Solutions to Topology by Conover

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1 Topological Spaces and Concepts in General

1.1 Exercise 2.6

1. Let X be a set. Verify that the indiscrete topology, the discrete topology and the finite-complement topology are in fact topologies on X.

Solution:
Indiscrete Topology:
$\emptyset$ and X are open sets, and any intersection/union of open sets are obviously either empty of X.
Discrete Topology:
$\emptyset$ and X are open sets. Since any subset is open, any intersection/union of open sets must be a subset, and hence is open.
Finite-complement Topology:
set and X are open sets. Let A_i denote open sets. Then $X - \bigcup_i A_i \subseteq X - A_i$, where $X - A_i$ has a finite cardinality by definition. Therefore, $X - \bigcup_i A_i$ also has a finite cardinality, so $\bigcup_i A_i$ is open. Now let B_i denote finitely many open sets. $X - B_i$ is finite, and so is $\bigcup_i (X - B_i)$ (finite union of finite sets is finite). Since $\bigcup_i (X - B_i) = X - \bigcap_i B_i$ which is finite, $\bigcap_i B_i$ is an open set.

2. (a) Verify that Sierpinski space is a topological space.

   Solution: It contains $\emptyset$ and X. Since there are only 3 open sets, brute forcing through all possible unions/intersections show that they are also open sets.

   (b) We said that there are only three different topologies that can be assigned to the 2 point set $\{0, 1\}$. Is the collection of $\{\emptyset, \{1\}, \{0, 1\}\}$ one of those three topologies on $\{0, 1\}$?

   Solution: Yes. The same approach for (a) can be used here, since this is essentially the Sierpinski space with 0 and 1 reversed.

   (c) What is $\{0, 1\}$ with the finite-complement topology?

   Solution: $\{\emptyset, \{0\}, \{1, \}, \{0, 1\}\}$

3. List all topologies that can be assigned to a 3 point set.

Solution:
$\{\emptyset, \{0, 1, 2\}\}$
$\{\emptyset, \{0\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0, 1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0, 2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1, 2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{0\}, \{0, 2\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1\}, \{0, 1\}, \{0, 1, 2\}\}$
$\{\emptyset, \{1\}, \{1, 2\}, \{0, 1, 2\}\}$

$\{\emptyset, \{2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset, \{1\}, \{0,1\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$
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$\{\emptyset, \{0\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{1\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

4. Verify that the Sorgenfrey topology defined on the real line is in fact a topology. Is the interval $(0,1)$ open in this topology? How about $(0,1]$? Is $[0,1]$ closed?

Solution:
The Sorgenfrey topology obviously contains $\emptyset$ and $\mathbb{R}$. Let A_i denote open sets. Then

$$\forall x \in \bigcup_i A_i, x \in A_i \Rightarrow x \in [a,b) \subseteq A_i \subseteq \bigcup_i A_i$$

so any union of open sets is open. Now let B_i denote finitely many open sets. Now if $x \in B_i$, we let $\{p_i, q_i\} \subset \mathbb{R}$ such that $x \in [p_i, q_i) \subseteq B_i$. Let $P = \{p_i\}$ and $Q = \{q_i\}$. Since both P and Q are finite, P has a maximum p and Q has a minimum Q. Now

$$x \in \bigcap_i B_i \Rightarrow x \in [p,q) \subseteq [p_i, q_i) \subseteq B_i \forall i$$

Since $[p,q)$ is a subset of $B_i \forall i$, it is a subset of $\bigcap_i B_i$, hence any finite intersection of open sets is open. This shows that the Sorgenfrey topology is a topology.
$(0,1)$ is open, because

$$\forall x \in (0,1), x \in [x,1) \subset (0,1)$$

$(0,1]$ is not open, because $1 \in (0,1]$, but if $1 \in [a,b)$, then $\frac{1+b}{2}$ is an element of $[a,b)$ but not $(0,1]$, so $1 \in [a,b) \subseteq (0,1]$ cannot be true.
Consider $A = \mathbb{R} - [0,1]$ and let $x \in A$. Either $x < 0$, and $x \in [x, \frac{x}{2}) \subset A$, or $x > 1$, and $x \in [x, x+1)$. Therefore, A is open, so $[0,1]$ is closed.

5. Consider the topological spaces $(\mathbb{R}, \mathcal{I}), (\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$ and with the finite-complement topology (where $\mathcal{U}$ denotes the usual topology on $\mathbb{R}$ s in Chapter 3).
   (a) If $p \in \mathbb{R}$, is $\{p\}$ open in any of these spaces? Which ones?

   Solution: $(\mathbb{R}, \mathcal{D})$.

   (b) If $p \in \mathbb{R}$, is $\{p\}$ closed in any of these spaces? Which ones?

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, and the finite-complement topology.

(c) In which of these spaces is (a, b) open? $[a, b)$? $(a, b]$? $[a, b]$?

Solution:
(a, b):
$(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$
$[a, b)$:
$(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{S})$
$(a, b]$:
$(\mathbb{R}, \mathcal{D})$
$[a, b]$:
$(\mathbb{R}, \mathcal{D})$

(d) Is the set $\{x \in \mathbb{R} : x \neq \frac{1}{n}\}$ open in any of the spaces? Is it closed in any of them?

Solution:
Open:
$(\mathbb{R}, \mathcal{D})$
Closed:
$(\mathbb{R}, \mathcal{D})$

(e) Is the set $\{x \in \mathbb{R} : x \neq \frac{1}{n} \text{ and } x \neq 0\}$ open in any of the spaces? Is it closed in any of them?

Solution:
Open:
$(\mathbb{R}, \mathcal{D})$
Closed:
$(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$

6. Consider the spaces of Problem 5 above again, together with the three spaces that can be defined on $\{0, 1\}$.

(a) In which of these spaces are true: If x and y are two distinct points in the space then either there exists an open set U such that $x \in U$ and $y \notin U$, or there exists an open set V such that $y \in V$ and $x \notin V$. (A space for which this statement holds is called a T_0-space.)

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, the finite-complement topology, and the Sierpinski space.

(b) In which of these spaces is the following statement true: If x and Y are two distinct points in the space, then there exists an open set U such that $x \in U$ and $y \notin U$, and there exists an open set v such that $y \in V$ and $x \notin V$. (A space for which this statement holds is called a T_1-space.)

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$, and the finite-complement topology.

(c) In which of these spaces is the following statement true: If x and y are two distinct points in the space, the there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. (A space for which this statement holds is called a T_2-space or a Hausdorff space.)

Solution: $(\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$

7. Show that every T_2-space is a T_1-space, and that every T_1-space is a T_0-space, and give an example of a T_0-space that is not a T_1-space, and an example of a T_1-space that is not a T_2-space.

Solution:
T_2-space $\Rightarrow$ T_1-space: Since $U \cap V \neq \emptyset$, $x \notin V$ and $y \notin U$, so V and U are open sets that make the space a T_1-space.
T_1-space $\Rightarrow$ T_0-space: either one of U and V make the space a T_0-space.
T_0-space that is not a T_1-space: the Sierpinski space
T_1-space that is not a T_2-space: the finite-complement topology

8. A topological space X is said to be **metrizable** if a metric can be defined on X so that a set is open in the metric topology induced by this metric if and only if it is open in the topology that is already on the space.

   (a) Let X be a set with more than one point. Prove that $(X, \mathcal{I})$ is *not* metrizable. Thus the indiscrete topology on a set with more than one point is an example of a topological space that is *not* a metric space.

   Solution:
   If X has more than one point, it has 2 distinct points x and y, where we let $r = d(x, y) > 0$ by the definition of a metric. $S_{\frac{r}{2}}(x)$ is an open space according to the metric. However, it is neither empty (contains x) nor the universe (does not contain y). This forms a contradiction.

