D_{i\perp}$$

Definition 3.1 (Thermal Diffusivity)**.**

$$nk\chi = K$$

Energy confinement time due to conduction:

$$\tau_E = \frac{3nkT(2\pi R\pi a^2)}{K_\pi(T/a)(2\pi R2\pi a)} \approx \frac{a^2}{\chi_\perp}$$

For $a = 1\text{m}$ and $\chi_\perp \approx 1\text{m}^2\,\text{s}^{-1}$, $\tau_E \approx 1\text{s}$.

4 Collisions

4.1 Kinetic Energy loss in elastic Collisions

We start with head-on collisions. Conservation of momentum gives

$$m_A v_{A1} = m_A v_{A2} + m_B v_{B2}$$

and conservation of energy gives

$$\frac{1}{2}m_A v_{A1}^2 = \frac{1}{2}m_A v_{A2}^2 + \frac{1}{2}m_B v_{B2}^2$$

change in kinetic energy is then

$$\frac{\Delta K_A}{K_A} = \frac{\frac{1}{2}m_A v_{A1}^2 - \frac{1}{2}m_A v_{A2}^2}{\frac{1}{2}m_A v_{A1}^2} = \frac{4m_A m_B}{(m_A + m_B)^2}$$

Recall from the momentum equation,

$$mn\frac{\partial \vec{v}}{\partial t} + mn\frac{\partial \vec{v}}{\partial x} = en(\vec{E} + \vec{v} \times \vec{B}) - \nabla p + C_{\text{coll}}$$

For a 90° collision,

$$\frac{\Delta K_A}{K_A} = \frac{2m_A m_B}{(m_A + m_B)^2}$$

Plugging in the values for protons and electrons, noting that $m_A = m_e << m_B = m_i$. Then

$$\frac{\Delta K_e}{K_e} = \frac{2m_e}{m_i} << 1$$

and

$$\tau_E > \tau_{\text{mom}}$$

4.2 Differential, Total, Momentum Transfer and Energy Transfer Cross Sections

$\sigma_{\text{diff}}(\theta)$ is the cross section for a scattering event with angle θ to $\theta + d\theta$. The number of scattering per volume per time in this range of angles is then $n_e v_e n_i \sigma_{\text{diff}}(\theta) d\theta$.

Total cross section is then

$$\sigma_T = \int_0^\pi \sigma_{\text{diff}}(\theta) d\theta$$

The Rutherford cross section is

$$\sigma_{\text{diff}}(\theta) \approx \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$

Fun Fact: if you integrate this from 0 to π, you get infinity. Hence nobody cares about total cross section.

Momentum Transfer Cross Section:

$$C_{\text{coll}} = mvn\nu_{\text{mom}}$$

where $\nu_{\text{mom}} = \sigma_{\text{mom}} n_e \langle \vec{v}_{r_1} - \vec{v}_{r_2} \rangle$. Change in forward momentum is

$$m_e v_e (1 - \cos\theta)$$

Now

$$\sigma_{\text{mom}} \equiv \int_0^\pi (1 - \cos\theta) \sigma_{\text{diff}}(\theta) d\theta$$

The energy transfer Cross-Section is

$$\frac{\Delta K}{K} = \frac{2m_e}{m_i}$$

for a 90° collision. Energy transfer is ineffective even if momentum transfer isn't. This also means

$$\sigma_{ei}^E \approx \frac{2m_e}{m_i} \sigma_{ei}^{\text{mom}} \Rightarrow \nu_{ei}^E \approx \frac{2m_e}{m_i} \nu_{ei}^{\text{mom}}$$

For σ_{ee} and σ_{ii} however, both are relatively similar, and we ignore the factor of 2.

Example 4.1 (Rate of electron cooling due to collisions with ions)**.** Now

$$\frac{dT_e}{dt} \approx -\frac{T_e}{\tau_{ei}^E} \approx -\nu_{ei}^E T_e \approx -\frac{2m_e}{m_i}\nu_{ei}^{\text{mom}} T_e$$

To obtain Cross-Section values, we consider 2 approaches, large angle collisions from a single impact and cumulative small angle collisions.

