Lecture 31

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1 Recap

Recall that given any rectangle $R \subseteq \mathbb{R}^n \exists$ a C^∞ function $\phi : \mathbb{R}^n \to \mathbb{R}$ with

$$\phi(x) = \begin{cases} 0 & x \notin R \\ > 0 & x \in \text{Int}(R) \end{cases}$$

In general consider any function $f : A \to \mathbb{R}$. The *support* is defined to be

Definition 1.1 (Support)**.** The **support** of a function is the closure of the set $\{x \in A | f(x) \neq 0\}$.

We start with a key lemma.

Lemma 1.1. *Let $\mathcal{A}$ be a collection of open sets and A their union. Then there exists a countable collection of rectangles R_i such that its union is A, and every R_i is contained in one open set in $\mathcal{A}$. Moreover, each $x \in A$ admits an open and bounded U such that U intersects finitely many of R_i.*

Proof. Let $D_1, D_2, \dots$ be a sequence of compact subsets whose union is A, and each is contained in the interior of the next. Define

$$B_i = D_i - \text{Int}D_{i-1}$$

where $D_{-1} = \emptyset$. Then B_i is contained in D_i, so it is bounded. It is the intersection of closed sets $D_i, \mathbb{R}^n - \text{Int}D_{i-1}$, so it is closed. Hence it is compact. Now $\forall x \in B_i$, let C_x be a closed cube centred at x small enough that it is disjoint from D_{i-2} and it is contained in an open set of $\mathcal{A}$. The union of C_x cover B_i. Since the latter is compact, we can pick finitely many

C_x that cover it; denote this collection of cubes by $\mathcal{C}_i$. Then the union of all $\mathcal{C}_i$ satisfy the lemma.
This is a countable union of finitely large sets, so it is countable. For $x \in A$, it is contained in some D_i, so it is contained in the interior of some D_i. Pick the smallest such i, then $x \in B_i$, so it lies in a cube in $\mathcal{C}_i$. Now any point in any rectangle is in an open set of $\mathcal{A}$, hence it is in A. Then the union of rectangles is equal to A. By construction, each rectangle is contained in an open set in A.
We check for the last condition. For an arbitrary $x \in A$, it is contained in the interior of some D_i. Pick any neighbourhood of x contained in the interior of D_i. This is always possible, since the latter is open. Then all cubes in $\mathcal{C}_{i+2}, \mathcal{C}_{i+3}, \dots$ are disjoint from the interior of D_i by construction, so they cannot intersect with x. Now x can only intersect with rectangles of $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{i+1}$. Each of these sets are finite, and there are finitely many sets, so there are (at most) finitely many rectangles. $\square$