Chapter 6

Symmetries and Conservation Laws

At a very fundamental level, physics is about identifying patterns of order in Nature. The field arguably begins with Tycho Brahe (1546-1601) - the first modern experimental physicist - and Johannes Kepler (1571-1630) - the first modern theoretical physicist. In the 16th century, Brahe painstakingly accrued huge quantities of astronomical data about the location of planets and stars with unprecedented accuracy - using impressive observing instruments that he had designed and set up in his castle. Kepler pondered for years over Brahe's long tables until he could finally identify patterns underlying planetary dynamics, summarized in what we now call Kepler's three laws. Later on, Isaac Newton (1643-1727) referred to these achievements, among others, in his famous quote: *"If I have seen further it is only by standing on the shoulders of giants"*.

Since then, physics has always been about identifying patterns in numbers, in measurements. And a pattern is simply an indication of a repeating rule, a constant attribute within seeming complexity, an underlying symmetry. In 1918, Emmy Noether published a seminal work that clarified the deep relations between symmetries and conserved quantities in Nature. In a sense, this work organizes physics into a clear diagram and gives us a bird's eye view of the myriad of branches of the field. Noether's theorem, as it is called, can change the way one thinks about physics in general. It is profound yet simple.

While one can study Noether's theorem in the context of Newton's formulation of mechanics, the methods in that case are quite cumbersome. Studying Noether's theorem using Lagrangian mechanics instead is an excellent demonstration of the power of the new formalism.

In this section we begin by recounting the connections between symmetries and conserved quantities already encountered in Chapter 4. We then use the variational principle to develop a statement of Noether's theorem, and go on to prove it and demonstrate its importance through examples.

6.1 Cyclic coordinates and generalized momenta

Already in Chapter 4 we defined a *cyclic* coordinate q_k as a generalized coordinate *absent* from the Lagrangian. The corresponding generalized velocity $\dot{q}_k$ is present in L, but not q_k itself. For example, a particle free to move in three dimensions x, y, z with uniform gravity in the z direction has Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \tag{6.1}$$

so both x and y are cyclic, but z is not. Then from Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \tag{6.2}$$

it follows that for each of the cyclic coordinates q_k the *generalized momentum*

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k} \tag{6.3}$$

is *conserved*, since $dp_k/dt = 0$. So in this example, both $p_x = m\dot{x}$ and $p_y = m\dot{y}$ are conserved but p_z is not.

Now the interesting question is: *Why* is a particular coordinate cyclic, from a physical point of view? Why are x and y cyclic in our example, but z is not?

The answer is quite clear. The physical environment is invariant under displacements in the x and y directions, but not under vertical displacements. Changes in z mean that we get closer or farther from the ground, but changes in x or y make no difference whatever. Everything looks exactly the same if we displace ourselves horizontally. We say there is a *symmetry* under horizontal displacements, but not under vertical displacements. So through

FIGURE 6.1 : An elliptical galaxy (NGC 1132) pulling on a star at the outer fringes.

Lagrange's equations we see that the momenta p_x, p_y are conserved but p_z is not. Two symmetries, those in the x and y directions, have led to two conserved quantities.

If a generalized coordinate q_i is missing from the Lagrangian, that means there is a symmetry under changes in that coordinate: the physical environment, whether a potential energy or a constraint, is independent of that coordinate. And if the environment possesses the symmetry, the corresponding generalized momentum will be conserved. This is very important, not least because the equation of motion for that coordinate will be only first order rather than second order, so becomes much easier to solve.

EXAMPLE 6-1: A star orbiting a spheroidal galaxy

A particular galaxy consists of an enormous sphere of stars somewhat squashed along one axis, so it becomes spheroidal in shape, as shown in equation (6.1). A star at the outer fringes of the galaxy experiences the general gravitational pull of the galaxy. Are there any conserved quantities in the motion of this star?

We can answer this question by identifying any symmetries in the environment of the star. There are no translational symmetries; any finite translation takes one nearer or farther from the galaxy, so no component of linear momentum is conserved.

There is, however, a symmetry under rotation about the squashed axis of the galaxy; if we imagine rotating about that axis the shape of the galaxy, and therefore its gravitational field, will be unchanged. Therefore we expect that the *angular momentum* p_φ about this axis will be conserved. No other component of angular momentum will be conserved, because if we rotate about any other axis, the galaxy will look different. Therefore we expect conservation of angular momentum about a single axis only, and no conservation of linear momentum in any direction.

Mathematically, using cylindrical coordinates (r, φ, z), the kinetic and gravitational potential energies of the star are

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2) \quad \text{and} \quad U = U(r, z) \tag{6.4}$$

where the symmetric shape of the galaxy means that the gravitational potential depends upon r and z, but not φ. Therefore the Lagrangian $L = T - U$ depends upon all three generalized velocities, but φ is cyclic. The corresponding generalized momentum p_φ, which is in fact the angular momentum about the z axis, is therefore conserved.

EXAMPLE 6-2: A charged particle moving outside a charged rod

An infinite straight dielectric rod is oriented in the z direction, and given a uniform electric charge per unit length. A point charge is free to move outside it. Are there any conserved quantities for the motion of the particle?

This time the environment of the particle has *two* symmetries: a symmetry corresponding to a rotation about the rod axis, and a symmetry corresponding to displacements in the z direction, along the rod axis. If φ is the angle about the rod in the x, y plane, then the generalized momentum p_φ, which is in fact the angular momentum of the particle about the z axis, is conserved, because of symmetry under rotation. The generalized momentum p_z is also conserved, because of symmetry under displacement in the z direction; this is the linear momentum of the particle in the z direction. The other components of linear momentum are not conserved, and the other components of angular momentum are also not conserved, because there is no symmetry under displacements in the x or y direction, or under rotations about any other axis.

Mathematically, the kinetic and potential energies in this case (using cylindrical coordinates) have the form

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2) \quad \text{and} \quad U = U(r). \tag{6.5}$$

Therefore, when the Lagrangian $L = T - U$ is assembled, both φ and z are cyclic, so the corresponding generalized momenta p_φ, p_z are conserved, as we already knew from our more "physical" analysis.

6.2 A less straightforward example

Consider the simple mechanics problem we saw in Example 6 of Chapter 4. Two particles, with masses m_1 and m_2, are constrained to move along a horizontal frictionless rail, as depicted in Figure 4.8. The action for the system is

$$S = \int dt \left(\frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - U(q_1 - q_2)\right) \tag{6.6}$$

with some interaction between the particles described by a potential $U(q_1 - q_2)$ that depends only on the distance between the particles.

Note that neither q_1 nor q_2 is cyclic, so it might seem that there are no symmetries in this problem. However, let us consider a simple *transformation* of the coordinates given by

$$q_1' = q_1 + \mathrm{C} \quad , \quad q_2' = q_2 + \mathrm{C} \tag{6.7}$$

where C is some arbitrary constant. So in this case, we have simply changed the origin of coordinates, which should have no effect on the physics. We then have

$$\dot{q}_1' = \dot{q}_1 \quad , \quad \dot{q}_2' = \dot{q}_2 \; . \tag{6.8}$$

Hence, the kinetic terms in the action are unchanged under this transformation. Furthermore, we also have

$$q_1' - q_2' = q_1 - q_2 \tag{6.9}$$

implying that the potential term is also unchanged. The action then preserves its overall structural form under the transformation

$$S \to \int dt \left(\frac{1}{2}m_1(\dot{q}_1')^2 + \frac{1}{2}m_2(\dot{q}_2')^2 - U(q_1' - q_2')\right) \; . \tag{6.10}$$

This means that the equations of motion, written in the primed transformed coordinates, are identical to the ones written in the unprimed original coordinates. We can then say that the transformation given by (6.7) is a symmetry of our system. Physically, we are simply saying that - since the interaction between the particles depends only on the distance between them - a constant shift of both coordinates leaves the dynamics unaffected. Indeed, we showed in Chapter 4 that in this problem a simple linear transformation of coordinates to center-of-mass and relative coordinates produced a cyclic coordinate, the position of the center of mass. As a result, the momentum of the center of mass, that is the total momentum, is conserved.

