Solutions to Topology by Conover

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1 Topological Spaces and Concepts in General

1.1 Exercise 2.6

1. Let X be a set. Verify that the indiscrete topology, the discrete topology and the finite-complement topology are in fact topologies on X.

> **Solution:**
> Indiscrete Topology:
> $\emptyset$ and X are open sets, and any intersection/union of open sets are obviously either empty of X.
> Discrete Topology:
> $\emptyset$ and X are open sets. Since any subset is open, any intersection/union of open sets must be a subset, and hence is open.
> Finite-complement Topology:
> *set* and X are open sets. Let A_i denote open sets. Then $X - \bigcup_i A_i \subseteq X - A_i$, where $X - A_i$ has a finite cardinality by definition. Therefore, $X - \bigcup_i A_i$ also has a finite cardinality, so $\bigcup_i A_i$ is open. Now let B_i denote finitely many open sets. $X - B_i$ is finite, and so is $\bigcup_i (X - B_i)$ (finite union of finite sets is finite). Since $\bigcup_i (X - B_i) = X - \bigcap_i B_i$ which is finite, $\bigcap_i B_i$ is an open set.

2. (a) Verify that Sierpinski space is a topological space.

> **Solution:** It contains $\emptyset$ and X. Since there are only 3 open sets, brute forcing through all possible unions/intersections show that they are also open sets.

(b) We said that there are only three different topologies that can be assigned to the 2 point set $\{0, 1\}$. Is the collection of $\{\emptyset, \{1\}, \{0, 1\}\}$ one of those three topologies on $\{0, 1\}$?

> **Solution:** Yes. The same approach for (a) can be used here, since this is essentially the Sierpinski space with 0 and 1 reversed.

(c) What is $\{0, 1\}$ with the finite-complement topology?

> **Solution:** $\{\emptyset, \{0\}, \{1, \}, \{0, 1\}\}$

3. List all topologies that can be assigned to a 3 point set.

> **Solution:**
> $\{\emptyset, \{0, 1, 2\}\}$
> $\{\emptyset, \{0\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{1\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{2\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{0, 1\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{0, 2\}, \{0, 1, 2\}\}$
> $\{\emptyset, \{1, 2\}, \{0, 1, 2\}\}$
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$\{\emptyset, \{1\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

4. Verify that the Sorgenfrey topology defined on the real line is in fact a topology. Is the interval $(0,1)$ open in this topology? How about $(0,1]$? Is $[0,1]$ closed?

Solution:
The Sorgenfrey topology obviously contains $\emptyset$ and $\mathbb{R}$. Let A_i denote open sets. Then

$$\forall x \in \bigcup_i A_i, x \in A_i \Rightarrow x \in [a,b) \subseteq A_i \subseteq \bigcup_i A_i$$

so any union of open sets is open. Now let B_i denote finitely many open sets. Now if $x \in B_i$, we let $\{p_i, q_i\} \subset \mathbb{R}$ such that $x \in [p_i, q_i) \subseteq B_i$. Let $P = \{p_i\}$ and $Q = \{q_i\}$. Since both P and Q are finite, P has a maximum p and Q has a minimum Q. Now

$$x \in \bigcap_i B_i \Rightarrow x \in [p,q) \subseteq [p_i, q_i) \subseteq B_i \forall i$$

Since $[p,q)$ is a subset of $B_i \forall i$, it is a subset of $\bigcap_i B_i$, hence any finite intersection of open sets is open. This shows that the Sorgenfrey topology is a topology.
$(0,1)$ is open, because

$$\forall x \in (0,1), x \in [x,1) \subset (0,1)$$

$(0,1]$ is not open, because $1 \in (0,1]$, but if $1 \in [a,b)$, then $\frac{1+b}{2}$ is an element of $[a,b)$ but not $(0,1]$, so $1 \in [a,b) \subseteq (0,1]$ cannot be true.
Consider $A = \mathbb{R} - [0,1]$ and let $x \in A$. Either $x < 0$, and $x \in [x, \frac{x}{2}) \subset A$, or $x > 1$, and $x \in [x, x+1)$. Therefore, A is open, so $[0,1]$ is closed.

5. Consider the topological spaces $(\mathbb{R}, \mathcal{I}), (\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$ and with the finite-complement topology (where $\mathcal{U}$ denotes the usual topology on $\mathbb{R}$ s in Chapter 3).
   (a) If $p \in \mathbb{R}$, is $\{p\}$ open in any of these spaces? Which ones?

   Solution: $(\mathbb{R}, \mathcal{D})$.

   (b) If $p \in \mathbb{R}$, is $\{p\}$ closed in any of these spaces? Which ones?

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, and the finite-complement topology.

(c) In which of these spaces is (a, b) open? $[a, b)$? $(a, b]$? $[a, b]$?

Solution:
(a, b):
$(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$
$[a, b)$:
$(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{S})$
$(a, b]$:
$(\mathbb{R}, \mathcal{D})$
$[a, b]$:
$(\mathbb{R}, \mathcal{D})$

(d) Is the set $\{x \in \mathbb{R} : x \neq \frac{1}{n}\}$ open in any of the spaces? Is it closed in any of them?

Solution:
Open:
$(\mathbb{R}, \mathcal{D})$
Closed:
$(\mathbb{R}, \mathcal{D})$

(e) Is the set $\{x \in \mathbb{R} : x \neq \frac{1}{n}$ and $x \neq 0\}$ open in any of the spaces? Is it closed in any of them?

