Solutions to Topology by Conover

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1 Topological Spaces and Concepts in General

1.1 Exercise 2.6

1. Let X be a set. Verify that the indiscrete topology, the discrete topology and the finite-complement topology are in fact topologies on X.

Solution:
Indiscrete Topology:
$\emptyset$ and X are open sets, and any intersection/union of open sets are obviously either empty of X.
Discrete Topology:
$\emptyset$ and X are open sets. Since any subset is open, any intersection/union of open sets must be a subset, and hence is open.
Finite-complement Topology:
set and X are open sets. Let A_i denote open sets. Then $X - \bigcup_i A_i \subseteq X - A_i$, where $X - A_i$ has a finite cardinality by definition. Therefore, $X - \bigcup_i A_i$ also has a finite cardinality, so $\bigcup_i A_i$ is open. Now let B_i denote finitely many open sets. $X - B_i$ is finite, and so is $\bigcup_i (X - B_i)$ (finite union of finite sets is finite). Since $\bigcup_i (X - B_i) = X - \bigcap_i B_i$ which is finite, $\bigcap_i B_i$ is an open set.

2. (a) Verify that Sierpinski space is a topological space.

   Solution: It contains $\emptyset$ and X. Since there are only 3 open sets, brute forcing through all possible unions/intersections show that they are also open sets.

   (b) We said that there are only three different topologies that can be assigned to the 2 point set $\{0, 1\}$. Is the collection of $\{\emptyset, \{1\}, \{0, 1\}\}$ one of those three topologies on $\{0, 1\}$?

   Solution: Yes. The same approach for (a) can be used here, since this is essentially the Sierpinski space with 0 and 1 reversed.

   (c) What is $\{0, 1\}$ with the finite-complement topology?

   Solution: $\{\emptyset, \{0\}, \{1, \}, \{0, 1\}\}$

3. List all topologies that can be assigned to a 3 point set.

Solution:
$\{\emptyset, \{0, 1, 2\}\}$
$\{\emptyset, \{0\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0, 1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0, 2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1, 2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0\}, \{0, 2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1\}, \{0, 1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1\}, \{1, 2\}, \{0, 1, 2\}\}$

$\{\emptyset, \{2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{1\}, \{0,1\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$
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$\{\emptyset, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{1\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

4. Verify that the Sorgenfrey topology defined on the real line is in fact a topology. Is the interval $(0,1)$ open in this topology? How about $(0,1]$? Is $[0,1]$ closed?

Solution:
The Sorgenfrey topology obviously contains $\emptyset$ and $\mathbb{R}$. Let A_i denote open sets. Then

$$\forall x \in \bigcup_i A_i, x \in A_i \Rightarrow x \in [a,b) \subseteq A_i \subseteq \bigcup_i A_i$$

so any union of open sets is open. Now let B_i denote finitely many open sets. Now if $x \in B_i$, we let $\{p_i, q_i\} \subset \mathbb{R}$ such that $x \in [p_i, q_i) \subseteq B_i$. Let $P = \{p_i\}$ and $Q = \{q_i\}$. Since both P and Q are finite, P has a maximum p and Q has a minimum Q. Now

$$x \in \bigcap_i B_i \Rightarrow x \in [p,q) \subseteq [p_i, q_i) \subseteq B_i \forall i$$

Since $[p,q)$ is a subset of $B_i \forall i$, it is a subset of $\bigcap_i B_i$, hence any finite intersection of open sets is open. This shows that the Sorgenfrey topology is a topology.
$(0,1)$ is open, because

$$\forall x \in (0,1), x \in [x,1) \subset (0,1)$$

$(0,1]$ is not open, because $1 \in (0,1]$, but if $1 \in [a,b)$, then $\frac{1+b}{2}$ is an element of $[a,b)$ but not $(0,1]$, so $1 \in [a,b) \subseteq (0,1]$ cannot be true.
Consider $A = \mathbb{R} - [0,1]$ and let $x \in A$. Either $x < 0$, and $x \in [x, \frac{x}{2}) \subset A$, or $x > 1$, and $x \in [x, x+1)$. Therefore, A is open, so $[0,1]$ is closed.

