Assignment 2

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1. Questions 1 and 6 from Dolan.

   (a) Find $\langle v_x^3 \rangle$ and $\langle v_x^4 \rangle$ for a Maxwellian distribution.

   Solution: The Maxwellian distribution function is

   $$f_M(\vec{x},\vec{v},t) = n(\vec{x},t)\left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2} \tag{1}$$

   Then

   $$\begin{aligned}
   \langle v_x^3 \rangle &= \frac{\int f_M(\vec{x},\vec{v},t) v_x^3 d\vec{v}}{f_M(\vec{x},\vec{v},t) d\vec{v}} \\
   &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n(\vec{x},t)\left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2} v_x^3 dv_x dv_y dv_z}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n(\vec{x},t)\left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2} dv_x dv_y dv_z} \\
   &= \frac{\int_{-\infty}^{\infty} e^{-\beta v_x^2} v_x^3 dv_x}{\int_{-\infty}^{\infty} e^{-\beta v_x^2} dv_x} \\
   &= 0
   \end{aligned}$$

   Because the numerator is the integral of an odd function. Similarly, we replace v_x^3 with v_x^4 and get

   $$\begin{aligned}
   \langle v_x^4 \rangle &= \frac{\int_{-\infty}^{\infty} e^{-\beta v_x^2} v_x^4 dv_x}{\int_{-\infty}^{\infty} e^{-\beta v_x^2} dv_x} \\
   &= \frac{3\sqrt{\pi}\beta^{-5/2}}{4\sqrt{\pi}\beta^{-1/2}} \\
   &= \frac{3}{4\beta^2}
   \end{aligned}$$

   (b) Find the mean value of x.

Solution:

$$\begin{aligned}
\overline{x} &= \int_0^\infty x p(x) dx \\
&= \int_0^\infty x \exp(-n_2 \sigma x) n_2 \sigma dx \\
&= -x \exp(-n_2 \sigma x)\Big|_0^\infty + \int_0^\infty \exp(-n_2 \sigma x) dx \\
&= -\frac{1}{n_2 \sigma} \exp(-n_2 \sigma x)\Big|_0^\infty \\
&= \frac{1}{n_2 \sigma}
\end{aligned}$$

2. Obtain the Maxwellian distribution for velocities in three dimensions.

Solution:

$$\begin{aligned}
f &\propto \exp\left(-\frac{E}{kT}\right) \\
&\propto \exp\left(-\frac{m||v||^2}{2kT}\right) \\
&\propto \exp\left(-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2kT}\right)
\end{aligned}$$

Then

$$f = C \exp\left(-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2kT}\right)$$

We know that

$$\int_{-\infty}^\infty e^{-kt^2} dt = \sqrt{\frac{\pi}{k}}$$

Hence integrating with respect to v_x, v_y, v_z, we get

$$C \left(\frac{2\pi kT}{m}\right)^{3/2} = n$$

Since there are n particles, the integral over f has to be n and not 1. Rearranging gives

$$C = n \left(\frac{m}{2\pi kT}\right)^{3/2}$$

The full form of f is then

$$f = n \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2kT}\right)$$

as desired.

3. Find, for a Maxwellian distribution, Γ, Q_x, and the energy density. Explain physically why Q is greater

than the product of Γ times the average particle energy of $\frac{3}{2}kT$.

Solution: Recall $\Gamma_x = nv_x$.

$$\begin{aligned}
\langle \Gamma_x \rangle &= \frac{n(\beta/\pi)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \Gamma_x \exp\left[-\beta\left(v_x^2+v_y^2+v_z^2\right)\right] dv_x dv_y dv_z}{n(\beta/\pi)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\beta\left(v_x^2+v_y^2+v_z^2\right)\right] dv_x dv_y dv_z} \\
&= \frac{\int_0^\infty nv_x \exp\left(-\beta v_x^2\right) dv_x}{\int_{-\infty}^{\infty} \exp\left(-\beta v_x^2\right) dv_x} \\
&= \frac{\frac{n}{2\beta}}{\sqrt{\frac{\pi}{\beta}}} \\
&= \frac{n}{2}\left(\frac{1}{\pi\beta}\right)^{1/2} \\
&= \frac{n}{4}\left(\frac{4}{\pi\beta}\right)^{1/2} \\
&= \frac{n}{4}\left(\frac{8kT}{\pi m}\right)^{1/2}
\end{aligned}$$

