Divide & Conquer

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Divide-and-conquer algorithms

- Strategy:
  - Divide the problem into smaller subproblems of the same type of problem
  - Solve the subproblems recursively
  - Combine the answers to solve the original problem
- The work is done in three places:
  - In partitioning the problem into subproblems
  - In solving the basic cases at the tail of the recursion
  - In merging the answers of the subproblems to obtain the solution of the original problem

Conventional product of polynomials

Example:

$$P(x) = 2x^3 + x^2 - 4$$

$$Q(x) = x^2 - 2x + 3$$

$$(P \cdot Q)(x) = 2x^5 + (-4+1)x^4 + (6-2)x^3 + 8x - 12$$

$$(P \cdot Q)(x) = 2x^5 - 3x^4 + 4x^3 + 8x - 12$$

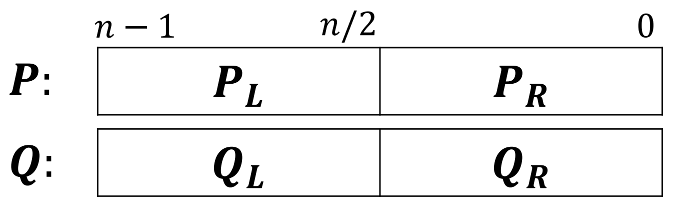
Conventional product of polynomials

```
function PolynomialProduct(P, Q)
    // P and Q are vectors of coefficients.
    // Returns R = P × Q.
    // degree(P) = size(P)-1, degree(Q) = size(Q)-1.
    // degree(R) = degree(P)+degree(Q).

    R = vector with size(P)+size(Q)-1 zeros;

    for each $P_i$
      for each $Q_j$
         $R_{i+j} = R_{i+j} + P_i \cdot Q_j$

    return R
```

Complexity analysis:
- Multiplication of polynomials of degree n: $O(n^2)$
- Addition of polynomials of degree n: $O(n)$

Product of polynomials: Divide&Conquer

Assume that we have two polynomials with n coefficients (degree $n-1$)



$$P(x)\cdot Q(x) = P_L(x)\cdot Q_L(x)\cdot x^n + (P_R(x)\cdot Q_L(x)+P_L(x)\cdot Q_R(x))\cdot x^{n/2} + P_R(x)\cdot Q_R(x)$$

$T(n) = 4\cdot T(n/2) + \mathrm{O}(n) = \mathrm{O}(n^2)$ ← Shown later

Product of complex numbers

- The product of two complex numbers requires four multiplications:

$$(a+bi)(c+di) = ac - bd + (bc+ad)i$$

- Carl Friedrich Gauss (1777-1855) noticed that it can be done with just three: ac, bd and $(a+b)(c+d)$

$$bc + ad = (a+b)(c+d) - ac - bd$$

- A similar observation applies for polynomial multiplication.

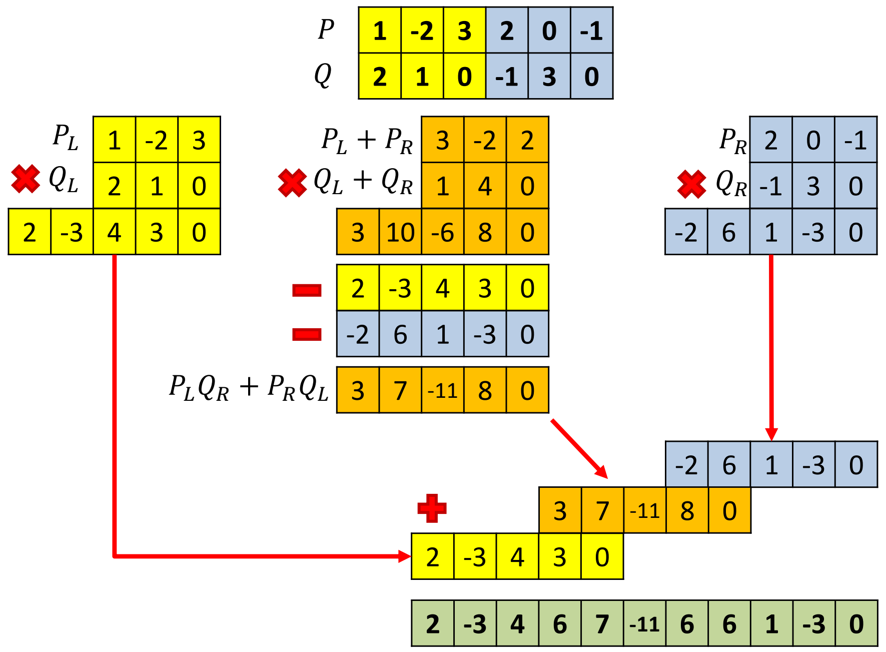
Product of polynomials with Gauss's trick

$$\begin{aligned} R_1 &= P_L Q_L \\ R_2 &= P_R Q_R \\ R_3 &= (P_L+P_R)(Q_L+Q_R) \end{aligned}$$

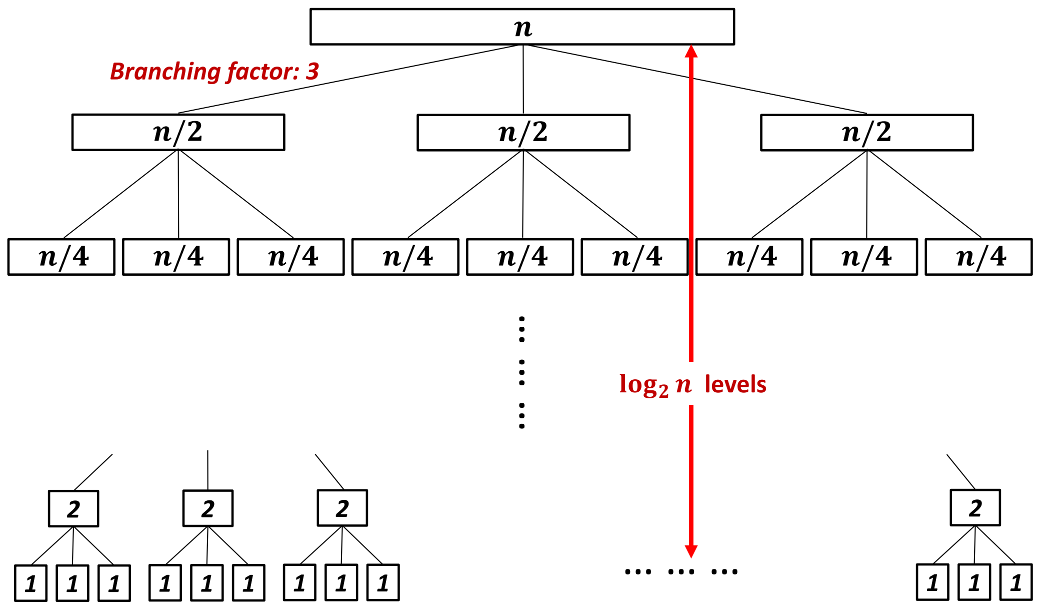
$$PQ = \underbrace{P_L Q_L}_{R_1} x^n + \underbrace{(P_R Q_L + P_L Q_R)}_{R_3 - R_1 - R_2} x^{n/2} + \underbrace{P_R Q_R}_{R_2}$$

