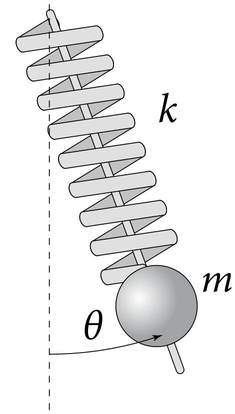
Project 1 A bead of mass m is attached to one end of a spring with force constant k, equilibrium length r_0, and negligible mass. The bead and spring are strung onto a rod of negligible mass. One end of the rod (and the other end of the spring) are hinged at a fixed point, so the system can oscillate in a vertical plane.



(a) Using generalized coordinates r and θ, write the Lagrangian of the system.

(b) Find Lagrange's equations.

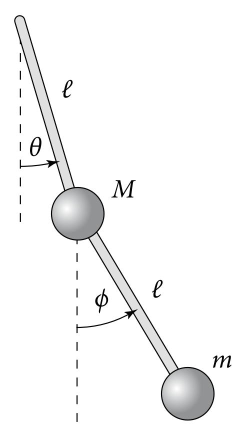
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 2 A double pendulum is constructed of two bobs of mass M and mass m, on two rods of negligible mass and length ℓ. The system is free to swing in a vertical plane. The angles of the two pendulums are θ and ϕ with respect to the vertical, as shown. The upper pendulum is hinged at the top and the lower pendulum is hinged at M.



(a) Using generalized coordinates θ and ϕ, write the Lagrangian of the system.

(b) Find Lagrange's equations.

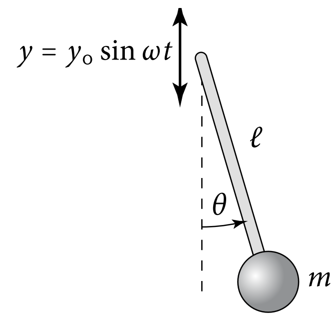
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 3 A pendulum has length ℓ and mass m, and is free to swing in a vertical plane. The upper end of the pendulum is forced to oscillate vertically according to $y = y_0 \sin \omega t$.



(a) Using θ as the generalized coordinate, write the Lagrangian of m.

(b) Find Lagrange's equations.

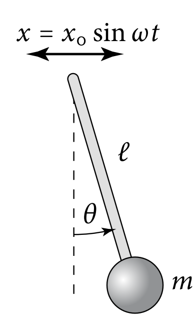
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 4 A pendulum has length ℓ and mass m, and is free to swing in a vertical plane. The upper end of the pendulum is forced to oscillate horizontally according to $x = x_0 \sin \omega t$.



(a) Using θ as the generalized coordinate, write the Lagrangian of m.

(b) Find Lagrange's equations.

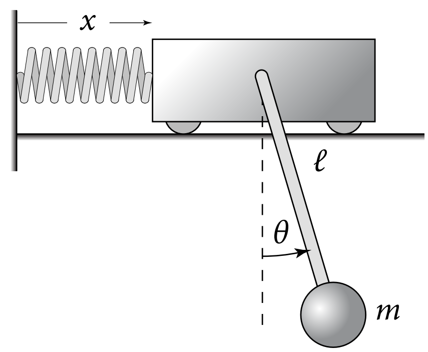
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 5 A cart of mass M rolls on a frictionless horizontal surface. The cart is attached to a spring of force constant k, equilibrium length d, and negligible mass. A pendulum of mass m and length ℓ hangs from the cart; the pendulum swings in a vertical plane, at angle θ to the vertical. Let x be the distance of the left end of the cart from the wall at the left.



(a) Using the generalized coordinates x and θ, write the Lagrangian of the system.

(b) Find Lagrange's equations.

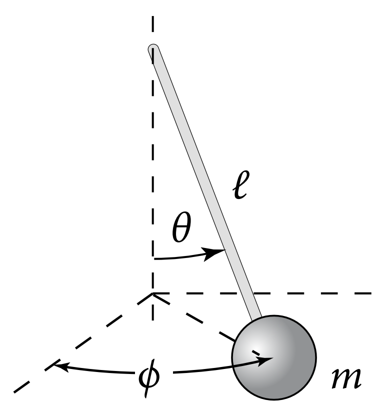
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 6 The orientation of a "spherical pendulum" of mass m and length ℓ is specified by angles θ and ϕ, as shown.



(a) Using the generalized coordinates θ and ϕ, write the Lagrangian of the system.

(b) Find Lagrange's equations.

