Beyond The Basics II

Chapter 10
From black holes to random forces

10.1 Beyond newtonian gravity

It was already clear to Einstein in 1905 that his special theory of relativity is consistent with Maxwell's electromagnetism, but *not* with Newton's gravitation. The fundamental problem is that while the equation of motion $\boldsymbol{F} = -(GMm/r^2)\hat{\boldsymbol{r}} = m\boldsymbol{a}$ is invariant under the **Galilean** transformation of Chapter 1, it is *not* invariant under the relativistic **Lorentz** transformation of Chapter 2.[1] So if the equation were true in one inertial frame of reference it would not be true in another using the correct transformation, and so could not be a fundamental law of physics according to the principle of relativity.

[1]As just one indication of this, the distance r between M and m is not invariant under a Lorentz boost. See examples in Chapter 6 for more details on how to explore the transformations and symmetries of Lagrangians.

Therefore Einstein set out to find a *relativistic* theory of gravitation. His decade-long effort finally culminated in his greatest single achievement, the **general theory of relativity**. There were several clues that gradually led him on. One was the apparently trivial fact that according to $\boldsymbol{F} = -GMm/r^2\hat{\boldsymbol{r}} = m\boldsymbol{a}$, the mass m cancels out on both sides: all masses m have the same acceleration in a given gravitational field according to Newton's theory and experiment as well. This is not as trivial as it seems, however, because the two m's have very different meanings. The m in GMm/r^2 is called the **gravitational mass**; it is the property of a particle that causes it to be attracted by another particle. The m in $m\boldsymbol{a}$ is called the **inertial mass**; it is the property of a particle that makes it sluggish, resistant to acceleration. The fact that these two kinds of mass appear to be the same is consistent with Newton's theory, but not explained by it. Einstein wanted a *natural* explanation. As discussed at the end of Chapter 3, this sort of thinking generated in Einstein his "happiest thought", the **principle of equivalence**. The equivalence of gravitational and inertial masses is an immediate consequence of the equivalence of (i) a uniformly accelerating frame without gravity, and (ii) an inertial reference frame containing a uniform gravitational field (See Problem 6.39). Einstein therefore saw a deep connection between *gravitation* and *accelerating reference frames*.

A second clue is the type of geometry needed within accelerating frames, as shown in the following thought experiment.

EXAMPLE 10-1: A thought experiment

A large horizontal turntable rotates with angular velocity ω: A reference frame rotating with the turntable is an accelerating frame, because every point on it is accelerating toward the center with $a = r\omega^2$. A colony of ants living on the turntable is equipped with meter sticks to make measurements (see Figure 10.1). A *second* colony of ants lives on a *non*rotating horizontal glass sheet suspended above the turntable; these ants are also equipped with meter sticks, and can make distance measurements on the glass while they are watching beneath them the rotating ants making similar measurements directly on the turntable itself.

Both ant colonies can measure the radius and circumference of the turntable, as shown in Figure 10.1. The *inertial* ants on the glass sheet, looking down on the turntable beneath them, lay out a straight line of sticks from the turntable's center to its rim, and so find

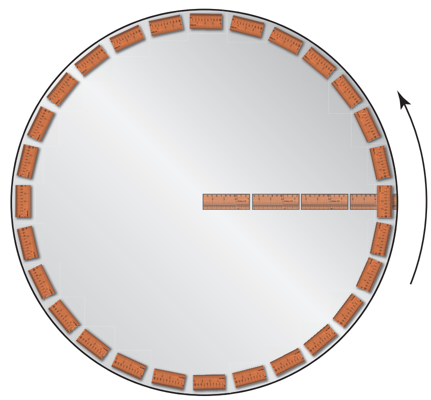

FIGURE 10.1 : An ant colony measures the radius and circumference of a turntable.

that the radius of the turntable (as measured on the glass sheet) is R_0. They also lay meter sticks end-to-end on the glass sheet around the rim of the turntable (as they see it through the glass), and find that the circumference of the turntable (according to their measurements) is $C_0 = 2\pi R_0$, verifying that Euclidean geometry is valid in their inertial frame.

The ants living on the turntable make similar measurements. Laying meter sticks along a radial line, they find that the radius is R. Laying sticks end-to-end around the rim, they find that the circumference of the turntable is C. Meanwhile the inertial ants, watching the rotating ants beneath them, find that the rotating-ant meter sticks laid out radially have no Lorentz contraction relative to their own inertial sticks, because the rotating sticks at each instant move *sideways* rather than *lengthwise*. So the inertial ants see that the rotating ants require exactly the same number of radial sticks as the inertial ants do themselves; in other words both sets of ants measure the same turntable radius, $R = R_0$.

However, the meter sticks laid out around the turntable *rim* by the rotating ants are moving with speed $v = R_0\omega$ in the direction of their lengths, and so will be Lorentz-contracted as observed in the inertial frame. More of these meter sticks will be needed by the rotating ants to go around the rim than is required by the inertial ants. Therefore it must be that the circumference C measured by the rotating ants is *greater* than the circumference C_0 measured by the inertial ants. Since the measured radius is the same, this means that in the accelerating frame $C > 2\pi R$. The logical deduction that $C > 2\pi R$ in the accelerating frame means that the geometry actually measured in the rotating frame is **non-Euclidean**, since the measurements are in conflict with Euclidean geometry. So this thought experiment shows that there appears to be a connection between *accelerating frames* and **non-Euclidean geometry**.

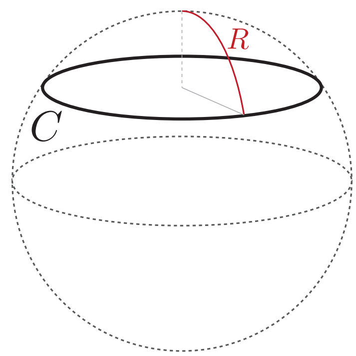

FIGURE 10.2 : Non-Euclidean geometry: circumferences on a sphere.

A third clue to a relativistic theory of gravity is that people have long known that Euclidean geometry is the geometry on a plane, while non-Euclidean geometries are the geometries on curved surfaces. Draw a circle on the curved surface of the Earth, for example, such as a constant-latitude line in the northern hemisphere (see Figure 10.2). Then the North Pole is at the center of the circle. The radius of the circle is a line on the sphere extending from the center to the circle itself; in the case of the Earth, this is a line of constant longitude. Then it is easy to show that the circumference and radius obey $C < 2\pi R$. The geometry on a two-dimensional curved space like the surface of the Earth is non-Euclidean. The opposite relationship holds if a circle is drawn on a saddle, with the center of the circle in the middle of the saddle; in that case one can show that $C > 2\pi R$, so that the geometry on a curved saddle is also non-Euclidean.

This suggests the question: *Are gravitational fields* associated with *curved spaces*? Let us emphasize how special is the gravitational force in this aspect. If we were to write Newton's second law for a probe of gravitational *and* inertial mass m near a larger mass M, we would get

$$m\,\boldsymbol{a} = -G\frac{M\,m}{r}\hat{\boldsymbol{r}} \tag{10.1}$$

as we already know. Because the m's on both sides of this equation are the same, they cancel, and we get

$$\boldsymbol{a} = -G\frac{M}{r}\hat{\boldsymbol{r}} \; ; \tag{10.2}$$

that is, all objects fall in gravity with the *same* acceleration that depends only on the source mass M and its location. An elephant experiences the same gravitational acceleration as does a feather. Contrast this with the Coulombic force:

$$m\,\boldsymbol{a} = \frac{1}{4\pi\varepsilon_0}\frac{Q\,q}{r}\hat{\boldsymbol{r}} \tag{10.3}$$

where the probe has mass m and charge q, and the source has charge Q. We see that m does not drop out, and the acceleration of a probe under the influence of the electrostatic force depends on the probe's attributes: its mass and charge. This dependence of a probes acceleration on its physical attributes is generic of all force laws except gravity! The gravitational force is very special in that it has a universal character — independent of the attributes of the object it acts on. Hence, gravity lends itself to be tucked into the fabric of space itself: all probes gravitate in the same way, and thus perhaps we can think of gravity as an attribute of space itself!