For the former, assume an incoming positron collides with an ion with charge ze. Then the potential energy is

$$\frac{ze^2}{4\pi\varepsilon_0 r} \equiv \frac{A}{r}$$

where A is defined as shown. Maximum velocity is then found by

$$\frac{1}{2}mv_\infty^2 = \frac{A}{r_0} \Rightarrow r_0 = \frac{2A}{mv_\infty^2}$$

The above analogy holds for head-on collisions. If the positron is $\frac{r_0}{2}$ away, then the deflection would be $\frac{\pi}{2}$. The cross-section would be related to

$$\pi r_0^2 = \frac{4\pi A^2}{m^2 v_\infty^2}$$

Using $E = \frac{3}{2}kT$ and $\nu_{\text{large}} = n_t \sigma_{\text{large}} v$, we get

$$\nu_{\text{large}} = \frac{4\pi A^2 n_i \sqrt{3kT/m}}{9(kT)^2}$$

Force acting on the electron is $F_{\max} = \frac{A}{b^2}$ for time $\Delta t = \frac{2b}{v}$. Impulse is

$$F\Delta t = \frac{2A}{bv}$$

For an electron passing through a cylindrical shell with height L, radius b and thickness db,

$$N_i = 2\pi b db L n_i$$
$$\Delta p_b = \sum_i \Delta p_i$$

By symmetry, $\overline{\Delta p} = 0$, but its square isn't. Then

$$(\Delta p)^2 = \sum_i (\Delta p_i)^2 + \sum_{i \neq j} \Delta p_i \Delta p_j$$

but the last term goes to 0 on average, as we increase the number of terms. Then

$$\begin{aligned}
\overline{(\Delta p_b)^2} &= \sum_i (\Delta p_i)^2 \\
&= N_i(\Delta p)^2 \\
&= 2\pi b db L n_i \frac{4A^2}{b^2 v^2} \\
\overline{(\Delta p)^2_{\text{TOT}}} &= \frac{8\pi A^2 L n_i}{v^2} \int_{b_{\min}}^{b_{\max}} \frac{db}{b} \\
&= \frac{8\pi A^2 L n_i}{v^2} \ln \Lambda
\end{aligned}$$

where $\Lambda = \frac{b_{\max}}{b_{\min}}$. If we choose $L = \lambda^{\text{mom}}$, then $\overline{(\Delta p)^2_{\text{TOT}}} \approx (mv)^2$. Substituting it in and noting that $\lambda = \frac{v}{\nu}$, we get

$$\nu^{\text{mom}} = \frac{8\pi A^2 n_i \ln \Lambda}{m^{1/2}(3kT)^{3/2}}$$

According to Spitzer, you get the above answer divided by 2. Plugging the constants,

$$\nu_{ei}^{\text{mom}} = 10^{-15} n T^{-3/2}$$

where T is in keV, $z = 1$, and $\ln \Lambda = 20$. $b_{\min} \approx r_0$, since that is where b ends for large angle collisions. $b_{\min} = \lambda_D$, the Debye length, because of plasma shielding. Substituting,

$$\Lambda \approx \frac{\lambda_D}{r_0} = \sqrt{\frac{\varepsilon_0 kT}{ne^2}} \times \frac{mv^2}{2A}$$

Example 4.2. For $z = 1, n = 10^{20}, T = 10\text{keV}$, we have

$$\begin{aligned}
\Lambda &= \sqrt{\frac{\varepsilon_0 kT}{ne^2}} \times \frac{3}{2}kT \times \frac{4\pi\varepsilon_0}{ze^2} \\
&= 7.75 \times 10^8
\end{aligned}$$

Whose logarithm is around 20.5.

Dolan uses $L = \ln \Lambda$, but we don't do that because there's an inundation of L's elsewhere.

