Lecture 3

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1 Limits and Convergence

Let (X, d) be a metric space. Consider a sequence $\{x_n\}$ with $n \in \mathbb{N}$ and $x_i \in X$. We say $\lim_{n\to\infty} x_n = x^*$ when

$$\lim_{n\to\infty} d(x_n, x^*) = 0$$

Example 1.1. If we have $X = \mathbb{R}^2 - \{0\}$, then $X_n = \left(\sin\left(\frac{1}{n}\right), \frac{1}{n}\right)$ tends to 0 under the normal distance functions.

1.1 Functions

Let $f : X \mapsto Y$. We say that f is continuous at x^* when $\forall x_n \to x^*$, then $f(x_n) \to f(x^*)$.

Proposition 1.1. *This is equivalent to the epsilon-delta definition.*

Proof. Let f be continuous at x^*, with $f(x^*) = y^*$. Then we prove this by contradiction. If it does not satisfy epsilon delta, then for a given ε, then we know $\forall\delta > 0$, there exists an "outlier" in X where $d(x, x^*) < \delta$ but $d(f(x), y^*) > \varepsilon$. Let $g(n)$ be any strictly decreasing sequence of positive reals that tend to 0. Then we define a_i to be the sequence of outliers where $\delta = g(i)$. Now we have $a_i \to x^*$ but $f(a_i)$ doesn't tend to y^*, which is a contradiction.
The converse is easy to prove. If f satisfies epsilon delta, then let x_n be any sequence that converges to x^*. Given any ε, we have a corresponding δ. Since x_n converges to x^*, it's distance with x^* will eventually be within δ starting from $n = N$, hence $f(x_n)$ will eventually be within ε of y^*. $\square$

Proposition 1.2. *We can also say that f is continuous iff for any open subset U of Y, then $f^{-1}(U)$ is also open.*

Proof. Let's say f is continuous. Let an open U be given, and let $u \in U$ be arbitrary. $\forall x$ such that $f(x) \in U$, by definition of continuity, for any ε centred at u which is contained in U, we have a similar δ ball in X which maps in the ε ball. Then this δ ball is in $f^{-1}(U)$. It is easy to show that the union of all δ balls for all $u \in U$ is equal to the preimage of U, hence it is open.

Conversely, let $x \in X$ be arbitrary. Then construct an arbitrary ε ball around $f(x)$. Its preimage in X is open, and obviously contains x. Since the preimage is open, we can find a δ ball centred at x which is contained in the preimage. This satisfies our epsilon delta definition, so f is continuous at any arbitrary x. $\square$

Remark. Let f, g be continuous functions mapping from X to Y and Y to Z respectively. Then their composite is continuous also. The proof is trivial using 1.2.

Definition 1.1. We define the interior of A to be

$$\mathrm{Int}(A) = \bigcup_{V \in A \text{ open}} V$$

The exterior is then

$$\mathrm{Ext}(A) = \mathrm{Int}(X - U)$$

And the boundary is

$$\mathrm{Bd}(A) = X - (\mathrm{Int}(A) \cup \mathrm{Ext}(A))$$