Solutions to Topology by Conover

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1 Some Familiar Topological Spaces and Basic Topological Concepts

1.1 Exercises 1.2

1. Let $A = (0, 1) \cup (1, 3)$. For the given $x \in A$, give a value of $r > 0$ such that $(x - r, x + r) \subseteq A$.
   (a) $x = \frac{3}{4}$
   (b) $x = 2$
   (c) $x = \frac{9}{8}$

   Solution: $\frac{1}{16}$

2. Prove that A $= (0, 1) \cup (1, 3)$ is an open subset of $\mathbb{R}$.

   Solution:
   Case 1: $x \in (0, 1)$
   Let $d = \min\{1 - x, x\}$. $(x - \frac{d}{2}, x + \frac{d}{2}) \subseteq A$ is an open set.
   Case 2: $x \in (1, 3)$
   Let $d = \min\{3 - x, x - 1\}$. The proof proceeds similarly as Case 1.
   As $\forall x \in A \exists$ an open interval $\subseteq A$ which contains x, A is an open set by definition.

3. Prove than an ordinary open interval is an open subset of $\mathbb{R}$ but that an open set need *not* be an open interval.

   Solution:
   All open intervals are open subsets:
   Replace 0 with $x - r$ and 1 with $x + r$ as in Case 1 from the question above.
   Open sets need not be an open interval:
   The empty set is an open set, but no open interval is empty, as it contains x (open intervals are in the form $(x - r, x + r)$).

4. State precisely what it means when a subset A of $\mathbb{R}$ is *not* open.

Solution: $\exists x \in A$ such that $(x-r, x+r) - A \neq \emptyset \forall r > 0$

5. Prove that the following subsets of $\mathbb{R}$ are not open.

   (a) The set of rational numbers

   Solution:
   An open interval is nonempty. As it is an open set, it cannot contain a single point only (next question) so it must contain at least 2 points. WLOG, let the 2 points be $p < q$. Using limits, $\exists n \in \mathbb{N}$ such that $r = p + \frac{\sqrt{2}}{n} < q$. Since all open intervals contain an irrational r, no open intervals can be a subset of the rationals, hence it is not open.

   (b) A set consisting of a single point

   Solution:
   $\forall$ open intervals $(x-r, x+r)$, it contains the distinct points x and $x+\frac{r}{2}$. Therefore, $\{x\}$ cannot be open.

   (c) An interval of the form $[a, b)$, where $a < b$

   Solution:
   $\forall$ open intervals $(a-r, a+r)$, $b = a - \frac{r}{2}$ is a point in that interval which is outside of $[a, b)$.

   (d) The set $A = \{x \in \mathbb{R} : x \neq \frac{1}{n}, \text{ for } n \in \mathbb{Z}^+\}$

   Solution:
   $\frac{1}{n}$ tends to 0. Therefore $\forall r > 0 \exists n \in \mathbb{N}$ such that $r > \frac{1}{n}$. Therefore all open intervals $(0-r, 0+r)$ contains a point outside of A.

1.2 Theorem 1.3

6. (a) The union of any collection of open subsets of the real line is also an open subset of the line

   Solution:
   Let $\bigcup A$ denote the union of subsets A. Then

   $$x \in \bigcup A \Rightarrow x \in A \Rightarrow x \in (x-r, x+r) \subseteq A \subseteq \bigcup A$$

   for some $r > 0$.
   As all points in $\bigcup A$ are in open intervals which are subsets of $\bigcup A$, the union is open by definition.

   (b) The intersection of any finite collection of open subsets of the real line is also an open subset of the line.

   Solution:
   This can be proven with induction.
   Let A and B be open sets. $x \in A \cap B \Rightarrow x \in A$. Since A is open, $\exists r_A > 0$ such that

   $$x \in (x - r_A, x + r_A) \subseteq A$$

The same holds for B. Letting $r = \min\{r_A, r_B\}$,

$$x \in (x - r, x + r) \subseteq A \cap B \forall x \in A \cap B$$

The same proof is used for the base case and the induction step.

(c) Both the empty set and $\mathbb{R}$ itself are open subsets of the real line.

Solution:
Empty set:

$$a \Rightarrow b$$

is defined to be true when a is false. The definition of an open set A involves the assumption $x \in A$, which is false, so it is vacuously true for the empty set.
$\mathbb{R}$:
$\forall x \in \mathbb{R}, (x - 1, x + 1) \in \mathbb{R}$