Notes on Classical Mechanics

Thierry Laurens

Last updated: January 23, 2022

Contents

Preface

Mechanics is a classical and thoroughly studied subject. Recently however, its presentation is often downplayed in physics curricula to make way for more active fields of research. I found this particularly inconvenient as a student learning mathematical tools that originated in the study of classical mechanics, because neither the physics nor math departments taught the corresponding physical motivation and intuition. Consequently, these notes are the product of a personal effort to collect these mathematical ideas and connect them to their inspiration.

The first objective of these notes is to present the core theory of classical mechanics designed for the mathematical palette. The presentation is intended to be self-contained and mathematically rigorous, while maintaining the example-driven physical mindset in order to cultivate intuition. From a physics viewpoint however, this excludes many topics that commonly appear in a first-year graduate course on the subject (e.g. rigid bodies). For a thorough study of the physical theory, techniques, and examples of classical mechanics, we refer the reader to the classic physics texts [Arn89, Gol51, LL76].

The second objective of these notes is to develop the resultant mathematical concepts alongside the physical inspiration, and then to recognize the physical system as an example. Such mathematical tools are foundational in the theory of ODEs, and they are ideas that can still be commonly found throughout mathematics today. Note that this does not include the rich field of mathematical methods used in physics, for which we refer the reader to the excellent mathematics texts [AKN06, SM95, MZ05].

In order to focus on mathematical tools which arose from mechanics, we will assume the reader possesses a beginner graduate or advanced undergraduate understanding of mathematics. Specifically, we assume the reader is familiar with multivariate real analysis throughout, with manifolds and tangent bundles in Part II, and with cotangent bundles and differential forms in Part III. Moreover, we will almost always work in the class of smooth (i.e. infinitely differentiable) functions. Although many of the results hold under weaker assumptions, we choose to focus on the proofs in the smooth case in order to highlight the physical motivation.

Part I

Newtonian Mechanics

The Newtonian framework is the most fundamental interpretation of the motion of mechanical systems. The physical theory is based on a few empirical observations which are taken as axioms, among which are Newton's equations of motion. The resulting systems of differential equations inherit special properties from their corresponding physical systems. Extracting mathematical statements from these properties about the behavior of solutions illustrates some basic mathematical ideas from the theory of ODEs. Along the way, we will also identify the key mathematical constructs that will allow us to include more general systems in the proceeding parts.

Chapter 1

Newton's equations

We begin the physical theory with the essential empirical observations and its immediate consequences. The material for this chapter is based on [Arn89, Ch. 1], [AKN06, Ch. 1], and [Gol51, Ch. 1].

1.1 Empirical assumptions

Classical mechanics is the study of the motion of particles in a Euclidean space $\mathbb{R}^d$. A **particle** (or **point mass**) consists of two pieces of information: the body's **mass** $m > 0$ and its **position** $\mathbf{x} \in \mathbb{R}^d$. This model for physical objects neglects the object's spatial dimensions. Given a system of N particles with positions $\mathbf{x}_i \in \mathbb{R}^d$, the collection of all possible positions $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$ constitute the **configuration space** $\mathbb{R}^d \times \cdots \times \mathbb{R}^d = \mathbb{R}^{Nd}$, whose dimension Nd is the system's **degrees of freedom**.

The evolution of a system is described by the particles' **trajectories**, N maps $\mathbf{x}_i : I \to \mathbb{R}^d$ for $I \subset \mathbb{R}$ an interval, which together constitute the **motion** of the system. In order to determine the system's evolution we will use the **velocity** $\dot{\mathbf{x}}_i$, **acceleration** $\ddot{\mathbf{x}}_i$, and **momentum**

$$\mathbf{p}_i = m_i \dot{\mathbf{x}}_i \tag{1.1}$$

of the ith particle. (We use the notation $\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}t}$ for time derivatives.) The positions and momenta together span the **phase space** $\mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$, whose dimension $2Nd$ is always twice the degrees of freedom. We will refer to the trajectories of the system plotted in phase space as the system's **phase portrait**.

Newton's principle of determinacy is the experimental observation that the initial state of a Newtonian system—all the positions $\mathbf{x}(t_0)$ and velocities $\dot{\mathbf{x}}(t_0)$ at some moment $t_0 \in \mathbb{R}$ in time—uniquely determines the system's motion. We formulate this mathematically as follows.

Definition 1.1 (Newton's principle of determinacy)**.** A **Newtonian system** is given by N particles and a function $\mathbf{F} : \mathbb{R} \times (\mathbb{R}^{Nd} \setminus \Delta) \times \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$, called

the **force**, such that

$$\dot{\mathbf{p}} = \mathbf{F}(t, \mathbf{x}, \dot{\mathbf{x}}). \tag{1.2}$$

Here, $\Delta = \bigcup_{i<j}\{\mathbf{x}_i = \mathbf{x}_j\}$ is the union of diagonals, so that we do not consider two particles occupying the same position.

Equation (1.2) is called **Newton's equation** (or **Newton's second law**). Unless otherwise noted, we will always assume the particle masses m_i are all constant. (1.2) then takes the form

$$m_i\ddot{\mathbf{x}}_i = \mathbf{F}_i(t, \mathbf{x}, \dot{\mathbf{x}}), \qquad i = 1, \dots, N. \tag{1.3}$$

Defining a new variable $\mathbf{p}$ by (1.1) might seem unnecessary at first glance. Experimentally however, we can observe that a particle's acceleration is inversely proportional to its mass according to (1.3), and so momentum truly is a natural variable.

In mathematics, it is common practice to reduce the system of N second-order ordinary differential equations (ODEs) on configuration space to first-order. To do this, we group the equations (1.1) for $\dot{\mathbf{x}}$ together with the equations (1.2) for $\dot{\mathbf{p}}$ to obtain a system of $2N$ first-order ODEs on phase space. For initial conditions, we then take the positions $\mathbf{x}(t_0)$ and momenta $\mathbf{p}(t_0)$ at some time $t_0 \in \mathbb{R}$. For convenience, we may assume that $t_0 = 0$ after replacing the variable t by $t - t_0$.

Unless otherwise stated, we will always assume that the force $\mathbf{F}$ is smooth (i.e. infinitely differentiable) on $\mathbb{R} \times (\mathbb{R}^{Nd} \setminus \Delta) \times \mathbb{R}^{Nd}$. It then follows (cf. Theorem A.5) that there exists a unique solution $\mathbf{x}(t)$ to the system of differential equations (1.3) for any initial state. However, the solution is only guaranteed to exist on a short time interval, and in general may not be able to be extended for all time (cf. Example A.4). From experience though, we expect that for naturally occurring mechanical systems that the time interval of existence can be extended to all of $\mathbb{R}$, and most of our rigorous statements will include a premise that ensures that the solutions to our mathematical model (1.3) exist for all time.

We will formulate one more mathematical assumption: that the physical laws governing the system's motion is independent of the choice of coordinates and origin we impose on $\mathbb{R}^d$. A transformation $\mathbb{R}_t \times \mathbb{R}^d \to \mathbb{R}_t \times \mathbb{R}^d$ is **Galilean** provided that it is an affine transformation (a linear transformation and a translation) that preserves time intervals and for any fixed $t \in \mathbb{R}$ is an isometry of $\mathbb{R}^d$; that is, if we write $g(t, \mathbf{x}) = (t', \mathbf{x}')$ then for each $t \in \mathbb{R}$ the function $\mathbf{x}'$ is an orthogonal transformation and a translation. It is straightforward to show that the set of Galilean transformations forms a group under function composition.

Example 1.2. The following are all Galilean transformations:

(1) Translations: $g_1(t, \mathbf{x}) = (t + t_0, \mathbf{x} + \mathbf{x}_0)$ for some fixed $\mathbf{x}_0 \in \mathbb{R}^d$, $t_0 \in \mathbb{R}$.

(2) Rotations and reflections: $g_2(t, \mathbf{x}) = (t, A\mathbf{x})$ for some fixed orthogonal transformation $A \in O(d)$.

(3) Galilean boosts: $g_3(t, \mathbf{x}) = (t, \mathbf{x} + \mathbf{v}t)$ for some fixed velocity $\mathbf{v} \in \mathbb{R}^d$.

In fact, these examples generate the entire Galilean group (cf. Exercise 1.1). **Galileo's principle of relativity** is the experimental observation that for an isolated system there exists a **reference frame**—a choice of origin and coordinate axes for $\mathbb{R}^d$—that is invariant under any Galilean transformation. Such a reference frame is called **inertial**, and the principle also asserts that all coordinate systems in uniform motion with constant velocity with respect to an inertial frame must also be inertial. (This is observed, for example, in a car traveling at constant velocity and noting that motion inside the car is as if the car were at rest.) We formulate this mathematically as follows.

Definition 1.3 (Galileo's principle of relativity)**.** A Newtonian system is **isolated** if there exists a reference frame so that Newton's equations (1.2) are invariant under any Galilean transformation.

Physically, this is requiring that the ambient space is both homogeneous and isotropic and that time is homogeneous. Geometrically, this principle requires that if we apply a Galilean transformation to a phase portrait, then the resulting graph still consists of trajectories.

If Newton's equations (1.2) hold in an inertial coordinate system, then they must be invariant with respect to the Galilean group. Let $\mathbf{x}(t)$ denote a solution in these coordinates. Applying the Galilean group generators of Example 1.2, we find the following conditions on $\mathbf{F}_i$:

(1) Translation invariance: $\mathbf{F}_i(t, \mathbf{x}, \dot{\mathbf{x}}) \equiv \mathbf{F}_i(\mathbf{x}_j - \mathbf{x}_k, \dot{\mathbf{x}})$.

(2) Rotation and reflection invariance: $\mathbf{F}_i(A\mathbf{x}, A\dot{\mathbf{x}}) = A\mathbf{F}_i(\mathbf{x}, \dot{\mathbf{x}})$ for $A \in O(d)$.

(3) Boost invariance: $\mathbf{F}_i(\mathbf{x}_j - \mathbf{x}_k, \dot{\mathbf{x}}) \equiv \mathbf{F}_i(\mathbf{x}_j - \mathbf{x}_k, \dot{\mathbf{x}}_j - \dot{\mathbf{x}}_k)$.

Note that the third type of transformation in Example 1.2 changes neither $\ddot{\mathbf{x}}$ nor $\mathbf{x}_i - \mathbf{x}_j$.

Proposition 1.4 (Newton's first law I)**.** *For an isolated Newtonian system of one particle, the particle's acceleration in an inertial coordinate system vanishes. In particular, the motion is* ***rectilinear****: uniform in time with constant velocity.*

Proof. Taking $N = 1$, the conditions (1)–(3) above require that $\mathbf{F}$ is independent of $\mathbf{x}$, $\dot{\mathbf{x}}$, t and is rotationally invariant. Therefore $\mathbf{F} \equiv 0$. □

1.2 Kinetic energy

The **kinetic energy** of the ith particle and the **total kinetic energy** are given by

$$T_i = \tfrac{1}{2} m_i |\dot{\mathbf{x}}_i|^2 = \tfrac{1}{2m_i} |\mathbf{p}_i|^2, \qquad T = \sum_{i=1}^{N} T_i \tag{1.4}$$

respectively. From observation, we know that the magnitude of the velocity and hence the kinetic energy can be increased and decreased by the force $\mathbf{F}_i$ acting on the ith particle, depending on the force's magnitude and direction. This is measured through the **work** W_i done by the force $\mathbf{F}_i$ on the ith particle from time 0 to t, defined by the line integral

$$W_i = \int_{\mathbf{x}(0)}^{\mathbf{x}(t)} \mathbf{F}_i \cdot \, \mathrm{d}\mathbf{s}_i = \int_0^t \mathbf{F}_i(\mathbf{x}(\tau)) \cdot \dot{\mathbf{x}}(\tau) \, \mathrm{d}\tau.$$

(We use $\mathrm{d}\mathbf{s}_i$ to denote the line element of the trajectory $\mathbf{x}_i(t)$, and so the second equality is the definition of path integration.) Although work is measured in the physical space $\mathbb{R}^d$, the **total work**

$$W = \int_{\mathbf{x}(0)}^{\mathbf{x}(t)} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s}$$

is naturally defined on configuration space $\mathbb{R}^{Nd}$.

The net change in kinetic energy is due to the work done by the tuple of forces $\mathbf{F}$ on the path $\mathbf{x}(t)$ in configuration space.

Proposition 1.5. *The increase in total kinetic energy is equal to the total work.*

Proof. Differentiating the kinetic energy (1.4), we obtain

$$\dot{T} = \sum_{i=1}^N m_i \dot{\mathbf{x}}_i \cdot \ddot{\mathbf{x}}_i = \sum_{i=1}^N \dot{\mathbf{x}}_i \cdot \mathbf{F}_i = \dot{\mathbf{x}} \cdot \mathbf{F}$$

by Newton's equations (1.3). Integrating, we have

$$T(t) - T(0) = \int_0^t \dot{T} \, \mathrm{d}t = \int_0^t \dot{\mathbf{x}} \cdot \mathbf{F} \, \mathrm{d}\tau = \int_{\mathbf{x}(0)}^{\mathbf{x}(t)} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s} = W. \qquad \square$$

1.3 Potential energy

For some systems there is also a potential energy. In physics, a Newtonian system is called **conservative** if the force $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) \equiv \mathbf{F}(\mathbf{x})$ depends only on the positions $\mathbf{x}$ and if the total work along any path connecting two points $\mathbf{y}, \mathbf{z}$ in configuration space,

$$W = \int_{\mathbf{y}}^{\mathbf{z}} \mathbf{F}(\mathbf{s}) \cdot \, \mathrm{d}\mathbf{s},$$

is independent of the path chosen for the line integral. (Note that the line integral path is allowed to be arbitrary, and is not limited to trajectories.) This independence is equivalent to the work around any simple closed path vanishing, since two paths with the same endpoints can be concatenated to form one closed path.

One way for this to be satisfied is if there is a **potential energy**, a function V such that $\mathbf{F} = -\nabla V$. Indeed, if this is the case, then by the fundamental theorem of calculus we have

$$W = -\int_{\mathbf{y}}^{\mathbf{z}} \nabla V(\mathbf{s}) \cdot \mathrm{d}\mathbf{s} = -V(\mathbf{z}) + V(\mathbf{y})$$

for all paths connecting $\mathbf{y}$ to $\mathbf{z}$.

Example 1.6. If the interaction forces depend only on particle distances,

$$\mathbf{F}_i = \sum_{j=1}^{N} \mathbf{F}_{ij}, \qquad \mathbf{F}_{ij} = f_{ij}(r_{ij})\mathbf{e}_{ij}, \qquad r_{ij} = |\mathbf{x}_i - \mathbf{x}_j|, \qquad \mathbf{e}_{ij} = \frac{\mathbf{x}_i - \mathbf{x}_j}{r_{ij}},$$

then the system is conservative with potential energy

$$V = \sum_{i<j} V_{ij}, \qquad V_{ij} = \int f_{ij}\, \mathrm{d}r.$$

In fact, every conservative system must have a potential energy.

Proposition 1.7. *A Newtonian system is conservative if and only if there exists a potential energy, i.e. a smooth function $V : \mathbb{R}^{Nd} \setminus \Delta \to \mathbb{R}$ such that $\boldsymbol{F} = -\nabla V$.*

Proof. First suppose that the total work integral is path independent. Fix some $\mathbf{x}_0 \in \mathbb{R}^{Nd} \setminus \Delta$. Then the line integral

$$V(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F}(\mathbf{s}) \cdot \mathrm{d}\mathbf{s}$$

is well-defined as a function of $\mathbf{x}$, because we do not need to choose a path from $\mathbf{x}_0$ to $\mathbf{x}$. We then have $\mathbf{F} = -\nabla V$ by the fundamental theorem of calculus.

Conversely, if there is a potential energy V, then the fundamental theorem of calculus tells us that

$$\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = -\int_{\mathbf{x}_0}^{\mathbf{x}} \nabla V(\mathbf{s}) \cdot \mathrm{d}\mathbf{s} = -V(\mathbf{x}) + V(\mathbf{x}_0).$$

That is, the work is independent of the path. □

In Proposition 1.7, it is assumed that $\mathbf{F}$ and V are defined on all of $\mathbb{R}^{Nd} \setminus \Delta$. The statement does not hold for arbitrary open subsets of the configuration space $\mathbb{R}^{Nd}$.

Example 1.8. Consider the vector field

$$\mathbf{F}(x, y) = \left(\frac{y}{x^2 + y^2}, -\frac{x}{x^2 + y^2} \right)$$

on $\mathbb{R}^2 \setminus \{0\}$. This is the negative gradient of the planar polar coordinate angle, which is defined on all of $\mathbb{R}^2 \setminus \{0\}$ unlike the angle coordinate. Consequently,

by the fundamental theorem of calculus the work done on a particle traveling around any simple closed curve not containing the origin is zero. Conversely, the work done on a particle traveling once clockwise about the unit circle is 2π, which reflects the fact that we cannot define an angular coordinate continuously on all of $\mathbb{R}^2 \setminus \{0\}$. Consequently, the system consisting of one particle subject to this force is an example of a nonconservative force.

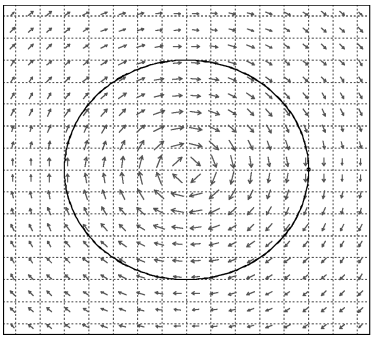


Figure 1.1: Configuration space for the system of Example 1.8.

From multivariate calculus we know that curl-free vector fields are not necessarily gradient fields for arbitrary subsets of $\mathbb{R}^3$, and so the fact that Proposition 1.7 holds for the proper subset $\mathbb{R}^{Nd} \setminus \Delta$ may initially be surprising. To understand why this is natural, let us specialize to $d = 3$, fix $N - 1$ particles, and consider only moving the ith particle. Let S be a smooth 2-dimensional surface in $\mathbb{R}^3$ with boundary. Using Green's theorem we write

$$0 = \int_{\partial S} \mathbf{F}_i \cdot \, \mathrm{d}\mathbf{s}_i = \int_S (\nabla_i \times \mathbf{F}_i) \cdot \mathbf{n} \, \mathrm{d}A,$$

where $\mathbf{n}$ is a unit vector field that is perpendicular to S. For this to hold for all such surfaces S, then we must have $\nabla_i \times \mathbf{F}_i \equiv 0$. So *the definition of a conservative force is an integral formulation of $\boldsymbol{F}_i$ being curl-free on all of* $\mathbb{R}^3$, which avoids the issue of $\mathbf{F}_i$ not being defined on $\bigcup_{j \neq i} \{\mathbf{x}_j\}$. As the domain is all of $\mathbb{R}^3$, we know from multivariable calculus that if $\nabla \times \mathbf{F}_i \equiv 0$ then $\mathbf{F}_i = -\nabla V_i(\mathbf{x}_i)$, with the caveat that our potential V_i may not be defined on $\bigcup_{j \neq i} \{\mathbf{x}_j\}$ because $\mathbf{F}$ is not defined on the diagonals Δ.

The same argument applies to arbitrary dimensions $d \geq 2$, provided that we use the language of differential geometry. If S is a smooth 2-dimensional submanifold of $\mathbb{R}^d$ then Stokes' theorem yields

$$0 = \int_{\partial S} \mathbf{F}_i \cdot \, \mathrm{d}\mathbf{s}_i = \int_S \mathrm{d}\mathbf{F}_i,$$

where $\mathrm{d}\mathbf{F}_i$ is the exterior derivative of $\mathbf{F}_i$ when viewed as a 1-form on $\mathbb{R}^d$. For this to hold for all such surfaces S, then we must have that $\mathbf{F}_i$ is closed (i.e.

$\mathrm{d}\mathbf{F}_i \equiv 0$) on $\mathbb{R}^d$. As $\mathbb{R}^d$ has no holes, we can conclude that $\mathbf{F}_i$ is exact on $\mathbb{R}^d$ (i.e. $\mathbf{F}_i = -\mathrm{d}V_i$) with the 0-form V_i may not be defined on $\bigcup_{j\neq i}\{\mathbf{x}_j\}$. In comparison, the 1-form $\mathbf{F}_i$ of Example 1.8 is closed on $\mathbb{R}^2 \setminus \{0\}$—indeed, if $S \subset \mathbb{R}^2 \setminus \{0\}$ then the work around ∂S vanishes—but it is not closed on all of $\mathbb{R}^2$.

1.4 Total energy

We now have two notions of mechanical energy: kinetic and potential. Their sum $E = T + V$ is the system's **total energy**.

Proposition 1.9 (Conservation of energy)**.** *For a conservative system, the total energy E is conserved under the motion: $E(t) = E(0)$ for all $t \in \mathbb{R}$.*

Proof. By Newton's equations (1.3) we have

$$\dot{E} = \dot{T} + \nabla V \cdot \dot{x} = \dot{x} \cdot \mathbf{F} - \mathbf{F} \cdot \dot{x} = 0. \qquad \square$$

In section 2.2 we will explore the implications of energy conservation more thoroughly, but as an immediate consequence we obtain global existence for coercive potentials:

Corollary 1.10. *If a conservative system has a smooth nonnegative potential energy $V(\boldsymbol{x})$ that satisfies $V(\boldsymbol{x}) \to +\infty$ as $|\boldsymbol{x}| \to \infty$, then every solution exists for all time.*

Proof. Fix initial data and let E_0 denote the corresponding initial energy. By Proposition 1.9, we know that the solution is confined to the region $\{(\mathbf{x}, \dot{\mathbf{x}}) : \frac{1}{2}|\dot{\mathbf{x}}|^2 + V(\mathbf{x}) = E_0\}$ for as long as it exists. This region is contained in the set $\{\frac{1}{2}|\dot{\mathbf{x}}|^2 \leq E_0\} \cap \{V(\mathbf{x}) \leq E_0\}$, which is bounded since $V(\mathbf{x}) \to +\infty$ as $|\mathbf{x}| \to \infty$, and hence our trajectory is bounded as well. In particular, the blowup condition of Corollary A.6 can never be satisfied, and so the maximal time of existence cannot be finite. $\square$

It is often helpful for physical intuition to picture a small ball rolling down the graph of $V(\mathbf{x})$. Suppose we have a solution $\mathbf{x}(t)$ to Newton's equations (1.3) with $E(\mathbf{x}(t)) \equiv E_0$. As kinetic energy is nonnegative, then a ball at position $V(\mathbf{x}(t))$ is confined to the region where $V(\mathbf{x}) \leq E_0$. A smaller potential energy yields a greater kinetic energy by Proposition 1.9, which implies a greater velocity; this tells us that the ball gains velocity as it rolls downhill. This picture makes some facts very intuitive, like that local minima and maxima of $V(\mathbf{x})$ are stable and unstable equilibria for the system, respectively. For a bounded potential well $V(\mathbf{x}) \leq E_0$ in a conservative system, the ball rolls right through any minima and up towards the boundary $V^{-1}(E_0)$.

1.5 Linear momentum

Suppose the force on the ith particle can be decomposed into

$$\mathbf{F}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{F}_{ij}(\mathbf{x}_i, \mathbf{x}_j, t) + \mathbf{F}'_i(\mathbf{x}_i, t), \tag{1.5}$$

where $\mathbf{F}_{ij}$ is the **interaction force** between the ith and jth particle and $\mathbf{F}'_i$ is the **external force** on the ith particle. A system is called **closed** if there are no external forces: $\mathbf{F}'_i \equiv 0$ for all i.

We will assume that the interaction forces obey the **law of action and reaction** (or **Newton's third law**): the experimental observation that the force two particles exert on each other are equal and opposite, i.e. $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. This is a common property of Newtonian systems, and we will often be working under this assumption. If we are also given an inertial frame, then this assumption implies that the interaction forces are collinear: $\mathbf{F}_{ij} = f_{ij}\mathbf{e}_{ij}$, where $\mathbf{e}_{ij}$ is the unit vector $(\mathbf{x}_j - \mathbf{x}_i)/|\mathbf{x}_j - \mathbf{x}_i|$ from the ith to the jth particle.

Example 1.11. Any system of the form in Example 1.6 obeys the law of action and reaction since $\mathbf{e}_{ij} = -\mathbf{e}_{ji}$.

Interaction forces for non-Newtonian systems generally do not obey this law. For example, a particle with electric charge q placed in an electromagnetic field is acted upon by the Lorentz force

$$\mathbf{F} = q\left[\mathbf{E} + \tfrac{1}{c}(\mathbf{v} \times \mathbf{H})\right],$$

where $\mathbf{E}, \mathbf{H}$ are the electric and magnetic fields (which satisfy the Maxwell system of equations) and c is the speed of light.

The effect of all external forces together can be observed through the **total (linear) momentum**,

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i.$$

Proposition 1.12 (Conservation of linear momentum)**.** *The increase in total momentum is equal to the total force. Moreover, if the forces can be decomposed as* (1.5)*, then the change in the total momentum is equal to the total external force* $\sum_i \boldsymbol{F}'_i$*. In particular, for a closed system the total linear momentum is conserved.*

Proof. By Newton's equations (1.2) we have

$$\dot{\mathbf{P}} = \sum_{i=1}^{N} \dot{\mathbf{F}}_i,$$

which yields the first claim. If we assume the decomposition (1.5), we obtain

$$\dot{\mathbf{P}} = \sum_{i=1}^{n} \dot{\mathbf{p}}_i = \sum_{i=1}^{n} \mathbf{F}_i = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbf{F}_{ij} + \sum_{i=1}^{n} \mathbf{F}'_i = \sum_{i=1}^{n} \mathbf{F}'_i,$$

which is the second claim. In the last equality, we note that the law of action and reaction (i.e. $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$) causes the double sum over interaction forces to cancel pairwise. □

By taking a dot product with the unit coordinate vectors, we also obtain component-wise conservation:

Corollary 1.13. *If the forces can be decomposed as* (1.5) *and the total external force is perpendicular to an axis, then the projection of the total momentum onto that axis is conserved.*

The system **center of mass** (or **barycenter**) is

$$\mathbf{X} = \frac{\sum_{i=1}^{N} m_i \mathbf{x}_i}{\sum_{i=1}^{N} m_i}. \tag{1.6}$$

This definition does not depend on the choice or origin, and it is characterized as the point with respect to which the total momentum vanishes (cf. Exercise 1.4). Moreover, the total momentum is equal to all of the mass lying at the center of mass:

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i = M\dot{\mathbf{X}}, \qquad \text{where} \quad M = \sum_{i=1}^{N} m_i. \tag{1.7}$$

Proposition 1.14 (Newton's first law II)**.** *The center of mass moves as if all masses were concentrated at it and all forces were applied to it. In particular, the center of mass of a closed system moves rectilinearly, i.e. uniformly with constant velocity.*

Proof. Differentiating (1.7) yields

$$M\ddot{\mathbf{X}} = \dot{\mathbf{P}} = \sum_{i=1}^{n} \mathbf{F}_i.$$

The right-hand side vanishes for a closed system. □

1.6 Angular momentum

In this section, we will specialize to the case $d = 3$ so that we may use the cross product on $\mathbb{R}^3$. The **angular momentum** (about the origin) of the ith particle and the **total angular momentum** are given by

$$\mathbf{L}_i = \mathbf{x}_i \times \mathbf{p}_i = m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i, \qquad \mathbf{L} = \sum_{i=1}^{N} \mathbf{L}_i$$

respectively. The **torque** (or **moment of force**) of the ith particle and the **total torque** are given by

$$\mathbf{N}_i = \mathbf{x}_i \times \mathbf{F}_i, \qquad \mathbf{N} = \sum_{i=1}^{N} \mathbf{N}_i$$

respectively.

When the forces can be decomposed as (1.5), we define the **external torque**

$$\mathbf{N}'_i = \mathbf{x}_i \times \mathbf{F}'_i.$$

The relationship between angular momentum and external torque is analogous to the linear case:

Proposition 1.15 (Conservation of angular momentum)**.** *The increase in total angular momentum is equal to the total torque. Moreover, if the forces can be decomposed as* (1.5)*, then the change in total angular momentum is equal to the total external torque* $\sum_i \boldsymbol{N}'_i$*. In particular, for a closed system the total angular momentum is conserved.*

Proof. By Newton's equations (1.2) we have

$$\dot{\mathbf{L}} = \sum_{i=1}^{N} \dot{\mathbf{L}}_i = \sum_{i=1}^{N} \left(\mathbf{x}_i \times \dot{\mathbf{p}}_i + \dot{\mathbf{x}}_i \times \mathbf{p}_i \right) = \sum_{i=1}^{N} (\mathbf{N}_i + 0) = \mathbf{N},$$

which yields the first claim. In the last equality we noted that the product $\dot{\mathbf{x}}_i \times \mathbf{p}_i$ vanishes because $\mathbf{p}_i$ is parallel to $\dot{\mathbf{x}}_i$. Assuming the decomposition (1.5) we have

$$\dot{\mathbf{L}} = \sum_{i=1}^{n} \mathbf{x}_i \times \mathbf{F}_i = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbf{x}_i \times \mathbf{F}_{ij} + \sum_{i=1}^{n} \mathbf{x}_i \times \mathbf{F}'_i = \sum_{i=1}^{n} \mathbf{N}'_i,$$

which is the second claim. In the last equality we used the law of action and reaction (i.e. $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$) to note that the double sum cancels pairwise:

$$\mathbf{x}_i \times \mathbf{F}_{ij} + \mathbf{x}_j \times \mathbf{F}_{ji} = (\mathbf{x}_i - \mathbf{x}_j) \times \mathbf{F}_{ij} = 0.$$

In particular, for a closed system the external torque vanishes and we have $\dot{\mathbf{L}} = 0$. □

Taking dot products, we also obtain component-wise conservation:

Corollary 1.16. *If the total external torque is perpendicular to an axis, then the projection of the total angular momentum onto that axis is conserved.*

As with linear momentum, from outside the system the momentum evolves as if it were all concentrated about the center of mass and all the external torques were applied to it.

Proposition 1.17. *The total angular momentum (total torque) about the origin is equal to the sum of the total angular momentum (total torque) about the center of mass and the angular momentum (torque) of the center of mass about the origin.*

Proof. Expanding about the center of mass we have

$$\begin{aligned}\mathbf{L} &= \sum_{i=1}^{N} [(\mathbf{x}_i - \mathbf{X}) + \mathbf{X}] \times m_i \big[(\dot{\mathbf{x}}_i - \dot{\mathbf{X}}) + \dot{\mathbf{X}}\big] \\ &= \sum_{i=1}^{N} (\mathbf{x}_i - \mathbf{X}) \times m_i (\dot{\mathbf{x}}_i - \dot{\mathbf{X}}) + \mathbf{X} \times M\dot{\mathbf{X}} \\ &\quad + \left[\sum_{i=1}^{N} m_i (\mathbf{x}_i - \mathbf{X})\right] \times \dot{\mathbf{X}} + \mathbf{X} \times \left[\sum_{i=1}^{N} m_i (\dot{\mathbf{x}}_i - \dot{\mathbf{X}})\right] \\ &= \sum_{i=1}^{N} (\mathbf{x}_i - \mathbf{X}) \times m_i (\dot{\mathbf{x}}_i - \dot{\mathbf{X}}) + \mathbf{X} \times \mathbf{P} + 0 + 0,\end{aligned}$$

which yields the claim for angular momentum. In the last equality we used the definition of total momentum (1.7) for the first term and Exercise 1.4 for the vanishing of the square-bracketed terms. The statement for torque follows by taking a time derivative. □

There is one last dynamical quantity of interest: the **moment of inertia** of the ith particle (about the origin), which is given by

$$I_i = m_i |\mathbf{x}_i|^2.$$

The moment of inertia plays the role for **angular velocity**,

$$\boldsymbol{\omega}_i = \frac{\mathbf{x}_i \times \dot{\mathbf{x}}_i}{|\mathbf{x}_i|^2},$$

analogous to mass for linear velocity, in the sense that

$$\mathbf{L}_i = \mathbf{x}_i \times \mathbf{p}_i = \mathbf{x}_i \times (m_i \boldsymbol{\omega}_i \times \mathbf{x}_i) = m_i \left[(\mathbf{x}_i \cdot \mathbf{x}_i)\boldsymbol{\omega}_i - (\mathbf{x}_i \cdot \boldsymbol{\omega}_i)\mathbf{x}_i\right] = I_i \boldsymbol{\omega}_i + 0.$$

As in the previous proposition, the formula $\mathbf{L} = I\boldsymbol{\omega}$ can also be translated to the center of mass.

Unlike mass, the moment of inertia evolves in time:

Proposition 1.18. *The total moment of inertia evolves according to $\ddot{I} = 4T + 2\boldsymbol{F} \cdot \boldsymbol{x}$. For a conservative system with homogeneous potential $V(\boldsymbol{x}) = c|\boldsymbol{x}|^k$, we have $\ddot{I} = 4E - 2(k+2)V$.*

Proof. The first statement is a straightforward calculation. For the second claim, we note that for a homogeneous potential we have

$$\mathbf{F} \cdot \mathbf{x} = -\nabla V \cdot \mathbf{x} = -kV, \qquad 4T = 4E - 4V.$$ □

1.7 Exercises

1.1 (Galilean group generators)**.** Show that every Galilean transformation g of the space $\mathbb{R}^d \times \mathbb{R}$ can be written uniquely as the composition $g_1 \circ g_2 \circ g_3$ of the three types of Galilean transformations in Example 1.2. Start by writing g as a general affine transformation on $\mathbb{R} \times \mathbb{R}^d$,

$$g(t, \mathbf{x}) = (\mathbf{b} \cdot \mathbf{x} + kt + t_0, A\mathbf{x} + \mathbf{v}t + \mathbf{x}_0),$$

and show that $\mathbf{b} = 0$, $k = 1$, and $A \in O(d)$. What is the dimension of the Galilean group for $d = 3$?

1.2. Suppose a Newtonian system of N particles in $\mathbb{R}^d$ is in some inertial frame and all of the initial velocities are zero. Show that if the particles are contained in a $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$ then they remain in this subspace for all time.

1.3 (Rotating reference frame)**.** Suppose we have a system in an inertial coordinate system $\mathbf{z} \in \mathbb{R}^3$ (e.g. coordinates relative to a stationary sun), so that Newton's equations (1.3) obey the conditions (1)–(3) of section 1.1. Consider non-inertial coordinates $\mathbf{x}$ (e.g. coordinates relative to a fixed point on Earth's surface) defined by

$$t \mapsto t, \qquad \mathbf{x} = B(t)\mathbf{z} + \mathbf{b}(t),$$

where $\mathbf{b}(t) \in \mathbb{R}^3$ is the new origin and $B(t) \in O(3)$ for all t.

(a) Show that the equations of motion in the new frame are

$$m_i \ddot{\mathbf{z}}_i = \mathbf{F}_i\big(\mathbf{z}_k - \mathbf{z}_j, B^{-1}\dot{B}(\mathbf{z}_k - \mathbf{z}_j) + (\dot{\mathbf{z}}_k - \dot{\mathbf{z}}_j)\big) + \mathbf{\Phi}_i + \mathbf{\Psi}_i,$$

where

$$\mathbf{\Phi}_i = -m_i\big(B^{-1}\ddot{B}\mathbf{z}_i + B^{-1}\ddot{\mathbf{b}}\big), \qquad \mathbf{\Psi}_i = -2m_i B^{-1}\dot{B}\dot{\mathbf{z}}_i.$$

The new forces $\mathbf{\Phi}$ and $\mathbf{\Psi}$ that appear on the right-hand side of the equations of motion for $\mathbf{z}$ are called **inertial** or **fictitious forces**.

(b) Differentiate the definition of an orthogonal matrix to show that $B^{-1}\dot{B}$ is skew-symmetric, and write $B^{-1}\dot{B}\mathbf{z} = \boldsymbol{\omega} \times \mathbf{z}$ for some vector $\boldsymbol{\omega}$ called the **angular velocity** of the moving frame. Now we have

$$\mathbf{\Psi}_i = -2m_i\boldsymbol{\omega} \times \dot{\mathbf{z}}.$$

$\mathbf{\Psi}_i$ is called the **Coriolis force**, and depends on the velocity. In the northern hemisphere of the earth it deflects every body moving along the earth to the right and every falling body eastward.

(c) Use the product rule for $B^{-1}\dot{B}$ and the skew-symmetry of $B^{-1}\dot{B}$ to show that

$$B^{-1}\ddot{B} = \frac{\mathrm{d}}{\mathrm{d}t}\big(B^{-1}\dot{B}\big) + B^{-1}\dot{B}B^{-1}\dot{B}.$$

Writing $\mathbf{w} = B^{-1}\mathbf{b}$ for the **acceleration** and $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ for the **angular acceleration** of the moving frame, obtain

$$\boldsymbol{\Phi}_i = -m_i\big[\mathbf{w} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{z}_i) + \boldsymbol{\alpha} \times \mathbf{z}_i\big].$$

The second term is called the **centrifugal force** and the third term is the **inertial rotation force** or the **Euler force**, both of which only depend on position. The former is always directed outward from the instantaneous axis of rotation and acts even on a body at rest in the coordinate system $\mathbf{z}$. The latter is only present for nonuniform rotation.

1.4. (a) Show that the center of mass (1.6) does not depend on the choice of origin.

(b) Show that both the "total position" and total momentum relative to the center of mass vanishes:

$$\sum_{i=1}^{N} m_i(\mathbf{x}_i - \mathbf{X}) = 0, \qquad \sum_{i=1}^{N} m_i(\dot{\mathbf{x}}_i - \dot{\mathbf{X}}) = 0.$$

1.5 (A way to compute π [Gal03])**.** In this example we will see a geometric aspect of phase space appear as a physically measurable quantity, which reinforces that phase space is an inherent object of a Newtonian system and not merely an abstract concept. Consider a frictionless horizontal ray with a vertical wall at the origin. One small block of mass m is initially at rest on the surface, and a big block of mass $M \gg m$ is pushed towards the small block so that the small block is sandwiched between the large block and the wall. We will count the number of collisions N the small block makes with the big block and the wall.

(a) Let v_1 and v_2 denote the velocities of the large and small blocks respectively, and consider the rescaling $y_1 = \sqrt{M}v_1$, $y_2 = \sqrt{m}v_2$. Plot the initial energy level set in the (y_1, y_2)-plane, to which the motion is confined since the surface is frictionless.

(b) Initially we have $v_1 < 0$, $v_2 = 0$—plot this point in the same (y_1, y_2)-plane. Assume that the collision is **purely elastic**, so that when the blocks collide the total momentum is conserved. Plot the total momentum level set which contains the initial point; the outcome velocities is determined by the other intersection of the level sets. After the first collision the small block will eventually hit the wall, and we will assume that this collision is also elastic so that $v_2 < 0$ is replaced by $-v_2 > 0$; plot this new point as well. Plot a few more iterates of this two-collision pattern in the (y_1, y_2)-plane.