Then $Q_x = \Gamma_x E = \frac{1}{2}nmv_x(v_x^2+v_y^2+v_z^2)$. The first term is equal to

$$\begin{aligned}
\langle \Gamma_x E_x \rangle &= \frac{nm}{2} \times \frac{n(\beta/\pi)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} v_x^3 \exp\left[-\beta\left(v_x^2+v_y^2+v_z^2\right)\right] dv_x dv_y dv_z}{n(\beta/\pi)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\beta\left(v_x^2+v_y^2+v_z^2\right)\right] dv_x dv_y dv_z} \\
&= \frac{nm}{2} \frac{\int_0^\infty v_x^3 \exp\left(-\beta v_x^2\right) dv_x}{\int_{-\infty}^{\infty} \exp\left(-\beta v_x^2\right) dv_x} \\
&= \frac{nm}{2} \times \frac{\frac{1}{2\beta^2}}{\sqrt{\frac{\pi}{\beta}}} \\
&= \frac{nm}{4} \times \frac{4k^2T^2}{m^2} \times \left(\frac{m}{2\pi kT}\right)^{1/2} \\
&= \frac{n}{4} \times 2kT \times \left(\frac{2kT}{m}\right)^{1/2} \\
&= kT\Gamma_x
\end{aligned}$$

The second and third term are the same. Their sum is thus

$$\begin{aligned}
\langle \Gamma_x E_{y,z} \rangle &= nm \times \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} v_x v_y^2 \exp\left[\beta\left(v_x^2+v_y^2+v_z^2\right)\right] dv_x dv_y dv_z}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} \exp\left[-\beta\left(v_x^2+v_y^2+v_z^2\right)\right] dv_x dv_y dv_z} \\
&= nm \times \frac{\int_0^{\infty} v_x \exp\left(-\beta v_x^2\right) v_x dx \int_{-\infty}^{\infty} v_y^2 \exp\left(-\beta v_y^2\right) dv_y}{\left(\int_{-\infty}^{\infty} \exp\left(-\beta v_y^2\right) dv_y\right)^2} \\
&= nm \times \frac{\frac{\sqrt{\pi}}{2\beta^{3/2}} \times \frac{1}{2\beta}}{\frac{\pi}{\beta}} \\
&= \frac{n}{4} \times m \times \left(\frac{1}{\pi\beta^3}\right)^{1/2} \\
&= \frac{n}{4} \times m \times \left(\frac{8k^3T^3}{\pi m^3}\right)^{1/2} \\
&= \frac{n}{4} kT \left(\frac{8kT}{\pi m}\right)^{1/2} \\
&= \Gamma_x kT
\end{aligned}$$

Their sum gives $\Gamma_x \times 2kT$ as desired. Q_x is greater than $\Gamma_x E$ because it only takes into account particles that pass through the surface, which has higher kinetic energy. Consider all particles to the left of the plane $x = 0$ that has a positive v_x. The average energy of such particles is $\frac{3}{2}kT$. We can approximate the flux by considering the particles that pass through $x = 0$ in time Δt. However, those with kinetic energy too low would not have enough velocity to pass $x = 0$ within the given time. This means that the average particle that crosses the surface has higher kinetic energy. Since Q_x is the product of Γ_x and the average energy of particles that *pass through the surface*, it is greater than simply the product of Γ_x and average energy.

4. Show that the fusion rate for colliding pairs with relative approach energy lying between E and E + dE is proportional to the given expression. Find $E_{\max}$. Estimate the number of particles in the distirbution that contribute effectively to fusion.

Solution: Reaction rate is proportional to $n_1 n_2 \langle \sigma v \rangle$. This means it is proportional to $f(E)\sigma v$. For convenience, we define

$$g(E) = \frac{-2^{3/2}\pi^2 M^{1/2} q_1 q_2}{4\pi\varepsilon_0 h E^{1/2}}$$

We know

$$\sigma \propto \frac{1}{E}\exp(g(E))$$

As for velocity,

$$E = \frac{1}{2}mv^2 \Rightarrow v \propto \sqrt{E}$$

As for $f(E)$,

$$\begin{aligned}
f(E) &\propto \int \exp(-\beta v^2) d\vec{v}^3 \\
&\propto \int \exp(-\beta v^2) v^2 dv \\
&\propto \int \exp\left(-\frac{E}{kT}\right)\sqrt{E} dE
\end{aligned}$$

Where the extra terms from using spherical coordinates are discarded, since they are constant multiples, and in the last expression, we make use of the facts that $dE \propto vdv$ and $\sqrt{E} \propto v$. Now multiplying all together, we get

$$\text{reaction rate } \propto \frac{1}{E}\exp(g(E)) \times \sqrt{E} \times \sqrt{E}\exp\left(\frac{E}{kT}\right) = \exp\left(g(E) - \frac{E}{kT}\right)$$

The maximum can be found when the derivative of the above is equal to 0.

$$\begin{aligned}
\frac{d}{dE}\exp\left(g(E) - \frac{E}{kT}\right) &= 0 \\
(g'(E) - \frac{1}{kT})\exp\left(g(E) - \frac{E}{kT}\right) &= 0 \\
\frac{\sqrt{2M}\pi q_1 q_2}{4\varepsilon_0 h E^{3/2}} - \frac{1}{kT} &= 0 \\
E &= \left(\frac{kT\sqrt{2M}\pi q_1 q_2}{4\pi\varepsilon_0 h}\right)^{2/3}
\end{aligned}$$