The next thought experiment is an illustration how this distortion of space due to gravity affect time as well! so that, in four-dimensional language, gravity can be packaged in *spacetime.*

EXAMPLE 10-2: Another thought experiment

Two clocks A and B are at rest in a uniform gravitational field g, with A on the ground and B at altitude h directly above. At time $t = t_0$ on A, A sends a light signal up to B, arriving at B at time t'_0 according to B, as shown in Figure 10.3. Later, when A reads t_1, A sends a second light signal to B, which arrives at t'_1 according to B. The time interval on A is $\Delta t_A = t_1 - t_0$, and the time interval on B is $\Delta t_B = t'_1 - t'_0$, which is greater than Δt_A, because from the principle of equivalence presented at the end of Chapter 3, high-altitude clocks run fast compared with low-altitude clocks. Now notice that the light signals together with the two clock world lines form a parallelogram in spacetime. Two of the sides are the parallel vertical timelike world lines of the clocks in the figure, and the other two are the slanted null lines, corresponding to light signals. In Euclidean geometry, opposite sides of a parallelogram have the same length. But in the spacetime parallelogram, the two parallel vertical lines, which are the clock world

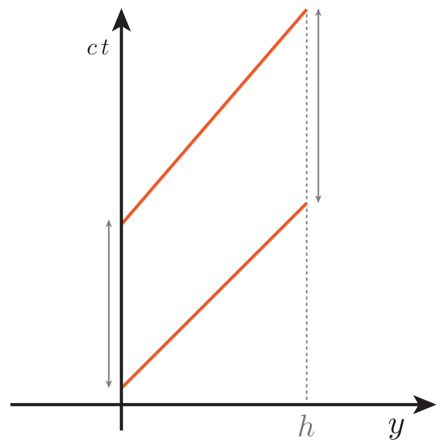

FIGURE 10.3 : Successive light rays sent to a clock at altitude h from a clock on the ground.

lines, have timelike "lengths" (measured by clock readings) $c\Delta t_A$ and $c\Delta t_B$, which are not equal (assuming c remains unchanged). Therefore when gravity is added to spacetime, the geometry becomes non-Euclidean, and so the spacetime has in some sense become *curved.* If there were no gravity the high and low clocks would tick at the same rate, so the two clock worldlines would have the same timelike length, and the parallelogram would obey the rules of Euclidean geometry, corresponding to a *flat*, Euclidean spacetime. *Gravity has curved the spacetime.*

Einstein knew that physics takes place in the arena of four-dimensional spacetime. From the clues that (i) *gravity is related to accelerating reference frames*, (ii) *accelerating reference frames are related to non-Euclidean geometries*, and (iii) *non-Euclidean geometries are related to curved spaces*, Einstein became convinced that gravity is an effect of curved four-dimensional spacetime. His quest to see how gravity is related to geometry ultimately led him to the famous Einstein Field Equations of 1915. These equations showed how the geometry was affected by the particles and fields within spacetime. Then given the geometry, particles that are affected only by gravity (that is, only by the curvature of spacetime) move on the straightest lines possible in the curved spacetime, *i.e.*, , along geodesics in four dimensions. General relativity has been summarized in a nutshell by the American physicist John Archibald Wheeler,

Matter tells space how to curve; space tells matter how to move.

Here we will simply present the solution of Einstein's equations for the curved spacetime surrounding a central spherically-symmetric mass M like the Sun. The solution takes the form of a spacetime **metric** analogous to the Minkowski metric of special relativity corresponding to flat spacetime. In spherical coordinates (r, θ, φ), the flat Minkowski metric is

$$ds^2 = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{10.4}$$

Remember that one can use this to measure Euclidean distance. For example, to compute the distance between two points measured simultaneously, we take $dt = 0$ and integrate $l = \int ds$ along the line joining the two points.

Consider the setup of a probe of mass m in the vicinity of a source mass $M \gg m$. In our Newtonian language, we have the reduced mass $\mu \simeq m$ and we focus on the dynamics of the probe. The curved spherically-symmetric metric around a mass M is the **Schwarzschild metric**

$$ds^2 = -(1-2\mathcal{M}/r)c^2dt^2 + (1-2\mathcal{M}/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{10.5}$$

where $\mathcal{M} \equiv GM/c^2$. Note that as $M \to 0$ the spacetime becomes flat.

In general relativity a particle subject to nothing but gravity moves on **geodesics** in four-dimensional spacetime, analogous to the geodesics on two-dimensional curved surfaces like the spherical surfaces discussed in Chapter 3. The interval obeys $ds^2 = -c^2d\tau^2$ along a timelike worldline, just as in flat spacetime as described in Chapter 2, and timelike geodesics in the four-dimensional Schwarzschild geometry are found by making stationary the proper time along the path,

$$I = c\int d\tau. \tag{10.6}$$

From spherical symmetry we know that the geodesics will be in a plane, which (as usual) we take to be the equatorial plane $\theta = \pi/2$, leaving the degrees of freedom r and φ. Therefore we seek to find the paths that make stationary the integral

$$\begin{aligned} I &= c\int d\tau = \int \sqrt{(1-2\mathcal{M}/r)c^2dt^2 - (1-2\mathcal{M}/r)^{-1}dr^2 - r^2d\varphi^2} \\ &= \int \sqrt{(1-2\mathcal{M}/r)c^2\dot{t}^2 - (1-2\mathcal{M}/r)^{-1}\dot{r}^2 - r^2d\dot{\varphi}^2}\, d\tau \end{aligned} \tag{10.7}$$

where $\dot{t} = dt/d\tau$, $\dot{r} = dr/d\tau$, etc. Note that while in non-relativistic physics the time t is an independent variable, and not a coordinate, in relativistic physics the time t has become one of the coordinates, and the independent variable is the proper time τ read by a clock carried along with the moving particle. This calculus of variations problem looks exactly like the principle of stationary action, with τ replacing t as the independent variable, and a Lagrangian that is the square-root integrand.

There is a way to simplify this form before making calculations. Note first of all that (since $ds^2 = -c^2 d\tau^2$)

$$(1 - 2\mathcal{M}/r)c^2\dot{t}^2 - (1 - 2\mathcal{M}/r)^{-1}\dot{r}^2 - r^2 d\dot{\varphi}^2 = c^2, \tag{10.8}$$

a constant along the world-line of the particle. This fact can be used to help show that making stationary the integral in equation (10.7) is equivalent to making stationary the same integral, with the same integrand but with the square root removed (See Problem 6.44). So our principle of stationary action becomes

$$\delta S = \delta \int L \, d\tau = 0 \tag{10.9}$$

where the effective Lagrangian is

$$L = (1 - 2\mathcal{M}/r)c^2\dot{t}^2 - (1 - 2\mathcal{M}/r)\dot{r}^2 - r^2\dot{\varphi}^2. \tag{10.10}$$

Of the three coordinates, t and φ are cyclic, so the corresponding generalized momenta $p = \partial L/\partial \dot{q}$ are conserved, giving us two first integrals of motion,

$$\begin{aligned} p^t &= \frac{\partial L}{\partial \dot{t}} = -2c^2(1 - 2\mathcal{M}/r)\dot{t} \equiv -2c^2\mathcal{E} \\ p^\varphi &= \frac{\partial L}{\partial \dot{\varphi}} = 2r^2\dot{\varphi} \equiv 2\mathcal{L} \end{aligned} \tag{10.11}$$

where $\mathcal{E}$ and $\mathcal{L}$ are constants. For the third first-integral we see that L is not an explicit function of the independent variable τ, so the analog of the Hamiltonian is conserved. However, it turns out that this is equivalent to equation (10.8)), already a first integral of motion. Then we can eliminate $\dot{t}$ and $\dot{\varphi}$ in equation (10.8 using the two other first integrals, giving

$$\dot{r}^2 - \frac{2\mathcal{M}c^2}{r} + \frac{\mathcal{L}^2}{r^2}\left(1 - \frac{2\mathcal{M}}{r}\right) = c^2(\mathcal{E}^2 - 1). \tag{10.12}$$

This looks more familiar if we divide by two, multiply by the mass m of the orbiting particle, recall that $\mathcal{M} \equiv GM/c^2$, and let $\mathcal{L} \equiv \ell/m$:

$$\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} + \frac{\ell^2}{2mr^2}\left(1 - \frac{2GM}{rc^2}\right) = \frac{mc^2}{2}(\mathcal{E}^2 - 1) \equiv E \tag{10.13}$$

which has the form of a one-dimensional conservation of energy equation (!)