(c) The pattern repeats until $v_1 > 0$ and $0 < v_2 < v_1$, so that the large block is moving away, and the small block will neither collide with the wall again nor catch up to the large block. Sketch the end zone region in the (y_1, y_2)-plane as well.

(d) Connect consecutive points occupied by the system in the (y_1, y_2)-plane. As the lines are either vertical or parallel, the angle θ any any point between any two consecutive lines is the same. Show that $\theta = \arctan\sqrt{m/M}$.

(e) For any point, the angle θ at that point subtends an arc on the circle opposite that point. Show that the angle at the origin which subtends the same arc is 2θ, and that the total number N of collisions that occur is the largest integer such that $N\theta < \pi$.

(f) Take $M = 100^n m$ for $n > 0$ an integer. Show that $10^{-n}N \to \pi$ as $n \to \infty$, and so the number of collisions N begins to spell out the digits of π for n large. In fact, N will be a number with $n + 1$ digits whose first n digits coincide with the first n decimal digits of the number π (starting with 3).

1.6 (Book stacking [Nah16])**.** We will stack books of mass 1 and length 1 on a table in an effort to produce the maximum amount of overhang.

(a) Place the first book with its left edge at $x = 0$ and its right edge lined up with the end of the table at $x = 1$. By considering the center of mass of the book, determine the distance $S(1)$ we can slide the book over the edge of the table before it falls.

(b) Starting with a stack of two books instead of one, we can reason as in part (a) and slide the top book forward a distance of $S(1)$ while keeping the bottom book stationary. By considering the center of mass of the two books, determine the distance $S(2)$ we can slide this two-book configuration before it falls.

(c) Now start with three books, slide the top one a distance of $S(1)$ and then the top two books as in part (b) in order to produce an overhang $S(2)$ from the edge of the bottom book. Determine the distance $S(3)$ we can slide the three-book configuration before it falls. Note that when multiplied by 2, the sequence $S(1), S(2), S(3), \ldots$ begins to spell out a familiar series.

(d) Postulate a formula for $S(n)$ and prove it by induction. Note that the overhang $S(n)$ tends to infinity as $n \to \infty$.

1.7. Consider a conservative system in $\mathbb{R}^3$ with coordinates so that the center of mass $\mathbf{X} = 0$ is the origin and is at rest, $\dot{\mathbf{X}} = 0$.

(a) Show that $|\mathbf{L}|^2 \leq 2IT$.

(b) Show that the trajectories in configuration space must be contained in the intersection of the hyperplane $\{\mathbf{x} \in \mathbb{R}^{3N} : \sum_{i=1}^N m_i \mathbf{x}_i = 0\}$ with the set $\{\mathbf{x} \in \mathbb{R}^{3N} : V(\mathbf{x}) + |\mathbf{L}|^2/(2I(\mathbf{x})) \leq E\}$, where $\mathbf{L}$ and E are constants of the motion.

Chapter 2

One degree of freedom

Newton's equations are differential equations that arise from mechanical systems, and as such they come with some special properties. In this chapter, we will see some of the implications these properties yield for solutions. While the definition of these properties extends to higher dimensions, we will focus on the case of one degree of freedom where the consequences are particularly clear (because trajectories in phase space have codimension 1). Sections 2.2 to 2.4 are based on [Str15, Ch. 5–7] and [JS07, Ch. 10], and section 2.5 is based on [LL76, Ch. 3] and [Arn89, Ch. 2].

2.1 Linear systems

Suppose we have a Newtonian system with one degree of freedom, and that the force is only a function of position:

$$m\ddot{x} = F(x).$$

Any such system is automatically conservative, because we can always find an antiderivative $-V(x)$ (unique up to an additive constant) so that

$$m\ddot{x} = -V'(x). \tag{2.1}$$

We want to understand the qualitative behavior of solutions near a given point x_0. We may assume that $x_0 = 0$ for convenience, after replacing the variable x by $x - x_0$ if necessary.

First we begin with the generic case $V'(0) \neq 0$. Let $a = V'(0)$. While x is small we have $-V'(x) \approx -a$, and so we expect $x(t)$ to be approximately equal to the solution of

$$m\ddot{x} = -a.$$

This can be integrated twice to obtain a quadratic function for $x(t)$, which describes motion with uniform acceleration. In phase space $\mathbb{R}_x \times \mathbb{R}_p$, the vector

field at (x_0, p_0) points in the direction

$$\dot{x} = \tfrac{1}{m} p_0, \qquad \dot{p} = -a.$$

So all trajectories near $x = 0$ flow in a direction dictated by a—downward if $a > 0$ and upward if $a < 0$. In fact, in a neighborhood of the origin there exists a smooth change of variables that straightens the flow into parallel lines (cf. Proposition A.18).

With this easy case out of the way, we will now assume $V'(0) = 0$ for the remainder of this section. The point $(x_0, p_0) = (0, 0)$ is then a **fixed point** (or **equilibrium**) of the flow, meaning that the constant function $x(t) \equiv 0$ (and $p(t) \equiv 0$) solves the equation. Our first step is to linearize: Taylor expanding about $x = 0$ and keeping only the linear term, we obtain

$$m\ddot{x} = -V'(x) \approx -V'(0) - V''(0)x = -V''(0)x. \tag{2.2}$$

The behavior of solutions to this approximate equation depends on the sign of $V''(0)$.

We begin with the case $V''(0) < 0$:

Example 2.1 (Saddle node)**.** Consider the linear system

$$m\ddot{x} = kx$$

for $k > 0$ a constant. This is a conservative system with potential and total energy

$$V(x) = -\tfrac{1}{2} k x^2, \qquad E(x, p) = \tfrac{1}{2m} p^2 - \tfrac{1}{2} k x^2.$$

The trajectories in phase space are confined to the level sets of E, which look like axes-symmetric hyperbolas. The origin is a **saddle node** for this linear system, and we have the explicit solutions

$$\begin{aligned} x(t) &= x_0 \cosh\left(\sqrt{\tfrac{k}{m}}\, t\right) + \tfrac{p_0}{\sqrt{km}} \sinh\left(\sqrt{\tfrac{k}{m}}\, t\right), \\ p(t) &= \sqrt{km}\, x_0 \sinh\left(\sqrt{\tfrac{k}{m}}\, t\right) + p_0 \cosh\left(\sqrt{\tfrac{k}{m}}\, t\right). \end{aligned}$$

We would like to know if the solutions to the approximate equation (2.2) provide an accurate prediction for the solutions to the actual equation (2.1). From ODE theory, the Hartman–Grobman theorem (**??**) tells us that there is a continuous change of variables from the nonlinear system to the linearized system, provided that all the eigenvalues of the linearized system have nonzero real parts. The theorem applies in this case, because we have the matrix

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ k & 0 \end{bmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

with eigenvalues $\pm\sqrt{\frac{k}{m}}$.

Next we consider the case $V''(0) > 0$:

Example 2.2 (Harmonic oscillator)**.** Consider the linear system

$$m\ddot{x} = -kx$$

for $k > 0$ a constant. This is a conservative system with potential and total energy

$$V(x) = \tfrac{1}{2}kx^2, \qquad E(x,p) = \tfrac{1}{2m}p^2 + \tfrac{1}{2}kx^2.$$

The trajectories in phase space are confined to the level sets of E, which look like axes-parallel ellipse centered at the origin. The origin is a **center** for this linear system, and we have the explicit solutions

$$\begin{aligned} x(t) &= x_0 \cos\big(\sqrt{\tfrac{k}{m}}\, t\big) + \tfrac{p_0}{\sqrt{km}} \sin\big(\sqrt{\tfrac{k}{m}}\, t\big), \\ p(t) &= -\sqrt{km}\, x_0 \sin\big(\sqrt{\tfrac{k}{m}}\, t\big) + p_0 \cos\big(\sqrt{\tfrac{k}{m}}\, t\big). \end{aligned}$$

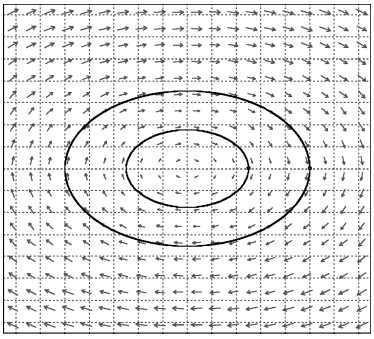


Figure 2.1: Phase portrait for the system of Example 2.2.

Unfortunately, we are unable to immediately conclude that this is an accurate description of the actual equation (2.1) because the Hartman–Grobman theorem no longer applies. Indeed, our matrix is

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{bmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

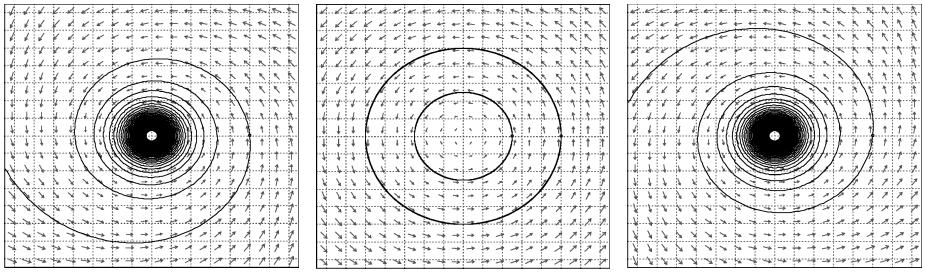
with purely imaginary eigenvalues $\pm i\sqrt{\frac{k}{m}}$.

We might hope that the conclusion of the Hartman–Grobman theorem still persists. This is not obvious however, because the premise of the theorem is necessary:

Example 2.3. Consider the system

$$\begin{matrix} \dot{x} = -y + ax(x^2+y^2) \\ \dot{y} = x + ay(x^2+y^2) \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{matrix}$$

on $\mathbb{R}^2$, where $a \in \mathbb{R}$ is a constant. When $a = 0$ we obtain the linearized system at $(x_*, y_*) = (0, 0)$, which has eigenvalues $\pm i$ and predicts that the origin is a center. However, when $a < 0$ ($a > 0$) we see that $r(t)$ is decreasing (increasing) monotonically and so the origin becomes a stable (unstable) spiral.



(a) $a < 0$ (b) $a = 0$ (c) $a > 0$

Figure 2.3: Phase portraits for the system of Example 2.3.

We will see in sections 2.2 and 2.4 that the prediction is indeed accurate for conservative systems, because the mechanical system (2.1) has special properties that general ODEs do not.

Finally, consider the case $V''(0) = 0$. Then the equation $m\ddot{x} = 0$ can be directly integrated to obtain a linear function for $x(t)$, which describes rectilinear motion (uniform motion with constant velocity). This is of course not a robust prediction since $\dot{p}$ is nonzero whenever $V' \neq 0$, and so we cannot draw any conclusions.

2.2 Conservative systems

In section 1.4 we saw that for conservative Newtonian systems the total mechanical energy is constant along trajectories. We will now give this mathematical phenomenon a name and examine its consequences. In addition to conservative mechanical systems, this also applies to some systems of ODEs which do not arise from mechanics; cf. Exercise 2.2.

Suppose we have the first-order ODE system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2.3}$$

defined for $\mathbf{x}$ in an open set $U \subset \mathbb{R}^n$. We can always reduce a degree-d system of ODEs to a larger first order system by treating the time derivatives $\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x}^{(d-1)}$ as independent variables from $\mathbf{x}$.

Definition 2.4. The ODE system (2.3) is called **conservative** if there exists a smooth function $E : U \to \mathbb{R}$ that is nonconstant on open subsets of U such that $\frac{\mathrm{d}}{\mathrm{d}t}(E(\mathbf{x}(t))) = 0$ for all solutions $\mathbf{x}(t)$.

Geometrically, this requires that trajectories $\mathbf{x}(t)$ lie in level sets of $E(\mathbf{x})$, and so the quantity E can restrict the directions in which trajectories may travel. We require that E is nonconstant on open sets so that E rules out some directions. For example, the constant function $E(\mathbf{x}) \equiv 0$ trivially obeys $\frac{\mathrm{d}}{\mathrm{d}t}(E(\mathbf{x}(t))) = 0$, but it does not reveal anything about the behavior of solutions.

A point $\mathbf{x}_*$ where $\mathbf{f}(\mathbf{x}_*) = 0$ is called a **fixed point** (or **equilibrium**) of (2.3). This implies that the constant function $\mathbf{x}(t) \equiv \mathbf{x}_*$ is a solution of (2.3).

A fixed point $\mathbf{x}_*$ is **attracting** if there exists an open ball $B_\epsilon(\mathbf{x}_*)$ centered at $\mathbf{x}_*$ so that for any initial condition $\mathbf{x}(0) \in B_\epsilon(\mathbf{x}_*)$ the corresponding solutions $\mathbf{x}(t)$ converge to $\mathbf{x}_*$ as $t \to \infty$. Likewise, a fixed point $\mathbf{x}_*$ is **repulsive** if the same statement holds with $-t$ in place of t.

Proposition 2.5. *If the ODE system* (2.3) *is conservative, then there are no attracting (or repulsive) fixed points.*

Proof. Suppose $\mathbf{x}_*$ were an attracting fixed point, and let $\epsilon > 0$ such that $\mathbf{x}(t) \to \mathbf{x}_*$ as $t \to \infty$ for all initial conditions $\mathbf{x}(0) \in B_\epsilon(\mathbf{x}_*)$. Note that E is continuous and is constant on the trajectory $\mathbf{x}(t)$, and so we have $E(\mathbf{x}(0)) = \lim_{t\to\infty} E(\mathbf{x}(t)) = E(\mathbf{x}_*)$. As $\mathbf{x}(0) \in B_\epsilon(\mathbf{x}_*)$ was arbitrary we conclude that E is constant on the open ball $B_\epsilon(\mathbf{x}_*)$, which contradicts the definition of E.

Substituting $t \mapsto -t$ yields the proof for repulsive fixed points. □

In order to understand the qualitative behavior of solutions near a fixed point $\mathbf{x}_*$, we can try to linearize about $\mathbf{x}_*$ as in section 2.1:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}_*) + D\mathbf{f}(\mathbf{x}_*)(\mathbf{x} - \mathbf{x}_*) = D\mathbf{f}(\mathbf{x}_*)(\mathbf{x} - \mathbf{x}_*), \tag{2.4}$$

where $D\mathbf{f}(\mathbf{x}_*)$ is the Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}_*$. Recall from Example 2.3 that when the linearized system predicts that $\mathbf{x}_*$ a center, we cannot conclude anything about the behavior of nearby solutions to the nonlinear system (2.3). This is because centers are delicate, in the sense that trajectories need to perfectly match up after one revolution and the neglected terms in the Taylor expansion (2.4) can prevent this by pushing them slightly inwards or outwards.

In two dimensions, the presence of a conserved quantity is enough to recover the prediction:

Theorem 2.6 (Nonlinear centers for conservative systems)**.** *Suppose that the 2-dimensional ODE system* (2.3) *has an isolated fixed point $\boldsymbol{x}_*$ and that $E(\boldsymbol{x})$ is a conserved quantity. If $\boldsymbol{x}_*$ is an isolated critical point of E and the Hessian matrix $E''(\boldsymbol{x}_*)$ is positive (or negative) definite, then all trajectories sufficiently close to $\boldsymbol{x}_*$ are closed.*

Proof. We will focus on the case where $\mathbf{x}_*$ is a minimum of E. Fix $\epsilon > 0$ sufficiently small so that $\mathbf{x}_*$ is the only fixed point of $\mathbf{f}$ in $B_\epsilon(\mathbf{x}_*) \setminus \{\mathbf{x}_*\}$, that $\mathbf{x}_*$ is the only critical point of E in this neighborhood, and that E has a strict global minimum on $B_\epsilon(\mathbf{x}_*)$ at $\mathbf{x}_*$.

Fix $\mathbf{x}_0 \in B_\epsilon(\mathbf{x}_*) \setminus \{\mathbf{x}_*\}$. Let $c = E(\mathbf{x}_0)$, and consider the component γ of the level set $E^{-1}(c)$ in $B_\epsilon(\mathbf{x}_*)$ containing $\mathbf{x}_0$. We know that γ is a smooth curve

by the implicit function theorem, because $\mathbf{x}_*$ is the only critical point of E in $B_\epsilon(\mathbf{x}_*)$ and hence ∇E is nonzero on this curve.

We claim that γ consists of only one trajectory. As E has a strict local minimum at $\mathbf{x}_*$, we know that $c = E(\mathbf{x}_0)$ is strictly greater than $E(\mathbf{x}_*)$ and hence $\mathbf{x}_*$ does not lie on γ. Also, there can be no other fixed points on γ because $\mathbf{f}$ is nonvanishing on $B_\epsilon(\mathbf{x}_*) \smallsetminus \{\mathbf{x}_*\}$. Therefore γ can consist only of nonconstant trajectories, and thus must be one single trajectory by connectedness.

Lastly, we claim that this trajectory $\mathbf{x}(t)$ is periodic. We know that $\mathbf{x}(t)$ exists for all $t \in \mathbb{R}$, because it is confined to the bounded set γ and so the blowup condition of Corollary A.6 can never be satisfied. Suppose for a contradiction that $\mathbf{x}(t)$ never repeats any value. Consider the sequence $\mathbf{x}(1), \mathbf{x}(2), \ldots$. It is contained in the closed and bounded set γ, and thus must admit a convergent subsequence. Along this sequence the derivative $\dot{\mathbf{x}} = \mathbf{f}$ is converging to zero. Since $\mathbf{f}$ is continuous then the value of $\mathbf{f}$ at the limit point must be zero, which contradicts that γ does not contain a fixed point.

Therefore the trajectory starting at $\mathbf{x}_0$ is periodic. As $\mathbf{x}_0 \in B_\epsilon(\mathbf{x}_*) \smallsetminus \{\mathbf{x}_*\}$ was arbitrary, we conclude that all trajectories in $B_\epsilon(\mathbf{x}_*) \smallsetminus \{\mathbf{x}_*\}$ are closed orbits.

Replacing E by $-E$ yields an analogous proof for local maxima of E. □

We need to assume that $\mathbf{x}_*$ is an isolated fixed point, otherwise there could actually be fixed points on the energy contour; cf. Exercise 2.3.

2.3 Nonconservative systems

In practice, we know that mechanical systems are never exactly conservative: dissipative forces (e.g. kinetic friction) dampen the motion and prevent trajectories from being perfectly periodic. However, we still have $\frac{\mathrm{d}}{\mathrm{d}t}E \leq 0$ in this case, which has new mathematical consequences for solutions.

Definition 2.7. Consider the two-dimensional ODE system (2.3) with a fixed point $\mathbf{x}_*$. Suppose there exists a smooth function $E(\mathbf{x})$ on a connected neighborhood $U = E^{-1}((-\infty, C))$ of the fixed point such that $\mathbf{x}_*$ is a strict global minimum with value zero, and for all $c \in (0, C)$ the level sets $E(\mathbf{x}) = c$ within U form simple closed curves. If also:

(1) $\frac{\mathrm{d}}{\mathrm{d}t}E(\mathbf{x}) \leq 0$ for all $\mathbf{x}$ in $U \smallsetminus \{\mathbf{x}_*\}$, then E is a **weak Lyapunov function**.

(2) $\frac{\mathrm{d}}{\mathrm{d}t}E(\mathbf{x}) < 0$ for all $\mathbf{x}$ in $U \smallsetminus \{\mathbf{x}_*\}$, then E is a **strong Lyapunov function**.

Example 2.8 (Damped harmonic oscillator)**.** Consider the one dimensional system

$$m\ddot{x} = -b\dot{x} - kx$$

with $b > 0$ a positive damping constant. The total energy is still

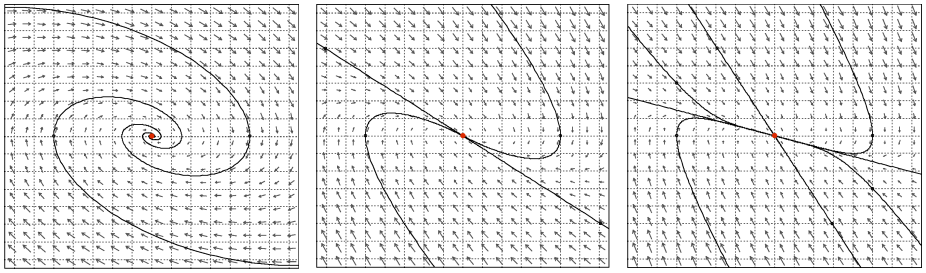
$$E = \tfrac{1}{2}(m\dot{x}^2 + kx^2),$$

but now

$$\frac{\mathrm{d}}{\mathrm{d}t}E = m\dot{x}\ddot{x} + kx\dot{x} = -b\dot{x}^2 \leq 0.$$

The total energy E is now a weak (but not strong) Lyapunov function. The origin is globally attracting with three qualitatively different phase portraits:

- $0 < b < 2\sqrt{km}$: **Under damped**. The origin is a stable spiral and the system oscillates infinitely many times with exponentially decaying amplitude.
- $b = 2\sqrt{km}$: **Critically damped**. The origin is a stable degenerate node. The oscillation and friction balance each other so that trajectories barely fail to make one complete oscillation. In fact, trajectories approach the origin faster than in the other two cases.
- $b > 2\sqrt{km}$: **Over damped**. The origin is a stable node and the system returns to the origin without oscillating.



(a) Under damped (b) Critically damped (c) Over damped

Figure 2.5: Phase portraits for the system of Example 2.8.

For our image of the ball rolling down the graph of the potential energy, the surface of the graph is now slightly sticky. The ball may still roll through a minimum, but does not have enough energy to approach the boundary $V^{-1}(E_0)$ again and so the permitted region for the ball continually shrinks. If V is shaped like a well about $\mathbf{x}_*$ as in the definition of a Lyapunov function then we intuitively expect that the ball tends to the bottom $\mathbf{x}_*$ of the well, and so $\mathbf{x}_*$ should be stable.

Theorem 2.9. *Consider the smooth two-dimensional ODE system* (2.3) *with a fixed point* $\boldsymbol{x}_*$.

(1) If there exists a weak Lyapunov function near the fixed point $\boldsymbol{x}_*$ *then the fixed point is Lyapunov stable: for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $|\boldsymbol{x}(0) - \boldsymbol{x}_*| < \delta$ *implies* $|\boldsymbol{x}(t) - \boldsymbol{x}_*| < \epsilon$ *for all* $t \geq 0$.

(2) *If there exists a strong Lyapunov function near the fixed point $\boldsymbol{x}_*$ then $\boldsymbol{x}_*$ is also asymptotically stable: for all initial conditions the corresponding solutions obey $\boldsymbol{x}(t) \to \boldsymbol{x}_*$ as $t \to \infty$.*

In particular, if there is a strong Lyapunov function then there can be no periodic solutions, unlike conservative systems.

Proof. (1) Fix $\epsilon > 0$. After shrinking ϵ if necessary, we may assume $B_\epsilon(\mathbf{x}_*) \subset U$. As E is smooth, there exists $c > 0$ sufficiently small so that the level set $E(\mathbf{x}) = c$ is contained in $B_\epsilon(\mathbf{x}_*)$. Pick $\delta > 0$ so that the ball $B_\delta(\mathbf{x}_*)$ is contained within the simple closed curve $E(\mathbf{x}) = c$.

Fix $\mathbf{x}(0) \in B_\delta(\mathbf{x}_*)$. We claim that the trajectory $\{\mathbf{x}(t) : t \geq 0\}$ can never enter the exterior of the curve $E(\mathbf{x}) = c$. Suppose for a contradiction that there exists $t > 0$ such that $\mathbf{x}(t)$ is in the exterior of $E(\mathbf{x}) = c$. Then $E(\mathbf{x}(t)) > E(\mathbf{x}(0))$. As $E(\mathbf{x}(t))$ is smooth, the mean value theorem guarantees that there is a time $t' \in [0, t]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\mathbf{x}(t))\bigg|_{t=t'} > 0.$$

This contradicts that E is a weak Lyapunov function.

In particular, the trajectory is contained in $B_\epsilon(\mathbf{x}_*)$ because it is contained in the level set $E(\mathbf{x}) = c$.

(2) Fix $\mathbf{x}(0)$, and let $c = E(\mathbf{x}(0)) > 0$ so that $\mathbf{x}(0)$ lies in the level set $E(\mathbf{x}) = c$.

We claim that there are no equilibrium points other than $\mathbf{x}_*$ and no closed orbits. In the former case, we would have $\frac{\mathrm{d}}{\mathrm{d}t} E(\mathbf{x}(t)) = 0$ for all $t \geq 0$, which contradicts that E is a strong Lyapunov function. In the latter case, there would exist $t > 0$ such that $\mathbf{x}(t) = \mathbf{x}(0)$, and hence $E(\mathbf{x}(t)) = E(\mathbf{x}(0))$. As $E(\mathbf{x}(t))$ is smooth, the mean value theorem guarantees that there is a time $t' \in [0, t]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\mathbf{x}(t))\bigg|_{t=t'} = 0,$$

which contradicts that E is a strong Lyapunov function.

By the the Poincaré–Bendixson theorem (cf. [CL55, Ch. 16 Th. 2.1]), the only remaining possibility for the trajectory $\{\mathbf{x}(t) : t \geq 0\}$ is that it approaches the origin. □

Next, we will examine the behavior of solutions in the limit where the friction term dominates. In this limit, the $\ddot{x}$ term is negligible and we may be left with an equation of the form $\dot{x} = -\nabla V(\mathbf{x})$. For the damped harmonic oscillator in Example 2.8 this is the limit $b \gg mk$ (i.e. $mk/b^2 \to 0$), which is rigorously justified in Exercise 2.4.

Definition 2.10. The ODE system (2.3) is a **gradient system** if there exists a smooth function V such that $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$.

Although conservative systems $\ddot{x} = -\nabla V(\mathbf{x})$ and gradient systems $\dot{x} = -\nabla V(\mathbf{x})$ look similar, they display entirely opposite behavior and should not be confused.

From multivariate calculus we know that the vector field $-\nabla V$ points in the direction of steepest descent for V and is orthogonal to the level sets of V. For our image of the ball rolling down the graph of the potential energy, the ball slows and never reaches the first minimum it encounters, as if the potential energy graph was a tank filled with water. Closed orbits are of course impossible again; cf. Exercise 2.5.

Example 2.11. In the over-damped limit for the harmonic oscillator we have

$$\dot{x} = -\tfrac{k}{b}x, \qquad V(x) = \tfrac{k}{2b}x^2.$$

This has the solution $x(t) = x_0 e^{-kt/b}$, which is the limiting (slow timescale) behavior for the over damped oscillator after the transient (fast timescale) behavior becomes negligible. In this limit, the trajectories in the phase portrait are confined to the line $p = m\dot{x} = -\frac{mk}{b}x$, which agrees with the fact that we can no longer take a second arbitrary initial condition $p(0)$ for the now one-dimensional system.

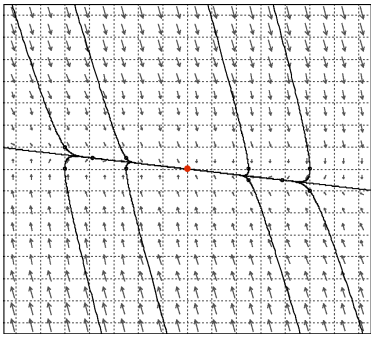


Figure 2.6: Phase portrait for the system of Example 2.11.

2.4 Time reversibility

In addition to a conserved quantity, conservative Newtonian systems also possess a symmetry in time. We will now give this phase space symmetry a name and study its mathematical consequences.

Definition 2.12. The ODE system (2.3) is **(time-)reversible** if there exists a smooth involution $\mathbf{R} : U \to U$ (i.e. $\mathbf{R}(\mathbf{R}(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in U$) such that the change of variables $t \mapsto -t$, $\mathbf{x} \mapsto \mathbf{R}(\mathbf{x})$ leaves the system (2.3) invariant (i.e. if $\mathbf{x}(t)$ is a solution, then so is $\mathbf{R}(\mathbf{x}(-t))$).

Example 2.13. Consider a Newtonian system of the form

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

that is independent of time and velocity (or if $\mathbf{F}$ is even in velocity and time). Note that the force $\mathbf{F}$ does not have to be conservative. This system is invariant under the change of variables $t \mapsto -t$ since $\ddot{\mathbf{x}}$ picks up two factors of -1. Consequently, if $\mathbf{x}(t)$ is a solution then so is $\mathbf{x}(-t)$.

As a first-order system of ODEs in phase space, the involution is $\mathbf{R}(\mathbf{x}, \mathbf{p}) = (\mathbf{x}, -\mathbf{p})$. Each plane $\{(x_i, p_i)\}$ in phase space is reflected about the position axis $\{(x_i, p_i) : p_i = 0\}$ and the arrows on trajectories are reversed. Consequently, the trajectories are symmetric about the position axes with the arrows reversed.

Recall from Example 2.3 that when the linearized system (2.4) predicts that $\mathbf{x}_*$ is a center, we cannot conclude anything about the behavior of nearby solutions to the nonlinear system (2.3). In two dimensions however, reversibility is enough to recover the prediction:

Theorem 2.14 (Nonlinear centers for reversible systems)**.** *Suppose that the 2-dimensional ODE system* (2.3) *is reversible with* $\boldsymbol{R}$ *a reflection across a line, and that* $\boldsymbol{x}_*$ *is an isolated fixed point lying on the line of reflection. If the linearized system about* $\boldsymbol{x}_*$ *predicts a center, then all trajectories sufficiently close to* $\boldsymbol{x}_*$ *are closed.*

Proof. As $\mathbf{f}$ is smooth, Taylor's theorem with remainder guarantees that

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{x}_*) \cdot (\mathbf{x} - \mathbf{x}_*) + \mathcal{O}(|\mathbf{x} - \mathbf{x}_*|^2).$$

By premise $\mathbf{x}_*$ is a linear center, and so solutions $\mathbf{y}(t)$ to the linearized equation

$$\dot{\mathbf{y}} = D\mathbf{f}(\mathbf{x}_*) \cdot (\mathbf{y} - \mathbf{x}_*)$$

are concentric ellipses centered at $\mathbf{x}_*$.

Fix $\mathbf{x}_0 \in B_\epsilon(\mathbf{x}_*)$ on the line of reflection. We will show that we can choose ϵ small enough so that $\mathbf{x}(t)$ is never on the opposite side of $\mathbf{x}_*$ as $\mathbf{y}(t)$. As the trajectory $\mathbf{y}(t)$ encircles the origin, it follows that the trajectory $\mathbf{x}(t)$ intersects the line of reflection on the other side of $\mathbf{x}_*$ at some time $t > 0$. By reversibility, we can reflect this trajectory to obtain a twin trajectory with the same endpoints but with its arrow reversed. Taking $\epsilon > 0$ smaller if necessary, we can know that $\mathbf{x}_*$ is the only fixed point in $B_\epsilon(\mathbf{x}_*)$, and so the two trajectories together form a closed orbit.

It only remains to justify that such an $\epsilon > 0$ exists. Let $T > 0$ denote the period of the periodic solutions $\mathbf{y}(t)$ to the linearized system. Given an initial condition $\mathbf{x}_0$, define the difference

$$\mathbf{h}_t(\mathbf{x}_0) = \mathbf{x}(t) - \mathbf{y}(t)$$

at time $t \in [0, T]$ between the nonlinear and linear solutions starting at $\mathbf{x}_0$. The differential equations for $\mathbf{x}$ and $\mathbf{y}$ match to first order, and so we have

$$\mathbf{h}_t(0) = 0, \qquad D\mathbf{h}_t(0) = 0,$$

where D denotes the gradient in the spatial coordinates. As $\mathbf{f}$ is smooth, there exists a constant c so that

$$|D\mathbf{h}_t(\mathbf{x}_0)| \leq c \qquad \text{for all } \mathbf{x}_0 \in B_\epsilon(\mathbf{x}_*),\ t \in [0,T].$$

Moreover, $c \to 0$ as $\epsilon \downarrow 0$. Using the mean value theorem we estimate

$$|\mathbf{h}_t(\mathbf{x}_0)| = |\mathbf{h}_t(\mathbf{x}_0) - \mathbf{h}_t(0)| \leq c|\mathbf{x}_0| \qquad \text{for all } \mathbf{x}_0 \in B_\epsilon(\mathbf{x}_*),\ t \in [0,T].$$

The linear solutions $\mathbf{y}(t)$ are ellipses, and so there exists a constant $a > 0$ (depending on the semi-major and semi-minor axes) so that

$$|\mathbf{y}(t)| \geq a|\mathbf{y}(0)| = a|\mathbf{x}_0| \qquad \text{for all } t \in [0,T].$$

Combing the previous two inequalities, we conclude that there exists $\epsilon > 0$ sufficiently small so that

$$|\mathbf{x}(t) - \mathbf{y}(t)| = |\mathbf{h}_t(\mathbf{x}_0)| \leq \tfrac{1}{2}|\mathbf{y}(t)| \qquad \text{for all } \mathbf{x}_0 \in B_\epsilon(\mathbf{x}_*),\ t \in [0,T].$$

In other words, for $\mathbf{x}(0) = \mathbf{y}(0) \in B_\epsilon(\mathbf{x}_*)$ we have $|\mathbf{x}(t) - \mathbf{y}(t)| < |\mathbf{y}(t)|$ for all $t \in [0,T]$, and so $\mathbf{x}(t)$ is never on the opposite side of $\mathbf{x}_*$ as $\mathbf{y}(t)$. □

This argument can also be applied to specific examples to show the existence of isolated closed, homoclinic, and heteroclinic orbits. The key input is establishing that the trajectory eventually reaches the hyperplane of symmetry, and then we can extend the trajectory using time-reversal symmetry.

We note that a general involution $\mathbf{R}$ can behave very differently in comparison to reflections, and so Theorem 2.14 does not easily generalize to nonlinear involutions. In particular, there may not be a subspace of fixed points; Exercise 2.8 provides a simple two-dimensional example where the symmetry only fixes one point. The analysis for nonlinear time symmetries depends strongly on the type of involution at hand.

2.5 Periodic motion

Consider a Newtonian system with one degree of freedom of the form

$$m\ddot{x} = F(x). \tag{2.5}$$

Any such system is conservative, because we can always find an antiderivative $-V(x)$ (unique up to an additive constant) so that $F = -V'(x)$. From Proposition 1.9 we know that the total energy

$$E = \tfrac{1}{2}m\dot{x}^2 + V(x) \tag{2.6}$$

is conserved for motions of the system. Notice that (2.6) provides a first-order equation for the motion $x(t)$ in place of the second-order equation (2.5). In this

section, we will use this observation to solve for the motion and record some consequences.

Suppose that the potential $V(x)$ is shaped like a well, in the sense that $V(x) \to +\infty$ as $x \to \pm\infty$. The total energy (2.6) is conserved by Proposition 1.9, and hence $E(t) \equiv E$ is constant. The kinetic energy $\frac{1}{2}m\dot{x}^2$ is nonnegative, and so the solution $x(t)$ is confined to the region $\{x : V(x) \leq E\}$ in configuration space $\mathbb{R}$. This set is bounded since $V(x) \to +\infty$ as $x \to \pm\infty$, and so the motion is bounded.

By conservation of energy, the velocity $\dot{x}(t)$ vanishes for values of x with $V(x) = E$; these values are called the **turning points** of the motion. They are the two endpoints of the interval of $\{x : V(x) < E\}$ containing x_0, and they are the extremal points of the motion.

Most trajectories are periodic and oscillate between the two turning points. Indeed, if we have $V'(x) \neq 0$ at the two turning points, then the trajectory $x(t)$ reaches the turning point in finite time and doubles back. This is not true if $V'(x)$ vanishes at one of the turning points. One possibility is that the trajectory starts at an equilibrium point (x_0, p_0) in phase space, in which case we have $p_0 = 0$ and $V'(x_0) = 0$ so that $\dot{x} = 0$ and $\dot{p} = 0$ respectively. Alternatively, it could be that the turning point is an equilibrium, in which case the motion $x(t)$ approaches the turning point as $t \to \infty$ but never reaches it; if $x(t)$ reached the turning point then we have $\dot{x}(t) = 0$ by conservation of energy, but this violates the uniqueness of the equilibrium solution.

Suppose $x(t)$ is periodic with turning points $x_1(E) < x_2(E)$. Solving the energy equation (2.6) for $\dot{x}$ we obtain

$$\dot{x} = \sqrt{\tfrac{2}{m}[E - V(x)]}. \tag{2.7}$$

This is a separable differential equation with solution

$$t(E) = \sqrt{\frac{m}{2}} \int \frac{\mathrm{d}x}{\sqrt{E - V(x)}} + t_0.$$

The period τ is given by the integral from x_1 to x_2 to x_1. The integral from x_2 to x_1 is -1 times the integral from x_1 to x_2, and so we obtain two times the integral from x_1 to x_2:

$$\tau(E) = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{\mathrm{d}x}{\sqrt{E - V(x)}}. \tag{2.8}$$

Altogether, we have proved the following:

Proposition 2.15. *Suppose the one-dimensional conservative Newtonian system* (2.5) *is such that the potential energy obeys $V(x) \to +\infty$ as $x \to \pm\infty$. Then the motion $x(t)$ of the system is bounded. Moreover, when the motion is periodic the period is given by the integral* (2.8)*, where E is the initial energy and $x_1(E) < x_2(E)$ are the turning points of the motion.*

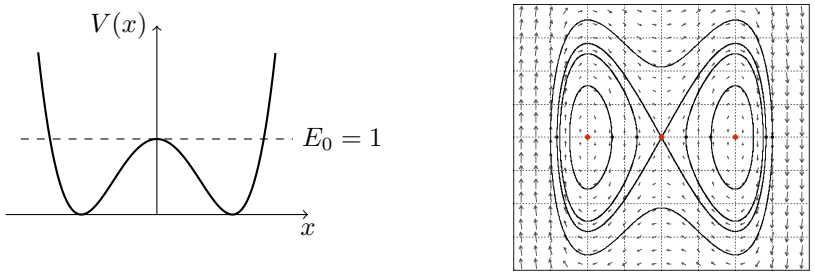
Example 2.16 (Double well)**.** Consider the system (2.5) with the double-well potential

$$V(x) = (x+1)^2(x-1)^2.$$

There are three equilibrium points in phase space: the two bottoms of the well, $(x_0, p_0) = (\pm 1, 0)$ with energy $E = 0$, and the unstable equilibrium $(x_0, p_0) = (0, 0)$ between the wells with energy $E = 1$.

For E in $(0, 1)$ or $(1, \infty)$ the motion is periodic. The turning points x_1, x_2 satisfy $V'(x_i) \neq 0$, and so $V(x) \approx E + V'(x_i)(x - x_i)$ nearby. Consequently the integrand of (2.8) has a singularity $(x - x_i)^{-1/2}$ at both endpoints, and so the integral converges.

The level set $E = 1$ in phase space is comprised of three trajectories: the equilibrium $(0, 0)$ and two homoclinic orbits which are each traversed in an infinite amount of time. We have $V'(0) = 0$, and so near the turning point $x = 0$ we have $V(x) \approx E + \frac{1}{2}V''(0)x^2$. Consequently the integrand of (2.8) has a singularity x^{-1} at the endpoint $x = 0$, and so the integral diverges to $+\infty$.