For a DT reaction with $T = 20\text{keV}$, this translates to $E_{\max} = 49\text{keV}$. For a D*-D* reaction at 20keV, $E_{\max} = 46.2\text{keV}$. Proportion of particles is then

$$\begin{aligned}
\frac{\int_{46.2}^{\infty}\sqrt{E}\exp\left(-\frac{E}{kT}\right)dE}{\int_0^{\infty}\sqrt{E}\exp\left(-\frac{E}{kT}\right)dE} &= \frac{\int_{46.2}^{\infty}\sqrt{E}\exp(-0.05E)dE}{\int_0^{\infty}\sqrt{E}\exp(-0.05E)dE} \\
&= 20\%
\end{aligned}$$

5. Question 5 is skipped because I don't have time :(
6. (a) Derive 2B21 from 2B17 in Dolan.

Solution: We can use spherical coordinates.

$$\begin{aligned}
\langle \sigma v \rangle &= \left(\frac{\beta}{\pi}\right)^{3/2} \int e^{-\beta v^2} \sigma(v) v d\vec{v} \\
&= \left(\frac{\beta}{\pi}\right)^{3/2} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} e^{-\beta v^2} \sigma(v) v * v^2 \sin\theta dv d\theta d\phi \\
&= \left(\frac{\beta}{\pi}\right)^{3/2} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} e^{-\beta v^2} \sigma(v) v^3 dv \\
&= \left(\frac{\beta}{\pi}\right)^{3/2} \times 2\pi \times 2 \int_0^{\infty} e^{-\beta v^2} \sigma(v) v^3 dv \\
&= \left(\frac{\beta}{\pi}\right)^{3/2} 4\pi \int_0^{\infty} e^{-\beta v^2} \sigma(v) v^3 dv
\end{aligned}$$

(b) Convert this expression for $\langle \sigma v \rangle$ from an integral over v to one over E where $E = \frac{1}{2} m_r v^2$ and $m_r \equiv \frac{m_1 m_2}{m_1 + m_2}$, the reduced mass.

Solution: The differential becomes $dE = m_r v dv$. Substituting,

$$\begin{aligned}
\langle \sigma v \rangle &= \left(\frac{\beta}{\pi}\right)^{3/2} 4\pi \int_0^{\infty} \exp\left(-\frac{2E\beta}{m_r}\right) \sigma(E) \times \frac{2E dE}{m_r^2} \\
&= \left(\frac{\beta}{\pi}\right)^{3/2} \frac{8\pi}{m_r^2} \int_0^{\infty} \exp\left(-\frac{2E\beta}{m_r}\right) E \sigma(E) dE
\end{aligned}$$

(c) Use the latter expression to confirm that the $\langle \sigma v \rangle$ value given in Fig. 2C3 of Dolan for 30 keV temperature D-T plasma is correct, using the σ value from Fig. 2C1 for D^+ on a stationary T target. Use an $n = 6$ midpoint approximation for the integral.

Solution: From Fig. 2C1, we see that $\sigma(E)$ has a maximum value on the order of 10^{-28}. Therefore in our approximation, we take $\sigma(E) = 10^{-28}$ to be the cutoff for integration from 0 and to infinity respectively, since $\sigma(E)$ being close to an order of magnitude lower would limit the error. The integral (ignoring the constants) becomes

$$\begin{aligned}
\int_a^b \exp\left(-\frac{2E\beta}{m_r}\right) E \sigma(E) dE = \overline{\sigma} \int_a^b \exp\left(-\frac{2E\beta}{m_r}\right) E dE \\
= \overline{\sigma} \left(-\frac{\exp\left(-\frac{2E\beta}{m_r}\right)\left(\frac{2E\beta}{m_r} + 1\right)}{\frac{4\beta^2}{m_r^2}} \right)_a^b
\end{aligned}$$

We then take

Energy (keV)	$\sigma\left(\frac{5}{3}E\right)$
0 - 40	2×10^{-29}
40 - 60	3×10^{-28}
60 - 100	4×10^{-28}
100 - 140	3×10^{-28}
140 - 300	2×10^{-28}
$300 - \infty$	6×10^{-29}

This gives an answer of $5.23 \times 10^{-22} \mathrm{m}^4 \, \mathrm{s}^{-2}$. This agrees with value given in Fig 2C3, 6×10^{22}.

(d) In carrying out (c) you will need to show first that the $\sigma(E)$ in your integral will have to be replaced by the σ for D^+ on stationary T evaluated for a deuterium energy $E_D = \frac{5}{3}E$. Explain why this relation holds.

Solution:

$$\begin{aligned}
\frac{E_D}{E} &= \frac{\frac{1}{2}m_D v^2}{\frac{1}{2}m_r v^2} \\
&= \frac{m_D}{m_r} \\
&= \frac{m_D + m_T}{m_T} \\
&\approx \frac{2+3}{3} \\
&= \frac{5}{3}
\end{aligned}$$