Assignment 6

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November 19, 2023

1. **Solution:** Substituting the oscillating expressions for n and v into the continuity equation,

$$\begin{aligned}
\dot{n} + \nabla \cdot (n\vec{v}) &= 0 \\
-i\omega\hat{n}e^{i(kz-\omega t)} + n_0\frac{\partial v}{\partial z} &= 0 \\
-i\omega\hat{n}e^{i(kz-\omega t)} + ikvn_0 &= 0 \\
-i\omega\hat{n} + ik\hat{v}n_0 &= 0 \\
i\omega\hat{n} &= ik\hat{v}n_0
\end{aligned}$$

When differentiating $n\vec{v}$, since

$$n = n_0 + \hat{n}e^{i(kz-\omega t)}, \hat{n} << n_0$$

we can assume that n takes a constant value n_0. The second order effects, i.e. $v\frac{\partial n}{\partial z}$ are negligible. The momentum equation is

$$mn\frac{\partial \vec{v}}{\partial t} + mn(\vec{v} \cdot \nabla)\vec{v} = -\nabla p + qn(\vec{E} + \vec{v} \times \vec{B}) - mn\vec{v}\nu$$

Neglecting the inertial and collisional terms, as well as the second order term $\vec{v} \times \vec{B}$,

$$mn\frac{\partial \vec{v}}{\partial t} = -\nabla p + qn\vec{E}$$

Using the ideal gas law,

$$\nabla p = kT\nabla n = kT\frac{\partial n}{\partial z}\hat{z}$$

and the density gradient can be expressed, using the previous result, as

$$\begin{aligned}
\nabla p &= kT\frac{\partial n}{\partial z}\hat{z} \\
&= m_e\alpha^2 ik\hat{n}e^{i(kz-\omega t)}\hat{z} \\
&= m_e\alpha^2\frac{ik^2\hat{v}n_0}{\omega}e^{i(kz-\omega t)}\hat{z}
\end{aligned}$$

where α is defined as in the problem statement. Substituting into the momentum equation, ignoring the second order term $\vec{v} \times \vec{B}$,

$$\begin{aligned}
-i\omega mnv &= -m_e\alpha^2\frac{ik^2\hat{v}n_0}{\omega}e^{i(kz-\omega t)} + qnE \\
-i\omega^2\hat{v} &= -\alpha^2 ik^2\hat{v} + \frac{q\omega\hat{E}}{m} \\
i\hat{v}(\alpha^2k^2 - \omega^2) &= \frac{q\omega\hat{E}}{m} \\
\frac{\hat{E}}{\hat{v}} &= \frac{i(\omega^2 - \alpha^2k^2)m}{e\omega}
\end{aligned}$$

From the wave equation,

$$\begin{aligned}\nabla \times (\nabla \times E) &= \nabla \times (\nabla \times (0, 0, \hat{E}_z) e^{i(kz-\omega t)}) \\ &= \nabla \times 0 \\ &= 0\end{aligned}$$

So

$$\begin{aligned}0 &= \left(\frac{\omega^2}{c^2}\hat{E} - i\omega\mu n_e e\hat{v}\right) \times e^{i(kz-\omega t)} \\ \frac{\omega^2}{a^2}\hat{E} &= i\omega\mu ne\hat{v} \\ \frac{\hat{E}}{\hat{v}} &= \frac{i\mu nea^2}{\omega} \\ \frac{m(\omega^2 - \alpha^2 k^2)}{e} &= \mu nea^2 \\ \omega^2 - \alpha^2 k^2 &= \frac{\mu ne^2 a^2}{m}\end{aligned}$$

We can express

$$\omega_p^2 = \frac{ne^2}{\varepsilon_0 m} = \frac{ne^2\alpha^2\mu_0}{m}$$

Substituting,

$$\begin{aligned}\omega^2 - \alpha^2 k^2 &= \frac{\mu ne^2 a^2}{m} \\ \omega^2 - \alpha^2 k^2 &= \omega_p^2 \\ \omega^2 &= \omega_p^2 + \alpha^2 k^2\end{aligned}$$

Giving the dispersion relation.
These waves do propogate, since $\omega > \omega_p$. Phase velocity is

$$v_p = \frac{\omega}{k} = \sqrt{\frac{\omega_p^2}{k^2} + \alpha^2}$$

and group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{\alpha^2 k}{\omega}$$

The wave does not propogate at $T_e = 0$ since all the energy is absorbed by the plasma. Some energy remains at a higher temperature. Wavelength increases with T_e because α, the speed of sound in plasma, increases with T_e.

2. Starting from $F = ma$, show that charged particles subject to a magnetic field $\vec{B} = (0, 0, B_z)$ and a constant force acting in a perpendicular direction, $\vec{F} = (F_x, 0, 0)$, drift at a velocity: $v_y = \frac{-F_x q}{B_z}$.