$$T(n) = 3T(n/2) + \mathrm{O}(n)$$

Polynomial multiplication: recursive step



Pattern of recursive calls



Useful reminders

- Sum of geometric series with ratio r:

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

$$S = a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r} + \frac{r}{r-1}ar^{n-1}$$

- Logarithms:

$$\log_b n = \log_b a \cdot \log_a n$$

$$a^{\log_b n} = a^{(\log_a n)(\log_b a)} = n^{\log_b a}$$

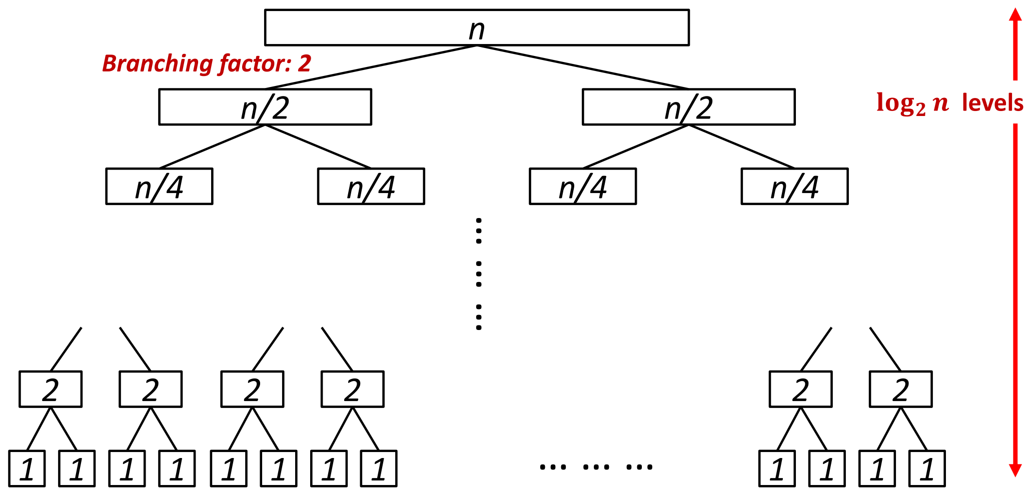
Complexity analysis

- The time spent at level k is

$$3^k \cdot \mathrm{O}\left(\frac{n}{2^k}\right) = \left(\frac{3}{2}\right)^k \cdot \mathrm{O}(n)$$

- For $k = 0$, runtime is $\mathrm{O}(n)$.
- For $k = \log_2 n$, runtime is $\mathrm{O}\left(3^{\log_2 n}\right)$, which is equal to $\mathrm{O}\left(n^{\log_2 3}\right)$.
- The runtime per level increases geometrically by a factor of 3/2 per level. The sum of any increasing geometric series is, within a constant factor, simply the last term of the series.
- Therefore, the complexity is $\mathrm{O}(n^{1.59})$.

A popular recursion tree



Example: efficient sorting algorithms.

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + \mathrm{O}(n)$$

Algorithms may differ on the amount of work done at each level: $\mathrm{O}(n^c)$

Examples

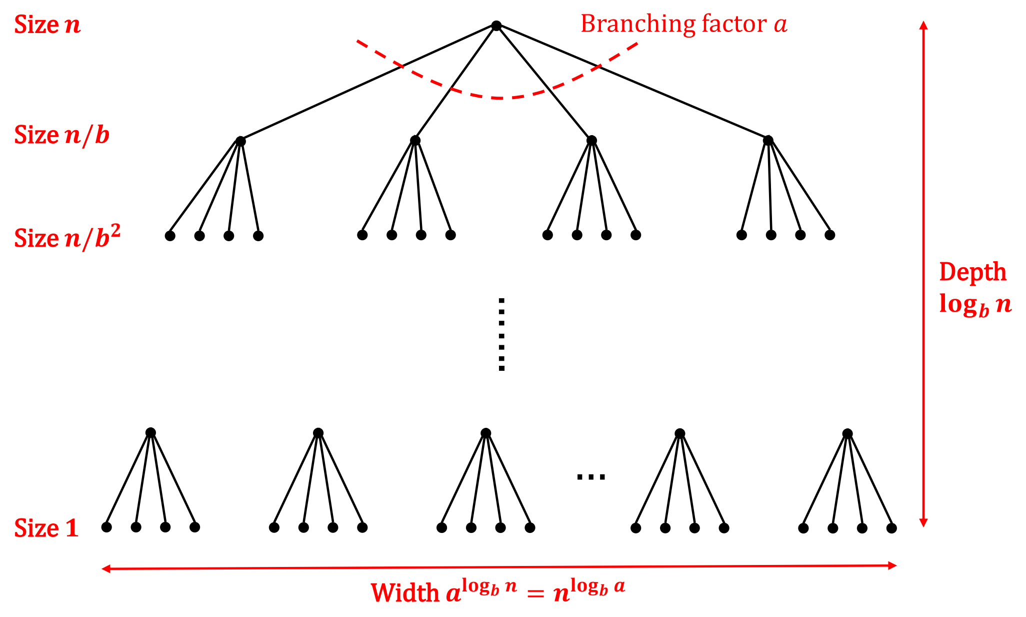
Algorithm	Branch	c	Runtime equation
Power (x^y)	1	0	$T(y) = T(y/2) + \mathrm{O}(1)$
Binary search	1	0	$T(n) = T(n/2) + \mathrm{O}(1)$
Merge sort	2	1	$T(n) = 2 \cdot T(n/2) + \mathrm{O}(n)$
Polynomial product	4	1	$T(n) = 4 \cdot T(n/2) + \mathrm{O}(n)$
Polynomial product (Gauss)	3	1	$T(n) = 3 \cdot T(n/2) + \mathrm{O}(n)$

