Solutions to Topology by Conover

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1 Some Familiar Topological Spaces and Basic Topological Concepts

1.1 Exercises 1.2

1. Let $A = (0, 1) \cup (1, 3)$. For the given $x \in A$, give a value of $r > 0$ such that $(x - r, x + r) \subseteq A$.
   (a) $x = \frac{3}{4}$
   (b) $x = 2$
   (c) $x = \frac{9}{8}$

   Solution: $\frac{1}{16}$

2. Prove that A $= (0, 1) \cup (1, 3)$ is an open subset of $\mathbb{R}$.

   Solution:
   Case 1: $x \in (0, 1)$
   Let $d = \min\{1 - x, x\}$. $(x - \frac{d}{2}, x + \frac{d}{2}) \subseteq A$ is an open set.
   Case 2: $x \in (1, 3)$
   Let $d = \min\{3 - x, x - 1\}$. The proof proceeds similarly as Case 1.
   As $\forall x \in A \exists$ an open interval $\subseteq A$ which contains x, A is an open set by definition.

3. Prove than an ordinary open interval is an open subset of $\mathbb{R}$ but that an open set need *not* be an open interval.

   Solution:
   All open intervals are open subsets:
   Replace 0 with $x - r$ and 1 with $x + r$ as in Case 1 from the question above.
   Open sets need not be an open interval:
   The empty set is an open set, but no open interval is empty, as it contains x (open intervals are in the form $(x - r, x + r)$).

4. State precisely what it means when a subset A of $\mathbb{R}$ is *not* open.

Solution: $\exists x \in A$ such that $(x - r, x + r) - A \neq \emptyset \forall r > 0$

5. Prove that the following subsets of $\mathbb{R}$ are not open.
   (a) The set of rational numbers

   Solution:
   An open interval is nonempty. As it is an open set, it cannot contain a single point only (next question) so it must contain at least 2 points. WLOG, let the 2 points be $p < q$. Using limits, $\exists n \in \mathbb{N}$ such that $r = p + \frac{\sqrt{2}}{n} < q$. Since all open intervals contain an irrational r, no open intervals can be a subset of the rationals, hence it is not open.

   (b) A set consisting of a single point

   Solution:
   $\forall$ open intervals $(x-r, x+r)$, it contains the distinct points x and $x+\frac{r}{2}$. Therefore, $\{x\}$ cannot be open.

   (c) An interval of the form $[a, b)$, where $a < b$

   Solution:
   $\forall$ open intervals $(a - r, a + r)$, $b = a - \frac{r}{2}$ is a point in that interval which is outside of $[a, b)$.

   (d) The set $A = \{x \in \mathbb{R} : x \neq \frac{1}{n},\ \text{for } n \in \mathbb{Z}^+\}$

   Solution:
   $\frac{1}{n}$ tends to 0. Therefore $\forall r > 0 \exists n \in \mathbb{N}$ such that $r > \frac{1}{n}$. Therefore all open intervals $(0-r, 0+r)$ contains a point outside of A.

1.2 Theorem 1.3

6. (a) The union of any collection of open subsets of the real line is also an open subset of the line

   Solution:
   Let $\bigcup A$ denote the union of subsets A. Then

   $$x \in \bigcup A \Rightarrow x \in A \Rightarrow x \in (x - r, x + r) \subseteq A \subseteq \bigcup A$$

   for some $r > 0$.
   As all points in $\bigcup A$ are in open intervals which are subsets of $\bigcup A$, the union is open by definition.

   (b) The intersection of any finite collection of open subsets of the real line is also an open subset of the line.

   Solution:
   This can be proven with induction.
   Let A and B be open sets. $x \in A \cap B \Rightarrow x \in A$. Since A is open, $\exists r_A > 0$ such that

   $$x \in (x - r_A, x + r_A) \subseteq A$$

The same holds for B. Letting $r = \min\{r_A, r_B\}$,

$$x \in (x-r, x+r) \subseteq A \cap B \forall x \in A \cap B$$

The same proof is used for the base case and the induction step.

(c) Both the empty set and $\mathbb{R}$ itself are open subsets of the real line.

Solution:
Empty set:

$$a \Rightarrow b$$

is defined to be true when a is false. The definition of an open set A involves the assumption $x \in A$, which is false, so it is vacuously true for the empty set.
$\mathbb{R}$:
$\forall x \in \mathbb{R}, (x-1, x+1) \in \mathbb{R}$

1.3 Exercise 1.4

7. Give an example of an infinite collection of open subsets of the real line whose intersection is not open, thus showing that the finiteness condition in Theorem 1.3(b) is necessary.

Solution:

$$\bigcap A \text{ where } A = \{(-r, r) | r > 0\}$$

Obviously $0 \in (-r, r) \forall r > 0$. However, the intersection does not contain any nonzero element, because $\forall x \neq 0$,

$$x \notin \{-\frac{|x|}{2}, \frac{|x|}{2}\}$$

The open intervals that form the intersection are open sets, but the intersection contains only 1 element, so it is not open.