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r) \tag{10.14}$$

where

$$U_{\text{eff}}(r) \equiv -\frac{GMm}{r} + \frac{\ell^2}{2mr^2}\left(1 - \frac{2GM}{rc^2}\right). \tag{10.15}$$

The first two terms of $U_{\text{eff}}(r)$, which are by far the largest, are *exactly the same as the corresponding effective potential for Newtonian gravity* (7.27) with $\mu \simeq m$![2] Note then that Einstein gravity's predictions for our probe's orbit can be approximated with those of Newtonian gravity when

$$\frac{GM}{rc^2} \ll 1\ ; \tag{10.16}$$

that is for small source masses, or large distances from the source. In short, for "weak gravity". In fact, Einstein gravity can become important in other regimes as well, ones involving even weak gravity under the right circumstances.

In Newtonian mechanics the first term $-GMm/r$ in equation (10.15) is the gravitational potential and the second term $\ell^2/2mr^2$ is the centrifugal potential. The *third* term is new and obviously relativistic, since it involves the speed of light. It has the effect of diminishing the centrifugal potential for small r, and can make the centrifugal term *attractive* rather than *repulsive*, as shown in Figure 10.4. This effect cannot be seen for the Sun or most stars, however, because their radii are larger than the radius at which the effective potential turns around and takes a nosedive at small r, and $U_{\text{eff}}(r)$ is only valid in the vacuum *outside* the central mass. The relativistic term can have a small but observable effect on the inner planets, however, as we will show in the next example.

EXAMPLE 10-3: The precession of Mercury's perihelion

[2]If this were not true, the theory would be dead in the water, because we know that Newtonian gravitation is extremely accurate, at least within the solar system.

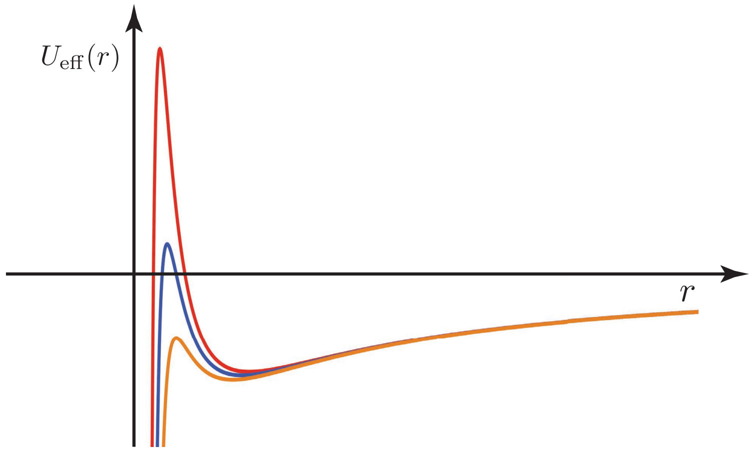

FIGURE 10.4 : Effective potential for the Schwarzschild geometry.

By the end of the nineteenth century astronomers knew there was a problem with the orbit of Mercury. In the Sun's inertial frame, the perihelion of Mercury's orbit does not keep returning to the same spot. The perihelion slowly *precesses*, so that each time Mercury orbits the Sun the perihelion occurs slightly later than it did on the previous revolution. The main reason for this is that the other planets pull slightly on Mercury, so the force it experiences is not purely central. Very accurate methods were worked out to calculate the total precession of Mercury's perihelion caused by the other planets, and although the calculations explained most of the precession, Mercury actually precesses by 43 seconds of arc per century more than the calculations predicted. Einstein was aware of this discrepancy when he worked on his general theory, and was intensely curious whether the effects of relativity might explain the 43 seconds/century drift.

We have already shown that the conservation equations we derived from the geodesics of the Schwarzschild geometry differ slightly from those for the nonrelativistic Kepler problem. Could the extra term in the effective potential cause the precession?

Begin with the conservation equations

$$E = \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} + \frac{\ell^2}{2mr^2}\left(1 - \frac{2GM}{rc^2}\right) \quad \text{and} \quad \mathcal{L} = \ell/m = r^2\dot{\varphi}. \tag{10.17}$$

Using the chain rule, and again defining the inverse radius $u = 1/r$ as coordinate,

$$\dot{r} \equiv \frac{dr}{d\tau} = \frac{dr}{du}\,\frac{du}{d\varphi}\,\frac{d\varphi}{d\tau} = -\frac{1}{u^2}\,u'\,\frac{\ell}{mr^2} = -\frac{\ell}{m}u' \tag{10.18}$$

where $u' \equiv du/d\varphi$. Substituting this result into equation (10.17) gives

$$E = \frac{\ell^2}{2m}(u'^2 + u^2) - GMmu - \frac{GM\ell^2}{mc^2}u^3. \tag{10.19}$$

Then differentiating with respect to φ, we get a second-order differential equation for the orbital shape $u(\varphi)$,

$$u'' + u = \frac{GMm^2}{\ell^2} + \frac{3GM}{c^2}u^2, \tag{10.20}$$

which is the same equation we found for the nonrelativistic Kepler problem, except for the second term on the right, which makes the equation nonlinear. We don't have to solve the equation exactly, however, because the new term is very small. We can solve the problem to sufficient accuracy using what is called first-order perturbation theory. Let $u = u_0 + u_1$, where u_0 is the solution of the linear equation without the new relativistic term, and u_1 is a new (small) contribution due to the relativistic term. Our goal is to find the small function u_1, to see whether the corrected solution leads to a precession of Mercury's orbit. Substituting $u = u_0 + u_1$ into equation (10.20),

$$u_0'' + u_0 + (u_1'' + u_1) = \frac{GMm^2}{\ell^2} + \frac{3GM}{c^2}(u_0 + u_1)^2. \tag{10.21}$$

The function u_0 already obeys the nonrelativistic Kepler equation

$$u_0'' + u_0 = \frac{GMm^2}{\ell^2}, \tag{10.22}$$

so the part left over is

$$u_1'' + u_1 = \frac{3GM}{c^2}(u_0 + u_1)^2. \tag{10.23}$$

Even the quantity $(3GM/c^2)u_0^2$ is already very small, so we neglect the u_1 function on the right, which would produce a doubly-small term. That leaves

$$u_1'' + u_1 = \frac{3GM}{c^2}u_0^2 \tag{10.24}$$

to first-order accuracy, which is a *linear* differential equation. We already know u_0 from the Kepler problem:

$$u_0 = A + B\cos\varphi = A(1 + \epsilon\cos\varphi) \tag{10.25}$$

where ϵ is the eccentricity of the orbit and $A \equiv [a(1-\epsilon^2)]^{-1}$. Therefore the u_1 equation becomes

$$u_1'' + u_1 = \frac{3GM}{c^2}A^2(1 + \epsilon\cos\varphi)^2 = C^2[1 + 2\epsilon\cos\varphi + \frac{\epsilon^2}{2}(1 + \cos 2\varphi)] \tag{10.26}$$

where we have set $C^2 \equiv (3GM/c^2)A^2$ and used the identity $\cos^2\varphi = (1/2)(1+\cos 2\varphi)$. This gives us three linearly independent terms on the right,

$$u_1'' + u_1 = C^2[(1 + \epsilon^2/2) + 2\epsilon\cos\varphi + \frac{\epsilon^2}{2}\cos 2\varphi.] \tag{10.27}$$