(a) Potential energy. (b) Phase portrait.

Figure 2.8: The system of Example 2.16.

The period of motion also has a geometric interpretation. Given a periodic trajectory at energy E, let $S(E)$ denote the area enclosed by the trajectory in the phase plane $\mathbb{R}_x \times \mathbb{R}_p$ with turning points $x_1 < x_2$. The system (2.5) is time-reversible, and so as in section 2.4 the trajectory in phase space is symmetric about the x-axis. Therefore, expressing the top half of the curve $p = m\dot{x}$ as a function of position using (2.7), we may write the area as twice the area under the curve:

$$S(E) = 2\int_{x_1(E)}^{x_2(E)} \sqrt{2m[E - V(x)]}\,\mathrm{d}x.$$

We can differentiate this using the Leibniz integral rule to get

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}E} &= 2\int_{x_1(E)}^{x_2(E)} \frac{\mathrm{d}}{\mathrm{d}E}\sqrt{2m[E - V(x)]}\,\mathrm{d}x \\ &\quad + 2\sqrt{2m[E - V(x_2)]}\frac{\mathrm{d}x_2}{\mathrm{d}E} - 2\sqrt{2m[E - V(x_1)]}\frac{\mathrm{d}x_1}{\mathrm{d}E}\end{aligned}$$

As x_1, x_2 are turning points we know $V(x_1) = V(x_2) = E$, and so the second and third terms vanish. This yields

$$\frac{\mathrm{d}S}{\mathrm{d}E} = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{\mathrm{d}x}{\sqrt{E - V(x)}}.$$

Comparing this expression to the period integral (2.8), we conclude:

Proposition 2.17. *For the system of Proposition 2.15, the period of a periodic trajectory is equal to the rate of change of the enclosed area with respect to energy:*

$$\tau(E) = \frac{\mathrm{d}S}{\mathrm{d}E}(E).$$

As a partial converse to Proposition 2.15, we will show that the period determines the shape of the potential.

Proposition 2.18. *For the system of Proposition 2.15, suppose that on an interval $(-r, r)$ the potential V has a minimum at $x = 0$, has no other critical points, and is even. Then all trajectories are periodic, and the potential V is uniquely determined (up to an additive constant) by the periods $\tau(E)$.*

Proof. By premise we have $V'(x) > 0$ for $x > 0$ and $V'(x) < 0$ for $x < 0$. This guarantees that all trajectories other than the equilibrium solution are periodic. As adding a constant does not change the equations of motion, we may assume that $V(0) = 0$. Each value of $V > 0$ corresponds to exactly two values of x, which we will notate by $x_-(V) \in (-r, 0)$ and $x_+(V) \in (0, r)$. Changing variables in the period integral (2.8), we obtain

$$\begin{aligned}
\tau(E) &= \sqrt{2m} \int_{x_1(E)}^{0} \frac{\mathrm{d}x}{\sqrt{[E - V(x)]}} + \sqrt{2m} \int_{0}^{x_2(E)} \frac{\mathrm{d}x}{\sqrt{[E - V(x)]}} \\
&= \sqrt{2m} \int_{E}^{0} \frac{\mathrm{d}x_-}{\mathrm{d}V} \frac{\mathrm{d}V}{\sqrt{E - V}} + \sqrt{2m} \int_{0}^{E} \frac{\mathrm{d}x_+}{\mathrm{d}V} \frac{\mathrm{d}V}{\sqrt{E - V}} \\
&= \sqrt{2m} \int_{0}^{E} \left(\frac{\mathrm{d}x_+}{\mathrm{d}V} - \frac{\mathrm{d}x_-}{\mathrm{d}V} \right) \frac{\mathrm{d}V}{\sqrt{E - V}}.
\end{aligned}$$

We divide both sides by $\sqrt{W - E}$ and integrate with respect to E from 0 to W:

$$\begin{aligned}
&\int_0^W \frac{\tau(E)\,\mathrm{d}E}{\sqrt{W - E}} = \sqrt{2m} \int_0^W \int_0^E \left(\frac{\mathrm{d}x_+}{\mathrm{d}V} - \frac{\mathrm{d}x_-}{\mathrm{d}V} \right) \frac{\mathrm{d}V\,\mathrm{d}E}{\sqrt{(W - E)(E - V)}} \\
&= \sqrt{2m} \int_0^W \left(\frac{\mathrm{d}x_+}{\mathrm{d}V} - \frac{\mathrm{d}x_-}{\mathrm{d}V} \right) \left[\int_V^W \frac{dE}{\sqrt{(W - E)(E - V)}} \right] \mathrm{d}V \\
&= \sqrt{2m} \int_0^W \left(\frac{\mathrm{d}x_+}{\mathrm{d}V} - \frac{\mathrm{d}x_-}{\mathrm{d}V} \right) \left[\tan^{-1} \left(\frac{W + V - 2E}{2\sqrt{(W - E)(E - V)}} \right) \right]_{E=V}^{E=W} \mathrm{d}V \\
&= \sqrt{2m} \int_0^W \left(\frac{\mathrm{d}x_+}{\mathrm{d}V} - \frac{\mathrm{d}x_-}{\mathrm{d}V} \right) \pi\,\mathrm{d}V = \pi\sqrt{2m}\,(x_+(W) - x_-(W)).
\end{aligned}$$

In the last equality we noted $x_+(0) = 0 = x_-(0)$. Taking $W = V$ we obtain

$$x_+(V) - x_-(V) = \frac{1}{\pi\sqrt{2m}} \int_0^V \frac{\tau(E)}{\sqrt{V-E}}\, \mathrm{d}E.$$

In this way, the period determines the potential energy. □

2.6 Exercises

2.1 (Pendulum)**.** Consider a mass m attached to the end of a rigid massless rod of length ℓ with the other end suspended at a fixed point. We allow the rod to rotate in a vertical plane, subject to a constant downward gravitational acceleration g.

(a) Let x denote the angle from the vertical directly below the pivot, and show that

$$\ddot{x} = -\tfrac{g}{\ell} \sin x.$$

(We have not allowed for configuration spaces other than Euclidean space yet, so at the moment we are simply viewing $x \in \mathbb{R}$.)

(b) Sketch the potential energy and the phase portrait, and convince yourself that the trajectories correspond to a small ball rolling down the graph of the potential. Note that near the origin in the phase plane the diagram looks like that of the harmonic oscillator (cf. Example 2.2), which is a consequence of the small angle approximation $\sin x \approx x$. Identify the equilibria and the **separatrix** (the eye-shaped boundary separating different modes of behavior). How many trajectories make up the separatrix from $-\pi \le x \le \pi$, and to what motion do they correspond?

(c) Now add a damping term:

$$\ddot{x} = -b\dot{x} - \tfrac{g}{\ell} \sin x$$

where $b > 0$. Sketch the new phase portrait, and show that E decreases monotonically along all trajectories that are not equilibria.

2.2 (A non-mechanical conservative system)**.** In dimensionless form, a Lotka–Volterra predator-prey model takes the form

$$\dot{x} = x(1-y), \qquad \dot{y} = \mu y(x-1),$$

where $\mu > 0$ is a constant.

(a) Use the chain rule to find a differential equation for $\mathrm{d}y/\mathrm{d}x$.

(b) Find a conserved quantity $E(x, y)$ for this system by integrating this separable differential equation.

(c) Show that all trajectories are periodic for initial conditions $x_0, y_0 > 0$.

2.3 (Minimum of a conservative system that is not a center)**.** Consider the system

$$\dot{x} = xy, \qquad \dot{y} = -x^2.$$

Show that $E(x, y) = x^2 + y^2$ is a conserved quantity and plot the phase portrait for this system. Although the origin is a minimum for E, it is not an isolated fixed point nor a center.

2.4 (Harmonic oscillator in the over damped limit)**.** This is an example of the over damped limit yielding a gradient system, as advertised in section 2.3. The objective is to find a limit for the damped harmonic oscillator

$$m\ddot{x} = -b\dot{x} - kx, \qquad b, k > 0$$

which justifies neglecting the $\ddot{x}$ term.

(a) By equating the units of all terms, determine the dimensions of b and k.

(b) Define a new dimensionless variable τ via $t = T\tau$ where T is a constant with units time to be chosen. Find the new differential equation in terms of τ. Divide the equation by the coefficient of the x term, pick T to cancel the coefficient of the $\dot{x}$ term, and check that T has units time.

(c) The coefficient of the $\ddot{x}$ term should be mk/b^2, and so the limit in which this term is negligible is $\epsilon = mk/b^2 \to 0$. Find the general solution $x(t) = c_1 e^{k_1 t} + c_2 e^{k_2 t}$ for the linear equation $\epsilon\ddot{x} + \dot{x} + x = 0$.

(d) Recall that $1/|k|$ is called the characteristic time of e^{kt}, since after a time $1/|k|$ the function has decreased (since $k < 0$) by a factor of $1/e$. Find the leading term in the Taylor expansion of $1/k_1$ and $1/k_2$ about $\epsilon = 0$; these are called the fast and slow timescales for the solution.

2.5. Show that nonconstant periodic solutions are impossible in a gradient system by considering the change in $V(\mathbf{x})$ around such an orbit. Conclude that any one-dimensional first order ODE has no periodic solutions.

2.6 (Low-regularity existence for gradient systems [San17])**.** The special form of gradient systems allows us to establish existence and uniqueness with less regularity than is needed. This is particularly useful for gradient PDE systems, where $\mathbb{R}^n$ is replaced by an infinite-dimensional function space.

Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is a convex function. This guarantees that the subdifferential

$$\partial F(\mathbf{x}) = \{\mathbf{v} \in \mathbb{R}^n : F(\mathbf{y}) \geq F(\mathbf{x}) + \mathbf{v} \cdot (\mathbf{y} - \mathbf{x}) \text{ for all } \mathbf{y} \in \mathbb{R}^n\}$$

is nonempty for each $\mathbf{x} \in \mathbb{R}^n$. Note that F is differentiable at $\mathbf{x}$ if and only if $\partial F(\mathbf{x}) = \{\nabla F(\mathbf{x})\}$. For $\mathbf{x} : [0, \infty) \to \mathbb{R}^n$ absolutely continuous, consider the problem

$$\dot{\mathbf{x}}(t) \in -\partial F(\mathbf{x}(t)) \quad \text{for a.e. } t > 0, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

(a) Prove that solutions are unique by differentiating the squared difference of two solutions.

(b) Fix $\tau > 0$, and recursively define the sequence

$$\mathbf{x}_0^\tau = \mathbf{x}_0, \qquad \mathbf{x}_{k+1}^\tau \text{ a minimizer of } F(\mathbf{x}) + \tfrac{1}{2\tau}|\mathbf{x} - \mathbf{x}_k^\tau|^2.$$

Show that

$$\tfrac{1}{\tau}\big[\mathbf{x}_{k+1}^\tau - \mathbf{x}_k^\tau\big] \in -\partial F(\mathbf{x}_{k+1}^\tau).$$

If F is differentiable then this is the implicit Euler scheme for $\dot{\mathbf{x}} = -\nabla F(\mathbf{x})$.

(c) In order to obtain convergence as $\tau \downarrow 0$ we need a compactness estimate. Use the definition of the sequence $\mathbf{x}_k^\tau$ show that

$$\sum_{k=0}^{K} \tfrac{1}{2\tau}|\mathbf{x}_{k+1}^\tau - \mathbf{x}_k^\tau|^2 \le F(\mathbf{x}_0) - F(\mathbf{x}_{K+1}^\tau).$$

(d) Define the piecewise constant and linear interpolations

$$\mathbf{x}^\tau(t) = \mathbf{x}_{k+1}^\tau, \quad \tilde{\mathbf{x}}^\tau(t) = \mathbf{x}_k^\tau + (t - k\tau)\tfrac{\mathbf{x}_{k+1}^\tau - \mathbf{x}_k^\tau}{\tau} \quad \text{for } t \in (k\tau, (k+1)\tau].$$

Use the previous part to show there exists a constant C such that

$$\int_0^T \tfrac{1}{2}|(\tilde{\mathbf{x}}^\tau)'(t)|^2 \,\mathrm{d}t \le C.$$

Conclude that

$$|\tilde{\mathbf{x}}^\tau(t) - \tilde{\mathbf{x}}^\tau(s)| \le C|t - x|^{1/2}, \qquad |\tilde{\mathbf{x}}^\tau(t) - \mathbf{x}^\tau(t)| \le C\tau^{1/2}.$$

(e) Assume that F is bounded below. For any $T > 0$, use Arzelà–Ascoli to show that $\tilde{\mathbf{x}}^\tau : [0, T] \to \mathbb{R}^n$ has a uniformly convergent subsequence as $\tau \downarrow 0$, and that $\mathbf{x}^\tau$ converges uniformly to the same limit. After passing to a further subsequence if necessary, show that $(\tilde{\mathbf{x}}^\tau)'$ converges weakly in $L^2([0, T])$. Conclude that the limit $\mathbf{x}(t)$ solves the gradient system for F.

2.7 (Gradient flows in PDE)**.** For $u : \mathbb{R}^n \to \mathbb{R}$ define the Dirichlet energy

$$E(u) = \begin{cases} \frac{1}{2}\int |\nabla u(x)|^2 \,\mathrm{d}x & \text{if } \nabla u \in L^2, \\ +\infty & \text{else.} \end{cases}$$

In analogy with directional derivatives for functions on $\mathbb{R}^n$, the gradient $\nabla E(u)$ of the functional E at u is defined by

$$\frac{\mathrm{d}}{\mathrm{d}s} E(u + sv)\bigg|_{s=0} = \langle \nabla E(u), v \rangle$$

when it exists. Different choices of inner products on the right-hand side yields different gradients.

(a) For the inner product

$$\langle u, v\rangle_{L^2} = \int u(x)v(x)\,\mathrm{d}x$$

on $L^2(\mathbb{R}^n;\mathbb{R})$, show that the gradient flow for the energy E is the heat equation

$$\frac{\partial u}{\partial t} = -\Delta u.$$

(b) For the inner product

$$\langle u, v\rangle_{H^1} = \int \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x$$

on $\dot{H}^1(\mathbb{R}^n;\mathbb{R}) = \{u : \mathbb{R}^n \to \mathbb{R} : \int |\nabla u|^2\,\mathrm{d}x < \infty\}$, show that the gradient flow for the energy E is the equation

$$\frac{\partial u}{\partial t} = -u.$$

Note that the higher regularity norm yields less regular solutions: it is well-known that solutions to the heat equation are smooth, while solutions $u(t,x) = e^{-t}u_0(x)$ to this equation are only as smooth as the initial data.

2.8 (A non-mechanical reversible system)**.** Show that the system

$$\dot{x} = -2\cos x - \cos y, \qquad \dot{y} = -2\cos y - \cos x$$

is reversible, by plotting the phase portrait and finding a symmetry. Note that the presence of stable and unstable nodes guarantees that this system is not conservative.

2.9 (Pendulum period)**.** Show that for the pendulum

$$\ddot{x} = -\tfrac{g}{\ell}\sin x$$

the motion with turning points $\pm\theta_0$ has period

$$\tau = 4\sqrt{\tfrac{\ell}{g}}K\left(\sin\tfrac{\theta_0}{2}\right),$$

where

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\xi}{\sqrt{1-k^2\sin^2\xi}}$$

is the complete elliptic integral of the first kind. Taylor expanding about $\theta_0 = 0$, find the expansion:

$$\tau \approx 2\pi\sqrt{\tfrac{\ell}{g}}\left(1 + \tfrac{1}{16}\theta_0^2 + \ldots\right).$$

Note that the zeroth order term is the constant period approximation obtained when we take $\sin x \approx x$ and replace the pendulum with a harmonic oscillator.

Chapter 3

Central fields

We will examine some examples of physical systems which are both standard and instructive. This limited selection is meant to provide a baseline intuition; a thorough study of classical mechanics should include many more examples, e.g. rigid bodies and the mechanical top. The material is based on [Arn89, Ch. 2], [LL76, Ch. 3], and [Gol51, Ch. 3].

3.1 Central fields

In this section we will solve for the motion of a single particle in $\mathbb{R}^3$ with a central force $\mathbf{F}$. A vector field $\mathbf{F}$ is called **central** (about the origin) if all of the vectors are radial and that the magnitude of the force is only a function of the radial coordinate $r = |\mathbf{x}|$; in other words, $\mathbf{F} \equiv F(r)\hat{\mathbf{r}}$. (This definition of course extends to $\mathbb{R}^d$, but we will shortly need to specialize to $\mathbb{R}^3$ in order to discuss angular momentum.)

A central field must be conservative, and the corresponding potential energy $V = V(r)$ depends only on the distance from the origin. This is because $F(r)$ is only a function of one variable and so we can always find an antiderivative $-V(r)$. Alternatively, if we write $\mathbf{F} = F(r)\hat{\mathbf{r}}$ then the work

$$\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s} = \int_{|\mathbf{x}_1|}^{|\mathbf{x}_2|} F(r) \, \mathrm{d}r$$

is path independent, and so from Proposition 1.7 we know the force is conservative. Consequently, we can write $\mathbf{F} = -\nabla V$ for a radial potential energy $V(r)$.

The torque of the particle is

$$\mathbf{N} = \mathbf{x} \times \mathbf{F} = F(r)r(\hat{r} \times \hat{r}) = 0,$$

and so by Proposition 1.15 we know the particle's angular momentum $\mathbf{L}$ is conserved. Therefore $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is constant in time, and since $\mathbf{L}$ is perpendicular

to $\mathbf{x}$ by definition then we know that the particle's motion is confined to the plane orthogonal to $\mathbf{L}$ provided that $\mathbf{L} \neq 0$ initially. When $\mathbf{L} = 0$, then $\mathbf{x}$ is parallel to the velocity $\dot{\mathbf{x}}$, and thus the particle's motion must be collinear. In both cases, the motion is coplanar.

Using polar coordinates (r, ϕ) on this plane, the velocity is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix} = \dot{r}\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} + r\dot{\phi}\begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}.$$

The magnitude L of the angular momentum $\mathbf{L}$ is then

$$L = |\mathbf{x} \times \mathbf{p}| = |(r\hat{\mathbf{r}}) \times (m\dot{\mathbf{x}})| = mr^2\dot{\phi}. \tag{3.1}$$

The area ΔS of an angular wedge swept by the radius vector $\mathbf{x}$ over an angle $\Delta\phi$ is to first order

$$\Delta S = \tfrac{1}{2}\mathbf{x} \cdot \mathbf{x}\Delta\phi + \mathcal{O}(\Delta\phi^2) = \tfrac{1}{2}r^2\dot{\phi}\Delta t + \mathcal{O}(\Delta t^2).$$

Therefore, the area S swept by the radius vector as a function of time obeys

$$\dot{S} = \tfrac{1}{2}r^2\dot{\phi} = \tfrac{1}{2m}L. \tag{3.2}$$

This proves the following consequence of the conservation of angular momentum:

Proposition 3.1 (Kepler's second law)**.** *In equal times the particle's radial vector sweeps out equal areas, i.e. $\dot{S}$ is constant.*

We can now use the constant L to help us write the total energy as

$$E = T + V = \tfrac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + V(r) = \tfrac{m}{2}\dot{r}^2 + \tfrac{L^2}{2m}r^{-2} + V(r). \tag{3.3}$$

This is the total energy for a one-dimensional Newtonian system in the coordinate r with the **effective potential** energy

$$V_{\text{eff}}(r) = V(r) + \tfrac{L^2}{2m}r^{-2}.$$

The added term in the effective potential is called the **centrifugal energy**. When the effective potential is equal to the total energy we have $\dot{r} = 0$, which is a turning point for the one-dimensional system. Unlike in the previous section, the actual particle is not at rest at such a point because the angle is changing (unless the angular momentum is zero).

Example 3.2. Kepler's problem seeks the equations of motion for a particle moving around a fixed gravitational mass, which is governed by the potential

$$V(r) = -kr^{-1}, \qquad k > 0.$$

This yields the effective potential

$$V_{\text{eff}}(r) = -kr^{-1} + \tfrac{L^2}{2m}r^{-2}.$$

The potential $V(r)$ tends to $-\infty$ as $r \downarrow 0$, but the added centrifugal energy makes the effective potential $V_{\text{eff}}(r)$ tend to $+\infty$.

If the initial energy E_0 is nonnegative, then the trajectory is unbounded. The particle comes in from infinity, swings around the central mass reaching a minimum radius where $V_{\text{eff}} = E_0$, and returns towards infinity.

If the initial energy E_0 is negative, then the trajectory is bounded. The particle moves within an annulus, oscillating between the inner and outer radii where $V_{\text{eff}} = E_0$. The trajectory could be periodic, or could never realign and (in fact) be dense in the annulus.

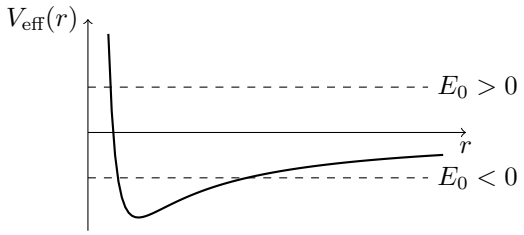


Figure 3.1: Effective potential for the system of Example 3.2.

Rearranging the total energy (3.3), we obtain

$$\dot{r} = \sqrt{\tfrac{2}{m}\left[E - V_{\text{eff}}(r)\right]}. \tag{3.4}$$

This is a separable differential equation, with solution

$$t = \int \frac{\mathrm{d}r}{\sqrt{\tfrac{2}{m}\left[E - V_{\text{eff}}(r)\right]}} + t_0. \tag{3.5}$$

We can also use the expression (3.4) for $\dot{r}$ to solve the separable differential equation (3.1) for ϕ:

$$\phi = \int \frac{L\,\mathrm{d}r}{r^2\sqrt{2m\left[E - V_{\text{eff}}(r)\right]}} + \phi_0. \tag{3.6}$$

This yields an equation for ϕ as a function of r. Note that the constancy of L requires that ϕ cannot change sign.

The original system had 6 degrees of freedom: 3 dimensions for the position $\mathbf{x} \in \mathbb{R}^3$ and 3 dimensions for the momentum $\mathbf{p} \in \mathbb{R}^3$. We then used 4 conserved quantities to reduce the system to 2 first-order equations, which could then be explicitly integrated. Indeed, the conservation of the 3 components of the angular momentum eliminated 2 degrees of freedom by restricting the motion to be coplanar, and then eliminated another degree of freedom by providing the first-order equation (3.1) for ϕ. The fourth conserved quantity was the total energy, which we used to obtain the first-order equation (3.3) for r.

As in the previous section, the motion is confined to the region $V_{\text{eff}} \leq E$. If the particle is unbounded for some finite energy E, then if the potential energy at infinity $V_\infty = \lim_{r\to\infty} V(r) = \lim_{r\to\infty} V_{\text{eff}}(r)$ exists it must be finite. If $E > V_\infty$, then we can define the **velocity at infinity** v_∞ via $E = \frac{1}{2}mv_\infty^2 + V_\infty$. Conversely, in the region $r \downarrow 0$, in order for the particle to reach the center the potential $V(r)$ must not outgrow the centrifugal energy:

$$\lim_{r\downarrow 0} r^2 V(r) < -\tfrac{L^2}{2m}.$$

Suppose now that for some given V_{eff} we have motion bounded by two turning points $r_{\min}$ and $r_{\max}$, which confines the particle within the annulus bounded by these two radii. Points where $r = r_{\min}$ are called **pericenters** and where $r = r_{\max}$ are called **apocenters**. The time symmetry for the one-dimensional system in r allow us to conclude that the trajectory will be symmetric about any ray from the origin through a pericenter or apocenter. According to the solution (3.6), the angle between successive pericenters (or apocenters) is

$$\Phi = 2\int_{r_{\min}}^{r_{\max}} \frac{L\,\mathrm{d}r}{r^2\sqrt{2m[E - V_{\text{eff}}(r)]}}$$

by symmetry. In general, Φ is not a rational multiple of 2π, and consequently the particle's orbit is not closed (and is in fact dense) in the annulus.

3.2 Periodic orbits

There are two standard examples of central fields: the three-dimensional harmonic oscillator potential

$$V(r) = kr^2, \qquad k > 0, \tag{3.7}$$

and the gravitational potential

$$V(r) = -kr^{-1}, \qquad k > 0. \tag{3.8}$$

The objective of this section is to show that these are the only two central fields for which all bounded orbits are periodic:

Theorem 3.3. *Suppose a particle moves in a smooth central field on* $\mathbb{R}^3 \setminus \{0\}$ *and there exists a bounded trajectory. If all bounded trajectories are periodic, then the potential is either the harmonic oscillator potential* (3.7) *or the gravitational potential* (3.8).

For $V(r) = -kr^p$ with p in the interval $(-2,-1)$ or $(-1,0)$, the effective potential still has the same qualitative shape as in the case $p = -1$ from Example 3.2: $V_{\text{eff}}(r) \to +\infty$ as $r \downarrow 0$, $V_{\text{eff}}(r) \to 0$ as $r \to \infty$, and there is a negative minimum in between. Theorem 3.3 implies that for $E_0 < 0$ the trajectories are not periodic, and in fact they are dense in the annulus defined by $\{r : V_{\text{eff}}(r) \leq E_0\}$.

Suppose $V(r)$ is a central field in which all bounded orbits are closed. The existence or closed orbits guarantees that $V(r)$ has a (strict) local minimum at some value $r > 0$. If $V'(r) < 0$ for all $r > 0$, then we eventually have $V(r) \leq E_0 - a$ for some $a > 0$ and all t large, and hence $\dot{r} \leq b > 0$ for all t large by conservation energy. If $V'(r) > 0$ for all $r > 0$, then we eventually have $V(r) \leq E_0 - a$ for some $a > 0$ and all t large, and hence $\dot{r} \leq -b < 0$ for all t large by conservation energy. In either case, we eventually have that $\dot{r}$ is single signed, and so no closed orbits could exist.

Let r_0 denote a local minimum of V. If the initial radius is sufficiently close to r_0 then the energy E_0 will be close to $V(r_0)$ and the motion will be confined to a bounded component of $\{r : V(r) \leq E_0\}$. By the solution (3.6) for the angular coordinate, given radial turning points $r_{\min} \leq r \leq r_{\max}$ the angle between successive pericenters or successive apocenters is

$$\Phi = 2 \int_{r_{\min}}^{r_{\max}} \frac{L}{r^2 \sqrt{2m[E - V(r) - L^2/2mr^2]}} \, \mathrm{d}r.$$

Substituting $x = L/mr$, we obtain

$$\Phi = \sqrt{2m} \int_{x_{\min}}^{x_{\max}} \frac{1}{\sqrt{E - V(L/mx) - mx^2/2}} \, \mathrm{d}x. \tag{3.9}$$

This is the period integral (2.8) for the one-dimensional system with potential

$$W(x) = V\left(\tfrac{L}{mx}\right) + \tfrac{m}{2}x^2.$$

Next, we compute the limiting period for small oscillations near minima for the one-dimensional system with potential W:

Lemma 3.4. *Consider a conservative one-dimensional Newtonian system with smooth potential $W(x)$. If W has a local minimum x_0 with value E_0, then*

$$\lim_{E \downarrow E_0} \tau(E) = 2\pi \sqrt{\tfrac{m}{W''(x_0)}}. \tag{3.10}$$

Proof. After substituting $W - E_0$ for W we may assume that $E_0 = 0$. As x_0 is a local minimum with value 0, the Taylor expansion of W at x_0 is

$$W(x) = \tfrac{1}{2}W''(x_0)(x - x_0)^2 + \mathcal{O}((x - x_0)^3).$$

This yields the equations of motion

$$m\ddot{x} = -W''(x_0)(x - x_0) + \mathcal{O}((x - x_0)^2).$$

Without the error term, this is a harmonic oscillator about x_0 with spring constant $k = W''(x_0)$, and in Example 2.2 we found that the solution can be expressed in terms of sine and cosine with period

$$2\pi \sqrt{\tfrac{m}{k}} = 2\pi \sqrt{\tfrac{m}{W''(x_0)}}.$$

This gives us the formula (3.10) for the leading term of $\tau(E)$ as $E \downarrow E_0$. The convergence of $\tau(E)$ as $E \downarrow E_0$ can be justified by passing the limit inside the integral expression (2.8). □

Let r_0 denote a local minimum of V. Then from the period integral (3.9) we know the system for W has a minimum at

$$x_0 = \tfrac{L}{mr_0}.$$

This yields the relation

$$0 = W'(x_0) = -\tfrac{LV'(r_0)}{mx_0^2} + mx_0 \quad \Longrightarrow \quad m^2x_0^3 = LV'(r_0).$$

Using this, we simplify the expression for $W''(x_0)$ to obtain

$$W''(x_0) = m\frac{r_0V''(r_0) + 3V'(r_0)}{V'(r_0)}.$$

Therefore, taking the limit $E \downarrow V(r_0)$ in the angle integral (3.9) and applying (3.10), we conclude that as the initial radius tends to r_0 the angle Φ tends to

$$\Phi_{\text{cir}} = 2\pi\sqrt{\frac{m}{W''(x_0)}} = 2\pi\sqrt{\frac{V'(r_0)}{r_0V''(r_0) + 3V'(r_0)}}. \tag{3.11}$$

All of the orbits near r_0 are bounded, and so by premise they must be closed. As Φ is the angle between successive pericenters or apocenters, some integer multiple of Φ must be equal to $2\pi n$ for some integer $n \geq 1$. Therefore Φ is a rational multiple of 2π, and since the integral is a continuous function of the initial condition in a neighborhood of r_0 then we must have $\Phi = \Phi_{\text{cir}}$ on this neighborhood of r_0. Setting (3.11) equal to a constant, we obtain the linear differential equation

$$cr_0V''(r_0) + (3c-1)V'(r_0) = 0$$

where $c \in \mathbb{R}$ is a constant. The solutions are

$$V(r) = ar^\alpha \text{ for } \alpha \in [-2,0) \cup (0,\infty), \quad \text{and} \quad V(r) = b\log r,$$

and Φ must be globally constant. Plugging this back into Φ_{cir}, we obtain

$$\Phi \equiv \Phi_{\text{cir}} = \tfrac{2\pi}{\sqrt{\alpha+2}} \quad \text{for } \alpha \geq -2. \tag{3.12}$$

The $\alpha = 0$ case corresponds to $V(r) = b\log r$. We will now split into cases.

We can rule out the case $V(r) = b\log r$. Taking $\alpha = 0$ in (3.12) we have $\Phi_{\text{cir}} = \sqrt{2}\pi$, and so Φ is not a rational multiple of 2π.

Consider the case $V(r) = ar^\alpha$ with $\alpha > 0$. The constant a must be positive so that there exists a bounded orbit, and hence $V(r) \to \infty$ as $r \to \infty$. Substituting $x = x_{\max}y$ in the Φ integral (3.9), we have

$$\Phi = \sqrt{2m}\int_{y_{\min}}^1 \frac{\mathrm{d}y}{\sqrt{W^*(1) - W^*(y)}}, \qquad W^*(y) = \frac{my^2}{2} + \frac{1}{x_{\max}^2}V\left(\frac{L}{mx_{\max}y}\right).$$

As $E \to \infty$ we have $x_{\max} \to \infty$ and the second term in W^* tends to zero. Moreover, in this limit we also have $y_{\min} \to 0$ and so we obtain

$$\lim_{E\to\infty} \Phi = \pi.$$

However $\Phi \equiv \Phi_{\text{cir}}$ is a constant, and so comparing to (3.12) we conclude that $\alpha = 2$.

Now consider the case $V(r) = ar^\alpha$ with $-2 \le \alpha < 0$. Taking the limit $E \downarrow 0$ in the integral (3.9), we obtain

$$\lim_{E\downarrow 0} \Phi = 2\int_0^1 \frac{\mathrm{d}x}{\sqrt{x^{-\alpha} - x^2}} = \frac{2\pi}{2+\alpha}.$$

Comparing this to (3.12), we conclude $\alpha = -1$.

Therefore, the only two possible potentials are the harmonic oscillator (3.7) and the gravitational potential (3.8). This concludes the proof of Theorem 3.3.

3.3 Kepler's problem

Kepler's problem seeks explicit solutions for the motion in the central field with potential

$$V(r) = -kr^{-1}, \qquad k > 0.$$

The original motivation was the motion of a celestial body in the gravitational field of a fixed mass at the origin, but this also describes the motion of an electrically charged particle attracted to a fixed charge at the origin.

From section 3.1 we know the motion is coplanar, and the radius r evolves according to the one-dimensional effective potential

$$V_{\text{eff}}(r) = -kr^{-1} + \tfrac{L^2}{2m} r^{-2}.$$

Note that $\lim_{r\downarrow 0} V_{\text{eff}}(r) = +\infty$ and $\lim_{r\to\infty} V_{\text{eff}}(r) = 0$. If $L \neq 0$ then the first derivative

$$V'_{\text{eff}}(r) = \tfrac{k}{r^2} - \tfrac{L^2}{mr^3}$$

has exactly one root for $r \in (0,\infty)$ at $r = L^2/mk$, and so V_{eff} has a unique global minimum with value

$$V_{\text{eff}}\left(\tfrac{L^2}{mk}\right) = -\tfrac{mk^2}{2L^2}.$$

As this is the only extremum, we conclude that $E \ge 0$ yields unbound motion, and $E < 0$, $L \neq 0$, requires that the motion is bounded with $E \ge -mk^2/2L^2$.

For this potential we can evaluate the general solution (3.6) for the angular coordinate:

$$\phi(r) = \cos^{-1}\left(\frac{\frac{L}{r} - \frac{mk}{L}}{\sqrt{2mE + \frac{m^2k^2}{L^2}}}\right).$$

Given an initial condition, we picked the origin for ϕ so that integration constant above is zero. Solving for r as a function of ϕ, we obtain

$$r = \frac{L^2/mk}{1 + \sqrt{1 + 2EL^2/mk^2}\,\cos\phi}.$$

Define the quantities

$$\ell = \tfrac{L^2}{mk}, \qquad \epsilon = \sqrt{1 + \tfrac{2EL^2}{mk^2}},$$

so that we may write

$$r(\phi) = \frac{\ell}{1 + \epsilon\cos\phi}.$$

This is the parametric equation for a **conic section** with one focus at the origin, **eccentricity** $\epsilon \in [0, \infty)$, and **latus rectum** 2ℓ. From planar geometry, the **semi-major** and **semi-minor axes** are determined by

$$a = \tfrac{\ell}{1-\epsilon^2} = \tfrac{k}{2|E|}\ , \qquad b = \tfrac{\ell}{\sqrt{1-\epsilon^2}} = \tfrac{L}{\sqrt{2m|E|}}, \tag{3.13}$$

respectively.

The eccentricities $\epsilon = 1$ and $\epsilon > 1$ correspond to parabolas and hyperbolas respectively, which agrees with the fact that $E \geq 0$ yields unbound orbits. Likewise $\epsilon = 0$ and $0 < \epsilon < 1$ correspond to circles and ellipses respectively, which agrees with the fact that $E \in [-mk^2/2L^2, 0)$ yields bounded orbits.

For the planets in our solar system, the eccentricities are very small and the trajectories are nearly circular. Consequently, before solving Kepler's problem, scientists (e.g. Copernicus) had thought that the planets' orbits were circular with the Sun at the center. Kepler corrected this, and **Kepler's first law** states that the planetary orbits are ellipses with the Sun at one focal point.

Now we will determine the period τ of a bounded elliptic orbit. Integrating Kepler's second law (3.2) over one orbit and recalling the area of an ellipse, we have

$$\pi ab = S = \tfrac{1}{2}L\tau.$$

This yields the explicit formula

$$\tau = \tfrac{2\pi mab}{L} = \pi k\sqrt{\tfrac{m}{2|E|^3}}$$

for the period as a function of the energy. Using the formula (3.13) for the semi-major axis a in terms of the energy E, we obtain

$$\tau = 2\pi\sqrt{\tfrac{ma^3}{k}}.$$

This demonstrates **Kepler's third law**: the squared period of a planet's orbit is proportional to its semi-major axis cubed.

In practice we know that the mass (or charge) sitting at the origin is perturbed by the particle's presence, which we will now address. The **two-body**

problem seeks the equations of motion for a closed system consisting of two gravitational bodies with positions $\mathbf{x}_i$ and masses m_i, for $i = 1, 2$. The system is conservative with potential

$$V(\mathbf{x}_1, \mathbf{x}_2) = -\frac{Gm_1m_2}{|\mathbf{x}_1 - \mathbf{x}_2|}, \tag{3.14}$$

where G is the **gravitational constant**.

By Newton's first law (cf. Proposition 1.14) we know that the center of mass moves with constant velocity. We choose the origin (i.e. a choice of reference frame) to be the center of mass, so that

$$m_1\mathbf{x}_1 + m_2\mathbf{x}_2 \equiv 0.$$

Define $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ to be the distance between the particles. We can recover the positions from $\mathbf{x}$ via

$$\mathbf{x}_1 = \tfrac{m_2}{m_1+m_2}\mathbf{x}, \qquad \mathbf{x}_2 = -\tfrac{m_1}{m_1+m_2}\mathbf{x}. \tag{3.15}$$

The total energy can be written solely in terms of $\mathbf{x}$ as

$$E = \tfrac{1}{2}m_1|\dot{\mathbf{x}}_1|^2 + \tfrac{1}{2}m_2|\dot{\mathbf{x}}_2|^2 + V(|\mathbf{x}_1 - \mathbf{x}_2|) = \tfrac{1}{2}\mu|\dot{\mathbf{x}}|^2 + V(|\mathbf{x}|), \tag{3.16}$$

where $\mu = m_1m_2/(m_1+m_2)$ is the **reduced mass**. That is, *the two-body system is equivalent to a single-particle of mass μ moving in the external central field $V(|\boldsymbol{x}|)$*. Solving for the motion of $\mathbf{x}$ in the reduced problem (3.16) yields the solutions to the original problem (3.14) using the relations (3.15). In particular, the motion of two gravitational bodies (or two attractive charges) will be two conic sections with a shared focus at the origin.

3.4 Virial theorem

The virial theorem is a formula for the time average of a system's kinetic energy, which admits a particularly simple relation for central fields.