1.4 Theorem 1.6

8. (a) The intersection of any collection of closed sets is closed.

Solution:
Let C_i be a closed set, and the corresponding open set be defined as $O_i = \mathbb{R} - C_i$.

$$\bigcap_{i \in I} C_i = \bigcap_{i \in I} \mathbb{R} - O_i = R - \bigcup_{i \in I} O_i$$

$\bigcup_{i \in I} O_i$ is a union of open sets, so it is open. Hence its complement (intersection of closed sets) is closed.

(b) The union of any finite collection of closed sets is closed.

Solution:

$$\bigcup_{n\in\mathbb{N}} C_n = \bigcup_{n\in\mathbb{N}} \mathbb{R} - O_n = R - \bigcap_{n\in\mathbb{N}} O_n$$

And the proof follows similar to the case above.

(c) $\emptyset$ and $\mathbb{R}$ itself are both closed

Solution: Their complements are each other, which are open.

1.5 Exercise 1.7

9. State precisely what it means when a subset of $\mathbb{R}$ is not closed. (Do this in term of points; saying that a set is not closed if its complement is not open is true, but is not what we want here.

Solution:
Let that subset be A. It is not closed when there is a point outside of it whose every open interval intersects with A.

$$\exists x \in \mathbb{R} - A \text{ such that } (x-r, x+r) \cap A \neq \emptyset \forall r > 0$$

10. Which of the following subsets of $\mathbb{R}$ are closed? Which are open?
    (a) The set $\mathbb{Z}$ of integers.
    (b) The set of rational numbers.
    (c) A set consisting of a single point.
    (d) An interval of the form $[a, b)$, where $a < b$.
    (e) The set $A = \{x \in \mathbb{R} : x \neq \frac{1}{n} \text{ for } n \in \mathbb{Z}^+\}$.
    (f) The set $A = \{x \in \mathbb{R} : x \neq \frac{1}{n} \text{ for } n \in \mathbb{Z}^+ \text{ and } x \neq 0\}$.

Solution:
Closed: a, c
Open: f

11. Prove that an ordinary closed interval is a closed subset of $\mathbb{R}$, but a closed set need not be a closed interval.

Solution:
Let the closed interval be $[a, b]$, where $a < b$. We want to prove that its complement is open. Let $x < a$ be in its complement. Then

$$(x-r, x+r) \text{ where } r = \frac{a-x}{2}$$

is an open interval in its complement. A similar open interval can be deduced for $x > b$. Therefore, the complement is open, and the closed interval is closed.

For the second part of the question, note that $\mathbb{R}$ is closed, but is not a closed interval. (Or else, let $\mathbb{R} = [a, b]$, and $\mathbb{R} - [a, b] \neq \emptyset$ forms a contradiction.)

12. Give an example of an infinite collection of closed subsets of whose union is not closed, thus showing tht the finiteness condition in theorem 1.6(b) is not necessary.

Solution:
Let the collection be

$$A = \{[-r, r] | 0 < r < 1\}$$

It is obvious that $\bigcup A = (-1, 1)$. The complement of $(-1, 1)$ contains 1, whose every open interval intersects with $(-1, 1)$. Hence the complement is not open, so $\bigcup A$ is not closed.

1.6 Exercise 2.3

13. Show that the absolute value formula, $d(x, y) = |x - y|$ is indeed a metric on the real line. Describe the **1-ball** centered at 0 in the topology induced by this metric.

Solution: Trivial.

1. $|x - y| \geq 0 \forall \{x, y\} \subset \mathbb{R}$
2. $|x - y| = 0$ iff $x = y$
3. $|x - z| + |y - z| = |x - z| + |z - y| \geq |x - z + z - y| = |x - y|$

The 1-ball is ordinary closed interval $(-1, 1)$.

14. Show that the distance formula is a metric on the Euclidean plane. Describe the 1-ball centered at $(0, 0)$ in the topology induced by this metric.

Solution:
In fact, this can be proven for all finite Euclidean spaces $\mathbb{R}^n$, where

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$d(x, y) \geq 0 \forall \{x, y\} \subset \mathbb{R}^n, d(x, y) = 0$ iff $x = y, d(x, y) = d(y, x) \forall \{x, y\} \subset \mathbb{R}^n$ are all trivial. What remains is the triangle inequality. Let $a_i = x_i - z_i$ and $b_i = z_i - y_i$. Therefore $a_i + b_i = x_i - y_i$.

$$d(x,z)+d(y,z)=\sqrt{\sum_i a_i^2}+\sqrt{\sum_i b_i^2}$$

$$(d(x,z)+d(y,z))^2=\sum_i(a_i^2+b_i^2)+2\sqrt{\left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)}$$

$$(d(x,z)+d(y,z))^2-(d(x,y))^2=2\left(\sqrt{\left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)}-\sum_i a_ib_i\right)$$

This is greater than 0 (from the Cauchy-Schwarz Inequality). Rearranging and taking the square root of both sides (we can do this because the distance formula is always positive) yields the desired inequality.
The 1-ball is the open disk with a radius of 1.

