Lecture 33

niceguy

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1 TE/TM modes in waveguides

For guided waves, we can identify the z dependance by

$$\tilde{\vec{E}}(x, y, z) = \tilde{\vec{e}}(x, y)e^{-j\beta z}$$

and similarly for $\tilde{\vec{H}}$. We can substitute into Maxwell's equations to solve for these unknowns.

$$\begin{aligned}
\vec{\nabla} \times \tilde{\vec{H}} &= j\omega\varepsilon\tilde{\vec{E}} \\
\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} &= j\omega\varepsilon\tilde{E}_x \\
\frac{\partial \tilde{h}_z}{\partial y} + j\beta\tilde{h}_y &= j\omega\varepsilon\tilde{e}_x
\end{aligned}$$

This is repeated for all 6 equations. Rearranging, $\tilde{E}_x, \tilde{E}_y, \tilde{H}_x, \tilde{H}_y$ can be rewritten as linear combinations of partial derivatives of $\tilde{E}_z, \tilde{H}_z$.

Example 1.1. Consider a cross section. $x \in [0, a], y \in [0, b]$, and the material outside is a perfect conductor. For rectangular waveguides in TM mode, $\tilde{E}_z = \tilde{e}_z(x, y)e^{-j\beta z}$. It has to satisfy the wave equation

$$\begin{aligned}
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\tilde{e}_z(x, y)e^{-j\beta z} &= 0 \\
\left(\frac{\partial^2 \tilde{e}_z}{\partial x^2} + \frac{\partial^2 \tilde{e}_z}{\partial y^2} - \beta^2\tilde{e}_z + k^2\tilde{e}_z\right)e^{-j\beta z} &= 0 \\
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 - \beta^2\right)\tilde{e}_z(x, y) &= 0
\end{aligned}$$

At the boundaries, e_z has to vanish. The solution is then

$$\tilde{e}_z(x,y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right), (n,m) \in \mathbb{N}^2$$

With

$$\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 = k^2 - \beta^2$$

Then the propogation constant is

$$\beta_{n,m} = \sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}$$

This can be rewritten as

$$\beta_{n,m} = k\sqrt{1 - \frac{\omega_{c,n,m}^2}{\omega^2}}, \omega_{c,n,m} = \frac{1}{\sqrt{\varepsilon\mu}}\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

We could likewise construct the TE mode.