Master theorem

- Typical pattern for Divide&Conquer algorithms:
  - Split the problem into a subproblems of size n/b
  - Solve each subproblem recursively
  - Combine the answers in $\mathrm{O}(n^c)$ time
- Running time: $T(n) = a \cdot T(n/b) + \mathrm{O}(n^c)$
- Master theorem:

$$T(n) = \begin{cases} \mathrm{O}(n^c) & \text{if } c > \log_b a \quad (a < b^c) \\ \mathrm{O}(n^c \log n) & \text{if } c = \log_b a \quad (a = b^c) \\ \mathrm{O}\left(n^{\log_b a}\right) & \text{if } c < \log_b a \quad (a > b^c) \end{cases}$$

Master theorem: recursion tree



Master theorem: proof

- For simplicity, assume n is a power of b.
- The base case is reached after $\log_b n$ levels.
- The kth level of the tree has a^k subproblems of size n/b^k.
- The total work done at level k is:

$$a^k \times \mathrm{O}\left(\frac{n}{b^k}\right)^c = \mathrm{O}(n^c) \times \left(\frac{a}{b^c}\right)^k$$

- As k goes from 0 (the root) to $\log_b n$ (the leaves), these numbers form a geometric series with ratio a/b^c. We need to find the sum of such a series.

$$T(n) = \mathrm{O}(n^c) \cdot \underbrace{\left(1 + \frac{a}{b^c} + \frac{a^2}{b^{2c}} + \frac{a^3}{b^{3c}} + \cdots + \frac{a^{\log_b n}}{b^{(\log_b n)c}}\right)}_{\log_b n \text{ terms}}$$

Master theorem: proof

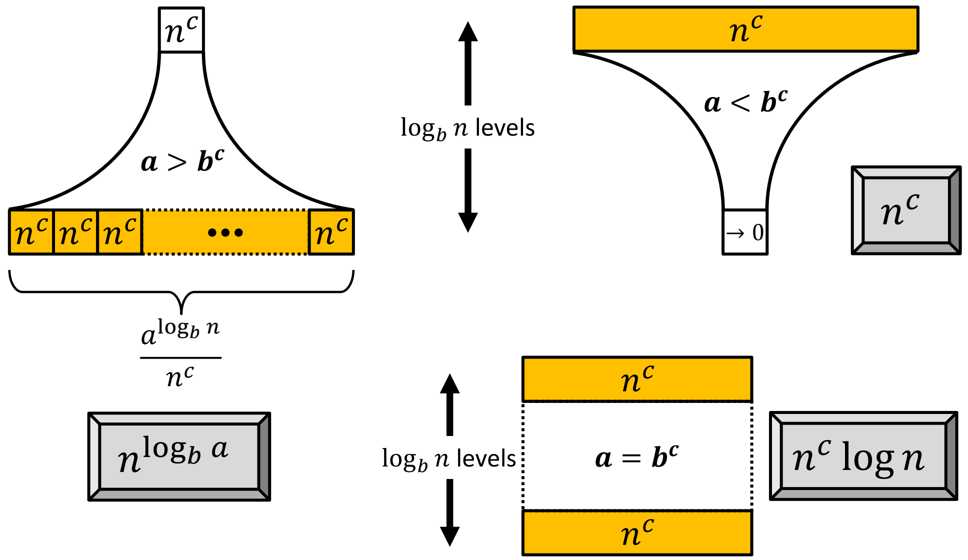
- Case $a/b^c < 1$. Decreasing series. The sum is dominated by the first term ($k = 0$): $\mathrm{O}(n^c)$.

- Case $a/b^c > 1$. Increasing series. The sum is dominated by the last term ($k = \log_b n$):

$$n^c \left(\frac{a}{b^c}\right)^{\log_b n} = n^c \left(\frac{a^{\log_b n}}{(b^{\log_b n})^c}\right) = a^{\log_b n} = n^{\log_b a}$$

- Case $a/b^c = 1$. We have $\mathrm{O}(\log n)$ terms all equal to $\mathrm{O}(n^c)$.

Master theorem: visual proof



Master theorem: examples

Running time: $T(n) = a \cdot T(n/b) + \mathrm{O}(n^c)$

$$T(n) = \begin{cases} \mathrm{O}(n^c) & \text{if } a < b^c \\ \mathrm{O}(n^c \log n) & \text{if } a = b^c \\ \mathrm{O}(n^{\log_b a}) & \text{if } a > b^c \end{cases}$$

Algorithm	**a**	**c**	**Runtime equation**	**Complexity**
Power (x^y)	1	0	$T(y) = T(y/2) + \mathrm{O}(1)$	$\mathrm{O}(\log y)$
Binary search	1	0	$T(n) = T(n/2) + \mathrm{O}(1)$	$\mathrm{O}(\log n)$
Merge sort	2	1	$T(n) = 2 \cdot T(n/2) + \mathrm{O}(n)$	$\mathrm{O}(n \log n)$
Polynomial product	4	1	$T(n) = 4 \cdot T(n/2) + \mathrm{O}(n)$	$\mathrm{O}(n^2)$
Polynomial product (Gauss)	3	1	$T(n) = 3 \cdot T(n/2) + \mathrm{O}(n)$	$\mathrm{O}(n^{\log_2 3})$

$b = 2$ for all the examples

Quick sort (Tony Hoare, 1959)

- Suppose that we know a number x such that one-half of the elements of a vector are greater than or equal to x and one-half of the elements are smaller than x.
  - Partition the vector into two equal parts ($n - 1$ comparisons)
  - Sort each part recursively

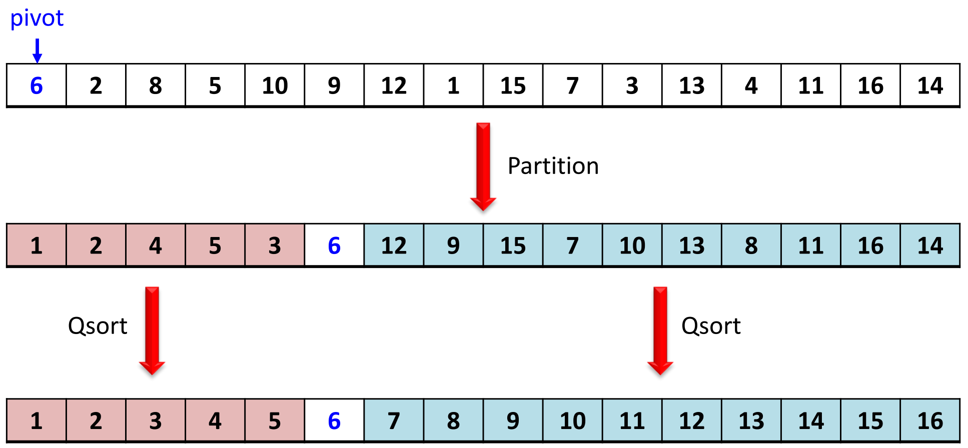
- Problem: we do not know x.

