Assignment 7

Daniel Chua

December 17, 2023

1. (a) Consider the case of a plasma being created at a constant rate S [ion pairs/m^3/s] in a region between two infinite planes, separated by distance L. In the first moments of setting up the plasma, the electrons rush ahead to the walls, charging them negatively. Then in steady-state, an electric field exists in the plasma pulling the ions toward the wall and repelling the electrons. The electrons find themselves in a potential well (it's minimum is at the midplane between the two walls). The electrons "dribble" over the top of the well to reach the walls at the same rate as the ions. The electron fluid is thus approximately static (its fluid velocity is very small compared to the electron sound speed) and therefore the electron density is related to the electric potential at each point in the plasma by the Boltzmann Relation. Write this relation.

Solution:

$$n = n_0 \exp\left(\frac{eV}{kT_e}\right)$$

(b) Assume the ions move towards the wall in a one-dimensional, steady-state way subject to (i) their own pressure gradient force, (ii) the electric field, (iii) collisions with a stationary neutral gas background. $B = 0$. Write down the momentum conservation equation for the ions. Assume $Te = Ti$ and that both are constant in space.

Solution:

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = \frac{q}{m_i} E - \frac{kT}{m_i n_i} \frac{dn_i}{dx} - \nu_{in} v_i - \frac{v_i}{n_i} S$$

(c) The conservation of ion mass equation is

$$\frac{d}{dx}(n_i v_i) = S$$

Use this with the momentum eqn. for the ions to find an expression for $\frac{dv_i}{dx}$ which shows that the plasma solution "blows up", ie., $\frac{dv_i}{dx} \to \infty$ when v_i reaches the ion acoustic sound speed: $c_s = \left(\frac{k(T_e+T_i)}{m_i}\right)^{1/2}$. (This singularity corresponds to the transition from the plasma to the sheath at the wall.)

Solution: Take $n_e = n_i = n$. Using the Boltzmann relation,

$$\begin{aligned} \frac{dn}{dx} &= \frac{ne}{kT_e} \frac{dV}{dx} \\ &= -E \frac{ne}{kT_e} \\ E &= -\frac{kT_e}{ne} \frac{dn}{dx} \end{aligned}$$

Substituting,

$$\begin{aligned}
\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} &= -\frac{e}{m_i} \times \frac{kT_e}{ne}\frac{dn}{dx} - \frac{kT}{m_i n}\frac{dn}{dx} - \nu_{in} v_i - \frac{v_i}{n} S \\
\frac{\partial v_i}{\partial x} &= -\frac{1}{v_i}\frac{\partial v_i}{\partial t} - \frac{k(T_e + T_i)}{n m_i v_i}\frac{dn}{dx} - \nu_{in} - \frac{S}{n} \\
&= -\frac{1}{v_i}\frac{\partial v_i}{\partial t} - \frac{c_s^2}{n v_i}\left(\frac{S}{v_i} - \frac{n}{v_i}\frac{dv_i}{dx}\right) - \nu_{in} - \frac{S}{n} \\
&= -\frac{1}{v_i}\frac{\partial v_i}{\partial t} - \frac{S}{n}\left(1 + \frac{c_s^2}{v_i^2}\right) - \nu_{in} + \frac{c_s^2}{v_i^2}\frac{dv}{dx} \\
&= -\frac{v_i}{v_i^2 - c_s^2}\frac{\partial v_i}{\partial t} - \frac{S}{n}\left(\frac{v_i^2 + c_s^2}{v_i^2 - c_s^2}\right) - \frac{\nu_{in} v_i^2}{v_i^2 - c_s^2}
\end{aligned}$$

As v_i reaches c_s, the denominators of the terms on the right go to zero, so $\frac{\partial v_i}{\partial x}$ blows up.

(d) Find the generalization to the Boltzmann Relation in terms of the electron Mach number. Clearly at the edge of the plasma $v_e = v_i = c_s$. What then is the largest error in using the uncorrected Boltzmann Relation for the electrons assuming $T_e = T_i$?

