Lecture 13

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1 Recap

We know that if $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^n$ is $\mathcal{C}^1$ near x, and it is invertible with a differentiable inverse, then

$$Df(x) \cdot Df^{-1}(f(x)) = I \Rightarrow Df^{-1}(y) = \left[Df\left(f^{-1}(y)\right)\right]^{-1}$$

As an immediate application, if $Df(x)$ is not of full rank, then f is either not invertible, or its inverse is not differentiable at $f(x)$. Examples of the former include $f(x) = x^2$, examples of the latter include $f(x) = x^3$.

2 Inverse Function Theorem

Theorem 2.1 (Inverse Function Theorem). *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n, f \in \mathcal{C}^r, Df(x)$ invertible. Then $\exists$ open $Y \subseteq V$ such that $f|_Y Y \to V$ is invertible with its inverse being differentiable on V.*

Lemma 2.1. *Suppose $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ in $\mathcal{C}^1$ and $Df(x_0)$ has full rank. Then $\exists \alpha > 0$ such that*

$$|f(x) - f(x_0) \geq \alpha |x - x_0| \forall x \in B(x_0, \varepsilon)$$

where ε is small enough.

Proof. First we prove this for linear functions. If a function is linear, by definition it is equal to its derivative. Since its derivative has full rank, the function itself is invertible. We know that

$$|A^{-1}w| \leq M|w|$$

where M is the norm of the matrix. Rearranging, we get

$$|A(x - x_0) \geq \alpha|x - x_0|$$

by substituting $\alpha = \frac{1}{M}$ and $w = A(x - x_0)$.
For a general function, we use the fact that $Df(x_0)$ is invertible, and denote it by A. We want to show that $\exists \alpha > 0$ such that

$$|A(x - x_0)| \geq \alpha|x - x_0|$$

We can define a $R(x)$ such that $|R(x)| \leq \varepsilon|x - x_0|$ since the derivative of f approximates f itself. Then picking $\varepsilon = \frac{1}{4}\left(||A^{-1}||\right)^{-1}$ and $\alpha = 2\varepsilon$ gives

$$|A(x - x_0) + R(x)| > |A(x - x_0)| - |R(x)| \geq \frac{3}{4}\left(||A^{-1}||\right)^{-1}|x - x_0|$$

which gives the new value α'. $\square$