- The algorithm also works no matter which x we pick for the partition. We call this number the **pivot**.

- **Observation:** the partition may be unbalanced.

Quick sort with Hungarian, folk dance

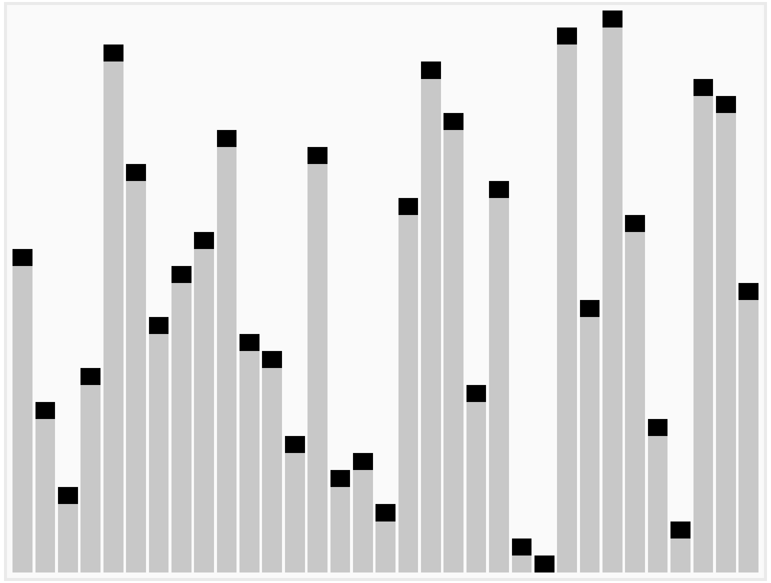
Quick sort: example



The key step of quick sort is the partitioning algorithm.

Question: how to find a good pivot?

Quick sort

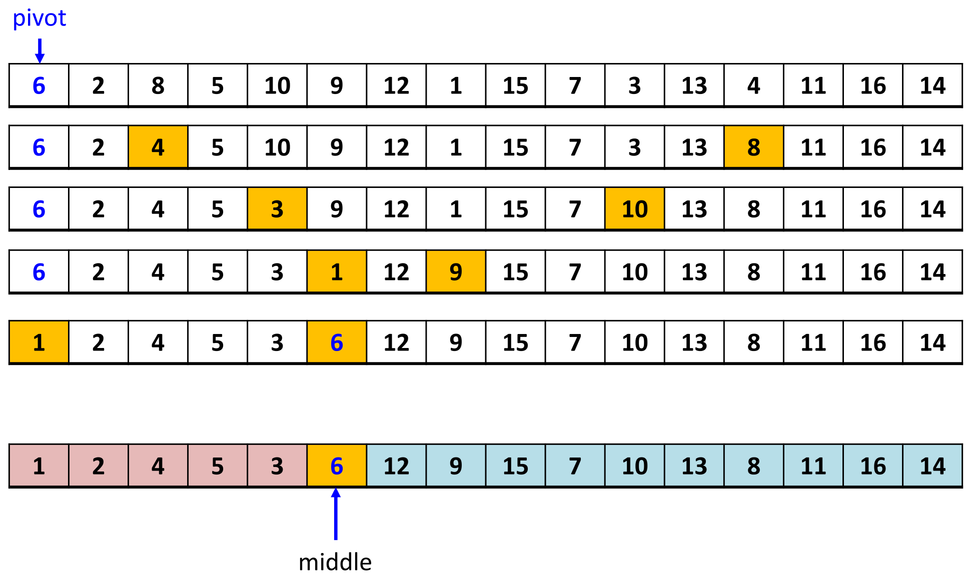


https://en.wikipedia.org/wiki/Quicksort

Quick sort: partition

```
function Partition(A, left, right)
  // A[left..right]: segment to be sorted
  // Returns the middle of the partition with
  //    A[middle] = pivot
  //    A[left..middle-1] ≤ pivot
  //    A[middle+1..right] > pivot

  x = A[left];  // the pivot
  i = left; j = right;

  while i < j do
    while i≤right and A[i]≤x do i = i+1;
    while j≥left and A[j]>x do j = j-1;
    if i < j then swap(A[i], A[j]);

  swap(A[left], A[j]);
  return j;
```

Quick sort partition: example



Quick sort: algorithm

```
function Qsort(A, left, right)

// A[left..right]: segment to be sorted

  if left < right then
    mid = Partition(A, left, right);
    Qsort(A, left, mid-1);
    Qsort(A, mid+1, right);
```

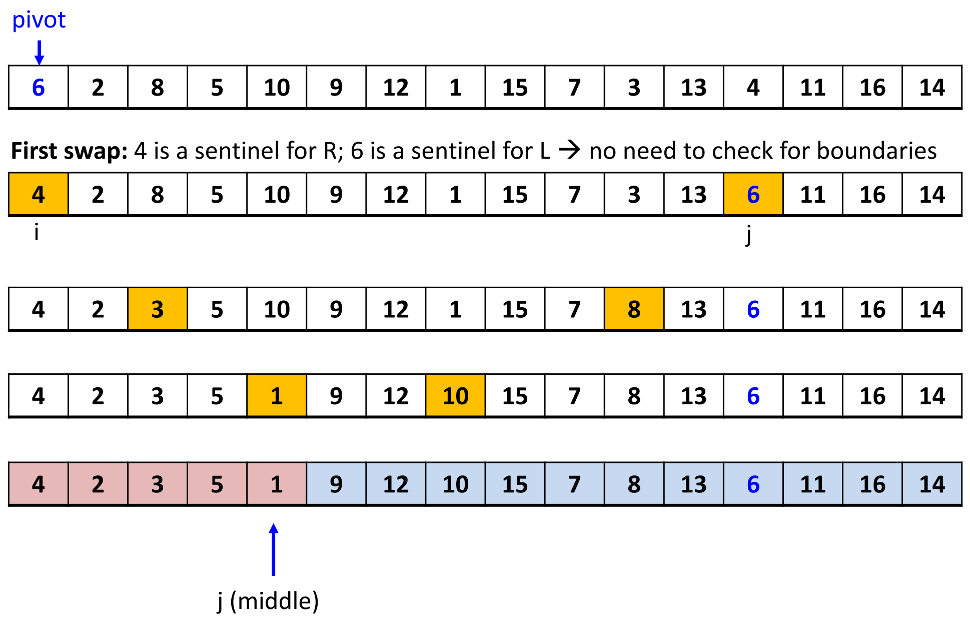
Quick sort: Hoare’s partition

```
function HoarePartition(A, left, right)

// A[left..right]: segment to be sorted.
// Output: The left part has elements ≤ than the pivot.
// The right part has elements ≥ than the pivot.
// Returns the index of the last element of the left part.

  x = A[left];  // the pivot
  i = left-1; j = right+1;

  while true do
    do i = i+1; while A[i] < x;
    do j = j-1; while A[j] > x;

    if i ≥ j then return j;

    swap(A[i], A[j]);
```

