Solutions to Topology by Conover

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1 Some Familiar Topological Spaces and Basic Topological Concepts

1.1 Exercises 1.2

1. Let $A = (0,1) \cup (1,3)$. For the given $x \in A$, give a value of $r > 0$ such that $(x - r, x + r) \subseteq A$.
   (a) $x = \frac{3}{4}$
   (b) $x = 2$
   (c) $x = \frac{9}{8}$

   Solution: $\frac{1}{16}$

2. Prove that A $= (0,1) \cup (1,3)$ is an open subset of $\mathbb{R}$.

   Solution:
   Case 1: $x \in (0,1)$
   Let $d = \min\{1 - x, x\}$. $(x - \frac{d}{2}, x + \frac{d}{2}) \subseteq A$ is an open set.
   Case 2: $x \in (1,3)$
   Let $d = \min\{3 - x, x - 1\}$. The proof proceeds similarly as Case 1.
   As $\forall x \in A \exists$ an open interval $\subseteq A$ which contains x, A is an open set by definition.

3. Prove than an ordinary open interval is an open subset of $\mathbb{R}$ but that an open set need *not* be an open interval.

   Solution:
   All open intervals are open subsets:
   Replace 0 with $x - r$ and 1 with $x + r$ as in Case 1 from the question above.
   Open sets need not be an open interval:
   The empty set is an open set, but no open interval is empty, as it contains x (open intervals are in the form $(x - r, x + r)$).

4. State precisely what it means when a subset A of $\mathbb{R}$ is *not* open.

Solution: $\exists x \in A$ such that $(x-r, x+r) - A \neq \emptyset \forall r > 0$

5. Prove that the following subsets of $\mathbb{R}$ are not open.

   (a) The set of rational numbers

   Solution:
   An open interval is nonempty. As it is an open set, it cannot contain a single point only (next question) so it must contain at least 2 points. WLOG, let the 2 points be $p < q$. Using limits, $\exists n \in \mathbb{N}$ such that $r = p + \frac{\sqrt{2}}{n} < q$. Since all open intervals contain an irrational r, no open intervals can be a subset of the rationals, hence it is not open.

   (b) A set consisting of a single point

   Solution:
   $\forall$ open intervals $(x-r, x+r)$, it contains the distinct points x and $x + \frac{r}{2}$. Therefore, $\{x\}$ cannot be open.

   (c) An interval of the form $[a, b)$, where $a < b$

   Solution:
   $\forall$ open intervals $(a-r, a+r)$, $b = a - \frac{r}{2}$ is a point in that interval which is outside of $[a, b)$.

   (d) The set $A = \{x \in \mathbb{R} : x \neq \frac{1}{n},\ \text{for } n \in \mathbb{Z}^+\}$

   Solution:
   $\frac{1}{n}$ tends to 0. Therefore $\forall r > 0 \exists n \in \mathbb{N}$ such that $r > \frac{1}{n}$. Therefore all open intervals $(0-r, 0+r)$ contains a point outside of A.

1.2 Theorem 1.3

1. (a) The union of any collection of open subsets of the real line is also an open subset of the line

   Solution:
   Let $\bigcup A$ denote the union of subsets A. Then

   $$x \in \bigcup A \Rightarrow x \in A \Rightarrow x \in (x-r, x+r) \subseteq A \subseteq \bigcup A$$

   for some $r > 0$.
   As all points in $\bigcup A$ are in open intervals which are subsets of $\bigcup A$, the union is open by definition.

   (b) The intersection of any finite collection of open subsets of the real line is also an open subset of the line.

   Solution:
   This can be proven with induction.
   Let A and B be open sets. $x \in A \cap B \Rightarrow x \in A$. Since A is open, $\exists r_A > 0$ such that

   $$x \in (x - r_A, x + r_A) \subseteq A$$

The same holds for B. Letting $r = \min\{r_A, r_B\}$,

$$x \in (x - r, x + r) \subseteq A \cap B \forall x \in A \cap B$$

The same proof is used for the base case and the induction step.

(c) Both the empty set and $\mathbb{R}$ itself are open subsets of the real line.

Solution:
Empty set:

$$a \Rightarrow b$$

is defined to be true when a is false. The definition of an open set A involves the assumption $x \in A$, which is false, so it is vacuously true for the empty set.
$\mathbb{R}$:
$\forall x \in \mathbb{R}, (x - 1, x + 1) \in \mathbb{R}$

1.3 Exercise 1.4

1. Give an example of an infinite collection of open subsets of the real line whose intersection is not open, thus showing that the finiteness condition in Theorem 1.3(b) is necessary.

Solution:

$$\bigcap A \text{ where } A = \{(-r, r) | r > 0\}$$

Obviously $0 \in (-r, r) \forall r > 0$. However, the intersection does not contain any nonzero element, because $\forall x \neq 0$,

$$x \notin \{-\frac{|x|}{2}, \frac{|x|}{2}\}$$

The open intervals that form the intersection are open sets, but the intersection contains only 1 element, so it is not open.

1.4 Theorem 1.6

1. (a) The intersection of any collection of closed sets is closed.

Solution:
Let C_i be a closed set, and the corresponding open set be defined as $O_i = \mathbb{R} - C_i$.

$$\bigcap_{i \in I} C_i = \bigcap_{i \in I} \mathbb{R} - O_i = R - \bigcup_{i \in I} O_i$$

$\bigcup_{i \in I} O_i$ is a union of open sets, so it is open. Hence its complement (intersection of closed sets) is closed.

(b) The union of any finite collection of closed sets is closed.

Solution:

$$\bigcup_{n\in\mathbb{N}} C_n = \bigcup_{n\in\mathbb{N}} \mathbb{R} - O_n = R - \bigcap_{n\in\mathbb{N}} O_n$$

And the proof follows similar to the case above.

(c) $\emptyset$ and $\mathbb{R}$ itself are both closed

Solution: Their complements are each other, which are open.

1.5 Exercise 1.7

1. State precisely what it means when a subset of $\mathbb{R}$ is not closed. (Do this in term of points; saying that a set is not closed if its complement is not open is true, but is not what we want here.

Solution:
Let that subset be A. It is not closed when there is a point outside of it whose every open interval intersects with A.

$$\exists x \in \mathbb{R} - A \text{ such that } (x-r, x+r) \cap A \neq \emptyset \forall r > 0$$

2. Which of the following subsets of $\mathbb{R}$ are closed? Which are open?
   (a) The set $\mathbb{Z}$ of integers.
   (b) The set of rational numbers.
   (c) A set consisting of a single point.
   (d) An interval of the form $[a, b)$, where $a < b$.
   (e) The set $A = \{x \in \mathbb{R} : x \neq \frac{1}{n} \text{ for } n \in \mathbb{Z}^+\}$.
   (f) The set $A = \{x \in \mathbb{R} : x \neq \frac{1}{n} \text{ for } n \in \mathbb{Z}^+ \text{ and } x \neq 0\}$.

Solution:
Closed: a, c
Open: f

3. Prove that an ordinary closed interval is a closed subset of $\mathbb{R}$, but a closed set need not be a closed interval.

Solution:
Let the closed interval be $[a, b]$, where $a < b$. We want to prove that its complement is open. Let $x < a$ be in its complement. Then

$$(x-r, x+r) \text{ where } r = \frac{a-x}{2}$$

is an open interval in its complement. A similar open interval can be deduced for $x > b$. Therefore, the complement is open, and the closed interval is closed.

For the second part of the question, note that $\mathbb{R}$ is closed, but is not a closed interval. (Or else, let $\mathbb{R} = [a, b]$, and $\mathbb{R} - [a, b] \neq \emptyset$ forms a contradiction.)

4. Give an example of an infinite collection of closed subsets of whose union is not closed, thus showing tht the finiteness condition in theorem 1.6(b) is not necessary.

Solution:
Let the collection be

$$A = \{[-r, r] | 0 < r < 1\}$$

It is obvious that $\bigcup A = (-1, 1)$. The complement of $(-1, 1)$ contains 1, whose every open interval intersects with $(-1, 1)$. Hence the complement is not open, so $\bigcup A$ is not closed.

1.6 Exercise 2.3

1. Show that the absolute value formula, $d(x, y) = |x - y|$ is indeed a metric on the real line. Describe the **1-ball** centered at 0 in the topology induced by this metric.

Solution: Trivial.