It is also useful to consider an *infinitesimal* version of such a transformation. Assume that the constant C is small, C $\to \epsilon$; and we write

$$q'_k - q_k \equiv \delta q_k = \epsilon \tag{6.11}$$

for $k = 1, 2$. We then say that $\delta q_k = \epsilon$ is a symmetry of our system. To make these ideas more useful, we want to extend this example by considering a general class of interesting transformations.

6.3 Infinitesimal transformations

There are two useful types of infinitesimal transformations: *direct and indirect*.

DIRECT TRANSFORMATIONS

A **direct transformation** deforms the generalized coordinates of a setup directly:

$$\delta q_k(t) = q'_k(t) - q_k(t) \equiv \Delta q_k(t, q) \ . \tag{6.12}$$

We use the notation Δ to distinguish a direct transformation. Note that $\Delta q_k(t, q)$ is possibly a function of time and all of the generalized coordinates in the problem. In the previous example, we had the special case where $\Delta q_k(t, q) =$ constant. But it need not be so. Figure 6.2(a) depicts a direct transformation: it is an arbitrary, but *small* shift in the

q_k's. In the example depicted, a particular generalized coordinate q_k is shifted to slightly larger variable values for early times, and then to slightly smaller variable values for later times.

INDIRECT TRANSFORMATIONS

In contrast, an **indirect transformation** affects the generalized coordinates indirectly - through the transformation of the time coordinate:

$$\delta t(t, q) \equiv t' - t \ . \tag{6.13}$$

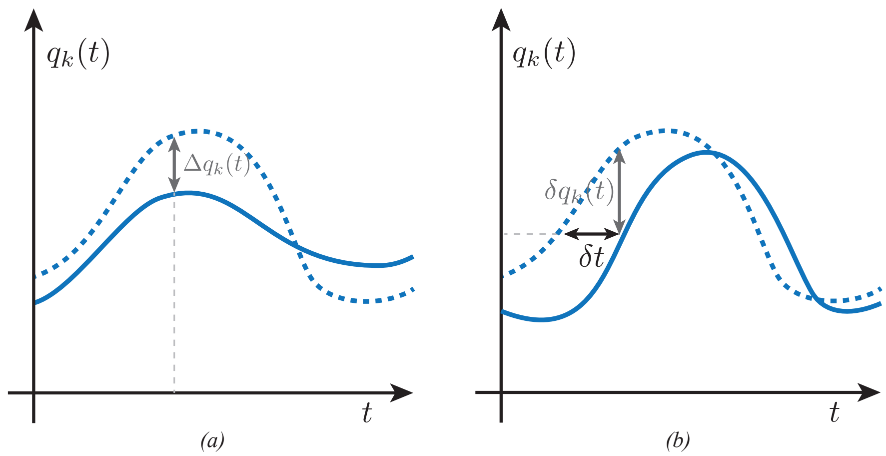

FIGURE 6.2 : The two types of transformations considered: direct on the left, indirect on the right.

Note again that the shift in time is assumed to be small, and this shift can itself depend on time. The small shifts in time can also bring about small shifts in the generalized coordinates; now they have been affected *indirectly*.

$$q_k(t) = q_k(t' - \delta t) \simeq q_k(t') - \frac{dq_k}{dt'}\delta t \simeq q_k(t') - \dot{q}_k \delta t \tag{6.14}$$

where we have used a Taylor expansion in δt to linear order since δt is small. We also have used $dq_k/dt' = dq_k/dt$ since this term already multiplies a power of δt: to linear order in δt, $\delta t\, dq_k/dt' = \delta t\, dq_k/dt \equiv \delta t\, \dot{q}_k$. We then see that shifting time results in a shift in the generalized coordinates

$$\delta q_k = q_k(t') - q_k(t) = \dot{q}\, \delta t(t, q) \ . \tag{6.15}$$

Figure 6.2(b) shows how you can think of this effect graphically: A small change in time has caused a small change in the generalized coordinate $\delta q_k(t)$.

COMBINED TRANSFORMATIONS

In general, we want to consider a transformation that may include *both* direct and indirect pieces. We write

$$\begin{aligned}
\delta q_k &= q_k'(t') - q_k(t) = q_k'(t') + [-q_k(t') + q_k(t')] - q_k(t) \\
&= [q_k'(t') - q_k(t')] + [q_k(t') - q_k(t)] = \Delta q_k(t', q) + \dot{q}\, \delta t(t, q) \\
&= \Delta q_k(t, q) + \dot{q}\, \delta t(t, q) \ .
\end{aligned} \tag{6.16}$$

where in the last line we have equated t and t' since the first term is already linear in the small parameters. To specify a particular transformation, we would then need to provide a set of functions

$$\delta t(t,q) \quad \text{and} \quad \delta q_k(t,q) \ . \tag{6.17}$$

Equation (6.16) then determines $\Delta q_k(t,q)$. For N degrees of freedom, that amounts to $N+1$ functions of time and the q_k's. Let us look at a few examples.

EXAMPLE 6-3: Translations

Consider a single particle in three dimensions, described by the three Cartesian coordinates $x_1 = x$, $x_2 = y$, and $x_3 = z$. We also have the time coordinate $x_0 = c\,t$. An infinitesimal spatial translation can be realized by

$$\delta x_i(t,x) = \epsilon_i \quad , \quad \delta t(t,x) = 0 \Rightarrow \Delta x_i(t,x) = \delta\epsilon_i \tag{6.18}$$

where $i = 1,2,3$, and the ϵ_i's are three small constants. A translation in space is then defined by

$$\{\delta t(t,x) = 0, \ \ \delta x_i(t,x) = \delta\epsilon_i\} \quad \text{Translation in space} \tag{6.19}$$

A translation in time on the other hand would be given by

$$\delta x_i(t,x) = 0 \quad , \quad \delta t(t,x) = \delta\epsilon \Rightarrow \Delta x_i(t,x) = -\dot{x}_i \delta\epsilon \tag{6.20}$$

for constant ε. Notice that for a translation purely in *time*, we require that the *total* shifts in the x_i's - the $\delta x_i(t,x)$'s - vanish. This then generates direct shifts, the Δx_i's, to compensate for the indirect effect on the spatial coordinates from the shifting of the time. A translation in time is then defined by

$$\{\delta t(t,x) = \delta\epsilon, \delta x_i(t,x) = 0\} \quad \text{Translation in time} \tag{6.21}$$

EXAMPLE 6-4: Rotations

To describe rotations, let us consider for simplicity a particle moving in two dimensions. We use the coordinates $x_1 = x$ and $x_2 = y$, and start by specifying

$$\delta t(t,x) = 0 \Rightarrow \delta x_i(t,x) = \Delta x_i(t,x) \ . \tag{6.22}$$

Next, we look at an arbitrary rotation angle θ using (2.21)

$$\begin{pmatrix} x_{1'} \\ x_{2'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ . \tag{6.23}$$

However, we need to focus on an infinitesimal version of this transformation: *i.e.*, we need to consider small angles $\delta\theta$. Using $\cos\delta\theta \sim 1$ and $\sin\delta\theta \sim \delta\theta$ to first order in $\delta\theta$, we then write [1]

$$\begin{pmatrix} x_{1'} \\ x_{2'} \end{pmatrix} = \begin{pmatrix} 1 & \delta\theta \\ -\delta\theta & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ . \tag{6.24}$$

This gives

$$\begin{aligned} \delta x_1(t,x) &= x_{1'}(t) - x_1(t) = \delta\theta\, x_2(t) = \Delta x_1(t,x) \\ \delta x_2(t,x) &= x_{2'}(t) - x_2(t) = -\delta\theta\, x_1(t) = \Delta x_2(t,x) \ . \end{aligned} \tag{6.25}$$

We see we now have a less trivial transformation. Rotations can then be defined from

$$\{\delta t(t,x) = 0, \delta x_i(t,x) = \delta\theta\, \varepsilon_{\mathrm{ij}} x_j(t)\} \quad \text{Spatial rotation} \tag{6.26}$$

where j is summed over 1 and 2. We have also introduced a useful shorthand: $\varepsilon_{\mathrm{ij}}$ is the totally antisymmetric matrix in two dimensions, given by

$$\varepsilon_{11} = \varepsilon_{22} = 0 \quad , \quad \varepsilon_{12} = -\varepsilon_{21} = 1 \ . \tag{6.27}$$

It allows us to write the transformation in a more compact notation.