Solution: Neglecting collisions and the source term, the momentum equation is

$$\begin{aligned}
\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial x} &= \frac{qE}{m_e} - \frac{1}{m_e n_e}\frac{dP}{dx} \\
c_e \frac{\partial M}{\partial t} + c_e^2 M \frac{\partial M}{\partial x} + \frac{eE}{m_e} &= -\frac{kT}{m_e n_e}\frac{dn}{dx} \\
-\frac{m_e c_e dx}{kT}\left(\frac{\partial M}{\partial t} + c_e M \frac{\partial M}{\partial x}\right) + \frac{eE}{kT} dV &= \frac{dn}{n} \\
-\frac{m_e c_e}{kT}\left(\frac{dx}{dt} dM + c_e M dM\right) + \frac{eE}{kT} dV &= \frac{dn}{n} \\
-\frac{m_e c_e^2}{kT} \times 2M dM + \frac{eE}{kT} dV &= \frac{dn}{n} \\
-16 M dM + \frac{eE}{kT} dV &= \frac{dn}{n} \\
n &= n_0 \exp\left(\frac{eE}{kT} - 8M^2\right)
\end{aligned}$$

The error is the extra exponential term, i.e. $\exp(8M^2)$. At $v_i = v_e = c_s$, the square of the Mach number is

$$\begin{aligned}
M^2 &= \frac{c_s^2}{c_e^2} \\
&= \frac{2kT}{m_i} \div \frac{8kT}{m_e} \\
&= \frac{m_e}{4m_i}
\end{aligned}$$

and the error becomes $\exp\left(\frac{2m_e}{m_i}\right) \approx 1.00109$. Then the percentage error is 0.109%, which is not too significant, considering how many approximations were made in the derivations.

2. The electron temperature and density in a conventional low pressure discharge tube such as a neon light or fluorescent tube. Example of H^2 as working gas.

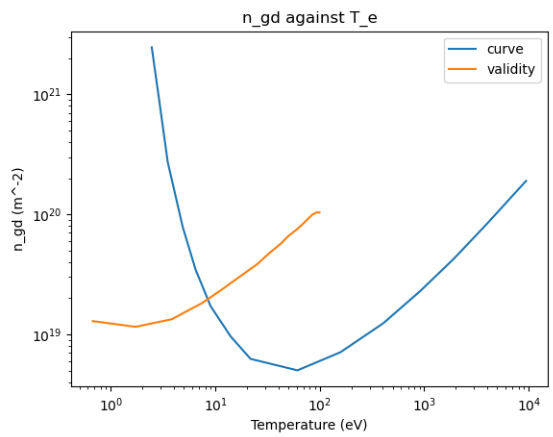
   (a) Show that the particle balance per unit length of tube leads to the following relation.

Solution: Production rate is equal to loss rate, so

$$\begin{aligned} V n_e n_g \langle \sigma v \rangle_{iz} &= \Gamma_e A \\ \frac{r}{2} n_e n_g \langle \sigma v \rangle_{iz} &= \frac{1}{2} n_e c_s \\ \frac{d}{2} n_g \langle \sigma v \rangle_{iz} &= c_s \\ n_g d &= \frac{2c_s}{\langle \sigma v \rangle_{iz}} \end{aligned}$$

Compared with the given equation, this is off by a factor of 2, which is fine according to Professor Davis. The curve is valid until

$$\begin{aligned} \lambda &= \frac{d}{2} \\ \frac{1}{n_g \sigma_{in}} &= \frac{d}{2} \\ \frac{2}{\sigma_{en}} &= n_g d \end{aligned}$$



From the graph, equality is at $T_e = 8.6$eV. This means the orange curve accurately describes the relationtion between T_e and $n_g d$ until it reaches the intersection. There is a minimum value of $n_g d$, since the graph was derived with the assumption that $T_e >> T_i$. This does not hold with n_g is too small.

   (b) Find the electrical conductivity assuming only e–n collisions. Thus find the ohmic power input, P_Ω to the electrons in W/m^3. Show that $P_\Omega \propto 1/n_e$. Show that $P_{e,\text{loss}} \propto n_e$, and so by equating electron energy gain to loss. Show that $n_e \propto j$, the current density. Find n_e for the specific example here. With this value of ne, or taking $n_e = 5 \times 10^{16}\text{m}^{-3}$ show that in fact it was correct to neglect e–i collisions in calculating P_Ω.