1. $|x - y| \geq 0 \forall \{x, y\} \subset \mathbb{R}$
2. $|x - y| = 0$ iff $x = y$
3. $|x - z| + |y - z| = |x - z| + |z - y| \geq |x - z + z - y| = |x - y|$

The 1-ball is ordinary closed interval $(-1, 1)$.

2. Show that the distance formula is a metric on the Euclidean plane. Describe the 1-ball centered at $(0, 0)$ in the topology induced by this metric.

Solution:
In fact, this can be proven for all finite Euclidean spaces $\mathbb{R}^n$, where

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$d(x, y) \geq 0 \forall \{x, y\} \subset \mathbb{R}^n, d(x, y) = 0$ iff $x = y, d(x, y) = d(y, x) \forall \{x, y\} \subset \mathbb{R}^n$ are all trivial. What remains is the triangle inequality. Let $a_i = x_i - z_i$ and $b_i = z_i - y_i$. Therefore $a_i + b_i = x_i - y_i$.

$$d(x,z)+d(y,z)=\sqrt{\sum_i a_i^2}+\sqrt{\sum_i b_i^2}$$

$$(d(x,z)+d(y,z))^2=\sum_i(a_i^2+b_i^2)+2\sqrt{\left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)}$$

$$(d(x,z)+d(y,z))^2-(d(x,y))^2=2\left(\sqrt{\left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)}-\sum_i a_ib_i\right)$$

This is greater than 0 (from the Cauchy-Schwarz Inequality). Rearranging and taking the square root of both sides (we can do this because the distance formula is always positive) yields the desired inequality.
The 1-ball is the open disk with a radius of 1.

3. Define the obvious metric for Euclidean 3-space, $E^3 = \mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x,y,z) : x,y,z \in \mathbb{R}\}$. Describe the 1-ball centered at the point $(0,0,0)$ in the topology induced by this metric. (In this topology, an r ball really is a ball - hence the name "r-ball.")

Solution: The metric is defined in the question above, and the 1-ball is an open sphere with a radius of 1.

4. Show that for any metric space (X,d),

(a) The union of any collection of open sets is open.

Solution:
Let O_i be open. Then

$$x\in\bigcup_i O_i \Rightarrow x\in O_i \Rightarrow x\in S_r(x)\subseteq O_i$$

Since $O_i \subseteq \bigcup O_i$, we have

$$x\in\bigcup_i O_i \Rightarrow x\in S_r(x)\subseteq\bigcup_i O_i$$

(b) The intersection of any finite collection of open sets is open.

Solution:
Similar to the metric case, this can be proven using induction.
Let A and B be open sets.

$$x\in A\cap B\Rightarrow x\in A\Rightarrow x\in S_{r_A}(x)\subseteq A$$

Similarly,

$$x\in S_{r_B}(x)\subseteq B$$

Letting $r=\min\{r_A,r_B\}$, we have

$$x\in A\cap B\Rightarrow x\in S_r(x)\subseteq A\cap B$$

(c) The empty set and X itself are open.

> **Solution:**
> The fact that the empty set is open is vacuously true, and the definition of $S_r(x)$ implies $S_r(x) \subseteq X$, so X is also open.

5. A set can have more than one metric defined on it, and different metrics may give rise to different topologies.

(a) Let $X = \mathbb{R}$ and define a metric on X by $d(x, y) = 1$ if $x \neq y$, $d(x, y) = 0$ if $x = y$. Prove that d is a metric on X. What is $S_{\frac{1}{2}}(0)$ in this metric? Is (X, d) the same space as $\mathbb{R}$ with its usual metric topology? In other words, does this metric give rise to the same topology on $\mathbb{R}$ as the usual metric does?

> **Solution:**
> The fact that d is a metric is trivial. $S_{\frac{1}{2}}(0) = \{0\}$. This metric gives rise to a different topology, as $\{0\}$ is open in this topology, but not the usual topology.

(b) Let X be the Euclidean place and define a metric on X by $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Prove that d is a metric on the plane, and describe the r-balls in this metric. Does this metric give rise to the same topology as the usual metric on this plane?

> **Solution:**
> $d(x, y) \geq 0 \forall \{x, y\} \subset \mathbb{R}^n, d(x, y) = 0$ iff $x = y, d(x, y) = d(y, x) \forall \{x, y\} \subset \mathbb{R}^n$ are all trivial. What remains is the triangle inequality.
>
> $$\begin{aligned} d(x,z) + d(y,z) &= |x_1 - z_1| + |x_2 - z_2| + |y_1 - z_1| + |y_2 - z_2| \\ &= |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| \\ &> |x_1 - y_1| + |x_2 - y_2| \\ &= d(x, y) \end{aligned}$$
>
> The r-balls of x are open squares centered at x with sides of length r. It is obvious that all open disks D centered at x contain open squares S also centered at x, (such that $x \in S \subset D$) and vice versa. If A is an open set under the usual topology, then
>
> $$x \in A \Rightarrow x \in D$$
>
> Combining this with
>
> $$x \in S \subset D$$
>
> we have
>
> $$x \in A \Rightarrow x \in S$$
>
> so A is an open set under this metric. Similarly, if B is an open set under this metric, B is also an open set under the usual metric. Therefore, both topologies are the same.

1.7 Exercise 3.3

1. Let $(\mathbb{Q}, d)$ be the space of rational numbers with the metric topology induced by the absolute value metric d.

(a) Show that the set $\{y \in \mathbb{Q} : 1 < y < 2\}$ is open in $(\mathbb{Q}, d)$.

> **Solution:**
> Denote the set as A. Let $r = \min\{\frac{2-y}{2}, \frac{y-1}{2}\}$. Then
> $$y \in (y-r, y+r) \cap A \subseteq A \forall y \in A$$

(b) Show that the set $[\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}$ is open in $(\mathbb{Q}, d)$.

> **Solution:** Noting that both $\sqrt{2}$ and $\sqrt{3}$ are not in the set, the proof can be obtained by replacing the rationals 1 and 2 in the solution above.

2. Let $S =$ [1,2] be given teh topology induced by the absolute value metric. Show that $[1, \frac{3}{2}), (\frac{3}{2}, 2]$ and $[1, 2]$ are all open in (S, d).

> **Solution:**
> For the first set, if $x \neq 1$, let $r = \min\{\frac{3}{2} - x, x - 1\}$. Then $S_r(x) \cap [1, \frac{3}{2}) = S_r(x)$, and so
> $$x \in [1, \frac{3}{2}) \Rightarrow x \in S_r(x) \subset [1, \frac{3}{2})$$
> If $x = 1$, then $r = \frac{5}{4}$ suffices.
> A similar approach is used for the second and third set, noting the special cases for $x = 1$ and $x = 2$.

3. Let $(\mathbb{Z}, d)$ be the space of integers with the metric topology induced by the absolute value metric. Show that *every* subset of $\mathbb{Z}$ is open in $(\mathbb{Z}, d)$. In particular, any set consisting of a single point is open in $(\mathbb{Z}, d)$.

> **Solution:**
> Let A be a subset, and let $r = \frac{1}{2}$. Then $\forall x \in A$,
> $$x \in S_r(x) \cap A = \{x\} \subseteq A$$

4. Let (X, d) be a discrete space. Show that every subset of X is open in (X, d).

> **Solution:**
> Let A be a subset. Then
> $$A = \bigcup \{\{x\} | x \in A\}$$
> which is the union of open sets, so A is also open.

1.8 Theorem 4.1

1. A subset A or $\mathbb{R}$ is open in the metric topology induced by the absolute value metric if and only if for each point $x \in A$, there exist real numbers a and b with $a < x < b$, such that $x \in (a, b) \subseteq A$.

Solution:
$\Leftarrow$:
Let $r = \min\{x-a, b-x\}$. Then $\forall x \in A$,

$$x \in (x-r, x+r) \subseteq (a,b) \subseteq A$$

$\Rightarrow$:
As A is open,

$$x \in A \Rightarrow x \in S_r(x)$$

so $a = x-r$ and $b = x+r$ suffices.

1.9 Exercise 4.3

1. Convince yourself that the real line is a totally ordered set. Give an example of a set with an order relation that satisfies (1), (2), and (3) of Definition 4.2, but does *not* satisfy (4). Such an ordered set is called **partially ordered** since not every pair of elements can be compared. [*Hint*: Consider $\mathcal{P}(X)$ with "$\leq$" thought as "subset of].