15. Define the obvious metric for Euclidean 3-space, $E^3=\mathbb{R}\times\mathbb{R}\times\mathbb{R}=\{(x,y,z):x,y,z\in\mathbb{R}\}$. Describe the 1-ball centered at the point $(0,0,0)$ in the topology induced by this metric. (In this topology, an r ball really is a ball - hence the name "r-ball.")

Solution: The metric is defined in the question above, and the 1-ball is an open sphere with a radius of 1.

16. Show that for any metric space (X,d),

(a) The union of any collection of open sets is open.

Solution:
Let O_i be open. Then

$$x\in\bigcup_i O_i\Rightarrow x\in O_i\Rightarrow x\in S_r(x)\subseteq O_i$$

Since $O_i\subseteq\bigcup O_i$, we have

$$x\in\bigcup_i O_i\Rightarrow x\in S_r(x)\subseteq\bigcup_i O_i$$

(b) The intersection of any finite collection of open sets is open.

Solution:
Similar to the metric case, this can be proven using induction.
Let A and B be open sets.

$$x\in A\cap B\Rightarrow x\in A\Rightarrow x\in S_{r_A}(x)\subseteq A$$

Similarly,

$$x\in S_{r_B}(x)\subseteq B$$

Letting $r=\min\{r_A,r_B\}$, we have

$$x\in A\cap B\Rightarrow x\in S_r(x)\subseteq A\cap B$$

(c) The empty set and X itself are open.

> **Solution:**
> The fact that the empty set is open is vacuously true, and the definition of $S_r(x)$ implies $S_r(x) \subseteq X$, so X is also open.

17. A set can have more than one metric defined on it, and different metrics may give rise to different topologies.

(a) Let $X = \mathbb{R}$ and define a metric on X by $d(x, y) = 1$ if $x \neq y$, $d(x, y) = 0$ if $x = y$. Prove that d is a metric on X. What is $S_{\frac{1}{2}}(0)$ in this metric? Is (X, d) the same space as $\mathbb{R}$ with its usual metric topology? In other words, does this metric give rise to the same topology on $\mathbb{R}$ as the usual metric does?

> **Solution:**
> The fact that d is a metric is trivial. $S_{\frac{1}{2}}(0) = \{0\}$. This metric gives rise to a different topology, as $\{0\}$ is open in this topology, but not the usual topology.

(b) Let X be the Euclidean place and define a metric on X by $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Prove that d is a metric on the plane, and describe the r-balls in this metric. Does this metric give rise to the same topology as the usual metric on this plane?

> **Solution:**
> $d(x, y) \geq 0 \forall \{x, y\} \subset \mathbb{R}^n, d(x, y) = 0$ iff $x = y, d(x, y) = d(y, x) \forall \{x, y\} \subset \mathbb{R}^n$ are all trivial. What remains is the triangle inequality.
>
> $$\begin{aligned} d(x, z) + d(y, z) &= |x_1 - z_1| + |x_2 - z_2| + |y_1 - z_1| + |y_2 - z_2| \\ &= |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| \\ &> |x_1 - y_1| + |x_2 - y_2| \\ &= d(x, y) \end{aligned}$$
>
> The r-balls of x are open squares centered at x with sides of length r. It is obvious that all open disks D centered at x contain open squares S also centered at x, (such that $x \in S \subset D$) and vice versa. If A is an open set under the usual topology, then
>
> $$x \in A \Rightarrow x \in D$$
>
> Combining this with
>
> $$x \in S \subset D$$
>
> we have
>
> $$x \in A \Rightarrow x \in S$$
>
> so A is an open set under this metric. Similarly, if B is an open set under this metric, B is also an open set under the usual metric. Therefore, both topologies are the same.

1.7 Exercise 3.3

18. Let $(\mathbb{Q}, d)$ be the space of rational numbers with the metric topology induced by the absolute value metric d.

(a) Show that the set $\{y \in \mathbb{Q} : 1 < y < 2\}$ is open in $(\mathbb{Q}, d)$.

> **Solution:**
> Denote the set as A. Let $r = \min\{\frac{2-y}{2}, \frac{y-1}{2}\}$. Then
> $$y \in (y - r, y + r) \cap A \subseteq A \forall y \in A$$

(b) Show that the set $[\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}$ is open in $(\mathbb{Q}, d)$.

> **Solution:** Noting that both $\sqrt{2}$ and $\sqrt{3}$ are not in the set, the proof can be obtained by replacing the rationals 1 and 2 in the solution above.

19. Let S = [1,2] be given teh topology induced by the absolute value metric. Show that $[1, \frac{3}{2}), (\frac{3}{2}, 2]$ and $[1, 2]$ are all open in (S, d).