Note that this equation is *linear*, and that its general solution is the sum of the solution of the homogeneous equation (with zero on the right), and a particular solution of the full equation. We do not need the solution of the homogeneous equation, however, because it is the same as that for the u_0 equation, so contributes nothing new. And (because of the linearity of the equation) the particular solution of the full equation is just the sum of the particular solutions due to each of the three terms on the right, taken one at a time. That is, $u_1 = u_1^{(1)} + u_1^{(2)} + u_1^{(3)}$, where[3]

$$\begin{aligned}
u_1'' + u_1 &= C^2(1+\epsilon^2/2), && \text{with solution } \ u_1^{(1)} = C^2(1+\epsilon^2/2) \\
u_1'' + u_1 &= C^2(\epsilon^2/2)\cos 2\varphi, && \text{with solution } \ u_1^{(3)} = -(\epsilon^2 C^2/6)\cos 2\varphi \\
u_1'' + u_1 &= 2C^2\epsilon\cos\varphi, && \text{with solution } \ u_1^{(2)} = \epsilon C^2 \varphi \sin\varphi
\end{aligned} \tag{10.28}$$

Altogether, the new contribution to the solution in first order is

$$u_1 = C^2[1 + \epsilon^2/2 - (\epsilon^2/6)\cos 2\varphi + \epsilon\varphi\sin\varphi]. \tag{10.29}$$

The only term here that can cause a perihelion precession is the $\varphi\sin\varphi$ term, the so-called **secular term**, since it is the only term that does not return to where it began after a complete revolution, *i.e.*, , as $\varphi \to \varphi + 2\pi$. The other terms can cause a slight change in shape, but not a precession. So including the secular term together with the zeroth-order terms,

$$u = u_0 + u_1 = A(1+\epsilon\cos\varphi) + C^2\epsilon\varphi\sin\varphi. \tag{10.30}$$

The perihelion corresponds to the minimum value of r, or the maximum value of u, so at perihelion,

$$\frac{du}{d\varphi} = 0 = -A\epsilon\sin\varphi + C^2\epsilon(\sin\varphi + \varphi\cos\varphi), \tag{10.31}$$

which has a solution at $\varphi = 0$, but *not* at $\varphi = 2\pi$. So we look for a solution at $\varphi = 2\pi + \delta$ for some small δ. For small δ, $\sin(2\pi+\delta) = \sin\delta \cong \delta$ and $\cos(2\pi+\delta) = \cos\delta \cong 1$ to first order in δ. Therefore at the end of one revolution, equation (10.31) gives

$$0 = -A\epsilon\delta + C^2\epsilon[\delta + (2\pi + \delta)]. \tag{10.32}$$

However, the δ's in the square brackets are small compared with 2π, so must be neglected for consistency, since C^2 is already very small. Thus we find that

$$\delta = 2\pi C^2/A = 2\pi\left(\frac{3GM}{c^2}\right)\frac{1}{a(1-\epsilon^2)} = \frac{6\pi GM}{c^2 a(1-\epsilon^2)}. \tag{10.33}$$

The data for Mercury's orbit is $a = 5.8\times10^{10}$ m, $\epsilon = 0.2056$, and $M = M_{\rm Sun} = 2.0\times10^{30}$ kg. The result is

$$\delta = 5.04\times10^{-7} \text{ radians/revolution}. \tag{10.34}$$

[3]See Problem 6.40 to work out the solutions.

We can convert this result to seconds of arc/century, using the facts that Mercury orbits the Sun every 88 days and that there are 60 × 60 = 3600 seconds of arc in one degree,

$$\begin{aligned}\delta &= 5.04\times 10^{-7}\ \frac{\mathrm{rad}}{\mathrm{rev}}\left(\frac{360\ \mathrm{deg}}{2\pi\ \mathrm{rad}}\right)\left(\frac{3600\ \mathrm{s}}{\mathrm{deg}}\right)\left(\frac{1\ \mathrm{rev}}{88\ \mathrm{d}}\right)\left(\frac{365\ \mathrm{d}}{\mathrm{yr}}\right)\left(\frac{100\ \mathrm{yr}}{\mathrm{cent}}\right)\\ &= 43\ \frac{\text{seconds of arc}}{\text{century}}\ ! \end{aligned} \tag{10.35}$$

After the extraordinary efforts and frequent frustrations leading up to his discovery of general relativity, here was Einstein's payoff. He had successfully explained a well-known and long standing conundrum. Later he wrote to a friend, "For a few days, I was beside myself with joyous excitement." And in the words of his biographer Abraham Pais, "This discovery was, I believe, by far the strongest emotional experience in Einstein's scientific life, perhaps in all his life. Nature had spoken to him. He had to be right."

EXAMPLE 10-4: Magnetic gravity

It is possible to capture the leading relativistic correction to Newtonian gravity using an analogy with electromagnetism. This is because similar symmetry principle – gauge symmetry – underlies both force laws. We start with Maxwell's equations from Chapter 8

$$\begin{aligned}&\nabla\cdot\boldsymbol{E}=4\pi\rho \quad \nabla\times\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}\\ &\nabla\cdot\boldsymbol{B}=0 \quad\ \ \nabla\times\boldsymbol{B}=\frac{4\pi}{c}\boldsymbol{J}+\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\ ;\end{aligned} \tag{10.36}$$

with the Lorentz force law on a probe of charge Q given by

$$\boldsymbol{F}_{\mathrm{em}}=Q\boldsymbol{E}+\frac{Q}{c}\boldsymbol{v}\times\boldsymbol{B}\ . \tag{10.37}$$

We already encountered the Coulomb fields that solve Maxwell's equations

$$\boldsymbol{E}=\frac{q}{r^2}\hat{\boldsymbol{r}}\quad ,\quad \boldsymbol{B}=0\ ; \tag{10.38}$$

and noted the similarity between the corresponding electrostatic force law and newtonian gravity

$$\boldsymbol{F}_{\mathrm{em}}=\frac{q\,Q}{r^2}\hat{\boldsymbol{r}}\quad ,\quad \boldsymbol{F}_{\mathrm{g}}=-G\frac{m\,M}{r^2}\hat{\boldsymbol{r}}\ . \tag{10.39}$$

We are now ready to go beyond Newtonian gravity. Inspired by the full form of (10.36) and (10.37), we propose a **magnetic gravitational vector field $\boldsymbol{b}$** such that we have the equations

$$\begin{aligned} \nabla \cdot \boldsymbol{g} = -4\pi G\,\rho_{\mathrm{m}} \quad & \nabla \times \boldsymbol{g} = -\frac{1}{c}\frac{\partial \boldsymbol{b}}{\partial t} \\ \nabla \cdot \boldsymbol{b} = 0 \qquad & \nabla \times \boldsymbol{b} = -\frac{4\pi G}{c}\boldsymbol{J}_m + \frac{1}{c}\frac{\partial \boldsymbol{g}}{\partial t}\ ; \end{aligned} \tag{10.40}$$

where ρ_{m} is volume mass density of some source mass distribution, and $\boldsymbol{J}_{\mathrm{m}}$ is the mass current density. And the force law on a probe of mass M and velocity $\boldsymbol{v}$ is defined as

$$\boldsymbol{F}_{\mathrm{g}} = M\boldsymbol{g} + \frac{M}{c}\boldsymbol{v} \times \boldsymbol{b}\ . \tag{10.41}$$

This means we now have a velocity dependent gravitational force arising from the motion of charges! Notice that the new effect is always multiplied by a factor of v/c; hence, it is negligible at small speeds. This modification of Newtonian gravity is indeed correct to linear order in v/c. It can be derived from General Relativity, even though the derivation is a subtle one with regards to truncating the corrections at the v/c linear level. This modified force law can for example be used to understand a gravitational gyroscope precession effect.

