Lecture 43

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1 Manifolds

$M^k \subseteq \mathbb{R}^n$ is a k manifold without boundary if $\forall P \in M^k \exists$ an open $U \subseteq \mathbb{R}^k$ and a 1 to 1 C^r function $\alpha : U \to \mathbb{R}^n$ with a continuous inverse, derivative with rank $k \forall x \in U$, and $P \in U$.

Example 1.1 (Graphs of Functions)**.** Graphs of C^r functions over open sets in $\mathbb{R}^n$ are manifolds. The function itself acts as α.

Definition 1.1 (Level Set)**.** Let $F(x_1, \ldots, x_n) : \Omega \subseteq \mathbb{R}^n$ have an open domain Ω. A level set is a set $M = F^{-1}(g) \subseteq \Omega$.

Example 1.2 (Level Set)**.** Consider a level set of a C^r function. We will show in tutorial that $M = F^{-1}(g)$ is a manifold of dimension $n-1$ provided $DF(x) \neq 0 \forall x \in M$.

The above example shows that ellipsoids are manifolds.

We want to build towards manifolds possibly with boundary. It should be a set $M^k \subseteq \mathbb{R}^n$ which is locally modeled on *either* open sets in $\mathbb{R}^n$ or $\mathbb{R}^k_+$, which is $\mathbb{R}^k$ with the last coordinate being positive.

Definition 1.2. Consider a function $f : S \to \mathbb{R}^k$, where $S \subseteq \mathbb{R}^k$. Then we say f is of class C^r if $\exists$ an extension $\tilde{f}$ of f to an open superset $U \supseteq S$ such that $\tilde{f} : U \to \mathbb{R}^n$ is C^r and $\tilde{f} = f$ whenever the latter is defined.

Lemma 1.1. *Suppose $f : U \to \mathbb{R}^n$ is of class C^r, with $U \subseteq \mathbb{R}^k_+$ being relatively open. Then $D\tilde{f}(x)$ is independent of $\tilde{f} \forall x \in U$.*

Proof. Case 1: $x \in \mathrm{Int}(U) \subseteq \mathbb{R}^k$. This is immediately true, since f and $\tilde{f}$ agree. If x is in the boundary, then the derivative depends only on f by continuity.

$$\begin{aligned}
\frac{\partial \tilde{f}_i}{\partial x_j} &= \lim_{h \to 0} \frac{\tilde{f}_i(x + he_j) - \tilde{f}_i(x)}{h} \\
&= \lim_{h \to 0^+} \frac{\tilde{f}_i(x + he_j) - \tilde{f}_i(x)}{h} \\
&= \lim_{h \to 0^+} \frac{f_i(x + he_j) - f_i(x)}{h}
\end{aligned}$$

which depends only on f. $\square$

Then a general k manifold has a similar definition, only that we accept either $U \subseteq \mathbb{R}^k$ of $U \subseteq \mathbb{R}^k_+$ being open.