Suppose we have a system of N particles in $\mathbb{R}^d$. Consider the quantity $\sum_i \mathbf{x}_i \cdot \mathbf{p}_i$ (which we might recognize from section 1.6 as $2\dot{I}$ where I is the moment of inertia). We calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^N \mathbf{x}_i \cdot \mathbf{p}_i\right) = \sum_{i=1}^N \dot{\mathbf{x}}_i \cdot \mathbf{p}_i + \sum_{i=1}^N \mathbf{x}_i \cdot \dot{\mathbf{p}}_i = 2T + \sum_{i=1}^N \mathbf{F}_i \cdot \mathbf{x}_i.$$

We then take the long-time average

$$\overline{f} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\tau)\,\mathrm{d}\tau \tag{3.17}$$

of both sides. This yields

$$\overline{\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^N \mathbf{x}_i \cdot \mathbf{p}_i\right)} = 2\overline{T} + \overline{\sum_{i=1}^N \mathbf{F}_i \cdot \mathbf{x}_i}. \tag{3.18}$$

For any bounded function f we have

$$\overline{\frac{\mathrm{d}f}{\mathrm{d}t}} = \lim_{T\to\infty} \frac{f(T)-f(0)}{T} = 0.$$

Therefore, if all particle motion is bounded, then the quantity $\sum_i \mathbf{x}_i \cdot \mathbf{p}_i$ must also be bounded and so the left-hand side of (3.18) vanishes. Altogether, we conclude:

Theorem 3.5 (Virial theorem)**.** *If the motion of a Newtonian system of N particles is bounded, then the long-time average* (3.17) *of the kinetic energy is given by*

$$\overline{T} = -\tfrac{1}{2}\overline{\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \boldsymbol{x}_i}.$$

Consider a single particle moving in a central field as in section 3.1. The force $\mathbf{F} = -V'(r)\hat{\mathbf{r}}$ is conservative and radial, and so we obtain

$$\overline{T} = \tfrac{1}{2}\overline{V'(r)r}.$$

Moreover, if we have a power-law force $F(r) = ar^n$, then $V(r) = br^{n+1}$ and so

$$V'(r)r = (n+1)V \quad \Longrightarrow \quad \overline{T} = \tfrac{n+1}{2}\overline{V}.$$

In particular, for Kepler's problem we have $n = -2$ and we obtain the simple relation

$$\overline{T} = -\tfrac{1}{2}\overline{V}.$$

3.5 Exercises

3.1. Show that for the central field with potential

$$V(r) = -kr^{-3},$$

the motion falls into the origin in finite time.

3.2 (Method of similarity)**.** Suppose that the potential energy of a central field is a homogeneous function of degree ν:

$$V(\alpha r) = \alpha^\nu V(r) \qquad \text{for all } \alpha > 0.$$

Show that if a curve γ is the orbit of a motion, then the homothetic curve $\alpha\gamma$ is also an orbit (under the appropriate initial conditions). Determine the ratio of the circulation times along these orbits. Conclude from this the isochronicity of the harmonic oscillator ($\nu = 2$) and Kepler's third law ($\nu = -1$).

3.3 (Escape velocity)**.** Let r_0 denote the radius of the Earth, and $g \approx 9.8\,\mathrm{m/sec}^2$ the gravitational acceleration at the Earth's surface. The gravitational potential energy of the Earth is

$$V(r) = -\tfrac{gr_0^2}{r}.$$

Determine the minimum velocity a particle must be given on the surface of the Earth in order for it to travel infinitely far away.

3.4 (Cosmic velocities)**.** Consider the gravitational potential energy of the Earth as in the previous problem. The escape velocity v_2 is sometimes called the second cosmic velocity. The first cosmic velocity is the velocity of motion on a circular orbit of radius equal to the radius of the earth. Find the magnitude of the first cosmic velocity v_1 and show that $v_2 = \sqrt{2}v_1$.

3.5 (Geosynchronous orbit [Nah16])**.** It is useful for communication satellites to be in geosynchronous orbit, so that their orbital period is that of the Earth's and the satellite appears to hover in the sky. We will calculate the height of this orbit for Earth in two different ways.

(a) Let m be the satellite's mass, $M = 5.98 \times 10^{24}\,\mathrm{kg}$ be the Earth's mass, v be the satellite's velocity, and R_s the radius of the satellite's circular orbit. First equate the gravitational and centripetal accelerations of a circular orbit, then substitute $v = 2\pi R_s/T$ where $T = 86.400\,\mathrm{s}$ is the period of the orbit, and solve for R_s.

(b) Use Kepler's third law to calculate the same value for R_s.

3.6 (Satellite paradox [Nah16])**.** Satellites in low Earth orbit experience significant atmospheric drag, which counter-intuitively increases the speed of the satellite.

(a) For a circularly orbiting satellite in Earth's gravitational potential, conclude from the virial theorem that the satellite's total energy is

$$E = -\tfrac{1}{2}mv^2.$$

Alternatively, this relation can be obtained by equating the gravitational and centripetal accelerations, solving for v, and substituting v into the kinetic energy.

(b) Differentiate the result from part (a) to determine $\dot{v}$ in terms of $\dot{E}$. Assuming the energy loss rate (dissipated power) of the satellite is

$$\dot{E} = -vf_d, \qquad f_d > 0,$$

show that $\dot{v} > 0$.

3.7 (Solar and lunar tides [Nah16])**.** If we differentiate the gravitational potential (3.14), we find the gravitational force between two bodies of masses m_1 and m_2 has magnitude

$$F(r) = Gm_1m_2r^{-2},$$

where r is the distance between the bodies' centers and G is the universal gravitational constant. For the Earth, Sun, and Moon, we have

$$\begin{aligned} M_s &= \text{mass of the Sun} = 2 \times 10^{30}\,\text{kg}, \\ M_m &= \text{mass of the Moon} = 7.35 \times 10^{22}\,\text{kg}, \\ R_s &= \text{Earth–Sun separation} = 93 \times 10^{6}\ \text{miles}, \\ R_m &= \text{Earth–Moon separation} = 2.39 \times 10^{5}\ \text{miles}. \end{aligned}$$

(a) Show that, even though the Sun is much farther from the Earth, the Sun's gravitational force on the Earth is greater than the Moon's. Find their ratio.

(b) As the Earth is not a point mass, then the Sun's gravitational force is stronger (weaker) on the side of the Earth closest (farthest) from the Sun. This causes water to bulge at the points closest an furthest from the Sun, which is called the solar high tides. Calculate the maximum difference in gravitational force in terms of Earth's radius R for both the Sun and the Moon.

(c) Extract the leading term in the limit $R/R_s \ll 1$ and $R/R_m \ll 1$ for each expression in part (b). Calculate their ratio and conclude that, although the Sun's gravitational force is stronger, the lunar tides are more than twice as large as the solar tides.

3.8 (Energy of the ocean tides [Nah16])**.** The high tides are not directly in line with the centers of the Earth and Moon, but are rather carried ahead slightly by the Earth's rotation and frictional forces. This means that the Moon's gravitational pull on both tides produce torque: the Moon's pull on the farther tide increases the Earth's rotational speed, but the stronger pull on the nearer tide is counter-rotational, and so the overall effect decreases the Earth's rotational speed. Atomic clocks have measured that the length of a day is increasing at the rate of about 2 milliseconds per century.

(a) Let Ω denote the angular rotation rate of the Earth and let T denote the length of a day in seconds, so that $\Omega T = 2\pi$. By integrating the kinetic energy of an infinitesimal mass $\mathrm{d}m$ over the volume of the Earth, show that the rotational energy E is given by

$$E = \tfrac{1}{2}\Omega^2 I,$$

where

$$I = \iiint_{r \le R} (x^2 + y^2)\,\mathrm{d}m$$

is the moment of inertia.

(b) Show that for a solid sphere of radius R and constant mass density ρ the moment of inertia is

$$I = \tfrac{8\pi}{15} R^3 \rho,$$

or in terms of the total mass M,

$$I = \tfrac{2}{5} MR^2.$$

The Earth is not a constant-density sphere, and so rather than 0.4 the coefficient is approximately 0.3444.

(c) Write the rotational energy E as a function of the period T, and show that

$$\frac{\mathrm{d}E}{\mathrm{d}T} = -\frac{4 \cdot 0.3444 \cdot M\pi^2 R^2}{T^3}.$$

Taking $T = 86.400\,\mathrm{s}$, the length of a day, and $\Delta T = 2 \times 10^{-3}\,\mathrm{s}$, $M = 5.98 \times 10^{24}\,\mathrm{kg}$, and $R = 6.38 \times 10^{6}\,\mathrm{m}$, find the change in the Earth's rotational energy ΔE over a century. Dividing ΔE by the number of seconds in a century, conclude that the power of the ocean tides is $3,260$ gigawatts or 4.37 billion horsepower.

3.9 (Moon recession rate [Nah16])**.** As in Exercise 3.8, tidal friction decreases the Earth's rotational angular momentum. Consequently, the Moon's orbital angular momentum increases to conserve total angular momentum, which results in the Moon drifting away from the Earth. We will estimate this recession rate, assuming that all of the momentum is transferred to Moon's orbit (and not rotation).

(a) Consider the Moon as a point mass m orbiting circularly about the Earth at a radius r, with speed v and angular speed ω radians per second. Show that the Moon's orbital angular momentum is $L_m = mrv$.

(b) The gravitational force on the Moon by the Earth has magnitude

$$F = GMmr^{-2},$$

where M is the mass of the Earth and G is the universal gravitational constant. Equating the gravitational and centripetal accelerations of the Moon, find v as a function of r and use this to find the angular momentum L_m as a function of r.

(c) Using part (b) of Exercise 3.8, the spin angular momentum of the Earth is $L_e = 0.3444MR^2\Omega$ where Ω is Earth's rotation rate. Expressing Ω in terms of the day length $T = 86.400\,\mathrm{s}$, find L_e as a function of T and calculate $\mathrm{d}L_e/\mathrm{d}T$.

(d) Using the daily change $\Delta T = 2 \times 10^{-5}/365$ seconds in the length of a day, approximate the daily change and then the yearly change in ΔL_e.

(e) Equating change in the Moon's orbital momentum ΔL_m with the change in Earth's rotational momentum $|\Delta L_e|$, find the yearly change in the

Moon's orbital radius. Using the values

$$\begin{aligned} M &= \text{mass of the Earth} = 5.98 \times 10^{24}\,\text{kg}, \\ m &= \text{mass of the Moon} = 7.35 \times 10^{22}\,\text{kg}, \\ r &= \text{radius of Moon's orbit} = 3.84 \times 10^{8}\,\text{m}, \\ R &= \text{Earth's radius} = 6.37 \times 10^{6}\,\text{m}, \\ G &= \text{gravitational constant} = 6.67 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}, \end{aligned}$$

find that the Moon is receding from the Earth at a rate of 3.75 centimeters of 1.48 inches per year. This value is in outstanding agreement with measurements made by a laser on Earth and corner cube reflectors on the Moon.

3.10 (Central field scattering)**.** In this problem we consider a classical model for a beam of charged particles passing near a repulsive central charge. Consider a repulsive central field in $\mathbb{R}^3$ that tends to zero as $|\mathbf{x}| \to \infty$. Suppose we have a uniform beam of particles all of the same mass and energy whose motion begins and ends colinearly at infinity. The **intenstiy** I of the beam is the number of particles crossing unit area normal to the initial direction of travel per unit time.

(a) Define the **impact parameter** s for a particle of mass m and initial velocity v_0 via

$$L = mv_0 s = s\sqrt{2mE}.$$

Using the facts from section 3.1, show that for the solid angle $\Omega \subset S^2$ the scattering **cross section** $\sigma(\Omega)$—the number of particles scattered per unit solid angle per unit time divided by the incident intensity—is given by

$$\sigma(\Theta) = -\frac{s}{\sin\Theta}\frac{\mathrm{d}s}{\mathrm{d}\Theta},$$

where Θ is the angle between the incident and scattered beams.

(b) Suppose the incident particles have charge $-q < 0$ and the fixed particle has charge $-Q < 0$. The motion is dictated by the Coulomb force

$$\mathbf{F}(r) = qQr^{-2}\hat{\mathbf{r}} = \nabla V, \qquad V(r) = qQr^{-1}.$$

Although in section 3.3 we assumed that $V = -kr^{-1}$ with $k > 0$ in order to model celestial motion, we never required that k be positive and so the explicit solutions are still valid. By relating the total angle of deflection Θ to the eccentricity ϵ of the hyperbolic trajectories, show that

$$\sin\tfrac{\Theta}{2} = \tfrac{1}{\epsilon}, \qquad \cot\tfrac{\Theta}{2} = \tfrac{2Es}{qQ}.$$

Using the previous part, conclude that the **Rutherford scattering cross section** is given by

$$\sigma(\Theta) = \frac{1}{4}\left(\frac{qQ}{2E}\right)^2 \frac{1}{\sin^4(\Theta/2)}.$$

This formula is a classical approximation of a quantum system, and thus has limitations. For example, if we were to integrate this expression over the sphere we would obtain an infinite total scattering cross section, which is of course impossible.

(c) As with Kepler's problem, the central charge is not truly fixed but instead recoils as a result of the scattering. Assume the central particle is initially stationary. Let the subscript 1 refer to the scattered particle, primed coordinates denote coordinates relative to the center of mass, and $\mathbf{X}$ represent the position of the center of mass in laboratory coordinates so that $\mathbf{x}_1 = \mathbf{X} + \mathbf{x}_1'$, $\mathbf{v}_1 = \dot{\mathbf{x}}_1 = \dot{\mathbf{X}} + \mathbf{v}_1'$. To replace the angle Θ between the initial and final vectors between the charges, consider the deflection angle ϑ between the initial and final directions of the scattered particles. Use the conservation of total momentum to show that

$$\tan\vartheta = \frac{|\mathbf{v}_1'|\sin\Theta}{|\mathbf{v}_1'|\cos\Theta + |\dot{\mathbf{X}}|}.$$

Then by considering the initial relative velocity, show that

$$\tan\vartheta = \frac{\sin\Theta}{\cos\Theta + \frac{m_1}{m_2}}.$$

As we would expect experimentally, when $m_2 \gg m_1$ the initially stationary particle does not experience significant recoil and $\vartheta \approx \Theta$.

(d) The number of particles scattered into a fixed element of solid angle must be the same in both the laboratory and relative to the center of mass. Consequently, show that the cross section $\sigma'(\vartheta)$ in the laboratory system is given by

$$\sigma'(\vartheta) = \sigma(\Theta)\frac{\sin\Theta}{\sin\vartheta}\frac{\mathrm{d}\Theta}{\mathrm{d}\vartheta} = \sigma(\Theta)\frac{\mathrm{d}(\cos\Theta)}{\mathrm{d}(\cos\vartheta)}.$$

(e) Rutherford was interested in α-particle scattering, for which corrections to $m_1/m_2 = 1$ are negligible. When $m_1 = m_2$, show that $\vartheta = \Theta/2$; in particular, $0 \le \vartheta \le \pi/2$ and there can be no back scattering. Conclude

$$\sigma'(\vartheta) = 4\cos\vartheta\,\sigma(2\vartheta)\ , \qquad \frac{|\mathbf{v}_1|}{|\mathbf{v}_0|} = \cos\vartheta.$$

Part II

Lagrangian Mechanics

The Lagrangian perspective is based upon the Principle of Least Action, which is a coordinate-free reformulation of mechanics on configuration space (and its tangent bundle). Unpacking the Principle of Least Action yields the Euler–Lagrange equations of motion—a system with one second order differential equation for each degree of freedom—and showcases some fundamental ideas from the calculus of variations. The independence of coordinates allows us to work with any choice of coordinates or on any manifold. In particular, when the coordinates are chosen to align with symmetries of the system, the Euler–Lagrange equations are effective in identifying the corresponding conserved quantities, reducing the number of equations, and simplifying the system.

Chapter 4

Euler–Lagrange equations

We will reformulate Newton's equations as the principle of least action, the unpacking of which will demonstrate some of the fundamental techniques of variational calculus. This presentation of the calculus of variations follows [AKN06, Ch. 1], and in the rest of the material is based on [LL76, Ch. 1–2], [Arn89, Ch. 3], and [Gol51, Ch. 1–2].

4.1 Principle of least action

For a mechanical system with N particles, we replace the configuration space $(\mathbb{R}^d)^N$ by a smooth manifold M. While physics texts often focus on the derivation for Euclidean space, we will present the theory for general manifolds. The freedom in the choice of configuration space and coordinates is a key feature of Lagrangian mechanics, and thus this level of generality is not superfluous.

We will use q to denote any (local) coordinates on M (sometimes called **generalized coordinates** in physics) and v (often $\dot{q}$ in physics) to denote a tangent vector in the tangents space T_qM at q. The number of components $n := \dim M$ of q is called the **degrees of freedom** of the system.

Definition 4.1. A **Lagrangian system** $(M, \mathscr{L})$ is a smooth n-dimensional manifold M called the **configuration space** together with a smooth function $\mathscr{L}(q, v, t) : TM \times I \to \mathbb{R}$ called the **Lagrangian**, where TM is the tangent bundle of M and $I \subset \mathbb{R}$ is an interval.

As we will see in section 4.2, the following abstract statement encodes Newton's equations.

Definition 4.2 (Hamilton's principle of least action)**.** Given two times $t_0 < t_1$, a smooth curve $q(t)$ from $[t_0, t_1]$ into M is a **motion** of the Lagrangian system $(M, \mathscr{L})$ if it is a critical point of the **action** functional

$$S(q(t)) = \int_{t_0}^{t_1} \mathscr{L}(q(t), \dot{q}(t), t)\,\mathrm{d}t. \tag{4.1}$$

In this section, we will see how to extract equations of motion from the principle of least action. The principle is called "least" action because it turns out that the motion is often a minimum; however, we will not make use of this additional assumption.

The goal of the calculus of variations (or variational calculus) is to find the extrema of functionals. A **path** from $a_0 \in M$ to $a_1 \in M$ (not necessarily distinct) starting at time t_0 and ending at $t_1 > t_0$ is a smooth map $\gamma : [t_0, t_1] \to M$ with $\gamma(t_0) = a_0$ and $\gamma(t_1) = a_1$. Let Ω denote the collection of all such paths. A **functional** is a function Φ from Ω into $\mathbb{R}$.

Example 4.3. The arc length of the graph of $\gamma = \mathbf{x}(t)$ from $[t_0, t_1]$ into $M = \mathbb{R}^d$ is a functional. It takes as input a smooth function $\mathbf{x}(t)$ from $[t_0, t_1]$ into $\mathbb{R}^d$ and returns the value

$$\Phi(\mathbf{x}(t)) = \int_{t_0}^{t_1} \sqrt{1 + \dot{\mathbf{x}}^2} \, \mathrm{d}t.$$

Intuitively, we expect the path of minimum length between two points to be the line segment connecting those two points. Indeed, we see that $\Phi(\gamma) \geq \int_{t_0}^{t_1} \sqrt{1} \, \mathrm{d}t = t_1 - t_0$ with equality if and only if $\dot{\mathbf{x}} \equiv 0$. The more systematic machinery that we develop in this section should also return this answer.

In order to solve for the optimizer, we want to obtain an equation that must be satisfied by the extremum of a functional. From calculus we expect that the first derivative should vanish at an extremum, and so we want to define a notion of first derivative for functionals. For more concrete examples we do not need to assume arbitrary smoothness (cf. Exercise 4.2) or that an extremum exists (cf. Exercise 4.5), but as before we will focus on the smooth case and remain within the context of smooth manifolds.

A (fixed-endpoint) **variation** of a path $\gamma \in \Omega$ is a smooth map $H(s, t) = H_s(t)$ from $(-\epsilon, \epsilon) \times [t_0, t_1]$ to M for some $\epsilon > 0$ such that:

- $H_0 = \gamma$,
- $H_s \in \Omega$ for all $s \in (-\epsilon, \epsilon)$, and
- $H(s, t_i) = a_i$ for $i = 0, 1$ for all $s \in (-\epsilon, \epsilon)$.

In other words, the paths H_s for various s form a smooth deformation of γ, which is equal to γ at $s = 0$ and always connect a_1 to a_2 for other values of s.

In analogy with functions on Euclidean space, we will call a functional Φ **differentiable** at a path $\gamma \in \Omega$ with **derivative** (or **first variation**) $\mathrm{d}\Phi|_\gamma$ (or $\delta\Phi$) if the limit

$$\lim_{s \to 0} \frac{\Phi(H_s) - \Phi(\gamma)}{s} = \mathrm{d}\Phi|_\gamma \left(\frac{\partial H_s}{\partial s} \right)$$

exists for all variations H of γ. In other words, there should exist a function $\mathrm{d}\Phi|_\gamma$ so that the Taylor expansion

$$\Phi(H_s) = \Phi(\gamma) + s \, \mathrm{d}\Phi|_\gamma \left(\frac{\partial H_s}{\partial s} \right) + o(s)$$

holds. We expect the function $\mathrm{d}\Phi|_\gamma$ to be linear in $\frac{\partial H_s}{\partial s}$ (like $\mathbf{v} \mapsto \nabla f(\mathbf{x}_0) \cdot \mathbf{v}$ on Euclidean space). We need Φ to be a function of $\frac{\partial H_s}{\partial s}$ instead of H because the space of derivatives $\frac{\partial H_s}{\partial s}$ will be linear, unlike the space of variations H which will be nonlinear if M is not flat.

It remains to define $\mathrm{d}\Phi|_\gamma$ and $\frac{\partial H_s}{\partial s}$. These are technical details that do not appear in the Euclidean theory, but they are necessary in order to consider non-Euclidean spaces. We can think of Ω as a manifold (albeit infinite-dimensional), and a variation as a path on Ω through γ at $s = 0$. The derivative $\frac{\partial H_s}{\partial s}$ should be tangent to Ω at the path H_s. We define a tangent vector W to Ω at a path $\gamma \in \Omega$ to be a function which associates to each $t \in [t_0, t_1]$ a tangent vector $W_t \in T_{\gamma(t)}M$, and is smooth in the sense that for every smooth function $f : M \to \mathbb{R}$ the function $t \mapsto \mathrm{d}f(W_t) \in \mathbb{R}$ is smooth. The tangent space $T_\gamma\Omega$ is the space of all tangent vectors W at γ such that $W_{t_0} = 0 = W_{t_1}$. Now we can define the tangent vector $\frac{\partial H_s}{\partial s}(0)$ to the variation H_s at $s = 0$ to be the set of derivatives

$$\frac{\partial H}{\partial s}(0, t) \in T_{\gamma(t)}M \qquad \text{for } t \in [t_0, t_1].$$

As

$$\frac{\partial H}{\partial s}(0, t_0) = 0, \qquad \frac{\partial H}{\partial s}(0, t_1) = 0$$

by the fixed-endpoint requirement, then we indeed have $\frac{\partial H_s}{\partial s}(0) \in T_\gamma\Omega$. Now we define

$$\mathrm{d}\Phi|_\gamma \left(\frac{\partial H_s}{\partial s} \right) = \frac{\mathrm{d}}{\mathrm{d}s} \Phi \circ H_s \Big|_{s=0}.$$

As should be the case, one can check that $\mathrm{d}\Phi|_\gamma$ is well-defined and linear (cf. Exercise 4.1).

Example 4.4. Consider the arc-length functional Φ of Example 4.3. Given two points $\mathbf{a}_0, \mathbf{a}_1 \in \mathbb{R}^d$, the space Ω of paths from $\mathbf{a}_0$ to $\mathbf{a}_1$ is the set of smooth functions $[t_0, t_1] \to \mathbb{R}^d$ with $\mathbf{x}(t_i) = \mathbf{a}_i$ for $i = 0, 1$. For any path $\mathbf{x} : [t_0, t_1] \to \mathbb{R}^d$, a variation $H_s(t)$ of $\mathbf{x}$ takes the form $\mathbf{x}(t) + \mathbf{h}_s(t)$, where $\mathbf{h}_s : [t_0, t_1] \to \mathbb{R}^d$ is smooth satisfying $\mathbf{h}_s(t_0) = 0 = \mathbf{h}_s(t_1)$ for all $s \in (-\epsilon, \epsilon)$ and $\mathbf{h}_0(t) \equiv 0$.

The derivative $\frac{\partial H}{\partial s}$ is equal to $\frac{\partial \mathbf{h}_s}{\partial s}(t)$. As $\mathbf{h}_0(t) \equiv 0$, then the variation at $s = 0$ is equal to $\mathbf{x}(t)$, and so $\frac{\partial H}{\partial s}(0, t) = \frac{\partial \mathbf{h}_0}{\partial s}(t)$ is a vector centered at $\mathbf{x}(t)$ and pointing in the direction of the variation $\mathbf{h}_s(t)$ for small s. Moreover, $\frac{\partial H}{\partial s}(0, t_i) = \frac{\partial \mathbf{h}_0}{\partial s}(t_i) = 0$ since $\mathbf{h}_s(t_0) = 0 = \mathbf{h}_s(t_1)$ for all $s \in (-\epsilon, \epsilon)$.

Conversely, a tangent vector W to Ω at $\mathbf{x}(t)$ is a set of vectors $\{\mathbf{w}_t : t \in [t_0, t_1]\}$ such that $\mathbf{w}_t$ is centered at $\mathbf{x}(t)$, the vectors $\mathbf{w}_{t_0}$ and $\mathbf{w}_{t_1}$ at the endpoints vanish, and they depend smoothly on t (in the sense that $t \mapsto \mathbf{v} \cdot \mathbf{w}_t$ is smooth, or equivalently every component of $\mathbf{w}_t$ is a smooth function of t).

For the rest of this section, we will restrict our attention to the action functional S defined at the beginning of the section. If we take $\mathscr{L}(\mathbf{x}, \mathbf{y}, t) = \sqrt{1 + \mathbf{y}^2}$, we recover the functional of Example 4.3. The reason we insisted that $\mathscr{L}$ be differentiable is so that the action is differentiable. For the principle of least action, we seek a **critical point** γ for Φ, which we define to mean $\mathrm{d}\Phi|_\gamma \equiv 0$.

Proposition 4.5 (Euler–Lagrange equations)**.** *The action functional* (4.1) *is differentiable, with derivative*

$$\mathrm{d}S|_\gamma \left(\frac{\partial H_s}{\partial s}\right) = \int_{t_0}^{t_1} \left(\frac{\partial \mathscr{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) \cdot \frac{\partial H_s}{\partial s}\, dt. \tag{4.2}$$

Consequently, a path $q(t)$ is a critical point for the action if and only if $q(t)$ solves the differential equations

$$\frac{d}{dt}\left[\frac{\partial \mathscr{L}}{\partial \dot{q}}\big(q(t), \dot{q}(t), t\big)\right] - \frac{\partial \mathscr{L}}{\partial q}\big(q(t), \dot{q}(t), t\big) = 0. \tag{4.3}$$

The n-many second-order differential equations (4.3) are called the **Euler–Lagrange equations** for the functional S, or simply the **Lagrange equations** when applied to a mechanical system. Note that (4.3) must hold for any choice of coordinates q, where $\frac{\partial \mathscr{L}}{\partial q}$ is the gradient of $\mathscr{L}(q, v, t)$ in the q variables. The derivative $\frac{\partial \mathscr{L}}{\partial \dot{q}}$ is a convenient notation for the derivative of the Lagrangian $\mathscr{L}(q, v, t)$ in the velocity variables v, and should technically be notated as $\frac{\partial \mathscr{L}}{\partial v}(q, \dot{q}, t)$. Although (4.3) is often shortened to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{\partial \mathscr{L}}{\partial q} = 0,$$

we are meant to plug in q and $\dot{q}$ before taking the total time derivative $\frac{\mathrm{d}}{\mathrm{d}t}$, which for example will turn $\dot{q}$ into $\ddot{q}$.

Proof. Fix a set of coordinates q on M, let γ be a path, and let H be a variation of γ. By taking the variation of γ to be supported within the image of the coordinate patch q, we will work solely withing the domain of q within $\mathbb{R}^n$. Let $\mathbf{x}(t)$ denote the path in $\mathbb{R}^n$, $\mathscr{L}(\mathbf{x}, \dot{\mathbf{x}}, t)$ denote the Lagrangian in $\mathbb{R}^n$, and $\mathbf{x}(t) + \mathbf{h}(t)$ the variation $H_s(t)$ in $\mathbb{R}^n$ (we are suppressing the s-dependence of $\mathbf{h}$). Note that the fixed endpoint condition requires that $\mathbf{h}(t_0) = 0 = \mathbf{h}(t_1)$.

The following argument in Euclidean space lies at the heart of the variational argument, and is part of any presentation of the theory. As $\mathscr{L}$ is differentiable, we may Taylor expand to get

$$\mathscr{L}(\mathbf{x}+\mathbf{h}, \dot{\mathbf{x}}+\dot{\mathbf{h}}, t) = \mathscr{L}(\mathbf{x}, \dot{\mathbf{x}}, t) + \frac{\partial \mathscr{L}}{\partial \mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, t)\cdot\mathbf{h}(t) + \frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}}, t)\cdot\dot{\mathbf{h}}(t) + \mathcal{O}(|\mathbf{h}(t)|^2).$$

Therefore,

$$\begin{aligned} S(\gamma + \mathbf{h}) - S(\gamma) &= \int_{t_0}^{t_1} \Big[\mathscr{L}(\mathbf{x} + \mathbf{h}, \dot{\mathbf{x}} + \dot{\mathbf{h}}, t) - \mathscr{L}(\mathbf{x}, \dot{\mathbf{x}}, t)\Big]\, \mathrm{d}t \\ &= \int_{t_0}^{t_1} \left(\frac{\partial \mathscr{L}}{\partial \mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, t) \cdot \mathbf{h}(t) + \frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}}, t) \cdot \dot{\mathbf{h}}(t)\right) \mathrm{d}t + \mathcal{O}(|\mathbf{h}|^2). \end{aligned}$$

We pulled the term $\mathcal{O}(|\mathbf{h}(t)|^2)$ outside of the integral to get a term $\mathcal{O}(|\mathbf{h}|^2)$ which is bounded by maximum of $|\mathbf{h}(t)|^2$ because $\mathscr{L}$ is continuously differentiable. The

integral in the rightmost expression will be the derivative $\mathrm{d}S|_\gamma$. Integrating by parts yields

$$\int_{t_0}^{t_1} \frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} \cdot \dot{\mathbf{h}} \,\mathrm{d}t = -\int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}}\right) \cdot \mathbf{h} \,\mathrm{d}t + \left(\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} \cdot \mathbf{h}\right)\Bigg|_{t_0}^{t_1}.$$

As $\mathbf{h}(t_0) = 0 = \mathbf{h}(t_1)$ the second term on the RHS above must vanish, and plugging this back into $\mathrm{d}S_\gamma$ produces (4.2) as desired.

It is clear from the formula that if (4.3) is satisfied then the functional derivative (4.2) vanishes. Conversely, assuming that $\mathrm{d}S|_\gamma = 0$ for any variation, we know that

$$\int_{t_0}^{t_1} \left(\frac{\partial \mathscr{L}}{\partial \mathbf{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}}\right) \cdot \mathbf{h} \,\mathrm{d}t = 0$$

for all smooth $\mathbf{h} : [t_0, t_1] \to \mathbb{R}^n$ with $\mathbf{h}(t_0) = 0 = \mathbf{h}(t_1)$. We can conclude that the integrand without $\mathbf{h}$ vanishes identically, since otherwise we could pick $\mathbf{h}(t)$ to be a bump function that witnesses any nonzero value to obtain $\mathrm{d}S|_\gamma(\mathbf{h}) \neq 0$. □

Remark. In the proof of Proposition 4.5, we only need that the coordinates are spanning in order to have enough directions $\mathbf{h}(t)$ to conclude that the Euler–Lagrange equation holds. Therefore, there is no obstruction to extending Proposition 4.5 to more than n coordinates provided that they span the same space.

Corollary 4.6. *Given a Lagrangian $\mathscr{L}(q, \dot{q}, t)$, adding a total time derivative:*

$$\mathscr{L}'(q, \dot{q}, t) = \mathscr{L}(q, \dot{q}, t) + \frac{d}{dt}\left[f(q, t)\right]$$

leaves the Euler–Lagrange equations (4.3) *unchanged.*

Proof. The new action is given by

$$S' = \int_{t_0}^{t_1} L(q, \dot{q}, t) + \int_{t_0}^{t_1} \frac{\mathrm{d}f}{\mathrm{d}t} \,\mathrm{d}t = S + f(q(t_1), t_1) - f(q(t_0), t_0),$$

and the addition of a constant to S does not affect whether or not a path is a critical point of S. □

Example 4.7. We will apply the Euler–Lagrange equations (4.3) to determine the extrema of the arc length functional of Example 4.3. Using Cartesian coordinates $\mathbf{x}$ on $\mathbb{R}^d$, the Euler–Lagrange equations are

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} - \frac{\partial \mathscr{L}}{\partial \mathbf{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\mathbf{x}}}{\sqrt{1 + \dot{\mathbf{x}}^2}}\right) - 0.$$

This means

$$\frac{\dot{\mathbf{x}}}{\sqrt{1 + \dot{\mathbf{x}}^2}} = \mathbf{a}$$

for some constant $\mathbf{a} \in \mathbb{R}^n$. Squaring and rearranging yields

$$\dot{\mathbf{x}} = \mathbf{b}$$

for a new constant $\mathbf{b} \in \mathbb{R}^n$. Finally, integrating yields

$$\mathbf{x}(t) = \mathbf{b}t + \mathbf{c}$$

for $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$. That is, the extrema of the arc length functional are straight lines. If we had instead chosen, say, polar coordinates, the Euler–Lagrange equations (4.3) would yield different differential equations but the solutions would still describe straight lines.

4.2 Conservative systems

In the previous sections we saw how the principle of least action yields the $n = \dim M$ second-order Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{\partial \mathscr{L}}{\partial q} = 0. \tag{4.4}$$

In this section we will see how (4.4) is a formulation of Newton's equations, and hence Newton's principle of determinacy is implied by Hamilton's principle of least action. For this reason, some graduate physics texts opt to begin with the principle of least action and view Newton's equations as a consequence.

First we will see how to apply the Euler–Lagrange equations in order to obtain the Lagrange equations of motion. Let $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^d$ be Cartesian coordinates on Euclidean space. For a conservative system of N particles, we will see that the right choice of Lagrangian is

$$\mathscr{L}(\mathbf{x}, \dot{\mathbf{x}}, t) = T(\dot{\mathbf{x}}) - V(\mathbf{x}), \tag{4.5}$$

where T and V are the kinetic and potential energies. The principle of least action implies that the motion $\mathbf{x}(t) = (\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))$ of the system satisfies the Euler–Lagrange equations (4.4). For this Lagrangian, we obtain

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} &= \frac{\partial T}{\partial \dot{\mathbf{x}}_i} = \frac{\partial}{\partial \dot{\mathbf{x}}}\left(\sum_{i=1}^{N} \tfrac{1}{2} m_i \dot{\mathbf{x}}_i^2\right) = \mathbf{p}, \\
\frac{\partial \mathscr{L}}{\partial \mathbf{x}} &= -\frac{\partial V}{\partial \mathbf{x}} = -\nabla V = \mathbf{F}.
\end{aligned}$$

Therefore the Euler–Lagrange equations are simply Newton's equations (1.2), and so for conservative systems we take the Lagrangian (4.5). The advantage of the Lagrangian perspective is that we are now longer restricted to Euclidean space.

Example 4.8. Consider a pendulum consisting of a mass m attached to the end of a rigid massless rod of length ℓ with the other end fixed, allowed to rotate in a vertical plane subject to a constant downward gravitational acceleration g. Let θ denote the angle from the vertical directly below the pivot, which entirely describes the system. The configuration space is the circle S^1, and the Lagrangian is a (time-independent) function defined on $(\theta, \dot\theta) \in TS^1 = S^1 \times \mathbb{R}$. The kinetic energy is

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\ell^2\dot\theta^2,$$

and since the force acting on the mass is $F = ma = -mgl\sin\theta$ (cf. Exercise 2.1) then the potential energy is

$$V = -mg\ell\cos\theta.$$

(We picked our integration constant so that $V = 0$ when the mass is at the height of the pivot $\theta = \pi/2$.) The Lagrangian is given by

$$\mathscr{L} = T - V = \tfrac{1}{2}m\ell^2\dot\theta^2 + mg\ell\cos\theta,$$

and so the Euler–Lagrange equation is

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathscr{L}}{\partial\dot\theta} - \frac{\partial\mathscr{L}}{\partial\theta} = m\ell^2\ddot\theta + mg\ell\sin\theta \quad \Longrightarrow \quad \ddot\theta = -\frac{g}{\ell}\sin\theta,$$

which agrees with what we found with Newton's equation in Exercise 2.1.

The expression (4.5) of the Lagrangian is not arbitrary, and in fact can be derived from Galileo's principle of relativity (cf. Definition 1.3) applied to the Lagrangian $\mathscr{L}$ rather than the force. Consider a single particle in space whose position is denoted by the Cartesian coordinates $\mathbf{x} \in \mathbb{R}^d$. In an inertial frame, the Lagrangian $\mathscr{L}(\mathbf{x}, \dot{\mathbf{x}}, t)$ of this particle cannot be explicitly dependent on position or time by homogeneity: $\mathscr{L} \equiv \mathscr{L}(\dot{\mathbf{x}})$. Also $\mathscr{L}$ cannot depend on the direction of $\dot{\mathbf{x}}$ either since space is isotropic: $\mathscr{L} \equiv \mathscr{L}(|\dot{\mathbf{x}}|)$. In fact, let us write this as $\mathscr{L} \equiv \mathscr{L}(|\dot{\mathbf{x}}|^2)$ since we expect $\mathscr{L}$ to be smooth. As $\frac{\partial\mathscr{L}}{\partial\mathbf{x}} = 0$, then the Euler–Lagrange equation for this particle is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathscr{L}}{\partial\dot{\mathbf{x}}} = 0.$$

The quantity $\partial\mathscr{L}/\partial\dot{\mathbf{x}}$ is therefore constant in time, and since $\mathscr{L} \equiv \mathscr{L}(\dot{\mathbf{x}})$ then $\dot{\mathbf{x}}$ must be constant. This is the Lagrangian mechanics proof of Newton's first law (cf. Proposition 1.4): in an inertial frame, the free motion of particle is uniform in time with constant velocity.

Now we know that $\mathscr{L} \equiv \mathscr{L}(|\dot{\mathbf{x}}|^2)$, but we have yet to conclude that for our free particle we must have $\mathscr{L} = T$. Let $\mathbf{v} = \dot{\mathbf{x}}$ denote the velocity, and consider an inertial frame K' moving with small velocity $\boldsymbol{\epsilon}$ relative to an inertial frame K. Then $\mathbf{v}' = \mathbf{v} + \boldsymbol{\epsilon}$ and $|\mathbf{v}'|^2 = |\mathbf{v}|^2 + 2\mathbf{v}\cdot\boldsymbol{\epsilon} + |\boldsymbol{\epsilon}|^2$, and so

$$\mathscr{L}(|\mathbf{v}'|^2) = \mathscr{L}(|\mathbf{v}|^2) + 2\mathscr{L}'(|\mathbf{v}|^2)\mathbf{v}\cdot\boldsymbol{\epsilon} + \mathcal{O}(|\boldsymbol{\epsilon}|^2). \tag{4.6}$$

However, as both frames are inertial then the two Lagrangians should be equivalent for all $\boldsymbol{\epsilon}$. Therefore the linear term of (4.6) must be a total time derivative (cf. Corollary 4.6). As the Lagrangian can only be a function of $|\mathbf{v}|^2$, then this term could only be a total time derivative if it is linear in $\mathbf{v}$. We therefore have that $\mathscr{L}'(|\mathbf{v}|^2)$ is independent of $\mathbf{v}$, and hence $\mathscr{L}$ is proportional to $|\mathbf{v}|^2$. This allows us to write

$$\mathscr{L} = \tfrac{1}{2} m |\mathbf{v}|^2, \tag{4.7}$$

where m is the particle's mass. Experimentally, we would observe that a particle's acceleration is inversely proportional to its mass as in section 1.1. Note that m cannot be negative since from the action (4.1) we see that Hamilton's principle would yield maxima instead of minima. We did not use the third type of Galilean transformations, but the expression (4.7) is automatically invariant with respect to rectilinear motion $\mathbf{v}' = \mathbf{v} + \mathbf{v}_0$. Indeed, we have

$$\begin{aligned}\mathscr{L}'(\mathbf{v}') &= \tfrac{1}{2} m |\mathbf{v}'|^2 = \tfrac{1}{2} m |\mathbf{v} + \mathbf{v}_0|^2 = \tfrac{1}{2} m |\mathbf{v}|^2 + m \mathbf{v} \cdot \mathbf{v}_0 + \tfrac{1}{2} m \mathbf{v}_0^2 \\ &= \mathscr{L}(\mathbf{v}) + \frac{\mathrm{d}}{\mathrm{d}t} \left(m \mathbf{x} \cdot \mathbf{v}_0 + \tfrac{1}{2} m |\mathbf{v}_0|^2 t \right),\end{aligned}$$

and so the Euler–Lagrange equations are the same by Corollary 4.6.