In Lagrangian force, we now know how what to add to the Lagrangian of a probe so that we reproduce this force law

$$L \to L + m\int dx_\mu a_\nu \eta_{\mu\nu} \tag{10.42}$$

where a_μ is a new gravitational four-vector potential field defined as

$$a_\mu = (a_0, \boldsymbol{a}) \tag{10.43}$$

with

$$\boldsymbol{g} = -\nabla a_0 - \frac{1}{c}\frac{\partial \boldsymbol{a}}{\partial t} \quad , \quad \boldsymbol{b} = \nabla \times \boldsymbol{a} \tag{10.44}$$

as expected.

EXAMPLE 10-5: Cosmic string

Consider an infinite straight wire of constant linear mass density λ_0 moving along its length at speed V, as shown inFigure 10.5.

It may represent a model for a cosmic string – a theorized configuration of mass in a stringy formation that can arise soon after the Big Bang. We want to find the force on a

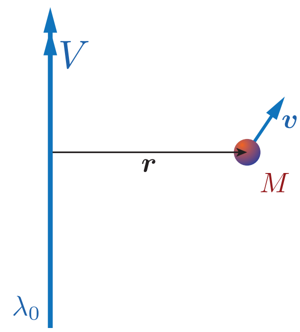

FIGURE 10.5 : An infinite linear mass distribution moves upward with speed V while a probe of mass M ventures nearby.

spaceship of mass M and velocity $\boldsymbol{v}$ that has ventured nearby. Given the cylindrical symmetry of the mass configuration, to find $\boldsymbol{g}$ we can integrate the first equation of equation (10.40) over the volume of a cylinder centered on the cosmic string of say radius r and height h

$$\int \nabla \cdot \boldsymbol{g}\, dV = -4\pi G \int \rho_{\mathrm{m}}\, dV \Rightarrow \oint \boldsymbol{g}.d\boldsymbol{A} = -4\pi G\, \lambda_0\, h \tag{10.45}$$

where we have used Gauss' theorem on the left hand side of the equation, and the resulting surface integral is over the lateral surface area of the cylinder. This gives

$$2\pi r\, h\, g = -4\pi G\, \lambda_0\, h \Rightarrow \boldsymbol{g} = -2\, G\, \frac{\lambda_0}{r} \hat{\boldsymbol{r}}\ . \tag{10.46}$$

To find $\boldsymbol{b}$, we integrate the last equation of equation (10.40) over a disc centered on the cosmic string of radius r through which the mass current protrudes

$$\int (\nabla \times \boldsymbol{b}).d\boldsymbol{A} = -\frac{4\pi\, G}{c} \int \boldsymbol{J}_m.d\boldsymbol{A} \Rightarrow \oint \boldsymbol{b}.d\boldsymbol{l} = -\frac{4\pi\, G}{c} \lambda_0 V \tag{10.47}$$

where we employed Stocke's theorem on the left hand side, and the resulting line integral is over a circle around the cosmic string. The right hand side is simply the mass current, mass per unit time, going through the disc. We then get

$$2\pi r\, b = -\frac{4\pi\, G}{c} \lambda_0 V \Rightarrow \boldsymbol{b} = -\frac{2\, G}{c} \frac{\lambda_0 V}{r} \hat{\boldsymbol{\varphi}}\ . \tag{10.48}$$

We can now compute the force on the probe using equation (10.41). We get

$$\boldsymbol{F}_{\mathrm{g}} = M\boldsymbol{g} + \frac{M}{c} \boldsymbol{v} \times \boldsymbol{b} = -2\, G\, \frac{M\, \lambda_0}{r} \hat{\boldsymbol{r}} - \frac{2\, G}{c^2} \frac{M\, \lambda_0 V}{r} \boldsymbol{v} \times \hat{\boldsymbol{\varphi}}\ . \tag{10.49}$$

The surprising result is that the second term, arising from the magnetic gravity effects, can generate a *repulsive* force since we can flip the sign of V at will by changing the direction of mass flow in the cosmic string. However, this is a small effect compared to the first attractive term that is more of a familiar nature. The reason for this is that the ratio of the second to the first term scales Vv/c^2: we need to move the cosmic string matter and the probe rather fast to get it to generate a repulsive magnetic gravitational force large enough to compete with the omnipresent attractive effect arising from the first term. Indeed, as speeds approach c, the electromangnetic gravity framework we developed in these last two examples breaks down and the full theory of general relativity is needed to reach correct conclusions.

10.2 Relativistic effects and the electromagnetic force

Our discussion of the dynamics of a probe charge in background electromagnetic fields focused on the non-relativistic regime, where the probe's speed stays much smaller than the speed of light. While this is an adequate approximation in many situations, relativity and the speed of light are central to electromagnetism and there are many interesting situations where relativistic effects play a central role. In this section, we revisit the problems of a probe in uniform electric and magnetic fields – presenting a full relativistic treatment to illustrate some of the new features due to Relativity.

EXAMPLE 10-6: Probe in a uniform magnetic field

We start with the example of a probe of charge q and mass m moving in the background of a uniform magnetic field $\boldsymbol{B}$. We already know that the Lagrangian of the *free* relativistic particle leads to the equations of motion

$$\frac{d\boldsymbol{p}}{dt} = \frac{d}{dt}\left(\gamma\, m\, \boldsymbol{v}\right) = 0 \quad , \quad \frac{dE}{dt} = \frac{d}{dt}\left(\gamma\, m\, c^2\right) = 0 \; . \tag{10.50}$$

When we add the term $q\boldsymbol{A}.\boldsymbol{v}/c$ to the Lagrangian, we end up modifying the right hand side of these equations to

$$\frac{d}{dt}\left(\gamma\, m\, \boldsymbol{v}\right) = \frac{q}{c}\boldsymbol{v}\times\boldsymbol{B} \quad , \quad \frac{dE}{dt} = \frac{d}{dt}\left(\gamma\, m\, c^2\right) = 0 \; . \tag{10.51}$$

where $\boldsymbol{B}$ is a constant vector, and the electric field $\boldsymbol{E} = 0$. The conservation of relativistic energy $\gamma\, m\, c^2$ from the second equation tells us the speed of the particle remains constant! This then allows to immediately write the first equation as

$$\gamma\, m \frac{d\boldsymbol{v}}{dt} = \frac{q}{c}\boldsymbol{v} \times \boldsymbol{B} \Rightarrow \frac{d\boldsymbol{v}}{dt} = \left(\frac{q\, c}{E}\right) \boldsymbol{v} \times \boldsymbol{B}\ . \tag{10.52}$$

Since the coefficient $q\, c/E$ is constant with constant energy E, the trajectory of the probe is qualitatively the same as in the non-relativistic case: spiral around the direction of $\boldsymbol{B}$. The difference is that the angular frequency of rotation is now

$$\omega_0 = \frac{q\, c}{E} B\ . \tag{10.53}$$

In the slow speed limit, we have

$$E = \gamma\, m\, c^2 \simeq m\, c^2 + \frac{1}{2} m\, v^2 \simeq m\, c^2 \tag{10.54}$$

yielding the approximate angular frequency

$$\omega_0 \simeq \frac{q\, B}{m\, c}\ , \tag{10.55}$$

which is the expression we obtained earlier. As the speed of the probe approaches that of light, the denominator of equation (10.53) becomes larger than the non-relativistic approximate counterpart. This implies that relativistic effects make the spiraling angular frequency smaller as compared to the incorrect non-relativistic estimate. In the limit where $v \to c$, we have $E \to \infty$, implying the $\omega_0 \to 0$: the probe does not spiral at all. This would then be the case for a massless charged particle, that is if one existed in Nature...