Admire a unique piece of art by Hoare:
The first swap creates two sentinels.
After that, the algorithm flies …

Quick sort partition: example

First swap: 4 is a sentinel for R; 6 is a sentinel for L → no need to check for boundaries



Quick sort with Hoare's partition

```
function Qsort(A, left, right)

// A[left..right]: segment to be sorted

  if left < right then
    mid = HoarePartition(A, left, right);
    Qsort(A, left, mid);
    Qsort(A, mid+1, right);
```

Quick sort: hybrid approach

```
function Qsort(A, left, right)
  // A[left..right]: segment to be sorted.
  // K is a break-even size in which insertion sort is
  // more efficient than quick sort.
  if right - left ≥ K then
    mid = HoarePartition(A, left, right);
    Qsort(A, left, mid);
    Qsort(A, mid+1, right);


function Sort(A):
  Qsort(A, 0, A.size()-1);
  InsertionSort(A);
```

Quick sort: complexity analysis

- The partition algorithm is $\mathrm{O}(n)$.
- Assume that the partition is balanced:

$$T(n) = 2 \cdot T(n/2) + \mathrm{O}(n) = \mathrm{O}(n \log n)$$

- Worst case runtime: the pivot is always the smallest element in the vector → $\mathrm{O}(n^2)$
- Selecting a good pivot is essential. There are different strategies, e.g.,
  – Take the median of the first, last and middle elements
  – Take the pivot at random

Quick sort: complexity analysis

- Let us assume that x_i is the ith smallest element in the vector.
- Let us assume that each element has the same probability of being selected as pivot.
- The runtime if x_i is selected as pivot is:

$$T(n) = n - 1 + T(i - 1) + T(n - i)$$

Quick sort: complexity analysis

$$T(n) = n - 1 + \frac{1}{n}\sum_{i=1}^{n}(T(i-1) + T(n-i))$$

$$T(n) = n - 1 + \frac{1}{n}\sum_{i=1}^{n}T(i-1) + \frac{1}{n}\sum_{i=1}^{n}T(n-i)$$

$$T(n) = n - 1 + \frac{2}{n}\sum_{i=0}^{n-1}T(i) \leq 2(n+1)(H(n+1) - 1.5)$$

$H(n) = 1 + 1/2 + 1/3 + \cdots + 1/n$ is the Harmonic series, that has a simple approximation: $H(n) = \ln n + \gamma + O(1/n)$.
$\gamma = 0.577 \ldots$ is Euler's constant. **[see the appendix]**

$$T(n) \leq 2(n+1)(\ln n + \gamma - 1.5) + O(1) = O(n \log n)$$

Quick sort: complexity analysis summary

- Runtime of quicksort:

$$T(n) = O(n^2)$$
$$T(n) = \Omega(n \log n)$$
$$T_{\text{avg}}(n) = O(n \log n)$$

- Be careful: some malicious patterns may increase the probability of the worst case runtime, e.g., when the vector is sorted or almost sorted.
- Possible solution: use random pivots.

The selection problem

- Given a collection of N elements, find the kth smallest element.
- Options:
  – Sort a vector and select the kth location: $O(N \log N)$
  – Read k elements into a vector and sort them. The remaining elements are processed one by one and placed in the correct location (similar to insertion sort). Only k elements are maintained in the vector. Complexity: $O(kN)$. Why?

The selection problem using a heap

- Algorithm:
  – Build a heap from the collection of elements: $O(N)$
  – Remove k elements: $O(k \log N)$
  – Note: heaps will be seen later in the course
- Complexity:
  – In general: $O(N + k \log N)$
  – For small values of k, i.e., $k = O(N/\log N)$, the complexity is $O(N)$.
  – For large values of k, the complexity is $O(k \log N)$.

Quick sort with Hoare's partition

```
function Qsort(A, left, right)

// A[left..right]: segment to be sorted

  if left < right then
    mid = HoarePartition(A, left, right);
    Qsort(A, left, mid);
    Qsort(A, mid+1, right);
```

Quick select with Hoare's partition

```
// Returns the element at location $k$ assuming
// A[left..right] would be sorted in ascending order.
// Pre: left $\leq k \leq$ right.
// Post: The elements of A have changed their locations.

function Qselect(A, left, right, k)
  if left == right then return A[left];

  mid = HoarePartition(A, left, right);
  // We only need to sort one half of A
  if k $\leq$ mid then return Qselect(A, left, mid, k);
  else return Qselect(A, mid+1, right, k);
```

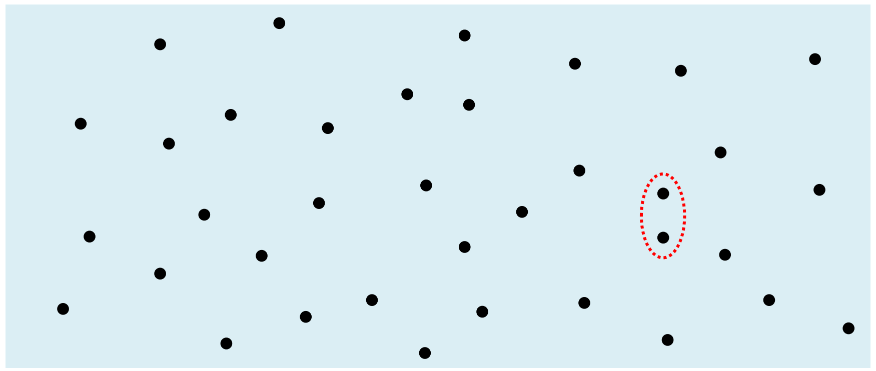
Quick Select: complexity

- Assume that the partition is balanced:
  - Quick sort: $T(n) = 2T(n/2) + O(n) = O(n \log n)$
  - Quick select: $T(n) = T(n/2) + O(n) = O(n)$

- The average linear time complexity can be achieved by choosing good pivots (similar strategy and complexity computation to qsort).

