# Tutorial 5

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## 1 Lagrange Multipliers

If we have a $g : \mathbb{R}^n \to \mathbb{R}^n$, and $M = g^{-1}(0), Dg(a)$ has full rank for $a \in M$, then $M$ is a manifold. (We have shown this before). The tangent space is

$$T_aM = \ker \nabla g(a)$$

Consider a function $f$ over a closed manifold $M$, where $M = g^{-1}(0)$ and we want to minimise/maximise $f$ over $M$. Think of expanding/contracting level sets of $f$. When they first intersect $M$, there is only one intersection point locally (usually), so it has to be on a tangent plane of the level curve. The same holds for $M$. Then the derivatives of $f$ and $g$ at $a$ are parallel.

**Theorem 1.1.** *Given $U \subseteq \mathbb{R}^n$ open, with $g : U \to \mathbb{R}^p$ be $C^1$, $f : U \to \mathbb{R}$ be a differentiable function. Suppose $f$ has a local extrema on $g^{-1}(0)$ at a point $a$. Then $\exists \lambda_1, \dots, \lambda_p \in \mathbb{R}$ such that*

$$Df(a) = \lambda_i \nabla g_i(a)$$

*or*

$$\frac{\partial f}{\partial x_j} = \lambda_i \frac{\partial g_i}{\partial x_j}$$

*at $a \forall j = 1, \dots, n$ and $g_i(a) = 0$.*

*Manifolds are level sets locally; use the inverse coordinate chart.*

**Example 1.1.** Maximise $xy$ subject to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\begin{aligned} Df &= \lambda Dg \\ (y, x) &= \lambda \left( \frac{2x}{a^2}, \frac{2y}{b^2} \right) \\ \frac{ya^2}{2x} &= \frac{xb^2}{2y} \\ a^2y^2 &= x^2b^2 \end{aligned}$$

Multiplying the constraint by $a^2b^2$,

$$x^2b^2 + y^2a^2 = a^2b^2 = 2x^2b^2 \Rightarrow x = \pm\frac{a}{\sqrt{2}}, y = \pm\frac{b}{\sqrt{2}}$$

**Example 1.2.** Prove that

$$uv \leq \frac{1}{\alpha}u^\alpha + \frac{1}{\beta}v^\beta \forall u, v \geq 0, \alpha, \beta > 0$$

such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.
If either vanishes, then this is true, so we can assume both are strictly positive. Note that the inequality holds for $U, V$ iff it holds for $u = Ut^{\frac{1}{\alpha}}, v = Vt^{\frac{1}{\beta}} \forall t > 0$. If it holds for $UV$, then

$$\begin{aligned} uv &= UVt^{\frac{1}{\alpha}+\frac{1}{\beta}} \\ &= UVt \\ &\leq t\left(\frac{1}{\alpha}U^\alpha + \frac{1}{\beta}V^\beta\right) \\ &= \frac{1}{\alpha}u^\alpha + \frac{1}{\beta}v^\beta \end{aligned}$$

If it holds for $uv$, putting $t = 1$ suffices. This means it suffices to show this for $uv = 1$, since if $uv = t, UV = 1$. We want to show

$$1 \leq \frac{1}{\alpha}u^\alpha + \frac{1}{\beta}v^\beta$$

We can start by minimising the right hand side with the constraint that $uv = 1$. Then

$$\begin{aligned} Df &= \lambda Dg \\ (u^{\alpha-1}, v^{\beta-1}) &= \lambda(v, u) \\ \frac{u^\alpha}{uv} &= \frac{v^\beta}{uv} \\ u^\alpha &= v^\beta \end{aligned}$$

In fact, this is equal to $\lambda$. Then $1 = uv = \lambda^{\frac{1}{\alpha}}\lambda^{\frac{1}{\beta}} = \lambda$. So $u = v = 1$, which gives the equality.

**Example 1.3.** Let $M = g^{-1}(0)$ be a manifold, where

$$g(x, y, z, w) = (x^2 + y^2 + z^2 - 2, y^2 + z^2 + w^2 - 3)$$

Find points of $M$ closest to $(0, \sqrt{2}, \sqrt{2}, 0)$.
First we find the gradients

$$Df = (2x, 2(y - \sqrt{2}), 2(z - \sqrt{2}), 2w)$$

and

$$Dg_1 = (2x, 2y, 2z, 0), Dg_2 = (0, 2y, 2z, 2w)$$

Now $Dg_1, D_g 2$ must be linearly independent, else $x = w = 0$, but then this point cannot lie in $M$. Comparing the first and last coefficients, $\lambda_1 = \lambda_2 = 1$ if $x, y \neq 0$. Then $y = z = -\sqrt{2}$, but $w \notin \mathbb{R}$. If $w = 0$, then $y^2 + z^2 = 3 \Rightarrow x^2 + 1 = 0$. If $x = 0$, then $y^2 + z^2 = 2 \Rightarrow w^2 = 1$. The points are

$$(0, \pm 1, \pm 1, \pm 1), (0, \pm 1, \pm 1, \mp 1)$$