Solution: Consider the component of force in both directions. In the x direction,

$$m\dot{v}_x = F_x + qv_y B_z$$

In the y direction,

$$m\dot{v}_y = -qv_x B_z$$

Differentiating the first with respect to time,

$$\begin{aligned} m\ddot{v}_x &= q\dot{v}_y B_z \\ m\ddot{v}_x &= q\left(-\frac{qv_x B_z}{m}\right) B_z \\ \ddot{v}_x &= -\frac{q^2 B_z^2}{m^2} v_x \end{aligned}$$

Then we can let a solution be

$$v_x = A\sin\Omega t + B\cos\Omega t$$

where $\Omega = \frac{qB_z}{m}$. Substituting this into the first equation gives

$$\begin{aligned} m\dot{v}_x &= F_x + qv_y B_z \\ m(A\Omega\cos\Omega t - B\Omega\sin\Omega t) &= F_x + qv_y B_z \\ v_y &= A\cos\Omega t - B\sin\Omega t - \frac{F_x}{qB_z} \end{aligned}$$

The drift velocity is the average v_y. Since the first 2 terms are sinusoidal, they have no effect on the average velocity. The drift velocity is then the remaining term $-\frac{F_x}{qB_z}$.

3. The polarization drift velocity.

Solution: Similarly to the previous question, we start from Newton's Third Law in the x and y equations. For electrons,

$$m\dot{v}_x = -qv_y B_z - q\hat{E}_x \sin\omega t = -q(v_y B_z + \hat{E}_x \sin\omega t)$$

$$m\dot{v}_y = qv_x B_z$$

Differentiating the first,

$$\begin{aligned} m\ddot{v}_x &= -q(\dot{v}_y B_z + \hat{E}_x \omega\cos\omega t) \\ \ddot{v}_x &= -\frac{q}{m}(qv_x \frac{B_z^2}{m} + \hat{E}_x \omega\cos\omega t) \\ &= -\frac{q^2 B_z^2}{m^2} v_x - \frac{q\hat{E}_x \omega}{m}\cos\omega t \end{aligned}$$

Defining $\Omega = \frac{qB_z}{m}$,

$$\ddot{v}_x = -\Omega\left(\Omega v_x + \frac{\hat{E}_x \omega}{B_z}\cos\omega t\right)$$

Differentiating the second,

$$\begin{aligned} m\ddot{v}_y &= q\dot{v}_x B_z \\ \ddot{v}_y &= -q\Omega\left(v_y B_z + \hat{E}_x \sin\omega t\right) \\ &= -\Omega^2\left(v_y - \frac{\hat{E}_x}{B_z}\sin\omega t\right) \end{aligned}$$

Assume the solution

$$v_y = A\sin\Omega t + B\cos\Omega t + \frac{\hat{E}_x}{B_z}\sin\omega t$$

Then substituting into the equation for $\ddot{v}_y$,

$$\begin{aligned}
\ddot{v}_y &= -\Omega^2(A\sin\Omega t + B\cos\Omega t) - \omega^2\frac{E_x}{B_z}\sin\omega t \\
&\approx -\Omega^2(A\sin\Omega t + B\cos\Omega t) \\
&= -\Omega^2\left(A\sin\Omega + B\cos\Omega t + \frac{\hat{E}_x}{B_z}\sin\omega t - \frac{\hat{E}_x}{B_z}\sin\omega t\right) \\
&= -\Omega^2(v_y - \frac{\hat{E}_x}{B_z}\sin\omega t)
\end{aligned}$$

This means that the solution we guessed approximately satisfies the ODE for $\ddot{v}_y$. We can then solve for v_x by

$$\begin{aligned}
m\dot{v}_y &= qv_xB_z \\
\Omega(A\cos\Omega t - B\sin\Omega t) + \omega\frac{\hat{E}_x}{B_z}\cos\omega t &= \Omega v_x \\
v_x &= A\cos\Omega t - B\sin\Omega t + \frac{\omega}{\Omega}\frac{\hat{E}_x}{B_z}\cos\omega t
\end{aligned}$$

We can check that this approximately solves the ODE for $\ddot{v}_x$.

$$\begin{aligned}
\ddot{v}_x &= -A\Omega^2\cos\Omega t + B\Omega^2\sin\Omega t + \frac{\omega}{\Omega}(\omega^2)\frac{\hat{E}_x}{B_z}\cos\omega t \\
&= -\Omega^2 v_x + \frac{\omega}{\Omega}(\Omega^2 - \omega^2)\frac{\hat{E}_x}{B_z}\cos\omega t \\
&\approx -\Omega^2 v_x + \frac{\omega}{\Omega}\Omega^2\frac{\hat{E}_x}{B_z}\cos\omega t \\
&= -\Omega^2 v_x + \Omega\frac{\hat{E}_x\omega}{B_z}\cos\omega t \\
&= -\Omega\left(\Omega v_x + \frac{\hat{E}_x\omega}{B_z}\cos\omega t\right)
\end{aligned}$$

We have found solutions for v_x and v_y assuming $\Omega >> \omega$. The drift velocities are

$$v_{d,x} = \frac{\omega}{\Omega}\frac{\hat{E}_x}{B_z}\cos\omega t$$

$$v_{d,y} = \frac{\hat{E}_x}{B_z}\sin\omega t$$