   (b) Let X be a set. Define a function from $X \times X = \{(x, y) : x, y \in X\}$ to $\mathbb{R}$ by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 \text{if } x = y \end{cases}$$

   Prove that d is a metric on X. What is the metric topology induced by d?

   Solution:
   Obviously, $d(x, y) = 0$ if and only if $x = y$, $d(x, y) \geq 0$, $d(x, y) = d(y, x)$. For the triangle inequality $d(x, y) \leq d(x, z) + d(y, z)$, note that it is trivial if $x = y$. If not, at least one of $d(x, z)$ and $d(y, z)$ must be nonzero, so the inequality holds. Therefore, $d(x, y)$ is a metric.
   $\forall x \in X, S_{0.5}(x) = \{x\}$. Since all singleton sets are open, the metric topology induced by d is the discrete topology.

1.2 Theorem 3.2

1. Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$. Then f is continuous at $x_0 \in X$ if and only if whenever V is an open subset of Y with $f(x_0) \in V$, then there exists an open subset U of X such that $x_0 \in U$ and $f(U) \subseteq V$.

Solution:
$\Rightarrow$:
Let V be open. Since V is open, $\forall v = f(x_0) \in V$, $\exists \epsilon > 0$ where $S_\epsilon(v) \subseteq V$. Since continuity is implied, $\exists \delta > 0$ where $f(S_\delta(x_0)) \subseteq S_\epsilon(v)$. Therefore $S_\delta(x_0)$ is the desired open U.
$\Leftarrow$:
Let $f(x_0) = v$. $\forall \epsilon > 0$, $V = S_\epsilon(v)$ is open. Then an open U exists where $x_0 \in U$. By definition, U is open, so $\exists \delta > 0$ such that $S_\delta(x_0) \in U$. Then

$$f(S_\delta(x_0)) \subseteq f(U) \subseteq V = S_\epsilon(v)$$

This demonstrates that f is continuous by the $\epsilon - \delta$ definition.

1.3 Theorem 3.4

1. Let X and Y be topological spaces and let $f : X \to Y$. Then f is continuous on X if and only if whenever V is an open subset of Y, then $f^{-1}(V)$ is open in X.

Solution:
$\Rightarrow$:
Based on Theorem 3.2, we have an open $U_x \forall f(x) \in V$ such that $f(U_x) \subseteq V$. Therefore, $U_x \subseteq f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

Since $f^{-1}(V)$ is a union of open sets, it is open.
$\Leftarrow$:
Similar to the second part of Theorem 3.2, let $f(x_0) = v$. $\forall \epsilon > 0$, $V = S_\epsilon(v)$ is open. Then $f^{-1}(V)$ is open (and contains x_0), so $\exists \delta > 0$ such that $S_\delta(x_0) \in f^{-1}(V)$. Then

$$f(S_\delta(x_0)) \subseteq V = S_\epsilon(v)$$

This demonstrates that f is continuous by the $\epsilon - \delta$ definition.

1.4 Exercise 3.5

1. Consider $(\mathbb{R}, \mathcal{I}), (\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$ and $(\mathbb{R}, \mathcal{F})$, the real line with the indiscrete topology, the discrete topology, the usual metric topology, the Sorgenfrey topology and the finite-complement topology, respectively. Let $f : \mathbb{R} \to \mathbb{R}$ be the identity function defined by $f(r) = r$ for all real numbers r. Determine all possible choices for $\mathcal{J}_1$ and $\mathcal{J}_2$ from $\mathcal{I}, \mathcal{D}, \mathcal{U}, \mathcal{S}, \mathcal{F}$ so that $f : (\mathbb{R}, \mathcal{J}_1) \to (\mathbb{R}, \mathcal{J}_2)$ is continuous.

Solution:

	$\mathcal{I}$	$\mathcal{D}$	$\mathcal{U}$	$\mathcal{S}$	$\mathcal{F}$
$\mathcal{I}$	✓	✗	✗	✗	✗
$\mathcal{D}$	✓	✓	✓	✓	✓
$\mathcal{U}$	✓	✗	✓	✗	✓
$\mathcal{S}$	✓	✗	✓	✓	✓
$\mathcal{F}$	✓	✗	✗	✗	✓

Where the rows denote $\mathcal{J}_1$ and the columns denote $\mathcal{J}_2$.

2. Let x be a set and let $\mathcal{J}$ be any topology on X. There is a topology $\mathcal{J}$' that can be assigned to X so that the identity function from $(X, \mathcal{J}')$ to $(X\mathcal{J})$ is always continuous no matter what $\mathcal{J}$ s. What is it?

Solution: The discrete topology $\mathcal{D}$.

3. Let X be any set and let $\mathcal{J}$ be any topology on X. There is a topology $\mathcal{J}'$ that can be given to X so that the identity function from $(X, \mathcal{J})$ to $(X, \mathcal{J}')$ is always continuous no matter what $\mathcal{J}$ is. What is it?

Solution: The indiscrete topology $\mathcal{I}$.

4. A function that preserves open sets is called an **open function**. More precisely, a function $f : X \to Y$ is an open function if whenever U is open in X, then $f(U)$ is open in Y.

   (a) Give an example of a continuous function that is not open.

   Solution: $f(r) = r$ where X is the real line with the discrete topology and Y is the real line with the indiscrete topology.

   (b) Give an example of an open function that is not continuous.

   Solution: $f(r) = r$ where X is the real line with the indiscrete topology and Y is the real line with the discrete topology.

1.5 Exercise 4.2

1. Consider the spaces $(\mathbb{R}, \mathcal{I}), (\mathbb{R}, \mathcal{D}), (\mathbb{R}, \mathcal{U}), (\mathbb{R}, \mathcal{S})$ and $(\mathbb{R}, \mathcal{F})$. In which of these spaces is:

   (a) $(0, 2)$ a neighbourhood of 1?

   (b) $[0, 2]$ a neighbourhood of 1?

   (c) $[0, 2]$ a neighbourhood of 0?

   (d) $\{0\}$ a neighbourhood of 0?

   Solution:
   (a): $\mathcal{D}$,$\mathcal{U}, \mathcal{S}$
   (b): $\mathcal{D}$,$\mathcal{U}, \mathcal{S}$
   (c): $\mathcal{D}, \mathcal{S}$
   (d): $\mathcal{D}$

2. In the plane with its usual topology, is the unit square

$$\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

   a neighbourhood of any point in it? Which points?

Solution:
Any point that is not the boundary, or more explicitly,

$$\{(x, y) : 0 < x < 1, 0 < y < 1\}$$

1.6 Theorem 4.3

1. Let X be a topological space. Then $U \subseteq X$ is open if and only if U is a neighbourhood of each point $x \in U$.

Solution:
$\Rightarrow$:
Since U is open, $x \in U$ implies $x \in O \subseteq U$ for some open set O. Since U is a superset of an open O that contains x, U is a neighbourhood of x by definition.
$\Leftarrow$:
Since U is a neighbourhood, $x \in U$ implies $x \in O \subseteq U$ where O is some open set. This means U is open by definition.

1.7 Exercise 4.5

1. Let X be the real line with the indiscrete topology and consider the sequence $\{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Prove that if r is any point of X, then this sequence converges to r.

Solution:
Let O be an open set containing r. Since this is the indiscrete topology, $O = X$ is the only possibility. Then $\forall n > 0$, $f(n) \in O$, so the sequence converges to r.

2. Let $X = \mathbb{R}$ with the finite-complement topology. Prove that $\{\frac{1}{n} : n \in \mathbb{Z}^+\}$ converges to every point in the space.