Solution: Use $T_e = 10\text{eV}$. Electrical conductivity is given by

$$\begin{aligned}\sigma &= \frac{ne^2}{m_e \nu_e} \\ &= \frac{ne^2}{m_e \sigma_{en} \overline{c_e} n_g}\end{aligned}$$

From Figure 3B2, $\sigma_{en} = 10^{-19}$ at $T_e = 10\text{eV}$. Mean speed of electrons is

$$\begin{aligned}\overline{c}_e &= \sqrt{\frac{8kT_e}{\pi m_e}} \\ &= 2.12 \times 10^6\end{aligned}$$

Then

$$\sigma = \frac{n_e e^2}{m_e \sigma_{en} \overline{c_e} n_g} = 2.66 \times 10^{-16} n_e$$

Ohmic power is then

$$\begin{aligned}P_\Omega &= \frac{j^2}{\sigma} \\ &= \frac{16 I^2}{d^4 \pi^2 \sigma} \\ &= \frac{6.78 \times 10^{22}}{n_e} \mathrm{W\,m^{-3}}\end{aligned}$$

The ohmic power output is then inversely proportional to n_e.
To find energy loss, we are given that $5kT_e$ is lost per electron. Rate of electron loss is flux density scaled by surface area. Then

$$\begin{aligned}P_e &= 5kT_e \Gamma_e A \\ &= 5kT_e \times \frac{1}{4} n_e \overline{c}_e \pi d L \\ &= 4.00 \times 10^{-13} L n_e\end{aligned}$$

This is proportional to n_e.
Equating ohmic power input and power loss,

$$P_\Omega \propto \frac{j^2}{n_e} \propto P_e \propto n_e \Rightarrow j^2 \propto n_e^2 \Rightarrow n_e \propto j$$

$$\begin{aligned}\frac{\pi d^2 L}{4} P_\Omega &= 4.00 \times 10^{-13} L n_e \\ 4.79 \times 10^{19} &= 4.00 \times 10^{-13} n_e^2 \\ n_e^2 &= 1.20 \times 10^{32} \\ n_e &= 1.10 \times 10^{16}\end{aligned}$$

Finally, we attempt to justify why we ignored ν_{ei}. Using $n_e = 5 \times 10^{16}$,

$$\begin{aligned}\frac{\nu_{ei}}{\nu_{en}} &= \frac{10^{-15} n_e T_e^{3/2}}{\sigma_{en} \overline{c_e} n_g} \\ &= 4.72 \times 10^{-4}\end{aligned}$$

which is very small. Since e-i collisions are negligible compared to e-n collisions, it is okay to neglect the former.

(c) To show $T_e >> T_i, T_g$. Calculate the energy transfer rate from electrons to ions and neutrals for $n_e = 10^{16}\text{m}^{-3}$ and $n_g = 5\times10^{20}\text{m}^{-3}$, and by comparing these rates to P_Ω confirm that $T_e >> T_i, T_g$.

Solution: For e-n transfers, transfer time is

$$\begin{aligned}\tau^E &= \frac{1}{\nu^E} \\ &= \frac{m_g}{m_e \nu^{\text{mom}}} \\ &= \frac{m_g}{m_e \sigma_{en} \overline{c_e} n_g} \\ &= 3.47 \times 10^{-5}\text{s}\end{aligned}$$

Power transfer rate is then

$$\begin{aligned}P &= \frac{E}{\tau} \\ &= \frac{m_e \left(\overline{c_e}\right)^2 n_e}{2\tau} \\ &= 587\text{W m}^{-3}\end{aligned}$$

For electron ion transfers, transfer time is

$$\begin{aligned}\tau^E &= \frac{m_i}{m_e \nu^{\text{mom}}} \\ &= \frac{m_i}{m_e \times 10^{-15} n T_e^{-3/2}} \\ &= 0.367\text{s}\end{aligned}$$

Similarly, power transfer rate is

$$P = \frac{E}{\tau} = 0.02\text{W m}^{-3}$$

Using the given n_e, $P_\Omega = 6.78 \times 10^6\text{W m}{-}3$, which is much greater than the other power losses. Therefore, there is not a lot of energy transfer from electrons, hence $T_e >> T_i, T_g$.

(d) Consider next the collisional regime where ions experience collisions with neutrals in distances less than d. The radial plasma flow is now ambipolar. Take $T_i = T_g = 500$ K, constant. Assuming $T_e >> T_i$, find the ambipolar diffusion coefficient as a function of T_e and by equating particle loss rates radially to the ionization rate find a new relation for $T_e(n_g d)$. Plot on the same graph as collisionless case.