> **Solution:**
> For the first set, if $x \neq 1$, let $r = \min\{\frac{3}{2} - x, x - 1\}$. Then $S_r(x) \cap [1, \frac{3}{2}) = S_r(x)$, and so
> $$x \in [1, \frac{3}{2}) \Rightarrow x \in S_r(x) \subset [1, \frac{3}{2})$$
> If $x = 1$, then $r = \frac{5}{4}$ suffices.
> A similar approach is used for the second and third set, noting the special cases for $x = 1$ and $x = 2$.

20. Let $(\mathbb{Z}, d)$ be the space of integers with the metric topology induced by the absolute value metric. Show that *every* subset of $\mathbb{Z}$ is open in $(\mathbb{Z}, d)$. In particular, any set consisting of a single point is open in $(\mathbb{Z}, d)$.

> **Solution:**
> Let A be a subset, and let $r = \frac{1}{2}$. Then $\forall x \in A$,
> $$x \in S_r(x) \cap A = \{x\} \subseteq A$$

21. Let (X, d) be a discrete space. Show that every subset of X is open in (X, d).

> **Solution:**
> Let A be a subset. Then
> $$A = \bigcup \{\{x\} | x \in A\}$$
> which is the union of open sets, so A is also open.

1.8 Theorem 4.1

22. A subset A or $\mathbb{R}$ is open in the metric topology induced by the absolute value metric if and only if for each point $x \in A$, there exist real numbers a and b with $a < x < b$, such that $x \in (a, b) \subseteq A$.

Solution:
$\Leftarrow$:
Let $r = \min\{x - a, b - x\}$. Then $\forall x \in A$,

$$x \in (x - r, x + r) \subseteq (a, b) \subseteq A$$

$\Rightarrow$:
As A is open,

$$x \in A \Rightarrow x \in S_r(x)$$

so $a = x - r$ and $b = x + r$ suffices.

1.9 Exercise 4.3

23. Convince yourself that the real line is a totally ordered set. Give an example of a set with an order relation that satisfies (1), (2), and (3) of Definition 4.2, but does *not* satisfy (4). Such an ordered set is called **partially ordered** since not every pair of elements can be compared. [*Hint*: Consider $\mathcal{P}(X)$ with "$\leq$" thought as "subset of].

Solution:
Intuitively, real numbers satisfy (1) through (4). A rigorous proof involves Dedekind cuts, which is (probably) out of the scope of this book.
$\subseteq$ in $\mathcal{P}(X)$ is one such example. (1) and (3) are trivial, and (2) is true by definition. However, $\{1, 2\}$ and $\{3, 4\}$ are elements of $\mathcal{P}(\mathbb{N})$ where both elements cannot be compared.

1.10 Exercise 4.5

24. Every open subset of $\mathbb{R}$ has cardinality c

Solution:
Let A be an open subset. $\forall x \in A$,

$$x \in (x - r, x + r) \subseteq A$$

The function

$$f(y) = \frac{y - x}{r} \times \frac{\pi}{2}$$

is a bijection from $(x - r, x + r)$ to $\mathbb{R}$, so they share the same cardinality c. As

$$(x - r, x + r) \subseteq A \subseteq \mathbb{R}$$

$|A| = c$ by the Cantor-Bernstein Theorem.

25. For any positive integer n, there are open subsets of $[0, \Omega)$ with cardinality n. In particular, there are points x in $[0, \Omega)$ such that the singleton set $\{x\}$ is an open set. But $[0, \Omega)$ is not discrete - not every singleton set is open.

Solution:
$[1, n]$ is a set with cardinality n. To show that it is open, $\forall x \in [1, n]$,

$$0 < x < n + 1 \text{ such that } x \in (0, n + 1) \subseteq A$$

Putting $n = 1$ gives us a singleton set that is open. However, not all singleton sets are open, e.g. $\{\omega\}$. For it to be open, $a < \omega < b$ such that

$$\omega \in (a, b) \subseteq \{\omega\}$$

Since ω is a limit ordinal, $a < a + 1 < \omega < b$, so

$$a + 1 \in (a, b) - \{\omega\}$$

which is a contradiction.

26. There are open subsets of $[0, \Omega)$ of cardinality $\aleph_0$ and there are open subsets of $[0, \Omega)$ of cardinality $\aleph_1$.

