Tutorial 11

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1 Orientable Manifolds

We want to integrate k forms on the k manifold M. If there is only a single coordinate chart $\alpha : U \to V$, define

$$\int_M \omega = \int_{\mathrm{Int}(U)} \alpha^* \omega$$

Definition 1.1. Let $g : A \to B$ be a diffeomorphism of open sets in $\mathbb{R}^k$. We say g is orientation preserving if $\det Dg > 0$. Otherwise, it is orientation-reversing.

Associated there is a linear transformation

$$g_* : T_x(\mathbb{R}^k) \to T_{g(x)}(\mathbb{R}^k)$$

Definition 1.2. Let M be a k manifold in $\mathbb{R}^n$ with coordinate patches $\alpha_i : U_i \to V_i$ on M. If M can be covered by a collection of coordinate charts such that each $V_i \cap V_j \neq \emptyset$, we have $\alpha_i^{-1} \circ \alpha_j$ is orientation-preserving, then we say M is orientable. Otherwise it is non-orientable.

Definition 1.3. Given a collection A of orientation preserving coordinate charts for M, extend this collection to all coordinate charts β such that $\beta^{-1} \circ \alpha$ is oreintation preserving $\forall \alpha \in A$. This extended collection defines an orientation on M. A manifold M together with orientation is called an orientation manifold.

Definition 1.4. Let M be a $(n-1)$ manifold in $\mathbb{R}^n$. If $p \in M$, let $(p; n)$ be a unit vector in the n dimensional vector space $T_p(\mathbb{R}^n)$ that is orthogonal to the subspace $T_p(M)$. So n uniquely determined up to sign. Given orientation of M, choose coordinate patch $\alpha : U \to V$ on M about p belong to this orientation; let $\alpha(x) = p$. Then $\frac{\partial \alpha}{\partial x_i}$ is a basis for $T_p(M)$. So $\begin{pmatrix} n & D\alpha(x) \end{pmatrix}$ gives a basis for $T_p(\mathbb{R}^n)$ and we take n such that its determinant is positive.

Example 1.1 (Non Orientable Manifolds)**.** Möbius strip.

Definition 1.5. Let M be an n manifold in $\mathbb{R}^n$. If $\alpha : U \to V$ is a coordinate chart ons where M then $D\alpha$ is an $n \times n$ matrix. Define natural orientation on M to be all coordinate chart where $\det D\alpha > 0$.

To check this, M may be covered by such coordinate charts. Given $p \in M$, let $\alpha : U \to V$ be a coordinate patch about p, with U open in $\mathbb{R}^n$ or $\mathbb{H}^n$. WLOG U is connected, so $\det D\alpha$ is either positive or negative on all of U. If positive, then α is our coordinate chart. Else, $\alpha \circ r$ is our coordinate chart, where r is the reflection

$$r(x_1, \ldots, x_n) = (-x_1, \ldots, -x_n)$$

2 Induced Orientation of ∂M

Theorem 2.1. *Let $k > 1$. If M is an orientable k manifold with nonempty boundary, then ∂M is orientable.*

Proof. Let $p \in \partial M$. Let $\alpha : U \to V$ be a coordinate patch about p. There is a corresponding coordinate patch α_0 on ∂M that is said to be obtained by restricting α. Formally, define $b : \mathbb{R}^{k-1} \to \mathbb{R}^k$ by the equation

$$b(x_1, \ldots, x_{k-1}) = (x_1, \ldots, x_{k-1}, 0)$$

Then $\alpha_0 = \alpha \circ b$. We show that if α, β are such that $\det D(\beta^{-1} \circ \alpha) > 0$, then so is their restrictions. Let $g : W_0 \to W_1$ be the transfer function. Then $\det Dg > 0$. Now if $x \in \partial \mathbb{H}^k$ then the derivative Dg at x has the last row $Dg_k = \begin{pmatrix} 0 & 0 & \frac{\partial g_k}{\partial x_k} \end{pmatrix}$ where the last element is positive. Changing $x_1, \ldots, x_{k-1}$ by a little does not change the value of g_k. But increasing x_k a little changes value of g_k non-negatively, so $\frac{\partial g_k}{\partial x_j} = 0$ for $j < k$ and $\frac{\partial g_k}{\partial x_k} > 0$. $\square$

Definition 2.1. Let M be an orientable k manifold with nonempty boundary. Given orientation of M, the corresponding orientation is: if k is even, orientation is given by restrictions, else we restrict the opposite orientation.