Solution:
Let O be an open set containing r. Since this is the finite-complement topology, $X - O$ is finite. Let $A = \{a \in X - O | a > 0\}$. If A is empty, let $N = 0$. Or else, A must have a minimum m that is a positive number. Since $\frac{1}{n}$ tends to 0, $\exists N \in \mathbb{N}$ such that $\frac{1}{n} < m \forall n > N$. Now $\forall n > N$, $f(n) \in O$, so the sequence tends to r.

3. Recall that a topological space X is a Hausdorff space if distinct points of X are contained in disjoint open sets.
   (a) Prove that a space is a Hausdorff space if and only if distinct points are contained in disjoint neighbourhoods.

> **Solution:**
> $\Rightarrow$:
> Open sets are also neighbourhoods, so distinct points in the space are contained in disjoint neighbourhoods.
> $\Leftarrow$:
> Let the distinct points be x and y, and the disjoint neighbourhoods N_x and N_y. Since N_x is a neighbourhoods of x, $x \in O_x \subseteq N_x$, and the same holds for y. Then x and y are in disjoint open sets O_x and O_y, so the space is a Hausdorff space.

(b) Prove that every metric space is a Hausdorff space.

> **Solution:**
> Let x and y be two distinct points, and let $R = d(x, y), r = \frac{R}{2}$. Consider a point $p \in S_r(x)$. Rearranging the triangle inequality,
>
> $$d(p, y) \geq R - d(p, x) > R - r = r$$
>
> so p cannot be in $S_r(y)$. The same holds if $p \in S_r(y)$. This means for all distinct x, y, there exists disjoint open sets $S_r(x)$ and $S_r(y)$ which contain x and y respectively, making it a Hausdorff space.

(c) Prove that in a Hausdorff space, if a sequence converges, then it converges to exactly one point. Deduced that the real line with the finite complement topology is not a Hausdorff space.

> **Solution:**
> Proof by contradiction. Let the sequence converge to distinct points x and y. Let O_x and O_y be a pair of disjoint open sets that contain x and y respectively. Since the sequence converges to x, $\exists N_x \in N$ such that $n > N \Rightarrow f(n) \in O_x$. There exists a similar N_y. Letting $N = \max\{N_x, N_y\}$, then $\forall n > N, x_n$ is in both O_x and O_y at the same time, which is a contradiction as both sets are disjoint.

4. Consider $X = [0, \Omega]$ with the order topology as defined in Section 4 of Chapter 3. We proved that in a metric space, a set S is closed if and only if whenever a sequence of points of S converges to a point x in the space, then $x \in S$.

(a) Prove, usig the definition of closed set, that $S = [0, \Omega)$ is not a closed subset of $X = [0, \Omega]$ with the order topology.

> **Solution:**
> If it is a closed set, then $\{\Omega\}$ must be open in the order topology, so
>
> $$\exists a \in X : \Omega \in (a, \Omega] \subseteq \{\Omega\}$$
>
> For the interval to make sense, $a < \Omega$. Noting that Ω is a limit ordinal,
>
> $$a < \Omega \Rightarrow a + 1 < \Omega$$
>
> so $a + 1 \neq \Omega$ is an element of $(a, \Omega]$, which is a contradiction, since its superset $\{\Omega\}$ does not contain $a + 1$.

(b) Prove that if a sequence of points of $S = [0, \Omega)$ converges to a point $x \in [0, \Omega]$, then $x \in S$.

Solution:
The sequence x_n cannot tend to Ω if it has a finite length, since it then as a maximum m, and $x_n \notin (m, \Omega] \forall n \in \mathbb{N}$. and If not, consider

$$l = \bigcap_{n \in \mathbb{N}} (x_n, \Omega]$$

Obviously $\Omega \in l$, but if $y \neq \Omega \in l$, then

$$y > x_n \forall n \in N \Rightarrow x_n \notin [y, \Omega] \forall n \in N$$

Loosely speaking, x_n can never reach y, so it can never "enter" the open set $[y, \Omega]$. However, such a y always exists by this.

(c) Deduce that $X = [0, \Omega]$ with the order topology is not a metric space, but

Solution:
If it is a metric space, then consider

$$A = \bigcap_{n \in \mathbb{N}} S_{\frac{1}{n}}(\Omega)$$

By this, $\exists a \in A : a \neq \Omega$. Let $d(a, \Omega) = r$. Since $\frac{1}{n}$ tends to 0, we know that $r > \frac{1}{i}$ for some $i \in \mathbb{N}$. Then $a \notin S_{\frac{1}{i}}(\Omega)$, so $a \notin A$, which is a contradiction.

(d) Prove that $X = [0, \Omega]$ with the order topology is a Hausdorff space.

Solution: Let $a < b$ be two distinct points. Then $[0, a]$ and $(a, \Omega]$ are disjoint open sets that contain a and b respectively.

1.8 Theorem 4.6

1. Let X and Y be topological spaces and let $f : X \to Y$. Then the funtion f is continuous at a point $x_0 \in X$ if and only if for every neighbourhood N_2 of $f(x_0)$ in Y, there is a neighbourhood N_1 of x_0 in X such that $f(N_1) \subseteq N_2$.

Solution:
$\Rightarrow$:
If N_2 is a neighbourhood, then $\exists$ an open $O_2 \subseteq N_2$ which contains $f(x_0)$. Since f is continuous, $f^{-1}(O_2)$ is open, and acts as the desired N_1.
$\Leftarrow$:
Let O_2 be an open set containing $f(x_0)$. Then there exists a N_1, and $x_0 \in O_1 \subseteq N_1$ for some open O_1 by the definition of a neighbourhood. Since

$$f(O_1) \subseteq O_2$$

f is continuous by definition.

2 Closed Sets and Closure

2.1 Theorem 5.2

1. Let X be a topological space. Then

   1. X and $\emptyset$ are closed.

   2. The intersection of an arbitrary collection of closed sets is closed.

   3. The union of a finite collection of closed sets is closed.

Solution:
Since the complements of X and $\emptyset$ are each other (which are open), they are also closed.
Let A be a set where $a \in A$ is closed. Then

$$X - \bigcap A = \bigcup_{a \in A} X - a$$

which is open. So the intersection of closed sets is closed. Now let B be a finite set where $b \in B$ is closed. Similarly,

$$X - \bigcup B = \bigcap_{b \in B} X - b$$

which is open, so a finite union of closed subsets is closed.

2.2 Exercise 5.3

1. Consider again the spaces $(\mathbb{R}, \mathbb{I}), (\mathbb{R}, \mathbb{D}), (\mathbb{R}, \mathbb{U}), (\mathbb{R}, \mathbb{S}), (\mathbb{R}, \mathbb{F})$. In which of these spaces is:
   (a) $[0, 1]$ closed?
   (b) $(0, 1)$ closed?
   (c) $[0, 1)$ closed?
   (d) $(0, 1]$ closed?
   (e) $\{0\}$ closed?

Solution:
(a): $\mathbb{D}, \mathbb{U}, \mathbb{S}$
(b): $\mathbb{D}$
(c): $\mathbb{D}, \mathbb{S}$
(d): $\mathbb{D}$
(e): $\mathbb{D}, \mathbb{U}, \mathbb{S}, \mathbb{F}$

2. Recall that a space is a T_1-space if for every pair of distinct points x and y in the space, there is an open set U such that $x \in V$ and $y \notin U$, and there is an open set V such that $y \in V$ and $x \notin V$.

(a) Restate the definition of a T_1-space in terms of neighbourhoods.

Solution: A space is a T_1-space if and only if for every distinct x, y pair, there exists a neighbourhood of x that does not contain y, and there exists a neighbourhood of y that does not contain x.