Solution:
Intuitively, real numbers satisfy (1) through (4). A rigorous proof involves Dedekind cuts, which is (probably) out of the scope of this book.
$\subseteq$ in $\mathcal{P}(X)$ is one such example. (1) and (3) are trivial, and (2) is true by definition. However, $\{1,2\}$ and $\{3,4\}$ are elements of $\mathcal{P}(\mathbb{N})$ where both elements cannot be compared.

1.10 Exercise 4.5

1. Every open subset of $\mathbb{R}$ has cardinality c

Solution:
Let A be an open subset. $\forall x \in A$,

$$x \in (x-r, x+r) \subseteq A$$

The function

$$f(y) = \frac{y-x}{r} \times \frac{\pi}{2}$$

is a bijection from $(x-r, x+r)$ to $\mathbb{R}$, so they share the same cardinality c. As

$$(x-r, x+r) \subseteq A \subseteq \mathbb{R}$$

$|A| = c$ by the Cantor-Bernstein Theorem.

2. For any positive integer n, there are open subsets of $[0, \Omega)$ with cardinality n. In particular, there are points x in $[0, \Omega)$ such that the singleton set $\{x\}$ is an open set. But $[0, \Omega)$ is not discrete - not every singleton set is open.

Solution:
$[1,n]$ is a set with cardinality n. To show that it is open, $\forall x \in [1,n]$,

$$0 < x < n+1 \text{ such that } x \in (0, n+1) \subseteq A$$

Putting $n = 1$ gives us a singleton set that is open. However, not all singleton sets are open, e.g. $\{\omega\}$. For it to be open, $a < \omega < b$ such that

$$\omega \in (a,b) \subseteq \{\omega\}$$

Since ω is a limit ordinal, $a < a+1 < \omega < b$, so

$$a+1 \in (a,b) - \{\omega\}$$

which is a contradiction.

3. There are open subsets of $[0,\Omega)$ of cardinality $\aleph_0$ and there are open subsets of $[0,\Omega)$ of cardinality $\aleph_1$.

Solution:
$\forall n \in \mathbb{N}$, $[n+1,\omega)$ is a subset of cardinality $\aleph_0$. To show that it is open, $\forall x \in [n+1,\omega)$,

$$n < x < \omega \text{ such that } x \in (n,\omega) \subseteq [n+1,\omega)$$

Similarly, $[n+1,\Omega)$ is a subset of cardinality $\aleph_1$. The bijective function from $[0,\Omega)$ to $[n+1,\Omega)$ is given by

$$f(x) = \begin{cases} x+n+1 & x \in \mathbb{N} \\ x & \text{else} \end{cases}$$

To show that it is open, $\forall x \in [n+1,\Omega)$,

$$n < x < \Omega \text{ such that } x \in (n,\Omega) \subseteq [n+1,\Omega)$$

4. Supplementary: No countable collection of open subsets of $[0,\Omega]$ intersect at $\{\Omega\}$ only.

Solution:
Let A_n be an open set in the collection. Since $\Omega \in A_n$,

$$\Omega \in (a_n, \Omega] \subseteq A_n$$

Where $(a_n,\Omega] = [a_n+1,\Omega]$. Let $b_n = a_n + 1$, and consider

$$b = \bigcup_{n\in\mathbb{N}} b_n$$

The union of ordinals (b) is an ordinal that is greater than or equal to all of the ordinals in the union ($b \geq b_n$). Moreover, this is a countable union of countable sets, so it is still countable, hence $b < \Omega$. Since we have subsets of A_n,

$$\bigcap_{n\in\mathbb{N}} [b_n, \Omega] \subseteq \bigcap_{n\in\mathbb{N}} A_n$$

Since $b \geq b_n \forall n \in \mathbb{N}$, b is an element of the set in the left, meaning the countable ordinal b is an element of the intersection, which is a contradiction.

1.11 Exercise 4.6

1. Show that $\omega = [0, \omega)$ with the order topology is a discrete space, so is the same as $\mathbb{Z}^+ \cap \{0\}$ when given the metric topology induced by the absolute value metric on $\mathbb{R}$.

Solution:
First we prove that all singleton sets $\{x\}$ are open.

$$x = 0 \Rightarrow x \in [0, 1) \subseteq \{0\}$$

$$x \neq 0 \Rightarrow x \in (x - 1, x + 1) \subseteq \{x\}$$

Note that $x - 1$ exists $\forall x \neq 0$, as x is a nonzero natural number.
Since

$$\omega = \mathbb{Z}^+ \cup \{0\}$$

and both are discrete, they are the same.

2. The definition of the order topology is given in terms of open intervals. Let $(X, \leq)$ be a totally ordered set and give it the order topology. Show that a subset of x is open in the order topology if and only if it is a union of open intervals. Because of this, we say that the collection of open intervals is a *basis* for the order topology on a totally ordered set.

Solution:
Let A be an open subset. Since A is open, $\forall x \in A$,

$$x \in O_x \subseteq A$$

where O_x denotes an open interval containing x that is a subset of A. Then

$$A = \bigcup_{x \in A} O_x$$

So all open sets are unions of open intervals O_x. Since open intervals are open sets (this can easily be observed), any union of open intervals are open sets.

1.12 Exercise 5.3

1. State precisely what it means when a sequence in (X, d) does not converge to the point $x \in X$.

Solution: $\forall N \in \mathbb{N} > 0, r > 0, \exists r > 0, n > N$ such that $x_n \notin S_r(x)$

2. Let $\{x_n : n \in \mathbb{Z}^+\}$ be the sequence in $\mathbb{R}$ defined by $x_n = \frac{1}{n}$. Does this sequence converge? To what? Prove it.

Solution:
It converges to 0. We take for granted that natural numbers are unbounded above, i.e. $\forall r \in \mathbb{R} \exists n \in \mathbb{N}$ such that $n > r$. Then $\forall r > 0$, $\exists N \in \mathbb{N} > \frac{1}{r}$, so $r > x_n \forall n \in \mathbb{N} > N$, which is equivalent to $x_n \in S_r(0) \forall n \in \mathbb{N} > N$.

3. Let $\{x_n : n \in \mathbb{Z}^+\}$ be the sequence in $\mathbb{R}$ defined by $x_n = (-1)^n(\frac{1}{n})$. Does this sequence converge? To what? Prove it.

Solution: It converges to 0. Note that $|x_n| \in S_r(0) \Rightarrow x_n \in S_r(0)$. Since $|x_n|$ converges by the previous question, so does x_n.

4. Let $\{x_n : n \in \mathbb{Z}^+\}$ be the sequence in $\mathbb{R}$ defined by $x_n = (-1)^n$. Does this sequence converge? To what? Prove it.

Solution:
It does not converge to $a \forall a \in \mathbb{R}$.
If $a > 1$, let $r = a - 1$, then x_n never enters $S_r(a)$.
If $a < -1$, let $r = -1 - a$, then x_n never enters $S_r(a)$.
If $a = 1$ or $a = -1$, let $r = 1$, then x_n never enters $S_r(a)$.
Else, let $r = \min\{1 - a, a + 1\}$, then x_n never enters $S_r(a)$.

1.13 Theorem 5.4

1. A sequence in a metric space can converge to at most one point.

Solution:
Proof by contradiction. Let it converge to two distinct points a and b. Let $r = \frac{d(a,b)}{2}$, so $S_r(a)$ and $S_r(b)$ are disjoint by the triangle inequality.
Since the sequence converges to both points, $\exists\{N_a, N_b\} \subseteq \mathbb{N}$ such that

$$n \in \mathbb{N} > N_a \Rightarrow x_n \in S_r(a)$$

and similarly for b.
Let $N = \max\{N_a, N_b\}$. Then $x_{N+1} \in S_r(a)$ and $x_{N+1} \in S_r(b)$ which is a contradiction.

1.14 Theorem 5.5

1. A subset S of a metric space X is closed if and only if whenever a sequence of point of S converges to a point $x \in X$, then $x \in S$.

Solution:
$\Rightarrow$:
Assume the opposite. Then $x \in X - S$ but $S_r(x)$ intersects with S $\forall r > 0$. Then $X - S$ is not open (it has no open subset containing x), so S is not closed, which is a contradiction.
$\Leftarrow$:
Assume the opposite. Then $X - S$ is not open, so $\exists x \in X - S$ such that $S_r(x)$ intersects with S $\forall r > 0$. Let $y_n \in S_{\frac{1}{n}}(x) \cap S$. Then y is a sequence in S that tends to a point x outside of S. This is a contradiction.