EXAMPLE 6-5: Lorentz transformations

To find the infinitesimal form of Lorentz transformations, we can start with the general transformation equations (2.15) and take β small. We need to be careful however to keep the leading order terms in β in all expansions. Given our previous example, it is easier to map the

[1] Recall that the Taylor series expansions of $\cos\delta\theta$ and $\sin\delta\theta$ are $\cos\delta\theta = 1 - \delta\theta^2/2! + \delta\theta^4/4! - ...$ and $\sin\delta\theta = \delta\theta - \delta\theta^3/3! + \delta\theta^5/5! - ...$

problem onto a rotation with hyperbolic trig functions and the rapidity ξ, using (2.27) from Chapter 2:

$$\begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,t' \\ x' \\ y' \\ z' \end{pmatrix} , \tag{6.28}$$

For simplicity, let us consider a particle in one dimension, with two relevant coordinates $x_0 = c\,t$ and x_1. We take the rapidity $\xi \to \delta\xi$ to be small and use $\cosh\delta\xi \sim 1$ and $\sinh\delta\xi \sim \delta\xi$ to linear order in $\delta\xi$. Along the same steps of the previous example, we quickly get

$$\delta x_0(t,x) = \delta\xi\, x_1 \quad , \quad \delta x_1(t,x) = \delta\xi\, x_0 \ . \tag{6.29}$$

Using equation (6.16), we then have

$$\Delta x_1(t,x) = \delta x_1(t,x) - \dot{x}_1 \delta t(t,x) = \delta\xi\, c\,t - \frac{\delta\xi}{c}\dot{x}_1 x_1 \ . \tag{6.30}$$

Note that we have

$$\sinh\delta\xi = \gamma\beta \Rightarrow \sinh\delta\xi \sim \delta\xi \sim \beta \ . \tag{6.31}$$

Lorentz transformations can then be written using

$$\{\delta x_0 = \beta\, x_1 \quad , \quad \delta x_1 = \beta\, x_0 \quad , \quad \delta x_2 = \delta x_3 = 0\} \quad \text{Lorentz tansformation} \tag{6.32}$$

where we added the transverse directions to the game as well.

6.4 Symmetry

We define a **symmetry** to be a transformation that leaves the action unchanged in form. In the particular example of two interacting particles on a wire, it was simple to see that the transformation was indeed a symmetry. Now that we have a general class of transformations, we want to find a general condition that can be used to test whether a particular transformation, possibly a complicated one, is or is not a symmetry. We then need to look at how the action changes under a general transformation; for a symmetry, this change should vanish. We start with the usual form for the action

$$S = \int dt\, L(q, \dot{q}, t) \ . \tag{6.33}$$

And we apply a general transformation given by $\Delta q_k(t,q)$ and $\delta t(t,q)$. We then have

$$\delta S = \int \delta(dt)\, L + \int dt\, \delta(L) \ , \tag{6.34}$$

where we used the Leibniz rule of derivation $\delta(ab) = (\delta a)b + a(\delta b)$ since δ is an infinitesimal change. The first term is the change in the *measure* of the integrand

$$\delta(dt) = dt\, \frac{\delta(dt)}{dt} = dt\, \frac{d}{dt}(\delta t) \ , \tag{6.35}$$

where in the last bit we exchanged the order of derivations, since derivations commute. The second term has two pieces,

$$\delta(L) = \Delta(L) + \delta t\, \frac{dL}{dt} \ . \tag{6.36}$$

The first piece is the change in L resulting from its dependence on the q_k's and $\dot{q}_k$'s. Hence, we labeled it as a direct change with a Δ. The second piece is the change in L to linear order in δt due to the change in t. This comes from changes in the q_k's on which L depends, as well as changes in t directly since t can make an explicit appearance in L. This is an identical situation to the linear expansion encountered for q_k in (6.16): there's a piece from direct changes in the degrees of freedom, plus a piece from the transformation of time. We can further write, using multivariable calculus,

$$\Delta(L) = \frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial \dot{q}_k}\Delta(\dot{q}_k) = \frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\left(\Delta q_k\right) \ . \tag{6.37}$$

Note that in the last term, we exchanged the orders of derivations, Δ and d/dt, since they commute. We can now put everything together and write

$$\begin{aligned}
\delta S &= \int dt\, \left(\frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\left(\Delta q_k\right) + \delta t \frac{dL}{dt} + L\,\frac{d}{dt}(\delta t)\right) \\
&= \int dt\, \left(\frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\left(\Delta q_k\right) + \frac{d}{dt}(L\,\delta t)\right) \ .
\end{aligned} \tag{6.38}$$

Given L, $\delta t(t,q)$, and $\Delta q_k(t,q)$, we can substitute these in (6.38) and check whether the expression vanishes. If it does vanish, we conclude that the given transformation $\{\delta t(t,q), \Delta q_k(t,q)\}$ is a symmetry of our system. This shall be our notion of symmetry. A bit later, we will revisit this statement and generalize it further. For now, this is enough to move onto the heart of the topic, *Noether's theorem*.

6.5 Noether's theorem

We start by simply stating the theorem:

For every symmetry $\{\delta t(t,q), \Delta q_k(t,q\}$, there exists a quantity that is conserved under time evolution.

A symmetry implies a conservation law. This is important for two reasons:

1. First, a conservation law identifies a rule in the laws of physics. Virtually everything we have a name for physics - mass, momentum, energy, charge, etc.. - is tied by definition to a conservation law. Noether's theorem then states that fundamental physics is founded on the principle of identifying the symmetries of Nature. If one wants to know all the laws of physics, one needs to ask: what are all the symmetries in Nature. From there, you find conservation laws and associated interesting conserved quantities. You then can study how these conservation laws can be violated. This leads you to equations you can use to predict the future. It's all about symmetries.

2. Second, conservation laws have the form

$$\frac{d}{dt}(\text{Something}) = 0 \Rightarrow \text{Something} = \text{Constant} \ . \tag{6.39}$$

   The 'Something' is typically a function of the degrees of freedom and the first derivatives of the degrees of freedom. The conservation statement then inherently leads to first-order differential equations. First-order differential equations are much nicer than second or higher order ones. Thus, technically, finding the symmetries and corresponding conservation laws in a problem helps a great deal in solving and understanding the physical system.

The easiest way to understand Noether's theorem is to prove it, which is a surprisingly simple exercise.

PROOF OF THE THEOREM

The premise of the theorem is that we have a given symmetry $\{\delta t(t,q), \Delta q_k(t,q\}$. This then implies, using (6.38), that

$$\delta S = 0 = \int dt \left(\frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}(\Delta q_k) + \frac{d}{dt}(L\,\delta t) \right) \ . \tag{6.40}$$

Note that we know that this equation is satisfied for *any* set of curves $q_k(t)$ by virtue of $\{\delta t(t,q), \Delta q_k(t,q\}$ constituting a symmetry. And now comes the crucial step: what if the $q_k(t)$'s satisfy the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \; ? \tag{6.41}$$

Of all possible curves $q_k(t)$, we pick the ones that satisfy the equations of motion. Given this additional statement, we can rearrange the terms in δS such that

$$0 = \int dt \, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\,\delta t\right) \; . \tag{6.42}$$

Since the integration interval is arbitrary, we then conclude that

$$\frac{d}{dt}\left(Q\right) = 0 \tag{6.43}$$

with

$$Q \equiv \frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\,\delta t \; ; \tag{6.44}$$

We have a conserved quantity Q called the **Noether charge**. Note a few important points:

- We used the equations of motion to prove the conservation law. However, we did not use the equations of motion to conclude that a particular transformation is a symmetry. The symmetry exists at the level of the action for any $q_k(t)$. The conservation law exists for physical trajectories that satisfy the equations of motion.

- The proof identifies explicitly the conserved quantity through (6.44). Knowing L, $\delta t(t,q)$, and $\Delta q_k(t,q)$, this equation tells us immediately the conserved quantity associated with the given symmetry.