(b) Prove that a space is a T_1-space if and only if for each point p in the space, the singleton set $\{p\}$ is a closed set.

Solution:
$\Rightarrow$:
Let the space be X. $\forall q \neq p$, let O_q be an open set that contains q but not p. Such a set always exists because the space is T_1. Therefore $\bigcup_{q \neq p \in X} O_q = X - \{p\}$ is open, so the singleton set $\{p\}$ is closed.
$\Leftarrow$:
Let the space be X, and x, y be a pair of distinct points. Then $X - \{x\}$ and $X - \{y\}$ are open sets which contain exactly one of x, y respectively.

2.3 Theorem 5.4

1. Let X and Y be topological spaces and let $f : X \to Y$. Then f is continuous (on X) if whenever F is a closed set in Y, then $f^{-1}(F)$ is closed in X.

Solution:
Since open sets and closed sets are complements, f^{-1} preserving closed sets it equivalent to it preserving open sets.
$\Rightarrow$:
Let F be an closed set in Y. Let $G = Y - F$, where G is open. Since f is continuous, $f^{-1}(G)$ is also open. Then $f^{-1}(F) = X - f^{-1}(G)$ is closed.
$\Leftarrow$:
Let G be an open set in Y. Let $F = Y - G$, where F is closed. Since $f^{-1}(F)$ is closed, $f^{-1}(G) = X - f^{-1}(F)$ is open.

2.4 Exercise 5.5

A function $f : X \to Y$ is a closed function if whenever F is closed in X then $f(F)$ is closed in Y.

1. Give an example of a closed function that is not continuous.

Solution: $f(r) = r$ from $(\mathbb{R}, \mathbb{U})$ to $(\mathbb{R}, \mathbb{D})$ is a closed function that is not continuous.

2. Give an example of a continuous function that is not closed.

Solution: $f(r) = r$ from $(\mathbb{R}, \mathbb{U})$ to $(\mathbb{R}, \mathbb{I})$ is a continuous function that is not closed.

2.5 Theorem 5.7

The closure of a set A is defined as

$$\overline{A} = \bigcap\{F : F \text{ is closed in } X \text{ and } A \subseteq F\}$$

1. Let X be a topological space and let $A \subseteq X$. Then
   (a) $\overline{A}$ is a closed set.

   Solution: $\overline{A}$ is an intersection of closed sets, so it too is closed.

   (b) $A \subseteq \overline{A}$

   Solution: $x \in A \Rightarrow x \in F \forall F \iff x \in \overline{A}$

   (c) $\overline{A}$ is the smallest closed set that contains A in the sense than if S is closed and $A \subseteq S$, then $\overline{A} \subseteq S$ also.

   Solution: Such an S corresponds to F as stated above, so $x \in S \Rightarrow x \in S$ by definition.

2.6 Theorem 5.8

1. Let X be a topological space and let $A \subseteq X$. Then if $x \in X, x \in \overline{A}$ if and only if $N \cap A \neq \emptyset$ for all neighbourhoods N of x.

Solution:
$\Rightarrow$:
Proof by contradiction: if there exists a neighbourhood N of x which is disjoint with A, then there exists an open set O_x containing x that is disjoint with A, and $C_x = X - O_x$ is a closed set containing A that does not contain x. Then $x \in \overline{A} \Rightarrow x \in C_x$, which is a contradiction.
$\Leftarrow$:
Proof by contradiction: let $x \notin \overline{A}$. Then there exists a closed set $C_x \in X$ that contains A but not x. Then $O_x = X - C_x$ is an open set containing x that is disjoint with A. Since all open sets are neighbourhoods, this means there exists a neighbourhood of x that is disjoint with A, which contradicts the assumption.

2.7 Exercise 5.9

1. In the spaces on the real line obtained by giving it the indiscrete topology, the discrete topology, the usual metric topology, the Sorgenfrey topology, and the finite-complement topology, what is $\overline{A}$ if A is
   (a) $(0, 1)$.

   Solution:
   $\mathbb{I} : \mathbb{R}$
   $\mathbb{D} : (0, 1)$
   $\mathbb{U} : [0, 1]$
   $\mathbb{S} : [0, 1)$

$\mathbb{F} : \mathbb{R}$

(b) $[0, 1]$.

Solution:
$\mathbb{I} : \mathbb{R}$
$\mathbb{D} : [0, 1]$
$\mathbb{U} : [0, 1]$
$\mathbb{S} : [0, 1]$
$\mathbb{F} : \mathbb{R}$

(c) $[0, 1)$.

Solution:
$\mathbb{I} : \mathbb{R}$
$\mathbb{D} : [0, 1)$
$\mathbb{U} : [0, 1]$
$\mathbb{S} : [0, 1)$
$\mathbb{F} : \mathbb{R}$

(d) $(0, 1]$.

Solution:
$\mathbb{I} : \mathbb{R}$
$\mathbb{D} : (0, 1]$
$\mathbb{U} : [0, 1]$
$\mathbb{S} : [0, 1]$
$\mathbb{F} : \mathbb{R}$

(e) $\{0\}$.

Solution:
$\mathbb{I} : \mathbb{R}$
$\mathbb{D} : \{0\}$
$\mathbb{U} : \{0\}$
$\mathbb{S} : \{0\}$
$\mathbb{F} : \{0\}$

(f) $\mathbb{Q}$

Solution:
$\mathbb{I} : \mathbb{R}$
$\mathbb{D} : \mathbb{Q}$
$\mathbb{U} : \mathbb{R}$
$\mathbb{S} : \mathbb{R}$

$\mathbb{F} : \mathbb{R}$

(g) $\{\frac{1}{n} : n \in \mathbb{Z}^+\}$

Solution:
$\mathbb{I} : \mathbb{R}$
$\mathbb{D} : \{\frac{1}{n} : n \in \mathbb{Z}^+\}$
$\mathbb{U} : \{\frac{1}{n} : n \in \mathbb{Z}^+\} \cup \{0\}$
$\mathbb{S} : \{\frac{1}{n} : n \in \mathbb{Z}^+\} \cup \{0\}$
$\mathbb{F} : \mathbb{R}$

(h) $\emptyset$

Solution:
$\mathbb{I} : \emptyset$
$\mathbb{D} : \emptyset$
$\mathbb{U} : \emptyset$
$\mathbb{S} : \emptyset$
$\mathbb{F} : \emptyset$

2. Instead of giving you a specific problem, this exercise asks you to formulate the problem and then to solve them.

(a) How does closure behave with respect to unions?

Solution:
A finite union of closures is equal to a closure of finite unions, or

$$\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$$

where I is finite.
LHS $\subseteq$ RHS:
Proof by contradiction. Let $x \in$ LHS but $x \notin$ RHS. Let O_i be an open set containing x but disjoint with A_i. Then $\bigcap_{i \in I} O_i$ is open, contains x, but disjoint with $\bigcup_{i \in I} A_i$, meaning $x \notin$ LHS, which is a contradiction.
RHS $\subseteq$ LHS:
Let $x \in RHS$, and N be an arbitrary neighbourhood of x. Then $N \cup \bigcup_{i \in} A_i$ is nonempty, so $x \in LHS$. This also shows that RHS $\subseteq$ LHS even when I is not finite.

(b) How does closure behave with respect to intersections?

Solution:
A closure of intersections is a subset of an intersection of closures, or

$$\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$$

$x \in$ LHS means than for an arbitrary neighbourhood N of x, $N \cap \bigcap_{i \in I} A_i$ is not empty.