For partially ionized gases,

$$\begin{aligned}\nu_e &= \nu_{ei} + \nu_{en} \\ &= 10^{-15} n_i T_e^{-3/2} + \sigma_{en}\overline{c_e} n_n\end{aligned}$$

Example 4.3 (Low Temperature Plasma)**.** For $T_e = 1\text{eV}, \sigma_{en} = 10^{-9}, \overline{c_e} = 6 \times 10^5, n_n = n_e = n_i$, then

$$\frac{\nu_{ei}}{\nu_{en}} = \frac{10^{-15}(10^{-3})^{-3/2}}{10^{-9} \times 6 \times 10^5} = 500$$

5 Plasma Relaxation Times

Definition 5.1 (Plasma Relaxation Time)**.** This is defined by the time required to reach a Maxwellian distribution. This can be species dependent.

For electron equilibria,

$$\nu_{ee}^E \approx \nu_{ee}^{\text{mom}} \approx \nu_{ei}^{\text{mom}}$$

thus

$$\tau_{ee}^E \approx \frac{1}{n_e 10^{-15} T_e^{3/2}}$$

Example 5.1. For $n = 10^{20}, T = 10\text{keV}$, $\tau_{ee}^E = 300\mu\text{s}$.

For ion equilibria,

$$\tau_{ii}^E \approx \tau_{ii}^{\text{mom}} \approx \sqrt{\frac{2m_i}{m_e}}\tau_{ee}^E$$

where we assume $T_i = T_e$. This results in an answer around 100 times larger, or 30 ms.

For electron-ion equilibria,

$$\tau_{ee}^E = \tau_{ie}^E \approx \tau_{ie}^{\text{mom}} \approx \left(\frac{m_i}{m_e}\right)\tau_{ei}^{\text{mom}}$$

This is 4000 times larger than τ_{ee}^E, which is around 1 second.

6 Plasma Waves

6.1 Introduction

Plasma waves depend on n, T, B, E, ν, etc. They are really complex. The oscillate like sine waves, e.g.

$$n = n_0 + \hat{n}\exp(i(kz - \omega t))$$

$k = \frac{2\pi}{\lambda}$ is the wavenumber, where λ is wavelength. $\omega = 2\pi f$ is the angular frequency.

Definition 6.1 (Dispersion Relation)**.** The dispersion relation is $\omega(k)$, since usually there is some relationship between ω and k.

We have a wavefunction

$$\Psi = A\exp(i(kz - \omega t))$$

Definition 6.2 (Phase Velocity)**.** Phase velocity is defined as

$$v_p = \frac{dz}{dt} = \frac{\omega}{k}$$

In practice, there is a wavepacket consisting of wavefunctions of different frequencies. This gives

$$\Psi \int A(k)e^{i(kz-\omega t)}dk$$

Let κ be the width of $A(k)$ at half the maximum height $A(k_0)$. We only care about the integral where A is not too small. We can then let

$$\begin{aligned}
\omega(k) &= \omega(k_0 + \kappa) \\
&= \omega_0 + \frac{\partial \omega}{\partial k}\Big|_{k_0}\kappa + \frac{1}{2}\frac{\partial^2 \omega}{\partial k^2}\Big|_{k_0}\kappa^2 + \ldots \\
kz - \omega t &\approx k_0 z - \omega_0 t + \kappa z - \frac{\partial \omega}{\partial k}\kappa t \\
\Psi &\approx e^{i(k_0 z - \omega_0 t)}\int A(k)e^{i\kappa\left(z - \frac{\partial \omega}{\partial k}t\right)}dk
\end{aligned}$$

The envelope moves with $\frac{dz}{dt}$, setting the phase to be constant. This gives the group velocity

Definition 6.3 (Group Velocity)**.**

$$v_g = \frac{d\omega}{dk}$$

6.2 Waves in Cold Plasma

Langevin Equation

$$mn\frac{\partial \vec{v}}{\partial t} + mn(\vec{v}\cdot\nabla)\vec{v} = -\nabla p + qn(\vec{E} + \vec{v}\times\vec{B}) - mn\vec{v}\nu$$

If we use a cold plasma approximation, we can assume $T = 0$, which is cold. Then $p = 0$, and it has no gradient. We ignore the inertial term (you can't always do that). Then we can simplify this to

$$mn\dot{\vec{v}} = -en(\vec{E} + \vec{v}\times\vec{B}) - mn\vec{v}\nu$$

If we consider electron oscillations,

$$\vec{v} = \hat{v}e^{i(kz-\omega t)}$$

As an aside, this gives the inertial term to be

$$\frac{v\cdot\nabla v}{\dot{v}} = \frac{ikv^2}{i\omega v} = \frac{vk}{\omega} = \frac{v}{v_p}$$