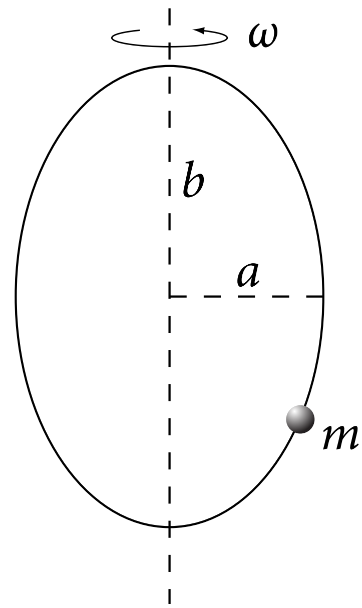
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 7 A bead of mass m is free to slide on an elliptical hoop forced to turn about a vertical axis with constant angular velocity ω. The ellipse is defined by $(x/a)^2 + (y/b)^2 = 1$.



(a) Choose a generalized coordinate for the bead, and write the Lagrangian.

(b) Find Lagrange's equations.

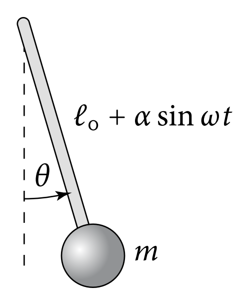
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 8 A pendulum of mass m and length $\ell = \ell_0 + \alpha \sin \omega t$ is free to oscillate in a vertical plane.



(a) Using the pendulum angle θ as the generalized coordinate, write the Lagrangian.

(b) Find Lagrange's equations.

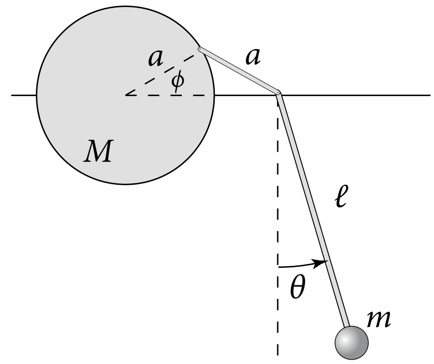
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 9 The point of suspension of a plane simple pendulum of mass m and length ℓ is constrained to move along a horizontal track and is connected to a point on the circumference of a uniform flywheel of mass M and radius a through a massless connecting rod also of length a. The flywheel can rotate freely about a center fixed on the track.



(a) Using the angles θ and ϕ as the generalized coordinate, as shown, find the Lagrangian of the system.

(b) Find Lagrange's equations.

(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 10 Our sun is probably oblate rather than spherical, because it rotates about an axis. Therefore the gravitational force it exerts on a planet should not be perfectly inverse-squared, but should instead be a central force

$$F = -\frac{GMm}{r^2} - GQ\frac{m}{r^4}$$

where Q, a constant, is proportional to the oblateness.

(a) Using plane polar coordinates r, θ of a planet around this sun, find the Lagrangian of the planet.

(b) Find Lagrange's equations.

(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 11 The Lagrangian of a particle of mass m and charge q in an electromagnetic field is $\mathscr{L} = \frac{1}{2}mv^2 - q\phi + q\mathbf{v} \cdot \mathbf{A}$, where $\mathbf{v}$ is the velocity of the particle, ϕ is the scalar potential, and $\mathbf{A}$ is the vector potential. In terms of ϕ and $\mathbf{A}$, the electric and magnetic fields are

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

In a particular magnetic mirror machine, $\phi = 0$ and $\mathbf{A} = \dfrac{B_0 + \alpha z^2}{2} r\hat{\boldsymbol{\theta}}$ in cylindrical coordinates r, θ, z for $-z_0 < z < z_0$, where α and z_0 are positive constants.

(a) Sketch $\mathbf{B}$.

(b) Show that Lagrange's equations are equivalent to $\mathbf{F} = m\mathbf{a}$ for $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$.

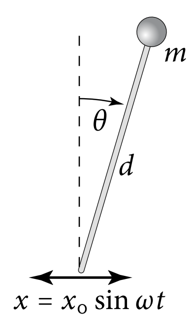
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 12 A small ball of mass m is placed on the end of a massless rod of length d. The end of the rod with the ball is placed above the other end, at angle θ to the vertical, while the lower end of the rod is forced to move back and forth horizontally with displacement $x = x_0 \sin \omega t$, where ω is a constant.



(a) Find the Lagrangian of the bead, using the parameter θ as the generalized coordinate.

(b) Find Lagrange's equations.