1.15 Exercise 5.6

1. Use Theorem 5.5 to decide if the following subsets of the real line are closed (in the metric topology induced by the absolute value metric).

   (a) $[0, 1]$

   (b) $[1, \infty)$

   (c) $\{x \in \mathbb{R} : x = \frac{1}{n} \text{ for } n \in \mathbb{Z}^+\}$

   (d) $\{x \in \mathbb{R} : x = \frac{1}{n} \text{ for } n \in \mathbb{Z}^+, \text{ or } x = 0\}$

   (e) $\{x \in \mathbb{R} : x = \frac{1}{\sqrt{n}} \text{ for } n \in \mathbb{Z}^+\}$

   Solution: Closed, closed, not closed, closed, not closed.

2. Is$(\mathbb{R} - \mathbb{Q}) \cap \{x \in \mathbb{R} : x = \frac{1}{\sqrt{n}} \text{ for } n \in \mathbb{Z}^+\}$ a closed subset of the *irrationals* with the metric topology induced by the absolute value metric?

   Solution:
   Yes. Whenever the sequence tends to an irrational point x, x is in the set. Proof by contradiction:
   If $x < 0$, let $r = -x$. The sequence never enters $S_r(x)$, so it cannot converge to x.
   $If x > 1$, let $r = x - 1$. The sequence never enters $S_r(x)$, so it cannot converge to x.
   If $0 < x < 1$, then $\frac{1}{\sqrt{n+1}} < x < \frac{1}{\sqrt{n}}$. Let $r = \min\{x - \frac{1}{\sqrt{n+1}}, \frac{1}{\sqrt{n}} - x\}$. The sequence never enters $S_r(x)$, so it cannot converge to x.
   In all other cases, x is either rational or in the subset itself.

3. Let $[p_n = (x_n, y_n) : n \in \mathbb{Z}^+\}$ be a sequence in E^2, the Euclidean plane. Prove that $p_n \to p = (x, y)$ if and only if both $x_n \to x$ and $y_n \to y$. (Thus a sequence in the plane converges if and only if it converges "coordinate-wise")

   Solution:
   $\Rightarrow$:
   Obviously, $d(p_n, p) \geq d(x_n, x)$, similarly for y. Hence $p_n \in S_r(p) \Rightarrow x_n \in S_r(x)$ and similarly for y. Since p_n converges to p, x_n converges to x and so does y.
   $\Leftarrow$:
   $\forall r > 0$, let $r' = \frac{r}{\sqrt{2}}$. Then $\exists N_x \in \mathbb{N}$ such that $n \in \mathbb{N} > N_x \Rightarrow x_n \in S_{r'}(x)$. A similar N_y exists for y. Letting $N = \max\{N_x, N_y\}$ implies that $\forall n > N$, $p_n \in S_r(p)$. As this holds $\forall r > 0$, we can conclude that $p_n \to p$.

4. Prove that every real number can be written as the limit of a convergent sequence of *rational* numbers, i.e., if $r \in \mathbb{R}$, exhibit a sequence $\{x_n : n \in \mathbb{Z}^+\} \subseteq \mathbb{Q}$ such that $x_n \to r$.

   Solution: This can be done by extending the decimal expansion of said real number, e.g.

   $$\pi = 3, 3.1, 3.14, 3.141, \ldots$$

   The maximum error decreases by one-tenth every term, so this sequence converges to the real number.

5. A real number r is called the **least upper bound** or **supremum** of a set $S \subseteq \mathbb{R}$ if
   (a) $r \geq s$ for all $s \in S$, and
   (b) If $t \in \mathbb{R}$ is a real number such that $t \geq s$ for all $s \in S$, then $t \geq r$.

   The least upper bound of a set S is denoted by $\sup S$. Prove that if $S \subseteq \mathbb{R}$ and $\sup S$ exists, then there exists a sequence of points of S that converges to $\sup S$.

Solution:
Denote a sequence with $r_0 = 1, r_{i+1} = \frac{r_i}{2}$. Let s denote the upper bound. Since $s - r_i$ is not the upper bound, $\exists s_i \in S$ where $s - r_i < s_i \leq s$. Then $s_i \in S_{r_i}(s)$. We now show that s_i tends to s. Since r_i obviously tends to 0, $\exists r_i < r \forall r > 0$, then $s_i \in S_{r_i}(s) \subseteq S_r(s)$. Therefore there is a sequence in S that converges to its supremum. This is probably the intended solution of the question above.

6. Prove that if $S \subseteq \mathbb{R}$ and $\sup S$ exists, then if S is closed, $\sup S \in S$. Is the converse true?

Solution:
There is a sequence in S that tends to $\sup S$. Since S is closed, this implies $\sup S \in S$ by Theorem 5.5.
The converse is not true, because $(0, 1]$ is not closed.

1.16 Exercise 5.8

1. If a sequence converges to a point x, then x is an accumulation point of the sequence. (So "ultimately" implies "frequently.")

Solution:
This follows from the definition. Since the sequence converges to x, $\forall r > 0$,

$$\exists N' \in \mathbb{N} \text{ such that } n \in \mathbb{N} > N' \Rightarrow x_n \in S_r(x)$$

Letting $n = \max\{N' + 1, N\}$, we have $x_n \in S_r(x)$ for arbitrary r and N.

2. If x is an accumulation point of a sequence, the sequence need not converge to x. (So "frequently" does not imply "ultimately".)

Solution:
Let $x_n = (-1)^n$. Then 1 is an accumulation point, because for arbitrary r and N, we know $2N \geq N$ so that

$$x_{2N} = 1 \in S_r(1) \forall r > 0$$

However, the sequence does not converge to 1. see this problem

3. A subset F of a metric space X is closed if and only if whenever $x \in X$ is an accumulation point of a sequence of points of F, then $x \in F$.

Solution:
$\Rightarrow$:
Let $r_0 = 1$ and $r_{i+1} = \frac{r_i}{2}$. Define x_0 to be the first point in the sequence in F to be in $S_{r_0}(x)$. Define x_{i+1} to be the first point after x_i in the sequence to be in $S_{r_{i+1}}(x)$. Then x_n tends to x, as

$$\forall r > 0 \exists r_i < r \text{ such that } \forall n > i, x_n \in S_{r_i}(x) \subseteq S_r(x)$$

Since F is closed and x_n is a sequence in F, we can conclude that $x \in F$ by Theorem 5.5.

$\Leftarrow$:
For all sequences in F that converge to x, x is an accumulation point, so $x \in F$. This implies F is closed by Theorem 5.5.

4. State precisely what it means when the point $x \in X$ is *not* an accumulation point of the sequence $\{x_n : n \in \mathbb{Z}^+\} \subseteq X$.

Solution: $\exists r \in \mathbb{R} > 0, N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \notin S_r(x)$

1.17 Exercise 6.2

1. State precisely what it means when a function f is *not* continuous at a point x_0 in its domain.

Solution: $\exists \epsilon > 0$ where $\forall \delta > 0, \exists x \in S_\delta(x_0)$ such that $f(x) \notin S_\epsilon(f(x_0))$

2. Prove that the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 2 \\ -1 & \text{if } x < 2 \end{cases}$$

is not continuous at $x = 2$, but is continuous everywhere else.

Solution:
If $x \neq 2$, setting $\delta = |x - 2|$ suffices.
If $x = 2$, then for $\epsilon = 1$, $\forall \delta > 0$,

$$x + \frac{\delta}{2} \in S_\delta(x) \text{ but } f\left(x + \frac{\delta}{2}\right) = -1 \notin S_1(1)$$

3. Prove that the function g defined by $g(x) = \frac{1}{x}$ for $x \in \mathbb{R}$ and $x > 0$ is continuous on its domain.

