Tutorial 1

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Theorem 0.1. *Let A be a $n \times n$ matrix and $h : \mathbb{R}^n \to \mathbb{R}^n$ be given by $h(x) = Ax$. Let S be a rectifiable set in $\mathbb{R}^n$ and $T = h(S)$. Then $v(T) = |\det A| v(S)$.*

Proof. First consider if A is nonsingular. Then h is a diffeomorphism of $\mathbb{R}^n$ onto itself, so

$$\begin{aligned} v(T) &= \int_T 1 \\ &= \int_S |\det Dg| \\ &= \int_S |\det A| \\ &= |\det A| \int_S 1 \\ &= |\det A| v(S) \end{aligned}$$

Or else, $\det A = 0$, so the dimension of T is less than n. V has measure zero in $\mathbb{R}^n$, hence

$$v(T) = 0 = |\det A| = |\det A| v(S)$$

□

Definition 0.1. Let $a_1, \ldots, a_k$ be linearly independent vectors in $\mathbb{R}^n$. We define the k dimensional parallelpiped $P = P(a_1, \ldots, a_k)$ be the set of all $x \in \mathbb{R}^n$ such that

$$x = \sum_i c_i a_i, 0 \leq c_i \leq 1$$

Theorem 0.2. *Let $a_1, \ldots, a_n$ be n linearly independent vectors in $\mathbb{R}^n$. Let $A = [a_1, \ldots, a_n]$ be an $n \times n$ matrix. Then $v(P) = |\det A|$.*

Proof. Consider the linear transformation $h(x) = Ax$. Then $h(e_i) = a_i$, so h carries the unit cube to P. Then the previous theorem shows that

$$v(P) = |\det A| v(I) = |\det A|$$

$\square$

Definition 0.2. (Probably useless) Let V be an n dimentional vector space. An n tuple $(a_1, \ldots, a_n)$ of linearly independent vectors in V is called an n-*frame* in V. In $\mathbb{R}^n$, we call it right-handed if $\det[a_1, \ldots, a_n] > 0$, and vice versa. An **orientation** is a choice of either the set of right- or left-handed frames.

More generally, choose a linear isomorphism $T : \mathbb{R}^n \to V$ and define one orientation of V to consist of all frame $(T(a_1), \ldots, T(a_n))$ for $(a_1, \ldots, a_n)$, a right-handed frame in $\mathbb{R}^n$.

Example 0.1. In $\mathbb{R}$, a frame is just one number, whose orientation depends on its sign. In $\mathbb{R}^2$, it is when a_2 is 0 to π counterclockwise from a_1. In $\mathbb{R}^3$, this is if $a_1 \times a_2$ points "in the direction of" a_3, i.e. the dot product is positive.

Theorem 0.3. *Let C be an $n \times n$ nonsingular matrix. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be $h(X) = Cx$. Let $(a_1, \ldots, a_n)$ be a frame in $\mathbb{R}^n$. If $\det C > 0$, then $(a_1, \ldots, a_n)$ and $(h(a_1), \ldots, h(a_n))$ have the same orientation. If $\det C < 0$, they have opposite orientation.*

Proof. Let $b_i = h(a_i)$. Then $C[a_1, \ldots, a_n] = [b_1, \ldots, b_n]$, and $\det(C)\det(A) = \det(B)$. The rest is trivial. $\square$