5. Consider the topological spaces $(\mathbb{R}, \mathcal{I}), (\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$ and with the finite-complement topology (where $\mathcal{U}$ denotes the usual topology on $\mathbb{R}$ s in Chapter 3).
   (a) If $p \in \mathbb{R}$, is $\{p\}$ open in any of these spaces? Which ones?

   Solution: $(\mathbb{R}, \mathcal{D})$.

   (b) If $p \in \mathbb{R}$, is $\{p\}$ closed in any of these spaces? Which ones?

Solution: $(\mathbb{R},\mathcal{D}),(\mathbb{R},\mathcal{U}),(\mathbb{R},\mathcal{S})$, and the finite-complement topology.

(c) In which of these spaces is (a,b) open? $[a,b)$? $(a,b]$? $[a,b]$?

Solution:
(a,b):
$(\mathbb{R},\mathcal{D}),(\mathbb{R},\mathcal{U}),(\mathbb{R},\mathcal{S})$
$[a,b)$:
$(\mathbb{R},\mathcal{D}),(\mathbb{R},\mathcal{S})$
$(a,b]$:
$(\mathbb{R},\mathcal{D})$
$[a,b]$:
$(\mathbb{R},\mathcal{D})$

(d) Is the set $\{x\in\mathbb{R} : x\neq\frac{1}{n}\}$ open in any of the spaces? Is it closed in any of them?

Solution:
Open:
$(\mathbb{R},\mathcal{D})$
Closed:
$(\mathbb{R},\mathcal{D})$

(e) Is the set $\{x\in\mathbb{R} : x\neq\frac{1}{n} \text{ and } x\neq 0\}$ open in any of the spaces? Is it closed in any of them?

Solution:
Open:
$(\mathbb{R},\mathcal{D})$
Closed:
$(\mathbb{R},\mathcal{D}),(\mathbb{R},\mathcal{U}),(\mathbb{R},\mathcal{S})$

6. Consider the spaces of Problem 5 above again, together with the three spaces that can be defined on $\{0,1\}$.

(a) In which of these spaces are true: If x and y are two distinct points in the space then either there exists an open set U such that $x\in U$ and $y\notin U$, or there exists an open set V such that $y\in V$ and $x\notin V$. (A space for which this statement holds is called a T_0-space.)

Solution: $(\mathbb{R},\mathcal{D}),(\mathbb{R},\mathcal{U}),(\mathbb{R},\mathcal{S})$, the finite-complement topology, and the Sierpinski space.

(b) In which of these spaces is the following statement true: If x and Y are two distinct points in the space, then there exists an open set U such that $x\in U$ and $y\notin U$, and there exists an open set v such that $y\in V$ and $x\notin V$. (A space for which this statement holds is called a T_1-space.)

Solution: $(\mathbb{R},\mathcal{D}),(\mathbb{R},\mathcal{U}),(\mathbb{R},\mathcal{S})$, and the finite-complement topology.

(c) In which of these spaces is the following statement true: If x and y are two distinct points in the space, the there exist open sets U and V such that $x\in U$, $y\in V$ and $U\cap V=\emptyset$. (A space for which this statement holds is called a T_2-space or a Hausdorff space.)

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$

7. Show that every T$_2$-space is a T$_1$-space, and that every T$_1$-space is a T$_0$-space, and give an example of a T$_0$-space that is not a T$_1$-space, and an example of a T$_1$-space that is not a T$_2$-space.

Solution:
T$_2$-space $\Rightarrow$ T$_1$-space: Since $U \cap V \neq \emptyset$, $x \notin V$ and $y \notin U$, so V and U are open sets that make the space a T$_1$-space.
T$_1$-space $\Rightarrow$ T$_0$-space: either one of U and V make the space a T$_0$-space.
T$_0$-space that is not a T$_1$-space: the Sierpinski space
T$_1$-space that is not a T$_2$-space: the finite-complement topology

8. A topological space X is said to be **metrizable** if a metric can be defined on X so that a set is open in the metric topology induced by this metric if and only if it is open in the topology that is already on the space.