Solution:
We want to show that $g(x)$ is continuous at an arbitrary point c. In other words, $\forall \epsilon > 0, \exists \delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$

$$
\begin{aligned}
|f(x) - f(c)| &< \epsilon \\
\left| \frac{1}{x} - \frac{1}{c} \right| &< \epsilon \\
\frac{|x-c|}{cx} &< \epsilon \\
|x - c| &< cx\epsilon
\end{aligned}
$$

Now we can impose a limit for δ, where $\delta \leq \frac{c}{2}$, so that $\frac{c}{2} \leq x \leq \frac{3c}{2}$. Substituting this into the inequality above yields

$$|x - c| < \frac{c^2}{2}\epsilon$$

Therefore, if we let $\delta = \min\{\frac{c^2}{2}\epsilon, \frac{c}{2}\}$, then

$$|x-c| < \delta \Rightarrow |x-c| < \frac{c^2}{2}\epsilon \Rightarrow |x-c| < cx\epsilon \Rightarrow \frac{|x-c|}{cx} < \epsilon \Rightarrow \left| \frac{1}{x} - \frac{1}{c} \right| < \epsilon$$

which completes the proof.

1.18 Theorem 6.4

1. Let (X, d_1) and (Y, d_2) be two metric spaces and let $f : D \subseteq X \to Y$. The f is **continuous at** point $x_0 \in D$ if and only if whenever $\{x_n : n \in \mathbb{Z}^+\}$ is a sequence in D that converges to x_0, then the sequence $\{f(x_n) : n \in \mathbb{Z}^+\}$ converges to $f(x_0)$ in Y.

Solution:
$\Rightarrow$:
Since f is continuous, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $f(S_\delta(x_0)) \subseteq S_\epsilon(f(x_0))$. Since x_n converges to x_0, $\exists N \in \mathbb{N}$ such that $n \in \mathbb{N} > N \Rightarrow x_n \in S_\delta(x_0)$. This also means that $f(x_n) \in S_\epsilon(f(x_0))$. Therefore

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } n \in \mathbb{N} > N \Rightarrow f(x_n) \in S_\epsilon(f(x_0))$$

so that $f(x_n)$ tends to $f(x_0)$.
$\Leftarrow$:
Assume f is not continuous. Then $\exists x_0 \in X$ where f is not continuous. That means $\exists \epsilon > 0$ where $\forall \delta > 0, A(\delta_i) = f(S_{\delta_i}(x_0)) - S_\epsilon(x) \neq \emptyset$. Since the set is nonempty, $\exists x_i \in A(\delta_i)$. Now let $\delta_i = 2^{-i}$. Then the sequence x_i tends to x_0, but $f(x_i)$ does not tend to $f(x_0)$, as it never enters $S_\epsilon(f(x_0))$. This is a contradiction, so f must be continuous.

1.19 Exercise 6.5

1. Let S be a sequence in $\mathbb{R}$, and let $\mathbb{Z}^+$ have the discrete topology that it gets by saying that every point in $\mathbb{Z}^+$ is open. Prove that S is continuous.

Solution: Set $\delta = 0.5$. Then $S_\delta(x) = \{x\}$, so $f(S_\delta(x)) = \{f(x)\} \in S_\epsilon(f(x)) \forall \epsilon > 0$

2. Let p be a point, and let $Y = \{p\}$. Make Y into a topological space by declaring that the sets $\emptyset$ and $Y = \{p\}$ are open. Let $f : \mathbb{R} \to Y$ be defined by $f(x) = p$ for all $x \in \mathbb{R}$. Prove that f is continuous. This is a special case of a theorem that says that any constant function is continuous.

Solution:
(Note: we have not defined convergence in this case where there is no metric)
For any sequence x_n that converges to x, it is obvious that $f(x_n)$ is always p. Therefore, whenever x_n is a sequence in $\mathbb{R}$ that converges to $x \in \mathbb{R}$, $f(x_n)$ is a (constant) sequence that converges to $f(x) = p$. By Theorem 6.4, f is continuous.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$
Prove that f is not continuous on $\mathbb{R}$. Is f continuous at any point of $\mathbb{R}$?

Solution:
f is not continuous at any point of $\mathbb{R}$ (hence it is not continuous).
Let $\epsilon = 1$. We take for granted that between any two numbers, one can always find a rational number and an irrational number. This implies $\forall \delta > 0$, $S_\delta(x)$ contains both rational and irrational x. Therefore $f(S_\delta(x))$ contains both 1 and -1, and so it is not contained in $S_{0.5}(f(x)) \forall x \in \mathbb{R}$.

4. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$
Is f continuous? Is f continuous at any point of $\mathbb{R}^+$?

Solution: At any $x \in \mathbb{Q}$, setting $\epsilon = \frac{f(x)}{2}$, one can see that $0 \notin S_\epsilon(f(x))$. However, since irrationals are dense in the reals (basically the same assumption as the previous question), $S_\delta(x)$ contains irrational numbers $\forall \delta > 0$. Then $f(S_\delta(x))$ contains 0, so it cannot be a subset of $S_\epsilon(f(x))$. Hence f is not continuous for all rational x.
However, f is continuous at any irrational point. The proof will be described verbally, as it is cumbersome to construct the sets required, yet it adds nothing to what a verbal explanation provides. First, given that $\epsilon > 0$, we can find the finite number of natural numbers where $\frac{1}{n} \geq \epsilon$. For each of these natural numbers n, we can list all of the finitely many fractions with a natural number as numerator and n as denominator. Repeat this for all n, and we have a finite set of fractions A. Let a denote the distance between x and the largest fraction in A that is smaller than x. Let b denote the distance between x and the smallest fraction in A that is greater than x. Finally, we can set $\delta = \min\{a, b\}$. Consider $y \in S_\delta(x)$, where x is irrational. If y is rational, it is not in A (or else the distance between x and y is smaller that both a and b, which is a contradiction). Therefore its denominator q in lowest terms must be so great that $\frac{1}{q} < \epsilon$, so $f(y) \in S_\epsilon(0)$. Otherwise, y is irrational, and $f(y) = 0 \in S_\epsilon(0)$. Therefore, we have proven that $f(S_\delta(x)) \subseteq S_\epsilon(0)$, which means that f is continuous at irrational x.

1.20 Exercise 7.3

1. Consider a closed interval $[a, b]$ in the real line with a and b both finite, and $a < b$. Make this interval into a topological space by giving it the metric topology induced by the absolute value metric $d(x, y) = |x - y|$, as in Section 3. Prove that $[0, 1], \cong [2, 4]$. This shows that length is *not* a topological property - it is not necessarily preserved by a homeomorphism.

   Solution: $f(x) = 2x + 2$ is a homeomorphism between the spaces.

2. Consider the open unit interval $(0, 1)$ as a topological space by giving it the metric topology induced by the absolute value metric $d(x, y) = |x - y|$ as in Section 3. Prove that $(0, 1) \cong \mathbb{R}$. This shows again that length is not a topological property - it is not necessarily preserved by a homeomorphism. In fact, we have here an interval of finite length homeomorphic to one whose length is infinite!

   Solution: $f(x) = \pi(x - 0.5)$ is a homeomorphism.

3. Prove that the unit circle centered at the origin is homeomorphic to a square with unit perimeter centered at the origin.

   Solution:
   We can use polar coordinates. Let the radius of the circle be R. For the square, given an angle θ, draw a line in that direction, and let $s(\theta)$ be the supremum of all the lengths the line can take without exiting the square. Then

   $$f(r, \theta) = \left(r \times \frac{s(\theta)}{R}, \theta\right)$$

   is such a homeomorphism from the circle to the square. Using the same method, all convex polygons are homeomorphic with the circle, and hence each other.

4. It was state in the introduction that a circle is not topologically the same as a straight line segment. The reason for this is not because the circle is curved and the line is not ("curveness" is not topological). Rather , it is because of the points of the line that are not end points.

   (a) Let $[a, b]$ be a closed interval on the real line with a and b both finite and $a < b$ (so $[a, b]$ is a line segment). Convince yourself that the property of not being an end point is a topological property; in other words, if $h : [a, b] \to Y$ is a homeomorphism then $x \neq a$ and $x \neq b$, $h(x)$ cannot be an "end point" of Y.

   Solution:
   Let $h(a) = p$, $h([a, b]) = Y'$. If there is an open set containing p in Y', intuitively, there should also be an open set that contains a in $[a, b]$. Since this is false, intuition tells us that any open set around p cannot be contained in Y'. This shouldn't be possible if p is not an endpoint.

   (b) Convince yourself that a line segment and a circle are not homeomorphic.

