Solutions to Topology by Conover

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1 Some Familiar Topological Spaces and Basic Topological Concepts

1.1 Exercises 1.2

1. Let $A = (0, 1) \cup (1, 3)$. For the given $x \in A$, give a value of $r > 0$ such that $(x - r, x + r) \subseteq A$.
   (a) $x = \frac{3}{4}$
   (b) $x = 2$
   (c) $x = \frac{9}{8}$

   Solution: $\frac{1}{16}$

2. Prove that A $= (0, 1) \cup (1, 3)$ is an open subset of $\mathbb{R}$.

   Solution:
   Case 1: $x \in (0, 1)$
   Let $d = \min\{1 - x, x\}$. $(x - \frac{d}{2}, x + \frac{d}{2}) \subseteq A$ is an open set.
   Case 2: $x \in (1, 3)$
   Let $d = \min\{3 - x, x - 1\}$. The proof proceeds similarly as Case 1.
   As $\forall x \in A \exists$ an open interval $\subseteq A$ which contains x, A is an open set by definition.

3. Prove than an ordinary open interval is an open subset of $\mathbb{R}$ but that an open set need *not* be an open interval.

   Solution:
   All open intervals are open subsets:
   Replace 0 with $x - r$ and 1 with $x + r$ as in Case 1 from the question above.
   Open sets need not be an open interval:
   The empty set is an open set, but no open interval is empty, as it contains x (open intervals are in the form $(x - r, x + r)$).

4. State precisely what it means when a subset A of $\mathbb{R}$ is *not* open.

Solution: $\exists x \in A$ such that $(x-r, x+r) - A \neq \emptyset \forall r > 0$

5. Prove that the following subsets of $\mathbb{R}$ are not open.
   (a) The set of rational numbers

   Solution:
   An open interval is nonempty. As it is an open set, it cannot contain a single point only (next question) so it must contain at least 2 points. WLOG, let the 2 points be $p < q$. Using limits, $\exists n \in \mathbb{N}$ such that $r = p + \frac{\sqrt{2}}{n} < q$. Since all open intervals contain an irrational r, no open intervals can be a subset of the rationals, hence it is not open.

   (b) A set consisting of a single point

   Solution:
   $\forall$ open intervals $(x-r, x+r)$, it contains the distinct points x and $x+\frac{r}{2}$. Therefore, $\{x\}$ cannot be open.

   (c) An interval of the form $[a, b)$, where $a < b$

   Solution:
   $\forall$ open intervals $(a-r, a+r)$, $b = a - \frac{r}{2}$ is a point in that interval which is outside of $[a, b)$.

   (d) The set $A = \{x \in \mathbb{R} : x \neq \frac{1}{n},\ \text{for } n \in \mathbb{Z}^+\}$

   Solution:
   $\frac{1}{n}$ tends to 0. Therefore $\forall r > 0 \exists n \in \mathbb{N}$ such that $r > \frac{1}{n}$. Therefore all open intervals $(0-r, 0+r)$ contains a point outside of A.

1.2 Theorem 1.3

6. (a) The union of any collection of open subsets of the real line is also an open subset of the line

   Solution:
   Let $\bigcup A$ denote the union of subsets A. Then

   $$x \in \bigcup A \Rightarrow x \in A \Rightarrow x \in (x-r, x+r) \subseteq A \subseteq \bigcup A$$

   for some $r > 0$.
   As all points in $\bigcup A$ are in open intervals which are subsets of $\bigcup A$, the union is open by definition.

   (b) The intersection of any finite collection of open subsets of the real line is also an open subset of the line.

   Solution:
   This can be proven with induction.
   Let A and B be open sets. $x \in A \cap B \Rightarrow x \in A$. Since A is open, $\exists r_A > 0$ such that

   $$x \in (x - r_A, x + r_A) \subseteq A$$

The same holds for B. Letting $r = \min\{r_A, r_B\}$,

$$x \in (x-r, x+r) \subseteq A \cap B \forall x \in A \cap B$$

The same proof is used for the base case and the induction step.

(c) Both the empty set and $\mathbb{R}$ itself are open subsets of the real line.

Solution:
Empty set:

$$a \Rightarrow b$$

is defined to be true when a is false. The definition of an open set A involves the assumption $x \in A$, which is false, so it is vacuously true for the empty set.
$\mathbb{R}$:
$\forall x \in \mathbb{R}, (x-1, x+1) \in \mathbb{R}$

1.3 Exercise 1.4

7. Give an example of an infinite collection of open subsets of the real line whose intersection is not open, thus showing that the finiteness condition in Theorem 1.3(b) is necessary.

Solution:

$$\bigcap A \text{ where } A = \{(-r, r) | r > 0\}$$

Obviously $0 \in (-r, r) \forall r > 0$. However, the intersection does not contain any nonzero element, because $\forall x \neq 0$,

$$x \notin \{-\frac{|x|}{2}, \frac{|x|}{2}\}$$

The open intervals that form the intersection are open sets, but the intersection contains only 1 element, so it is not open.

1.4 Theorem 1.6

8. (a) The intersection of any collection of closed sets is closed.

Solution:
Let C_i be a closed set, and the corresponding open set be defined as $O_i = \mathbb{R} - C_i$.

$$\bigcap_{i \in I} C_i = \bigcap_{i \in I} \mathbb{R} - O_i = R - \bigcup_{i \in I} O_i$$

$\bigcup_{i \in I} O_i$ is a union of open sets, so it is open. Hence its complement (intersection of closed sets) is closed.

(b) The union of any finite collection of closed sets is closed.

Solution:

$$\bigcup_{n\in\mathbb{N}} C_n = \bigcup_{n\in\mathbb{N}} \mathbb{R} - O_n = R - \bigcap_{n\in\mathbb{N}} O_n$$

And the proof follows similar to the case above.

(c) $\emptyset$ and $\mathbb{R}$ itself are both closed

Solution:
Their complements are each other, which are open.