Solution:
Open:
$(\mathbb{R}, \mathcal{D})$
Closed:
$(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$

6. Consider the spaces of Problem 5 above again, together with the three spaces that can be defined on $\{0, 1\}$.

(a) In which of these spaces are true: If x and y are two distinct points in the space then either there exists an open set U such that $x \in U$ and $y \notin U$, or there exists an open set V such that $y \in V$ and $x \notin V$. (A space for which this statement holds is called a T_0-space.)

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, the finite-complement topology, and the Sierpinski space.

(b) In which of these spaces is the following statement true: If x and Y are two distinct points in the space, then there exists an open set U such that $x \in U$ and $y \notin U$, and there exists an open set v such that $y \in V$ and $x \notin V$. (A space for which this statement holds is called a T_1-space.)

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, and the finite-complement topology.

(c) In which of these spaces is the following statement true: If x and y are two distinct points in the space, the there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. (A space for which this statement holds is called a T_2-space or a Hausdorff space.)

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$

7. Show that every T_2-space is a T_1-space, and that every T_1-space is a T_0-space, and give an example of a T_0-space that is not a T_1-space, and an example of a T_1-space that is not a T_2-space.

Solution:
T_2-space $\Rightarrow$ T_1-space: Since $U \cap V \neq \emptyset$, $x \notin V$ and $y \notin U$, so V and U are open sets that make the space a T_1-space.
T_1-space $\Rightarrow$ T_0-space: either one of U and V make the space a T_0-space.
T_0-space that is not a T_1-space: the Sierpinski space
T_1-space that is not a T_2-space: the finite-complement topology

8. A topological space X is said to be **metrizable** if a metric can be defined on X so that a set is open in the metric topology induced by this metric if and only if it is open in the topology that is already on the space.

   (a) Let X be a set with more than one point. Prove that $(X, \mathcal{I})$ is *not* metrizable. Thus the indiscrete topology on a set with more than one point is an example of a topological space that is *not* a metric space.

   Solution:
   If X has more than one point, it has 2 distinct points x and y, where we let $r = d(x, y) > 0$ by the definition of a metric. $S_{\frac{r}{2}}(x)$ is an open space according to the metric. However, it is neither empty (contains x) nor the universe (does not contain y). This forms a contradiction.

   (b) Let X be a set. Define a function from $X \times X = \{(x, y) : x, y \in X\}$ to $\mathbb{R}$ by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 \text{if } x = y \end{cases}$$

   Prove that d is a metric on X. What is the metric topology induced by d?

   Solution:
   Obviously, $d(x, y) = 0$ if and only if $x = y$, $d(x, y) \geq 0$, $d(x, y) = d(y, x)$. For the triangle inequality $d(x, y) \leq d(x, z) + d(y, z)$, note that it is trivial if $x = y$. If not, at least one of $d(x, z)$ and $d(y, z)$ must be nonzero, so the inequality holds. Therefore, $d(x, y)$ is a metric.
   $\forall x \in X, S_{0.5}(x) = \{x\}$. Since all singleton sets are open, the metric topology induced by d is the discrete topology.

1.2 Theorem 3.2

9. Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$. Then f is continuous at $x_0 \in X$ if and only if whenever V is an open subset of Y with $f(x_0) \in V$, then there exists an open subset U of X such that $x_0 \in U$ and $f(U) \subseteq V$.

Solution:
$\Rightarrow$:
Let V be open. Since V is open, $\forall v = f(x_0) \in V$, $\exists \epsilon > 0$ where $S_\epsilon(v) \subseteq V$. Since continuity is implied, $\exists \delta > 0$ where $f(S_\delta(x_0)) \subseteq S_\epsilon(v)$. Therefore $S_\delta(x_0)$ is the desired open U.
$\Leftarrow$:
Let $f(x_0) = v$. $\forall \epsilon > 0$, $V = S_\epsilon(v)$ is open. Then an open U exists where $x_0 \in U$. By definition, U is open, so $\exists \delta > 0$ such that $S_\delta(x_0) \in U$. Then

$$f(S_\delta(x_0)) \subseteq f(U) \subseteq V = S_\epsilon(v)$$

This demonstrates that f is continuous by the $\epsilon - \delta$ definition.

1.3 Theorem 3.4

10. Let X and Y be topological spaces and let $f : X \to Y$. Then f is continuous on X if and only if whenever V is an open subset of Y, then $f^{-1}(V)$ is open in X.

Solution:
$\Rightarrow$:
Based on Theorem 3.2, we have an open $U_x \forall f(x) \in V$ such that $f(U_x) \subseteq V$. Therefore, $U_x \subseteq f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

Since $f^{-1}(V)$ is a union of open sets, it is open.
$\Leftarrow$:
Similar to the second part of Theorem 3.2, let $f(x_0) = v$. $\forall \epsilon > 0$, $V = S_\epsilon(v)$ is open. Then $f^{-1}(V)$ is open (and contains x_0), so $\exists \delta > 0$ such that $S_\delta(x_0) \in f^{-1}(V)$. Then

$$f(S_\delta(x_0)) \subseteq V = S_\epsilon(v)$$

This demonstrates that f is continuous by the $\epsilon - \delta$ definition.