   Solution:
   If they are homeomorphic, a circle should then have two distinct end points. There doesn't seem to be any "intuitive" way to map a line segment continuously onto the circle while preserving two end points.

5. We said in the discussion of Cartesian product (Section 3 of Chapter I) that X is not a subset of $X \times Y$. However, X is a "topological subset" of $X \times Y$ in the sense that X is homeomorphic to a subset of $X \times Y$. Illustrate this case of $\mathbb{R} \times \mathbb{R}$ by convincing yourself that $\mathbb{R}$ is homeomorphic to the x-axis ($= \mathbb{R} \times \{0\}$) when the x-axis is given a topology as follows: a subset U of the x-axis is open if and only if for each point $(x, 0) \in U$, there is an $r > 0$ such tthat $S_r(x, 0) \cap$ (x-axis) is contained in U. Is $\mathbb{R}$ homeomorphic to any other subset of $\mathbb{R} \times \mathbb{R}$ when that subset is topologized analogously?

Solution:
This is very similar to our definition of open sets using the absolute value metric. Hence $f(x) = (x, 0)$ is a homeomorphism. Similarly, $\mathbb{R}$ is homeomorphic the y-axis and any other straight line in $\mathbb{R}^2$.

1.21 Exercise 7.5

1. Show that for any point $p \in \mathbb{R}$, $\{p\}$ is a retract of $\mathbb{R}$.

Solution: $f(x) = p \forall x \in \mathbb{R}$ is a retraction of $\mathbb{R}$ onto $\{p\}$.

2. Show that $(0, 1)$ is a retract of $\mathbb{R}$.

Solution:
I probably did something wrong here
It seems that $(0, 1)$ is not a retract (?) Proof:
Assume such a function r exists, and let $r(0) = a$. Since $a \in (0, 1)$, we have $0 < a < 1$. Let $\epsilon = \frac{a}{2}$. $\forall \delta > 0$, let $b = \min\{\epsilon, \frac{\delta}{2}\}$. Note that $b > 0$. Then $b \in S_\delta(0)$, but $b \notin S_\epsilon(a)$. Since this is true $\forall \delta > 0$, r is not continuous, which is a contradiction.
If $(0, 1)$ was a typo and the author meant $[0, 1]$, then

$$r(x) = \begin{cases} x & x \in [0,1] \\ 0 & x < 0 \\ 1 & x > 1 \end{cases}$$

can be a retraction.

3. Show that $A \subseteq \mathbb{R}$ is a retract of $\mathbb{R}$ if and only if the identity function $\mathrm{id}_A : A \to A$, defined by $\mathrm{id}_A(a) = a$ for $a \in A$, can be *extended* over $\mathbb{R}$ in the following sense. A function $f : A \to A$ can be *extended* over $\mathbb{R}$ if there is a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F(a) = f(a)$ for all $a \in A$.

Solution: If A is a retract, r is the required extension. If F exists, it is a retraction.

4. Do you think that $\{0, 1\}$ is a retract of $\mathbb{R}$? Why or why not?

Solution:
Let $r(x) = 0$. Let $A = \{a \in \mathbb{R} | f(S_a(x)) = \{0\}\}$. If A is empty, r is not continuous at x (let $\epsilon = 0.5$). Or else, A must have a supremum , since $S_a(x)$ cannot contain 1. Then at least one of $S_\delta(x + a)$ or $S_\delta(x - a)$ must contain points which map to 0 and 1 $\forall \delta > 0$. Setting $\epsilon = 0.5$, r cannot be continuous at both $x + a$ and $x - a$. Therefore, a retraction does not exists, and $\{0, 1\}$ is not a retract.

1.22 Exercise 8.3

Note: Someone who actually knows topology should look into this.

1. Determine all topologically different connected 1-manifolds that you can (*connected* means all in one piece). Try to determine any essential differences between them.

> **Solution:**
> $[a,b],[a,b),(a,b],(a,b)$ and a circle. They differ by the number of boundary points they contain, and whether or not a boundary point exists (in the universe but outside of the set).

2. Is a space that looks like a Y a topological 1-manifold? Why or why not?

> **Solution:** Consider the point where all three branches of the Y meet. Any open set containing that point is essentially a smaller Y, and there is no homeomorphism between that and any open subset of $\mathbb{R}$ or $\mathbb{R}^+$.

3. Give as many different examples of connected topological 2-manifolds as you can and indicate why you think they are different.

> **Solution:**
> (There are probably more)
> Those without boundary and those with boundary, e.g. $\{(x,y)|x^2+y^2<1\}$ and $\{(x,y)|x^2+y^2\leq 1\}$.

4. Find out what a Möbius strip and a Klein bottle are. Do you want to add anything to your answers to Problems 1 or 3 above?

> **Solution:** See "list_of_skipped_questions.txt"

2 Topological Spaces and Concepts in General

2.1 Exercise 2.6

1. Let X be a set. Verify that the indiscrete topology, the discrete topology and the finite-complement topology are in fact topologies on X.

> **Solution:**
> Indiscrete Topology:
> $\emptyset$ and X are open sets, and any intersection/union of open sets are obviously either empty of X.
> Discrete Topology:
> $\emptyset$ and X are open sets. Since any subset is open, any intersection/union of open sets must be a subset, and hence is open.
> Finite-complement Topology:
> *set* and X are open sets. Let A_i denote open sets. Then $X - \bigcup_i A_i \subseteq X - A_i$, where $X - A_i$ has a finite cardinality by definition. Therefore, $X - \bigcup_i A_i$ also has a finite cardinality, so $\bigcup_i A_i$ is open. Now let B_i denote finitely many open sets. $X - B_i$ is finite, and so is $\bigcup_i (X - B_i)$ (finite union of finite sets is finite). Since $\bigcup_i (X - B_i) = X - \bigcap_i B_i$ which is finite, $\bigcap_i B_i$ is an open set.

2. (a) Verify that Sierpinski space is a topological space.

   > **Solution:** It contains $\emptyset$ and X. Since there are only 3 open sets, brute forcing through all possible unions/intersections show that they are also open sets.

   (b) We said that there are only three different topologies that can be assigned to the 2 point set $\{0, 1\}$. Is the collection of $\{\emptyset, \{1\}, \{0, 1\}\}$ one of those three topologies on $\{0, 1\}$?

   > **Solution:** Yes. The same approach for (a) can be used here, since this is essentially the Sierpinski space with 0 and 1 reversed.

   (c) What is $\{0, 1\}$ with the finite-complement topology?

   > **Solution:** $\{\emptyset, \{0\}, \{1, \}, \{0, 1\}\}$

3. List all topologies that can be assigned to a 3 point set.

> **Solution:**
> $\{\emptyset, \{0, 1, 2\}\}$
> $\{\emptyset, \{0\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{1\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{2\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{0, 1\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{0, 2\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{1, 2\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{0\}, \{0, 2\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{1\}, \{0, 1\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{1\}, \{1, 2\}, \{0, 1, 2\}\}$

$\{\emptyset, \{2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{1\}, \{0,1\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$
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$\{\emptyset, \{0\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{1\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

4. Verify that the Sorgenfrey topology defined on the real line is in fact a topology. Is the interval $(0,1)$ open in this topology? How about $(0,1]$? Is $[0,1]$ closed?

Solution:
The Sorgenfrey topology obviously contains $\emptyset$ and $\mathbb{R}$. Let A_i denote open sets. Then

$$\forall x \in \bigcup_i A_i, x \in A_i \Rightarrow x \in [a,b) \subseteq A_i \subseteq \bigcup_i A_i$$

so any union of open sets is open. Now let B_i denote finitely many open sets. Now if $x \in B_i$, we let $\{p_i, q_i\} \subset \mathbb{R}$ such that $x \in [p_i, q_i) \subseteq B_i$. Let $P = \{p_i\}$ and $Q = \{q_i\}$. Since both P and Q are finite, P has a maximum p and Q has a minimum Q. Now

$$x \in \bigcap_i B_i \Rightarrow x \in [p,q) \subseteq [p_i, q_i) \subseteq B_i \forall i$$

Since $[p,q)$ is a subset of $B_i \forall i$, it is a subset of $\bigcap_i B_i$, hence any finite intersection of open sets is open. This shows that the Sorgenfrey topology is a topology.
$(0,1)$ is open, because

$$\forall x \in (0,1), x \in [x,1) \subset (0,1)$$

$(0,1]$ is not open, because $1 \in (0,1]$, but if $1 \in [a,b)$, then $\frac{1+b}{2}$ is an element of $[a,b)$ but not $(0,1]$, so $1 \in [a,b) \subseteq (0,1]$ cannot be true.
Consider $A = \mathbb{R} - [0,1]$ and let $x \in A$. Either $x < 0$, and $x \in [x, \frac{x}{2}) \subset A$, or $x > 1$, and $x \in [x, x+1)$. Therefore, A is open, so $[0,1]$ is closed.

