Lecture 34

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1 Waveguides Summary and Examples

Example 1.1. A TM wave propagates in a rectangular waveguide, with $H_y = 6\cos(25\pi x)\sin(100\pi y)\sin(1.5\pi\times 10^{10}t - 109\pi z)\mathrm{mA\,m^{-1}}$. Its dimensions along x, y are 4 cm and 2 cm respectively.
The modes are as follows

$$k_x = \frac{m\pi}{a} = 25\pi \Rightarrow m = 25a = 25 \times 0.04 = 1$$

$$k_y = \frac{n\pi}{b} = 100\pi \Rightarrow n = 100b = 100 \times 0.02 = 2$$

So this is $\mathrm{TM}_{1,2}$ mode.
If $\mu = \mu_0, \varepsilon = \varepsilon_0\varepsilon_r$, find ε_r.
Recall the final term has the form $\sin(\omega t - \beta z)$. Comparing, we find that the frequency is 7.5×10^9, and

$$\beta = 109\pi = k\sqrt{1 - \left(\frac{f_{c,1,2}}{f}\right)^2}$$

To find ε_r,

$$k = \omega\sqrt{\varepsilon\mu} = 2\pi f\sqrt{\varepsilon_0\varepsilon_r\mu_0} = \frac{2\pi f}{3 \times 10^8}\sqrt{\varepsilon_r} = 50\pi\sqrt{\varepsilon_r}$$

To find f_c,

$$f_{c,m,n} = \frac{1}{2\sqrt{\varepsilon\mu}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{3 \times 10^8}{2\sqrt{\varepsilon_r}}\sqrt{25^2 + 100^2}$$

Substituting, we have

$$
\begin{aligned}
109\pi &= 50\pi\sqrt{\varepsilon_r}\sqrt{1-\frac{25^2+100^2}{2500\varepsilon_r}} \\
2.18 &= \sqrt{\varepsilon_r}\sqrt{1-\frac{1+16}{4\varepsilon_r}} \\
\varepsilon_r &= 9
\end{aligned}
$$

Find the phase velocity v_p.

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 7.5 \times 10^9}{109\pi} = 1.38 \times 10^8 > \frac{1}{\sqrt{\varepsilon\mu}}$$

Find E_x.

$$Z_{\mathrm{TM}} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

The values in the expression are

$$f_{c,1,2} = \frac{3\times 10^8}{2\times 3}\sqrt{25^2+100^2} = 5.15\times 10^9$$

and $\eta = \frac{\mu}{\varepsilon} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \div \sqrt{\varepsilon_r} = \frac{120\pi}{3} = 40\pi$. Substituting,

$$Z_{\mathrm{TM}} = 40\pi\sqrt{1-\left(\frac{5.15}{7.5}\right)^2} = 91.3$$

Then $\tilde{E}_x = Z_{\mathrm{TM}}\tilde{H}_y$, or

$$E_x = 0.55\cos(25\pi x)\sin(100\pi y)\sin(1.5\pi\times 10^{10}t - 109\pi z)\,\mathrm{V\,m^{-1}}$$

Analyzing this in terms of plane waves,

$$H_y \propto \cos(25\pi x)\sin(100\pi y)$$

so

$$
\begin{aligned}
\tilde{H}_y &= H_0\cos(25\pi x)\sin(100\pi y)e^{-j\beta z} \\
&= H_0\frac{e^{j25\pi x}+e^{-j25\pi x}}{2}\frac{e^{j100\pi y}-e^{-j100\pi y}}{2j}e^{-j\beta z} \\
&= \frac{H_0}{4j}\left(e^{j(2.5\pi x+100\pi y-\beta z)} + e^{j(25\pi x-100\pi y-\beta z)} + e^{j(-25\pi x+100\pi y-\beta z)} + ej(-25\pi x-100\pi y-\beta z)\right)
\end{aligned}
$$