For a system of free noninteracting particles, the Lagrangian for each individual particle cannot be dependent on the coordinates of any other, and so $\mathscr{L}$ must be additive:

$$\mathscr{L} = \sum_{i=1}^{N} \tfrac{1}{2} m_i |\mathbf{v}_i|^2 = T,$$

which is the kinetic energy as claimed. Formally we have

$$|\dot{\mathbf{x}}|^2 = \left(\frac{\mathrm{d}s}{\mathrm{d}t} \right)^2 = \frac{(\mathrm{d}s)^2}{(\mathrm{d}t)^2},$$

and so in practice we only need to know the line element $\mathrm{d}s$ or metric $\mathrm{d}s^2$ in order to transform the kinetic energy into other coordinate systems. If we wish to express the Cartesian coordinates x_i as functions of generalized coordinates $q = (q_1, \ldots, q_n)$, then

$$T = \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij}(q) \dot{q}_i \dot{q}_j \tag{4.8}$$

where a_{ij} are functions of the coordinates only. That is, the kinetic energy T in generalized coordinates is still a quadratic function of velocities, but may also depend on the other coordinates. Mathematically, a conservative Lagrangian system is determined by a Riemann manifold—where the metric determines the kinetic energy—and a potential function.

For a general system of particles which may interact we add a function to the Lagrangian, which for a conservative system is the potential energy:

$$\mathscr{L}(q, \dot{q}) = T(q, \dot{q}) - V(q).$$

Now that we are no longer in Euclidean space, we must add the assumption that for a conservative system on a manifold M the force is exact, i.e. $\mathbf{F} = -\nabla V$, and not merely closed (i.e. that work is path independent). Now the time reversibility (cf. section 2.4) is easily seen as the time independence of the Lagrangian—time reversal $t \mapsto -t$ in the quadratic kinetic energy (4.8) preserves each product $\dot{q}_i \dot{q}_j$. In particular, when a time-independent Lagrangian is of the above form then the total energy is conserved.

4.3 Nonconservative systems

Thus far we have considered Lagrangians of the form $\mathscr{L} = T - V$, and the resulting system is automatically conservative since it is derived from the potential. In this section we will extend Lagrangian mechanics to include nonconservative systems. We will see that the correct choice is

$$\mathscr{L} = T + W, \tag{4.9}$$

where W is the total work of the (generalized) forces:

$$W = \sum_{i=1}^{n} Q_i q_i. \tag{4.10}$$

It suffices to consider the case of Euclidean space as in the reduction at the beginning of the proof of Proposition 4.5, but we will continue to use the notation q for convenience. Consider a system whose motion is described by the n coordinates $q_j(t) \in \mathbb{R}^d$ from time t_0 to t_1, and write a fixed-endpoint variation of the motion as $q_j(t) + h_j(t)$ with $h_j(t_0) = h_j(t_1) = 0$. Repeating the integration by parts procedure from the proof of Proposition 4.5, we get

$$\begin{aligned} \mathrm{d}\Big(\int T \,\mathrm{d}t \Big)\Big|_{q(t)}(h) &= \int_{t_0}^{t_1} \sum_{j=1}^{n} \left(\frac{\partial T}{\partial q_j} h_j + \frac{\partial T}{\partial \dot{q}_j} \dot{h}_j \right) \mathrm{d}t \\ &= \left[\sum_{j=1}^{n} \frac{\partial T}{\partial \dot{q}_j} h_j \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \sum_{j=1}^{n} \left(\frac{\partial T}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_j} \right) h_j \,\mathrm{d}t \\ &= \int_{t_0}^{t_1} \sum_{j=1}^{n} \left(\frac{\partial T}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_j} \right) h_j \,\mathrm{d}t, \end{aligned} \tag{4.11}$$

where in the last equality we noted that $h_j(t_0) = h_j(t_1) = 0$ for a fixed-endpoint variation.

From (4.10) we compute the variation of the work to be

$$\mathrm{d}\Big(\int W \,\mathrm{d}t \Big)\Big|_{q(t)}(h) = \int_{t_0}^{t_1} \sum_{j=1}^{n} Q_j h_j \,\mathrm{d}t. \tag{4.12}$$

Adding this in, the principle of least action for the new Lagrangian (4.9) yields

$$0 = \mathrm{d}\bigg(\int (T+W)\,\mathrm{d}t \bigg)\bigg|_{q(t)}(h) = \int_{t_0}^{t_1} \sum_{j=1}^{n} \left(\frac{\partial T}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_j} + Q_j \right) h_j \,\mathrm{d}t.$$

As this must be true for all variations h_j, then we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j, \qquad j = 1, \ldots, n. \tag{4.13}$$

These are **Lagrange's equations for nonconservative forces**. In other words, the motion of a nonconservative system is given by Lagrange's equations (4.4) for the Lagrangian (4.9).

Nowhere did we assume that the generalized forces are not conservative, and so eq. (4.13) must reduce to the familiar formulation of Lagrange's equations when the forces are conservative. In the case $Q = -\mathrm{d}V$, we can use the same integration by parts procedure in reverse to obtain

$$\begin{aligned}
\mathrm{d}\bigg(\int W\,\mathrm{d}t \bigg)\bigg|_{q(t)}(h) &= \int_{t_0}^{t_1} \sum_{j=1}^{n} Q_j h_j \,\mathrm{d}t = -\int_{t_0}^{t_1} \sum_{j=1}^{n} \left[\frac{\partial V}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial \dot{q}_j} \right] h_j \,\mathrm{d}t \\
&= \left[\sum_{j=1}^{n} \frac{\partial V}{\partial \dot{q}_j} h_j \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_{j=1}^{n} \left(\frac{\partial V}{\partial q_j} h_j + \frac{\partial V}{\partial \dot{q}_j} \dot{h}_j \right) \mathrm{d}t \\
&= -\int_{t_0}^{t_1} \sum_{j=1}^{n} \left(\frac{\partial V}{\partial q_j} h_j + \frac{\partial V}{\partial \dot{q}_j} \dot{h}_j \right) \mathrm{d}t = -\mathrm{d}\bigg(\int V\,\mathrm{d}t \bigg)\bigg|_{q(t)}(h).
\end{aligned}$$

Therefore, the new Lagrangian (4.9) generates the same motion as the conservative Lagrangian $\mathscr{L} = T - V$.

Even when the forces are nonconservative, all we needed for the previous paragraph is the ability to write the jth component of the generalized force as

$$Q_j = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial \dot{q}_j} - \frac{\partial V}{\partial q_j}. \tag{4.14}$$

In this case, we recover the familiar form of Lagrange's equations (4.4) with no right-hand side, but now with a velocity-dependent potential $V(q, \dot{q}, t)$.

The quantity $V = T - \mathscr{L}$ is called the **potential energy** even for nonconservative systems, and is generally time-dependent. A common example is a system A with coordinates q_A is not closed, but it moves in an external field due to a system B with coordinates $q_B(t)$ independent of q_A and the entire system $A + B$ is closed. This system has a Lagrangian of the form

$$\mathscr{L} = T_A(q_A, \dot{q}_A) - V(q_A, q_B(t)). \tag{4.15}$$

We may ignore T_B since it depends only on time and is thus a complete time derivative. (4.15) is a Lagrangian of the usual type, but with V being possibly time-dependent. If system A is a single particle, then the Euler–Lagrange

equations yield

$$m\ddot{q} = -\frac{\partial V}{\partial q}(q,t) = F(q,t). \tag{4.16}$$

As an example, if $F \equiv F(t)$ is uniform (i.e. independent of position) then $V = \mathbf{F} \cdot \mathbf{x}$ in Euclidean space.

4.4 Equivalence to Newton's equations

Now we will see how to obtain the Euler–Lagrange equations (4.4) from Newtonian mechanics, so that Hamilton's principle of least action is equivalent to Newton's principle of determinacy.

A mechanical system with a configuration manifold M can always be—and in experiment is automatically—embedded in some Euclidean space $\mathbb{R}^N$. Within M, the motion of the system is dictated by some known force $\mathbf{F}$. The effect of constraining the motion to the manifold M can be thought of as a force $\mathbf{N}$ orthogonal to M, called the **constraint force**. Newton's equations for this system is

$$m_i\ddot{\mathbf{x}}_i = \mathbf{F}_i + \mathbf{N}_i.$$

Rearranging, we see that $m_i\ddot{\mathbf{x}}_i - \mathbf{F}_i = \mathbf{N}_i$ is orthogonal to M, and so

$$(m_i\ddot{\mathbf{x}}_i - \mathbf{F}_i) \cdot \boldsymbol{\xi}_i = 0$$

for all vectors $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N)$ tangent to M. This is Newton's equation in the tangent plane to the surface M. Summing over all particles, we get the **d'Alembert–Lagrange principle**:

$$\sum_{i=1}^{N} (m_i\ddot{\mathbf{x}}_i - \mathbf{F}_i) \cdot \boldsymbol{\xi}_i = 0 \tag{4.17}$$

for all vectors $\boldsymbol{\xi} \in \mathbb{R}^N$ tangent to M. In section 5.1 we will see that this principle more generally dictates the motion of a system with constraints. Note that for a free system $M = \mathbb{R}^n$ we may take any vector $\boldsymbol{\xi} \in \mathbb{R}^n$, and so we recover Newton's equations.

Let $q = (q_1, \ldots, q_n)$ be local coordinates on M. Then by the chain rule we have

$$\dot{x}_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j}\dot{q}_j,$$

and so we may write the kinetic energy

$$T(q,\dot{q}) = \sum_{i=1}^{N} \tfrac{1}{2} m_i |\dot{x}_i|^2 = \sum_{i,j=1}^{n} a_{ij}(q)\dot{q}_i\dot{q}_j$$

as a positive definite quadratic form on M.

Expressing the force in terms of the coordinates q_j, the covectors Q_j defined by the one-form equation

$$\sum_{i=1}^{N} F_i \,\mathrm{d}x_i = \sum_{j=1}^{n} Q_j \,\mathrm{d}q_j,$$

or equivalently

$$Q_j = \sum_{i=1}^{N} F_i \frac{\partial x_i}{\partial q_j}, \tag{4.18}$$

are called the **generalized forces**. They dictate the evolution of the kinetic energy via the following expression.

Proposition 4.9. *The Newtonian motion $q(t)$ of the system satisfies*

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = Q. \tag{4.19}$$

Proof. We repeat the argument from section 4.3. The calculation (4.11) of the variation of the kinetic energy contribution still holds, since we did not use any equations of motion. Taking a dot product with an arbitrary tangent vector $\boldsymbol{\xi}$ to M, we can replace the coordinates $x \in \mathbb{R}^N$ with $q \in M$. Similarly, the calculation (4.12) still holds on M by the definition (4.18) and the d'Alembert–Lagrange principle (4.17)—that is, Newton's equations hold on the manifold M in terms of the generalized forces. Adding these together, we obtain (4.19) as desired. □

We defer the proof to the next section, where we will derive (4.19) from scratch under more general assumptions.

For a conservative system we have that the one-form $Q\,\mathrm{d}q$ is exact and may be written as $-\mathrm{d}V$ for a potential energy $V(q)$. For such a system we have the Lagrangian $\mathscr{L} = T - V$, and hence (4.19) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{\partial \mathscr{L}}{\partial q} = 0.$$

In section 4.1, we saw that this implies $q(t)$ is a critical point for the action functional. As $q(t)$ was an arbitrary motion of the system, we conclude that the principle of least action must hold.

4.5 Momentum and conservation

In this section we will investigate two special mathematical cases when we can replace a second order Euler–Lagrange equation by a first order equation, which correspond to two physically important conservation laws.

For a Lagrangian system, the **momentum** (sometimes called **generalized momentum** in physics) of a particle is defined to be

$$p_i = \frac{\partial \mathscr{L}}{\partial \dot{q}_i}.$$

If $q_i = x_i$ is a Cartesian coordinate, the kinetic energy part of the Lagrangian has a term $\frac{1}{2} m_i \dot{x}_i^2$ and so $p_i = m_i x_i$ is the linear momentum along the x_i-axis. If $q_i = \phi_i$ is the azimuthal angular coordinate in $\mathbb{R}^3$, then the kinetic energy about the z-axis is $\frac{1}{2} m_i r_i^2 \dot{\phi}_i^2$ and so $p_i = m_i r_i^2 \dot{\phi}_i$ is the angular momentum about the z-axis.

We can similarly define the **(generalized) force** as

$$F_i = \frac{\partial \mathscr{L}}{\partial q_i}.$$

This way, Lagrange's equations (4.4) imply

$$F_i = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}_i} = \frac{\mathrm{d}p_i}{\mathrm{d}t},$$

and so Newton's equation is still satisfied for these generalized quantities. This definition agrees with the one-form definition (4.18) for Q_i, and so both forces F_i and momenta p_i should be interpreted as covectors; conversely, the velocities $\dot{q}_i$ are by definition tangent vectors. This fact is built into the kinetic energy:

$$T_i = \tfrac{1}{2} \mathbf{v}_i \cdot \mathbf{p}_i = \tfrac{1}{2} \langle \mathbf{v}_i, \mathbf{p}_i \rangle,$$

and so it is indeed natural.

These definitions immediately illuminate a special case when we can replace a second order Euler–Lagrange equation by a first order equation.

Proposition 4.10 (Conservation of momentum)**.** *If the Lagrangian is independent of a variable q_i, then the corresponding momentum is conserved:*

$$\frac{\partial \mathscr{L}}{\partial \dot{q}_i} = \textit{constant}.$$

Proof. The Euler–Lagrange equations (4.4) imply

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}_i} = \frac{\partial \mathscr{L}}{\partial q_i} = 0,$$

where in the last equality we used that the Lagrangian is independent of q_i. □

Any such coordinate q_i is called **cyclic**, and so Proposition 4.10 says that if a coordinate is cyclic then the corresponding momentum is conserved. This fact includes the conservation laws we saw in Propositions 1.12 and 1.15. One of the strengths of Lagrangian mechanics is that if we can find a cyclic coordinate then there is one less equation of motion to solve. For example, in section 3.1

we picked the z-axis to align with the initial angular momentum, which eliminated the equations for the z-coordinate of the position and momentum. We can identify cyclic coordinates geometrically—Proposition 4.10 requires that the trajectories $q(t)$ in configuration space M lie in the level sets of p_i, and so the system has a translation symmetry in the q_i direction.

We can extend this heuristic to include the time coordinate: when the cyclic coordinate is time, then the conserved quantity is the total energy. We define the **total energy** (or **Hamiltonian** H) of a system as

$$E = \sum_{i=1}^{n} \dot{q}_i p_i - \mathscr{L}. \tag{4.20}$$

This includes our previous definition, since for a conservative system on Euclidean space we have

$$\begin{aligned} E &= \sum_{i=1}^{n} v_i \frac{\partial}{\partial v_i} \left[\sum_{j=1}^{n} \tfrac{1}{2} m_j v_j^2 - V(\mathbf{x}) \right] - \sum_{j=1}^{n} \left[\tfrac{1}{2} m_i v_i^2 - V(\mathbf{x}) \right] \\ &= \sum_{i=1}^{n} m_i v_i^2 - \sum_{i=1}^{n} \tfrac{1}{2} m_i v_i^2 + V = T + V. \end{aligned}$$

Proposition 4.11 (Conservation of energy)**.** *The solution $q(t)$ to the Euler–Lagrange equations* (4.3) *satisfies*

$$H(q) := \dot{q} \cdot \frac{\partial \mathscr{L}}{\partial \dot{q}}(q, \dot{q}, t) - \mathscr{L}(q, \dot{q}, t) = \int_{t_0}^{t} \frac{\partial \mathscr{L}}{\partial t}(q(s), \dot{q}(s), s) \, \mathrm{d}s + \textit{constant}.$$

In particular, if the Lagrangian $\mathscr{L}$ is independent of t, then the Hamiltonian $H(q)$ is constant.

In the context of variational calculus this is sometimes called the **second Euler–Lagrange equation**, which replaces the second order equation (4.3) with a first order equation.

Proof. Using the chain rule and the Euler–Lagrange equation (4.3), we have

$$\begin{aligned} \frac{\mathrm{d}\mathscr{L}}{\mathrm{d}t} &= \frac{\partial \mathscr{L}}{\partial t} + \frac{\partial \mathscr{L}}{\partial q_i} \cdot \dot{q}_i + \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \cdot \ddot{q}_i \\ &= \frac{\partial \mathscr{L}}{\partial t} + \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \right) \cdot \dot{q}_i + \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \cdot \ddot{q}_i = \frac{\partial \mathscr{L}}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{q}_i \cdot \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \right). \end{aligned}$$

Integrating in time gives the desired result. □

4.6 Noether's theorem

The conservation of momentum (Proposition 4.10) says that a system that is continuously symmetric along the coordinate q_i possesses the corresponding

conserved quantity p_i. This turns out to be a general phenomenon—every one-parameter group of symmetries which preserves a Lagrangian system has a conserved quantity—which we will now present.

Consider a Lagrangian system consisting of a smooth n-dimensional manifold M with a time-independent Lagrangian $\mathscr{L}(q, v, t) : TM \to \mathbb{R}$. We say that a diffeomorphism $h : M \to M$ is a **symmetry** of the system if

$$\mathscr{L}(h_*(q, v)) = \mathscr{L}(q, v) \quad \text{for all } (q, v) \in TM. \tag{4.21}$$

Here, $h_*(q, v) = \mathrm{d}h(q, v)$ is the **pushforward** or **differential** of the point $q \in M$ and tangent vector $v \in T_qM$, and is equal to the point $h(q) \in M$ with the tangent vector $\mathrm{d}h|_q(v) \in T_{h(q)}M$.

Proposition 4.12 (Noether's theorem)**.** *If the Lagrangian system $(M, \mathscr{L})$ is time-independent and has a differentiable one-parameter group of diffeomorphisms $h^s : M \to M$, $s \in \mathbb{R}$, then the Lagrangian motion of the system has a conserved quantity (or "integral") $I : TM \to \mathbb{R}$. In local coordinates,*

$$I(q, v) = \frac{\partial \mathscr{L}}{\partial \dot{q}}(q, v) \cdot \frac{\mathrm{d}}{\mathrm{d}s} h^s(q)\bigg|_{s=0}. \tag{4.22}$$

Remark. The quantity I is independent of the choice of local coordinates q. Indeed, I measures the rate of change of $\mathscr{L}(q, v)$ as v is varied in T_qM in the direction of the tangent vector $\frac{\mathrm{d}}{\mathrm{d}s} h^s(q)|_{s=0}$, which is why the formula (4.22) looks like the chain rule for $\frac{\mathrm{d}}{\mathrm{d}s} \mathscr{L}((h^s)_*(q, v))|_{s=0}$, and this does not intrinsically involve a choice of coordinates.

Proof. As in the set up for the proof of Proposition 4.5, we may fix a coordinate patch on M and take the variation to be supported within the image in order to reduce the statement to Euclidean space $\mathbb{R}^n$. Let $q(t) : \mathbb{R} \to \mathbb{R}^n$ denote a solution to Lagrange's equations. As h^s is a symmetry for each s then $h^s \circ q$ will also be a solution, because $(h^s \circ q)(t) = (h^s)_*(q(t))$ is the pushforward of the point $q(t)$, $\frac{\mathrm{d}}{\mathrm{d}t}(h^s \circ q)(t) = \mathrm{d}(h^s)\,|_{q(t)}(\dot{q}(t)) = (h^s)_*(\dot{q}(t))$ is the pushforward of the tangent vector $\dot{q}(t)$, and so by (4.21) the Lagrangian evaluated at $(h^s \circ q)(t)$ is the same as the Lagrangian evaluated at $q(t)$.

Consider the map $\Phi : \mathbb{R}_s \times \mathbb{R}_t \to \mathbb{R}^n$ given by $\Phi(s, t) = (h^s \circ q)(t)$. As all of the symmetries h^s preserve the Lagrangian $\mathscr{L}$, then

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} \mathscr{L}(\Phi, \dot{\Phi}) = \frac{\partial \mathscr{L}}{\partial q} \cdot \frac{\partial \Phi}{\partial s} + \frac{\partial \mathscr{L}}{\partial \dot{q}} \cdot \frac{\partial \dot{\Phi}}{\partial s}, \tag{4.23}$$

where everything on the right-hand side is evaluated at $(\Phi(s, t), \dot{\Phi}(s, t)) \in T\mathbb{R}^n$. As we have already argued, for fixed s the map $\Phi(s, \cdot) : \mathbb{R} \to \mathbb{R}^n$ satisfies Lagrange's equation:

$$\frac{\partial}{\partial t}\left[\frac{\partial \mathscr{L}}{\partial \dot{q}}\big(\Phi(s, t), \dot{\Phi}(s, t)\big)\right] = \frac{\partial \mathscr{L}}{\partial q}\big(\Phi(s, t), \dot{\Phi}(s, t)\big)$$

for each $s \in \mathbb{R}$. Plugging this into the right-hand side of (4.23), we see that

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\left[\frac{\partial \mathscr{L}}{\partial \dot{q}}\big(\Phi, \dot{\Phi}\big)\right] \cdot \frac{\partial \Phi}{\partial s}\big(\Phi, \dot{\Phi}\big) + \frac{\partial \mathscr{L}}{\partial \dot{q}}\big(\Phi, \dot{\Phi}\big) \cdot \frac{\partial \dot{\Phi}}{\partial s}\big(\Phi, \dot{\Phi}\big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathscr{L}}{\partial \dot{q}} \cdot \frac{\partial \Phi}{\partial s}\right]\big(\Phi, \dot{\Phi}\big) = \frac{\mathrm{d}I}{\mathrm{d}t}\big(\Phi, \dot{\Phi}\big) \end{aligned}$$

by the chain rule. □

Example 4.13 (Translational symmetry)**.** Consider the conservative N-particle Lagrangian

$$\mathscr{L}(\mathbf{x}, \mathbf{v}) = \sum_{i=1}^{N} \tfrac{1}{2} m_i |\mathbf{v}_i|^2 - V(\mathbf{x}), \tag{4.24}$$

where $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$, $\mathbf{x}_i \in \mathbb{R}^d$ and similarly for $\mathbf{v}$. If the potential energy is invariant under translations along the first coordinate axis $\mathbf{e}_1 \in \mathbb{R}^n$, then the system is symmetric with respect to the N translations

$$h_j^s : (\mathbb{R}^d)^N \to (\mathbb{R}^d)^N, \qquad h_j^s(\mathbf{x}_1, \ldots, \mathbf{x}_N) = (\mathbf{x}_1, \ldots, \mathbf{x}_j + s\mathbf{e}_1, \ldots, \mathbf{x}_N)$$

for $j = 1, \ldots, N$. Noether's theorem yields N conserved quantities

$$I_j(\mathbf{x}, \mathbf{v}) = \frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} \cdot (0, \ldots, 0, \mathbf{e}_1, 0, \ldots, 0) = m_j v_{j1},$$

which we recognize as the first component of the jth particle's momentum $\mathbf{p}_j = m_j \mathbf{v}_j$ (cf. Corollary 1.13).

Example 4.14 (Rotational symmetry)**.** Now suppose $d = 3$, and that the conservative N-particle Lagrangian (4.24) is invariant under rotations about the first coordinate axis $\mathbf{e}_1 \in \mathbb{R}^3$. Then the system is symmetric with respect to the N rotations

$$h_j^s(\mathbf{x}_1, \ldots, \mathbf{x}_N) = (\mathbf{x}_1, \ldots, \cos(s)\mathbf{x}_j + \sin(s)\mathbf{e}_1 \times \mathbf{x}_j + (1 - \cos s)x_{j1}\mathbf{e}_1, \ldots, \mathbf{x}_N)$$

for $j = 1, \ldots, N$. Noether's theorem returns the N conserved quantities

$$I_j(\mathbf{x}, \mathbf{v}) = \frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} \cdot (0, \ldots, 0, \mathbf{e}_1 \times \mathbf{x}_j, 0, \ldots, 0) = \mathbf{p}_j \cdot (\mathbf{e}_1 \times \mathbf{x}_j) = (\mathbf{x}_j \times \mathbf{p}_j) \cdot \mathbf{e}_1,$$

which we recognize as the first component of the jth particle's angular momentum $\mathbf{L}_j = \mathbf{x}_j \times \mathbf{p}_j$ (cf. Corollary 1.16).

4.7 Exercises

4.1 (Functional derivative is well-defined)**.** Show that if Φ is a differentiable functional, then its differential is linear and is independent of the choice in variation of γ.

4.2. Repeat the proof of Proposition 4.5 to prove the following stronger statement on Euclidean space: If $\mathscr{L} \in C^1(\mathbb{R}^n \times \mathbb{R}^n \times [t_0, t_1]; \mathbb{R})$ and $q \in C^1([t_0, t_1]; \mathbb{R}^d)$ is a (fixed-endpoint) critical point for the action functional defined in (4.1), then $\frac{\partial \mathscr{L}}{\partial \dot{q}}(q, \dot{q}, t)$ is a C^1 function on $[t_0, t_1]$ and q solves the Euler–Lagrange equations (4.3).

4.3 (Geodesics on the sphere [Tro96, Ch. 1])**.** In Example 4.7 we saw that the geodesics—paths of shortest length between two given points—in $\mathbb{R}^d$ are straight lines. We will repeat this procedure for the sphere S^2.

(a) Using coordinates (ϕ, θ), we can parameterize a path $\mathbf{x}(t)$ in $S^2 \subset \mathbb{R}^3$ as

$$\mathbf{x}(t) = (\cos\phi(t)\sin\theta(t), \sin\phi(t)\sin\theta(t), \cos\theta(t)), \quad t \in [0, 1]$$

(ϕ is the azimuth and θ is the zenith angle). Find the formula for the arc length functional $\Phi[\mathbf{x}(t)]$.

(b) After rotating the sphere we may assume that $\theta(0) = 0$, $\theta(1) = \theta_1$, and $\phi(1) = 0$. Find a simple lower bound for $\Phi[\mathbf{x}(t)]$ using the $\phi'(t)$ term. By considering when equality occurs, conclude that the geodesic connecting the north pole $\mathbf{x}(0)$ to the point $\mathbf{x}(1)$ is the shorter arc of the great circle (an equator, or circle of maximum circumference on the sphere) connecting the two points. This is another example where we can directly confirm that the critical path for the functional is a minimum.

4.4 (Brachistochrone [Tro96, Ch. 6])**.** The brachistochrone between two points in the plane is the curve on which a frictionless bead would traverse the quickest subject to a downward gravitational acceleration. Johann Bernoulli in 1696 challenged mathematicians to find the shape of the brachistochrone, and it was his brother Jakob Bernoulli who provided a solution which was later refined into the calculus of variations.

(a) After translating, we may assume that the initial point is the origin $(0, 0)$ and the second point is given by some (x_1, y_1) with $x_1 > 0$ and $y_1 < 0$. Explain why it is reasonable to assume that the brachistochrone is the graph of a function $y(x)$, $x \in [0, x_1]$ as opposed to a general parametric curve. Show that the time is takes the bead to traverse this curve is

$$\Phi[y(x)] = \int_0^{x_1} \frac{\sqrt{1 + y'(x)^2}}{v(x)}\, \mathrm{d}x$$

where $v(x)$ is the bead's speed.

(b) With constant downward acceleration g, show that $v(x) = \sqrt{2gy(x)}$.

(c) Using the conservation of total energy (4.20), find the first order differential equation

$$\sqrt{\frac{y}{c^2 - y}}\, y' = 1.$$

(d) Introducing a new dependent variable $\theta(x)$ so that

$$y = c^2 \sin^2 \tfrac{\theta}{2} = \tfrac{1}{2}c^2(1 - \cos\theta), \qquad 0 \le \theta < 2\pi,$$

show that

$$\tfrac{1}{2}c^2(1 - \cos\theta)\theta' = 1.$$

(e) By integrating the two equations of the previous part, obtain the parametric equations

$$\begin{aligned} x &= c^2(\theta - \sin\theta) + c_1, \\ y &= c^2(1 - \cos\theta), \end{aligned} \qquad 0 \le \theta \le \theta_1.$$

In order for $x(0) = 0 = y(0)$, we must have $c_1 = 0$. That is, the brachistochrone is a cycloid; these equations describe the path traced by a fixed point on a circle of radius c^2 as it rolls along the x-axis in the lower half-plane.

4.5 (Lagrangian PDE I [Eva10, Ch. 8])**.** In this exercise we will prove the existence of a solution to the elliptic divergence-form PDE

$$-\nabla \cdot (A(x)\nabla u(x)) = 0 \text{ for } x \in \Omega, \quad u(x) = 0 \text{ for } x \in \partial\Omega,$$

for an open set $\Omega \subset \mathbb{R}^n$. Here, $A(x) = (a_{ij}(x))$ is a symmetric $n \times n$ matrix with $a_{ij} \in H^2(\Omega)$ (the Sobolev space), and we also assume that A is uniformly elliptic:

$$\lambda I \le A(x) \le \Lambda I \text{ for } x \in \Omega$$

(in the sense of positive definite matrices).

(a) For $u \in H_0^1(\Omega)$ (the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$), show that the energy functional

$$E(u) = \tfrac{1}{2}\int_\Omega \nabla u(x) \cdot A(x)\nabla u(x)\,\mathrm{d}x$$

is finite. Show that for $\phi \in C_c^\infty(\Omega)$ the first variation at u is

$$\lim_{\epsilon \to 0} \frac{E(u + \epsilon\phi) - E(u)}{\epsilon} = \int_\Omega \nabla\phi \cdot A\nabla u$$

Consequently, if u is a minimum of E, then the above expression vanishes for all $\phi \in C_c^\infty(\Omega)$; such a function $u \in H_0^1(\Omega)$ is called a weak solution of the PDE and boundary condition, since one formal integration by parts would produce the PDE—but we do not assume u is twice differentiable.

(b) Let u_j be a sequence in $H_0^1(\Omega)$ such that

$$E(u_j) \to \inf_{H_0^1(\Omega)} E \text{ as } j \to \infty.$$

Using Poincaré's inequality, show that $E(u)$ is bounded below and hence our sequence is bounded. Conclude that there exists a weakly convergent subsequence $u_{j_k} \rightharpoonup u$ in $H^1(\Omega)$ using the Riesz representation theorem.

(c) Show that E is (sequentially) weakly lower semicontinuous:

$$E(u) \leq \liminf_{k\to\infty} E(u_{j_k}).$$

Conclude that $u \in H_0^1(\Omega)$ is a minimum for E, and hence is a weak solution to the PDE by part (a).

4.6 (Two pendulums connected by a spring)**.** Consider two pendulums of unit length and unit mass in a constant gravitational field g. Suppose they are connected by a massless spring with spring constant k whose resting length is the same as their distance of separation.

(a) Let θ_1 and θ_2 be the angles the pendulums make with the downward vertical. Find the Lagrangian for the system for small angles, so that $\sin\theta \approx \theta$.

(b) Define new variables

$$q_1 = \frac{\theta_1 + \theta_2}{\sqrt{2}}, \quad q_2 = \frac{\theta_1 - \theta_2}{\sqrt{2}},$$

and show that Lagrangian separates into two harmonic oscillators:

$$\mathscr{L} = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \tfrac{1}{2}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2).$$

Find ω_1 and ω_2. When $q_2 = 0$, we have $\theta_1 = \theta_2$ and so both pendulums swing in phase with each other with frequency ω_1. When $q_1 = 0$, we have $\theta_2 = -\theta_1$ and so the pendulums swing in exact opposite phase with frequency ω_2.

(c) For $k \ll 1$ we will see that an exchange of energy occurs. Suppose that at time $t = 0$ we have $\theta_1 = 0 = \theta_2$, $\dot{\theta}_2 = 0$, and $\dot{\theta}_1 = v_0$. Using part (b), show that the motion is given by

$$\theta_1(t) = \tfrac{v_0}{2}\left(\sin t + \tfrac{1}{\omega}\sin\omega t\right), \quad \theta_2(t) = \tfrac{v_0}{2}\left(\sin t - \tfrac{1}{\omega}\sin\omega t\right)$$

with $\omega = \omega_2$. For $k \ll 1$ we have $1/\omega \approx 1$, and so

$$\theta_1(t) \approx v_0 \cos\epsilon t \sin\omega' t, \quad \theta_2(t) \approx -v_0 \cos\omega' t \sin\epsilon t$$

for some ϵ and ω'. Show that to leading order as $k \to 0$ we have

$$\epsilon \approx \tfrac{k}{2}, \quad \omega' \approx 1,$$

and so after a time $T = \pi/2\epsilon \approx \pi/k$ the pendulums have switched roles and now essentially only the second pendulum is oscillating.

4.7 (Charged particle in an electromagnetic field)**.** The motion of a charged particle is an example of a nonconservative Lagrangian system. A particle with charge q moving through the vector fields E, D, B, H on $\mathbb{R}^3$ with the scalar

charge density ρ and vector current density j obeys Maxwell's equations (in Gaussian units):

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0, \qquad \nabla \cdot D = 4\pi\rho,$$
$$\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c} j, \qquad \nabla \cdot B = 0.$$

The force on the charge is given by the Lorentz law

$$F = q\left[E + \tfrac{1}{c}(v \times B)\right].$$

(a) The fourth Maxwell equation requires that B is divergence-free, and so we introduce a vector potential A for B so that $B = \nabla \times A$. Using the first equation, introduce a scalar potential ϕ so that the electric field becomes

$$E = -\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t}$$

and the Lorentz force is

$$F = q\left[-\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t} + \frac{1}{c}(v \times (\nabla \times A))\right].$$

(b) The Lorentz force is nonconservative, but if we can put it in the form (4.14) then we will have a Lagrangian for this system. The first term $q\phi$ is already of the desired form. Show that the x-component of the rightmost term $v \times (\nabla \times A)$ may be rewritten as

$$(v \times (\nabla \times A))_x = \frac{\partial}{\partial x}(v \cdot A) - \frac{\mathrm{d}A_x}{\mathrm{d}t} + \frac{\partial A_x}{\partial t}.$$

By symmetry, we get the same relation for the other components with x replaced by the respective variable.

(c) Show that the x-component of the Lorentz force can be written as

$$F_x = -\frac{\partial V}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial v_x},$$

for the potential energy

$$V = q\phi - \tfrac{q}{c} v \cdot A.$$

Symmetrically, the y- and z-components of the Lorentz force are also of this form if we replace x with the respective coordinates. Consequently, the Lagrangian for this system is

$$\mathscr{L} = T - V = \tfrac{1}{2}mv^2 - q\phi + \tfrac{q}{c} v \cdot A$$

where m is the particle's mass.

4.8 (Noether's theorem with time dependence)**.** Consider a time-dependent Lagrangian system $\mathscr{L} : TM \times \mathbb{R} \to \mathbb{R}$.

(a) Prove Noether's theorem for this system by applying Proposition 4.12 to the extended configuration space $M_1 = M \times \mathbb{R}$ with Lagrangian

$$\mathscr{L}_1\left(q, t, \frac{\mathrm{d}q}{\mathrm{d}\tau}, \frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = \mathscr{L}\left(q, \frac{\mathrm{d}q/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau}, t\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}$$

and the new "time" variable τ, to obtain a conserved quantity $I(q, \dot{q}, t)$ on M.

(b) Apply this to a time-independent Lagrangian $\mathscr{L}(q, v, t) \equiv \mathscr{L}(q, v)$ to conclude that the total energy is conserved (cf. Proposition 1.9).

4.9 (Lagrangian PDE II)**.** In this exercise, we will explore the formal Lagrangian structure associated with the wave equation. The Lagrangian formulation of the wave equation is preferable (as opposed to the Hamiltonian formulation) because it shares the Lorentz symmetries of the wave equation. For further reading, see [SS98].