   (a) Let X be a set with more than one point. Prove that $(X, \mathcal{I})$ is *not* metrizable. Thus the indiscrete topology on a set with more than one point is an example of a topological space that is *not* a metric space.

   Solution:
   If X has more than one point, it has 2 distinct points x and y, where we let $r = d(x, y) > 0$ by the definition of a metric. $S_{\frac{r}{2}}(x)$ is an open space according to the metric. However, it is neither empty (contains x) nor the universe (does not contain y). This forms a contradiction.

   (b) Let X be a set. Define a function from $X \times X = \{(x, y) : x, y \in X\}$ to $\mathbb{R}$ by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 \text{if } x = y \end{cases}$$

   Prove that d is a metric on X. What is the metric topology induced by d?

   Solution:
   Obviously, $d(x, y) = 0$ if and only if $x = y$, $d(x, y) \geq 0$, $d(x, y) = d(y, x)$. For the triangle inequality $d(x, y) \leq d(x, z) + d(y, z)$, note that it is trivial if $x = y$. If not, at least one of $d(x, z)$ and $d(y, z)$ must be nonzero, so the inequality holds. Therefore, $d(x, y)$ is a metric.
   $\forall x \in X, S_{0.5}(x) = \{x\}$. Since all singleton sets are open, the metric topology induced by d is the discrete topology.

1.2 Theorem 3.2

1. Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$. Then f is continuous at $x_0 \in X$ if and only if whenever V is an open subset of Y with $f(x_0) \in V$, then there exists an open subset U of X such that $x_0 \in U$ and $f(U) \subseteq V$.

Solution:
$\Rightarrow$:
Let V be open. Since V is open, $\forall v = f(x_0) \in V$, $\exists \epsilon > 0$ where $S_\epsilon(v) \subseteq V$. Since continuity is implied, $\exists \delta > 0$ where $f(S_\delta(x_0)) \subseteq S_\epsilon(v)$. Therefore $S_\delta(x_0)$ is the desired open U.
$\Leftarrow$:
Let $f(x_0) = v$. $\forall \epsilon > 0$, $V = S_\epsilon(v)$ is open. Then an open U exists where $x_0 \in U$. By definition, U is open, so $\exists \delta > 0$ such that $S_\delta(x_0) \in U$. Then

$$f(S_\delta(x_0)) \subseteq f(U) \subseteq V = S_\epsilon(v)$$

This demonstrates that f is continuous by the $\epsilon - \delta$ definition.

1.3 Theorem 3.4

1. Let X and Y be topological spaces and let $f : X \to Y$. Then f is continuous on X if and only if whenever V is an open subset of Y, then $f^{-1}(V)$ is open in X.

Solution:
$\Rightarrow$:
Based on Theorem 3.2, we have an open $U_x \forall f(x) \in V$ such that $f(U_x) \subseteq V$. Therefore, $U_x \subseteq f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

Since $f^{-1}(V)$ is a union of open sets, it is open.
$\Leftarrow$:
Similar to the second part of Theorem 3.2, let $f(x_0) = v$. $\forall \epsilon > 0$, $V = S_\epsilon(v)$ is open. Then $f^{-1}(V)$ is open (and contains x_0), so $\exists \delta > 0$ such that $S_\delta(x_0) \in f^{-1}(V)$. Then

$$f(S_\delta(x_0)) \subseteq V = S_\epsilon(v)$$

This demonstrates that f is continuous by the $\epsilon - \delta$ definition.