Solution:

$$\begin{aligned}D_A &= \frac{T_e}{T_i} D_i \\ &= \frac{T_e}{T_i} \times \frac{kT_i}{m_i \nu_{in}} \\ &= \frac{kT_e}{m_i \sigma_{in} \overline{c}_i n_g}\end{aligned}$$

Equating particle loss with ionization rate,

$$D_a \times \frac{n_e}{r}\pi d L = n_e n_g \langle \sigma v \rangle_{iz} \times \frac{\pi d^2}{4} L$$
$$\frac{2kT_e}{m_i \sigma_{in} \overline{c_i}} = n_g^2 \langle \sigma v \rangle_{iz} \times \frac{d^2}{4}$$
$$T_e = \frac{m_i \sigma_{in} \overline{c_i} n_g^2 \langle \sigma v \rangle_{iz} d^2}{8k}$$

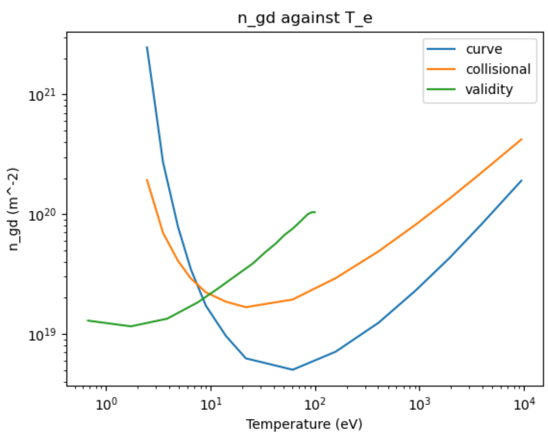
Mean speed is

$$\bar{c}_i = \sqrt{\frac{8kT_i}{\pi m_i}}$$
$$= 2292 \mathrm{m\,s^{-1}}$$

At 500 K, $\sigma_{en} \approx 10^{-19}$. Plugging in the constants,

$$T_e = 3.78 \times 10^{-24} n_g^2 d^2 \langle \sigma v \rangle_{iz}$$

where T_e can be completely determined by $n_g d$. The plot is then as follows (see orange line)



3. Tokamak Safety Factor.

Solution: Recall

$$B_p = \frac{\mu_0}{2\pi r} \int_0^r j(\rho) 2\pi \rho d\rho$$

For a uniform current profile,

$$B_p = \frac{\mu_0}{2\pi r} \times j_0 \pi r^2 = \frac{\mu_0 j_0 r}{2}$$

Substituting into q,

$$
\begin{aligned}
q &= \frac{B_T r}{B_p R} \\
&= \frac{2B_T r}{\mu_0 j_0 r R} \\
&= \frac{2B_T}{\mu_0 j_0 R}
\end{aligned}
$$

which has no r dependence. For the more realistic profile,

$$
\begin{aligned}
B_p &= \frac{\mu_0}{2\pi r} \times \int_0^r j_0 \exp\left(-\frac{3\rho^2}{a^2}\right) 2\pi\rho d\rho \\
&= \frac{\mu_0 j_0}{2r} \int_0^r \exp\left(-\frac{3\rho^2}{a^2}\right) d\left(\rho^2\right) \\
&= \frac{\mu_0 j_0}{2r} \frac{a^2}{3} \left(1 - \exp\left(-\frac{3r^2}{a^2}\right)\right)
\end{aligned}
$$

Then at $r = a$,

$$
\begin{aligned}
B_p &= \frac{\mu_0 j_0 a}{6}(1 - e^{-3}) \\
q &= \frac{B_T r}{B_p R} \\
&= \frac{6B_T a}{\mu_0 j_0 a(1 - e^{-3})R} \\
&\approx \frac{6B_T}{\mu_0 j_0 R}
\end{aligned}
$$

where we approximate $1 - e^{-3} \approx 1$. More care is needed for the $r = 0$ case. Since we cannot divide by 0, we write q in terms of r and take the limit as $r \to 0$.

$$
\begin{aligned}
q &= \frac{B_T r}{B_p R} \\
&= \frac{6B_T r^2}{\mu_0 j_0 a^2 \left(1 - \exp\left(-\frac{3r^2}{a^2}\right)\right) R}
\end{aligned}
$$

As r tends to 0, both the numerator and denominator tend to 0. Using L'Hôpital's rule,

$$
\begin{aligned}
q(r \to 0) &= \lim_{r \to 0} \frac{12B_T r}{6\mu_0 j_0 r R} \\
&= \frac{2B_T}{\mu_0 j_0 R} \\
&\approx \frac{q(r = 3)}{3}
\end{aligned}
$$

4. (a) A fully-ionized toroidal plasma, as in Fig. 5B2 of Dolan, is to have its current maintained by a gradually changing magnetic induction. It is desired to maintain a current of 20 MA approximately uniformly distributed over a plasma cross sectional area 30 m^2 at $R = 10$ m, with $T_e = 5$ keV, $n = 5 \times 10^{20}$m^{-3}. Estimate the plasma resistivity and required dB/dt which must be provided in

a transformer with core area 4 m^2. (Hint: Use Ohm's Law and Faraday's Law.)