This proof also highlights a route to generalize the original definition of symmetry. All that was needed was to have

$$\delta S = \int dt \frac{d}{dt}\left(K\right) \tag{6.45}$$

where K is some function that you would find out by using (6.38). If K turns out to be a constant, we would get $\delta S = 0$ and we are back to the situation at hand. However, if K is non-trivial, we would get

$$\delta S = \int dt \frac{d}{dt}(K) = \int dt\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\,\delta t\right) \ . \tag{6.46}$$

This now means that

$$\frac{d}{dt}(Q - K) = 0 \tag{6.47}$$

and hence the conserved quantity is $Q-K$ instead of Q. Since the interesting conceptual content of a symmetry is its associated conservation law, we want to turn the problem on its back: we want to define a symmetry through a conservation statement. So, here's a revised, more general statement:

$$\{\delta t(t,q), \Delta q_k(t,q\} \text{ is a symmetry if } \delta S = \int dt\, \frac{dK}{dt} \text{ for some } K \ . \tag{6.48}$$

Noether's theorem then states that for every such symmetry, there is a conserved quantity given by $Q - K$.

To summarize, here's then the general prescription:

1. Given a Lagrangian L and a candidate symmetry $\{\delta t(t,q), \Delta q_k(t,q\}$, use (6.38) to find δS. If $\delta S = \int dt\, dK/dt$ for some K you are to determine, $\{\delta t(t,q), \Delta q_k(t,q\}$ is indeed a symmetry.

2. If $\{\delta t(t,q), \Delta q_k(t,q\}$ was found to be a symmetry with some K, we can find an associated conserved quantity $Q - K$, with Q given by (6.44).

Let us look at a few examples.

EXAMPLE 6-6: Space translations and momentum

We start with the spatial translation transformation in the i^{th} direction, as in our previous example (6.18)

$$\delta x_i(t,x) = \delta\epsilon_i \quad , \quad \delta t(t,x) = 0 \Rightarrow \Delta x_i(t,x) = \delta\epsilon_i \ . \tag{6.49}$$

We next need a Lagrangian to test this transformation against. Consider first a particle moving freely in the i^{th} direction; its Lagrangian is

$$L = \frac{1}{2} m \dot{x}_i \dot{x}_i \quad (\text{no sum on } i) \tag{6.50}$$

Substitute (6.49)) and (6.50) into (6.38:

$$\begin{aligned} \delta S &= \int dt \left(\frac{\partial L}{\partial q_k} \Delta q_k + \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt}(\Delta q_k) + \frac{d}{dt}(L \delta t) \right) \\ &= \int \left(0 \cdot \Delta q_k + \dot{x}_i \frac{d}{dt} \delta \epsilon_i + \frac{d}{dt}(0) \right) = 0, \end{aligned} \tag{6.51}$$

since $\delta\epsilon_i$ is a constant. Therefore $K =$ Constant, so we have a symmetry at hand. To find the associated conserved charge, we use (6.44) and find

$$Q_i = m \dot{x}_i \delta \epsilon_i \quad (\text{no sum on } i). \tag{6.52}$$

We then have three charges for the three possible directions for translation. An overall additive or multiplicative constant is arbitrary since it does not affect the statement of conservation $\dot{Q}_i = 0$. Writing the conserved quantities as P_i instead, we state

$$P_i = m \dot{x}_i \tag{6.53}$$

i.e., **momentum** is the Noether charge associated with the symmetry of spatial translational invariance. If we have a physical system set up on a table and we notice that we can move the table by any amount in any of the three spatial directions *without* affecting the dynamics of the system, we can conclude that there is a quantity - called momentum by definition - that remains constant in time.

We can then try to find the conditions under which this symmetry, and hence conservation law, is violated. For example, we can add a simple harmonic oscillator potential to the Lagrangian,

$$L = \frac{1}{2} m \dot{x}_i \dot{x}_i - \frac{1}{2} k\, x_i x_i \quad (\text{no sum on } i). \tag{6.54}$$

Using (6.38), we now get

$$\delta S = \int dt\, (-k\, x_i \delta \epsilon_i) \neq \int \frac{d}{dt}(K) \ . \tag{6.55}$$

Hence, momentum is not conserved and we can write

$$\dot{P}_i \neq 0 \Rightarrow \dot{P}_i \equiv F_i \tag{6.56}$$

thus introducing the concept of **force**. We now see that the second law of Newton is nothing but a reflection of the existence or non-existence of a certain symmetry in Nature.

Newton's third law is also related to this idea: action-reaction pairs cancel each other, so that the total force on an isolated system is zero and hence the total momentum is conserved.

To see this, look back at the two particle system on a rail described by the Lagrangian (6.6)). Using once again (6.38 with $\delta t = 0$ and $\delta q_i = \delta\epsilon$, we get

$$\delta S = \int dt \left(-\frac{\partial U}{\partial q_1}\delta\epsilon - \frac{\partial U}{\partial q_2}\delta\epsilon \right) = 0 \tag{6.57}$$

since

$$\frac{\partial U(q_1 - q_2)}{\partial q_1} = -\frac{\partial U(q_1 - q_2)}{\partial q_2} \ . \tag{6.58}$$

The forces on each particle are $-\partial U/\partial q_1$ and $-\partial U/\partial q_2$, which are equal in magnitude but opposite in sign since we have $U(q_1 - q_2)$ - note the relative minus sign between q_1 and q_2. These two forces form the action-reaction pair. The cancelation of forces arises because of the dependence of the potential and force on the distance $q_1 - q_2$ between the particles - which is what makes the problem translationally invariant as well. We now see that the third law is intimately tied to the statement of translation symmetry. The associated conserved quantity determined from (6.44) is

$$P_i = m_1\dot{q}_1 + m_2\dot{q}_2 \ ; \tag{6.59}$$

Thus, the *total* momentum of the system is our Noether charge and it is conserved. ———

EXAMPLE 6-7: Time translation and the Hamiltonian

Next, let us consider time translational invariance. Due to its particular usefulness, we want to treat this example with greater generality. We focus on a system with an arbitrary number of degrees of freedom labeled by q_k's with a general Lagrangian $L(q, \dot{q}, t)$. We propose the transformation

$$\delta t = \delta\epsilon \quad , \quad \delta q_k = 0 \ . \tag{6.60}$$

Hence, the degrees of freedom are left unchanged, but the time is shifted by a constant $\delta\epsilon$. This means that we need a direct shift

$$\delta q_k = 0 = \Delta q_k + \dot{q}_k \delta t = \Delta q_k + \delta\epsilon\, \dot{q}_k \Rightarrow \Delta q_k = -\delta\epsilon\, \dot{q}_k \tag{6.61}$$

to compensate for the indirect change in q_k induced by the shift in time. We then use (6.38) to find the condition for time translational symmetry

$$\delta S = \int dt \left(-\delta\epsilon \dot{q}_k \frac{\partial L}{\partial q_k} - \delta\epsilon \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \delta\epsilon \frac{dL}{dt} \right) \ . \tag{6.62}$$

But we also know

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k \ . \tag{6.63}$$

We then get

$$\delta S = \int dt\, \frac{\partial L}{\partial t}\delta\epsilon \ . \tag{6.64}$$

In general, since L depends on the q_k's as well, we need to consider the more restrictive condition for symmetry $\delta S = 0$, *i.e.*, we have $K =$ constant. This implies that we have time translational symmetry if

$$\frac{\partial L}{\partial t} = 0 \tag{6.65}$$

i.e., if the Lagrangian does not depend on time *explicitly*. If this is the case, we then have a conserved quantity given by (6.44)

$$Q = -\delta\epsilon\, \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \delta\epsilon\, L \ . \tag{6.66}$$

Dropping an overall constant term $-\delta\epsilon$ and rearranging, we write

$$Q \to \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \ . \tag{6.67}$$

which we recognize as the **Hamiltonian** H of the system, already introduced in Chapter 4. Consider for example the two-particle problem in one dimension described by the Lagrangian (4.56). One then finds

$$H = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 + U(q_1 - q_2); \tag{6.68}$$

which is obviously also the energy $E = T + U$ in this case. As shown in Appendix A of Chapter 4, the Hamiltonian H is equal to the energy E if the transformation $\boldsymbol{r} = \boldsymbol{r}(q_k, t)$ from the generalized coordinates to Cartesian coordinates, is not an explicit function of time. Hence, if we note that the results of a series of experiments do not depend upon when the experiments are performed, we expect that the Hamiltonian of the system will be conserved. In many circumstances $H = E$, so we often say that invariance under displacements in time indicates that the energy of the system is conserved. We can then look at dissipative effects involving loss of energy and learn about new physics through the non-conservation of energy. ———