Therefore $N \cap A_i \forall i \in I$ is non empty, so $x \in \overline{A_i} \forall i \in I$, equivalent to saying $x \in \bigcap_{i \in I} \overline{A_i}$.
LHS does not have to equal RHS, as seen in $\{x \in \mathbb{R} | x < 0\}$ and $\{x \in \mathbb{R} | x > 0\}$ in the space $\mathbb{R}$.

(c) How does closure behave with respect to complementation?

Solution:
The complement of the closure is a subset of the closure of the complement, or

$$X - \overline{A} \subseteq \overline{X - A}$$

$x \in \text{LHS} \Rightarrow x \in X - A \Rightarrow x \in \overline{X - A}$
LHS does not have to equal RHS, as seen in $A = \{x \in \mathbb{R} | x < 0\}$.

(d) How does closure behave with respect to Cartesian products?

Solution:
Let A_i be a finite collection of sets with metrices d_i. Then

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}$$

If the metric for the cartesian products is defined as

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i \in I} d_i^2(x_i, y_i)}$$

First note that "neighbourhoods" can be replaced by "open sets" in Theorem 5.8. Also, an open hypercube can replace an open sphere in defining open sets under the usual topology.
LHS $\subseteq$ RHS:
Let $x \in$ LHS, and C_x be an open hypercube centred at x with length $2l$. Since $x \in$ LHS, we know any open set containing x intersects A_i, meaning $\exists y_i \in S_l(x_i) \cap A_i$. There exists a point y in the open cube C_x whose ith component is in $\overline{A_i}$. Therefore, for an arbitrary x in LHS and an arbitrary open set containing it, the open set also intersects with $\prod_{i \in I} A_i$, meaning $x \in$ RHS.
RHS $\subseteq$ LHS:
Let $x \in$ RHS, and $S_l(x_i)$ be an arbitrary open interval containing x_i. Consider an open hypercube C_x centred at x with length $2l$. Since RHS is a closure, this cube contains $y \in \prod_{i \in I} A_i$, meaning $y_i \in S_l(x_i) \cap A_i$. Since an arbitrary open set containing x_i intersects A_i, we have $x_i \in \overline{A_i} \forall i \in I$.

(e) How do functions affect closure?

Solution:
For a continuous function, the function of the closure is a subset of the closure of the function, or

$$f(\overline{A}) \subseteq \overline{f(A)}$$

Let $f : X \to Y$, and $y \in$ LHS. Then $\exists x \in \overline{A} : f(x) = y$. Let O_y be an arbitrary open set in Y containing y. Then $O_x = f^{-1}(O_y)$ is an open set containing x. Since $x \in$ LHS, O_x must contain an $a \in A$, so $f(a) \in O_y$, meaning O_y intersects $f(A)$. Since this holds $\forall O_y$, $y \in \overline{f(A)}$.
LHS does not have to equal RHS, e.g. when $f(x) = x, X = (\mathbb{R}, \mathbb{U}), Y = (\mathbb{R}, \mathbb{I})$.

2.8 Theorem 5.10

1. Let X and Y be topological spaces and let $f : X \to Y$. Then
   (a) f is continuous if and only if whenever $A \subseteq X$, then $f(\overline{A}) \subseteq \overline{f(A)}$

Solution:
$\Rightarrow$:
See here.
$\Leftarrow$:
Let B be a closed set in Y, and let $A = f^{-1}(B)$. Since B is closed, we have

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq B$$

This means

$$x \in \overline{A} \Rightarrow f(x) \in B \Rightarrow x \in A$$

by definition of A. Combining this with the fact that A is a subset of $\overline{A}$, we can conclude $A = \overline{A}$, which means A is closed. Under f, the inverse of any closed set is still closed, so it is continuous.

   (b) f is continuous if and only if whenever $B \subseteq Y$, then $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$

Solution:
$\Rightarrow$:
Let $x \in \overline{f^{-1}(B)}$. Consider an open set $O_y \subseteq Y$ containing $y = f(x)$. Since f is continuous, $O_x = f^{-1}(O_y)$ is open. O_x is an open set that contains x, so $\exists b \in O_x : f(b) \in B$. Then O_y intersects B. Since O_y is arbitrary, any open set containing y intersects B, so $y \in \overline{B}$. Hence $x \in f^{-1}(\overline{B})$.
$\Leftarrow$:
Let B be a closed set in Y. Then

$$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B)$$

so

$$x \in \overline{f^{-1}(B)} \Rightarrow f(x) \in B \Rightarrow x \in f^{-1}(B)$$

Similar to (a), we have shown that the inverse of B is equal to its closure, i.e. it is closed. Since the inverse of any closed set is closed, f is continuous.

2.9 Exercise 5.11

1. Show that, fora subset A of a topological space, $A = \overline{A}$ if and only if A is closed.

Solution:
$\Rightarrow$:
Let A be closed. Then A is the smallest closed set that contains A itself, so it is its own closure.
$\Leftarrow$:
$\overline{A}$ is closed by definition. If A is not closed, $A = \overline{A}$ becomes a contradiction.

2. $x \in X$ is a cluster point of $A \subseteq X$ if every neighbourhood of x meets A in at least one point other than x itself. The set of all cluster points of A is called the derived set of A and is denoted A'.

(a) Let $\mathbb{R}$ have its usual topology, $a, b \in \mathbb{R}$ with $a < b$. What is $(a, b)'$? $[a, b)'$? $[a, b]'$? $\{a\}'$?

Solution: $[a, b], [a, b], [a, b], \{a\}$

(b) Show that if X is any topological space, then $\overline{A} = A \cup A'$. Is $A' = \overline{A} - A$?

Solution: Proof for $\overline{A} = A \cup A'$
LHS $\subseteq$ RHS
Let $x \in$ LHS. Either $x \in A$, or any neighbourhood of x intersects A at a point other than x, since $x \notin A$. Hence $x \in A$ or $x \in A'$.
RHS $\subseteq$ LHS
Let $x \in$ RHS. If $x \in A$, the $x \in$ LHS by definition. Or else, any neighbourhood of x intersects with A, meaning $x \in \overline{A}$.
$A' = \overline{A} - A$ is false. Consider the usual topology where $A = [0, 1]$.

(c) Let $X = \{a, b, c, d\}$ with the topology $\mathcal{J} = \{\emptyset, X, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$. What are $\{p\}'$ and $\overline{\{p\}}$ when $p = a, b, c,$ or d? What is $\{a, d\}'$? $\overline{\{a, d\}}$? What is $\{b, d\}'$? $\overline{\{b, d\}}$?

Solution:
$p = a$: $\{b\}, \{a, b\}$
$p = b$: $\emptyset, \{b\}$
$p = c$: $\{d\}, \{c, d\}$
$p = d$: $\{c\}, \{c, d\}$
$\{a, d\}' = \{b, c\}, \overline{\{a, d\}} = X, \{b, d\}' = \{c\}, \overline{\{b, d\}} = \{b, c, d\}$

3. The largest open set that is contained in a subset A of a topolotical space X is called the interior of A, is denoted by A°, and is defined by

$$A^\circ = \bigcup \{U \subseteq X : U \text{ is open and } U \subseteq A\}$$

(a) Show that A° is open.

Solution: A° is open because it is a union of open sets.

(b) Show that A° is the largest open set contained in A by showing that $A^\circ \subseteq A$ and if $U \subseteq X$ is open and $U \subseteq A$, then $U \subseteq A^\circ$.

Solution: $A^\circ \subseteq A$ because $x \in A^\circ \Rightarrow x \in U \subseteq A \Rightarrow x \in A$ For the second part of the question, note that the U defined is exactly the U whose union forms A°, so $U \subseteq A^\circ$.

(c) Show that $A = A^\circ$ if and only if A is open.