(a) Let

$$g = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = (g_{\alpha\beta})_{\alpha,\beta=0,\ldots,d}$$

denote the $(d+1) \times (d+1)$ matrix associated with the Minkowski metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2$ on $\mathbb{R}_t \times \mathbb{R}_x^d$, with coordinates $(x_0, \ldots, x_d) = (t, x_1, \ldots, x_d)$. Consider the Lagrangian

$$\mathscr{L}(u, \nabla u) := \sum_{\alpha,\beta=0}^{d} (g^{-1})_{\alpha\beta} \frac{\partial u}{\partial x_\alpha}\frac{\partial u}{\partial x_\beta} + F(u) = -\tfrac{1}{2}\left|\tfrac{\partial u}{\partial t}\right|^2 + \tfrac{1}{2}|\nabla u|^2 + F(u).$$

Show formally that a function $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is stationary for the action

$$S(u) := \int_{\mathbb{R}\times\mathbb{R}^d} L(u, \nabla u)\sqrt{-\det g}\,\mathrm{d}x\,\mathrm{d}t$$

if and only if u solves the *(semilinear) wave equation*

$$-\tfrac{\partial^2 u}{\partial t^2} + \Delta u = F'(u).$$

(b) Now we will take $F \equiv 0$ and derive the conservation laws of the linear wave equation. Consider the same Lagrangian $\mathscr{L}(u, \nabla u, g)$ and action $S(u, g)$ to be functions of the metric g. Given $\tau : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d$ a smooth and

compactly supported diffeomorphism, let g_s, $s \in \mathbb{R}$ be the pullback of the metric g by the map $\mathrm{id} + s\tau$, so that

$$(g_s)_{\alpha\beta}\big|_{s=0} = g_{\alpha\beta}, \qquad \tfrac{\mathrm{d}}{\mathrm{d}s}(g_s)_{\alpha\beta}\big|_{s=0} = \tfrac{\partial \tau_\beta}{\partial x_\alpha} + \tfrac{\partial \tau_\alpha}{\partial x_\beta} =: \pi_{\alpha\beta}.$$

Similarly, let u_s be the pullback of the metric g by the map $\mathrm{id} + s\tau$, so that $u_s|_{s=0} = u$. Using that $\frac{\mathrm{d}}{\mathrm{d}s} S(u_s, g_s)|_{s=0} = 0$ for all diffeomorphisms τ (because the Lagrangian is invariant under the change of variables $(t, x) \mapsto (t, x) + s\tau(t, x)$), show formally that if u is stationary for $S(u, g)$ then

$$\int_{\mathbb{R}\times\mathbb{R}^d} \sum_{\alpha,\beta=0}^{d} T^{\alpha\beta} \pi_{\alpha\beta} \sqrt{-\det g}\, \mathrm{d}x\, \mathrm{d}t,$$

where

$$T^{\alpha\beta} := \tfrac{\partial \mathscr{L}}{\partial (g^{-1})_{\alpha\beta}} - \tfrac{1}{2} g_{\alpha\beta} \mathscr{L}$$

is the *stress-energy tensor*. Conclude that $T^{\alpha\beta}$ is divergence-free: i.e.,

$$\sum_{\alpha=0}^{d} \tfrac{\partial T^{\alpha\beta}}{\partial x_\alpha} = 0 \quad \text{for } \beta = 0, \dots, d.$$

These yield the *microscopic conservation laws*

$$\tfrac{\partial T^{00}}{\partial t} + \sum_{k=1}^{d} \tfrac{\partial T^{0k}}{\partial x_k} = 0, \qquad \tfrac{\partial T^{0j}}{\partial t} + \sum_{k=1}^{d} \tfrac{\partial T^{jk}}{\partial x_k} = 0 \quad \text{for } j = 1, \dots, d.$$

(c) Integrate in space to formally show that the total energy

$$E(u) := \int_{\mathbb{R}^d} T^{00}(t, x)\, \mathrm{d}x = \int_{\mathbb{R}^d} \left(\tfrac{1}{4} \left| \tfrac{\partial u}{\partial t} \right|^2 + \tfrac{1}{4} |\nabla u|^2 \right) \mathrm{d}x$$

and the components of the momentum

$$p_j(u) := \int_{\mathbb{R}^d} T^{0j}(t, x)\, \mathrm{d}x = -\tfrac{1}{2} \int_{\mathbb{R}^d} \mathrm{Re}\left(\tfrac{\partial u}{\partial t} \overline{\tfrac{\partial u}{\partial x_j}} \right) \mathrm{d}x \quad \text{for } j = 1, \dots, d$$

are conserved. These are the corresponding *macroscopic conservation laws*.

Chapter 5

Constraints

In this chapter we explore some physical and mathematical formulations of constraints. This will not include methods of solving for the equations of constrained motion; see for example [AKN06, Sec. 1.6]. The treatment of holonomic constraints is based on [AKN06, Ch. 1], the treatment of nonholonomic constraints is based on [Gol51, Ch. 1–2], and section 5.5 is based on [Str08, §1.6].

5.1 D'Alembert–Lagrange principle

A **holonomic constraint** on a Lagrangian system $(M, \mathscr{L})$ is the requirement that the system's motion is confined to a submanifold S of the phase space TM that can be locally expressed in the form

$$f_1(q, \dot{q}, t) = \cdots = f_k(q, \dot{q}, t) = 0. \tag{5.1}$$

A constraint that is not holonomic is called a **nonholonomic constraint**.

Example 5.1. Rigid-body motion in Euclidean space $\mathbb{R}^d$, in which the distances between all particles $\mathbf{x}_i \in \mathbb{R}^d$ are fixed, is a holonomic constraint. This constraint may be expressed as

$$|\mathbf{x}_i - \mathbf{x}_j|^2 - c_{ij}^2 = 0, \tag{5.2}$$

where c_{ij} are the inter-particle distances. Conversely, the jth particle confined to a rigid spherical container of radius R centered at the origin must obey the equation

$$|\mathbf{x}_j|^2 - R^2 \leq 0,$$

which is nonholonomic. These two examples happen to be time-independent, but this need not be the case.

We will assume that we have a holonomic constraint for the this section. A vector $\xi \in T_qM$ is tangent to the submanifold S if

$$\frac{\partial f_1}{\partial \dot{q}}(q, \dot{q}, t) \cdot \xi = \cdots = \frac{\partial f_k}{\partial \dot{q}}(q, \dot{q}, t) \cdot \xi = 0. \tag{5.3}$$

Such a vector ξ is called a **virtual velocity** of the constrained motion at the state $(q, \dot{q}) \in TM$ and time t. We can ensure that the motion $q(t)$ is constrained to S by insisting that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{\partial \mathscr{L}}{\partial q} \right) \cdot \xi = 0 \tag{5.4}$$

for all virtual velocities ξ at the state $(q(t), \dot{q}(t))$ (cf. the constraint forces of section 4.4).

Definition 5.2 (d'Alembert–Lagrange principle)**.** A motion of the Lagrangian system $(M, \mathscr{L})$ subject to the holonomic constraints (5.1) is a smooth trajectory solving eq. (5.4) for all virtual velocities ξ.

By the definition (5.3) of the virtual velocities, this requires that the left-hand side of the Euler–Lagrange equations be within the span of the derivatives $\frac{\partial f_i}{\partial \dot{q}}$. In other words, there exist constants $\mu_1, \ldots, \mu_k$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{\partial \mathscr{L}}{\partial q} = \sum_{i=1}^{k} \mu_i \frac{\partial f_i}{\partial \dot{q}} \tag{5.5}$$

These are called **Lagrange's equations with multipliers**. Mathematically, we recognize the μ_i as Lagrange multipliers: in order for the action functional to obtain a minimum on the submanifold S, the gradient of the action (the left-hand side of eq. (5.5)) must be orthogonal to the submanifold S. Physically, if we imagine that the particle system is confined to the submanifold S via fictitious **constraint forces** given by the right-hand side of (5.5), then (5.4) states that the constraint forces must be orthogonal to the motion and hence do no work.

We can also observe the constraint effect via the principle of least action and the admissible path variations. Given a path $\gamma \in \Omega$, let Γ be the subspace of the tangent space $T_\gamma \Omega$ consisting of vectors W where W_t is a virtual velocity for each t. For holonomic constraints, the following proposition states that the constrained Lagrangian system on M is given by a the principle of least action on S, and hence the system's degrees of freedom are effectively reduced.

Proposition 5.3 (Hölder's principle)**.** *A path $\gamma \in \Omega$ is a motion of the Lagrangian system with the holonomic constraints S if and only if the functional derivative $\mathrm{d}S|_\gamma$ of the action vanishes on the subspace Γ.*

Proof. By Proposition 4.5, the first derivative of the action S at the path $q(t)$ in the direction of the fixed-endpoint variation $H_s(t)$ is

$$\mathrm{d}S|_\gamma \left(\frac{\partial H_s}{\partial s} \right) = \int_{t_0}^{t_1} \left(\frac{\partial \mathscr{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}} \right) \cdot \frac{\partial H_s}{\partial s} \, \mathrm{d}t.$$

Restricting $\mathrm{d}S|_\gamma$ to the subspace Γ is equivalent to requiring that $\frac{\partial H_s}{\partial s}$ is a virtual velocity. As every virtual velocity can be attained by some variation $\frac{\partial H_s}{\partial s}$, we see from the above variation that $\mathrm{d}S|_\gamma$ vanishes on Γ if and only if the d'Alembert–Lagrange principle (5.4) is satisfied. □

Example 5.4. We can think of the pendulum of Example 4.8 as a particle of mass m in a vertical plane (x, y) subject to the downward gravitational force with potential $V(x, y) = mgy$ subject to the holonomic constraints

$$x^2 + y^2 - \ell^2 = 0, \qquad 2x\dot{x} + 2y\dot{y} = 0,$$

where ℓ is the length of the pendulum arm. Note that the second condition is a time derivative of the first equation, but is necessary to specify the two-dimensional submanifold S in the four-dimensional tangent space $T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$. If we let θ denote the angle from the downward vertical and r the distance from the pivot, then the Lagrangian becomes

$$\mathscr{L} = \tfrac{1}{2}m(r^2\dot{\theta}^2 + \dot{r}^2) + mgr\cos\theta,$$

and the holonomic conditions are

$$f_1(r, \theta, \dot{r}, \dot{\theta}) = r^2 - \ell^2 = 0, \qquad f_2(r, \theta, \dot{r}, \dot{\theta}) = \dot{r} = 0.$$

The first constraint does not place any restrictions on the virtual velocities $\xi = (\xi_r, \xi_\theta)$ since $\frac{\partial f_1}{\partial(\dot{r},\dot{\theta})} = 0$, but the second condition yields

$$0 = \frac{\partial f_2}{\partial(\dot{r}, \dot{\theta})} \cdot \xi = \xi_r.$$

The d'Alembert–Lagrange principle (5.4) then yields the condition

$$(m\ddot{r} - mg\cos\theta)\xi_r + (mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta)\xi_\theta = 0.$$

As $\xi_r = 0$ and ξ_θ is arbitrary, we conclude that the second parenthetical term above vanishes. After using the constraints f_1 and f_2, we obtain the familiar equation of motion

$$\ddot{\theta} + \tfrac{g}{\ell}\sin\theta = 0.$$

The holonomic constraints have effectively discarded the r equation and reduced the degrees of freedom from two to one.

5.2 Gauss' principle of least constraint

Unlike the principle of least action, the d'Alembert–Lagrange principle is not a optimization problem because the virtual velocities are not given by a functional variation. Gauss sought a rephrasing as a optimization problem, in which the actual constrained motion is the possible constrained motion which deviates the least from the unconstrained motion.

The state of the system at time t_0 is determined by the position $q_0 = q(t_0)$ and velocity $v_0 = \dot{q}(t_0)$ at a fixed time, and then the acceleration $a(t_0) = \ddot{q}(t_0)$ is determined by the laws of motion and constraints. Consequently, we will consider q_0 and v_0 as fixed while we vary a_0. For a fixed state $(q_0, v_0) \in TM$ and time t_0, we will refer to all paths $q(t)$ allowed by the constraints with $q(t_0) = q_0$ and $\dot{q}(t_0) = v_0$ as the **conceivable motions**; these lie in the constraint submanifold S but do not necessarily satisfy the equations of motion. A **released motion** satisfies the unconstrained Euler–Lagrange equations (4.4) (and does not lie in S), and an **actual motion** is a conceivable motion satisfying the d'Alembert–Lagrange principle (5.4) (and hence lies in S).

By considering only variations supported in a fixed coordinate patch, we may reduce to a neighborhood about the point q_0 and work in Euclidean coordinates and use the vector notation $q(t) = \mathbf{x}(t)$. We will write an actual motion of the system $\mathbf{x}_a(t) = \mathbf{x}_r(t) + \delta\mathbf{x}(t)$ as a deviation from the released motion $\mathbf{x}_r(t)$, and we assume that the initial positions $\mathbf{x}_a(t_0) = \mathbf{x}_r(t_0) = \mathbf{x}_0$ and velocities $\dot{\mathbf{x}}_a(t_0) = \dot{\mathbf{x}}_r(t_0) = \mathbf{v}_0$ are fixed. Taylor expanding we have

$$\begin{aligned}\mathbf{x}_r(t) &= \mathbf{x}_0 + (t-t_0)\mathbf{v}_0 + \tfrac{1}{2}(t-t_0)^2\ddot{\mathbf{x}}_r(t_0) + \mathcal{O}\big((t-t_0)^3\big),\\ \mathbf{x}_a(t) &= \mathbf{x}_0 + (t-t_0)\mathbf{v}_0 + \tfrac{1}{2}(t-t_0)^2\ddot{\mathbf{x}}_a(t_0) + \mathcal{O}\big((t-t_0)^3\big).\end{aligned}$$

Therefore we obtain

$$\delta\mathbf{x}(t) = \tfrac{1}{2}(t-t_0)^2\delta\ddot{\mathbf{x}}(t_0) + \mathcal{O}\big((t-t_0)^3\big). \tag{5.6}$$

So far, our considerations have been independent of the constraints.

In our local coordinates the d'Alembert–Lagrange principle requires

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} - \frac{\partial \mathscr{L}}{\partial \mathbf{x}}\right)\cdot \boldsymbol{\xi} = 0 \tag{5.7}$$

for all virtual displacements $\boldsymbol{\xi}$. For conservative systems, $\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}}$ is the system's force $-\nabla V$ and $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}}$ is the total force $m\ddot{\mathbf{x}}$ (the system's force and the fictitious constraint forces), and so $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} - \frac{\partial \mathscr{L}}{\partial \mathbf{x}}$ is the force $m\,\delta\ddot{\mathbf{x}}$ due to the constraints. We will now realize this more generally by evaluating $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} - \frac{\partial \mathscr{L}}{\partial \mathbf{x}}$ at $\mathbf{x}_a(t) = \mathbf{x}_r(t) + \delta\mathbf{x}(t)$. Taylor expanding about $\mathbf{x}_r(t)$ we have

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} - \frac{\partial \mathscr{L}}{\partial \mathbf{x}}\right)\bigg|_{\mathbf{x}=\mathbf{x}_a} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} - \frac{\partial \mathscr{L}}{\partial \mathbf{x}}\right)\bigg|_{\mathbf{x}=\mathbf{x}_r}\\ &+ \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^2 \mathscr{L}}{\partial \dot{\mathbf{x}}^2}\delta\dot{\mathbf{x}} + \frac{\partial^2 \mathscr{L}}{\partial \dot{\mathbf{x}}\partial \mathbf{x}}\delta\mathbf{x}\right) - \frac{\partial^2 \mathscr{L}}{\partial \mathbf{x}^2}\delta\mathbf{x} - \frac{\partial^2 \mathscr{L}}{\partial \dot{\mathbf{x}}\partial \mathbf{x}}\delta\dot{\mathbf{x}}\right]_{\mathbf{x}=\mathbf{x}_r} + \mathcal{O}(|\delta\mathbf{x}|^2).\end{aligned}$$

The first term on the right-hand side vanishes since the actual motion $\mathbf{x}_a$ solves Lagrange's equations. For the second term, we use eq. (5.6) and take the limit $t \to t_0$ to obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{x}}} - \frac{\partial \mathscr{L}}{\partial \mathbf{x}}\right)\bigg|_{\substack{\mathbf{x}=\mathbf{x}_a\\ t=t_0}} = \frac{\partial^2 \mathscr{L}}{\partial \dot{\mathbf{x}}^2}\bigg|_{\substack{\mathbf{x}=\mathbf{x}_r\\ t=t_0}}\delta\ddot{\mathbf{x}}(t_0).$$

Therefore, we interpret eq. (5.7) as saying that

$$\left.\frac{\partial^2 \mathscr{L}}{\partial \dot{q}^2}\right|_{\substack{q=q_r \\ t=t_0}} (\ddot{q}_a(t_0) - \ddot{q}_r(t_0)) \tag{5.8}$$

is orthogonal to the constraint submanifold S. On the other hand, (5.8) is the gradient of the functional

$$Z(q(t)) = \tfrac{1}{2}(\ddot{q} - \ddot{q}_r) \cdot A(\ddot{q} - \ddot{q}_r)\big|_{t=t_0} \tag{5.9}$$

for the Hessian matrix $A = \frac{\partial^2 \mathscr{L}}{\partial \dot{q}^2}$ of the kinetic energy quadratic form at $\mathbf{x} = \mathbf{x}_r$ and $t = t_0$. For conservative systems, A is a positive definite matrix containing the particle masses. The quantity (5.9) is called Gauss' **compulsion** and it measures how much the motion $q(t)$ deviates from the released motion.

As the gradient (5.8) of the compulsion functional (5.9) evaluated at the actual motion q_a is orthogonal to the submanifold S, we conclude that the actual motion q_a is a critical point for the compulsion functional. All we needed to prove this was that the multipliers in the equations (5.5) are Lagrange multipliers. In fact, the converse of this statement is also true. Let $f(q)$ be a smooth functional on M with a critical point q_a when restricted to the submanifold S. This happens if and only if $D(f \circ \phi)|_{q_a} = 0$ for all coordinates ϕ from Euclidean space to a neighborhood of q_a in S. Writing $D(f \circ \phi) = \nabla f \cdot D\phi$ and noting that the columns of the matrix $D\phi$ span the tangent space $T_q S \subset T_q M$, we conclude that q_a is a critical point on S if and only if the gradient $\nabla f(q_a) \in T_q M$ is orthogonal $T_q S$.

Altogether, we have proven the following:

Theorem 5.5 (Gauss' principle)**.** *Among the conceivable motions, the actual motion $q(t)$ is a critical point for the compulsion* (5.9) *with respect to the released motion. In particular, if A is positive definite then the actual motion is the global minimum for the compulsion with respect to the released motion.*

See Exercise 5.1 for an example of Gauss' principle.

Gauss' principle is analogous to the method of least squares in regression analysis (which is another discovery due to Gauss). In the method of least squares, there is an unknown data correlation function determined by n parameters and a larger number $N \gg n$ of observations. The observations deviate slightly from the desired function's exact values due to observation error, and hence the overdetermined system for the function is inconsistent. The remedy is to construct the square sum error between the function and the data, and then find the desired function as a error minimizer in the n parameters. Here, the compulsion is determined by the $2n$ initial conditions $(q_0, v_0) \in TM$ where $n = \dim M$ is the number of degrees of freedom, and the system is overdetermined due to the extra constraint conditions. Here, the actual motion $q(t)$ plays the role of the function we seek determined by the smaller number of conditions $\dim TS < 2n$, the released motion is the measured data which overdetermines the function with error, and the compulsion is the square sum error. Moreover,

the matrix A of masses can statistically be interpreted as weights in the method of least squares, which are modified based on the assumed reliability of the data accuracy.

5.3 Integrability

Some authors also require that holonomic constraints are integrable (in the sense of Frobenius). In this section, we will explore what this additional assumption yields.

In this section, we will assume that the motion is constrained within a k-dimensional **distribution** $\Delta \subset TM$: for all $q_0 \in M$, Δ_{q_0} is a k-dimensional subspace of $T_{q_0}M$ and there exists smooth vector fields $X_1, \dots, X_k$ defined on a neighborhood $U \subset M$ of q_0 such that Δ_q is given by the span of $X_1(q), \dots, X_k(q)$ for all $q \in U$. For holonomic constraints, we take Δ to be the space of virtual velocities. We will also assume that Δ is **integrable**: there exists an embedding $i : N \to M$ such that $\mathrm{d}i(T_qN) = \Delta_q$ for all $q \in N$; in other words, Δ is given by the tangent bundle of a submanifold.

Frobenius' theorem gives us a condition to verify if the distribution Δ is integrable:

Theorem 5.6 (Frobenius' theorem)**.** *The distribution Δ is integrable if and only if Δ is **involutive**—that is, for any vector fields $X, Y \in \Delta$, the Lie bracket $[X, Y]$ is also in Δ.*

This theorem tells us that a system of differential equations can be solved locally if and only if they are involutive. For example, consider a function $u = (u_1, \dots, u_m)$ of the variables $q = (q_1, \dots, q_k)$ which solves the system of equations

$$\frac{\partial u}{\partial q_1}(q) = F_1(q, u), \quad \dots, \quad \frac{\partial u}{\partial q_k}(q) = F_k(q, u).$$

Of course, if there exists a solution u, then the right-hand sides $F_1, \dots, F_k$ must be consistent:

$$\begin{aligned} \frac{\partial}{\partial q_j} F_i(q, u(q)) &= \frac{\partial^2 u}{\partial q_i \partial q_j}(q) = \frac{\partial}{\partial q_i} F_j(q, u(q)) \\ \implies \quad \frac{\partial F_i}{\partial q_j} + \frac{\partial F_i}{\partial u} \cdot F_j &= \frac{\partial F_j}{\partial q_i} + \frac{\partial F_j}{\partial u} \cdot F_i \end{aligned}$$

for all i and j. Frobenius' theorem also gives us the converse: if the right-hand sides $F_1, \dots, F_k$ are consistent, then there exists a local solution. For a full proof of Frobenius' theorem, see [Lee13, Th. 19.12].

If the holonomic constraints are integrable, then the motion $q(t)$ must lie in a submanifold of M, not merely a submanifold of TM. Now that we know that the holonomic constraints have a corresponding smooth integrable submanifold $N \subset M$ of dimension $d-k$, we can show that the d'Alembert–Lagrange condition is equivalent to the principle of least action holding on N:

Proposition 5.7. *Suppose the Lagrangian system $(M, \mathscr{L})$ has integrable holonomic constraints. Then a constrained path is a motion of the system if and only if the path is a motion for the system $(N, \mathscr{L}|_N)$.*

Proof. From Hölder's principle (Proposition 5.3) we know that the d'Alembert–Lagrange condition is equivalent to insisting that the action variation $\mathrm{d}S|_\gamma$ vanishes on a subspace Γ of conceivable variations. Here, Γ is a subspace of the tangent space $T_{q(t)}\Omega_M$ to the paths Ω_M on M. On the other hand, the motion of the Lagrangian system on N is given by taking the action S on the paths Ω_N on N, and then insisting that its variation vanishes on the tangent space $T_{q(t)}\Omega_N$. The key observation is that the tangent space $T_{q(t)}\Omega_N$ is equal to the subspace Γ, and so the two conditions are identical. □

In particular, the motion for integrable holonomic constraints is determined by the restriction of the Lagrangian to the constraint submanifold N. In this way, a system with holonomic constraints is like a new mechanical system with fewer degrees of freedom; this is a characteristic feature of holonomic constraints, and is not true for nonholomic constraints.

5.4 Integral constraints

In this section, we will assume we have a Lagrangian system $(M, \mathscr{L})$ subject to the constraint

$$\int_{t_0}^{t_1} G(q(t))\,\mathrm{d}t = 0, \tag{5.10}$$

for some smooth function $G : M \to \mathbb{R}$ that is not constant.

Proposition 5.8. *If $q(t)$ is a critical point of the action subject to the constraint* (5.10) *and $G(q(t))$ is not constant, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ so that $q(t)$ solves*

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{\partial \mathscr{L}}{\partial q} = \lambda \frac{\partial G}{\partial q}. \tag{5.11}$$

Proof. As in the set up for the proof of Proposition 4.5, we may fix a coordinate patch on M and take the variation to be supported within the image in order to reduce the statement to Euclidean space $\mathbb{R}^n$. Let $q(t) : \mathbb{R} \to \mathbb{R}^n$ be a critical point, and let $g = \frac{\partial G}{\partial q}$ denote the gradient of G. By premise, we know that $g(q) \not\equiv 0$. In particular, there exists a smooth function $v(t) : \mathbb{R} \to \mathbb{R}^n$ so that

$$\int_{t_0}^{t_1} g(q(t)) \cdot v(t)\,\mathrm{d}t \neq 0. \tag{5.12}$$

Let $h(t)$ be a fixed-endpoint variation. To obtain a differential equation for $q(t)$ we would like to perturb $q(t)$ by the variation $h(t)$. However, the perturbed

path $q(t) + h(t)$ will not satisfy the constraint for arbitrary $h(t)$, and so we will need to modify our perturbation. Let

$$J(\sigma, \tau) := \int_{t_0}^{t_1} G\big(q(t) + \sigma h(t) + \tau v(t)\big)\,\mathrm{d}t.$$

We claim there is a smooth function $\tau(\sigma)$ defined on a neighborhood of zero such that $\tau(0) = 0$ and

$$J(\sigma, \tau(\sigma)) \equiv 0 \quad \text{for all } \sigma. \tag{5.13}$$

This follows from the implicit function theorem once we note that $J(0,0) = 0$ and that

$$\frac{\partial J}{\partial \tau}(0,0) = \int_{t_0}^{t_1} g(q(t)) \cdot v(t)\,\mathrm{d}t$$

is nonzero by (5.12).

The perturbation $q + \sigma h + \tau(\sigma)v$ now satisfies the constraint for all σ sufficiently small. As $q(t)$ is a critical point of the constrained action functional, then we must have

$$0 = \frac{\mathrm{d}}{\mathrm{d}\sigma} \int_{t_0}^{t_1} \mathscr{L}\big(q + \sigma h + \tau(\sigma)v, \dot{q} + \sigma\dot{h} + \tau(\sigma)\dot{v}, t\big)\,\mathrm{d}t.$$

Using the variation of the action functional (4.2), we obtian

$$= \int_{t_0}^{t_1} \left[\frac{\partial \mathscr{L}}{\partial q}(q) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{q}}(q)\right] \cdot (h + \tau'(0)w)\,\mathrm{d}t.$$

Differentiating (5.13) with respect to σ, we see that

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}\sigma}\big[J(\sigma, \tau(\sigma))\big]_{\sigma=0} &= \frac{\partial J}{\partial \sigma}(0,0) + \frac{\partial J}{\partial \tau}(0,0)\,\tau'(0) \\ &= \int_{t_0}^{t_1} g(q(t)) \cdot h(t)\,\mathrm{d}t + \tau'(0)\int_{t_0}^{t_1} g(q(t)) \cdot v(t)\,\mathrm{d}t, \end{aligned}$$

and so

$$\tau'(0) = -\frac{\int_{t_0}^{t_1} g(q(t)) \cdot h(t)\,\mathrm{d}t}{\int_{t_0}^{t_1} g(q(t)) \cdot v(t)\,\mathrm{d}t}.$$

Plugging this into the functional variation we arrive at

$$0 = \int_{t_0}^{t_1} \left[\frac{\partial \mathscr{L}}{\partial q}(q) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{q}}(q) + \lambda g(q)\right] \cdot h\,\mathrm{d}t,$$

where

$$\lambda = -\frac{\int_{t_0}^{t_1} [\frac{\partial \mathscr{L}}{\partial q}(q) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{q}}(q)] \cdot w(t)\,\mathrm{d}t}{\int_{t_0}^{t_1} g(q(t)) \cdot v(t)\,\mathrm{d}t}$$

is independent of h. As $h(t)$ was an arbitrary fixed-endpoint variation, we conclude that (5.11) holds. □

5.5 Duality

Another mathematical technique to deal with constraints is to build them into the functional using duality. We will illustrate this technique via the proof of the following theorem:

Theorem 5.9. *Suppose we have a conservative system with a potential* $V : \mathbb{R}^n \to \mathbb{R}$ *that is smooth, strictly convex, and nonnegative, satisfying* $V(0) = 0$ *and* $V(\boldsymbol{x}) \to \infty$ *as* $|\boldsymbol{x}| \to \infty$*. Then for any* $\alpha > 0$ *there is a periodic solution* $\boldsymbol{x}(t)$ *to Newton's equations* (1.3) *satisfying*

$$E(\boldsymbol{x}(t), \boldsymbol{v}(t)) := \sum_{i=1}^{n} \tfrac{1}{2} m_i v_i^2 + V(\boldsymbol{x}) \equiv \alpha$$

for all $t \in \mathbb{R}$*, where* $\boldsymbol{v} := \dot{\boldsymbol{x}}$*.*

The constraint in Theorem 5.9 is that the motion is confined to the energy surface $E^{-1}(\alpha)$ and is periodic. For $n = 1$ we know from section 2.5 that many trajectories are periodic; however, the following example illustrates that periodic trajectories can be quite exceptional when $n \geq 2$. Nevertheless, Theorem 5.9 still guarantees the existence of a periodic trajectory.

Example 5.10. Consider two particles of mass $m = 1$ moving according to the two-dimensional harmonic oscillator potential

$$V(x_1, x_2) = \tfrac{1}{2}(k_1 x_1^2 + k_2 x_2^2),$$

where $k_1, k_2 > 0$ are constants. Then Newton's system of equations decouple, and x_1 and x_2 are independent harmonic oscillators with constants k_1 and k_2 respectively. We found the equations of motion in Example 2.2, and $x_1(t)$ and $x_2(t)$ are each periodic with periods $2\pi/\sqrt{k_1}$ and $2\pi/\sqrt{k_2}$ respectively. Therefore, when $k_1 = k_2$ then all trajectories of $\mathbf{x} = (x_1, x_2)$ are periodic, and when $\sqrt{k_1/k_2}$ is only rational then there are still infinitely many periodic trajectories. However, when $\sqrt{k_1/k_2}$ is irrational, only trajectories with $x_1 \equiv p_1 \equiv 0$ or $x_2 \equiv p_2 \equiv 0$ are periodic, and every other trajectory is aperiodic.

To prove Theorem 5.9, we will extract the trajectory $\mathbf{x}(t)$ as a minimizer of a functional. The action functional (4.1) is not suitable for this, because the Lagrangian $\mathscr{L} = T - V$ is not bowl-shaped and so it is hard to guarantee the existence of a minimizer.

Our next guess is the functional

$$\Phi(\mathbf{x}(t), \mathbf{v}(t)) = \int_0^1 \sum_{i=1}^{n} \tfrac{1}{2} m_i (x_i \dot{v}_i - \dot{x}_i v_i)\, \mathrm{d}t$$

on the domain

$$C_\alpha = \left\{ (\mathbf{x}, \mathbf{v}) : \mathbb{R} \to \mathbb{R}^{2n} : \text{smooth, 1-periodic, and } \int_0^1 E(\mathbf{x}(t), \mathbf{v}(t))\, \mathrm{d}t = \alpha \right\}.$$

We are treating $\mathbf{x}$ and $\mathbf{v}$ as independent variables and we have written the integrand of Φ in a symmetric way, but after formally integrating by parts we see that Φ is proportional to the kinetic energy. We have removed the potential energy from Φ and instead put it as an integral constraint in C_α. Suppose $(\mathbf{x}(t), \mathbf{v}(t)) \in C_\alpha$ is a critical point of Φ. Then by Proposition 5.8 the critical point satisfies (5.11), which reads

$$\begin{aligned} -(m_1\dot{v}_1, \dots, m_n\dot{v}_n) &= \lambda \frac{\partial E}{\partial \mathbf{x}} = \lambda \nabla V(\mathbf{x}), \\ (m_1\dot{x}_1, \dots, m_n\dot{x}_n) &= \lambda \frac{\partial E}{\partial \mathbf{v}} = \lambda (m_1 v_1, \dots, m_n v_n). \end{aligned}$$

Without the factor of λ, the second equation is the definition of the velocity and the first equation is Newton's equation. Consequently, the rescaling $\mathbf{x}(t/|\lambda|)$ solves Newton's equations with period $|\lambda|^{-1}$.

However, the functional Φ is also not viable, because it is not bounded below over the set of 1-periodic functions. Indeed, if we take

$$\mathbf{x}(t) = (\cos(2\pi k t), 0, \dots, 0),$$

then

$$\Phi(\mathbf{x}, \dot{\mathbf{x}}) = -2m_1(\pi k)^2,$$

which tends to $-\infty$ as $k \to +\infty$. In order to prove Theorem 5.9, we will find a new functional with the desired trajectory $(\mathbf{x}, \dot{\mathbf{x}})$ as a minimizer.

Proof of Theorem 5.9. Let

$$E^*(\mathbf{y}, \mathbf{w}) = \sup_{(\mathbf{x},\mathbf{v}) \in \mathbb{R}^{2n}} \left[(\mathbf{x}, \mathbf{v}) \cdot (\mathbf{y}, \mathbf{w}) - E(\mathbf{x}, \mathbf{v}) \right]$$

denote the Legendre transform of the energy E. We will study this transformation more thoroughly in section 7.2 when it arises naturally in the study of Hamiltonian mechanics, and so we defer the proofs of some basic facts which we will borrow. The energy E is convex because V is, and so the expression inside the supremum above is concave and so the supremum is finite and is obtained at most once. Moreover, $E \to \infty$ as $|(\mathbf{x}, \mathbf{v})| \to \infty$ by premise, and so the supremum is obtained at exactly one point, which we can solve for by differentiation. This yields

$$E^*(\mathbf{y}, \mathbf{w}) = \sum_{i=1}^{n} \frac{w_i^2}{2m_i} + V^*(\mathbf{y}).$$

In particular, $E^*(\mathbf{y}, \mathbf{w})$ is also convex. (This is no coincidence; cf. Theorem 7.2).

Consider the new functional

$$\Psi(\mathbf{y}, \mathbf{w}) := \int_0^1 \left[E^*(\mathbf{y}, \mathbf{w}) - \tfrac{1}{2}\mathbf{y} \cdot I\mathbf{w} + \tfrac{1}{2}\mathbf{w} \cdot I\mathbf{y} \right] \mathrm{d}t$$

on the domain

$$M_0 := \left\{ (\mathbf{y}, \mathbf{w}) \in L^p([0,1] \to \mathbb{R}^{2n}) : \int_0^1 \mathbf{y}(t)\,\mathrm{d}t = \int_0^1 \mathbf{w}(t)\,\mathrm{d}t = 0 \right\}$$

for some $p \in (2, \infty)$ to be chosen later. Here, I is the integration operator

$$[I\mathbf{y}](t) := \int_0^t \mathbf{y}(s)\,\mathrm{d}s.$$

Like differentiation, the operator I is antisymmetric. Together with the convexity of E^*, we shall see that this makes the functional Ψ actually bounded below.

Suppose we have found a minimizer $(\mathbf{y}, \mathbf{w})$ of Ψ on M_0. Then by Proposition 5.8 the critical point solves (5.11), which reads

$$-\nabla(V^*)(\mathbf{y}) + I\mathbf{w} = \lambda, \qquad -(\tfrac{w_1}{m_1}, \dots, \tfrac{w_n}{m_n}) - I\mathbf{y} = \lambda.$$

A short computation shows that $\nabla(V^*) = (\nabla V)^{-1}$ (cf. (7.10)). Therefore, the functions

$$\mathbf{x} := \tfrac{1}{m}(I\mathbf{w} - \lambda), \qquad \mathbf{v} := -I\mathbf{y} - \lambda$$

satisfy

$$-m\dot{\mathbf{v}} + \nabla V(\mathbf{x}) = 0, \qquad -\dot{\mathbf{x}} + \mathbf{v} = 0.$$

So $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity and $\mathbf{x}$ solves Newton's equation. We also need to ensure that $E(\mathbf{x}, \dot{\mathbf{x}}) \equiv \alpha$, but we will come back to this.

Next we want to show that there exists a minimizer of Ψ in M_0. To prove this, we will first consider the special case where E is positive homogeneous of degree q for some $q \in (1, 2)$, i.e.

$$E(r\mathbf{x}, r\mathbf{v}) = r^q E(\mathbf{x}, \mathbf{v}) \quad \text{for all } r > 0 \text{ and } \mathbf{x}, \mathbf{v} \in \mathbb{R}^n.$$

Then E^* is positive homogeneous of degree $p := \frac{q}{q-1} \in (2, \infty)$, since

$$\begin{aligned}
\frac{E^*(\mathbf{y}, \mathbf{w})}{|(\mathbf{y}, \mathbf{w})|^p} &= \sup_{(\mathbf{x},\mathbf{v}) \in \mathbb{R}^{2n}} \left[\frac{(\mathbf{x}, \mathbf{v})}{|(\mathbf{y}, \mathbf{w})|^{p-1}} \cdot \frac{(\mathbf{y}, \mathbf{w})}{|(\mathbf{y}, \mathbf{w})|} - \frac{E(\mathbf{x}, \mathbf{v})}{|(\mathbf{y}, \mathbf{w})|^p} \right] \\
&= \sup_{(\mathbf{x},\mathbf{v}) \in \mathbb{R}^{2n}} \left[\frac{(\mathbf{x}, \mathbf{v})}{|(\mathbf{y}, \mathbf{w})|^{p-1}} \cdot \frac{(\mathbf{y}, \mathbf{w})}{|(\mathbf{y}, \mathbf{w})|} - E\left(\frac{(\mathbf{x}, \mathbf{v})}{|(\mathbf{y}, \mathbf{w})|^{p-1}} \right) \right] = E^*\left(\frac{(\mathbf{y}, \mathbf{w})}{|(\mathbf{y}, \mathbf{w})|} \right).
\end{aligned}$$

We claim that Ψ is coercive, in the sense that

$$\Psi(\mathbf{y}, \mathbf{w}) \to +\infty \quad \text{as } \|(\mathbf{y}, \mathbf{w})\|_{L^p} \to \infty.$$

As $p > 2$, then E^* grows faster than quadratically:

$$E^*(\mathbf{y}, \mathbf{w}) \geq C|(\mathbf{y}, \mathbf{w})|^p \quad \text{for } C := \inf_{|(\mathbf{y},\mathbf{w})|=1} E^*(\mathbf{y}, \mathbf{w}).$$

We know that $C > 0$ because

$$E^*(\mathbf{y}, \mathbf{w}) \geq \big[(\mathbf{x}, \mathbf{v}) \cdot (\mathbf{y}, \mathbf{w}) - E(\mathbf{x}, \mathbf{v})\big]_{(\mathbf{x},\mathbf{v}) = a(\mathbf{y},\mathbf{w})} = a|(\mathbf{y}, \mathbf{w})|^2 - a^q E^*\left(\frac{(\mathbf{y}, \mathbf{w})}{|(\mathbf{y}, \mathbf{w})|} \right),$$

and the RHS is positive if we choose $a > 0$ sufficiently small. For the other term of Ψ, we have

$$\left| \int_0^1 \left[-\mathbf{y} \cdot I\mathbf{w} + \mathbf{w} \cdot I\mathbf{y} \right] \mathrm{d}t \right| \leq \int_0^1 \left(|\mathbf{y}|^2 + |\mathbf{w}|^2 \right) \mathrm{d}t \leq \left(\int_0^1 |(\mathbf{y}, \mathbf{w})|^p \, \mathrm{d}t \right)^{2/p}$$

by Hölder's inequality, since $p > 2$. Together, we conclude

$$\begin{aligned} \Psi(\mathbf{y}, \mathbf{w}) &= \int_0^1 \left[E^*(\mathbf{y}, \mathbf{w}) - \tfrac{1}{2} \mathbf{y} \cdot I\mathbf{w} + \tfrac{1}{2} \mathbf{w} \cdot I\mathbf{y} \right] \mathrm{d}t \\ &\geq C \int_0^1 |(\mathbf{y}, \mathbf{w})|^p \, \mathrm{d}t - \tfrac{1}{2} \left(\int_0^1 |(\mathbf{y}, \mathbf{w})|^p \, \mathrm{d}t \right)^{2/p}. \end{aligned}$$

As $\frac{2}{p} < 1$, we see that the RHS tends to $+\infty$ as $\|(\mathbf{y}, \mathbf{w})\|_{L^p} \to \infty$.