The problem is however more interesting that just portrayed. As is the case in all such problems, we ignored the electromagnetic fields put out by the probe charge. But as a spiraling particle accelerates, it leaks energy by emitting electromagnetic waves and this leakage can become significant for a relativistic particle. The rate of energy loss for a probe undergoing circular motion is found to be

$$\frac{dE}{dt} = \frac{2}{3} \frac{q^2}{m^2 c^3} \gamma^2 \left(\frac{d\boldsymbol{p}}{dt}\right)^2\ . \tag{10.56}$$

Note in particular the γ^2 factor. As the probe speed is increased, the power loss to electromagnetic radiation will quickly become significant enough that the electromagnetic fields from the probe cannot be ignored in determining its dynamics. Such effects are called *back-reaction effects* – the probe's own fields react back onto its dynamics. In this case, as energy is drained out of the probe faster and faster, we expect that the probe will slow down – its spiraling radius getting smaller and smaller. We can write a differential equation for the speed using equation (10.56)

$$\frac{d}{dt}\left(\gamma\, m\, c^2\right) = \frac{2}{3} \frac{q^4}{m^2 c^5} \gamma^2 \left(\boldsymbol{v} \times \boldsymbol{B}\right)^2 = \frac{2}{3} \frac{q^4}{m^2 c^5} \gamma^2 \left(v^2 B^2 - \boldsymbol{v}.\boldsymbol{B}\right) \tag{10.57}$$

where we used the vector identity

$$(\boldsymbol{a} \times \boldsymbol{b}).(\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a}.\boldsymbol{c})(\boldsymbol{b}.\boldsymbol{d}) - (\boldsymbol{a}.\boldsymbol{d})(\boldsymbol{b}.\boldsymbol{c}) \; . \tag{10.58}$$

If we choose a scenario where $\boldsymbol{B}$ is oriented such that $\boldsymbol{v}.\boldsymbol{B} = 0$, perpendicular to the plane of circular motion, we get

$$\frac{d\gamma}{dt} = \frac{2}{3}\frac{q^4 B^2}{m^3 c^7}\gamma^2 v^2 = \frac{v}{c^2}\gamma^3\frac{dv}{dt} \tag{10.59}$$

where in the last step we used the chain rule on $d\gamma/dt$. We then end up with a differential equation for the speed of the probe given by

$$\frac{dv}{dt} = \frac{2}{3}\frac{q^4 B^2}{m^3 c^5} v\sqrt{1 - \frac{v^2}{c^2}} \; . \tag{10.60}$$

One can find an exact solution to this equation and determine $v(t)$. We can see from this that the characteristic decay time for v is set by the combination m^3c^5/q^4B^2. However, this analysis is not fully complete: in arriving at this result, we assumed that the first equation of equation (10.51) holds. But the outgoing electromagnetic waves carry momentum as well and hence we may expect a modification of $d\boldsymbol{p}/dt$ as well.

EXAMPLE 10-7: Probe in crossed uniform electric and magnetic fields

We next look at the problem of a probe particle of mass m and charge q moving in the background of uniform electric *and* magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, that are perpendicular to each other $\boldsymbol{E}.\boldsymbol{B} = 0$. We now want to treat the full relativistic problem where the probe's speed may not be much smaller than that of light. Along the same line of approach as in the previous example, we quickly arrive at the equations of motion

$$\frac{d\boldsymbol{p}}{dt} = q\boldsymbol{E} + \frac{q}{c}\boldsymbol{v} \times \boldsymbol{B} \; . \tag{10.61}$$

Here, we have the momentum $\boldsymbol{p} = \gamma\, m\, \boldsymbol{v}$. This is a complicated problem which can however be simplified significantly through a physical trick: knowing that these equations of motion are Lorentz covariant, we switch to a reference frame where the background field is either entirely electric or entirely magnetic. Let us assume that $E < B$, and leave the opposite case as an exercise to the reader. Looking at the Lorentz transformations of the electric and magnetic fields equation (8.32) and equation (8.33), we see that if we choose a reference frame – call it $\mathcal{O}'$ – moving with the velocity

$$\frac{\boldsymbol{V}}{c} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} \tag{10.62}$$

then the primed electric and magnetic fields become

$$\boldsymbol{E}' = 0 \quad , \quad \boldsymbol{B}' = \frac{1}{\gamma_V}\boldsymbol{B} \ , \tag{10.63}$$

where we used the vector identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c} \ . \tag{10.64}$$

Note that requiring $V < c$ implies that $E < B$. We then have mapped the problem onto the previous example, a probe moving in a uniform *purely* magnetic background field. This is because the equations of motion of the probe in the new reference frame are simply

$$\frac{d\boldsymbol{p}'}{dt} = q\boldsymbol{E}' + \frac{q}{c}\boldsymbol{v}' \times \boldsymbol{B}' = \frac{q}{c}\boldsymbol{v}' \times \boldsymbol{B}' \tag{10.65}$$

since the equations of motion are Lorentz covariant and do not change structural form under change of inertial reference frame. From this reference frame's perspective, the particle then circles around the new $\boldsymbol{B}'$ field which is a factor of γ smaller in magnitude than the original one. Switching back to the unprimed reference frame, we then superimpose on this circling motion a drift perpendicular to $\boldsymbol{E}$ and $\boldsymbol{B}$ given by the velocity $\boldsymbol{V}$. Hence, the relativistic problem is qualitatively very similar to the non-relativistic version – except for a few factors of γ here and there.

10.3 Gauge symmetry

The central symmetry of Maxwell's equations, gauge symmetry, is one of the most profound principles underlying the laws of Nature. Indeed, positing the symmetry as a starting point is enough information to *derive* Maxwell's equations and the Lorentz force law. While this beautiful derivation is simple and elegant, it requires either quantum mechanics or classical field theory to appreciate. Hence, we will not be able to expose the reader to it in this book. However, it is worthwhile noting that such gauge symmetry principles in fact underly all known forces of Nature. As a brief exposition of the workings of this symmetry principle in electromagnetism, we here demonstrate the process of using this symmetry to "fix a gauge" – a technique that can be very handy in practical problem solving in the classical dynamics of probe charges.

EXAMPLE 10-8: Fixing a gauge

The gauge symmetry (8.23) provides a freedom to choose a particular scalar and vector potential for a given electric and magnetic fields. That is, one may have several profiles for the potentials correspond to one physical electromagnetic field profile. Given this freedom, it is customary to **fix the gauge** so as to make the manipulation of the potentials more convenient. For example, we may choose the **static gauge**

$$\phi = 0 \quad \text{Static gauge} \tag{10.66}$$

We can see that this is always possible as follows. Imagine you start with some ϕ and $\boldsymbol{A}$ such that $\phi \neq 0$. Then apply a gauge transformation (8.23) – which we know does not change the electromagnetic fields and the associated physics – such that

$$\phi' = \phi - \frac{1}{c}\frac{\partial f}{\partial t} = 0\ . \tag{10.67}$$

That is, find a function f such that this equation is satisfied. For any ϕ, this equation indeed has a solution f. This is a rather strange *gauge choice* since it sets the electric potential to zero. But this is entirely legal. Note that it does *not* imply that the electric field is zero since we still have

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \tag{10.68}$$

in this gauge choice. Another interesting aspect of gauge fixing is that, typically, the process does not necessarily fix *all* the gauge freedom. In the case of the static gauge, we can still apply a gauge transformation f_0 such that

$$\phi' = 0 \to \phi'' = 0 = \phi' - \frac{1}{c}\frac{\partial f_0}{\partial t} = -\frac{1}{c}\frac{\partial f_0}{\partial t} \tag{10.69}$$

without changing the gauge condition that the electric potential is zero. We see from this expression that this is possible if

$$\frac{\partial f_0}{\partial t} = 0\ ; \tag{10.70}$$

that is, if the gauge transformation function f_0 is time independent. Hence, some of the original freedom of the gauge symmetry is still left even after gauge fixing. This is known as **residual gauge freedom** for obvious reasons.

Another very common gauge choice is the **Coulomb gauge**

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0\ ; \tag{10.71}$$

and a Lorentz invariant version known as the **Lorentz gauge**

$$\frac{\partial A_\mu}{\partial x_\nu}\eta_{\mu\nu} = 0\ . \tag{10.72}$$

In this manner, depending on the details of a problem at hand, one can choose the most convenient gauge choice so that the Lagrangian of a probe particle becomes most easy to tackle.