Solution: Using Ohm's Law,

$$E = \frac{j}{\sigma} = \frac{I}{A\sigma} = \frac{20 \times 10^6}{30\sigma} = \frac{6.67 \times 10^5}{\sigma}$$

Given the cross sectional area, the minor radius is found to be

$$r = \sqrt{\frac{30}{\pi}}$$

From Faraday's Law,

$$\begin{aligned}
\nabla \times \vec{E} &= -\frac{\partial B}{\partial t} \\
\oint \nabla \times \vec{E} \cdot d\vec{S} &= \oint \frac{dB}{dt} dS \\
\int \vec{E} \cdot d\vec{l} &= 4\frac{dB}{dt} \\
\frac{dB}{dt} &= \frac{1}{4} 2\pi RE \\
&= \frac{2\pi \times 10}{4} \times \frac{6.67 \times 10^5}{\sigma} \\
&= \frac{1.05 \times 10^6}{\sigma}
\end{aligned}$$

Now conductivity is

$$\begin{aligned}
\sigma &= \frac{e^2 n_e}{m_e \nu_{ei}} \\
&= \frac{n_e e^2}{m_e 10^{-15} n_e T^{-3/2}} \\
&= \frac{e^2}{10^{-15} m_e \times 5^{-3/2}} \\
&= 3.14 \times 10^8
\end{aligned}$$

Then resistivity is

$$\rho = \frac{1}{\sigma} = 3.18 \times 10^{-9} \Omega\,\mathrm{m}$$

and

$$\frac{dB}{dt} = 0.0333 \mathrm{T\,s^{-1}} = 33.3 \mathrm{mT\,s^{-1}}$$

Solution: For the primary circuit, $B \propto NI$. For there to be a constant $\frac{dB}{dt}$, I needs to grow linearly. It would not help to have two of them, since that is simply superposition. You would get the same result by putting the 2 power supplies in series. Inductors are governed by

$$v = L\frac{dI}{dt}$$

Current needs to increase to maintain a voltage, but there are obviously numerous restrictions on maximum current (e.g. power consumption). Hence a given voltage can only be maintained for so much time until this limit is hit. Therefore, it only makes sense to characterize power supply by voltage seconds. A higher voltage can be reached with the trade-off of less time, since current takes less time to reach its maximum. In fact,

$$Vt = L\frac{dI}{dt}t \approx LI$$

where I is the maximum current.

5. (a) Evaluate the Greenwald density limit for C-mod, JT-60U and ASDEX-Upgrade.

Solution: The Greenwald density limit is given by

$$\overline{n}_e \leq n_G = \frac{I}{\pi a^2} \times 10^{20}$$

where I has units of MA and a has units of metres.
For C-mod, $I = 2, a = 0.22$, so the density limit is $n_G = 1.32 \times 10^{21}\text{m}^{-3}$. For JT-60U, $I = 1.4, a = 0.5 - 0.8$. Taking the "average" value of a^2, i.e.

$$E(a^2) = \frac{1}{0.3}\int_{0.5}^{0.8} x^2 dx = 0.43$$

which gives $n_G = 1.04 \times 10^{20}\text{m}^{-3}$. Finally, for the ASDEX-Upgrade, $I = 5, a = 1.0$, so $n_G = 1.59 \times 10^{20}\text{m}^{-3}$.

(b) Using the L-mode empirical scaling for τ_E , estimate the maximum possible Lawson parameter for these 3 tokamaks. Additional input power will improve both n_G and τ_E.

Solution: The maximum Lawson parameter is given by $n_G\tau_E$, where the former is calculated above, and the latter is given by

$$\tau_L = 0.048\frac{I^{0.85}R_0^{1.2}a^{0.3}\kappa^{0.5}\overline{n}^{0.1}B_0^{0.2}A^{0.5}}{P^{0.5}}$$