Solution:
$\forall n \in \mathbb{N}$, $[n + 1, \omega)$ is a subset of cardinality $\aleph_0$. To show that it is open, $\forall x \in [n + 1, \omega)$,

$$n < x < \omega \text{ such that } x \in (n, \omega) \subseteq [n + 1, \omega)$$

Similarly, $[n + 1, \Omega)$ is a subset of cardinality $\aleph_1$. The bijective function from $[0, \Omega)$ to $[n + 1, \Omega)$ is given by

$$f(x) = \begin{cases} x + n + 1 & x \in \mathbb{N} \\ x & \text{else} \end{cases}$$

To show that it is open, $\forall x \in [n + 1, \Omega)$,

$$n < x < \Omega \text{ such that } x \in (n, \Omega) \subseteq [n + 1, \Omega)$$

27. Supplementary: No countable collection of subsets of $[0, \Omega]$ intersect at $\{\Omega\}$ only.

Solution:
Let A_n be an open set in the collection. Since $\Omega \in A$,

$$\Omega \in (a_n, \Omega] \subseteq A$$

Where $(a_n, \Omega] = [a_n + 1, \Omega]$. Let $b_n = a_n + 1$, and consider

$$b = \bigcup_{n \in \mathbb{N}} b_n$$

The union of ordinals (b) is an ordinal that is greater than all of the ordinals in the union ($b > b_n$). Since we have subsets of A_n,

$$\bigcap_{n \in \mathbb{N}} [b_n, \Omega] \subseteq \bigcap_{n \in \mathbb{N}} A_n$$

Since $b > b_n \forall n \in \mathbb{N}$, b is an element of the set in the left, meaning the countable ordinal b is an element of the intersection, which is a contradiction.

1.11 Exercise 4.6

28. Show that $\omega = [0, \omega)$ with the order topology is a discrete space, so is the same as $\mathbb{Z}^+ \cap \{0\}$ when given the metric topology induced by the absolute value metric on $\mathbb{R}$.

Solution:
First we prove that all singleton sets $\{x\}$ are open.

$$x = 0 \Rightarrow x \in [0, 1) \subseteq \{0\}$$

$$x \neq 0 \Rightarrow x \in (x - 1, x + 1) \subseteq \{x\}$$

Note that $x - 1$ exists $\forall x \neq 0$, as x is a nonzero natural number.
Since

$$\omega = \mathbb{Z}^+ \cup \{0\}$$

and both are discrete, they are the same.

29. The definition of the order topology is given in terms of open intervals. Let $(X, \leq)$ be a totally ordered set and give it the order topology. Show that a subset of x is open in the order topology if and only if it is a union of open intervals. Because of this, we say that the collection of open intervals is a *basis* for the order topology on a totally ordered set.

Solution:
Let A be an open subset. Since A is open, $\forall x \in A$,

$$x \in O_x \subseteq A$$

where O_x denotes an open interval containing x that is a subset of A. Then

$$A = \bigcup_{x \in A} O_x$$

So all open sets are unions of open intervals O_x. Since open intervals are open sets (this can easily be observed), any union of open intervals are open sets.

1.12 Exercise 5.3

30. State precisely what it means when a sequence in (X, d) does not converge to the point $x \in X$.

Solution: $\forall N \in \mathbb{N} > 0, r > 0, \exists r > 0, n > N$ such that $x_n \notin S_r(x)$

31. Let $\{x_n : n \in \mathbb{Z}^+\}$ be the sequence in $\mathbb{R}$ defined by $x_n = \frac{1}{n}$. Does this sequence converge? To what? Prove it.

Solution:
It converges to 0. We take for granted that natural numbers are unbounded above, i.e. $\forall r \in \mathbb{R} \exists n \in \mathbb{N}$ such that $n > r$. Then $\forall r > 0$, $\exists N \in \mathbb{N} > \frac{1}{r}$, so $r > x_n \forall n \in \mathbb{N} > N$, which is equivalent to $x_n \in S_r(0) \forall n \in \mathbb{N} > N$.

32. Let $\{x_n : n \in \mathbb{Z}^+\}$ be the sequence in $\mathbb{R}$ defined by $x_n = (-1)^n(\frac{1}{n})$. Does this sequence converge? To what? Prove it.

> **Solution:** It converges to 0. Note that $|x_n| \in S_r(0) \Rightarrow x_n \in S_r(0)$. Since $|x_n|$ converges by the previous question, so does x_n.

33. Let $\{x_n : n \in \mathbb{Z}^+\}$ be the sequence in $\mathbb{R}$ defined by $x_n = (-1)^n$. Does this sequence converge? To what? Prove it.

> **Solution:**
> It does not converge to $a \forall a \in \mathbb{R}$.
> If $a > 1$, let $r = a - 1$, then x_n never enters $S_r(a)$.
> If $a < -1$, let $r = -1 - a$, then x_n never enters $S_r(a)$.
> If $a = 1$ or $a = -1$, let $r = 1$, then x_n never enters $S_r(a)$.
> Else, let $r = \min\{1 - a, a + 1\}$, then x_n never enters $S_r(a)$.

1.13 Theorem 5.4

34. A sequence in a metric space can converge to at most one point.

> **Solution:**
> Proof by contradiction. Let it converge to two distinct points a and b. Let $r = \frac{d(a,b)}{2}$, so $S_r(a)$ and $S_r(b)$ are disjoint by the triangle inequality.
> Since the sequence converges to both points, $\exists\{N_a, N_b\} \subseteq \mathbb{N}$ such that
>
> $$n \in \mathbb{N} > N_a \Rightarrow x_n \in S_r(a)$$
>
> and similarly for b.
> Let $N = \max\{N_a, N_b\}$. Then $x_{N+1} \in S_r(a)$ and $x_{N+1} \in S_r(b)$ which is a contradiction.