10.4 Stochastic forces

Consider a particle of dust floating in air, tracing an irregular trajectory as it is bumped around by air molecules around it; or a bacterium swimming against the random forces of water molecules around it; or a minuscule spring with randomly fluctuating spring constant describing the forces between two nanoparticles. Such random forces abound in physics and can significantly affect the dynamics of small enough particles. Fundamentally, these are not new force laws: they are indeed mostly electromagnetic in origin. However, they are characterized by a level of randomness that requires us to treat them differently.

Randomness or the lack of determinism in physics arises from two possible sources:

- Quantum mechanics portrays a probabilistic picture of the world: every state of a system is allowed and all that Nature keeps track of is the likelihood of one state or another. Hence, in quantum mechanics, degrees of freedom fluctuate and we talk about the average values of measurements. Such fluctuations are called **quantum fluctuations**. They are important to the dynamics whenever the characteristic scale of the action – that is, characteristic energy scale times the characteristic timescale – is of order $\hbar$.

- In the context of deterministic classical physics or probabilistic quantum mechanics, a high level complexity in the dynamics of a system can arise from the involvement of a very large number of particles that are interacting through complex non-linear force laws. The 10^{25} molecules of air may each individually be well described by classical deterministic trajectories, yet such a description is in practice impossible and in essence undesirable. The complex interactions between the molecules results in *ergodic dynamics*: an evolution that explores the various possible configurations of the whole system in a pseudo-random (often chaotic) pattern. The setup is then best quantified by average values of observables and statistical fluctuations. Such fluctuations then originate from complexity – whether within classical or quantum mechanics. Their size is determined by a macroscopic parameter known as *temperature*: the higher the temperature of the complex system, the larger the statistical fluctuations from ergodicity. Fluctuations of this

> type are referred to as **thermal fluctuations**. If a probe particle is tracked as it interacts with such an ergodic system – *i.e* a dust particle in air – the probe's dynamics is significantly affected by the background complexity if the energy of the probe is of order of $k_B T$, where k_B is Boltzmann's constant and T is the background's temperature.

Whether quantum or thermal in origin, the effect of fluctuations on the *classical* dynamics of a probe or particle can be described in the same language. Let us focus on the example of a dust particle floating in air – subject to thermal fluctuations. The central idea is that there are two timescales in the problem. The background system – in this case the air molecules – forms a large *thermal reservoir* whose dynamics is not effected much by the dust particle; its dynamics is associated with a very short time scale as compared to the time scale of evolution of the dust particle. The latter on the other hand can be treated classically, but is then subject to randomly and quickly fluctuating forces from the background molecules. We are interested in describing the trajectory of the dust particle in such a setting.

Say we have a dust particle of mass m – moving in one dimension for simplicity – evolving according to the equation of motion

$$m\,\ddot{x} = m\,\dot{v} = -\alpha\,v + \sigma f(t) \tag{10.73}$$

where $\alpha\,v$ is an effective frictional force arising from the particles interaction with the background fluid molecules; and $f(t)$ is a random force of the same origin changing in time very quickly as compared to the evolution of the x coordinate. This problem is known as the Ornstein-Uhlenbeck process; a simpler form of it without the $\alpha\,v$ frictional term is the celebrated Brownian motion problem of Einstein. In fact, through a more fundamental statistic treatment, one can *derive* the
$alpha\,v$ term from the effect of the background fluctuations. Looking at this equation, we are immediately lead to a couple of mathematical puzzles. First, if $f(t)$ is randomly fluctuating, we need to clarify what do we mathematically mean by randomness. Second, we expect $x(t)$ and $v(t)$ to have zigzagging profiles, as shown in Figure 10.6, which would make them very different from the nice differentiable smooth functions we are used to in differential equations. In fact, we may suspect that $x(t)$ and $v(t)$ are not strictly speaking *functions* in the usual sense. Let us tackle each of these issues separately.

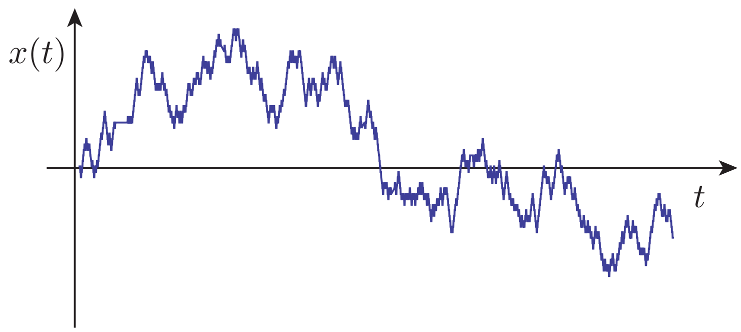

FIGURE 10.6 : A stochastic evolution of the position of particle subject to random forces.

If $f(t)$ is a randomly fluctuating variable, we can talk about its statistical moments to describe it. For example, the average $\overline{f(t)}$ would be a well defined quantity. We may write for example

$$\overline{f(t)} = 0 \tag{10.74}$$

for a force fluctuating about a zero value. Here, by an average we mean a *time average* over timescales much longer than the short timescale associated with the fluctuations. Another quantity that quantifies the randomness of this force is

$$\overline{f(t_0)(f(t_0+t))} = C(t) \ . \tag{10.75}$$

This quantity measures how correlated are the fluctuations over time. Expecting that the system is time translationally invariant on timescales of interest, we have written the right hand side as a function of t only, and not t_0. Typically, $C(t)$ is an exponentially decaying function, implying that as we look at fluctuations at larger separations in time t, they quickly appear more and more uncorrelated. The Fourier transform of $C(t)$ is known as the *spectral density* of the force

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} C(t) dt \ . \tag{10.76}$$

A particularly interesting random force law arises when $S(\omega)$ is a constant independent of ω. We then say that $f(t)$ describes **white noise**: its spectral density is frequency independent. From equation (10.76), this implies we must have

$$C(t) = \delta(t) \ . \tag{10.77}$$

That is, the force fluctuations are totally uncorrelated over time, or the time scale of fluctuations is essentially zero – much smaller than any other timescale in the problem. This is an idealization that is strictly speaking unrealistic but it turns out the be a good approximation in many situations.

Now that we have a rigorous definition of the randomness of $f(t)$, we need to tackle the issue that $x(t)$ and $v(t)$ are not well behaved "functions". We have the equation

$$\dot{v} = -\alpha\, v + \sigma f(t) \tag{10.78}$$

where $f(t)$ is meant to be a random white noise force – which is then expected to lead to a jagged trajectory for the probe. Instead of this equation, we propose to write

$$\frac{dV}{dt} = -\alpha\, V + \sigma f(t) \ . \tag{10.79}$$

with the following caveat: the capitalized variable $V(t)$ represents many velocities of the probe *in an ensemble of many realizations of the evolution*. $V(t)$ is *not* a function, but a placeholder for many possible values of $V(t)$ realized when the experiment is ran many times – all measured at time t. Such an object is called a **stochastic process**; and equation (10.79) is known as a **stochastic differential equation**. It describes the evolution of an ensemble of systems under the influence of a random force law.

Much like regular differential equations, one can develop mathematical machinery to manipulate and solve such stochastic equations. All such techniques can be derived by going back to the premise that a capitalized variable represents a measurement in an ensemble with a given probability distribution inherited from the random force $f(t)$. Then basic statistical methods can be used to compute statistical moments of for example $V(t)$.