EXAMPLE 6-8: Rotations and angular momentum

Consider the problem of a non-relativistic particle of mass m moving in two dimensions, on a plane labeled by $x_1 = x$ and $x_2 = y$. We add to the problem a central force and write a Lagrangian of the form

$$L = \frac{1}{2} m \dot{x}_i \dot{x}_i - U(x_i x_i) \quad \text{summed on } i \text{, with } i = 1,\, 2 \tag{6.69}$$

Note that the potential depends only on the distance of the particle from the origin of the coordinate system $\sqrt{x_i x_i}$. Rotations are described by (6.26)). We can then use (6.38 to test for rotational symmetry

$$\begin{aligned} \delta S &= \int dt \left(\frac{\partial U}{\partial x_i} \Delta x_i + m\, \dot{x}_i \frac{d}{dt}(\Delta x^i) \right) \\ &= \int dt \left(2\, U' \delta\theta\, \varepsilon_{\mathrm{ij}} x^i x^j + m\, \delta\theta\, \varepsilon_{\mathrm{ij}} \dot{x}_i \dot{x}^j \right) \; . \end{aligned} \tag{6.70}$$

In the second line, we wrote

$$\frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial u} \frac{\partial u}{\partial x_i} = \frac{\partial U}{\partial u} (2\, x_i) = 2U' x_i \tag{6.71}$$

where $u \equiv x_i x_i$ and we used the chain rule. We now want to show that $\delta S = 0$. Focus on the first term in (6.70)

$$2\, U' \delta\theta\, \varepsilon_{\mathrm{ij}} x_i x_j = 2\, U' \delta\theta\, \varepsilon_{12} x_1 x_2 + 2\, U' \delta\theta\, \varepsilon_{21} x_2 x_1 = 2\, U' \delta\theta\, x_1 x_2 - 2\, U' \delta\theta\, x_2 x_1 = 0 \, . \tag{6.72}$$

Let us do this one more time, with more grace and elegance:

$$\begin{aligned} & 2\, U' \delta\theta\, \varepsilon_{\mathrm{ij}} x_i x_j \\ & = 2\, U' \delta\theta\, \varepsilon_{\mathrm{ji}} x_j x_i \\ & = 2\, U' \delta\theta\, \varepsilon_{\mathrm{ji}} x_i x_j \\ & = -2\, U' \delta\theta\, \varepsilon_{\mathrm{ij}} x_i x_j \; . \end{aligned} \tag{6.73}$$

In the second line, we just relabeled the indices $i \to j$ and $j \to i$: since they are summed indices, it does not matter what they are called. In the third line, we use the fact that multiplication is commutative $x_j x_i = x_i x_j$. Finally, in the third line, we used the property $\varepsilon_{\mathrm{ij}} = -\varepsilon_{\mathrm{ji}}$ from (6.27). Hence, we have shown

$$2\, U' \delta\theta\, \varepsilon_{\mathrm{ij}} x_i x_j = -2\, U' \delta\theta\, \varepsilon_{\mathrm{ij}} x_i x_j \Rightarrow 2\, U' \delta\theta\, \varepsilon_{\mathrm{ij}} x_i x_j = 0 \; . \tag{6.74}$$

The key idea is that $\varepsilon_{\mathrm{ij}}$ is antisymmetric in its indices while $x_i x_j$ is symmetric under the same indices. The sum of their product then cancels. The same is true for the second term in (6.70)

$$m\, \delta\theta\, \varepsilon_{\mathrm{ij}} \dot{x}_i \dot{x}_j = -m\, \delta\theta\, \varepsilon_{\mathrm{ij}} \dot{x}_i \dot{x}_j \Rightarrow m\, \delta\theta\, \varepsilon_{\mathrm{ij}} \dot{x}_i \dot{x}_j = 0 \; . \tag{6.75}$$

We will use this trick occasionally later on in other contexts. We thus have shown that our system is rotational symmetric

$$\delta S = 0 \Rightarrow K = \text{Constant} \ . \tag{6.76}$$

We can then determine the conserved quantity using (6.44)

$$Q = \frac{\partial L}{\partial \dot{x}_i} \Delta x_i = m\, \dot{x}_i \delta\theta\, \varepsilon_{\mathrm{ij}} x_j \ . \tag{6.77}$$

Dropping a constant term $\delta\theta$, we write

$$l = m\, \varepsilon_{\mathrm{ij}} \dot{x}_i x_j = m\, (\dot{x}_1 x_2 - \dot{x}_2 x_1) = (\boldsymbol{r} \times m\, \boldsymbol{v})_z \tag{6.78}$$

i.e., this is the z-component of the *angular momentum* of the particle; it points perpendicular to the plane of motion as expected. Rotational symmetry implies conservation of angular momentum. Rotation about the z axis corresponds to angular momentum along the z axis. -

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EXAMPLE 6-9: Galilean Boosts

How about a Lorentz transformation? Special relativity *requires* the Lorentz transformation as a symmetry of any physical system: it is not a question of whether it is a symmetry of a given system; it better be! We could then use (6.38), with Lorentz transformations, as a *condition* for sensible Lagrangians. Noether's theorem can be used to construct theories consistent with the required symmetries. In general, an experiment would identify a set of symmetries in a newly discovered system. Then the theorist's task is to build a Lagrangian that describes the system; and a good starting point would be to assure that the Lagrangian has all the needed symmetries. We now see the power of Noether's theorem: it allows us to mold equations and theories to our needs.

Returning to Lorentz transformations, let us look at an explicit example and find the associated conserved charge. We consider a relativistic system, say a free relativistic particle with action

$$S = -mc^2 \int d\tau = -mc^2 \int \sqrt{1 - \frac{\dot{x}_i \dot{x}_i}{c^2}} \ . \tag{6.79}$$

A particular Lorentz transformation is given by (6.32)) which we can then substitute in (6.38 to show that we have a symmetry. We will come back to this case in the next example. For now, we consider instead the small speed limit, *i.e.*, we consider a non-relativistic system with Galilean symmetry. We take a single free particle in one dimension with Lagrangian

$$L = \frac{1}{2} m \dot{x}^2 \ . \tag{6.80}$$

The expected symmetry is Galilean, given by (1.1) which we quote again for convenience

$$x = x' + Vt' \quad , \quad y = y' \quad , \quad z = z' \quad , \quad t = t' \ . \tag{6.81}$$

We can then write the infinitesimal version quickly

$$\begin{aligned} &\delta t = \delta y = \delta z = 0 \quad , \quad \delta x = -V\,t \\ &\Rightarrow \Delta t = \Delta y = \Delta z = 0 \text{ and } \Delta x = -V\,t \ . \end{aligned} \tag{6.82}$$

Note that we want to think of this transformation near $t \sim 0$, the instant in time the two origins coincide, to keep it as a small deformation for any V. Using (6.38), we then get

$$\delta S = \int dt\, m\, V\, \dot{x} \ . \tag{6.83}$$

Before we panic from the fact that this did not vanish, let us remember that all that is needed is $\delta S = \int dt\, dK/dt$ for some K. That is indeed the case

$$\delta S = \int dt\, \frac{d}{dt}\left(m\, V\, x\right) \ . \tag{6.84}$$

We then have

$$K = m\, V\, x + \text{Constant} \ . \tag{6.85}$$

The system then has Galilean symmetry. We look at the associated Noether charge using (6.44)

$$Q = \frac{\partial L}{\partial \dot{x}} \Delta x = m\, \dot{x}\, V\, t \ . \tag{6.86}$$

