Lecture 6

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1 Connected Spaces

Theorem 1.1. *If $X \subseteq \mathbb{R}^n$ is closed and bounded, then X is compact.*

Proof. Let A be a collection of open sets covering X. Since X is closed, add the open set $\mathbb{R}^n - X$ to our collection. Since X is bounded, there exists a rectangle Q such that $X \subseteq Q$. By compactness of Q, there exists some finite subcover of Q. Note that this is because the union of A and $\mathbb{R}^n - X$ covers $\mathbb{R}^n$ and hence Q. Then this is a finite subcover of X. $\square$

Definition 1.1. If X is a metric space, we say X is connected if X cannot be written as $A \cup B$ where A and B are open, nonempty, and disjoint.

Remark. $\mathbb{R}$ is connected. The only connected subspaces of $\mathbb{R}$ are open intervals.

Proof. Let $\mathcal{U} \subseteq \mathbb{R}$ such that $\mathcal{U}$ is not an interval. Then by definition, there exists $a, b, c \in \mathbb{R}$ with $a < c < b$ and $a, b \in \mathcal{U}$, $c \notin \mathcal{U}$. Then $A = (-\infty, c) \cap \mathcal{U}, B = (c, \infty) \cap \mathcal{U}$, but their union is equal to $\mathcal{U}$. $\square$

Theorem 1.2 (Intermediate Value Theorem)**.** *Let X be connected. If $f : X \mapsto Y$ is continuous, then $f(X)$ is a connected subspace of Y.*

Proof. Suppose $f(X) = A \cup B$, where A, B are disjoint, nonempty and open, then $f^{-1}(A)$ and $f^{-1}(B)$ are open, disjoint, and nonempty. By contradiction, $f(X)$ is connected. $\square$

Proposition 1.1. *If $f : X \mapsto \mathbb{R}$ is continuous and if $f(x_0) < r < f(x_1)$, then $f(x) = r$ for some $x \in X$.*

Proof. Given f, let $A = \{y < r | y \in \mathbb{R}\}$ and $B = \{y > r | y \in \mathbb{R}\}$. Note that both sets are open, disjoint, and nonempty. If $r \notin f(X)$, then $f(X)$ is not connected. $\square$

Definition 1.2 (Line Segment)**.** If $a, b \in \mathbb{R}^n$, the line segment joining them is

$$\{x = a + t(b - a) | 0 \leq t \leq 1\}$$

Any line segment is connected as it is the continuous image of

$$t \mapsto a + t(b - a), t \in [0, 1]$$

Definition 1.3 (Convex Set)**.** A subset $U \subseteq \mathbb{R}^n$ is convex if for all $a, b \in U$, the line segment between them is in U.

Any convex subset $A \subseteq \mathbb{R}^n$ is connected. If $A = U \cup V$ where both are disjoint, nonempty, and open, then there exists $u \in U, v \in V$ such that the line between them can be written as

$$L = (L \cap U) \cup (L \cap V)$$

which contradicts the connectivity of L.