Lecture 27

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1 Extended Integral

Consider the function $f : A \to \mathbb{R}$ not necessarily bounded, with $A \subseteq \mathbb{R}^n$ being open. In single variable calculus, you can say for $f(x) \geq 0$,

$$\begin{aligned}\int_0^1 f(x)dx &= \sup_{\epsilon>0} \int_\epsilon^1 f(x)dx \\ &= \lim_{n\to\infty} \int_{\frac{1}{n}}^1 f(x)dx\end{aligned}$$

However, we don't want to enforce a particular sequence such as $\frac{1}{n}$. Consider a general $A \subseteq \mathbb{R}^n$ which is open, and $f(x)$ is continuous on A. Then we can define $f(x) = f_+(x) - f_-(x)$ where

$$f_+(x) = \begin{cases} f(x) & f(x) \geq 0 \\ 0 & f(x) < 0 \end{cases}$$

$$f_-(x) = \begin{cases} 0 & f(x) > 0 \\ -f(x) & f(x) \leq 0 \end{cases}$$

It is easy to show that f_+, f_- are continuous functions over A. Also $|f|(x) = f_+(x) + f_-(x)$. We then say $f(x)$ is extended-integrable over A iff

$$\sup_{D\in\mathcal{D}} \int_D f_+(x) + \int_D f_-(x) < \infty$$

Where $\mathcal{D}$ is the set of compact and rectifiable subsets of A. Define the extended integral Ext$\int$ by

$$\text{Ext}\int_A f(x) = \text{Ext}\int_A f_+ - \text{Ext}\int_A f_-$$

Remark. If A is bounded and rectifiable, $v(A)$ is well defined. Then $\forall\epsilon > 0 \exists D \in \mathcal{D}$ such that $v(D) > v(A) - \epsilon$.

Proof. Start with $\chi_A(x)$ defined over $Q \supseteq A$. χ_A is integrable, so $\forall\epsilon > 0\exists$ a partition P of Q such that

$$\sum_{R\in P} m_R(\chi_A)v(R) > v(A) - \epsilon$$

m_R is either 0 or 1. Taking only the terms where it is 1, one can see that

$$\sum_{R\subseteq A} v(R) > v(A) - \epsilon$$

since $m_R = 1 \Leftrightarrow R \subseteq A$. Then define $D = \bigcup_{R\in P, R\subseteq A} R$. $\square$

Lemma 1.1. *If $A \subseteq \mathbb{R}^n$ is open, there exists a sequence $D_n \subseteq A$ such that $D_n \subseteq IntD_{n+1}$ and $\bigcup_{n=1}^{\infty} D_n = A$.*

Proof. Define

$$C_n = \{x \in A | d(x, A^c \geq \frac{1}{n})\}$$

Note that C_n is the preimage of $y \geq \frac{1}{n}$, a closed set, under a continuous distance function. Hence C_n is closed. It is also obvious that it is contained in the interior of C_{n+1}. Its union is a subset of A. But any point in A admits an ε ball contained in A, so its distance from A^c is strictly positive, and is containedi n some C_n. So A is also a subset of the union. Then both are equal, i.e.

$$\bigcup_{n=1}^{\infty} C_n = A$$

Construct $D_n's$ out of the $C_n's$ to preserve all properties, but also get that D_n is rectifiable (this implies D_n is bounded, and hence closed). More explicitly, construct

$$C_n \subseteq D_n \subseteq \text{Int}C_{n+1}$$

$\forall x \in C_n \exists Q(x, \varepsilon) \subseteq \text{Int}C_{n+1}$. The union of such Q $\forall x \in C_n$ is a superset of C_n. Define it to be D_n. $\square$