But this is not the conserved charge: $Q - K$ is the conserved quantity

$$Q - K = m\, \dot{x}\, V\, t - m\, V\, x = \text{Constant} \ . \tag{6.87}$$

Rewriting things, we have a simple first order differential equation

$$\dot{x}\, t - x = \text{Constant} \ . \tag{6.88}$$

Integrating this gives the expected linear trajectory $x(t) \propto t$. Unlike momentum, energy, and angular momentum, this conserved quantity $Q - K$ is not given its own glorified name. Since Galilean (or the more general Lorentz) symmetry is expected to be prevalent in all systems, this does not add any useful distinguishing physics ingredient to a problem. Perhaps if we were to discover a fundamental phenomenon that breaks Galilean/Lorentz symmetry, we could then revisit this conserved quantity and study its non-conservation. For now, this conserved charge gets relegated to second rate status... ——————————

EXAMPLE 6-10: Lorentz invariance

We now come back to the case of Lorentz invariance. We saw in Chapter 2 that Lorentz transformations are a kind of rotations of space and time – using hyperbolic trigonometric functions. Let us start by writing infinitesimal Lorentz transformations (6.32) in a more compact form along (6.26). We write

$$\delta x_\mu = \beta \varepsilon_{\mu\nu} x_\lambda \eta_{\nu\lambda} \tag{6.89}$$

where $\varepsilon_{\mu\nu}$ is non-zero only for

$$\varepsilon_{01} = +1 \quad , \quad \varepsilon_0 = -1 \tag{6.90}$$

as for the case of rotations. Note that the metric $\eta_{\nu\lambda}$ factor is there to flip the sign of the time transformation to the correct form. This transformation corresponds to an infinitesimal boost in the x direction. More generally, Lorentz boosts can be in any of three spatial directions. By symmetry, we can guess that the most general infinitesimal Lorentz boost *and* spatial rotation must have the form

$$\delta x_\mu = \omega_{\mu\nu} x_\lambda \eta_{\nu\lambda} \quad \text{Lorentz boosts and rotations} \tag{6.91}$$

where the $\omega_{\mu\nu}$'s are a set of boost or rotational angle parameters and $\omega\mu\nu$ is antisymmetric

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \ . \tag{6.92}$$

For example, if all $\omega\mu\nu$'s are zero except for $\omega_{12} = -\omega_{21} = \delta\theta$, we get rotation about the z axis as in (6.26); on the other hand, if all $\omega\mu\nu$'s are zero except for $\omega_{01} = -\omega_{10} = \beta$, we get (6.89). Correspondingly, $\omega_{23} = -\omega_{32}$ rotates about the x axis, $\omega_{13} = -\omega_{31}$ rotates about the y axis, $\omega_{02} = -\omega_{20}$ boosts in the y direction, and $\omega_{03} = -\omega_{30}$ boosts in the z direction. We have then a total of three rotations and three boosts as needed, packaged into one antisymmetric matrix $\omega_{\mu\nu}$.

Notice that we cannot have *general* Lorentz invariance without rotational invariance: otherwise, we can always break the Lorentz boost symmetry by simply realigning the axes by a rotation. To see where the requirement of Lorentz invariance leads us to, consider the general change in the action given by (6.38)

$$\delta S = \int d\tau \left(\frac{\partial \mathcal{L}}{\partial q_k} \Delta q_k + \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \frac{d}{d\tau} (\Delta q_k) + \frac{d}{d\tau} (\mathcal{L}\, \delta\tau) \right) \tag{6.93}$$

where we have traded the t dependence to a dependence on proper time τ instead, expecting that this will make the analysis cleaner. This means that the action is

$$S = \int d\tau\, \mathcal{L}; \tag{6.94}$$

and the dot in $\dot{q}_k$ denotes a derivative with respect to τ. We can rearrange (6.93) into

$$\delta S = \int d\tau \left(\frac{\partial \mathcal{L}}{\partial q_k} \delta q_k + \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \frac{d}{d\tau} (\delta q_k) \right) \tag{6.95}$$

where we have used $\delta\tau = 0$ and $\delta q_k = \Delta q_k + \dot{q}_k \delta\tau = \Delta q_k$. Noting that $q_k \to x_\mu$ for the spatial components, we can then write instead

$$\delta S = \int d\tau \left(\frac{\partial \mathcal{L}}{\partial x_\mu} \delta x_\mu + \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} \frac{d}{d\tau} \left(\delta x_\mu \right) \right) . \tag{6.96}$$

We now can read off the condition of Lorentz invariance. For example, consider *any* function of $s \equiv x_\mu x_\nu \eta_{\mu\nu}$; that is, say $\mathcal{L}$ contains a piece $\mathcal{L} \to \mathcal{L}(x_\mu x_\nu \eta_{\mu\nu})$. We then get

$$\frac{\partial \mathcal{L}}{\partial x_\mu} \delta x_\mu = \frac{\partial \mathcal{L}}{\partial s} \frac{\partial s}{\partial x_\mu} \delta x_\mu = \frac{\partial \mathcal{L}}{\partial s} 2\, x_\nu \eta_{\mu\nu} \delta x_\mu = \frac{\partial \mathcal{L}}{\partial s} 2\, x_\nu \eta_{\mu\nu} \omega_{\mu\alpha} \eta_{\alpha\beta} x_\beta = 0 . \tag{6.97}$$

The last statement follows because the product of x's is symmetric while the $\omega_{\mu\nu}$ matrix is antisymmetric: the argument is the same one as we saw in (6.73). This mechanism of cancellation is so general that it will apply whenever the Lagrangian depends on mathematical objects that are properly Lorentz contracted: there are no "hanging" Lorentz indices, and pairs of Lorentz indices μ and ν are always summed over through $\eta_{\mu\nu}$. We can then easily build Lagrangians $\mathcal{L}$ out of x_μ's and $dx_\mu/d\tau$'s, as long as we contract indices properly. The result is guaranteed to be Lorentz invariant.

EXAMPLE 6-11: Sculpting Lagrangians from symmetry

Let us take the previous example one step further. What if we were to *stipulate* a particular symmetry and ask for all possible Lagrangians that fit the mold? To be more specific, consider a one-dimensional system with one degree of freedom, denoted as $q(t)$. We want to ask: what are all possible theories that can describe this system with the following conditions: they are to be Galilean invariant and invariant under time translations. The Galilean symmetry is obtained from (6.82) in the previous example

$$\delta t = 0 \text{ and } \delta x = \Delta x = -V\, t . \tag{6.98}$$

Substituting this into (6.38), we get

$$\delta S = \int dt \left(-\frac{\partial L}{\partial x} V\, t - \frac{\partial L}{\partial \dot{x}} V \right) = \int dt \frac{d}{dt} (K) . \tag{6.99}$$

The question is then to find the most general L that does the job for some K. This means we need

$$\frac{\partial L}{\partial x} t + \frac{\partial L}{\partial \dot{x}} \propto \frac{d}{dt} (K) \to \frac{d}{dt} \left(\widetilde{K} \right) , \tag{6.100}$$

where we absorbed the proportionality constant inside the yet-to-be-determined function $\widetilde{K}$. Note also that we are *not* allowed to use the equations of motion! Using the chain rule with $\widetilde{K}(t, x, \dot{x}, \ddot{x}, \ldots)$, we can write

$$\frac{d}{dt}\left(\widetilde{K}\right) = \frac{\partial \widetilde{K}}{\partial t} + \frac{\partial \widetilde{K}}{\partial x}\dot{x} + \frac{\partial \widetilde{K}}{\partial \dot{x}}\ddot{x} + \ldots \tag{6.101}$$

Comparing this to (6.100), we see that we need $\widetilde{K}(t, x)$ - a function of t and x only - since we know L is a function of t, x, and $\dot{x}$ only; hence we have

$$\frac{\partial L}{\partial x}t + \frac{\partial L}{\partial \dot{x}} = \frac{\partial \widetilde{K}}{\partial t} + \frac{\partial \widetilde{K}}{\partial x}\dot{x} \ . \tag{6.102}$$