1.4 Exercise 3.5

1. Consider $(\mathbb{R}, \mathcal{I}), (\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$ and $(\mathbb{R}, \mathcal{F})$, the real line with the indiscrete topology, the discrete topology, the usual metric topology, the Sorgenfrey topology and the finite-complement topology, respectively. Let $f : \mathbb{R} \to \mathbb{R}$ be the identity function defined by $f(r) = r$ for all real numbers r. Determine all possible choices for $\mathcal{J}_1$ and $\mathcal{J}_2$ from $\mathcal{I}, \mathcal{D}, \mathcal{U}, \mathcal{S}, \mathcal{F}$ so that $f : (\mathbb{R}, \mathcal{J}_1) \to (\mathbb{R}, \mathcal{J}_2)$ is continuous.

Solution:

	$\mathcal{I}$	$\mathcal{D}$	$\mathcal{U}$	$\mathcal{S}$	$\mathcal{F}$
$\mathcal{I}$	✓	✗	✗	✗	✗
$\mathcal{D}$	✓	✓	✓	✓	✓
$\mathcal{U}$	✓	✗	✓	✗	✓
$\mathcal{S}$	✓	✗	✓	✓	✓
$\mathcal{F}$	✓	✗	✗	✗	✓

Where the rows denote $\mathcal{J}_1$ and the columns denote $\mathcal{J}_2$.

2. Let x be a set and let $\mathcal{J}$ be any topology on X. There is a topology $\mathcal{J}$' that can be assigned to X so that the identity function from $(X, \mathcal{J}')$ to $(X\mathcal{J})$ is always continuous no matter what $\mathcal{J}$ s. What is it?

Solution: The discrete topology $\mathcal{D}$.

3. Let X be any set and let $\mathcal{J}$ be any topology on X. There is a topology $\mathcal{J}'$ that can be given to X so that the identity function from $(X, \mathcal{J})$ to $(X, \mathcal{J}')$ is always continuous no matter what $\mathcal{J}$ is. What is it?

Solution: The indiscrete topology $\mathcal{I}$.

4. A function that preserves open sets is called an **open function**. More precisely, a function $f : X \to Y$ is an open function if whenever U is open in X, then $f(U)$ is open in Y.
   (a) Give an example of a continuous function that is not open.

   Solution: $f(r) = r$ where X is the real line with the discrete topology and Y is the real line with the indiscrete topology.

   (b) Give an example of an open function that is not continuous.

   Solution: $f(r) = r$ where X is the real line with the indiscrete topology and Y is the real line with the discrete topology.

1.5 Exercise 4.2

1. Consider the spaces $(\mathbb{R}, \mathcal{I}), (\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$ and $(\mathbb{R}, \mathcal{F})$. In which of these spaces is:
   (a) $(0, 2)$ a neighbourhood of 1?
   (b) $[0, 2]$ a neighbourhood of 1?
   (c) $[0, 2]$ a neighbourhood of 0?
   (d) $\{0\}$ a neighbourhood of 0?

   Solution:
   (a): $\mathcal{D}, \mathcal{U}, \mathcal{S}$
   (b): $\mathcal{D}, \mathcal{U}, \mathcal{S}$
   (c): $\mathcal{D}, \mathcal{S}$
   (d): $\mathcal{D}$

2. In the plane with its usual topology, is the unit square

$$\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

   a neighbourhood of any point in it? Which points?

Solution:
Any point that is not the boundary, or more explicitly,

$$\{(x, y) : 0 < x < 1, 0 < y < 1\}$$

1.6 Theorem 4.3

1. Let X be a topological space. Then $U \subseteq X$ is open if and only if U is a neighbourhood of each point $x \in U$.

Solution:
$\Rightarrow$:
Since U is open, $x \in U$ implies $x \in O \subseteq U$ for some open set O. Since U is a superset of an open O that contains x, U is a neighbourhood of x by definition.
$\Leftarrow$:
Since U is a neighbourhood, $x \in U$ implies $x \in O \subseteq U$ where O is some open set. This means U is open by definition.

1.7 Exercise 4.5

1. Let X be the real line with the indiscrete topology and consider the sequence $\{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Prove that if r is any point of X, then this sequence converges to r.

