

Solutions to Topology by Conover

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1 Topological Spaces and Concepts in General

1.1 Exercise 2.6

1. Let X be a set. Verify that the indiscrete topology, the discrete topology and the finite-complement topology are in fact topologies on X .

Solution:

Indiscrete Topology:

\emptyset and X are open sets, and any intersection/union of open sets are obviously either empty or X .

Discrete Topology:

\emptyset and X are open sets. Since any subset is open, any intersection/union of open sets must be a subset, and hence is open.

Finite-complement Topology:

Let A_i denote open sets. Then $X - \bigcup_i A_i \subseteq X - A_i$, where $X - A_i$ has a finite cardinality by definition. Therefore, $X - \bigcup_i A_i$ also has a finite cardinality, so $\bigcup_i A_i$ is open. Now let B_i denote finitely many open sets. $X - B_i$ is finite, and so is $\bigcup_i (X - B_i)$ (finite union of finite sets is finite). Since $\bigcup_i (X - B_i) = X - \bigcap_i B_i$ which is finite, $\bigcap_i B_i$ is an open set.

2. (a) Verify that Sierpinski space is a topological space.

Solution: It contains \emptyset and X . Since there are only 3 open sets, brute forcing through all possible unions/intersections show that they are also open sets.

- (b) We said that there are only three different topologies that can be assigned to the 2 point set $\{0, 1\}$. Is the collection of $\{\emptyset, \{1\}, \{0, 1\}\}$ one of those three topologies on $\{0, 1\}$?

Solution: Yes. The same approach for (a) can be used here, since this is essentially the Sierpinski space with 0 and 1 reversed.

- (c) What is $\{0, 1\}$ with the finite-complement topology?

Solution: $\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$

3. List all topologies that can be assigned to a 3 point set.

Solution:

$\{\emptyset, \{0, 1, 2\}\}$

$\{\emptyset, \{0\}, \{0, 1, 2\}\}$

$\{\emptyset, \{1\}, \{0, 1, 2\}\}$

$\{\emptyset, \{2\}, \{0, 1, 2\}\}$

$\{\emptyset, \{0, 1\}, \{0, 1, 2\}\}$

$\{\emptyset, \{0, 2\}, \{0, 1, 2\}\}$

$\{\emptyset, \{1, 2\}, \{0, 1, 2\}\}$

$\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}\}$

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$\{\emptyset, \{1\}, \{1, 2\}, \{0, 1, 2\}\}$

$\{\emptyset, \{2\}, \{0, 2\}, \{0, 1, 2\}\}$
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 $\{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

4. Verify that the Sorgenfrey topology defined on the real line is in fact a topology. Is the interval $(0, 1)$ open in this topology? How about $(0, 1]$? Is $[0, 1]$ closed?

Solution:

The Sorgenfrey topology obviously contains \emptyset and \mathbb{R} . Let A_i denote open sets. Then

$$\forall x \in \bigcup_i A_i, x \in A_i \Rightarrow x \in [a, b) \subseteq A_i \subseteq \bigcup_i A_i$$

so any union of open sets is open. Now let B_i denote finitely many open sets. Now if $x \in B_i$, we let $\{p_i, q_i\} \subset \mathbb{R}$ such that $x \in [p_i, q_i) \subseteq B_i$. Let $P = \{p_i\}$ and $Q = \{q_i\}$. Since both P and Q are finite, P has a maximum p and Q has a minimum q . Now

$$x \in \bigcap_i B_i \Rightarrow x \in [p, q) \subseteq [p_i, q_i) \subseteq B_i \forall i$$

Since $[p, q)$ is a subset of $B_i \forall i$, it is a subset of $\bigcap_i B_i$, hence any finite intersection of open sets is open. This shows that the Sorgenfrey topology is a topology.

$(0, 1)$ is open, because

$$\forall x \in (0, 1), x \in [x, 1) \subset (0, 1)$$

$(0, 1]$ is not open, because $1 \in (0, 1]$, but if $1 \in [a, b)$, then $\frac{1+b}{2}$ is an element of $[a, b)$ but not $(0, 1]$, so $1 \in [a, b) \subseteq (0, 1]$ cannot be true.

Consider $A = \mathbb{R} - [0, 1]$ and let $x \in A$. Either $x < 0$, and $x \in [x, \frac{x}{2}) \subset A$, or $x > 1$, and $x \in [x, x+1)$. Therefore, A is open, so $[0, 1]$ is closed.

5. Consider the topological spaces $(\mathbb{R}, \mathcal{I})$, $(\mathbb{R}, \mathcal{D})$, $(\mathbb{R}, \mathcal{U})$, $(\mathbb{R}, \mathcal{S})$ and with the finite-complement topology (where \mathcal{U} denotes the usual topology on \mathbb{R} in Chapter 3).

- (a) If $p \in \mathbb{R}$, is $\{p\}$ open in any of these spaces? Which ones?

Solution: $(\mathbb{R}, \mathcal{D})$.

- (b) If $p \in \mathbb{R}$, is $\{p\}$ closed in any of these spaces? Which ones?