An important theorem of stochastic differential equations can help us solve equation (10.79):

Theorem: The stochastic differential equation

$$\frac{dZ(t)}{dt} = a(t)Z(t) + b(t)f(t) + c(t) \tag{10.80}$$

where $f(t)$ is a random noise, and $Z(t)$ is a stochastic process with initial condition

$$Z(0) = C \tag{10.81}$$

where C is a process with a given probability distribution, is solved by

$$Z(t) = \varphi(t)\left(C + \int_0^t \frac{c(s)}{\varphi(s)}ds + \int_0^t \frac{b(s)}{\varphi(s)}f(s)ds\right) \tag{10.82}$$

with

$$\varphi(t) = e^{\int_0^t a(s)ds} \; . \tag{10.83}$$

This is a powerful theorem that can be used to solve many stochastic differential equations. The proof of the theorem is beyond the scope of this book. For now, we want to use it to solve our physics problem described by equation (10.79).

Mapping equation (10.79) onto equation (10.80), we identify

$$Z(t) \to V(t) \quad , \quad a(t) \to -\alpha \quad , \quad b(t) \to \sigma \quad , \quad c(t) \to 0 \; . \tag{10.84}$$

We then have

$$\varphi(t) = e^{-\alpha\, t} \; . \tag{10.85}$$

And we get the full solution

$$V(t) = e^{-\alpha\, t}C + \sigma \int_0^t e^{-\alpha\,(t-s)} f(s)ds \; . \tag{10.86}$$

We can now use the statistical properties of the white noise force given by equation (10.74),(10.75), and (10.77) to compute statistical properties of the probe's velocity process $V(t)$. For example, we immediately get

$$\overline{V(t)} = e^{-\alpha\, t}\overline{C} \; , \tag{10.87}$$

and

$$\begin{aligned} \overline{V(s)V(t)} &= e^{-2\alpha t}\overline{C^2} + \sigma^2 \int_0^t \int_0^s e^{-\alpha(s-s')} e^{-\alpha(t-s'')} \overline{f(s')f(s'')} ds' ds'' \\ &= e^{-2\alpha t}\overline{C^2} + \frac{\sigma^2}{2\,\alpha} e^{-\alpha(t+s)} \left(e^{2\,\alpha \min(t,s)} - 1\right) \; . \end{aligned} \tag{10.88}$$

For concreteness, let us specify a specific boundary condition: let the initial velocity C represent a process with probability distribution

$$\mathrm{Prob}(C) = \delta(c) \ , \tag{10.89}$$

implying we start with zero initial velocity with no statistical spread at all. We then have

$$\overline{C} = 0 \quad , \quad \overline{C^2} = 0 \ . \tag{10.90}$$

This then implies that

$$\overline{V(t)} = 0 \quad , \quad \overline{V(t)^2} = \frac{\sigma^2}{2\,\alpha}\left(1 - e^{-2\,\alpha\,t}\right) \ . \tag{10.91}$$

We can compute higher moments of $V(t)$ to check that $V(t)$ has a gaussian probability profile, and hence is entirely quantified by these first two statistical moments. From this result, we see that as $t \to \infty$, we have

$$\overline{V(t)^2} \to \frac{\sigma^2}{2\,\alpha} \tag{10.92}$$

exponentially in time. Hence, the initial zero velocity of the probe gets "fuzzed out" along a gaussian distribution which, at large times, has a standard deviation of $\sigma/\sqrt{2\,\alpha}$. Note that σ tunes the strength of the random forces, while α tunes the strength of the frictional forces. The effect of the random forces is then to spread the velocity of the probe from zero to a fixed range; and this range is larger from smaller frictional forces!

From these results, we can also determine the statistical properties of the position of the probe. We have

$$X(t) = X(0) + \int_0^t V(s) ds \ . \tag{10.93}$$

Using equation (10.86), we can then easily find $\overline{X(t)}$ and $\overline{X(s)X(t)}$. We leave this problem as an exercise for the reader. The evolution of the position of the probe can be shown to be related to the celebrated random walk problem of statistical physics.

Problems

PROBLEM 10-1: Verify the particular solutions given of the inhomogeneous first-order equations for the perihelion precession, as given in equation (10.19).

PROBLEM 10-2: Find the general-relativistic precession of Earth's orbit around the Sun, in seconds of arc per century. Earth's orbital data is $a = 1.50 \times 10^6$ m and $\epsilon = 0.0167$.

PROBLEM 10-3: The metric of flat, Minkowski spacetime in Cartesian coordinates is $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. Show that the geodesics of particles in this spacetime correspond to motion in straight lines at constant speed.

PROBLEM 10-4: Show from the effective potential corresponding to the Schwarzschild metric that if U_{eff} can be used for arbitrarily small radii, there are actually *two* radii at which a particle can be in a circular orbit. The outer radius corresponds to the usual stable, circular orbit such as a planet would have around the Sun. Find the radius of the inner circular orbit, and show that it is unstable, so that if the orbiting particle deviates slightly outward from this radius it will keep moving outward, and if it deviates slightly inward it will keep moving inward.

PROBLEM 10-5: The geodesic problem in the Schwarzschild geometry is to make stationary the integral

$$I = \int \sqrt{(1 - 2\mathcal{M}/r)c^2\dot{t}^2 - (1 - 2\mathcal{M}/r)^{-1}\dot{r}^2 - r^2d\dot{\theta}^2}\ d\tau.$$

Use this integrand in the Euler-Lagrange equations to show that one obtains exactly the same differential equations in the end if the square root is removed, *i.e.*, , if we make stationary instead the integral

$$I = \int [(1 - 2\mathcal{M}/r)c^2\dot{t}^2 - (1 - 2\mathcal{M}/r)^{-1}\dot{r}^2 - r^2d\dot{\theta}^2]\ d\tau.$$

You may use the fact that $(1 - 2\mathcal{M}/r)c^2\dot{t}^2 - (1 - 2\mathcal{M}/r)^{-1}\dot{r}^2 - r^2d\dot{\theta}^2 = c^2$, a constant along the path of the particle.

PROBLEM 10-6: Show that there are no stable circular orbits of a particle in the Schwarzschild geometry with a radius less than $6GM/c^2$.

PROBLEM 10-7: Show that if the initial condition C of the linear stochastic differential equation introduced in the text has a gaussian distribution, so does the solution of the stochastic differential equation.

PROBLEM 10-8: From the problem of a dust particle in a fluid considered in the text, find $\overline{X(t)}$ and $\overline{X(s)X(t)}$.

PROBLEM 10-9: Find the speed as a function of time $v(t)$ for a relativistic particle of mass m and charge q circling in a uniform magnetic field B while radiating electromagnetic energy.

PROBLEM 10-10: Find the motion of a probe of mass m and charge q in a background of uniform electric and magnetic fields that are perpendicular to each other such that $E > B$.

BEYOND EQUATIONS

William Rowan Hamilton (1805 - 1865)

W. R. Hamilton was born in Dublin. He was a child prodigy, especially in languages; by the age of thirteen he had mastered one language for each year of his life. At thirteen he also entered Trinity College, Dublin, studying classics and science. He made important contributions to mathematics and physics, working particularly on systems of light rays, inventing and applying mathematics to analyze bundles of rays. He was appointed Professor of Astronomy in Dublin at age 22.

During his career, Hamilton worked on classical mechanics and geometrical optics, making important contributions to each and finding a way to unify the two fields. We have already met Hamilton's principle; in this chapter we will display Hamilton's equations and the Hamilton-Jacobi equations. They are not new physics, but they serve to illustrate once again (as with Lagrange's equations) that new ways of looking at older physics can be very fruitful. For example, in the twentieth century Hamilton's methods served as a natural bridge to quantum mechanics.

Hamilton made many other contributions to mathematics, especially the invention of quaternians, which he believed to be his masterpiece. Quaternians involve an algebra of rotations in three-dimensional space in which the commutative law of multiplication does not hold. Although they are now used in a variety of fields, including computer graphics and signal processing, quaternians have not turned out to be as universally useful as Hamilton hoped and expected.

Throughout his career, Hamilton's desire was for his work to bring glory to Ireland. It did; in 2005, the 200th anniversary of Hamilton's birth, the Irish government proudly designated 2005 as the *Hamilton Year, celebrating Irish science.*