We want a general form for L, yet $\widetilde{K}(t, x)$ is also arbitrary. Since the right-hand side is linear in $\dot{x}$, so must be the left-hand side. This implies we need L to be a quadratic polynomial in $\dot{x}$

$$L = f_1(t, x)\dot{x}^2 + f_2(t, x)\dot{x} + f_3(t, x) \tag{6.103}$$

with three unknown functions $f_1(t, x)$, $f_2(t, x)$, and $f_3(t, x)$. Looking at the $\partial L/\partial x$ term, we can immediately see that we need $f_1(t, x) = C_1$, a constant independent of x and t: otherwise we generate a term quadratic in $\dot{x}$ that does not exist on the right-hand side of (6.102). Our Lagrangian now looks like

$$L = C_1\dot{x}^2 + f_2(t, x)\dot{x} + f_3(t, x) \ . \tag{6.104}$$

But we forgot about the time translational symmetry. That, we know, requires that $\partial L/\partial t = 0$. We then should write instead

$$L = C_1\dot{x}^2 + f_2(x)\dot{x} + f_3(x) \ . \tag{6.105}$$

But the second term is irrelevant to the dynamics. This is because L will appear in the action integrated over time; and this second term can be integrated out

$$\int dt\, f_2(x)\dot{x} = \int dt\, \frac{d}{dt}\left(F_2(x)\right) = \left. F_2(x)\right|_{\text{boundaries}} \tag{6.106}$$

for some function $F_2(x) = \int^x d\xi\, f_2(\xi)$. Hence the term does not depend on the shape of paths plugged into the action functional and cannot contribute to the statement of stationarity - otherwise known as the equation of motion. We are now left with

$$L \to C_1\dot{x}^2 + f_3(x) \ . \tag{6.107}$$

The condition (6.102) on L now looks like

$$\frac{\partial f_3(x)}{\partial x}t + 2\,C_1\dot{x} = \frac{\partial \widetilde{K}}{\partial t} + \frac{\partial \widetilde{K}}{\partial x}\dot{x} \ . \tag{6.108}$$

Picking out the $\dot{x}$ dependences on either sides, this implies

$$2\,C_1 = \frac{\partial \widetilde{K}}{\partial x} \quad , \quad \frac{\partial f_3(x)}{\partial x}t = \frac{\partial \widetilde{K}}{\partial t} \ . \tag{6.109}$$

Since we know that

$$\frac{\partial^2 \widetilde{K}}{\partial x \partial t} = \frac{\partial^2 \widetilde{K}}{\partial t \partial x} \ , \tag{6.110}$$

differentiating the two equations in (6.109) leads to the condition

$$\frac{\partial^2 f_3(x)}{\partial x^2} = 0 \Rightarrow f_3(x) = C_2 x + C_3 \tag{6.111}$$

for some constants C_2 and C_3. The Lagrangian now looks like

$$L = C_1 \dot{x}^2 + C_2 x \ , \tag{6.112}$$

where we set $C_3 = 0$ since a constant shift of L does not affect the equation of motion. We can now solve for $\widetilde{K}$ as well if we wanted to using (6.109)

$$\widetilde{K} = 2\, C_1 x + \frac{C_2}{2} t^2 + \text{Constant} \ . \tag{6.113}$$

Now, let us stare back at the important point, equation (6.112). We write the constants C_1 and C_2 as $C_1 = m/2$ and $C_2 = -m\, g$

$$L = \frac{1}{2} m \dot{x}^2 - m\, g\, x \ . \tag{6.114}$$

With tears of joy in our eyes, we just showed that the most general Galilean and time translation-invariant mechanics problem in one dimension necessarily looks like a particle in uniform gravity. We were able to *derive* the canonical kinetic energy term and gravitational potential from a symmetry requirement. This is just a hint at the power of symmetries and conservation laws in physics. Indeed, all the known forces of Nature can be derived from first principles from symmetries[2]!

6.6 Some comments on symmetries

Let us step back for a moment and comment on several additional issues about symmetries and conservation laws.

- If a system has N degrees of freedom, then the typical Lagrangian leads to N second-order differential equations (provided the Lagrangian depends on at most first derivatives of the variables). If we were lucky enough to solve these equations, we would parameterize the solution

[2] See for example *Quantum Field Theory in a Nutshell* by A. Zee, and references therein.

with $2\,N$ constants related to the boundary conditions. If our system has M symmetries, it would then have M conserved quantities. Each of the symmetries leads to a first-order differential equation, and hence a total of M constants of motion. In total, the conservation equations will gives has $2\,M$ constants to parameterize the solution with: M from the constants of motion, and another M for integrating the first-order equations. These $2\,M$ constants would necessarily be related to the $2\,N$ constants mentioned earlier. What if we have $M = N$? The system is then said to be *integrable*. This means that all one needs to do is to write the conservation equations and integrate them. We need not even stare at any second-order differential equations to find the solution to the dynamics. In general, we will have $M \leq N$. And the closer M is to N, the easier will be to understand the given physical problem. As soon as a good physicist sees a physics problem, he or she would first count degrees of freedom; then he or she would instinctively look for the symmetries and associated conserved charges. This immediately lays out a strategy how to tackle the problem based on how many symmetries one has versus the number of degrees of freedom.

- Noether's theorem is based on infinitesimal transformations: symmetries that can be built up from small incremental steps of deformations. There are other symmetries in Nature that do not fit this prescription. For example, discrete symmetries are rather common. Reflection transformations, *e.g. time reflection $t \to -t$ or discrete rotations of a lattice*, can be very important for understanding the physics of a problem. Noether's theorem does not apply to these. However, such symmetries are also often associated with conserved quantities. Sometimes, these are called *topological* conservation laws.

- Infinitesimal transformations can be catalogued rigorously in mathematics. A large and useful class of such transformations fall under the topic of *Lie groups* of Group theory. The Lie group catalogue (developed by E. Cartan) is exhaustive. Many if not all of the entries in the catalogue are indeed realized in Nature in various physical systems. In addition to Lie groups, physicists also flirt with other more exotic symmetries such as supersymmetry. Albeit mathematically very beautiful, unfortunately none of these will be of interest to us in mechanics.

Problems

PROBLEM 6-1:

Consider a particle of mass m moving in two dimensions in the $x-y$ plane, constrained to a rail-track whose shape is describe by an arbitrary function $y = f(x)$. There is NO GRAVITY acting on the particle.

(a) Write the Lagrangian in terms of the x degree of freedom only.

(b) Consider some general transformation of the form

$$\delta x = g(x) \quad , \quad \delta t = 0 \; ; \tag{6.115}$$

where $g(x)$ is an arbitrary function of x. Assuming that this transformation is a symmetry of the system such that $\delta S = 0$, show that this implies the following differential equation relating $f(x)$ and $g(x)$

$$\frac{g'}{g} = -\frac{1}{2(1+f'^2)} \frac{d}{dx} \left(f'^2 \right) \; ; \tag{6.116}$$

where prime stands for derivative with respect to x (not t).

(c) Write a general expression for the associated conserved charge in terms of $f(x)$, $g(x)$, and $\dot{x}$.

(d) We will now specify a certain $g(x)$, and try to find the laws of physics obeying the prescribed symmetry; *i.e.*, for given $g(x)$, we want to find the shape of the rail-track $f(x)$. Let

$$g(x) = \frac{g_0}{\sqrt{x}} \; ; \tag{6.117}$$

where g_0 is a constant. Find the corresponding $f(x)$ such that this $g(x)$ yields a symmetry. Sketch the shape of the rail-track. (*Hint*: $h(x) = f'^2$.)

PROBLEM 6-2: One of the most important symmetries in Nature is that of *scale Invariance*. This symmetry is very common (e.g. arises whenever a substance undergoes phase transition), fundamental (e.g. it is at the foundation of the concept of *renormalization group* for which a physics Nobel Prize was awarded in 1982), and entertaining (as you will now see in this problem).

Consider the action

$$S = \int dt \; \sqrt{h} \dot{q}^2 \tag{6.118}$$

of two degrees of freedom $h(t)$ and $q(t)$.

(a) Show that the following transformation (known as a scale transformation or dilatation)

$$\delta q = \alpha q \quad , \quad \delta h = -2\alpha h \quad , \quad \delta t = \alpha t \tag{6.119}$$

is a symmetry of this system.

(b) Find the resulting constant of motion.

PROBLEM 6-3:

A massive particle moves under the acceleration of gravity and without friction on the surface of an inverted cone of revolution with half angle α.

(a) Find the Lagrangian in polar coordinates.

