Problem Set 1

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1. Find the group velocity of a wavepacket associated with a relativistic particle in terms of the velocity of the associated particle.

Solution: Note that

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1}$$

and

$$p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \tag{2}$$

Then squaring both,

$$\begin{aligned}
E^2 &= \frac{p^2c^4}{v^2} && (3)\\
&= c^2p^2 \times \frac{c^2}{v^2} && (4)\\
&= c^2p^2\left(1+\frac{c^2-v^2}{v^2}\right) && (5)\\
&= c^2p^2 + \frac{m^2c^2(c^2-v^2)}{1-\frac{v^2}{c^2}} && (6)\\
&= c^2p^2 + m^2c^4 && (7)
\end{aligned}$$

From particle properties,

$$E = h\nu = \hbar\omega$$

and

$$p = \frac{h}{\lambda} = \hbar k$$

Then

$$\begin{aligned}
V &= \frac{d\omega}{dk}\\
&= \frac{\frac{1/\hbar}{d}E}{\frac{1/\hbar}{d}p}\\
&= \frac{dE}{dp}
\end{aligned}$$

This hold regardless if the particle is relativistic or not. Hence for a relativistic particle, differentiating both sides of 3 with respect to p, noting that m and c are constants,

$$\begin{aligned} 2E \times \frac{dE}{dp} &= 2c^2 p \\ \frac{dE}{dp} &= c^2 \times \frac{p}{E} \\ V &= v \end{aligned}$$

2. Prove the given identity.

Solution: We define $f(k)$ such that

$$\psi(x,t) = \int f(k) dk$$

for convenience. Then the left hand side of the identity becomes

$$\begin{aligned} i\hbar \frac{\partial \psi(x,t)}{\partial t} &= i\hbar \int -i\omega(k) f(k) dk \\ &= \frac{\hbar^2}{2m} \int k^2 f(k) dk \end{aligned}$$

And the right hand side becomes

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} &= -\frac{\hbar^2}{2m} \int -k^2 f(k) dk \\ &= \frac{\hbar^2}{2m} \int k^2 f(k) dk \end{aligned}$$

which is equal to the right hand side.

3. Prove the given identity.

Solution: First note that

$$\nabla \cdot (f\vec{g}) = f(\nabla \cdot \vec{g}) + \vec{g} \cdot \nabla f \tag{8}$$

Also note that differential operators and the complex conjugate function commute. Now starting from the second term,

$$\begin{aligned} \nabla \cdot j &= -\frac{i\hbar}{2m} \left[\psi^*(\nabla^2 \psi) + (\nabla \psi) \cdot (\nabla^2 \psi)^* - \psi(\nabla^2 \psi)^* - (\nabla \psi)^* \cdot (\nabla \psi) \right] \\ &= -\frac{i\hbar}{2m} \left[\psi^*(\nabla^2 \psi) - \psi(\nabla^2 \psi)^* \right] \end{aligned}$$

Now rearranging the given equation on top, the Laplacian of the wavefunction is given by

$$-\frac{2mi}{\hbar} \frac{\partial \psi}{\partial t} + \frac{2m}{\hbar^2} V\psi \tag{9}$$

Substituting, the second term equals

$$
\begin{aligned}
-\frac{i\hbar}{2m}\left[\psi^*(\nabla^2\psi) - \psi(\nabla^2\psi)^*\right] &= -\frac{i\hbar}{2m}\left[-\frac{2mi}{\hbar}\psi^*\frac{\partial\psi}{\partial t} + \frac{2m}{\hbar^2}V|\psi|^2 - \frac{2mi}{\hbar}\psi\frac{\partial\psi^*}{\partial t} - \frac{2m}{\hbar^2}V|\psi|^2\right] \\
&= -\frac{i\hbar}{2m} \times \frac{2mi}{\hbar}\left(\psi\frac{\partial\psi^*}{\partial t} + \psi^*\frac{\partial\psi}{\partial t}\right) \\
&= -\left(\psi\frac{\partial\psi^*}{\partial t} + \psi^*\frac{\partial\psi}{\partial t}\right) \\
&= -\frac{\partial\psi^*\psi}{\partial t} \\
&= -\frac{\partial\rho}{\partial t}
\end{aligned}
$$

Hence the sum of the two terms vanish.