Solution:
$\Rightarrow$:
Since A° is always open, $A = A^\circ$ implies A is open also.
$\Leftarrow$:
We have proven that $A^\circ \subseteq A$. Since A itself satisfies the definition of U, A is part of the union that forms A°, hence $A \subseteq A^\circ$. As both sets are subsets or each other, they are equal by definition.

3 Basis for a Topology

3.1 Theorem 6.1

1. A subset of the real line with the usual topology is open if and only if it is the union of a collection of open intervals.

> **Solution:**
> $\Rightarrow$:
> Let A be an open subset. For it to be open, $\forall a \in A$ there exists an open interval O_a that is contained in A. Hence
> $$A = \bigcup_{a \in A} O_a$$
> which is a union of open intervals.
> $\Leftarrow$:
> Open intervals are open sets, so a union of them is always open.

3.2 Theorem 6.2

1. A subset of a metric space (X, d) is open if and only if it is the union of a collection of r-balls.

> **Solution:** Replace "interval" with "ball" in Theorem 6.1.

3.3 Exercise 6.4

1. Give a basis for $\mathbb{R}$ when it has the
   (a) usual metric topology
   (b) the indiscrete topology
   (c) the discrete topology
   (d) the Sorgenfrey topology
   (e) the finite-complement topology

> **Solution:**
> (a): $\mathcal{B} = \{(r - \epsilon, r + \epsilon) | r \in \mathbb{R}, \epsilon \in \mathbb{R}^+\}$
> (b): $\mathcal{B} = \{\mathbb{R}\}$
> (c): $\mathcal{B} = \{\{r\} | r \in \mathbb{R}\}$
> (d): $\mathcal{B} = \{[r, r + \epsilon) | r \in \mathbb{R}, \epsilon \in \mathbb{R}^+\}$
> (e): $\mathcal{B} = \{O | \mathbb{R} - O \text{ is odd}\}$

2. (a) What is a basis for the order topology on $\mathbb{Z}^+$? On $\mathbb{R}$?

   > **Solution:** $\{\{n\} | n \in \mathbb{Z}^+\}, \{(r - \epsilon, r + \epsilon) | r \in \mathbb{R}, \epsilon \in \mathbb{R}^+\}$

(b) What is a basis for the order topology on $[0, \Omega)$?

> **Solution:** $\{(a, b)|a < b \text{ and } a, b \in \Omega\} \cup \{0\}$

(c) Answer all three questions in (a) and (b) by giving a basis for the order topology on an arbitrary totally ordered set X.

> **Solution:**
> If X has no maximum nor minimum, the basis is $\mathcal{B} = \{(a, b)|a < b \text{ and } a, b \in X\}$. If X has a maximum M and/or minimum m, the basis is the union between $\mathcal{B}$ and $\{(a, M]|a \neq M \in X\}$ and/or $\{[m, b)|b \neq m \in X\}$.

3. A given topology may have more than one basis.

(a) Show that the collection of open squares is a basis for the usual metric topology on the plane.

> **Solution:**
> In fact, this can be proven for all $\mathbb{R}^n$. Firstly, all open squares are open. For any x in an open square O_s, let r be the minimum distance between x and any of the bounds of the square. Then $x \in S_r x \subseteq O_s$. Secondly, the union of open squares produce all open sets. Let the centre of the square be x, and the side length l. Denote the open square by $S'_l(x)$. Given the dimensions, $\exists k \in \mathbb{R}$ such that the distance from x to any corner is kl. Then all open spheres contain open squares, namely $S'_{\frac{r}{k}}(x) \subseteq S_r(x)$. Similarly, all open squares contain open spheres, namely $S_{\frac{l}{2}}(x) \subseteq S'_l(x)$. Define functions $f(S_r(x)) = S'_{\frac{r}{k}}(x)$ and $g(S'_l(x)) = S_{\frac{l}{2}}(x)$. Any open set $O = \bigcup_{x \in O} O_x$ defined using open spheres can be rewritten as $\bigcup_{x \in O} f(O_x)$. Similarly, if the same open set is defined using open squares O_x, it can be rewritten as $\bigcup_{x \in O} g(O_x)$. Therefore, the topology defined by both open squares and open spheres are the same.

(b) Can you think of any other basis for the usual metric topology on the plane?

> **Solution:** Open equilateral triangles.

3.4 Theorem 6.5

1. Let (X, d) be a metric space. Then the collection $\mathcal{B} = \{S_{\frac{1}{n}}(x) : n \in \mathbb{Z}^+ \text{ and } x \in X\}$ is a basis for the metric topology on (X, d).

> **Solution:**
> $\frac{1}{n}$ tends to 0, so $\forall x > 0 \exists n \in \mathbb{N} : x > \frac{1}{n}$. This can be written as a function, where $f(x) = n$ where if x is positive, the inverse of n is smaller than x. Now define $g(S_r(x)) = S_{f(r)}(x)$. If O is an open set, it can be written as $\bigcup_{x \in O} O_x$ where O_x are open spheres centred at x. The same set can be rewritten as $\bigcup_{x \in O} g(O_x)$. Any union of open sets in $\mathcal{B}$ is obviously open, while any open set has shown to be a union of open sets in $\mathcal{B}$, thus the collection is a basis.

3.5 Theorem 6.8

1. Let X be a 1^{st} countable topological space. Then a subset $F \subseteq X$ is closed if and only if whenever $\{x_n : n \in \mathbb{Z}^+\}$ is a sequence of points of F that converges to $x \in X$, then $x \in F$.

Solution:
$\Rightarrow$:
Proof by contradiction: let there be a sequence x_n that converges to $x \in G = X - F$, which is open. For the sequence to converge to x, it must enter any open set containing x, e.g. G. This is false, leading to a contradiction.
$\Leftarrow$:
First we prove $G = X - F$ is open by contradiction. If it is not open, then $\exists g \in G$ where every open set containing g is not contained in G. Let B_n be an enumeration of the local basis, and define $O_n = \bigcap_{i=1}^n B_i$. Since O_n is a finite intersection of open sets containing g, O_n itself is also an open set that contains g, so it is not contained in G, i.e. $O_n - G \neq \emptyset$. Define a sequence x_n where $x_i \in O_i - G$. For any open set O containing g, it is a superset of B_k for some $k \in \mathbb{N}$. Then for $n > k, x_n \in O_n - G \subset O_n \subseteq B_k \subseteq O$. Since this holds for all open O containing g, the sequence x_n tends to g. However, x_n is a sequence in F that tends to $g \notin F$, which leads to a contradiction. Hence G must be open, and F is closed.

3.6 Exercise 6.9

1. Give an example of a topological space which is not 1^{st} countable.

Solution:
The finite-complement topology on $\mathbb{R}$. This can be proven by contradiction. Let x be the point, and A_n the enumeration of its local basis. $\mathbb{R} - A_n$ is then finite, and $A = \bigcup_{n \in \mathbb{N}} R - A_n$ is at most countable. Since $\mathbb{R}$ is uncountable, $\mathbb{R} - A$ is uncountable, meaning it contains more than 1 element, ie $y \neq x \in \mathbb{R}$. But $\mathbb{R} - A = \bigcap_{n \in \mathbb{N}} A_n$, which means $y \in A_n \forall n \in \mathbb{N}$. Then none of A_n can be the subset of the open set $\mathbb{R} - \{y\}$, leading to a contradiction.

2. Let $(X, \mathcal{J})$ be a topological space and let $\mathcal{B}_x$ be a local basis at $x \in X$. Can $\bigcap \mathcal{B}_x = \{x\}$? Must $\bigcap \mathcal{B}_x = \{x\}$?

