Lecture 26

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1 Rectifiable Sets

Note: what we call a rectifiable set in this course is different from what they mean conventionally. We use textbook naming for consistency.

Definition 1.1 (Rectifiable Set)**.** A bounded set $S \subseteq \mathbb{R}^n$ is rectifiable if its boundary has measure zero.

Remark. S is rectifiable iff

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

is integrable.

Define for $Q \supseteq S$

$$V(S) = \int_S 1 = \int_Q \chi_S$$

Recall we will exclusively be considering continuous functions for now on.

1.1 Properties of Rectifiable Sets

1. If S_1, S_2 are rectifiable, then $S_1 \cup S_2, S_1 \cap S_2$ are rectifiable
2. If $f : S \to \mathbb{R}$ is a bounded continuous function, then f is integrable and S is rectifiable
3. If S_1, S_2 are rectifiable, then

$$v(S_1 \cup S_2) = v(S_1) + v(S_2) - v(S_1 \cap S_2)$$

Proof. The first is easy to prove. For the union, the boundary of the union is a subset of the union of the boundaries. The union of sets with measure zero has measure zero, and a subset of a set with measure zero also has measure zero.

For the second, consider the extension of $f_S(x)$ over a rectangle $Q \supset S$ and show that the set of discontinuities has measure zero. The set of discontinuities of f_S is contained in the boundary of S. $\square$

We will be considering continuous functions over open sets which need not be bounded. Our next goal is to define integrability for such functions. Consider $f(x) = \frac{1}{x^p}, p > 0$ fixed. Define integrability by studying

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{1} \frac{1}{x^p} dx$$

Integrability wants that the limit exists and is finite. Then define

$$\int_0^1 f(x)dx = \sup_{\epsilon > 0} \int_{\epsilon}^{1} f(x)dx = \lim_{n \to \infty} \int_{\frac{1}{n}}^{1} f(x)dx$$

To define integrability of continuous but unbounded functions over an opne set S, we seek to construct:

Consider the set $\mathcal{D}$ of all rectifiable, compact sets, where all elements D satisfy

$$D \subseteq S$$

We take $\sup_{D \in \mathcal{D}} \int_D f$.