Lecture 5

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Definition 0.1. Let $X \subseteq \mathbb{R}^n$. Given $\varepsilon > 0$, the set $\bigcup_{x\in X} \mathcal{U}(x;\varepsilon)$ is called the ε neighbourhood of X.

Theorem 0.1 (The ε neighbourhood theorem)**.** *Let $X \subseteq \mathbb{R}^n$ be compact. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$ containing X. Then there is an varepsilon > 0 such that the ε neighbourhood of X is contained in $\mathcal{U}$.*

Proof. The ε neighbourhood of X in the euclidean metric is contained in that of the sup metrix, so it suffices to show that this holds for the sup metric.
First, fix a set $C \subseteq \mathbb{R}^n$ for each $x \in \mathbb{R}^n$. Define

$$d(x, C) = \inf\{d(x, c)|c \in C\}$$

We claim that $d(x, C)$ is continuous in x. Using the sequence definition, it is enough to show that

$$d(x, C) - d(y, C) \leq d(x, y)$$

Letting $c \in C$,

$$d(x, C) - d(x, y) \leq d(x, c) - d(x, y) \leq d(y, c)$$

Taking the infimum of both sides with free choice of c, we get

$$d(x, C) - d(x, y) \leq d(y, C) \Rightarrow d(x, C) - d(y, C) \leq d(x, y)$$

We can reverse the argument to show that this holds if we switch x and y. This is enough to prove our claim.
Given $\mathcal{U}$, define $f : X \mapsto \mathbb{R}$ by

$$f(x) = d(x, \mathcal{U}^C)$$

We know f is continuous and $f(x) \geq 0 \forall x \in X$ since the δ ball of x is contained in $\mathcal{U}$ (because it is open). Because X is compact, $f(x)$ has a minimum value which gives the ε. (Finitely many open balls centred at a point in X cover X). $\square$

Lemma 0.1. *The rectangle* $Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \in \mathbb{R}^n$ *is compact.*

Proof. We prove this by induction. For the induction step, let $Q = X \times [a_{n+1}, b_{n+1}]$ where we assume X is compact. Let A be an open cover. For any $t \in [a_{n+1}, b_{n+1}]$, it is obvious that $X \times \{t\}$ is compact, since it is isomorphic with X. Let $\mathcal{U}$ be a finite subcover, and by 0.1, the set $X \times [t - \varepsilon, t + \varepsilon]$ is contained in $\mathcal{U}$. Then for any t, we can find an open V_t defined similarly such that $X \times V_t$ is compact. Since $n = 1$ holds, we only need finitely many V_t to cover $[a_{n+1}, b_{n+1}]$, and hence it holds for $n + 1$. $\square$