Solution: It can be true, e.g. for $\mathcal{J} = \mathcal{D}$, the discrete topology. It doesn't have to be true, e.g. for $\mathcal{J} = \mathcal{I}$, the indiscrete topology.

3. (a) Let $(X, \mathcal{J})$ be a topological space and, for each point $x \in X$, let $\mathcal{B}_x$ be a local basis at x. Show that $\mathcal{B} = \bigcup \{\mathcal{B}_x : x \in X\}$ is a basis for the topology on X.

Solution:
Let O be an open set. Let f be defined on O such that $f(x)$ maps to an open subset of O containing x. This is possible as all points have a local basis. Then $O = \bigcup_{x \in O} f(x)$. All open sets in $\mathcal{J}$ can be written as unions of some sets in $\mathcal{B}$, and all sets in it are obviously open in $\mathcal{J}$. Therefore it is a basis for $\mathcal{J}$.

(b) Let $(X, \mathcal{J})$ be a topological space and let $\mathcal{B}$ be a basis for $\mathcal{J}$. Show that for each point $x \in X$, the collection

$$\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$$

is a local basis at x.

Solution:
Obviously all sets in $\mathcal{B}_x$ are open sets containing x. Let O_x be an open set containing x. Since it is open, it is a union of sets in $\mathcal{B}$, one of which must contain x. Denote that set by B_x. $B_x \in \mathcal{B}_x$ since it is an open set containing x. Then B_x is a set in the local basis that is also a subset of O_x. Hence $\mathcal{B}_x$ is a local basis.

3.7 Theorem 6.10

1. Let $(X, \mathcal{J}_1)$ and $(Y, \mathcal{J}_2)$ be topological spaces, let $\mathcal{B}$ be a basis for the topology $\mathcal{J}_2$ on Y, and let $F : X \to Y$. Then f is continuous on X if and only if for any $B \in \mathcal{B}$, $f^{-1(B)}$ is open in $(X, \mathcal{J}_1)$.

Solution:
$\Rightarrow$:
Any $B \in \mathcal{B}$ is open, so if f is continuous, $f^{-1}(B)$ is also continuous by definition.
$\Leftarrow$:
Let O_y be an open set in Y. Then it is a union of sets in $\mathcal{B}$, ie $O_y = \bigcup_{i\in I} B_i$, where all $B_i \in \mathcal{B}$. Define $A_i = f^{-1}B_i$, which is open by assumption. Observe that $f^{-1}(O_y) = \bigcup_{i\in I} A_i$ which is open. This holds for arbitrary open sets $O_y \in Y$, so f is continuous.

3.8 Exercise 6.11

1. Let $\mathbb{R}^+ \cup \{0\}$ be the set of non-negative real numbers and give it the usual metric topology induced by the absolute value metric.
   (a) Describe a basis for this topology.
   (b) Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ such that there is a basic open set in $\mathbb{R}^+ \cup \{0\}$ whose inverse is not a basic open set in $\mathbb{R}$.

Solution: $\{(a, b)|a, b \in \mathbb{R}^+ \cup \{0\} \text{ and } a < b\}$, $f(x) = x^2$

3.9 Theorem 6.12

1. A subset N of a topological space X with basis $\mathcal{B}$ is a neighbourhood of a point $x \in X$ if and only if there is a basic open set $B \in \mathcal{B}$ such that $x \in B \subseteq N$.

Solution:
$\Rightarrow$:
If N is a neighhbourhood of x, it contains an open set O_x containing x. Since O_x is open, it is a union of basic open sets, including at least one that contains x. Or else, x will be in O_x but not in the union of basic open sets, leading to a contradiction. Therefore, such a basic open set exists, and it is the required set such that $x \in B_x \subseteq N$
$\Leftarrow$:
A basic open set is an open set. This is true by definition of a neighbourhood.

3.10 Exercise 6.14

1. Show that the collection of sets $\{[a,b] : a < 0 < b\}, \{[a,b) : a < 0 < b\}, \{(a,b] : a < 0 < b\}$, and $\{(a,b) : a < 0 < b\}$ are all local neighbourhood bases at 0 in the usual topology on the real line. Which of these collection is a local basis at 0 in the usual topology on $\mathbb{R}$?

Solution:
$\mathcal{B}_0 = \{(a,b) : a < 0 < b\}$ is a local basis at 0. Any open set containing 0 must contain an open interval centred at 0, which is in $\mathcal{B}_0$ in the form of $(-\epsilon, \epsilon)$. Any set in any collection has a subset in $\mathcal{B}_0$ by substituting the values of a, b. Therefore, they all are local neighbourhood bases.

4 Topology Generated by a Basis

4.1 Exercise 7.2

1. Let $\mathcal{B}$ be the collection of all closed intervals on the real line. Show that $\mathcal{B}*$, the collection of all possible unions of elements of $\mathcal{B}$, together with the empty set, is not a topology on $\mathbb{R}$.

Solution:
I probably did something wrong here
$\{r\} \forall r \in \mathbb{R}$ are closed intervals. This means $\mathcal{B}*$ is the discrete topology.

2. Let $\mathbb{R}$ have the usual topology and consider the collection $\mathcal{B}$ of subsets of $\mathbb{R}$ defined by

$$\mathcal{B} = \{\{x\} : x \in \mathbb{R}\}$$

   (a) Show that $\mathcal{B}*$ is a topology on the set $\mathbb{R}$, but
   (b) Show that $\mathcal{B}*$ is not the usual topology on $\mathbb{R}$

Solution: Similar to above, $\mathcal{B}*$ is the discrete topology, which is different from the usual topology.

4.2 Theorem 7.3

1. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. Let $\mathcal{B}*$ be the collection of all possible unions of members of $\mathcal{B}$, together with the empty set. Then $\mathcal{B}*$ is a topology on X if and only if
   (a) $\bigcup \mathcal{B} = X$, and
   (b) For B_1 and B_2 in $\mathcal{B}$ and any point $x \in B_1 \cap B_2$ there exists a set $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Solution: $\Rightarrow$:
We first prove that $\bigcup \mathcal{B} = X$. Note that LHS is obviously a subset of RHS. If both sets are not equal, $\exists x \in X - \bigcup \mathcal{B}$. However, X is an open set by definition, ie a union of some sets in $\mathcal{B}$, and $x \in X$, so x must be in some open set in $\mathcal{B}$, which is a contradiction. Hence equality must hold. Secondly, if B_1 and B_2 are open sets, their intersection must be open, so it is a union of some basic

open sets. Since the intersection contains x, at least one of those basic open sets must contain x. Denote that basic open set as B_3, and we have $x \in B_3 \subseteq B_1 \cap B_2$ as desired.
$\Leftarrow$:
We know that $\emptyset \in \mathcal{B}$, and that X is a union of all basic open sets. Therefore $\mathcal{B}*$ contains both, which is required for any topology. Let $A_i = \bigcup_{j \in f(A_i)} B_j$ be an arbitrary open set, where B_j denote basic open sets. Then any union of open sets can be represented as a union of basic open sets

$$\bigcup_{i \in I} A_i = \bigcup_{j \in \bigcup_{i \in I} f(A_i)} B_j$$

ie the union of open sets is open. Then, we show that the intersection of any two open sets P and Q is open. Then any finite intersection of open sets is open by induction. Let $x \in P \cap Q$. Since $x \in P$, we know x is in at least one of the basic open sets whose union form P, meaning $x \in B_p \subseteq P$, where B_p is a basic open set. A similar B_q exists for Q. By the second condition, there exists a $B_3 \in \mathcal{B}$ which is in the intersection of B_p and B_q. Let C_x be a set of all such B_3 in the case of x. Then

$$P \cap Q = \bigcup \bigcup_{x \in P \cap Q} C_x$$

So the intersection of any two open sets is a union of basic open sets, which is open.