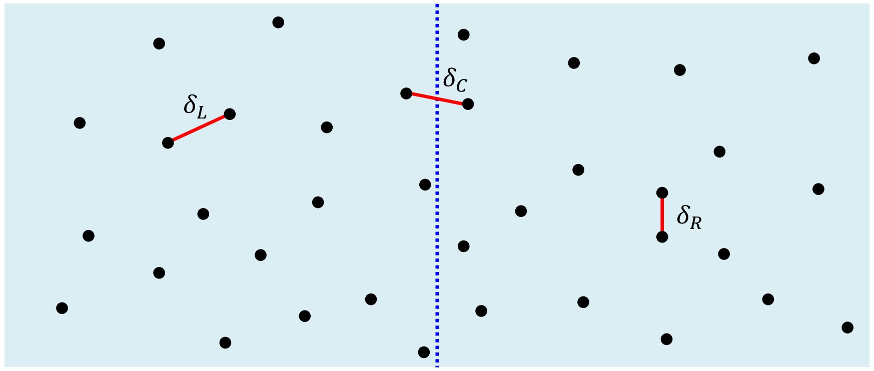
The Closest-Points problem

- **Input:** A list of n points in the plane
  $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$
- **Output:** The pair of closest points
- **Simple approach:** check all pairs → $O(n^2)$
- We want an $O(n \log n)$ solution !



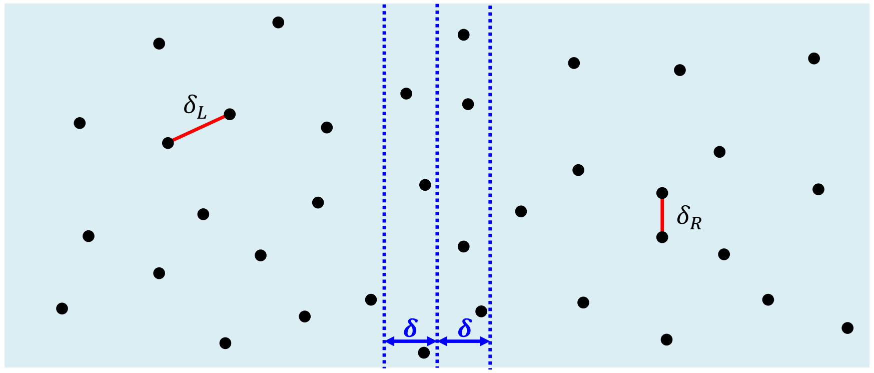
The Closest-Points problem

- We can assume that the points are sorted by the x-coordinate. Sorting the points is free from the complexity standpoint ($O(n \log n)$).
- Split the list into two halves. The closest points can be both at the left, both at the right or one at the left and the other at the right (center).
- The left and right pairs are easy to find (recursively). How about the pairs in the center?



The Closest-Points problem

- Let $\delta = min(\delta_L, \delta_R)$. We only need to compute δ_C if it improves δ.
- We can define a strip around de center with distance δ at the left and right. If δ_C improves δ, then the points must be within the strip.
- In the worst case, all points can still reside in the strip.
- But how many points do we really have to consider?



The Closest-Points problem

Let us take all points in the strip and sort them by the y-coordinate. We only need to consider pairs of points with distance smaller than δ.

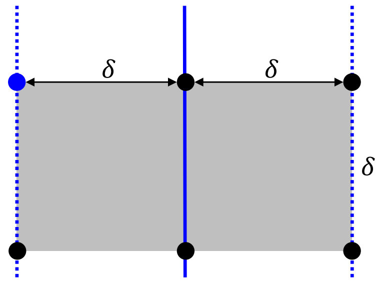
Once we find a pair (p_i, p_j) with y-coordinates that differ by more than δ, we can move to the next p_i.

```
for (i=0; i < NumPointsInStrip; ++i)
  for (j=i+1; j < NumPointsInStrip; ++j)

    if ($p_i$ and $p_j$'s $y$-coordinate differ by
        more than $\delta$) break; // Go to next $p_i$

    if ($dist(p_i, p_j) < \delta$)  $\delta = dist(p_i, p_j)$;
```

But, how many pairs (p_i, p_j) do we need to consider?

The Closest-Points problem

- For every point p_i at one side of the strip, we only need to consider points from p_{i+1}.



- The relevant points only reside in the $2\delta \times \delta$ rectangle below point p_i. There can only be 8 points at most in this rectangle (4 at the left and 4 at the right). Some points may have the same coordinates.

The Closest-Points problem: algorithm

- Sort the points according to their x-coordinates.
- Divide the set into two equal-sized parts.
- Compute the min distance at each part (recursively).
  Let δ be the minimal of the two minimal distances.
- Eliminate points that are farther than δ from the separation line.
- Sort the remaining points according to their y-coordinates.
- Scan the remaining points in the y order and compute the distances of each point to its 7 neighbors.

The Closest-Points problem: complexity

- Initial sort using x-coordinates: $O(n \log n)$.
  It comes for free.
- Divide and conquer:
  - Solve for each part recursively: $2T(n/2)$
  - Eliminate points farther than δ: $O(n)$
  - Sort remaining points using y-coordinates: $O(n \log n)$
  - Scan the remaining points in y order: $O(n)$

$$T(n) = 2T(n/2) + O(n) + O(n \log n) = O(n \log^2 n)$$

- Can we do it in $O(n \log n)$? Yes, we need to sort by y in a smart way.