Solution:
Let O be an open set containing r. Since this is the indiscrete topology, $O = X$ is the only possibility. Then $\forall n > 0$, $f(n) \in O$, so the sequence converges to r.

2. Let $X = \mathbb{R}$ with the finite-complement topology. Prove that $\{\frac{1}{n} : n \in \mathbb{Z}^+\}$ converges to every point in the space.

Solution:
Let O be an open set containing r. Since this is the finite-complement topology, $X - O$ is finite. Let $A = \{a \in X - O | a > 0\}$. If A is empty, let $N = 0$. Or else, A must have a minimum m that is a positive number. Since $\frac{1}{n}$ tends to 0, $\exists N \in \mathbb{N}$ such that $\frac{1}{n} < m \forall n > N$. Now $\forall n > N$, $f(n) \in O$, so the sequence tends to r.

3. Recall that a topological space X is a Hausdorff space if distinct points of X are contained in disjoint open sets.
   (a) Prove that a space is a Hausdorff space if and only if distinct points are contained in disjoint neighbourhoods.

Solution:
$\Rightarrow$:
Open sets are also neighbourhoods, so distinct points in the space are contained in disjoint neighbourhoods.
$\Leftarrow$:
Let the distinct points be x and y, and the disjoint neighbourhoods N_x and N_y. Since N_x is a neighbourhoods of x, $x \in O_x \subseteq N_x$, and the same holds for y. Then x and y are in disjoint open sets O_x and O_y, so the space is a Hausdorff space.

(b) Prove that every metric space is a Hausdorff space.

Solution:
Let x and y be two distinct points, and let $R = d(x, y), r = \frac{R}{2}$. Consider a point $p \in S_r(x)$. Rearranging the triangle inequality,

$$d(p, y) \geq R - d(p, x) > R - r = r$$

so p cannot be in $S_r(y)$. The same holds if $p \in S_r(y)$. This means for all distinct x, y, there exists disjoint open sets $S_r(x)$ and $S_r(y)$ which contain x and y respectively, making it a Hausdorff space.

(c) Prove that in a Hausdorff space, if a sequence converges, then it converges to exactly one point. Deduced that the real line with the finite complement topology is not a Hausdorff space.

Solution:
Proof by contradiction. Let the sequence converge to distinct points x and y. Let O_x and O_y be a pair of disjoint open sets that contain x and y respectively. Since the sequence converges to x, $\exists N_x \in N$ such that $n > N \Rightarrow f(n) \in O_x$. There exists a similar N_y. Letting $N = \max\{N_x, N_y\}$, then $\forall n > N, x_n$ is in both O_x and O_y at the same time, which is a contradiction as both sets are disjoint.

4. Consider $X = [0, \Omega]$ with the order topology as defined in Section 4 of Chapter 3. We proved that in a metric space, a set S is closed if and only if whenever a sequence of points of S converges to a point x in the space, then $x \in S$.

(a) Prove, usig the definition of closed set, that $S = [0, \Omega)$ is not a closed subset of $X = [0, \Omega]$ with the order topology.

Solution:
If it is a closed set, then $\{\Omega\}$ must be open in the order topology, so

$$\exists a \in X : \Omega \in (a, \Omega] \subseteq \{\Omega\}$$

For the interval to make sense, $a < \Omega$. Noting that Ω is a limit ordinal,

$$a < \Omega \Rightarrow a + 1 < \Omega$$

so $a + 1 \neq \Omega$ is an element of $(a, \Omega]$, which is a contradiction, since its superset $\{\Omega\}$ does not contain $a + 1$.

(b) Prove that if a sequence of points of $S = [0, \Omega)$ converges to a point $x \in [0, \Omega]$, then $x \in S$.