1.14 Theorem 5.5

35. A subset S of a metric space X is closed if and only if whenever a sequence of point of S converges to a point $x \in X$, then $x \in S$.

> **Solution:**
> $\Rightarrow$:
> Assume the opposite. Then $x \in X - S$ but $S_r(x)$ intersects with S $\forall r > 0$. Then $X - S$ is not open (it has no open subset containing x), so S is not closed, which is a contradiction.
> $\Leftarrow$:
> Assume the opposite. Then $X - S$ is not open, so $\exists x \in X - S$ such that $S_r(x)$ intersects with S $\forall r > 0$. Let $y_n \in S_{\frac{1}{n}}(x) \cap S$. Then y is a sequence in S that tends to a point x outside of S. This is a contradiction.

1.15 Exercise 5.6

36. Use Theorem 5.5 to decide if the following subsets of the real line are closed (in the metric topology induced by the absolute value metric).
    (a) $[0, 1]$
    (b) $[1, \infty)$
    (c) $\{x \in \mathbb{R} : x = \frac{1}{n} \text{ for } n \in \mathbb{Z}^+\}$
    (d) $\{x \in \mathbb{R} : x = \frac{1}{n} \text{ for } n \in \mathbb{Z}^+, \text{ or } x = 0\}$
    (e) $\{x \in \mathbb{R} : x = \frac{1}{\sqrt{n}} \text{ for } n \in \mathbb{Z}^+\}$

Solution: Closed, closed, not closed, closed, not closed.

37. Is$(\mathbb{R} - \mathbb{Q}) \cap \{x \in \mathbb{R} : x = \frac{1}{\sqrt{n}} \text{ for } n \in \mathbb{Z}^+\}$ a closed subset of the *irrationals* with the metric topology induced by the absolute value metric?

Solution:
Yes. Whenever the sequence tends to an irrational point x, x is in the set. Proof by contradiction:
If $x < 0$, let $r = -x$. The sequence never enters $S_r(x)$, so it cannot converge to x.
$If x > 1$, let $r = x - 1$. The sequence never enters $S_r(x)$, so it cannot converge to x.
If $0 < x < 1$, then $\frac{1}{\sqrt{n+1}} < x < \frac{1}{\sqrt{n}}$. Let $r = \min\{x - \frac{1}{\sqrt{n+1}}, \frac{1}{\sqrt{n}} - x\}$. The sequence never enters $S_r(x)$, so it cannot converge to x.
In all other cases, x is either rational or in the subset itself.

38. Let $[p_n = (x_n, y_n) : n \in \mathbb{Z}^+\}$ be a sequence in E^2, the Euclidean plane. Prove that $p_n \to p = (x, y)$ if and only if both $x_n \to x$ and $y_n \to y$. (Thus a sequence in the plane converges if and only if it converges "coordinate-wise")

Solution:
$\Rightarrow$:
Obviously, $d(p_n, p) \geq d(x_n, x)$, similarly for y. Hence $p_n \in S_r(p) \Rightarrow x_n \in S_r(x)$ and similarly for y. Since p_n converges to p, x_n converges to x and so does y.
$\Leftarrow$:
$\forall r > 0$, let $r' = \frac{r}{\sqrt{2}}$. Then $\exists N_x \in \mathbb{N}$ such that $n \in \mathbb{N} > N_x \Rightarrow x_n \in S_{r'}(x)$. A similar N_y exists for y. Letting $N = \max\{N_x, N_y\}$ implies that $\forall n > N$, $p_n \in S_r(p)$. As this holds $\forall r > 0$, we can conclude that $p_n \to p$.

39. Prove that every real number can be written as the limit of a convergent sequence of *rational* numbers, i.e., if $r \in \mathbb{R}$, exhibit a sequence $\{x_n : n \in \mathbb{Z}^+\} \subseteq \mathbb{Q}$ such that $x_n \to r$.

Solution: This can be done by extending the decimal expansion of said real number, e.g.

$$\pi = 3, 3.1, 3.14, 3.141, \ldots$$

The maximum error decreases by one-tenth every term, so this sequence converges to the real number.

40. A real number r is called the **least upper bound** or **supremum** of a set $S \subseteq \mathbb{R}$ if

    (a) $r \geq s$ for all $s \in S$, and

    (b) If $t \in \mathbb{R}$ is a real number such that $t \geq s$ for all $s \in S$, then $t \geq r$.

    The least upper bound of a set S is denoted by sup S. Prove that if $S \subseteq \mathbb{R}$ and sup S exists, then there exists a sequence of points of S that converges to sup S.