The Closest-Points problem: complexity

- Let Y a vector with the points sorted by the y-coordinates. This can be done initially for free.
- Each time we partition the set of points by the x-coordinate, we also partition Y into two sorted vectors (using an "*unmerging*" procedure with linear complexity)

```
$Y_L = Y_R = \emptyset$   // Initial lists of points
foreach $p_i \in Y$ in ascending order of $y$ do
  if $p_i$ is at the left part then $Y_L$.push_back($p_i$)
  else  $Y_R$.push_back($p_i$)
```

- Now, sorting the points by the y-coordinate at each iteration can be done in linear time, and the problem can be solved in $O(n \log n)$

Subtract and Conquer

- Sometimes we may find recurrences with the following structure:

$$T(n) = a \cdot T(n-b) + O(n^c)$$

- Examples:

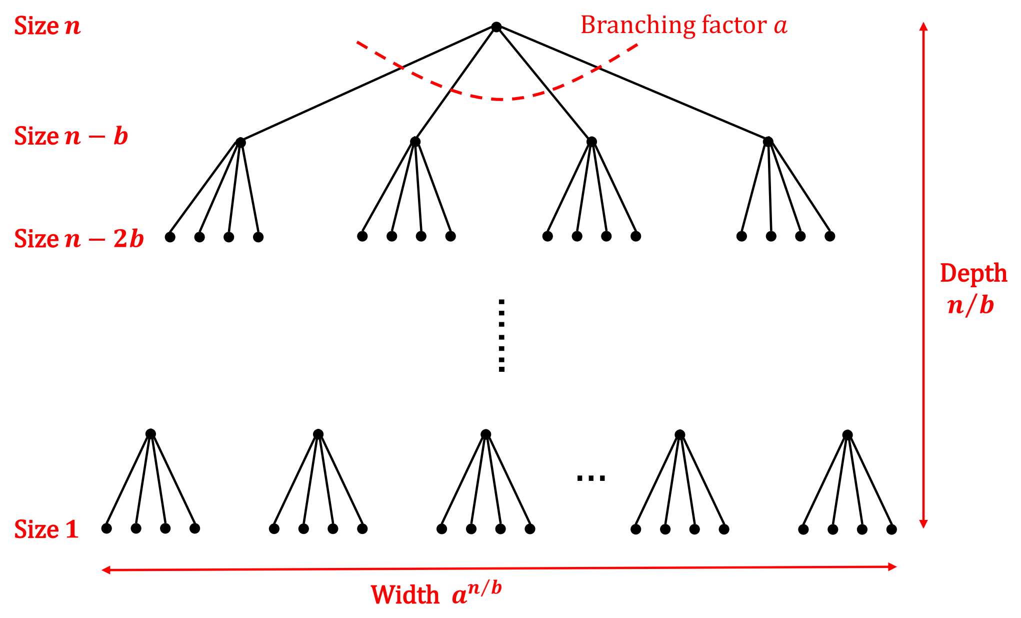
$$\text{Hanoi}(n) = 2 \cdot \text{Hanoi}(n-1) + O(1)$$

$$\text{Sort}(n) = \text{Sort}(n-1) + O(n)$$

- *Muster* theorem:

$$T(n) = \begin{cases} O(n^c) & \text{if } a < 1 \quad \text{(never occurs)} \\ O(n^{c+1}) & \text{if } a = 1 \\ O(n^c a^{n/b}) & \text{if } a > 1 \end{cases}$$

Muster theorem: recursion tree



Muster theorem: proof

- Expanding the recursion (assume that $f(n)$ is $O(n^c)$)

$$\begin{aligned} T(n) &= aT(n-b)+f(n) \\ &= a\big(aT(n-2b)+f(n-b)\big)+f(n) \\ &= a^2T(n-2b)+af(n-b)+f(n) \\ &= a^3T(n-3b)+a^2f(n-2b)+af(n-b)+f(n) \end{aligned}$$

- Hence:

$$T(n)=\sum_{i=0}^{n/b} a^i\cdot f(n-ib)$$

- Since $f(n-ib)$ is in $O\big((n-ib)^c\big)$, which is in $O(n^c)$, then

$$T(n)=O\left(n^c\sum_{i=0}^{n/b} a^i\right)$$

- The proof is completed by this property:

$$\sum_{i=0}^{n/b} a^i = \begin{cases} O(1), & \text{if } a<1 \\ O(n), & \text{if } a=1 \\ O\big(a^{n/b}\big), & \text{if } a>1 \end{cases}$$

Muster theorem: examples

- **Hanoi**: $T(n)=2T(n-1)+O(1)$

  We have $a=2$ and $c=0$, thus $T(n)=O(2^n)$.

- **Selection sort** (recursive version):
  - Select the min element and move it to the first location
  - Sort the remaining elements

  $T(n)=T(n-1)+O(n) \qquad (a=c=1)$

  Thus, $\qquad T(n)=O(n^2)$

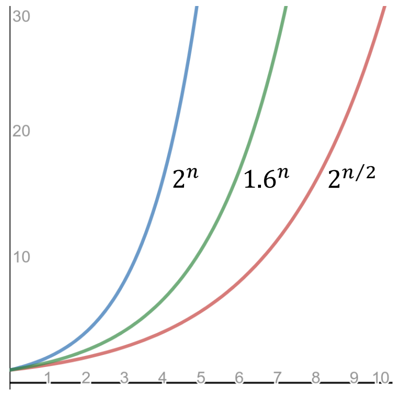
Muster theorem: examples

Fibonacci: $T(n)=T(n-1)+T(n-2)+O(1)$

We can compute bounds:

$$2T(n-2)+O(1)\le T(n)\le 2T(n-1)+O(1)$$

Thus, $\qquad O\big(2^{n/2}\big)\le T(n)\le O(2^n)$



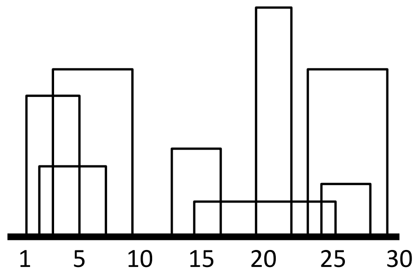
EXERCICES

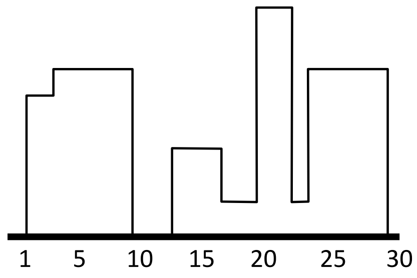
The skyline problem

Given the exact locations and shapes of several rectangular buildings in a city, draw the skyline (in two dimensions) of these buildings, eliminating hidden lines (source: Udi Manber, *Introduction to Algorithms*, Addison-Wesley, 1989).

Input:
(1,**11**,5) (2,**6**,7) (3,**13**,9) (12,**7**,16) (14,**3**,25) (19,**18**,22) (23,**13**,29) (24,**4**,28)

Output:
(1,**11**,3,**13**,9,**0**,12,**7**,16,**3**,19,**18**,22,**3**,23,**13**,29,**0**)
(numbers in boldface represent heights)





Describe (in natural language) two different algorithms to solve the skyline problem:

- By induction: assume that you know how to solve it for $n-1$ buildings.
- Using Divide&Conquer: solve the problem for $n/2$ buildings and combine.