Solution:
The sequence x_n cannot tend to Ω if it has a finite length, since it then as a maximum m, and $x_n \notin (m, \Omega] \forall n \in \mathbb{N}$. and If not, consider

$$l = \bigcap_{n \in \mathbb{N}} (x_n, \Omega]$$

Obviously $\Omega \in l$, but if $y \neq \Omega \in l$, then

$$y > x_n \forall n \in N \Rightarrow x_n \notin [y, \Omega] \forall n \in N$$

Loosely speaking, x_n can never reach y, so it can never "enter" the open set $[y, \Omega]$. However, such a y always exists by this.

(c) Deduce that $X = [0, \Omega]$ with the order topology is not a metric space, but

Solution:
If it is a metric space, then consider

$$A = \bigcap_{n \in \mathbb{N}} S_{\frac{1}{n}}(\Omega)$$

By this, $\exists a \in A : a \neq \Omega$. Let $d(a, \Omega) = r$. Since $\frac{1}{n}$ tends to 0, we know that $r > \frac{1}{i}$ for some $i \in \mathbb{N}$. Then $a \notin S_{\frac{1}{i}}(\Omega)$, so $a \notin A$, which is a contradiction.

(d) Prove that $X = [0, \Omega]$ with the order topology is a Hausdorff space.

Solution: Let $a < b$ be two distinct points. Then $[0, a]$ and $(a, \Omega]$ are disjoint open sets that contain a and b respectively.

1.8 Theorem 4.6

1. Let X and Y be topological spaces and let $f : X \to Y$. Then the funtion f is continuous at a point $x_0 \in X$ if and only if for every neighbourhood N_2 of $f(x_0)$ in Y, there is a neighbourhood N_1 of x_0 in X such that $f(N_1) \subseteq N_2$.

Solution:
$\Rightarrow$:
If N_2 is a neighbourhood, then $\exists$ an open $O_2 \subseteq N_2$ which contains $f(x_0)$. Since f is continuous, $f^{-1}(O_2)$ is open, and acts as the desired N_1.
$\Leftarrow$:
Let O_2 be an open set containing $f(x_0)$. Then there exists a N_1, and $x_0 \in O_1 \subseteq N_1$ for some open O_1 by the definition of a neighbourhood. Since

$$f(O_1) \subseteq O_2$$

f is continuous by definition.

2 Closed Sets and Closure

2.1 Theorem 5.2

1. Let X be a topological space. Then

   1. X and $\emptyset$ are closed.

   2. The intersection of an arbitrary collection of closed sets is closed.

   3. The union of a finite collection of closed sets is closed.

Solution:
Since the complements of X and $\emptyset$ are each other (which are open), they are also closed.
Let A be a set where $a \in A$ is closed. Then

$$X - \bigcap A = \bigcup_{a \in A} X - a$$

which is open. So the intersection of closed sets is closed. Now let B be a finite set where $b \in B$ is closed. Similarly,

$$X - \bigcup B = \bigcap_{b \in B} X - b$$

which is open, so a finite union of closed subsets is closed.

2.2 Exercise 5.3

1. Consider again the spaces $(\mathbb{R}, \mathbb{I}), (\mathbb{R}, \mathbb{D}), (\mathbb{R}, \mathbb{U}), (\mathbb{R}, \mathbb{S}), (\mathbb{R}, \mathbb{F})$. In which of these spaces is:
   (a) $[0, 1]$ closed?
   (b) $(0, 1)$ closed?
   (c) $[0, 1)$ closed?
   (d) $(0, 1]$ closed?
   (e) $\{0\}$ closed?

Solution:
a: $\mathbb{D}, \mathbb{U}, \mathbb{S}$
b: $\mathbb{D}$
c: $\mathbb{D}, \mathbb{S}$
d: $\mathbb{D}$
e: $\mathbb{D}, \mathbb{U}, \mathbb{S}, \mathbb{F}$

2. Recall that a space is a T_1-space if for every pair of distinct points x and y in the space, there is an open set U such that $x \in V$ and $y \notin U$, and there is an open set V such that $y \in V$ and $x \notin V$.