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, and the finite-complement topology.

- (c) In which of these spaces is (a, b) open? $[a, b)$? $(a, b]$? $[a, b]$?

Solution:

(a, b) :

$(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$

$[a, b)$:

$(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{S})$

$(a, b]$:

$(\mathbb{R}, \mathcal{D})$

$[a, b]$:

$(\mathbb{R}, \mathcal{D})$

- (d) Is the set $\{x \in \mathbb{R} : x \neq \frac{1}{n}\}$ open in any of the spaces? Is it closed in any of them?

Solution:

Open:

$(\mathbb{R}, \mathcal{D})$

Closed:

$(\mathbb{R}, \mathcal{D})$

- (e) Is the set $\{x \in \mathbb{R} : x \neq \frac{1}{n} \text{ and } x \neq 0\}$ open in any of the spaces? Is it closed in any of them?

Solution:

Open:

$(\mathbb{R}, \mathcal{D})$

Closed:

$(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$

6. Consider the spaces of Problem 5 above again, together with the three spaces that can be defined on $\{0, 1\}$.

- (a) In which of these spaces are true: If x and y are two distinct points in the space then either there exists an open set U such that $x \in U$ and $y \notin U$, or there exists an open set V such that $y \in V$ and $x \notin V$. (A space for which this statement holds is called a T_0 -space.)

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, the finite-complement topology, and the Sierpinski space.

- (b) In which of these spaces is the following statement true: If x and Y are two distinct points in the space, then there exists an open set U such that $x \in U$ and $y \notin U$, and there exists an open set v such that $y \in V$ and $x \notin V$. (A space for which this statement holds is called a T_1 -space.)

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, and the finite-complement topology.

- (c) In which of these spaces is the following statement true: If x and y are two distinct points in the space, then there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. (A space for which this statement holds is called a T_2 -space or a Hausdorff space.)

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$

7. Show that every T_2 -space is a T_1 -space, and that every T_1 -space is a T_0 -space, and give an example of a T_0 -space that is not a T_1 -space, and an example of a T_1 -space that is not a T_2 -space.

Solution:

T_2 -space $\Rightarrow T_1$ -space: Since $U \cap V \neq \emptyset$, $x \notin V$ and $y \notin U$, so V and U are open sets that make the space a T_1 -space.

T_1 -space $\Rightarrow T_0$ -space: either one of U and V make the space a T_0 -space.

T_0 -space that is not a T_1 -space: the Sierpinski space

T_1 -space that is not a T_2 -space: the finite-complement topology

8. A topological space X is said to be **metrizable** if a metric can be defined on X so that a set is open in the metric topology induced by this metric if and only if it is open in the topology that is already on the space.

- (a) Let X be a set with more than one point. Prove that (X, \mathcal{I}) is *not* metrizable. Thus the indiscrete topology on a set with more than one point is an example of a topological space that is *not* a metric space.

Solution:

If X has more than one point, it has 2 distinct points x and y , where we let $r = d(x, y) > 0$ by the definition of a metric. $S_{\frac{r}{2}}(x)$ is an open space according to the metric. However, it is neither empty (contains x) nor the universe (does not contain y). This forms a contradiction.

- (b) Let X be a set. Define a function from $X \times X = \{(x, y) : x, y \in X\}$ to \mathbb{R} by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Prove that d is a metric on X . What is the metric topology induced by d ?

Solution:

Obviously, $d(x, y) = 0$ if and only if $x = y$, $d(x, y) \geq 0$, $d(x, y) = d(y, x)$. For the triangle inequality $d(x, y) \leq d(x, z) + d(y, z)$, note that it is trivial if $x = y$. If not, at least one of $d(x, z)$ and $d(y, z)$ must be nonzero, so the inequality holds. Therefore, $d(x, y)$ is a metric.

$\forall x \in X, S_{0.5}(x) = \{x\}$. Since all singleton sets are open, the metric topology induced by d is the discrete topology.

1.2 Theorem 3.2

1. Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \rightarrow Y$. Then f is continuous at $x_0 \in X$ if and only if whenever V is an open subset of Y with $f(x_0) \in V$, then there exists an open subset U of X such that $x_0 \in U$ and $f(U) \subseteq V$.

Solution: \Rightarrow :

Let V be open. Since V is open, $\forall v = f(x_0) \in V$, $\exists \epsilon > 0$ where $S_\epsilon(v) \subseteq V$. Since continuity is implied, $\exists \delta > 0$ where $f(S_\delta(x_0)) \subseteq S_\epsilon(v)$. Therefore $S_\delta(x_0)$ is the desired open U .

 \Leftarrow :

Let $f(x_0) = v$. $\forall \epsilon > 0$, $V = S_\epsilon(v)$ is open. Then an open U exists where $x_0 \in U$. By definition, U is open, so $\exists \delta > 0$ such that $S_\delta(x_0) \in U$. Then

$$f(S_\delta(x_0)) \subseteq f(U) \subseteq V = S_\epsilon(v)$$

This demonstrates that f is continuous by the $\epsilon - \delta$ definition.