(b) Provide a complete analysis of the trajectory problem. Use Noether charges when useful.

PROBLEM 6-4:

Consider a particle of mass m moving in two dimensions in the $x-y$ plane, constrained to a rail-track whose shape is describe by an arbitrary function $y = f(x)$. There is NO GRAVITY acting on the particle.

(a) Write the Lagrangian in terms of the x degree of freedom only.

(b) Consider some general transformation of the form

$$\delta x = g(x) \quad , \quad \delta t = 0 \; ; \tag{6.120}$$

where $g(x)$ is an arbitrary function of x. Assuming that this transformation is a symmetry of the system such that $\delta S = 0$, show that this implies the following differential equation relating $f(x)$ and $g(x)$

$$\frac{g'}{g} = -\frac{1}{2(1+f'^2)}\frac{d}{dx}\left(f'^2\right) \; ; \tag{6.121}$$

where prime stands for derivative with respect to x (not t).

(c) Write a general expression for the associated conserved charge in terms of $f(x)$, $g(x)$, and $\dot{x}$.

(d) We will now specify a certain $g(x)$, and try to find the laws of physics obeying the prescribed symmetry; *i.e.*, for given $g(x)$, we want to find the shape of the rail-track $f(x)$. Let

$$g(x) = \frac{g_0}{\sqrt{x}} \; ; \tag{6.122}$$

where g_0 is a constant. Find the corresponding $f(x)$ such that this $g(x)$ yields a symmetry. Sketch the shape of the rail-track. (HINT: $h(x) = f'^2$.)

PROBLEM 6-5: For the two body central force problem with a Newtonian potential, the effective two dimensional orbit dynamics can be describe by the Lagrangian

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{k}{r} = \frac{1}{2}\mu\left(\dot{x}^2 + \dot{y}^2\right) + \frac{k}{\sqrt{x^2+y^2}} \tag{6.123}$$

where $k > 0$, and we have chosen to write things in cartesian coordinates.

(a) Show that the equations of motion become

$$\mu\ddot{x} = -k\frac{x}{(x^2+y^2)^{3/2}} \quad , \quad \mu\ddot{y} = -k\frac{y}{(x^2+y^2)^{3/2}} \; . \tag{6.124}$$

(b) Consider the rotation

$$\delta x = \alpha\, y \quad , \quad \delta y = -\alpha\, x \quad , \quad \delta t = 0 \tag{6.125}$$

for small α. Show that this is a symmetry of the action.

PROBLEM 6-6: For the previous problem, show that the conserved Noether charge associated with the symmetry (6.125) is indeed the angular momentum $|\boldsymbol{r} \times (\mu \boldsymbol{v})|$, which is naturally entirely in the z direction.

PROBLEM 6-7: The two body central force problem we have been dealing with in the last two problem also has another unexpected and amazing symmetry. Consider the transformation

$$\delta x = -\frac{\beta}{2}\mu\, y\, \dot{y} \quad , \quad \delta y = \frac{\beta}{2}\mu\, (2\, x\, \dot{y} - y\, \dot{x}) \quad , \delta t = 0 \tag{6.126}$$

for some constant β. This horrific velocity dependent transformation is a symmetry if and only if the equations of motion (6.124) are satisfied - unlike other symmetries we've seen where the equations of motion need not be satisfied. It is said that it is an *on-shell symmetry*. Show that the change in the Lagrangian resulting from this transformation is given by

$$\delta L = \beta\, \mu\, k\, \frac{d}{dt}\left(\frac{x}{\sqrt{x^2+y^2}}\right) \ . \tag{6.127}$$

Hence, it's a total derivative and generates a symmetry under our generalized definition of a symmetry. (HINT: You will need to use (6.124) to get at this result.)

PROBLEM 6-8: For the previous problem, show that the conserved charge associated with the symmetry is

$$Q \propto \mu^2 x \dot{y}^2 - \mu^2 y \dot{x} \dot{y} - \mu k \frac{x}{\sqrt{x^2+y^2}} \ . \tag{6.128}$$

PROBLEM 6-9: The hidden symmetry of the previous few problems is part of a two fold transformation - one of which given by (6.126), and another similar one that we have not shown; together, they result in the conservation of a vector known as the Laplace-Runge-Lenz vector

$$\boldsymbol{A} = \mu \boldsymbol{v} \times (\boldsymbol{r} \times \mu \boldsymbol{v}) - \mu k \frac{\boldsymbol{r}}{r} \tag{6.129}$$

Show that (6.128) is the x-component of this more general vector quantity. (HINT: You may find it useful to use the Lagrange identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$.)

PROBLEM 6-10: Show using (6.129) that

$$\frac{d\boldsymbol{A}}{dt} = 0 \tag{6.130}$$

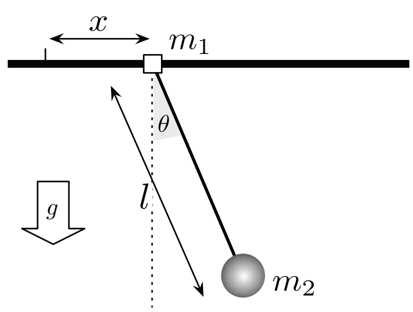

FIGURE 6.3 : Sliding pendulum.

Draw an elliptical orbit in the x-y plane and show on it the Laplace-Runge-Lenz vector $\boldsymbol{A}$. The existence of this conserved vector quantity is the reason why one can smoothly deform ellipses into a circle without changing the energy of the system. Mathematically, this additional hidden symmetry implies that the Newtonian problem is equivalent to a free particle on a three dimensional sphere embedded in an abstract four dimensional world. It is believed that this is a mathematical accident; no physical significance of this fourth dimension has yet been identified...

PROBLEM 6-11: Consider a simple pendulum of mass m_2 and arm length l having its pivot on a point of support of mass m_1 that is free to move horizontally on a frictionless rail.

(a) Find the Lagrangian of the system in terms of the two degrees of freedom x and θ shown on the figure. Do *NOT* assume small displacements. (b) Identify two symmetries and the two corresponding conservation laws. Write two first order differential equations that describe the dynamics of the two degrees of freedom x and θ. Correspondingly, write one nasty integral for $\theta(t)$.

PROBLEM 6-12: Consider the Lagrangian $L' = m\,\dot{x}\,\dot{y} - k\,x\,y$ for a particle free to move in two dimensions, where x and y are Cartesian coordinates, and m and k are constants. (a) Find the equations of motion for the system. (b) Confirm that the answer to part (a) is the same if we were to use instead the Lagrangian for the Harmonic Oscillator $L = (1/2)m(\dot{x}^2 + \dot{y}^2) - (1/2)k(x^2 + y^2)$. (c) Show that L and L' do *not* differ by a total derivative!

Johannes Kepler (1571 - 1630)

Johannes Kepler was born in a small town in Swabia, now part of southwestern Germany. He attended the University of Tübingen, where he studied mathematics, astronomy, Greek, and Hebrew. He then obtained a position teaching mathematics in Graz, now in Austria. Kepler was convinced that the Copernican model of the Sun and planets was basically correct, with the Sun at the center, not the official view at that time. To explain planetary orbit radii, he showed that (to a pretty good approximation) the five regular solids (tetrahedon, cube, octahedron, etc.) fit neatly between spheres whose radii are those of the six planets known in his day. He presented these ideas and others in his book *Mysterium Cosmographicum*.

Kepler sent a copy of his book in 1600 to the great Danish astronomer Tycho Brahe, who was then working in Prague; Tycho hired Kepler to be his assistant. Tycho had made a huge number of careful measurements of planetary positions, so Kepler gained access to all this wonderful data. Tycho died a year later, and Kepler replaced him as Imperial Mathematician. Kepler declared "War on Mars", using Tycho's data to understand the motion of Mars about the Sun. His "war" required years of extremely careful hand calculations, which led to his first and second laws. This work he published in a 1609 book, *Astronomia Nova*.

Kepler moved to Linz (now in Austria), and wrote a second book, *Harmonices Mundi*, which presented what we now call his third law. He also spent a great deal of time (successfully) defending his mother against charges of witchcraft. Kepler represents a remarkable bridge between the world of mysticism and geometrical idealism and the newer world of using careful observations to understand nature.