Solutions to Topology by Conover

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1 Topological Spaces and Concepts in General

1.1 Exercise 2.6

1. Let X be a set. Verify that the indiscrete topology, the discrete topology and the finite-complement topology are in fact topologies on X.

Solution:
Indiscrete Topology:
$\emptyset$ and X are open sets, and any intersection/union of open sets are obviously either empty of X.
Discrete Topology:
$\emptyset$ and X are open sets. Since any subset is open, any intersection/union of open sets must be a subset, and hence is open.
Finite-complement Topology:
set and X are open sets. Let A_i denote open sets. Then $X - \bigcup_i A_i \subseteq X - A_i$, where $X - A_i$ has a finite cardinality by definition. Therefore, $X - \bigcup_i A_i$ also has a finite cardinality, so $\bigcup_i A_i$ is open. Now let B_i denote finitely many open sets. $X - B_i$ is finite, and so is $\bigcup_i (X - B_i)$ (finite union of finite sets is finite). Since $\bigcup_i (X - B_i) = X - \bigcap_i B_i$ which is finite, $\bigcap_i B_i$ is an open set.

2. (a) Verify that Sierpinski space is a topological space.

   Solution: It contains $\emptyset$ and X. Since there are only 3 open sets, brute forcing through all possible unions/intersections show that they are also open sets.

   (b) We said that there are only three different topologies that can be assigned to the 2 point set $\{0, 1\}$. Is the collection of $\{\emptyset, \{1\}, \{0, 1\}\}$ one of those three topologies on $\{0, 1\}$?

   Solution: Yes. The same approach for (a) can be used here, since this is essentially the Sierpinski space with 0 and 1 reversed.

   (c) What is $\{0, 1\}$ with the finite-complement topology?

   Solution: $\{\emptyset, \{0\}, \{1, \}, \{0, 1\}\}$

3. List all topologies that can be assigned to a 3 point set.

Solution:
$\{\emptyset, \{0, 1, 2\}\}$
$\{\emptyset, \{0\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0, 1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0, 2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1, 2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0\}, \{0, 2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1\}, \{0, 1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1\}, \{1, 2\}, \{0, 1, 2\}\}$

$\{\emptyset, \{2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{1\}, \{0,1\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{2\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$
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$\{\emptyset, \{1\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

4. Verify that the Sorgenfrey topology defined on the real line is in fact a topology. Is the interval $(0,1)$ open in this topology? How about $(0,1]$? Is $[0,1]$ closed?

Solution:
The Sorgenfrey topology obviously contains $\emptyset$ and $\mathbb{R}$. Let A_i denote open sets. Then

$$\forall x \in \bigcup_i A_i, x \in A_i \Rightarrow x \in [a,b) \subseteq A_i \subseteq \bigcup_i A_i$$

so any union of open sets is open. Now let B_i denote finitely many open sets. Now if $x \in B_i$, we let $\{p_i, q_i\} \subset \mathbb{R}$ such that $x \in [p_i, q_i) \subseteq B_i$. Let $P = \{p_i\}$ and $Q = \{q_i\}$. Since both P and Q are finite, P has a maximum p and Q has a minimum Q. Now

$$x \in \bigcap_i B_i \Rightarrow x \in [p,q) \subseteq [p_i, q_i) \subseteq B_i \forall i$$

Since $[p,q)$ is a subset of $B_i \forall i$, it is a subset of $\bigcap_i B_i$, hence any finite intersection of open sets is open. This shows that the Sorgenfrey topology is a topology.
$(0,1)$ is open, because

$$\forall x \in (0,1), x \in [x,1) \subset (0,1)$$

$(0,1]$ is not open, because $1 \in (0,1]$, but if $1 \in [a,b)$, then $\frac{1+b}{2}$ is an element of $[a,b)$ but not $(0,1]$, so $1 \in [a,b) \subseteq (0,1]$ cannot be true.
Consider $A = \mathbb{R} - [0,1]$ and let $x \in A$. Either $x < 0$, and $x \in [x, \frac{x}{2}) \subset A$, or $x > 1$, and $x \in [x, x+1)$. Therefore, A is open, so $[0,1]$ is closed.