1.3 Theorem 3.4

- Let X and Y be topological spaces and let $f : X \rightarrow Y$. Then f is continuous on X if and only if whenever V is an open subset of Y , then $f^{-1}(V)$ is open in X .

Solution: \Rightarrow :

Based on Theorem 3.2, we have an open $U_x \forall f(x) \in V$ such that $f(U_x) \subseteq V$. Therefore, $U_x \subseteq f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

Since $f^{-1}(V)$ is a union of open sets, it is open.

 \Leftarrow :

Similar to the second part of Theorem 3.2, let $f(x_0) = v$. $\forall \epsilon > 0$, $V = S_\epsilon(v)$ is open. Then $f^{-1}(V)$ is open (and contains x_0), so $\exists \delta > 0$ such that $S_\delta(x_0) \in f^{-1}(V)$. Then

$$f(S_\delta(x_0)) \subseteq V = S_\epsilon(v)$$

This demonstrates that f is continuous by the $\epsilon - \delta$ definition.

1.4 Exercise 3.5

- Consider $(\mathbb{R}, \mathcal{I})$, $(\mathbb{R}, \mathcal{D})$, $(\mathbb{R}, \mathcal{U})$, $(\mathbb{R}, \mathcal{S})$ and $(\mathbb{R}, \mathcal{F})$, the real line with the indiscrete topology, the discrete topology, the usual metric topology, the Sorgenfrey topology and the finite-complement topology, respectively. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the identity function defined by $f(r) = r$ for all real numbers r . Determine all possible choices for \mathcal{J}_1 and \mathcal{J}_2 from $\mathcal{I}, \mathcal{D}, \mathcal{U}, \mathcal{S}, \mathcal{F}$ so that $f : (\mathbb{R}, \mathcal{J}_1) \rightarrow (\mathbb{R}, \mathcal{J}_2)$ is continuous.

Solution:

	\mathcal{I}	\mathcal{D}	\mathcal{U}	\mathcal{S}	\mathcal{F}
\mathcal{I}	✓	✗	✗	✗	✗
\mathcal{D}	✓	✓	✓	✓	✓
\mathcal{U}	✓	✗	✓	✗	✓
\mathcal{S}	✓	✗	✓	✓	✓
\mathcal{F}	✓	✗	✗	✗	✓

Where the rows denote \mathcal{J}_1 and the columns denote \mathcal{J}_2 .

2. Let x be a set and let \mathcal{J} be any topology on X . There is a topology \mathcal{J}' that can be assigned to X so that the identity function from (X, \mathcal{J}') to (X, \mathcal{J}) is always continuous no matter what \mathcal{J} is. What is it?

Solution: The discrete topology \mathcal{D} .

3. Let X be any set and let \mathcal{J} be any topology on X . There is a topology \mathcal{J}' that can be given to X so that the identity function from (X, \mathcal{J}) to (X, \mathcal{J}') is always continuous no matter what \mathcal{J} is. What is it?

Solution: The indiscrete topology \mathcal{I} .

4. A function that preserves open sets is called an **open function**. More precisely, a function $f : X \rightarrow Y$ is an open function if whenever U is open in X , then $f(U)$ is open in Y .

- (a) Give an example of a continuous function that is not open.

Solution: $f(r) = r$ where X is the real line with the discrete topology and Y is the real line with the indiscrete topology.

- (b) Give an example of an open function that is not continuous.

Solution: $f(r) = r$ where X is the real line with the indiscrete topology and Y is the real line with the discrete topology.

1.5 Exercise 4.2

1. Consider the spaces $(\mathbb{R}, \mathcal{I})$, $(\mathbb{R}, \mathcal{D})$, $(\mathbb{R}, \mathcal{U})$, $(\mathbb{R}, \mathcal{S})$ and $(\mathbb{R}, \mathcal{F})$. In which of these spaces is:

- (a) $(0, 2)$ a neighbourhood of 1?
- (b) $[0, 2]$ a neighbourhood of 1?
- (c) $[0, 2]$ a neighbourhood of 0?
- (d) $\{0\}$ a neighbourhood of 0?

Solution:

- (a): $\mathcal{D}, \mathcal{U}, \mathcal{S}$
- (b): $\mathcal{D}, \mathcal{U}, \mathcal{S}$
- (c): \mathcal{D}, \mathcal{S}
- (d): \mathcal{D}

2. In the plane with its usual topology, is the unit square

$$\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

a neighbourhood of any point in it? Which points?

Solution:

Any point that is not the boundary, or more explicitly,

$$\{(x, y) : 0 < x < 1, 0 < y < 1\}$$

1.6 Theorem 4.3

1. Let X be a topological space. Then $U \subseteq X$ is open if and only if U is a neighbourhood of each point $x \in U$.

Solution:

\Rightarrow :

Since U is open, $x \in U$ implies $x \in O \subseteq U$ for some open set O . Since U is a superset of an open O that contains x , U is a neighbourhood of x by definition.

\Leftarrow :

Since U is a neighbourhood, $x \in U$ implies $x \in O \subseteq U$ where O is some open set. This means U is open by definition.