5. Consider the topological spaces $(\mathbb{R}, \mathcal{I}), (\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$ and with the finite-complement topology (where $\mathcal{U}$ denotes the usual topology on $\mathbb{R}$ s in Chapter 3).
   (a) If $p \in \mathbb{R}$, is $\{p\}$ open in any of these spaces? Which ones?

   Solution: $(\mathbb{R}, \mathcal{D})$.

   (b) If $p \in \mathbb{R}$, is $\{p\}$ closed in any of these spaces? Which ones?

> **Solution:** $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, and the finite-complement topology.

(c) In which of these spaces is (a, b) open? $[a, b)$? $(a, b]$? $[a, b]$?

> **Solution:**
> (a, b):
> $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$
> $[a, b)$:
> $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{S})$
> $(a, b]$:
> $(\mathbb{R}, \mathcal{D})$
> $[a, b]$:
> $(\mathbb{R}, \mathcal{D})$

(d) Is the set $\{x \in \mathbb{R} : x \neq \frac{1}{n}\}$ open in any of the spaces? Is it closed in any of them?

> **Solution:**
> Open:
> $(\mathbb{R}, \mathcal{D})$
> Closed:
> $(\mathbb{R}, \mathcal{D})$

(e) Is the set $\{x \in \mathbb{R} : x \neq \frac{1}{n}$ and $x \neq 0\}$ open in any of the spaces? Is it closed in any of them?

> **Solution:**
> Open:
> $(\mathbb{R}, \mathcal{D})$
> Closed:
> $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$

6. Consider the spaces of Problem 5 above again, together with the three spaces that can be defined on $\{0, 1\}$.

(a) In which of these spaces are true: If x and y are two distinct points in the space then either there exists an open set U such that $x \in U$ and $y \notin U$, or there exists an open set V such that $y \in V$ and $x \notin V$. (A space for which this statement holds is called a T_0-space.)

> **Solution:** $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, the finite-complement topology, and the Sierpinski space.

(b) In which of these spaces is the following statement true: If x and Y are two distinct points in the space, then there exists an open set U such that $x \in U$ and $y \notin U$, and there exists an open set v such that $y \in V$ and $x \notin V$. (A space for which this statement holds is called a T_1-space.)

> **Solution:** $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, and the finite-complement topology.

(c) In which of these spaces is the following statement true: If x and y are two distinct points in the space, the there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. (A space for which this statement holds is called a T_2-space or a Hausdorff space.)

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$

7. Show that every T_2-space is a T_1-space, and that every T_1-space is a T_0-space, and give an example of a T_0-space that is not a T_1-space, and an example of a T_1-space that is not a T_2-space.

Solution:
T_2-space $\Rightarrow$ T_1-space: Since $U \cap V \neq \emptyset$, $x \notin V$ and $y \notin U$, so V and U are open sets that make the space a T_1-space.
T_1-space $\Rightarrow$ T_0-space: either one of U and V make the space a T_0-space.
T_0-space that is not a T_1-space: the Sierpinski space
T_1-space that is not a T_2-space: the finite-complement topology

8. A topological space X is said to be **metrizable** if a metric can be defined on X so that a set is open in the metric topology induced by this metric if and only if it is open in the topology that is already on the space.

   (a) Let X be a set with more than one point. Prove that $(X, \mathcal{I})$ is *not* metrizable. Thus the indiscrete topology on a set with more than one point is an example of a topological space that is *not* a metric space.

   Solution:
   If X has more than one point, it has 2 distinct points x and y, where we let $r = d(x, y) > 0$ by the definition of a metric. $S_{\frac{r}{2}}(x)$ is an open space according to the metric. However, it is neither empty (contains x) nor the universe (does not contain y). This forms a contradiction.

   (b) Let X be a set. Define a function from $X \times X = \{(x, y) : x, y \in X\}$ to $\mathbb{R}$ by

   $$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 \text{if } x = y \end{cases}$$

   Prove that d is a metric on X. What is the metric topology induced by d?

   Solution:
   Obviously, $d(x, y) = 0$ if and only if $x = y$, $d(x, y) \geq 0$, $d(x, y) = d(y, x)$. For the triangle inequality $d(x, y) \leq d(x, z) + d(y, z)$, note that it is trivial if $x = y$. If not, at least one of $d(x, z)$ and $d(y, z)$ must be nonzero, so the inequality holds. Therefore, $d(x, y)$ is a metric.
   $\forall x \in X, S_{0.5}(x) = \{x\}$. Since all singleton sets are open, the metric topology induced by d is the discrete topology.

2.2 Theorem 3.2

1. Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$. Then f is continuous at $x_0 \in X$ if and only if whenever V is an open subset of Y with $f(x_0) \in V$, then there exists an open subset U of X such that $x_0 \in U$ and $f(U) \subseteq V$.

Solution:
$\Rightarrow$:
Let V be open. Since V is open, $\forall v = f(x_0) \in V$, $\exists \epsilon > 0$ where $S_\epsilon(v) \subseteq V$. Since continuity is implied, $\exists \delta > 0$ where $f(S_\delta(x_0)) \subseteq S_\epsilon(v)$. Therefore $S_\delta(x_0)$ is the desired open U.
$\Leftarrow$:
Let $f(x_0) = v$. $\forall \epsilon > 0$, $V = S_\epsilon(v)$ is open. Then an open U exists where $x_0 \in U$. By definition, U is open, so $\exists \delta > 0$ such that $S_\delta(x_0) \in U$. Then

$$f(S_\delta(x_0)) \subseteq f(U) \subseteq V = S_\epsilon(v)$$

This demonstrates that f is continuous by the $\epsilon - \delta$ definition.

2.3 Theorem 3.4

1. Let X and Y be topological spaces and let $f : X \to Y$. Then f is continuous on X if and only if whenever V is an open subset of Y, then $f^{-1}(V)$ is open in X.

Solution:
$\Rightarrow$:
Based on Theorem 3.2, we have an open $U_x \forall f(x) \in V$ such that $f(U_x) \subseteq V$. Therefore, $U_x \subseteq f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

Since $f^{-1}(V)$ is a union of open sets, it is open.
$\Leftarrow$:
Similar to the second part of Theorem 3.2, let $f(x_0) = v$. $\forall \epsilon > 0$, $V = S_\epsilon(v)$ is open. Then $f^{-1}(V)$ is open (and contains x_0), so $\exists \delta > 0$ such that $S_\delta(x_0) \in f^{-1}(V)$. Then

$$f(S_\delta(x_0)) \subseteq V = S_\epsilon(v)$$

This demonstrates that f is continuous by the $\epsilon - \delta$ definition.

2.4 Exercise 3.5

1. Consider $(\mathbb{R}, \mathcal{I}), (\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$ and $(\mathbb{R}, \mathcal{F})$, the real line with the indiscrete topology, the discrete topology, the usual metric topology, the Sorgenfrey topology and the finite-complement topology, respectively. Let $f : \mathbb{R} \to \mathbb{R}$ be the identity function defined by $f(r) = r$ for all real numbers r. Determine all possible choices for $\mathcal{J}_1$ and $\mathcal{J}_2$ from $\mathcal{I}, \mathcal{D}, \mathcal{U}, \mathcal{S}, \mathcal{F}$ so that $f : (\mathbb{R}, \mathcal{J}_1) \to (\mathbb{R}, \mathcal{J}_2)$ is continuous.

Solution:

	$\mathcal{I}$	$\mathcal{D}$	$\mathcal{U}$	$\mathcal{S}$	$\mathcal{F}$
$\mathcal{I}$	✓	✗	✗	✗	✗
$\mathcal{D}$	✓	✓	✓	✓	✓
$\mathcal{U}$	✓	✗	✓	✗	✓
$\mathcal{S}$	✓	✗	✓	✓	✓
$\mathcal{F}$	✓	✗	✗	✗	✓

Where the rows denote $\mathcal{J}_1$ and the columns denote $\mathcal{J}_2$.