Next, we claim that Ψ is weakly continuous on $L^p([0,1] \to \mathbb{R}^{2n})$. Let $(\mathbf{y}_n, \mathbf{w}_n) \rightharpoonup (\mathbf{y}, \mathbf{w})$ be a weakly convergent sequence. As E^* is continuous and convex, its integral converges to the value at $(\mathbf{y}, \mathbf{w})$. Notice that $I\mathbf{y}_n \rightharpoonup I\mathbf{y}$ and $I\mathbf{w}_n \rightharpoonup I\mathbf{w}$ in $W^{1,p}$. We write

$$\begin{aligned} &\int_0^1 \left[-\tfrac{1}{2} \mathbf{y}_n \cdot I\mathbf{w}_n + \tfrac{1}{2} \mathbf{w}_n \cdot I\mathbf{y}_n \right] \mathrm{d}t - \int_0^1 \left[-\tfrac{1}{2} \mathbf{y} \cdot I\mathbf{w} + \tfrac{1}{2} \mathbf{w} \cdot I\mathbf{y} \right] \mathrm{d}t \\ &= \int_0^1 \left[-\tfrac{1}{2} (\mathbf{y}_n - \mathbf{y}) \cdot I\mathbf{w}_n + \tfrac{1}{2} (\mathbf{w}_n - \mathbf{w}) \cdot I\mathbf{y}_n \right] \mathrm{d}t \\ &\quad + \int_0^1 \left[-\tfrac{1}{2} \mathbf{y} \cdot (I\mathbf{w}_n - I\mathbf{w}) + \tfrac{1}{2} \mathbf{w} \cdot (I\mathbf{y}_n - I\mathbf{y}) \right] \mathrm{d}t. \end{aligned}$$

The first integral on the RHS converges to zero since $\mathbf{y}_n \rightharpoonup \mathbf{y}$ and $I\mathbf{y}_n$ is bounded in $L^p \subset L^q$, and similarly for $\mathbf{w}_n$. The second integral on the RHS converges to zero since $\mathbf{y}$ is in $L^p \subset L^q$ and $I\mathbf{y}_n \rightharpoonup I\mathbf{y}$ in L^p, and similarly for $\mathbf{w}_n$. Together, we conclude that the RHS converges to zero as desired.

Now that we know that Ψ is coercive, M_0 is a weakly closed subspace, and Ψ is weakly continuous on $L^p([0,1] \to \mathbb{R}^{2n})$, we conclude that there exists a minimizer in M_0. (This is because a minimizing sequence must be bounded, hence admit a weak subsequential limit, and this limit is a minimum since Ψ is weakly continuous.) This minimizer yields a solution $\mathbf{x}(t)$ to Newton's equation. (For PDEs this only yields a weak solution, but for ODEs weak solutions are automatically classical solutions; see for example [Tao06, Lem. 1.3].) Then $E(\mathbf{x}, \dot{\mathbf{x}})$ is equal to some constant $\beta \geq 0$. We cannot have $\beta = 0$; indeed, if $\beta = 0$ then $0 \equiv \mathbf{x}(t) = \frac{1}{m}(I\mathbf{w} - \lambda)$ and so $\mathbf{w} \equiv 0 \equiv \mathbf{y}$, which is a contradiction because Ψ obtains negative values on M_0 (since the quadratic terms dominate for $(\mathbf{y}, \mathbf{w})$ small). As $\beta > 0$, we can then scale $\mathbf{x}$ to make $E(\mathbf{x}, \dot{\mathbf{x}}) = \alpha$ since E is positive homogeneous.

Lastly, we consider the general case when E is not positive homogeneous. As E is strictly convex with minimum zero, then for each radial vector $\nu \in \mathbb{R}^{2n}$, $|\nu| = 1$ there is a unique $r(\nu)$ so that

$$E(r(\nu)\nu) = \alpha.$$

Fix $q \in (1,2)$ and define

$$\tilde{E}(\mathbf{x},\mathbf{v}) = \begin{cases} \alpha|(\mathbf{x},\mathbf{v})|^q r(\nu)^{-q} & \text{for } (\mathbf{x},\mathbf{v}) \neq 0, \\ 0 & \text{for } (\mathbf{x},\mathbf{v}) = 0. \end{cases}$$

Then $\tilde{E}$ is positive homogeneous of degree q, and so we may apply the special case to obtain a minimizer $\tilde{\mathbf{x}}$ for $\tilde{E}$. Now $\nabla E(\mathbf{x},\mathbf{v}) = \lambda(\mathbf{x},\mathbf{v})\nabla\tilde{E}(\mathbf{x},\mathbf{v})$ for some $\lambda(\mathbf{x},\mathbf{v}) > 0$, since E and $\tilde{E}$ share the same level sets. Then the rescaling

$$\mathbf{x}(t) := \tilde{\mathbf{x}}(s(t)), \quad \text{where} \quad \dot{s}(t) = \lambda(\tilde{\mathbf{x}}(t), \dot{\tilde{\mathbf{x}}}(t)),$$

solves

$$\dot{\mathbf{v}} = \dot{\tilde{\mathbf{v}}}\lambda(\tilde{\mathbf{x}}(t),\tilde{\mathbf{v}}(t)) = -\lambda(\tilde{\mathbf{x}}(t),\tilde{\mathbf{v}}(t))\nabla_{\mathbf{x}} E = -\nabla_{\mathbf{x}}\tilde{E} = -\nabla V$$

as desired. □

5.6 One-form constraints

In the derivation of Lagrange's equations (Proposition 4.5), the last step relied upon the generalized coordinates q_j being independent. As discussed in section 5.3, such q_j can always be chosen for a system subject to (integrable) holonomic constraints. This is not always true in the nonholonomic case however, and so we wish to develop a new tool for this situation.

We will consider the special case of m nonholonomic constraints that can be expressed as the vanishing of one-forms:

$$\sum_{k=1}^{n} a_{\ell k}(q,t)\,\mathrm{d}q_k + a_{\ell t}(q,t)\,\mathrm{d}t = 0, \quad \ell = 1,\dots,m. \tag{5.14}$$

Note that velocity-independent holonomic constraints also fit this requirement, because if the condition (5.1) is independent of $\dot{q}$ then taking the differential of both sides yields

$$\sum_{k=1}^{n} \frac{\partial f}{\partial q_k}\,\mathrm{d}q_k + \frac{\partial f}{\partial t}\,\mathrm{d}t = 0.$$

However, the constraint (5.14) also includes some nonholonomic constraints; for example, see (5.23) of Exercise 5.4.

Consider fixed-endpoint variations $\delta q(t)$ of a path $q(t)$ between the times t_0 and t_1 as in the proof of Proposition 4.5. The constraint (5.14) is satisfied for $q(t)$, and so Taylor expansion yields

$$\sum_{k=1}^{n} a_{\ell k}(q,t)\,\delta q_k = \mathcal{O}(\delta q^2), \quad \ell = 1,\dots,m. \tag{5.15}$$

If there were no constraints, the principle of least action would require

$$\int_{t_0}^{t_1} \sum_{k=1}^{n} \left(\frac{\partial \mathscr{L}}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{q}_k} \right) \delta q_k\,\mathrm{d}t = 0. \tag{5.16}$$

To make (5.15) look like this, we multiply by coefficients λ_ℓ to be determined, sum over ℓ, and integrate from t_0 to t_1:

$$\int_{t_0}^{t_1} \sum_{k=1}^{n} \sum_{\ell=1}^{m} \lambda_\ell a_{\ell k} \delta q_k \, \mathrm{d}t = \mathcal{O}(\delta q^2).$$

Adding this to eq. (5.16) we obtain

$$\int_{t_0}^{t_1} \sum_{k=1}^{n} \left(\frac{\partial \mathscr{L}}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}_k} + \sum_{\ell=1}^{m} \lambda_\ell a_{\ell k} \right) \delta q_k \, \mathrm{d}t = \mathcal{O}(\delta q^2). \tag{5.17}$$

The generalized coordinates q_k are not independent since they are related by the m constraints. However, the first $d - m$ coordinates can be chosen independently, and the remaining m coordinates are determined by the conditions (5.15). Pick the multipliers λ_ℓ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}_k} - \frac{\partial \mathscr{L}}{\partial q_k} = \sum_{\ell=1}^{m} \lambda_\ell a_{\ell k}, \quad d - m < k \leq d. \tag{5.18}$$

This causes the last m terms of the summation in the variation (5.17) to vanish, leaving us with

$$\int_{t_0}^{t_1} \sum_{k=1}^{n-m} \left(\frac{\partial \mathscr{L}}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}_k} + \sum_{\ell=1}^{m} \lambda_\ell a_{\ell k} \right) \delta q_k \, \mathrm{d}t = \mathcal{O}(\delta q^2).$$

As the first $d - m$ coordinates q_k are independent, then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}_k} - \frac{\partial \mathscr{L}}{\partial q_k} = \sum_{\ell=1}^{m} \lambda_\ell a_{\ell k}, \quad k = 1, \ldots, n - m.$$

Including our choice (5.18) of the last m coordinates, we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}_k} - \frac{\partial \mathscr{L}}{\partial q_k} = \sum_{\ell=1}^{m} \lambda_\ell a_{\ell k}, \quad k = 1, \ldots, d. \tag{5.19}$$

This is the extension of **Lagrange's equations for nonholonomic constraints**. Note that these are n equations in $d+m$ unknowns—the d coordinates q_k and the m multipliers λ_ℓ. To make the system not under-determined, we must also consider the m equations:

$$\sum_{k=1}^{d} a_{\ell k} \dot{q}_k + a_{\ell t} = 0, \quad \ell = 1, \ldots, m,$$

obtained from the one-form constraints (5.14). Comparing these new equations of motion (5.19) to the generalization for nonconservative forces (4.13), we observe that the quantities $\sum_\ell \lambda_\ell a_{\ell k}$ are a manifestation of the constraint forces.

5.7 Exercises

5.1. For the pendulum of Example 5.4, explicitly show that minimizing the compulsion Z leads to the familiar equation of motion.

5.2 (Hoop rolling down an inclined plane)**.** Consider a circular disk of mass M and radius r rolling without slipping due to gravity down a stationary inclined plane of fixed inclination ϕ.

(a) In a vertical plane, the disk requires three coordinates: for example, two Cartesian coordinates (x, y) for the center of mass and an angular coordinate to measure the disk's rotation. If we pick the origin such that the surface of the inclined plane is the line $y = r - (\tan\phi)x$, obtain a holonomic constraint of the form (5.1) for the center of mass corresponding to the disk sitting on the plane.

(b) Consequently, we now can pick two generalized coordinates to describe the disk's motion: let x denote the distance of the disk's point of contact and the top of the inclined plane, and θ the angle through which the disk has rotated from its initial state. By considering the arc length through which the disk has rolled, show that rolling without slipping poses another holonomic constraint.

(c) In this case, it is easier to treat rolling without slipping as a nonholonomic constraint of the type in section 5.6:

$$r\,\mathrm{d}\theta - \mathrm{d}x = 0.$$

Show that the Lagrangian for this system is

$$\mathscr{L} = \tfrac{1}{2}M(\dot{x}^2 + r^2\dot{\theta}^2) + Mgx\sin\phi.$$

(d) Apply Lagrange's equations of the form (5.19) to determine the equations of motion. Here, λ is the force of friction that causes the disk to roll without slipping. Substituting in the differential equation of constraint:

$$r\dot{\theta} = \dot{x},$$

conclude that

$$\ddot{x} = \tfrac{g}{2}\sin\phi, \qquad \ddot{\theta} = \tfrac{g}{2r}\sin\phi, \qquad \lambda = \tfrac{Mg}{2}\sin\phi.$$

5.3 (Catenary [Tro96, Ch. 3])**.** The catenary is the shape taken by a cable hung by both ends under its own weight.

(a) Consider a cable of length L hung between two equal height supports separated by a distance $H < L$. Let y denote the vertical coordinate with $y = 0$ at the point where the cable is fastened, and let $y(s)$ denote

the shape of the cable where s is the arc length along the cable, so that $y(0) = 0 = y(L)$. If the weight per unit length is a constant W, explain why the cable shape $y(s)$ minimizes the mass integral

$$F[y(s)] = W \int_0^L y(s)\,\mathrm{d}s.$$

If we instead chose the horizontal coordinate x as the independent variable, it turns out that the resulting functional would not be convex.

(b) The functional with Lagrangian $(y, y', t) \mapsto Wy$ is only convex—not strictly convex—and consequently may not have a unique minimizer. Show that in order to span the supports, the cable must satisfy the constraint

$$\int_0^L \sqrt{1 - y'(s)^2}\,\mathrm{d}s = H.$$

(Note that $|y'(s)| = 1$ requires $x'(s) = 0$, which would produce a cusp.) The new Lagrangian

$$\mathscr{L}(y, y', t) = Wy - \lambda\sqrt{1 - y'^2}$$

is strictly convex for $\lambda > 0$.

(c) Apply the Euler–Lagrange equation and integrate once to obtain the first order differential equation

$$\frac{\lambda y'(s)}{\sqrt{1 - y'^2(s)}} = s + c$$

for the catenary, where c is a constant.

(d) As we expect $y(s)$ to be unique by convexity, then we may place additional assumptions. Assume that the cable shape is symmetric about the midpoint $\ell = L/2$, and conclude that $y'(\ell) = 0$ and $c = -\ell$. Solve for $y'(s)$ and integrate on $[0, s]$ to obtain

$$y(s) = \sqrt{\lambda^2 + (l - s)^2} - \sqrt{\lambda^2 + l^2} \quad \text{on } [0, \ell].$$

(e) Show that the constraint of part (b) yields

$$\int_0^\ell \frac{\lambda}{\sqrt{\lambda^2 + (\ell - s)^2}}\,\mathrm{d}s = \frac{H}{2}.$$

Make the substitution $(\ell - s) = \lambda \sinh\theta$, evaluate the integral, and conclude that

$$\lambda = \frac{\ell}{\sinh(H/L)}.$$

(f) Show that $x(s)$ is given by

$$x(s) = \int_0^s \sqrt{1 - y'(t)^2}\,\mathrm{d}t = \frac{H}{2} - \lambda \sinh^{-1}\left(\frac{\ell - s}{\lambda}\right).$$

Together with $y(s)$ from part (d), we have parametric equations for the catenary. Eliminate the variable s and conclude

$$y(x) = \lambda \cosh\left(\frac{x - \frac{H}{2}}{\lambda}\right) - \sqrt{\lambda^2 + \ell^2}$$

for $x \in [0, H]$. That is, the catenary is the graph of hyperbolic cosine.

(g) Obtain the same expression for $y(x)$ using Proposition 5.8.

5.4 (Solving Kepler's problem using harmonic oscillators [KS65])**.** Consider the system of section 3.3 in which a particle $\mathbf{x} \in \mathbb{R}^3$ of mass m under the influence of a central potential

$$V(|\mathbf{x}|) = -\frac{Mm}{|\mathbf{x}|}.$$

The squaring function on $\mathbb{C} \simeq \mathbb{R}^2$ given by

$$u := u_1 + iu_2 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto x := u^2 = \begin{pmatrix} u_1^2 - u_2^2 \\ 2u_1u_2 \end{pmatrix} \tag{5.20}$$

is conformal (except at the origin) and maps conic sections centered at the origin to conic sections with one focus at the origin. As Kepler's motion is given by conic sections, we are inspired to apply a transformation with similar properties to our system in order to turn the elliptic orbits into simple harmonic oscillation. (As a problem solving method this may seem *ad hoc*, but an analogous transformation was used by Feynman to solve a long-standing open problem in physics.)

Associated with eq. (5.20) is the the linear transformation

$$\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{pmatrix} = 2 \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} \begin{pmatrix} \mathrm{d}u_1 \\ \mathrm{d}u_2 \end{pmatrix}$$

of differentials. The matrix of this transformation has the following key properties:

- the entries are linear homogeneous functions of the u_i;
- the matrix is orthogonal, in the sense that:
  - the dot product of any two different rows vanishes;
  - each row has norm $u_1^2 + u_2^2 + \cdots + u_n^2$.

We would like to find such a transformation of differentials on $\mathbb{R}^n$. It turns out such transformations can only exist for $n = 1$, 2, 4, or 8. Ultimately we would like a transformation $\mathbb{R}^n \to \mathbb{R}^3$, and so we take $n = 4$ and choose such a matrix

$$A = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix}$$

which satisfies these properties. (This matrix also can be obtained from quaternion multiplication.) Consequently, we set

$$\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \\ 0 \end{pmatrix} = 2A \begin{pmatrix} \mathrm{d}u_1 \\ \mathrm{d}u_2 \\ \mathrm{d}u_3 \\ \mathrm{d}u_4 \end{pmatrix} = 2 \begin{pmatrix} u_1\,\mathrm{d}u_1 - u_2\,\mathrm{d}u_2 - u_3\,\mathrm{d}u_3 + u_4\,\mathrm{d}u_4 \\ u_2\,\mathrm{d}u_1 + u_1\,\mathrm{d}u_2 - u_4\,\mathrm{d}u_3 - u_3\,\mathrm{d}u_4 \\ u_3\,\mathrm{d}u_1 + u_4\,\mathrm{d}u_2 + u_1\,\mathrm{d}u_3 + u_2\,\mathrm{d}u_4 \\ u_4\,\mathrm{d}u_1 - u_3\,\mathrm{d}u_2 + u_2\,\mathrm{d}u_3 - u_1\,\mathrm{d}u_4 \end{pmatrix}. \tag{5.21}$$

Sadly, only the first three of these are complete differentials for three quantities:

$$\begin{aligned} x_1 &= u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ x_2 &= 2(u_1u_2 - u_3u_4) \\ x_3 &= 2(u_1u_3 + u_2u_4) \end{aligned} \tag{5.22}$$

and the fourth line of (5.21) yields the nonholonomic constraint:

$$u_4\,\mathrm{d}u_1 - u_3\,\mathrm{d}u_2 + u_2\,\mathrm{d}u_3 - u_1\,\mathrm{d}u_4 = 0. \tag{5.23}$$

Explicit formulas exist for both the one-dimensional kernel and the inverse of the transformation (5.22).

(a) Let $r = |\mathbf{x}|$ denote the distance to the origin in $\mathbb{R}^3$. Show that

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = r.$$

Using the orthogonality of A, invert the transformation (5.21) of differentials and conclude that for a fixed point $u \in \mathbb{R}^4$ our transformation conformally maps the space orthogonal to the kernel at u onto $\mathbb{R}^3$.

(b) Let $u, v \in \mathbb{R}^4$ be two orthonormal vectors which satisfy the condition

$$u_4v_1 - u_3v_2 + u_2v_3 - u_1v_4 = 0,$$

inspired by the constraint (5.23). From part (a) we know that the plane $\operatorname{span}\{u, v\}$ is mapped conformally onto a plane of $\mathbb{R}^3$. Show that when this transformation is restricted to $\operatorname{span}\{u, v\}$ onto its image, distances from the origin are squared and angles at the origin are doubled. (The calculation of the image x of a point in $\operatorname{span}\{u, v\}$ is rather lengthy, and to reach the conclusion it may help to compare the formula for x to the formula for Cayley–Klein parameters (see [Gol51, Sec. 4.5]).)

(c) In particular, it follows that a conic section centered at the origin in the plane $\operatorname{span}\{u, v\} \subset \mathbb{R}^4$ gets mapped to another conical section in $\mathbb{R}^3$ with one focus at the origin, the latter of which describes Kepler's motion. Using part (b) and polar coordinates in the plane $\operatorname{span}\{u, v\}$, show that ellipses and hyperbolas in the plane $\operatorname{span}\{u, v\}$ centered at the origin are mapped to ellipses and hyperbolas respectively, and from the limit case conclude that a line in $\operatorname{span}\{u, v\}$ is mapped to a parabola.

(d) We want to take the motion $x(t) \in \mathbb{R}^3$ subject to the force $P = (P_1, P_2, P_3)$ and transpose it into motion $u(t)$ in $\mathbb{R}^4$ subject to the force $Q = (Q_1, Q_2, Q_3, Q_4)$ and the constraint (5.23). Use the transformation of differentials (5.21) to determine the kinetic energy T and the force Q in terms of the coordinates u and forces P, and use the formula for the force Q to show that

$$u_4 Q_1 - u_3 Q_2 + u_2 Q_3 - u_1 Q_4 = 0. \tag{5.24}$$

(e) Apply Lagrange's equations (4.13) for nonconservative forces to obtain the equations of motion for $u(t)$. These equations have Q_i on the right-hand side, so add them together according to the identity (5.24), simplify, and integrate once to obtain

$$r(u_4 \dot{u}_1 - u_3 \dot{u}_2 + u_2 \dot{u}_3 - u_1 \dot{u}_4) = \text{constant}. \tag{5.25}$$

The parenthetical term above is exactly the constraint (5.23) divided by $\mathrm{d}t$, and so to ensure that the constraint is upheld we pick the initial conditions for u and $\dot{u}$ so that the parenthetical term vanishes initially, and hence for all time by the conserved quantity (5.25). (Conversely, it can be shown that the equations of motion for $u(t)$ and the condition $u_4 \dot{u}_1 - u_3 \dot{u}_2 + u_2 \dot{u}_3 - u_1 \dot{u}_4 = 0$ yield $m\ddot{x} = P$.)

(f) The equations of motion for u have a singularity at $r = 0$. To deal with this, substitute the **regularizing time**

$$s = \int_0^t \frac{\mathrm{d}t}{r}, \quad \text{so that} \quad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}s},$$

for the time variable t to regularize the equations of motion for u. Plug in the Kepler forces $Q_i = -\partial V(r)/\partial u_i$ and the (signed) semi-major axis

$$a_0 = \left(\frac{2}{r} - \frac{v^2}{M}\right)^{-1}$$

to arrive at

$$\frac{\partial^2 u_i}{\partial s^2} + \frac{M}{4a_0} u_i = 0.$$

That is, the preimage of bounded orbits ($a_0 > 0$) under this transformation is simple harmonic motion in $\mathbb{R}^4$ with frequency $\omega = \sqrt{M/4a_0}$. The harmonic motion can be (painstakingly) transformed into a solution $u(t)$ by computing and substituting the physical time $t = \int_0^s r(s)\,\mathrm{d}s$.

Chapter 6

Hamilton–Jacobi equation

Rather than n second-order ODEs, the Hamilton–Jacobi equation is one partial differential equation in $2d + 1$ variables. This perspective is sometimes less powerful when solving for the motion, but it provides a physical interpretation of the previously abstract action functional. The material for this chapter is based on [LL76, Ch. 7] and [Gol51, Ch. 9].

6.1 Hamilton–Jacobi equation

Let us return to the action functional. In section 4.1 we considered the variation of the action $S(q)$ at the systems motion $q(t)$ for paths with fixed endpoints. As in the set up for the proof of the Euler–Lagrange equations (Proposition 4.5) we can reduce to an open subset of Euclidean space, where the variation of the motion $q(t)$ between times t_1 and t_2 is given by $q(t) + \delta q(t)$. Previously we assumed $\delta q(t_1) = \delta q(t_2) = 0$ and then prescribed the resulting variation δS to vanish in accordance with the principle of least action. Now we will consider the action of the true progression of the system $S[q(t)]$ as a function of the coordinates at time t_2 by allowing $\delta q(t_2)$ (and consequently δS) to vary.

Repeating the calculation of the action variation from Proposition 4.5, we now have

$$\delta S(q; \delta q) = \left[\frac{\partial \mathscr{L}}{\partial \dot{q}} \cdot \delta q \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left(\frac{\partial \mathscr{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}} \right) \cdot \delta q \, \mathrm{d}t = \frac{\partial \mathscr{L}}{\partial \dot{q}}(t_2) \cdot q(t_2).$$

The integral vanishes due to Lagrange's equations of motion, but now only one of the boundary terms vanish since $\delta q(t_1) = 0$. If we let $t := t_2$ vary now, then we conclude

$$\frac{\partial S}{\partial q_i} = \frac{\partial \mathscr{L}}{\partial \dot{q}_i} = p_i \tag{6.1}$$

after recognizing $\partial \mathscr{L} / \partial \dot{q}_i$ as the momentum p_i.

By definition of the action we have

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mathscr{L}. \tag{6.2}$$

On the other hand we know $S = S[t, q(t)]$, and so the chain rule insists

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q} \cdot \dot{q} = \frac{\partial S}{\partial t} + p \cdot \dot{q} \tag{6.3}$$

after using the derivative (6.1). Setting (6.2) and (6.2) equal yields

$$\frac{\partial S}{\partial t} = \mathscr{L} - p \cdot \dot{q} = -H,$$

where H is the Hamiltonian (or total energy) (4.20). Using the derivative (6.1) we recognize this identity as a first order partial differential equation (PDE) for $S(t, q)$:

$$0 = \frac{\partial S}{\partial t} + H\left(t, q, \frac{\partial S}{\partial q}\right). \tag{6.4}$$

This is the **Hamilton–Jacobi equation**, and for a system with n degrees of freedom there are $n + 1$ independent variables $(t, q_1, \ldots, q_n)$. It is often the case in practice that the form of the action $S(t, q)$ is unknown, and cannot be determined from (6.4) alone.

The solution, or **complete integral**, of this equation has $n + 1$ integration constants corresponding to the number of independent variables. Calling these constants $\alpha_1, \ldots, \alpha_n$, and A (which play a physical role as we will see), we can write

$$S = f\left(t, q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n\right) + A;$$

we know one of these constants can be additive since the action appears in the PDE (6.4) only through its partial derivatives and hence is invariant under the addition of a constant. Mathematically speaking, a solution $S(t, q, \alpha)$ to the Hamilton–Jacobi equation (6.4) should be required to satisfy

$$\det \frac{\partial^2 S}{\partial q \, \partial \alpha} \neq 0$$

to be a complete integral, in order to avoid incomplete solutions.

The function $f(t, q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n)$ (called a **generating function**) induces a change of coordinates (a **canonical transformation**)—we will develop this idea more from the Hamiltonian perspective. Think of $\alpha_1, \ldots, \alpha_n$ as new momenta, and let $\beta_1, \ldots, \beta_n$ denote new coordinates to be chosen. Note that by the chain rule,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \cdot \dot{q} + \frac{\partial f}{\partial \alpha} \cdot \dot{\alpha}. \tag{6.5}$$

Set

$$\beta_i := -\frac{\partial f}{\partial \alpha_i}, \tag{6.6}$$

and note that $p_i = \partial f/\partial q_i$ by (6.1). Consider the new Hamiltonian

$$H'(\alpha, \beta, t) := H + \frac{\partial f}{\partial t} = H + \frac{\partial S}{\partial t} = 0,$$

which has corresponding Lagrangian

$$\begin{aligned}\mathscr{L}'(\beta, \alpha, t) := \alpha \cdot \dot{\beta} - H' = \alpha \cdot \dot{\beta} &= -\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \cdot \dot{q} \\ &= -\frac{\mathrm{d}f}{\mathrm{d}t} - H + p \cdot \dot{q} = \mathscr{L} - \frac{\mathrm{d}f}{\mathrm{d}t}\end{aligned}$$

by the chain rule (6.5). As this new Lagrangian $\mathscr{L}'$ differs from the old $\mathscr{L}$ by a complete time derivative, then they generate the same motion by Corollary 4.6.

For any Lagrangian $\mathscr{L}$, the definition of the Hamiltonian (4.20) requires

$$\begin{aligned}\frac{\partial H}{\partial p} &= \frac{\partial}{\partial p}\left(p \cdot \dot{q} - \mathscr{L}(q, \dot{q}, t)\right) = \dot{q}, \\ \frac{\partial H}{\partial q} &= \frac{\partial}{\partial q}\left(p \cdot \dot{q} - \mathscr{L}(q, \dot{q}, t)\right) = -\frac{\partial \mathscr{L}}{\partial q} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{q}} = -\dot{p}.\end{aligned}$$

This calculation may appear questionable at first (is the velocity $\dot{q}$ really independent of the position q and momentum p), but it is indeed correct; in fact, we will later give it thorough justification as it is the basis of Hamiltonian mechanics. Applying this to our new Hamiltonian $H'(\beta, \alpha, t)$, we observe that

$$\dot{\alpha} = 0, \quad \dot{\beta} = 0.$$

In other words, we have used the integration constants α to generate corresponding conserved quantities β. Recalling the definition (6.6) of the coordinates β, we conclude

$$\frac{\partial S}{\partial \alpha_i} = \text{constant} \tag{6.7}$$

in the n coordinates and time. Note that even if we can obtain a partial solution to the Hamilton–Jacobi equation involving only m constants α_i, we still get the corresponding m constants of motion.

Lastly, let us specialize to a conservative system. In this case, the Hamiltonian H is the total energy E and is constant and the action integral from times 0 to t becomes:

$$\begin{aligned}S[q(t)] = \int_{t_0}^{t} \mathscr{L}(q(t), \dot{q}(t), t)\,\mathrm{d}t &= \int_{t_0}^{t} \left(p \cdot \dot{q} - H\right)\,\mathrm{d}t \\ &= \int_{t_0}^{t} p \cdot \dot{q}\,\mathrm{d}t - \int_{t_0}^{t} E\,\mathrm{d}t = S_0\left(q_1, \ldots, q_n\right) - Et\end{aligned}$$

for some $S_0(q)$ which is only a function of the coordinates. As $\partial S_0/\partial t = 0$, the Hamilton–Jacobi equation then takes the special form

$$H\left(q, \frac{\partial S}{\partial q}, t\right) = E. \tag{6.8}$$

In particular, we are no longer required to know the formula for the action.

6.2 Separation of variables

Sometimes we can reduce the Hamilton–Jacobi equation by one coordinate using separation of variables. Suppose that for some system with n degrees of freedom, we have a coordinate q_1 with corresponding derivative $\partial S/\partial q_1$ that appear in the Hamilton–Jacobi equation only in some particular combination $\phi(q_1, \partial S/\partial q_1)$. That is, the Hamilton–Jacobi equation (6.4) takes the form

$$\Phi\left(t, q_2, \ldots, q_n, \frac{\partial S}{\partial q_2}, \ldots, \frac{\partial S}{\partial q_n}, \phi\left(q_1, \frac{\partial S}{\partial q_1}\right)\right) = 0 \tag{6.9}$$

after rearranging the independent variables. We take the (additive) separation of variables ansatz

$$S = S'(q_2, \ldots, q_n, t) + S_1(q_1)$$

for the solution. Plugging this back into our Hamilton–Jacobi equation (6.9) we get

$$\Phi\left(t, q_2, \ldots, q_n, \frac{\partial S'}{\partial q_2}, \ldots, \frac{\partial S'}{\partial q_n}, \phi\left(q_1, \frac{dS_1}{dq_1}\right)\right) = 0. \tag{6.10}$$

Note that q_1 only influences ϕ and is entirely independent from the rest of the expression. As the variables are independent, we conclude that ϕ must be constant:

$$\phi\left(q_1, \frac{dS_1}{dq_1}\right) = \alpha_1, \quad \Phi\left(t, q_2, \ldots, q_n, \frac{\partial S'}{\partial q_2}, \ldots, \frac{\partial S'}{\partial q_n}, \alpha_1\right) = 0. \tag{6.11}$$

We now have a first order ODE and the Hamilton–Jacobi equation in terms of the remaining $n-1$ coordinates. As the resulting form of Φ is coherent with what we found in the previous section, then we conclude that our ansatz is correct. The ability to remove a coordinate in the Hamilton–Jacobi consideration even when it is not cyclic is a virtue of this approach.

If we do have some cyclic coordinate q_1 for the system (i.e. S or $\mathscr{L}$ is independent of q_1), then the Hamilton–Jacobi equation (6.4) becomes

$$\frac{\partial S}{\partial t} + H\left(q_2, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, t\right) = 0. \tag{6.12}$$

Then q_1 is of the type (6.9) for the function

$$\phi\left(q_1, \frac{\partial S}{\partial q_1}\right) = \frac{\partial S}{\partial q_1}. \tag{6.13}$$

Our result (6.11) yields

$$\frac{\partial S}{\partial q_1} = \alpha_1, \quad S = S'(q_2, \ldots, q_n, t) + \alpha_1 q_1. \tag{6.14}$$

Here, $\alpha_1 = \partial S/\partial q_1$ is just the conserved momentum corresponding to q_1.

If this cyclic variable is time, then the system is conservative and we saw in (6.8) that the action is given by

$$S = W\left(q_1, \ldots, q_n\right) - Et. \tag{6.15}$$

Here, $-E$ is the conserved quantity associated with t, although it is not necessarily always the total energy. We have used W instead of S' to denote this special case, but they share the same purpose; W is known as the **Hamilton's characteristic function**. The Hamilton–Jacobi equation is now

$$H\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = -\frac{\partial S}{\partial t} = E, \tag{6.16}$$

and E is just one integration constant α_j of the motion of which S' is independent. The corresponding conserved quantity (6.7) will give the coordinates implicitly as functions of the constants α_i, β_i, and time:

$$\beta_i = \frac{\partial S}{\partial \alpha_i} = \begin{cases} \dfrac{\partial W}{\partial \alpha_i} - t & \text{for } i = j, \\ \dfrac{\partial W}{\partial \alpha_i} & \text{for } i \neq j. \end{cases} \tag{6.17}$$

Only the jth of these equations is time-dependent, and so one of the q_i can be chosen as an independent variable and the rest will be able to be written in terms of this coordinate. Such solutions for the motion are called **orbit equations**. For the central forces of section 3.1, we were able to solve for the angle ϕ as a function of the radius r.

6.3 Conditionally periodic motion

For this section, we will examine a system of n degrees of freedom with bounded motion, such that every variable can be separated using the method of the previous section. This means the action takes the form

$$S = \sum_{i=1}^{d} S_i(q_i) + S'(t), \tag{6.18}$$

where each coordinate S_i is related to the corresponding momentum via

$$p_i = \frac{\partial S_i}{\partial q_i}, \qquad S_i = \int p_i \, \mathrm{d}q_i. \tag{6.19}$$

The motion is bounded, and so this integral represents the area enclosed by a loop in the phase plane (q_i, p_i) (just as with the period integral (2.8)). Every time q_i returns to a value, S_i has incremented by an amount $2\pi I_i$ with

$$I_i = \frac{1}{2\pi} \oint p_i \, \mathrm{d}q_i. \tag{6.20}$$

These I_i are called the **action variables**.

As in section 6.1, the generating function S_i induces a canonical transformation of coordinates with the action variables I_i as the new momenta. We will see this process more systematically from the Hamiltonian perspective. The new position coordinates, called **angle variables**, are given by

$$w_i = \frac{\partial S}{\partial I_i} = \sum_{k=1}^{n} \frac{\partial S_k(q_k, I_1, \ldots, I_n)}{\partial I_i} \tag{6.21}$$

since this is the time integral of "$\partial H/\partial p = \dot{q}$". The generating functions S_i are time-independent, and so the new Hamiltonian $H = E$ is just the old in terms of the new coordinates. The (Hamiltonian) equations of motion require

$$\dot{I}_i = -\frac{\partial H(I_1, \ldots, I_n)}{\partial w_i} = 0, \quad \dot{w}_i = \frac{\partial H(I_1, \ldots, I_n)}{\partial I_i} = \frac{\partial E(I_1, \ldots, I_s)}{\partial I_i}, \tag{6.22}$$

which can be immediately integrated to yield

$$I_i = \text{constant}, \quad w_i = \frac{\partial E}{\partial I_i} t + \text{constant}. \tag{6.23}$$

As we have already observed, S_i increments by $2\pi I_i$ each time q_i returns to its original value, and so the angle variables w_i also increment by 2π. Consequently the derivative $\partial E/\partial I_i$ is the **frequency** of motion in the ith coordinate, which **we were able to identify without solving the entire system**.

As the motion in these variables is periodic, any single-valued function $F(q, p)$ of the system coordinates and momenta will be periodic in the angle variables with period 2π after being transformed to the canonical variables. Fourier expanding in each of the angle variables, we have

$$F = \sum_{\ell \in \mathbb{Z}^n} A_\ell e^{i\ell \cdot w} = \sum_{\ell_1 \in \mathbb{Z}} \cdots \sum_{\ell_n \in \mathbb{Z}} A_{\ell_1 \ldots \ell_n} e^{i(\ell_1 w_1 + \cdots + \ell_n w_n)}. \tag{6.24}$$

Using (6.23), we may write the angle variables as functions of time. Absorbing the integration constants of the w_i into the coefficients A_ℓ, we get

$$F = \sum_{\ell \in \mathbb{Z}^n} A_\ell \exp\left(it\ell \cdot \frac{\partial E}{\partial I} \right). \tag{6.25}$$

Each term of this sum is periodic with frequency $\ell \cdot (\partial E/\partial I)$. If the frequencies $\partial E/\partial I_i$ are not commensurable however, then the total quantity F is not periodic. In particular, the coordinates q, p are may not be periodic, and the system may not return to any given state that it instantaneously occupies. However, if we wait long enough the system will come arbitrarily close to any given occupied state—this phenomenon is referred to as **Poincaré's recurrence theorem**. Such motion is called **conditionally periodic**.

Two frequencies $\partial E/\partial I_i$ that are commensurable (i.e. their ratio is a rational number) are called a **degeneracy** of the system, and if all n are commensurable

the system is said to be **completely degenerate**. In the latter case, all motion is periodic, and so we must have a full set of $2n-1$ conserved quantities. Only n of these will be independent, and so they can be defined to be the action variables $I_1, \ldots, I_n$. The remaining $n-1$ constants may be chosen to be $w_i \partial E/\partial I_k - w_k \partial E/\partial I_i$ for distinct i, k, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(w_i \frac{\partial E}{\partial I_k} - w_k \frac{\partial E}{\partial I_i}\right) = \dot{w}_i \frac{\partial E}{\partial I_k} - \dot{w}_k \frac{\partial E}{\partial I_i} = \frac{\partial E}{\partial I_i}\frac{\partial E}{\partial I_k} - \frac{\partial E}{\partial I_k}\frac{\partial E}{\partial I_i} = 0.$$

Note, however, that since the angle variables are not single-valued, neither will be the $n-1$ constants of motion.

Consider a partial degeneracy, say, of frequencies 1 and 2. This means

$$k_1 \frac{\partial E}{\partial I_1} = k_2 \frac{\partial E}{\partial I_2} \tag{6.26}$$

for some $k_1, k_2 \in \mathbb{Z}$. The quantity $w_1 k_2 - w_2 k_1$ will then be conserved, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(w_1 k_1 - w_2 k_2\right) = \dot{w}_1 k_1 - \dot{w}_2 k_2 = \frac{\partial E}{\partial I_1} k_1 - \frac{\partial E}{\partial I_2} k_2 = 0. \tag{6.27}$$

Note that this quantity is single-valued modulus 2π, and so a trigonometric function of it will be an actual conserved quantity.

In general, for a system with n degrees of freedom whose action is totally separable and has n single-valued integrals of motion, the system state moves densely in a n-dimensional manifold in $2d$-dimensional phase space. For degenerate systems we have more than n integrals of motion, and consequently the system state is confined to a manifold of dimension less than n. When a system has less than n degeneracies, then there has fewer than n integrals of motion and the system state travels within a manifold of dimension greater than n.

6.4 Geometric optics analogy

In this section we will see that the level sets of the action propagate through configuration space mathematically similar to how light travels through a medium. This is not meant literally, but rather to bring a physical analogy to the formerly abstract notion of the action.

Suppose we have a system for which the Hamiltonian is conserved and equal to the total energy. For simplicity, suppose the system describes just one particle moving in three-dimensional Euclidean space using Cartesian coordinates $q = \mathbf{x} = (x, y, z) \in \mathbb{R}^3$; much of the results we will show however hold without these assumptions. Equation (6.15) tells us that the action is given by

$$S(q, t) = W(q) - Et. \tag{6.28}$$

As we have chosen Cartesian coordinates, rather than discussing the particle's motion we may discuss the motion of the action level surfaces $S(q, t) = b$ in

time within the same space. If we were to generalize this argument to multiple-particle systems, then instead of the particle's motion in Cartesian space we must consider the path that the system traces out in configuration space—the space spanned by the coordinates q. At time $t = 0$, we have an equation for Hamilton's characteristic function $W = b$, and after a time step Δt we then have $W + \Delta W = b + E\Delta t + \mathcal{O}(\Delta t^2)$.