Solution:
Denote a sequence with $r_0 = 1, r_{i+1} = \frac{r_i}{2}$. Let s denote the upper bound. Since $s - r_i$ is not the upper bound, $\exists s_i \in S$ where $s - r_i < s_i \leq s$. Then $s_i \in S_{r_i}(s)$. We now show that s_i tends to s. Since r_i obviously tends to 0, $\exists r_i < r \forall r > 0$, then $s_i \in S_{r_i}(s) \subseteq S_r(s)$. Therefore there is a sequence in S that converges to its supremum. This is probably the intended solution of the question above.

41. Prove that if $S \subseteq \mathbb{R}$ and sup S exists, then if S is closed, sup $S \in S$. Is the converse true?

Solution:
There is a sequence in S that tends to sup S. Since S is closed, this implies sup $S \in S$ by Theorem 5.5.
The converse is not true, because $(0, 1]$ is not closed.

1.16 Exercise 5.8

42. If a sequence converges to a point x, then x is an accumulation point of the sequence. (So "ultimately" implies "frequently.")

Solution:
This follows from the definition. Since the sequence converges to x, $\forall r > 0$,

$$\exists N' \in \mathbb{N} \text{ such that } n \in \mathbb{N} > N' \Rightarrow x_n \in S_r(x)$$

Letting $n = \max\{N' + 1, N\}$, we have $x_n \in S_r(x)$ for arbitrary r and N.

43. If x is an accumulation point of a sequence, the sequence need not converge to x. (So "frequently" does not imply "ultimately".)

Solution:
Let $x_n = (-1)^n$. Then 1 is an accumulation point, because for arbitrary r and N, we know $2N \geq N$ so that

$$x_{2N} = 1 \in S_r(1) \forall r > 0$$

However, the sequence does not converge to 1. see this problem

44. A subset F of a metric space X is closed if and only if whenever $x \in X$ is an accumulation point of a sequence of points of F, then $x \in F$.

Solution:
$\Rightarrow$:
Let $r_0 = 1$ and $r_{i+1} = \frac{r_i}{2}$. Define x_0 to be the first point in the sequence in F to be in $S_{r_0}(x)$. Define x_{i+1} to be the first point after x_i in the sequence to be in $S_{r_{i+1}}(x)$. Then x_n tends to x, as

$$\forall r > 0 \exists r_i < r \text{ such that } \forall n > i, x_n \in S_{r_i}(x) \subseteq S_r(x)$$

Since F is closed and x_n is a sequence in F, we can conclude that $x \in F$ by Theorem 5.5.

$\Leftarrow$:
For all sequences in F that converge to x, x is an accumulation point, so $x \in F$. This implies F is closed by Theorem 5.5.

45. State precisely what it means when the point $x \in X$ is *not* an accumulation point of the sequence $\{x_n : n \in \mathbb{Z}^+\} \subseteq X$.

Solution: $\exists r \in \mathbb{R} > 0, N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \notin S_r(x)$

1.17 Exercise 6.2

46. State precisely what it means when a function f is *not* continuous at a point x_0 in its domain.

Solution: $\exists \epsilon > 0$ where $\forall \delta > 0, \exists x \in S_\delta(x_0)$ such that $f(x) \notin S_\epsilon(f(x_0))$

47. Prove that the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 2 \\ -1 & \text{if } x < 2 \end{cases}$$

is not continuous at $x = 2$, but is continuous everywhere else.

Solution:
If $x \neq 2$, setting $\delta = |x - 2|$ suffices.
If $x = 2$, then for $\epsilon = 1$, $\forall \delta > 0$,

$$x + \frac{\delta}{2} \in S_\delta(x) \text{ but } f\left(x + \frac{\delta}{2}\right) = -1 \notin S_1(1)$$

48. Prove that the function g defined by $g(x) = \frac{1}{x}$ for $x \in \mathbb{R}$ and $x > 0$ is continuous on its domain.

Solution:
We want to show that $g(x)$ is continuous at an arbitrary point c. In other words, $\forall \epsilon > 0, \exists \delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$

$$|f(x) - f(c)| < \epsilon$$
$$\left|\frac{1}{x} - \frac{1}{c}\right| < \epsilon$$
$$\frac{|x - c|}{cx} < \epsilon$$
$$|x - c| < cx\epsilon$$

Now we can impose a limit for δ, where $\delta \leq \frac{c}{2}$, so that $\frac{c}{2} \leq x \leq \frac{3c}{2}$. Substituting this into the inequality above yields

$$|x - c| < \frac{c^2}{2}\epsilon$$

Therefore, if we let $\delta = \min\{\frac{c^2}{2}\epsilon, \frac{c}{2}\}$, then

$$|x - c| < \delta \Rightarrow |x - c| < \frac{c^2}{2}\epsilon \Rightarrow |x - c| < cx\epsilon \Rightarrow \frac{|x - c|}{cx} < \epsilon \Rightarrow \left|\frac{1}{x} - \frac{1}{c}\right| < \epsilon$$

which completes the proof.