Using Maxwell's Equations,

$$\nabla\times\vec{B} = \frac{1}{c^2}\vec{E} - ne\mu_0\vec{v}$$

Assume $\hat{n} <<< n_0$, or in other words, $n = n_0$. With no external fields, we can let both $\vec{B}$ and $\vec{E}$ oscillate by the usual equations. $\vec{v}\times\vec{B}$ is a second order term which we ignore. Combining gives us

$$\hat{v} = \frac{-e\hat{E}}{m(\nu - i\omega)}$$

From Maxwell's equations, we know $\nabla\times\vec{E}$ is in terms of $\vec{B}$, and we have $\nabla\times\vec{B}$. Hence we get the wave equation

$$\nabla\times(\nabla\times\vec{E}) = \left(\frac{\omega^2}{c^2}\hat{E} - i\omega\mu_0 n_e e\hat{v}\right)e^{k(kz-\omega t)}$$

In vacuum, $n_e = 0$, so

$$\nabla \times (\nabla \times \vec{E}) = \frac{\omega^2}{c^2}\vec{E}$$

Assume a plane wave, where $\vec{E} = E_x\hat{\imath} + E_z\hat{k}$, and only E_x oscillates. Then we can simplify ∇ to a partial derivative, and

$$\nabla \times (\nabla \times \vec{E}) = \left(-\frac{\partial^2 E_x}{\partial z^2}, 0, 0\right) = \left(k^2\hat{E}_x, 0, 0\right) e^{i(kz-\omega t)}$$

Plugging back into the equation, we get $\omega = \pm kc$, $\hat{E}_z = 0$. The phase and group velocities are both c, the speed of light. Now we consider a second case, plasma, with $n \neq 0$, but $B_0 = E_0 = \nu = T = 0$.

$$\begin{aligned}\nabla \times (\nabla \times \vec{E}) &= \left(\frac{\omega^2}{c^2}\hat{E} - \frac{i\omega\mu_0 ne^2}{im\omega}\hat{E}\right) e^{i(kz-\omega t)} \\ &= \frac{\omega^2 - \omega_p^2}{c^2}\hat{E}e^{i(kz-\omega t)}\end{aligned}$$

where $\omega_p = \sqrt{\frac{ne^2}{m_e\varepsilon_0}}$ is the plasma frequency. Note for electron sound wave, $\frac{d\omega}{dk} = 0$.

Transverse waves: $k^2c^2 = \omega^2 - \omega_p^2$. This requires $\omega > \omega_p$ for transmission. Else,

$$E \propto e^{-k_I z}$$

giving us dampened waves.

For transverse waves, using $k^2c^2 = \omega^2 - \omega_p^2$ as derived above,

$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1 - (\omega_p/\omega)^2}}$$

$$v_g = \frac{d\omega}{dk} = c\sqrt{1 - (\omega_p/\omega)^2}$$

Group velocity is always less than c, and the opposite holds for phase veliocity. They converge as $\omega \to \infty$, meaning at higher frequency, there is less divergence between different wave packets.

For Collisional electron plasma waves, where $T_e = \vec{B}_0 = \vec{E}_0 = 0, \nu \neq 0$, then the electron momentum conservation equation gives

$$mn\dot{\vec{v}} = -en(\vec{E} + \vec{v} \times \vec{B}) - m_e n\nu\vec{v}$$

We ignore $\vec{v}\times\vec{B}$ since they are second order. Letting $\vec{v}, \vec{E}, \vec{B}, n$ oscillate by $e^{i(kz-\omega t)}$ (where the average value for n is n_0 and 0 for the rest), substituting gives

$$\hat{v} = \frac{-e\hat{E}}{m(\nu - i\omega)}$$

where the hats refer to the maximum amplitudes.