4.3 Theorem 7.4

1. Let $(X, \mathcal{J}$ be a topological space, let $\mathcal{B}$ be a basis for a topology on X, and let $\mathcal{B}*$ denote the topology for which $\mathcal{B}$ is a basis (i.e., $\mathcal{B}$ generates $\mathcal{B}*$). Then $\mathcal{B}* = \mathcal{J}$ if and only if
   (a) For every set $B \in \mathcal{B}$ and every point $x \in B$, there is a $U \in \mathcal{J}$ such that $x \in U \subseteq B$, and
   (b) For every set $U \in \mathcal{J}$ and every point $x \in U$, there is a $B \in \mathcal{B}$ such that$x \in B \subseteq U$.

Solution:
$\Rightarrow$:
For the first condition, B is an open set, so $U = B$ suffices. For the second condition, U is a union of basic open sets, one of which must contain x as $x \in U$. Denote that basic open set as B, and $x \in B \in U$ as desired.

$\Leftarrow$:
Let O_b be an open set in $\mathcal{B}*$. Then$O_b = \bigcup_{i \in I} B_i$ for some basic open sets B_i. From the first condition, letting $f(B, x) = U$ as a mapping to the guaranteed $U \in \mathcal{J}$, it can be seen that $B_i = \bigcup_{x \in B_i} f(B_i, x)$. Substitution yields

$$O_b = \bigcup_{i \in I} \bigcup x \in B_i f(B_i, x)$$

meaning O_b is a union of sets in $\mathcal{J}$, which must also be in $\mathcal{J}$. Therefore all open sets in $\mathcal{B}*$ are also in $\mathcal{J}$.
Let O_j be an open set in $\mathcal{J}$. From the second condition we can let $g(U, x) = B$ which maps to the guaranteed open set in $\mathcal{B}$.

$$O_j = \bigcup_{x \in O_j} g(O_j, x)$$

meaning O_j is a union of open sets in $\mathcal{B}$, which must be in $\mathcal{B}*$. Since $\mathcal{J}$ and $\mathcal{B}*$ are subsets of each other, they are equal.

4.4 Exercise 7.5

1. Are the following statements true or false?
   (a) A topology on a set can have more than one basis.

   Solution: True. Consider $(\mathbb{R}, \mathcal{U})$. Both $\{(a,b)|a,b \in \mathbb{R} \text{ and } a < b\}$ and $\mathcal{U}$ are different bases for the same topology.

   (b) A basis for a topology on a set can generate more than one topology.

   Solution: No. Let A and B be two topologies generated. Any $a \in A$ is a union of certain open sets in the basis, so it is also in B, and vice versa. Thus $A = B$, and distinct topologies cannot be generated by the same basis.

4.5 Theorem 7.6

1. Let $\mathcal{B}$ be a basis for the topology $\mathcal{J}$ on a set X (i.e., $\mathcal{B}$ generates $\mathcal{J}$ so $\mathcal{J} = \mathcal{B}^*$). A subset $U \subset X$ is open in $\mathcal{J} = \mathcal{B}*$ if and only if for any $x \in U$ there exists a basic open neighbourhood $N \in \mathcal{B}$ of x such taht $x \in N \subset U$.

Solution:
$\Rightarrow$:
Setting $N = B$ as described in (2) of Theorem 7.4 suffices.
$\Leftarrow$:
Obviously,

$$U = \bigcup_{x \in U} N_x$$

where N_x is the basic open neighbourhood as described. Then U is open by definition.

4.6 Exercise 7.7

1. Let d be a metric on a set X. Then
   (a) the collection of all r-balls,

$$\mathcal{B} = \{S_r(x) : r \in \mathbb{R}, r > 0 \text{ and } x \in X\}$$

   is a basis for a topology on X, the metric topology induced by d.

   Solution: Using Theorem 7.4, let A be an open set under the metric topology. Then by definition, $x \in S_r(x) \subseteq A \forall x$ for some $r \in R$. Every r-ball is open, so the opposite also holds. By the theorem, the collection of all r-balls is a basis for the metric topology.

   (b) For a given $x \in X$, the collection of all r-balls centered at x,

$$\mathcal{B}_x = \{S_r(x) : r \in \mathbb{R}^+\}$$

   is a local basis at x in the metrc topology induced by d.

Solution: Let A be an open set containing x. By definition, for A to be open, for all points in A such as x, $x \in S_r(x) \subseteq A$ for some $r \in \mathbb{R}^+$.

(c) The collection $\{S_{\frac{1}{n}}(s) : n \in Z^+\}$ is a local basis at x in the metric topology induced by d.

Solution: As $\frac{1}{n}$ tends to 0 as n tends to infinity, $\forall r \in \mathbb{R}^+ \exists n \in N : \frac{1}{n} < r$. Then $S_{\frac{1}{n}}(x) \subset S_r(x)$, and the former can be used to substitute the latter in the proof of the part above.

2. Let $x = [-1, 1]$ and let

$$\mathcal{B} = \{[-1, b), (a, 1] : -1 < a, b < 1\} \cup \{0\}$$

Then $\mathcal{B}$ is a basis for a topology on $[-1, 1]$ which is different from any that we have considered previously. In this topology, the singleton set $\{0\}$ is an open set, but no other singleton set is open.

Solution: From Theorem 7.3, for $\mathcal{B}$ to be a basis,

$$\bigcup B = X$$

and for B_1 and B_2 in $\mathcal{B}$ with any $x \in B_1 \cap B_2 \exists B_3 \in \mathcal{B} : x \in B_3 \subseteq B_1 \cap B_2$. The first is obviously true considering $a = 0, b = 0.5$. The second is true considering $\{0\} \in \mathcal{B}$, and any pair of sets in $\mathcal{B}$ that intersect must intersect at $\{0\}$.

3. Let $\mathcal{J}([0, 1])$ denote the collection of all functions from $[0, 1]$ into $\mathbb{R}$ such that $\int_0^1 |f(x)|dx$ exists.

(a) We can make this collection of functions into a topological space in which the points of the space are functions by specifying a basis for the topology. For $f \in \mathcal{J}([0, 1])$, let

$$\mathcal{B}_f = \{B_r(f) : r \in \mathbb{R}^+\}$$

where

$$B_r(f) = \left\{ g \in \mathcal{J}([0, 1]) : \int_0^1 |f(x) - g(x)|dx < r \right\}$$

Let $\mathcal{B} = \bigcup_{f \in \mathcal{J}([0,1])} \mathcal{B}_f$. Show that $\mathcal{B}$ is a basis for a topology on $\mathcal{J}([0, 1])$.

Solution: We use Theorem 7.3 as above. Note that $f \in B_r(f) \forall f \in \mathcal{J}([0, 1])$. The first condition $\bigcup \mathcal{B} = \mathcal{J}$ obviously holds. Then for the second condition, for any x in any intersection of basic open sets, x is a function in $\mathcal{J}([0, 1])$, so $B_r(x)$ suffices.

(b) Show that the "distance function" implicitly defined above, namely

$$d(f, g) = \int_0^1 |f(x) - g(x)|dx$$

is not a metric on $\mathcal{J}([0, 1])$.

Solution: **I probably did something wrong here.** Obviously, $d(f, g) = d(g, f)$ and $d(f, g) \in \mathbb{R}^+$. Then, $|f(x) - g(x)| + |g(x) - h(x)| \geq |f(x) - h(x)|$ at every $x \in [0, 1]$, so the integrals inherit the inequality, i.e. $d(f, g) + d(g, h) \geq d(f, h)$.