Lecture 35

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1 Group Velocity

Even if we start out with a simple sinusoidal signal, it often ends up as wavepackets, where low frequency envelopes are modulated by high frequency waves. We then get

$$f(t) = g(t)\sin(\omega_0 t)$$

The bandwidth Δw of the envelope signal is much less than ω_0.

Example 1.1. Consider 2 waves with similar frequencies $\omega_0 \pm \frac{\Delta\omega}{2}$. The faster signal propogates with

$$\beta(\omega_+) \approx \beta(\omega_0) + \frac{\Delta\omega}{2}\beta'(\omega_0)$$

and the same holds for ω_-. For the first case,

$$\cos(\omega_+ t - \beta_+ z) = \cos\left((\omega_0 t - \beta(\omega_0)z) + \frac{\Delta\omega}{2}(t - \beta'(\omega_0)z)\right)$$

and similarly for the second case

$$\cos(\omega_- t - \beta_- z) = \cos\left((\omega_0 t - \beta(\omega_0)z) - \frac{\Delta\omega}{2}(t - \beta'(\omega_0)z)\right)$$

Using the identity

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y$$

their sum becomes

$$2\cos(\omega_0 t - \beta(\omega_0)z)\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta\omega}{2}\beta'(\omega_0)z\right)$$

The second term is the low frequency envelope, and the first term is the high frequency wave, which propagates at the same velocity as the initial waves.

From the above example, the velocity of the envelope itself is

$$\frac{\Delta\omega}{2} \div \frac{\Delta\omega}{2}\beta'(\omega_0) = \frac{1}{\beta'(\omega_0)}$$

The definition becomes obvious.

Definition 1.1. The group velocity is defined as

$$v_g = \frac{1}{\beta'(\omega_0)} = \frac{1}{\frac{\partial\beta}{\partial\omega}(\omega_0)}$$

For non-TEM waveguide modes,

$$\beta = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \sqrt{\varepsilon\mu}\sqrt{\omega^2 - \omega_c^2}$$

The phase velocity is

$$\begin{aligned} v_p &= \frac{\omega}{\beta} \\ &= 1/\sqrt{\varepsilon\mu} \times \frac{1}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} > \frac{1}{\sqrt{\varepsilon\mu}} \end{aligned}$$

and the group velocity is

$$\begin{aligned} \frac{\partial\beta}{\partial\omega} &= \sqrt{\varepsilon\mu}\frac{\omega}{\sqrt{\omega^2 - \omega_c^2}} \\ &= \sqrt{\frac{\varepsilon\mu}{1 - \left(\frac{\omega_c}{\omega}\right)^2}} \\ v_g &= \frac{1}{\frac{\partial\beta}{\partial\omega}} \\ &= \sqrt{\frac{1 - \left(\frac{\omega_c}{\omega}\right)^2}{\varepsilon\mu}} \end{aligned}$$

It is immediately seen that $v_g v_p = \frac{1}{\varepsilon\mu} = c^2$.

Example 1.2. For transmission lines, $\beta = \omega\sqrt{L'C'}$, so $v_p = \frac{1}{\sqrt{L'C'}} = v_g$. For a backwards wave line, $\beta = \pm\frac{1}{\omega\sqrt{L'C'}}$. In this case, $v_p = \omega^2\sqrt{L'C'}, v_g = -\omega^2\sqrt{L'C'}$. They have different directions! Similarly, we can have planar negative index lens.