From Maxwell's equations,

$$\begin{aligned}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t} + \mu_0\vec{J} \\
&= \varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t} - \mu_0 ne\vec{v} \\
\nabla \times \left(\frac{\partial \vec{B}}{\partial t}\right) &= \varepsilon_0\mu_0\frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 ne\frac{\partial \vec{v}}{\partial t} \\
\nabla \times \nabla \times \vec{E} = \qquad\qquad &= -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 en\frac{\partial \vec{v}}{\partial t} \\
&= \left(\frac{\omega^2}{c^2}\hat{E} - i\omega\mu en\hat{v}\right)e^{i(kz-\omega t)} \\
&= \left(\frac{\omega^2}{c^2} - \frac{i\omega\mu_0 e^2 n}{m(\nu - i\omega)}\right)\hat{E}e^{i(kz-\omega t)}
\end{aligned}$$

Consider $\vec{E} = (E_x, 0, E_z)$. Then

$$\nabla \times \nabla \times E = \left(-\frac{\partial^2 E_x}{\partial z^2}, 0, 0\right) = (k^2\hat{E}_x, 0, 0)e^{i(kz-\omega t)}$$

Substituting, in transverse waves

$$k^2\hat{E}_x = \left(\frac{\omega^2}{c^2} + \frac{i\omega\mu_0 e^2 n}{m(\nu - i\omega)}\right)\hat{E}_x$$

For longitudinal waves,

$$0 = \frac{\omega^2}{c^2} + \frac{i\omega\omega_p^2}{(\nu - i\omega)c^2}$$

which gives

$$\omega^2 + i\omega\nu = \omega_p^2$$

We see that ω is complex, so we let $\omega = \alpha + \beta i$. Solving,

$$\alpha = \pm\sqrt{\omega_p^2 - \frac{\nu^2}{4}}, \beta = -\frac{\nu}{2}$$

The negative value of β means that there is dampening of longitudinal oscillations through collisions. The same happens for transverse waves, since k is complex.

Here is a list of plasma wave applications.

- Measuring n_e
- Energy transport in laser fusion
- Radio-wave transmissions around the earth (out of sight)
- Satellite Communication requires $f > 10^7$Hz
- Spacecraft re-entry blackout
- Plasma confinement by RF radiation

7 Individual Particle Motions in $\vec{E}$ and $\vec{B}$ Fields

$$\Omega_e = \frac{eB}{m} = \frac{1.6 \times 10^{-19} \times 5}{9.1 \times 10^{-31}} \approx 10^{12}$$

$$\nu_e = 10^{-15} n_e T_e^{-3/2} = 10^{-15} \times 10^{20} \times 10^{-3/2} \approx 3000$$

7.1 Constant magnetic field; no electric field

$$m\dot{\vec{v}} = q(\vec{E} + \vec{v} \times \vec{B}) = q\vec{v} \times \vec{B}$$

Let $\vec{B} = (0, 0, B)$. Then solving in each coordinate,

$$\begin{aligned} m\dot{v}_x &= v_y qB \Rightarrow \dot{v}_x = \Omega v_y \\ m\dot{v}_y &= -v_x qB \Rightarrow \dot{v}_y = -\Omega v_x \\ m\dot{v}_z &= 0 \Rightarrow v_z = C \end{aligned}$$

where $\Omega \equiv \frac{qB}{m}$ is the gyrofrequency. Then $\ddot{v_x} = -\Omega^2 v_x, \ddot{v_y} = -\Omega^2 v_y$. Solving yields

$$v_x = v_\perp \sin \Omega t, v_y = v_\perp \cos \Omega t$$

where $v_\perp = v_x^2 + v_y^2$. Letting r_L be the Larimar radius,

$$\begin{aligned} 2\pi r_L &= \frac{v_\perp}{\Omega/2\pi} \\ r_L &= \frac{v_\perp}{\Omega} \\ &= \frac{mv_\perp}{qD} \end{aligned}$$

Using

$$\frac{1}{2}mv_\perp^2 = kT$$

we get

$$r_L = \frac{\sqrt{2mkT}}{qB}$$

so r_L is much greater for ions than electrons.

