Lecture 2

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Definition 1.1 (Closed Set)**.** A set $B \subseteq X$ is closed iff $X - B \subseteq X$ is open.

Example 1.1 (Sets that are both closed and open)**.** For any nonempty metric space X, the empty set and X itself are 2 unique sets that are both closed and open.

Remark. If d is a distance function of X, it is also a distance function for any $Y \subseteq X$.

Theorem 1.1 (Intersection of Open Sets is Open)**.** *Let A and B be open sets in X. Then $A \cap B$ is always open.*

Proof. Let A and B be open. Then their intersection is either empty or not. The empty set is open. If not, then $x \in A \cap B$. This implies $x \in A$, so there exists an open ball $\mathcal{U}(x; \varepsilon_1) \subseteq A$. Similarly, there exists an open ball ε_2 in B. Pick ε to be the smaller one, then it can be shown that $\mathcal{U}(x; \varepsilon)$ is a subset of both A and B. Since this holds for arbitrary $x \in A \cap B$, the intersection is open. By induction, this holds for any finite intersection of open sets. $\square$

Proposition 1.1. *If $A \subseteq X$ and $Y \subseteq X$ are both open, then $A \cap Y$ is open in Y under the same distance function.*

Proof. Follows from the theorem. $\square$

Recall we define d_1 as sum of $|x - y|$, d_2 to be Euclidean, and d_3 to be the supremum of $|x - y|$.

Theorem 1.2. *Any set open in d_i is open in d_1, d_2, d_3.*

Proof. If a set A is open in d_1, then any point $a \in A$ is associated with an open ball $\mathcal{U}(a, \varepsilon_a) \subseteq A$. Then setting $\varepsilon = \frac{\varepsilon_a}{n}$ shows it is open for d_3. $\square$

Definition 1.2 (Limit Point)**.** Let $A \subseteq X$. Then $x_0 \in X$ is a limit point of A if $\forall \varepsilon > 0, \mathcal{U}(x_0; \varepsilon) - \{x_0\}$ contains a point a point in A.

Proposition 1.2. *A set is closed iff it contains all of its limit points.*

Proof. If A is closed, then let $x \in X - A$. Since $X - A$ is open, there is an ε where $\mathcal{U}(x_0; \varepsilon) \in X - A$, which means it doesn't intersect with A. Hence no limit points are outside A; A contains all of its limit points.
If A contains all of its limit points, then for any $x \in X - A$, then there exists a ε such that $\mathcal{U}(x; \varepsilon) \in X - A$, or else x is a limit point outside of A. Then $X - A$ is open, and A is closed. $\square$

Definition 1.3. Define $\mathcal{L}(A)$ to be the set of limit points of A.

Definition 1.4. Define the closure of A to be

$$\overline{A} = A \cup \mathcal{L}(A)$$

Proposition 1.3. *A closure is always closed.*

Proof. Proof by contradiction. Let x be a limit point outside of $\overline{A}$. Then for some $\varepsilon > 0$, the epsilon-ball only intersects with $\mathcal{L}(A)$ and not A, or else it is in $\mathcal{L}(A)$. Since this $\mathcal{U}(x; \varepsilon)$ is open and contains $a \in \mathcal{L}(A)$, there is a smaller epsilon-ball centred at A that is in $\mathcal{U}(x; \varepsilon)$. Since a is a limit point, we know any epsilon-ball surrounding it contains $a' \neq a$ that is in A. Then $\mathcal{U}(x; \varepsilon)$ contains $a' \in A$ for all x and ε pairs. Now $x \in \mathcal{L}(A)$, which is a contradiction. Hence $\overline{A}$ contains all of its limit points, and it is closed. $\square$

Remark. We have proven that the intersection of 2 open sets are open. By induction, this holds for finitely many open sets. Now that union of any open sets denoted by A_i is obviously open, since any point a in the union is in A_i for some i. Using the fact that A_i is open, $\exists \varepsilon$ such that $U(a, \varepsilon) \subseteq A_i \subseteq \bigcup_i A_i$. Hence the union is open.
Taking the complement, the union of finitely many closed sets is closed, and the intersection of any closed sets is also closed.
Note: it is trivial to prove

$$\left(\bigcup_i A_i\right)' = \bigcap_i A_i'$$

and

$$\left(\bigcap_i A_i\right)' = \bigcup_i A_i'$$

Remark. A closure of A is also the smallest closed superset of A.

Proof. Note that a limit point of A is a limit point of any superset of A. Hence any closed set containing A must contain $\mathcal{L}(A)$. $\square$