Analyze the cost of each solution.

A, B or C?

Suppose you are choosing between the following three algorithms:

- Algorithm **A** solves problems by dividing them into five subproblems of half the size, recursively solving each subproblem, and then combining the solutions in linear time.
- Algorithm **B** solves problems of size n by recursively solving two subproblems of size $n-1$ and them combining the solutions in constant time.
- Algorithm **C** solves problems of size n by dividing them into nine subproblems of size $n/3$, recursively solving each subproblem, and then combining the solutions in $\mathrm{O}(n^2)$ time.

What are the running times of each of these algorithms (in big-O notation), and which one would you choose?

Source: Dasgupta, Papadimitriou and Vazirani, *Algorithms*, McGraw-Hill, 2008.

Crazy sorting

Let $T[i..j]$ be a vector with $n = j - i + 1$ elements. Consider the following sorting algorithm:

a) If $n \leq 2$ the vector is easily sorted (constant time).

b) If $n \geq 3$, divide the vector into three intervals $T[i..k-1]$, $T[k..l]$ and $T[l+1..j]$, where $k = i + \lfloor n/3 \rfloor$ and $l = j - \lfloor n/3 \rfloor$. The algorithm recursively sorts $T[i..l]$, then it sorts $T[k..j]$, and finally sorts $T[i..l]$.

- Proof the correctness of the algorithm.
- Analyze the asymptotic complexity of the algorithm (give a recurrence of the runtime and solve it).

The majority element

A majority element in a vector, A, of size n is an element that appears more than $n/2$ times (thus, there is at most one). For example, the vector $[3,3,4,2,4,4,2,4,4]$ has a majority element (4), whereas the vector $[3,3,4,2,4,4,2,2]$ does not. If there is no majority element, your program should indicate this. Here is a sketch of an algorithm to solve the problem:

> *First, a candidate majority element is found (this is the hardest part). This candidate is the only element that could possibly be the majority element. The second step determines if this candidate is actually the majority. This is just a sequential search through the vector. To find a candidate in the vector, A, form a second vector, B. Then compare A_0 and A_1. If they are equal, add one of these to B; otherwise do nothing. Then compare A_2 and A_3. Again if they are equal, add one of these to B; otherwise do nothing. Continue in this fashion until the entire vector is read. The recursively find a candidate for B; this is the candidate for A (why?).*

- How does the recursion terminate?
- What is the running time of the algorithm?
- How can we avoid using an extra array, B?
- Prove the correctness of the algorithm (hint: prove it for n even)
- How is the case where n is odd handled?

Source: Mark A. Weiss, *Data Structures and Algorithms in C++*, 4th edition, Pearson, 2014.

Breaking into pieces

Let us assume that f is $\Theta(1)$ and g has a runtime proportional to the size of the vector it has to process, i.e., $\Theta(j-i+1)$. What is the asymptotic cost of A and B as a function of n? (n is the size of the vector).

If both functions do the same, which one would you choose?

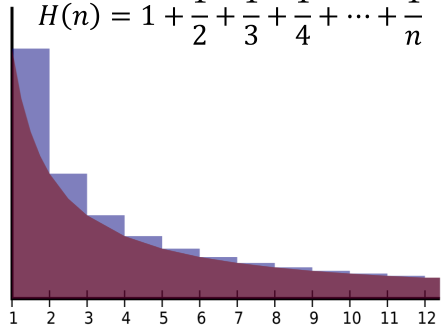
```
double A(vector<double>& v, int i, int j) {
  if (i < j) {
    int x = f(v, i, j);
    int m = (i+j)/2;
    return A(v, i, m-1) + A(v, m, j) + A(v, i+1, m) + x;
  } else {
    return v[i];
} }

double B(vector<double>& v, int i, int j) {
  if (i < j) {
    int x = g(v, i, j);
    int m1 = i + (j-i+1)/3;
    int m2 = i + (j-i+1)*2/3;
    return B(v, i, m1-1) + B(v, m1, m2-1) + B(v, m2, j) + x;
  } else {
    return v[i];
} }
```

APPENDIX

Logarithmic identities

$b^{\log_b a} = \log_b b^a = a$
$\log_b(xy) = \log_b x + \log_b y$
$\log_b \frac{x}{y} = \log_b x - \log_b y$
$\log_b x^c = c \log_b x$
$\log_b x = \frac{\log_c x}{\log_c b}$
$x^{\log_b y} = y^{\log_b x}$

$$H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$



$$\gamma = \lim_{n\to\infty}\left(-\ln n + \sum_{k=1}^{n}\frac{1}{k}\right) \Rightarrow \sum_{k=1}^{n}\frac{1}{k} \in \Theta(\log n)$$

$\gamma = 0.5772\ldots$ (Euler-Mascheroni constant) (Harmonic series)

Full-history recurrence relation

$$\boxed{T(n) = n - 1 + \frac{2}{n}\sum_{i=0}^{n-1} T(i)}$$

A recurrence that depends on all the previous values of the function.

$$nT(n) = n(n-1) + 2\sum_{i=0}^{n-1} T(i), \qquad (n+1)T(n+1) = (n+1)n + 2\sum_{i=0}^{n} T(i)$$

$$(n+1)T(n+1) - nT(n) = (n+1)n - n(n-1) + 2T(n) = 2n + 2T(n)$$

$$T(n+1) = \frac{n+2}{n+1}T(n) + \frac{2n}{n+1} \le \frac{n+2}{n+1}T(n) + 2$$

$$T(n) \le 2 + \frac{n+1}{n}\left(2 + \frac{n}{n-1}\left(2 + \frac{n-1}{n-2}\left(\cdots\frac{4}{3}\right)\right)\right)$$

$$T(n) \le 2\left(1 + \frac{n+1}{n} + \frac{n+1}{n}\frac{n}{n-1} + \frac{n+1}{n}\frac{n}{n-1}\frac{n-1}{n-2} + \cdots + \frac{n+1}{n}\frac{n}{n-1}\frac{n-1}{n-2}\cdots\frac{4}{3}\right)$$

$$T(n) \le 2\left(1 + \frac{n+1}{n} + \frac{n+1}{n-1} + \frac{n+1}{n-2} + \cdots + \frac{n+1}{3}\right) = 2(n+1)\left(\frac{1}{n+1} + \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{3}\right)$$

$$\boxed{T(n) \le 2(n+1)(H(n+1) - 1.5) = \Theta(n\log n)}$$