7.2 Constant $\vec{E}$ and $\vec{B}$; $\vec{E} \times \vec{B}$ drift

Letting $\vec{B} = (0, 0, B_z), \vec{E} = (0, E_y, 0)$, one obtains

$$\dot{v}_x = \frac{q}{m} v_y B_z$$
$$\dot{v}_y = -\frac{q}{m} v_y B_z + \frac{qE_y}{m}$$
$$\dot{v}_z = 0$$

Solving yields

$$v_x = v_\perp \sin \Omega t + \frac{E_y}{B_z}$$
$$v_y = v_\perp \cos \Omega t$$
$$v_z = C$$

so

$$\vec{v}_D = \frac{\vec{E} \times \vec{B}}{B^2}$$

7.3 Generalizing to other $\vec{F} \times \vec{B}$ drifts

$$\vec{v}_D = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$$

Example 7.1 (Gravity).

$$v_D = \frac{1}{q} \frac{F_q}{B}$$

Example 7.2 (Curvature Drift). The force is

$$\vec{F} = \frac{m v_\parallel^2 \vec{r}}{r^2}$$

Now

$$v_D = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2} = \frac{2E_\parallel}{q} \frac{\vec{r} \times \vec{B}}{r^2 B^2}$$

7.4 Inhomogeneous $\vec{B}$ Field ($\vec{E} = 0$)

With a constant $\vec{B}$ field, a charged particle will undergo circular motion. If $\vec{B}$ field has varying strength, it will spiral in a certain direction. We define

$$\delta x = 2(r_{L1} - r_{L2})$$
$$\tau = \frac{1}{2}(\tau_1 + \tau_2)$$

where

$$\tau_1 = \frac{2\pi}{\Omega_1} = \frac{2\pi m}{qB_1}, \tau_2 = \frac{2\pi m}{qB_2}$$

and

$$r_{L1} = \frac{v_\perp}{\Omega} = \frac{v_\perp m}{qB_1}, r_{L2} = \frac{v_\perp m}{qB_2}$$

so

$$\begin{aligned} v_D &= \frac{\Delta x}{\tau} \\ &= \frac{2(r_{L1} - r_{L2})}{\frac{1}{2}(\tau_1 + \tau_2)} \\ &= \frac{4\frac{v_\perp m}{q}(\frac{1}{B_2} - \frac{1}{B_1})}{\frac{2\pi m}{q}\left(\frac{1}{B_1} + \frac{1}{B_2}\right)} \\ &= \frac{2v_\perp}{\pi}\frac{B_1 - B_2}{B_1 + B_2} \end{aligned}$$

The gradient of $\vec{B}$ is approximately $\frac{B_1 - B_2}{r_{L1} + r_{L2}}$, so

$$B_1 - B_2 = (r_{L1} + r_{L2})\nabla B = \frac{mv_\perp}{q}\frac{B_1 + B_2}{B_1 B_2}\nabla B$$

Define B to be the average of B_1 and B_2.

$$\begin{aligned} v_D &\approx \frac{2v_\perp}{\pi}\frac{1}{B_1 + B_2}\frac{mv_\perp}{q}\frac{B_1 + B_2}{B^2}\nabla B \\ &\approx \frac{\frac{2}{\pi}mv_\perp^2}{qB^2}\nabla B \\ &\approx \frac{mv_\perp^2 \nabla B}{2qB^2} \end{aligned}$$

where we use the approximation $B_1 B_2 \approx B^2$. In a 3D case, this becomes

$$\vec{v}_D = \frac{E_\perp}{qB^3}\vec{B} \times \nabla B$$

Alternatively, write $\vec{F} = -\mu \nabla B$ where the magnetic moment is

$$\mu = \frac{\frac{1}{2}mv_\perp^2}{B}$$

Example 7.3. For $\nabla B \approx 1\mathrm{T\,m^{-1}}, B \approx 3\mathrm{T}, E_\perp \approx 10\mathrm{keV}$, we have

$$v_d = \frac{10 \times 1000 \times 11606}{1.6 \times 10^{-19}} \times \frac{1}{3^2} = 10^3 \mathrm{m\,s^{-1}}$$