(a) Restate the definition of a T_1-space in terms of neighbourhoods.

Solution: A space is a T_1-space if and only if for every distinct x, y pair, there exists a neighbourhood of x that does not contain y, and there exists a neighbourhood of y that does not contain x.

(b) Prove that a space is a T_1-space if and only if for each point p in the space, the singleton set $\{p\}$ is a closed set.

Solution:
$\Rightarrow$:
Let the space be X. $\forall q \neq p$, let O_q be an open set that contains q but not p. Such a set always exists because the space is T_1. Therefore $\bigcup_{q \neq p \in X} O_q = X - \{p\}$ is open, so the singleton set $\{p\}$ is closed.
$\Leftarrow$:
Let the space be X, and x, y be a pair of distinct points. Then $X - \{x\}$ and $X - \{y\}$ are open sets which contain exactly one of x, y respectively.

2.3 Theorem 5.4

1. Let X and Y be topological spaces and let $f : X \to Y$. Then f is continuous (on X) if whenever F is a closed set in Y, then $f^{-1}(F)$ is closed in X.

Solution:
Since open sets and closed sets are complements, f^{-1} preserving closed sets it equivalent to it preserving open sets.
$\Rightarrow$:
Let F be an closed set in Y. Let $G = Y - F$, where G is open. Since f is continuous, $f^{-1}(G)$ is also open. Then $f^{-1}(F) = X - f^{-1}(G)$ is closed.
$\Leftarrow$:
Let G be an open set in Y. Let $F = Y - G$, where F is closed. Since $f^{-1}(F)$ is closed, $f^{-1}(G) = X - f^{-1}(F)$ is open.

2.4 Exercise 5.5

A function $f : X \to Y$ is a closed function if whenever F is closed in X then $f(F)$ is closed in Y.

1. Give an example of a closed function that is not continuous.

Solution: $f(r) = r$ from $(\mathbb{R}, \mathbb{U})$ to $(\mathbb{R}, \mathbb{D})$ is a closed function that is not continuous.

2. Give an example of a continuous function that is not closed.

Solution: $f(r) = r$ from $(\mathbb{R}, \mathbb{U})$ to $(\mathbb{R}, \mathbb{I})$ is a continuous function that is not closed.

2.5 Theorem 5.7

The closure of a set A is defined as

$$\bar{A} = \bigcap \{F : F \text{ is closed in } X \text{ and } A \subseteq F\}$$

1. Let X be a topological space and let $A \subseteq X$. Then
   (a) $\bar{A}$ is a closed set.

   > **Solution:** $\bar{A}$ is an intersection of closed sets, so it too is closed.

   (b) $A \subseteq \bar{A}$

   > **Solution:** $x \in A \Rightarrow x \in F \forall F \iff x \in \bar{A}$

   (c) $\bar{A}$ is the smallest closed set that contains A in the sense than if S is closed and $A \subseteq S$, then $\bar{A} \subseteq S$ also.

   > **Solution:** Such an S corresponds to F as stated above, so $x \in S \Rightarrow x \in S$ by definition.

2.6 Theorem 5.8

1. Let X be a topological space and let $A \subseteq X$. Then if $x \in X, x \in \bar{A}$ if and only if $N \cap A \neq \emptyset$ for all neighbourhoods N of x.

> **Solution:**
> $\Rightarrow$:
> Proof by contradiction: if there exists a neighbourhood N of x which is disjoint with A, then there exists an open set O_x containing x that is disjoint with A, and $C_x = X - O_x$ is a closed set containing A that does not contain x. Then $x \in \bar{A} \Rightarrow x \in C_x$, which is a contradiction.
> $\Leftarrow$:
> Proof by contradiction: let $x \notin \bar{A}$. Then there exists a closed set $C_x \in X$ that contains A but not x. Then $O_x = X - C_x$ is an open set containing x that is disjoint with A. Since all open sets are neighbourhoods, this means there exists a neighbourhood of x that is disjoint with A, which contradicts the assumption.