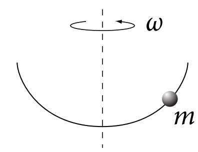
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 13 A wire is bent into the shape of a cycloid, defined by the parametric equations $x = A(\phi + \sin\phi)$ and $y = A(1 - \cos\phi)$, where ϕ is the parameter and A is a constant, with $-\pi < \phi < \pi$). The wire is in a vertical plane, and is spun at constant angular velocity ω about a vertical axis through its center. A bead of mass m is slipped onto the wire and slides without friction.



(a) Find the Lagrangian of the bead, using the parameter ϕ as the generalized coordinate.

(b) Find Lagrange's equations.

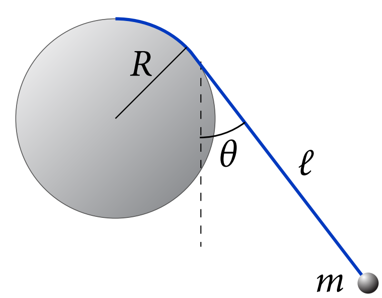
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 14 A pendulum is constructed by attaching a mass m to a string of length ℓ. The other end of the string is attached to the top of a horizontal cylinder of radius R, which satisfies $R < 2\ell/\pi$. The angle the string makes with respect to the vertical is θ, and the pendulum moves in a plane.



(a) Using θ as the generalized coordinate, find the Lagrangian.

(b) Find Lagrange's equations.

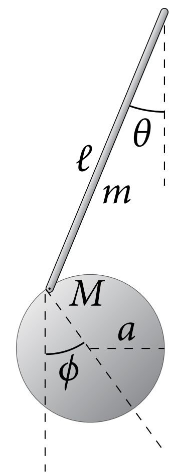
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 15 A plane pendulum consists of a uniform rod of mass m, length ℓ, and negligible thickness, suspended in a vertical plane by one end. At the other end a uniform disk of radius a and mass M is attached to the rod (with a pin) so it can rotate freely in the vertical plane.



(a) Use the angles θ and ϕ as the generalized coordinates of the system, where θ is the angle of the pendulum relative to the vertical and ϕ is the angle of the center of the disk relative to the vertical, beneath the pin. Write the Lagrangian of the system.

(b) Find Lagrange's equations.

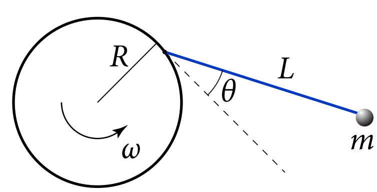
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 16 A hoop of radius R turns about its symmetry axis with constant angular velocity ω. A lever of length L and negligible mass is attached to a point on the hoop, and is free to swing without friction around that point. A weight of mass m is attached to the other end of the lever.



(a) Use the angle θ as the generalized coordinate of the weight, where θ is the angle of the lever relative to the tangent to the hoop at the point at which the lever is attached to the hoop, write the Lagrangian of the weight.

(b) Find Lagrange's equations.

(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 17 A particle of mass m is free to move in two dimensions under the central force $F = -kr^3$, where k is a constant.

(a) Using polar coordinates r, θ as generalized coordinates, write the Lagrangian.

(b) Find Lagrange's equations.

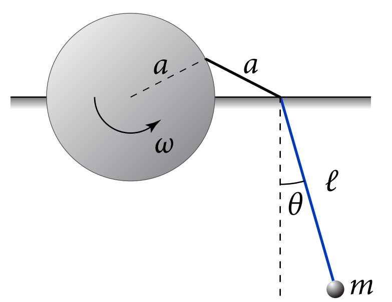
(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.

Project 18 The point of suspension of a plane simple pendulum of mass m and length ℓ is constrained to move along a horizontal track and is connected to a point on the circumference of a wheel of radius a through a massless connecting rod also of length a. The flywheel rotates with constant angular velocity ω about its center, which is fixed on the track.



(a) Using the pendulum angle θ as the generalized coordinate, as shown, find the Lagrangian of the pendulum.

(b) Find Lagrange's equations.

(c) Write and identify any first integrals of motion.

(d) Select a complete set of equations to integrate numerically, and rewrite them in dimensionless form, by scaling any coordinates having dimensions to obtain dimensionless coordinates. The resulting differential equations should be dimensionless; they may contain one or more dimensionless parameters.

(e) Solve the equations numerically for various choices of initial conditions and various choices of any dimensionless parameters.

(f) Present the results in a convincing and understandable way.