7.5 Tokamak Drifts

For velocity in the radial direction, and a $\vec{B}$ field in a tokamak in the angular direction, ions go up and electrons go down, by the right hand rule. In a toroidal solenoid, wires wrap around the toroid (parallel to cross section) to produce magnetic fields. By Ampere's Law,

$$\begin{aligned}
\nabla \times \vec{B} &= \mu_0 \vec{j}_{TOT} \\
\int \nabla \times \vec{B} d\vec{A} &= \mu_0 \int \vec{j}_{TOT} d\vec{A} \\
\int \vec{B} \cdot d\vec{l} &= \mu_0 N \int j dA \\
2\pi R B &= \mu_0 N I \\
B &= \frac{\mu_0 n I}{2\pi R}
\end{aligned}$$

If ∇B points inwards, ions go up and electrons go down, for the same configuration. If there is an $\vec{E}$ field pointing downwards, both ions and electrons go outwards.

7.6 Magnetic Mirrors

Due to symmetry, $B_\theta = 0, \frac{\partial}{\partial \theta} = 0$. Then

$$\nabla \cdot \vec{B} = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

Usually, there is a weak dependence of $\frac{\partial B_z}{\partial z}$ on r, so integrating,

$$\begin{aligned} rB_r &= -\int_0^r \rho \frac{\partial B_z}{\partial z} d\rho \\ &= -\frac{1}{2} r^2 \frac{\partial B_z}{\partial z} \end{aligned}$$

The Lorentz force is then

$$F_r = qv_\theta B_z$$
$$F_\theta = q(-v_{r_z} + v_z B_r)$$
$$F_z = -qv_\theta B_r$$

Substituting the above results,

$$F_z = \frac{1}{2} qv_\theta r \frac{\partial B_z}{\partial z}$$

On the axis, $v_\theta = -v_\perp$ and $r = r_L$. Further expanding,

$$\begin{aligned} F_z &= -\frac{1}{2} qv_\perp r_L \frac{\partial B_z}{\partial z} \\ &+ -\frac{1}{2} \frac{mv_\perp^2}{B} \frac{\partial B_z}{\partial z} \\ &= -\mu \frac{\partial B_z}{\partial z} \end{aligned}$$

Consider $F_z = ma_z = m\frac{dv_z}{dt} = -\mu\frac{\partial B_z}{\partial z}$. Now

$$\begin{aligned} mv_z \frac{dv_z}{dt} &= \frac{d}{dt}\left(\frac{1}{2} mv_z^2\right) \\ &= -\mu \frac{\partial B_z}{\partial z} \frac{dz}{dt} \\ &= -\mu \frac{dB_z}{dt} \end{aligned}$$

so

$$\frac{d}{dt}\left(\frac{1}{2}mv_z^2+\frac{1}{2}mv_\perp^2\right)=\frac{d}{dt}\left(\frac{1}{2}mv_z^2+\mu B_z\right)$$
$$-\mu\frac{dB}{dt}+\frac{d}{dt}(\mu B_z)=0$$
$$\frac{d\mu}{dt}=0$$

So μ is a constant. We can visualise the above derivative as follows. Total energy, kinetic and magnetic, is conserved. If kinetic energy goes to zero before we reach the maximum magnetic field, it is reflected. If not, it passes the magnetic mirror.

At a low field, $B_0 \Rightarrow v_\perp = v_{\perp 0}, v_z = v_{z0}$. At reflection, $B' \Rightarrow v_\perp = v'_\perp, v_z = 0$.

$$\frac{1}{2}\frac{mv_{\perp 0}^2}{B_0}=\frac{1}{2}\frac{mv_\perp^2}{B'}$$

and

$$v'_\perp = v_{\perp 0}^2 + v_{z_0}^2 = v_0^2$$

thus

$$\frac{B_0}{B}=\frac{v_\perp^2}{v_\perp'^2}=\frac{v_{v_\perp 0}^2}{v_0^2}\equiv\sin^2(\theta)$$

Defining θ_m such that particles within it do not get reflected,

$$\sin^2\theta_m=\frac{B_0}{B_{\max}}=\frac{1}{R_m}$$