2. Let x be a set and let $\mathcal{J}$ be any topology on X. There is a topology $\mathcal{J}$' that can be assigned to X so that the identity function from $(X, \mathcal{J}')$ to $(X\mathcal{J})$ is always continuous no matter what $\mathcal{J}$ s. What is it?

Solution: The discrete topology $\mathcal{D}$.

3. Let X be any set and let $\mathcal{J}$ be any topology on X. There is a topology $\mathcal{J}'$ that can be given to X so that the identity function from $(X, \mathcal{J})$ to $(X, \mathcal{J}')$ is always continuous no matter what $\mathcal{J}$ is. What is it?

Solution: The indiscrete topology $\mathcal{I}$.

4. A function that preserves open sets is called an **open function**. More precisely, a function $f : X \to Y$ is an open function if whenever U is open in X, then $f(U)$ is open in Y.
   (a) Give an example of a continuous function that is not open.

   Solution: $f(r) = r$ where X is the real line with the discrete topology and Y is the real line with the indiscrete topology.

   (b) Give an example of an open function that is not continuous.

   Solution: $f(r) = r$ where X is the real line with the indiscrete topology and Y is the real line with the discrete topology.

2.5 Exercise 4.2

1. Consider the spaces $(\mathbb{R}, \mathcal{I}), (\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$ and $(\mathbb{R}, \mathcal{F})$. In which of these spaces is:
   (a) $(0, 2)$ a neighbourhood of 1?
   (b) $[0, 2]$ a neighbourhood of 1?
   (c) $[0, 2]$ a neighbourhood of 0?
   (d) $\{0\}$ a neighbourhood of 0?

   Solution:
   (a): $\mathcal{D}$,$\mathcal{U}$, $\mathcal{S}$
   (b): $\mathcal{D}$,$\mathcal{U}$, $\mathcal{S}$
   (c): $\mathcal{D}$, $\mathcal{S}$
   (d): $\mathcal{D}$

2. In the plane with its usual topology, is the unit square

$$\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

   a neighbourhood of any point in it? Which points?

Solution:
Any point that is not the boundary, or more explicitly,

$$\{(x, y) : 0 < x < 1, 0 < y < 1\}$$

2.6 Theorem 4.3

1. Let X be a topological space. Then $U \subseteq X$ is open if and only if U is a neighbourhood of each point $x \in U$.

Solution:
$\Rightarrow$:
Since U is open, $x \in U$ implies $x \in O \subseteq U$ for some open set O. Since U is a superset of an open O that contains x, U is a neighbourhood of x by definition.
$\Leftarrow$:
Since U is a neighbourhood, $x \in U$ implies $x \in O \subseteq U$ where O is some open set. This means U is open by definition.

2.7 Exercise 4.5

1. Let X be the real line with the indiscrete topology and consider the sequence $\{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Prove that if r is any point of X, then this sequence converges to r.

Solution:
Let O be an open set containing r. Since this is the indiscrete topology, $O = X$ is the only possibility. Then $\forall n > 0$, $f(n) \in O$, so the sequence converges to r.

2. Let $X = \mathbb{R}$ with the finite-complement topology. Prove that $\{\frac{1}{n} : n \in \mathbb{Z}^+\}$ converges to every point in the space.

Solution:
Let O be an open set containing r. Since this is the finite-complement topology, $X - O$ is finite. Let $A = \{a \in X - O | a > 0\}$. If A is empty, let $N = 0$. Or else, A must have a minimum m that is a positive number. Since $\frac{1}{n}$ tends to 0, $\exists N \in \mathbb{N}$ such that $\frac{1}{n} < m \forall n > N$. Now $\forall n > N$, $f(n) \in O$, so the sequence tends to r.

3. Recall that a topological space X is a Hausdorff space if distinct points of X are contained in disjoint open sets.
   (a) Prove that a space is a Hausdorff space if and only if distinct points are contained in disjoint neighbourhoods.

Solution:
$\Rightarrow$:
Open sets are also neighbourhoods, so distinct points in the space are contained in disjoint neighbourhoods.
$\Leftarrow$:
Let the distinct points be x and y, and the disjoint neighbourhoods N_x and N_y. Since N_x is a neighbourhoods of x, $x \in O_x \subseteq N_x$, and the same holds for y. Then x and y are in disjoint open sets O_x and O_y, so the space is a Hausdorff space.

(b) Prove that every metric space is a Hausdorff space.

Solution:
Let x and y be two distinct points, and let $R = d(x, y), r = \frac{R}{2}$. Consider a point $p \in S_r(x)$. Rearranging the triangle inequality,

$$d(p, y) \geq R - d(p, x) > R - r = r$$

so p cannot be in $S_r(y)$. The same holds if $p \in S_r(y)$. This means for all distinct x, y, there exists disjoint open sets $S_r(x)$ and $S_r(y)$ which contain x and y respectively, making it a Hausdorff space.

(c) Prove that in a Hausdorff space, if a sequence converges, then it converges to exactly one point. Deduced that the real line with the finite complement topology is not a Hausdorff space.

Solution:
Proof by contradiction. Let the sequence converge to distinct points x and y. Let O_x and O_y be a pair of disjoint open sets that contain x and y respectively. Since the sequence converges to x, $\exists N_x \in N$ such that $n > N \Rightarrow f(n) \in O_x$. There exists a similar N_y. Letting $N = \max\{N_x, N_y\}$, then $\forall n > N, x_n$ is in both O_x and O_y at the same time, which is a contradiction as both sets are disjoint.

4. Consider $X = [0, \Omega]$ with the order topology as defined in Section 4 of Chapter 3. We proved that in a metric space, a set S is closed if and only if whenever a sequence of points of S converges to a point x in the space, then $x \in S$.

   (a) Prove, usig the definition of closed set, that $S = [0, \Omega)$ is not a closed subset of $X = [0, \Omega]$ with the order topology.

   Solution:
   If it is a closed set, then $\{\Omega\}$ must be open in the order topology, so

   $$\exists a \in X : \Omega \in (a, \Omega] \subseteq \{\Omega\}$$

   For the interval to make sense, $a < \Omega$. Noting that Ω is a limit ordinal,

   $$a < \Omega \Rightarrow a + 1 < \Omega$$

   so $a + 1 \neq \Omega$ is an element of $(a, \Omega]$, which is a contradiction, since its superset $\{\Omega\}$ does not contain $a + 1$.

   (b) Prove that if a sequence of points of $S = [0, \Omega)$ converges to a point $x \in [0, \Omega]$, then $x \in S$.

Solution:
The sequence x_n cannot tend to Ω if it has a finite length, since it then as a maximum m, and $x_n \notin (m, \Omega] \forall n \in \mathbb{N}$. and If not, consider

$$l = \bigcap_{n \in \mathbb{N}} (x_n, \Omega]$$

Obviously $\Omega \in l$, but if $y \neq \Omega \in l$, then

$$y > x_n \forall n \in N \Rightarrow x_n \notin [y, \Omega] \forall n \in N$$

Loosely speaking, x_n can never reach y, so it can never "enter" the open set $[y, \Omega]$. However, such a y always exists by this.

(c) Deduce that $X = [0, \Omega]$ with the order topology is not a metric space, but

Solution:
If it is a metric space, then consider

$$A = \bigcap_{n \in \mathbb{N}} S_{\frac{1}{n}}(\Omega)$$

By this, $\exists a \in A : a \neq \Omega$. Let $d(a, \Omega) = r$. Since $\frac{1}{n}$ tends to 0, we know that $r > \frac{1}{i}$ for some $i \in \mathbb{N}$. Then $a \notin S_{\frac{1}{i}}(\Omega)$, so $a \notin A$, which is a contradiction.

(d) Prove that $X = [0, \Omega]$ with the order topology is a Hausdorff space.

Solution: Let $a < b$ be two distinct points. Then $[0, a]$ and $(a, \Omega]$ are disjoint open sets that contain a and b respectively.

2.8 Theorem 4.6

1. Let X and Y be topological spaces and let $f : X \to Y$. Then the funtion f is continuous at a point $x_0 \in X$ if and only if for every neighbourhood N_2 of $f(x_0)$ in Y, there is a neighbourhood N_1 of x_0 in X such that $f(N_1) \subseteq N_2$.