1.18 Theorem 6.4

49. Let (X, d_1) and (Y, d_2) be two metric spaces and let $f : D \subseteq X \to Y$. The f is **continuous at** point $x_0 \in D$ if and only if whenever $\{x_n : n \in \mathbb{Z}^+\}$ is a sequence in D that converges to x_0, then the sequence $\{f(x_n) : n \in \mathbb{Z}^+\}$ converges to $f(x_0)$ in Y.

Solution:
$\Rightarrow$:
Since f is continuous, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $f(S_\delta(x_0)) \subseteq S_\epsilon(f(x_0))$. Since x_n converges to x_0, $\exists N \in \mathbb{N}$ such that $n \in \mathbb{N} > N \Rightarrow x_n \in S_\delta(x_0)$. This also means that $f(x_n) \in S_\epsilon(f(x_0))$. Therefore

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } n \in \mathbb{N} > N \Rightarrow f(x_n) \in S_\epsilon(f(x_0))$$

so that $f(x_n)$ tends to $f(x_0)$.
$\Leftarrow$:
Assume f is not continuous. Then $\exists x_0 \in X$ where f is not continuous. That means $\exists \epsilon > 0$ where $\forall \delta > 0, A(\delta_i) = f(S_{\delta_i}(x_0)) - S_\epsilon(x) \neq \emptyset$. Since the set is nonempty, $\exists x_i \in A(\delta_i)$. Now let $\delta_i = 2^{-i}$. Then the sequence x_i tends to x_0, but $f(x_i)$ does not tend to $f(x_0)$, as it never enters $S_\epsilon(f(x_0))$. This is a contradiction, so f must be continuous.

1.19 Exercise 6.5

50. Let S be a sequence in $\mathbb{R}$, and let $\mathbb{Z}^+$ have the discrete topology that it gets by saying that every point in $\mathbb{Z}^+$ is open. Prove that S is continuous.

Solution: Set $\delta = 0.5$. Then $S_\delta(x) = \{x\}$, so $f(S_\delta(x)) = \{f(x)\} \in S_\epsilon(f(x)) \forall \epsilon > 0$

51. Let p be a point, and let $Y = \{p\}$. Make Y into a topological space by declaring that the sets $\emptyset$ and $Y = \{p\}$ are open. Let $f : \mathbb{R} \to Y$ be defined by $f(x) = p$ for all $x \in \mathbb{R}$. Prove that f is continuous. This is a special case of a theorem that says that any constant function is continuous.

Solution:
(Note: we have not defined convergence in this case where there is no metric)
For any sequence x_n that converges to x, it is obvious that $f(x_n)$ is always p. Therefore, whenever x_n is a sequence in $\mathbb{R}$ that converges to $x \in \mathbb{R}$, $f(x_n)$ is a (constant) sequence that converges to $f(x) = p$. By Theorem 6.4, f is continuous.

52. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Prove that f is not continuous on $\mathbb{R}$. Is f continuous at any point of $\mathbb{R}$?

Solution:
f is not continuous at any point of $\mathbb{R}$ (hence it is not continuous).
Let $\epsilon = 1$. We take for granted that between any two numbers, one can always find a rational number and an irrational number. This implies $\forall \delta > 0$, $S_\delta(x)$ contains both rational and irrational x. Therefore $f(S_\delta(x))$ contains both 1 and -1, and so it is not contained in $S_{0.5}(f(x)) \forall x \in \mathbb{R}$.

53. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Is f continuous? Is f continuous at any point of $\mathbb{R}^+$?

Solution: At any $x \in \mathbb{Q}$, setting $\epsilon = \frac{f(x)}{2}$, one can see that $0 \notin S_\epsilon(f(x))$. However, since irrationals are dense in the reals (basically the same assumption as the previous question), $S_\delta(x)$ contains irrational numbers $\forall \delta > 0$. Then $f(S_\delta(x))$ contains 0, so it cannot be a subset of $S_\epsilon(f(x))$. Hence f is not continuous for all rational x.
However, f is continuous at any irrational point. The proof will be described verbally, as it is cumbersome to construct the sets required, yet it adds nothing to what a verbal explanation provides. First, given that $\epsilon > 0$, we can find the finite number of natural numbers where $\frac{1}{n} \geq \epsilon$. For each of these natural numbers n, we can list all of the finitely many fractions with a natural number as numerator and n as denominator. Repeat this for all n, and we have a finite set of fractions A. Let a denote the distance between x and the largest fraction in A that is smaller than x. Let b denote the distance between x and the smallest fraction in A that is greater than x. Finally, we can set $\delta = \min\{a, b\}$. Consider $y \in S_\delta(x)$, where x is irrational. If y is rational, it is not in A (or else the distance between x and y is smaller that both a and b, which is a contradiction). Therefore its denominator q in lowest terms must be so great that $\frac{1}{q} < \epsilon$, so $f(y) \in S_\epsilon(0)$. Otherwise, y is irrational, and $f(y) = 0 \in S_\epsilon(0)$. Therefore, we have proven that $f(S_\delta(x)) \subseteq S_\epsilon(0)$, which means that f is continuous at irrational x.