The propagation of this surface can be thought of as a wavefront. The change in the characteristic function W during the time interval $\mathrm{d}t$ is $\mathrm{d}W = E\,\mathrm{d}t$. If we call the distance traveled normal to the wavefront $\mathrm{d}s$, then we also have

$$\frac{\partial W}{\partial s} = |\nabla W|. \tag{6.29}$$

The velocity u of the wavefront is then

$$u = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{dW/|\nabla W|}{dW/E} = \frac{E}{|\nabla W|}. \tag{6.30}$$

As a conservative system, the Hamilton–Jacobi equation takes the form (6.16), which in Cartesian coordinates looks like

$$E = H\left(q, p = \frac{\partial W}{\partial q}\right) = T\left(p = \frac{\partial W}{\partial q}\right) + V = \frac{(\nabla W)^2}{2m} + V,$$

or after rearranging,

$$(\nabla W)^2 = 2m(E - V). \tag{6.31}$$

Plugging this into the velocity (6.29), we have

$$u = \frac{E}{\sqrt{2m(E-V)}} = \frac{E}{\sqrt{2mT}} = \frac{E}{p}. \tag{6.32}$$

The faster the particle moves, the slower the action level sets propagate.

The momentum is given by

$$p = \frac{\partial W}{\partial q} = \nabla W. \tag{6.33}$$

The gradient is of course normal to the level sets, and so this relation tells us that the particle always moves normal to the level sets of the characteristic function W.

We will now see how the level sets of the action propagate like waves. For some scalar-valued function ϕ, the wave equation of optics is

$$\nabla^2\phi - \frac{n^2}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{6.34}$$

If the refractive index n is constant, then there is a family of **plane wave** solutions:

$$\phi(r,t) = \phi_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{6.35}$$

where the vector $\mathbf{k} \in \mathbb{R}^3$ is the propagation direction and the magnitude of $k = 2\pi/\lambda = n\omega/c$ is the **wave number**.

In geometric optics, the refractive index $n \equiv n(\mathbf{x})$ is not assumed to be constant, but rather to be changing slowly on the scale of the wavelength. Consequentially, we now seek solutions to the wave equation (6.34) with the plane wave ansatz (6.35):

$$\phi(r,t) = e^{A(\mathbf{x})+ik_0(L(\mathbf{x})-ct)}. \tag{6.36}$$

where $A(\mathbf{x})$ is related to the amplitude of the wave, $k_0 = \omega/c$ is the wave number in vacuum ($n \equiv 1$), and $L(\mathbf{x})$ is called the **optical path length** of the wave. Plugging in (6.36), the wave equation (6.34) becomes

$$\phi\left\{\left[\nabla^2 A + (\nabla A)^2 - k_0^2(\nabla L)^2 + k_0^2 n^2\right] + ik_0\left[\nabla^2 L + 2\nabla A\cdot\nabla L\right]\right\} = 0. \tag{6.37}$$

In general, ϕ is nonzero and so the curly-bracketed expression must vanish. We want A and L to be real-valued by construction, and so we the real and imaginary parts in square brackets must also vanish:

$$\nabla^2 A + (\nabla A)^2 + k_0^2\left[n^2 - (\nabla L)^2\right] = 0, \quad \nabla^2 L + 2\nabla A\cdot\nabla L = 0. \tag{6.38}$$

Now comes the geometric optics approximation: the wavelength $\lambda = 2\pi/k$ is small compared to the rate of change of the medium. In particular, the wave number in vacuum $k_0 = 2\pi/\lambda_0$ must be considerably large compared to the derivative terms in the first equation of (6.38), and thus we require

$$(\nabla L)^2 = n^2. \tag{6.39}$$

This is called the **eikonal equation**.

Returning to (6.31), we see that the characteristic function W satisfies an eikonal equation for a "medium" of refractive index $\sqrt{2m(E-V)} = p$ (this equality holds for the single-particle case). This illustrates that the characteristic function is like a wavefront that propagates through the medium of configuration space with refractive index p, in the geometric optics limit.

6.5 Exercises

6.1 (Harmonic oscillator)**.** The harmonic oscillator Hamiltonian is given by

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2.$$

(a) Write down the Hamilton–Jacobi equation (6.4) for this system. As this system is conservative, we expect a solution (up to an arbitrary additive constant) of the form

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t$$

where the constant α is the total energy. Plug this ansatz into the Hamilton–Jacobi equation and conclude that

$$W = \pm m\omega \int \sqrt{\frac{2\alpha}{m\omega^2} - q^2}\, \mathrm{d}q.$$

This integral can be evaluated further, but it is not necessary for our purposes.

(b) The quantity β will implicitly give us the equation of motion $q(\alpha, \beta, t)$. Using the definition (6.6), show that

$$t + \beta = \frac{1}{\omega} \begin{Bmatrix} \sin^{-1}\left(\sqrt{\frac{m}{2\alpha}}\omega q\right) & \text{(plus sign)} \\ \cos^{-1}\left(\sqrt{\frac{m}{2\alpha}}\omega q\right) & \text{(minus sign)} \end{Bmatrix} + \text{constant}.$$

The integration constant may be absorbed into β which has yet to be determined, and since $\sin^{-1}(x) = \pi/2 - \cos^{-1}(x)$ we can take the second case and absorb the $\pi/2$ into β also. We should pick up a factor of -1 on one side of the equation due to this last step, but this will not matter since we have chosen cosine which is even. Altogether, we arrive at the familiar solution

$$q(t) = \sqrt{\frac{2\alpha}{m\omega^2}} \cos\left(\omega(t + \beta)\right).$$

(c) To find the constants we must apply the initial conditions $q(0) = q_0$, $p(0) = p_0$. Determine α and β using $p_0 = (\partial S/\partial q)_{t=0}$ and $q(0) = q_0$ respectively, and obtain the solution as a function of the initial values:

$$q(t) = \sqrt{q_0^2 + \frac{p_0^2}{m^2\omega^2}} \cos\left[\omega t + \cos^{-1}\left(\frac{q_0}{\sqrt{q_0^2 + \frac{p_0^2}{m^2\omega^2}}}\right)\right].$$

6.2 (Central field)**.** Consider the motion of a particle in a central field as in section 3.1. In polar coordinates, we have

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2), \quad V = V(r),$$
$$p_r = \frac{\partial \mathscr{L}}{\partial \dot{r}} = m\dot{r}, \quad p_\phi = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} = mr^2\dot{\phi},$$
$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + V(r).$$

(a) This Hamiltonian is both time-independent and cyclic in ϕ, and so we expect an action of the form

$$S = W_1(r) + \alpha_\phi \phi - Et.$$

Plug this into the Hamilton–Jacobi equation (6.16) for conservative systems and integrate to arrive at

$$W = W_1(r) + \alpha_\phi \phi = \pm \int \sqrt{2m(E - V(r)) - \frac{\alpha_\phi^2}{r^2}}\, \mathrm{d}r + \alpha_\phi \phi.$$

(b) Use (6.17) to obtain the implicit equations of motion

$$\beta_1 = \pm \int \frac{\frac{1}{2} 2m\, \mathrm{d}r}{\sqrt{2m(E - V(r)) - \frac{\alpha_\phi^2}{r^2}}} - t,$$
$$\beta_2 = \pm \int \frac{-\frac{1}{2}\frac{2\alpha_\phi}{r^2}\, \mathrm{d}r}{\sqrt{2m(E - V(r)) - \frac{\alpha_\phi^2}{r^2}}} + \phi.$$

These match what we found in equations (3.5) and (3.6), where the constant $\alpha_\phi = M$ is the angular momentum associated to the cyclic coordinate ϕ.

6.3 (Kepler's problem)**.** We will find the frequency of oscillations for Kepler's problem using action variables without solving the equations of motion. Consider a particle of mass m in an inverse-square central force field, as in section 3.3:

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2), \quad V = -\frac{k}{r},$$
$$p_r = \frac{\partial \mathscr{L}}{\partial \dot{r}} = m\dot{r}\ , \quad p_\theta = \frac{\partial \mathscr{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad p_\phi = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} = mr^2 \sin^2\theta \dot{\phi},$$
$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - \frac{k}{r}.$$

The constant k is positive, since we must have an attractive force for bounded motion.

(a) Write down the Hamilton–Jacobi equation (6.16) for conservative systems. Notice that all of the coordinates are separable, and so the characteristic function is of the form

$$W = W_r(r) + W_\theta(\theta) + W_\phi(\phi).$$

(b) The Hamiltonian is cyclic in ϕ, and so the derivative $\partial W/\partial\phi = \partial W_\phi/\partial\phi = \alpha_\phi$ is the constant that is angular momentum about the z-axis. Plug this in, group the terms involving only θ, and conclude that

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2, \quad \frac{1}{2m}\left[\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2}\right] - \frac{k}{r} = E.$$

The equations for W_ϕ, W_θ, and W_r demonstrate the conservations of angular momentum about the z-axis p_θ, total angular momentum p, and total energy E, and from here they could be integrated to obtain the equations of motion.

(c) Use the three differential equations of part (b) to obtain the action variables:

$$\begin{aligned} I_\phi &= \alpha_\phi = p_\phi, \\ I_\theta &= \frac{1}{2\pi} \oint \sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2\theta}}\, \mathrm{d}\theta, \\ I_r &= \frac{1}{2\pi} \oint \sqrt{2mE + \frac{2mk}{r} - \frac{\alpha_\theta^2}{r^2}}\, \mathrm{d}r. \end{aligned}$$

(d) Let us look at the second action variable I_θ. We know from section 3.1 that this motion is coplanar, so let ψ denote the angle in the plane of orbit. Set the momentum in the (r, θ, ϕ) variables and (r, ψ) variables equal, and conclude that $p_\theta\, \mathrm{d}\theta = p\, \mathrm{d}\psi - p_\phi\, \mathrm{d}\phi$. Conclude that

$$I_\theta = p - p_\phi = \alpha_\theta - \alpha_\phi.$$

(e) Now for the third action variable I_r. The integral for I_r is evaluated between two turning points r_1, r_2 for which the integrand $p_r = m\dot{r}$ must vanish. We can therefore integrate from r_2 to r_1 and back to r_2, for which the integrand is first negative then positive, corresponding to the sign of the momentum $p_r = m\dot{r}$. In the complex plane, this integrand is analytic everywhere but $r = 0$ and along the segment on the real axis connecting r_1 and r_2. Integrate around a counterclockwise simple path γ encompassing r_1 and r_2 to obtain

$$I_r = -\alpha_\theta^2 + \frac{mk}{\sqrt{-2mE}} = -(I_\theta + I_\phi) + \frac{mk}{\sqrt{-2mE}}.$$

The energy

$$E(I) = -\frac{mk^2}{2(I_r + I_\theta + I_\phi)^2}$$

is symmetric in the three action variables, and so the frequency of oscillations in each coordinate r, θ, ϕ is the same:

$$\frac{\partial E}{\partial I_r} = \frac{\partial E}{\partial I_\theta} = \frac{\partial E}{\partial I_\phi} = \frac{mk^2}{(I_r + I_\theta + I_\phi)^3},$$

as is expected since the force is central.

Part III

Hamiltonian Mechanics

Hamiltonian mechanics treats the position and momentum variables as independent coordinates on phase space. This yields Hamilton's equations of motion—a system with two first-order differential equations for each degree of freedom—which portrays the system's motion as the flow of a particular vector field on phase space. While we are still free to choose coordinates, the position and momentum variables are of course not truly independent quantities, and consequently permissible changes of variables must preserve the class of such vector fields. This underlying structure of these vector fields induces a geometry on phase space, which is the foundation of symplectic and contact structures in differential geometry.

Chapter 7

Hamilton's equations

We will develop the equations of motion and key tools of the Hamiltonian perspective, with a primary focus on Euclidean space before exploring general geometries. The material for this chapter is based on [Arn89, Ch. 3], [Gol51, Ch. 7–8], and [LL76, Ch. 7].

7.1 Hamilton's equations

Recall that for a system with coordinates $q = (q_1, q_2, \ldots, q_n)$ on a n-dimensional manifold M the Hamiltonian

$$H(q, p, t) := \dot{q} \cdot p - \mathscr{L}(q, \dot{q}, t) = \sum_{i=1}^{n} \dot{q}_i \frac{\partial \mathscr{L}}{\partial \dot{q}_i} - \mathscr{L} \tag{7.1}$$

is often the total energy of the system, and the components of p are the momenta $p_i := \partial \mathscr{L} / \partial \dot{q}_i$. In general, the Hamiltonian is a smooth function function $H(q, p, t) : T^*M \times I \to \mathbb{R}$ where $I \subset \mathbb{R}$ is an interval, since the momentum is a covector and not a vector. In Proposition 4.11 we saw that the Hamiltonian determines the system's motion, and that the quantity H is conserved whenever the Lagrangian $\mathscr{L}$ is time-independent.

When we defined the Hamiltonian H in section 4.5, we showed that it is equal to the total energy $T + V$ of the system in Cartesian coordinates. We will now show that the Hamiltonian $H(q, p, t)$ is equal to the total energy $T + V$ for any system with velocity-independent potential energy $V \equiv V(q, t)$. As in (4.8) the kinetic energy is a quadratic form in the velocity variables:

$$T = \tfrac{1}{2} \sum_{i,j} a_{ij}(q, t) \dot{q}_i \dot{q}_j. \tag{7.2}$$

So T is a homogeneous function of order 2 in $\dot{q}$: for any constant k we have

$$T(q, k\dot{q}, t) = \tfrac{1}{2} \sum_{i,j} a_{ij}(q, t) \frac{d}{dt}(kq_i) \frac{d}{dt}(kq_j) = k^2 T(q, \dot{q}, t). \tag{7.3}$$

This determines the derivative of T, since

$$2kT = \frac{d}{dk}(k^2 T) = \frac{d}{dk} T(q, k\dot{q}, t) = \left(\frac{\partial}{\partial(k\dot{q})} T(q, k\dot{q}, t)\right) \cdot \left(\frac{\partial}{\partial k} k\dot{q}\right) = \frac{\partial T(k\dot{q})}{\partial(k\dot{q})} \cdot \dot{q},$$

and so taking $k = 1$, we conclude

$$\frac{\partial T}{\partial \dot{q}} \cdot \dot{q} = 2T.$$

(This result is known as **Euler's homogeneous function theorem**.) As the potential energy is velocity independent, then we have

$$H = p \cdot \dot{q} - \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \dot{q}} \cdot \dot{q} - \mathscr{L} = \frac{\partial T}{\partial \dot{q}} \cdot \dot{q} - (T - V) = 2T - T + V = T + V$$

as desired.

Treating the position q, velocity $\dot{q}$, and momentum p as independent variables, the differential of the Hamiltonian must satisfy

$$\begin{aligned} \frac{\partial H}{\partial p}\,\mathrm{d}p + \frac{\partial H}{\partial q}\,\mathrm{d}q + \frac{\partial H}{\partial t}\,\mathrm{d}t = \mathrm{d}H &= \mathrm{d}(p \cdot \dot{q} - \mathscr{L}) \\ &= \dot{q}\,\mathrm{d}p + \left(p - \frac{\partial \mathscr{L}}{\partial \dot{q}}\right)\mathrm{d}\dot{q} - \frac{\partial \mathscr{L}}{\partial q}\,\mathrm{d}q - \frac{\partial \mathscr{L}}{\partial t}\,\mathrm{d}t \\ &= \dot{q}\,\mathrm{d}p + 0\,\mathrm{d}\dot{q} - \dot{p}\,\mathrm{d}q - \frac{\partial \mathscr{L}}{\partial t}\,\mathrm{d}t \end{aligned}$$

where in the last equality we used Lagrange's equations. Of course the momentum and velocity are not actually independent, but the coefficient of $\mathrm{d}\dot{q}$ will appear with a factor in the term $\mathrm{d}p$ and the same cancellation occurs. Matching terms we conclude

$$\frac{\partial H}{\partial p} = \dot{q}, \qquad \frac{\partial H}{\partial q} = -\dot{p}. \tag{7.4}$$

These $2n$ first-order differential equations are called **Hamilton's equations**. Note that equating the terms containing $\mathrm{d}t$ provides no information since this equality follows immediately from the definition of the Hamiltonian.

In Lagrangian mechanics we saw that a cyclic coordinate q_k resulted in the conservation of $p_k = \partial \mathscr{L}/\partial \dot{q}_k$, and thus p_k entered the remaining $n-1$ equations as a constant parameter. For Hamilton's equations, a cyclic variable q_k yields

$$\dot{p}_k = \frac{\partial H}{\partial q_k} = 0, \qquad \dot{q}_k = \frac{\partial H}{\partial p_k}.$$

The first of these equations expresses the conservation of p_k, and the second is independent of q_k and is thus easily integrable. Consequently, we are left with $2n - 2$ equations in terms of the constant parameter p_k.

To conclude this section, we will observe Hamilton's equations as a consequence of the principle of least action. Although we have defined momentum

and thus Hamilton's equations themselves in terms of the Lagrangian and the generalized coordinates q_i, we can also consider the momenta p_i as variables since we can still extract the Euler–Lagrange equations as long as we have at least n independent coordinates (cf. the remark at the end of section 4.1). For the Lagrangian of a system with n degrees of freedom, we must specify $2n$ initial conditions—n for the coordinates q_i and another n for the $\dot{q}_i$—and so deriving Hamilton's equations without the Lagrangian equations of motion allow us to regard the q_i and p_i both as independent coordinates linked only by the equations of motion. To demonstrate this, suppose we have a system whose motion is described by the coordinates $(q(t), p(t))$ from time t_0 to t_1. Within a coordinate patch on Euclidean space, we may write any other time progression of the system as $\mathbf{q}(t) + \delta\mathbf{q}(t)$ with $\delta\mathbf{q}(t_0) = \delta\mathbf{q}(t_1) = 0$. The principle of least action insists

$$\begin{aligned} 0 = \delta S &= \delta \int_{t_0}^{t_1} \mathscr{L} \,\mathrm{d}t = \delta \int_{t_0}^{t_1} \left(\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)\right) \mathrm{d}t \\ &= \int_{t_0}^{t_1} \sum_i \left(\dot{\mathbf{q}}\cdot\mathbf{p} + \mathbf{p}\cdot\delta\dot{\mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}}\cdot\delta\mathbf{q} - \frac{\partial H}{\partial \mathbf{p}}\cdot\delta\mathbf{p}\right) \mathrm{d}t. \end{aligned}$$

We may integrate the second term by parts:

$$\int_{t_0}^{t_1} p_i \delta\dot{q}_i dt = \int_{t_0}^{t_1} p_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\delta q_i\right) \mathrm{d}t = p_i \delta q_i \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{p}_i \delta q_i \,\mathrm{d}t \tag{7.5}$$

The boundary term vanishes by the fixed-endpoint condition $\delta q_i(t_0) = \delta q_i(t_1) = 0$. Plugging this back in to the action variation yields

$$0 = \int_{t_0}^{t_1} \left[\left(\dot{\mathbf{q}} - \frac{\partial H}{\partial \mathbf{p}}\right)\cdot\delta\mathbf{p} - \left(\dot{\mathbf{p}} + \frac{\partial H}{\partial \mathbf{q}}\right)\cdot\delta\mathbf{q}\right] \mathrm{d}t$$

As this integral vanishes for any choice of $\delta p_i(t)$ and $\delta q_i(t)$, and we are always able to pick paths such that these parameters are varied independently, then we conclude that the two parenthetical terms must vanish:

$$\dot{q}_i - \frac{\partial H}{\partial p_i} = 0, \qquad \dot{p}_i + \frac{\partial H}{\partial q_i} = 0.$$

This agrees with Hamilton's equations (7.4).

7.2 Legendre transformation

The Legendre transformation is an involution on the space of real-valued strictly convex functions, which naturally joins the Lagrangian and Hamiltonian perspectives. As is the case in physical application, the Lagrangian and Hamiltonian are quadratic in $\dot{q}$ due to the kinetic energy, and are Legendre transforms of each other in the variable $\dot{q}$.

Throughout this section, $f : U \to \mathbb{R}$ will denote a smooth function on an open convex set $U \subset \mathbb{R}^d$ that is (strictly) **convex** in the sense that the Hessian matrix is positive definite, which is written as $f''(\mathbf{x}) > 0$, and is defined by

$$f''(\mathbf{x})\mathbf{y} \cdot \mathbf{y} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) \right) \mathbf{y} \cdot \mathbf{y} > 0 \qquad \text{for all } 0 \neq \mathbf{y} \in \mathbb{R}^d$$

at every point $\mathbf{x} \in U$. We will now define the **Legendre transform** $f^*(\boldsymbol{\xi})$ of such a function $f(\mathbf{x})$. Consider the distance $F(\mathbf{x}, \boldsymbol{\xi}) = \mathbf{x} \cdot \boldsymbol{\xi} - f(\mathbf{x})$ between the hyperplane $\mathbf{x} \cdot \boldsymbol{\xi}$ of "slope" $\boldsymbol{\xi}$ and the graph of the function $f(\mathbf{x})$. As f is convex, then for fixed $\boldsymbol{\xi}$ the function F will be a concave function of $\mathbf{x}$, and so there is a unique point $\mathbf{x}^*(\xi)$ which maximizes $F(\mathbf{x}, \boldsymbol{\xi})$. We define the Legendre transform of f by $f^*(\xi) = F(\mathbf{x}^*(\boldsymbol{\xi}), \boldsymbol{\xi})$, which mathematically is the function

$$f^*(\boldsymbol{\xi}) = \sup_{\mathbf{x} \in U} (\mathbf{x} \cdot \boldsymbol{\xi} - f(\mathbf{x})) \tag{7.6}$$

on the domain U^* where this supremum is finite:

$$U^* = \sup \left\{ \boldsymbol{\xi} \in \mathbb{R}^d : \sup_{\mathbf{x} \in U} (\mathbf{x} \cdot \boldsymbol{\xi} - f(\mathbf{x})) < \infty \right\}. \tag{7.7}$$

Example 7.1. Consider the quadratic function

$$f(\mathbf{x}) = \tfrac{1}{2} A\mathbf{x} \cdot \mathbf{x} + \mathbf{b} \cdot \mathbf{x} + c$$

on $U = \mathbb{R}^d$, where A is a real symmetric positive definite matrix, $\mathbf{b} \in \mathbb{R}^d$, and $c \in \mathbb{R}$. In order to maximize the distance $F(\mathbf{x}, \boldsymbol{\xi})$, we differentiate:

$$0 = \frac{\partial F}{\partial \mathbf{x}}(\mathbf{x}^*) = \frac{\partial}{\partial \mathbf{x}} \left[\mathbf{x} \cdot \boldsymbol{\xi} - \tfrac{1}{2} A\mathbf{x} \cdot \mathbf{x} - \mathbf{b} \cdot \mathbf{x} - c \right]_{\mathbf{x} = \mathbf{x}^*} = \boldsymbol{\xi} - A\mathbf{x}^* - \mathbf{b}.$$

This equation has one critical point $\mathbf{x}^* = A^{-1}(\boldsymbol{\xi} - \mathbf{b})$, which must be our maximum. Plugging this back in, we get

$$\begin{aligned} f^*(\boldsymbol{\xi}) &= F(\mathbf{x}^*(\boldsymbol{\xi}), \boldsymbol{\xi}) \\ &= A^{-1}(\boldsymbol{\xi} - \mathbf{b}) \cdot \boldsymbol{\xi} - \tfrac{1}{2}(\boldsymbol{\xi} - \mathbf{b}) \cdot A^{-1}(\boldsymbol{\xi} - \mathbf{b}) - \mathbf{b} \cdot A^{-1}(\boldsymbol{\xi} - \mathbf{b}) - c \\ &= \tfrac{1}{2} A^{-1}(\boldsymbol{\xi} - \mathbf{b}) \cdot (\boldsymbol{\xi} - \mathbf{b}) - c \end{aligned}$$

on $U^* = \mathbb{R}^d$. In particular, we can take $A = mI$, $\mathbf{b} = 0$, and $c = 0$ so that $f(\mathbf{x}) = \frac{1}{2} m|\mathbf{x}|^2$ is the kinetic energy, and we get that $f^*(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2/2m$ is also the kinetic energy if we recognize $\boldsymbol{\xi}$ as the momentum.

Next, we will demonstrate the transformation's key properties.

Theorem 7.2 (Involutive property)**.** *The Legendre transform $f^* : U^* \to \mathbb{R}$ of $f : U \to \mathbb{R}$ is a convex function, whose Legendre transform $(f^*)^*$ is f. In other words, the Legendre transform is an involution.*

Proof. First we check that the transform f^* is also convex. As the unique maximizer of the distance $F(\mathbf{x}, \boldsymbol{\xi})$, we know that $\mathbf{x}^* = \mathbf{x}^*(\boldsymbol{\xi})$ is the unique solution to

$$0 = \frac{\partial F}{\partial \mathbf{x}}(\mathbf{x}^*) = \xi - Df(\mathbf{x}^*) \tag{7.8}$$

where $Df(\mathbf{x}^*) = \nabla f(\mathbf{x}^*)$ is the gradient. The function Df between open sets of $\mathbb{R}^d$ is invertible and so we can write $\mathbf{x}^* = (Df)^{-1}(\boldsymbol{\xi})$. By the inverse function theorem, the derivative of this function $\mathbf{x}^*(\boldsymbol{\xi})$ is

$$D_{\boldsymbol{\xi}}\mathbf{x}^*(\boldsymbol{\xi}) = D_{\boldsymbol{\xi}}(Df)^{-1}(\boldsymbol{\xi}) = (D^2 f)^{-1}(\mathbf{x}^*(\boldsymbol{\xi})) \tag{7.9}$$

where $D^2 f(\mathbf{x}) = f''(\mathbf{x})$ is the Hessian matrix whose entries are functions of $\mathbf{x}$. The first derivative of $f^*(\boldsymbol{\xi})$ is

$$\begin{aligned} Df^*(\boldsymbol{\xi}) &= D_{\boldsymbol{\xi}} \left\{ \boldsymbol{\xi} \cdot (Df)^{-1}(\boldsymbol{\xi}) - f\left[(Df)^{-1}(\boldsymbol{\xi})\right] \right\} \\ &= (Df)^{-1}(\boldsymbol{\xi}) + \boldsymbol{\xi} \cdot (D^2 f)^{-1}(\mathbf{x}^*(\boldsymbol{\xi})) - Df\left[(Df)^{-1}(\boldsymbol{\xi})\right] \cdot (D^2 f)^{-1}(\mathbf{x}^*(\boldsymbol{\xi})) \\ &= (Df)^{-1}(\boldsymbol{\xi}). \end{aligned} \tag{7.10}$$

Therefore, the second derivative of our Legendre transform is

$$(f^*)''(\boldsymbol{\xi}) = D^2 f^*(\boldsymbol{\xi}) = (D^2 f)^{-1}(\mathbf{x}^*(\boldsymbol{\xi})) > 0, \tag{7.11}$$

which is positive definite since $f''(\mathbf{x})$ is. That is, the Legendre transform f^* is also convex.

Now that we know

$$g(\boldsymbol{\xi}) := f^*(\boldsymbol{\xi}) = \mathbf{x}^*(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} - f(\mathbf{x}^*(\boldsymbol{\xi})) = Df^{-1}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} - f(Df^{-1}(\boldsymbol{\xi})) \tag{7.12}$$

is also convex, we may now consider its Legendre transform. Let ξ^* be the point which maximizes the distance function $G(\boldsymbol{\xi}, \mathbf{x})$ for g. This point is determined by the condition

$$0 = \frac{\partial G}{\partial \boldsymbol{\xi}}(\boldsymbol{\xi}^*) = \frac{\partial}{\partial \boldsymbol{\xi}} \left[\boldsymbol{\xi} \cdot \mathbf{x} - g(\boldsymbol{\xi})\right]_{\boldsymbol{\xi}=\boldsymbol{\xi}^*} = \mathbf{x} - Dg(\boldsymbol{\xi}^*), \tag{7.13}$$

which we can solve as

$$\boldsymbol{\xi}^* = (Dg)^{-1}(\mathbf{x}) = Df(\mathbf{x}) \tag{7.14}$$

(using our previous calculation of $Dg = Df^*$). Consequently, the transform of $g(\boldsymbol{\xi})$ is given by

$$\begin{aligned} g^*(\mathbf{x}) &= G(\boldsymbol{\xi}^*, \mathbf{x}) = \boldsymbol{\xi}^* \cdot \mathbf{x} - g(\boldsymbol{\xi}^*) \\ &= Df(\mathbf{x}) \cdot \mathbf{x} - Df^{-1}(Df(\mathbf{x})) \cdot Df(\mathbf{x}) + f(Df^{-1}(Df(\mathbf{x}))) = f(\mathbf{x}) \end{aligned}$$

as desired. For fixed $\mathbf{x}_0$, $G(\boldsymbol{\xi}, \mathbf{x}_0)$ geometrically represents the "y-coordinate" of the intersection of the line tangent to $f(\mathbf{x})$ with slope $\boldsymbol{\xi}$. As f is convex, all of the tangent lines of f lie below the curve, and so the maximum of $G(\boldsymbol{\xi}, \mathbf{x})$ will be at $f(\mathbf{x}_0)$. Therefore, the transformation of an already transformed function $g(\boldsymbol{\xi})$ of some smooth $f(\mathbf{x})$ must reproduce the original function $f(\mathbf{x})$. □

Returning to the context of mechanics, we make the following simple calculation which is a consequence of the kinetic energy being quadratic.

Proposition 7.3. *Let M be a real symmetric positive definite matrix. The Legendre transformation of the conservative Lagrangian*

$$\mathscr{L}(q, \dot{q}) = \tfrac{1}{2} M\dot{q} \cdot \dot{q} - V(q)$$

in the velocity $\dot{q}$ variables with dual variable p yields the corresponding Hamiltonian

$$H(q, p) = \tfrac{1}{2} M^{-1} p \cdot p + V(q).$$

As the velocity $\dot{q}$ is a tangent vector, then this duality and the corresponding natural paring $\mathbf{x} \cdot \boldsymbol{\xi} = \langle \dot{q}, p \rangle$ shows us that momentum p is naturally a covector.

Proof. Take $A = M$, $\mathbf{b} = 0$, and $c = V(q)$—which is constant with respect to the velocity $\dot{q}$—in Example 7.1. □

7.3 Liouville's theorem

Liouville's theorem states that the density of trajectories in phase space surrounding a system configuration as it evolves in time is constant.

Suppose the Hamiltonian $H(q, p)$ for a system with n degrees of freedom does not depend explicitly on time. We are interested in time progressions of the system in phase space, the $2n$-dimensional space of the coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n}$. Consider the vector field given by Hamilton's equations: at each point (q, p) in phase space we know that the trajectory of the system at this configuration is tangent to the vector $(-\partial H/\partial q, \partial H/\partial p)$ which is tangent to the trajectory passing through (q, p). We will (as always) suppose that the solution $(q(t), p(t))$ for any initial condition $(q(0), p(0)) = (q_0, p_0)$ of Hamilton's equations can be extended for all time. The **phase flow** is then the set of transformations $\phi^t : (q_0, p_0) \mapsto (q(t), p(t))$ with $t \in \mathbb{R}$ of phase space to itself. The phase flow forms a group under function composition $\circ$ parameterized by the additive group $t \in \mathbb{R}$ and a group action on phase space $\mathbb{R}^{2d} = \mathbb{R}^d_q \times \mathbb{R}^d_p$:

(a) At $t = 0$ the phase flow is the identity map: $\phi^0(q_0, p_0) = (q_0, p_0)$ for all $(q_0, p_0) \in \mathbb{R}^{2n}$.

(b) Composition corresponds to addition in $t \in \mathbb{R}$: $(\phi^{t_0} \circ \phi^{t_1})(q_0, p_0) = \phi^{t_0}(q(t_1), p(t_1)) = (q(t_0 + t_1), p(t_0 + t_1)) = \phi^{t_0 + t_1}(q_0, p_0)$.

(c) Inversion corresponds to negation in $t \in \mathbb{R}$: $(\phi^t \circ \phi^{-t})(q_0, p_0) = \phi^{t+(-t)}(q_0, p_0) = \phi^0(q_0, p_0)$ and vice versa.

(d) Function composition is associative and the group action is as well, because addition is associative in $\mathbb{R}$.

Theorem 7.4 (Liouville's theorem)**.** *The density of system points in phase space $\mathbb{R}^{2n} = \mathbb{R}^n_p \times \mathbb{R}^n_q$ around a phase curve remains constant along the curve. That is, for any measurable set $D \subset \mathbb{R}^{2n}$ in phase space, the volume of $\phi^t(D)$ is equal to that of D.*

Proof. First, let us simplify notation using the point $x := (q, p) \in \mathbb{R}^{2n}$ and the Hamiltonian vector field $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $f(x) := (-\partial H/\partial q, \partial H/\partial p)$, so that

$$\dot{x} = f(x), \qquad \phi^t(x) = x + f(x)t + \mathcal{O}(t^2). \tag{7.15}$$

Consider a region $D(0) \subset \mathbb{R}^{2n}$, and let $D(t) = \phi^t(D(0))$ denote its flow and $v(t)$ denote volume of $D(t)$. We will prove the more general statement

$$\text{if } \nabla \cdot f \equiv 0 \qquad \text{then } v(t) = v(0),$$

which is sufficient since for our vector field we have

$$\nabla \cdot f = -\frac{\partial^2 H}{\partial p \partial q} + \frac{\partial^2 H}{\partial q \partial p} = 0$$

by the symmetry of second derivatives.

The volume $v(t)$ may be expressed as the integral

$$v(t) = \int_{\phi^{t-t_0}(D(t_0))} \mathrm{d}x = \int_{D(t_0)} \det\left[\frac{\partial \phi_i^{t-t_0}}{\partial x_j}\right] \mathrm{d}x \tag{7.16}$$

by changing variables. Using the Taylor expansion (7.15), we can write

$$\left[\frac{\partial \phi_i^{t-t_0}}{\partial x_j}\right] = I + (t - t_0)\left[\frac{\partial f_i}{\partial x_j}\right] + \mathcal{O}\left((t-t_0)^2\right).$$

It is true for any matrix A and scalar k that

$$\det(I + hA) = 1 + h \operatorname{tr}(A) + \mathcal{O}(h^2),$$

which is just a rewriting of Jacobi's formula from matrix calculus. With this, the volume formula (7.16) becomes

$$\begin{aligned} v(t) &= \int_{D(t_0)} \left[1 + (t - t_0) \operatorname{tr}\left[\frac{\partial f_i}{\partial x_j}\right] + \mathcal{O}\left((t-t_0)^2\right)\right] \mathrm{d}x \\ &= \int_{D(t_0)} \left[1 + (t - t_0)\left(\nabla \cdot f\right) + \mathcal{O}\left((t-t_0)^2\right)\right] \mathrm{d}x. \end{aligned}$$

Differentiating this, we conclude

$$\left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=t_0} = \int_{D(t_0)} \nabla \cdot f \, \mathrm{d}x \tag{7.17}$$

The integrand is zero by premise, and so $v(t)$ is constant in time: $v(t) \equiv v(0)$. □

Liouville's theorem has many important consequences. For example, we now know that for a Hamiltonian system there can be no asymptotically stable equilibrium points or asymptotically stable closed trajectories in phase space, since these would require that the density of phase curves to increase around such phenomena. We also have the following phenomenon.

Corollary 7.5. *(Poincare's recurrence theorem) Fix $t \in \mathbb{R}$ and $D \subset \mathbb{R}^{2n}$ a bounded region of phase space, and let $\phi := \phi^t$ denote a phase flow group element. Then for any positive measure set $U \subset D$ there exists $x_0 \in U$ and a positive integer n such that $\phi^n(x_0) \in U$.*

For bounded motion—as is the case for a conservative system with potential energy $V(\mathbf{x}) \to +\infty$ as $|\mathbf{x}| \to \infty$, in which case a particle is confined to the region $D = \{(q,p) \in \mathbb{R}^{2d} : T + V \leq E\}$—this means that the system will return to an arbitrary vicinity of any given possible configuration $(q,p) \in \mathbb{R}^{2d}$ infinitely often, given enough time. If we were to open a connection between a chamber of gas and a chamber of vacuum then the gas molecules will eventually all return to the initial chamber, seemingly in violation of the second law of thermodynamics. Although it may appear that Poincare's theorem contradicts Liouville's, the time scales are often quite large—for a gas chamber it is longer than the age of the universe—and so there is no conflict.

Proof. As for a smooth Hamiltonian function Hamilton's equations are a smooth system of ODEs, then the subsequent flow $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is injective by uniqueness of solutions and continuous by well-posedness. Liouville's theorem tells us that the special form of this system that g is volume-preserving.

Consider the collection of sets $U, \phi(U), \phi^2(U), \phi^3(U), \cdots \subset D$. As ϕ is volume-preserving, all of these sets must have the same volume. D is bounded and thus has finite volume, and so it is impossible for all of these sets to be disjoint. That is, there exists some distinct $0 < j < k$ such that $\phi^j(U) \cap \phi^k(U) \neq \emptyset$. As ϕ is injective, this requires $\phi^{k-j}(U) \cap U \neq \emptyset$. Namely, we can pick some $x_0 \in U$ in this intersection, which gives $\phi^{k-l}(x_0) \in U$.

In fact, we can conclude that the set of points in U which do not return to U infinitely often has measure zero. □

Together, a measure space X with a finite measure μ and a measurable function $\phi : X \to X$ that is measure preserving (in the sense that $\mu(\phi^{-1}(A)) = \mu(A)$ for all measurable $A \subset X$) constitute a measure preserving space, which is the fundamental object of study in discrete dynamical systems.

7.4 Poisson bracket

For any two functions f and g of position q, momenta p, and time t we define the **Poisson bracket** of f and g to be

$$\{f, g\} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} \right). \tag{7.18}$$

(Note that there is another popular convention which differs by a factor of -1.) It arises naturally as the time evolution of any function $f(p,q,t)$; by the chain rule and Hamilton's equations we have

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q_k}\dot{q}_k + \frac{\partial f}{\partial p_k}\dot{p}_k \right) \\ &= \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial H}{\partial q_k} \right) = \frac{\partial f}{\partial t} + \{H, f\}\,.\end{aligned} \tag{7.19}$$

Consequently, a time-independent quantity $f(q,p)$ is conserved exactly when its bracket with the Hamiltonian $\{H,f\} = 0$ vanishes. Taking the quantity f to be a position or momentum coordinate, we see that Hamilton's equations may be written as

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \{H, q_i\}\,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{H, p_i\}\,. \tag{7.20}$$

This perspective generalizes nicely to the Heisenberg formulation of quantum mechanics, with the Poisson brackets replaced by operator commutators; an observable without explicit time dependence form a Lax pair with the Hamiltonian.

Next, we record some properties. It is easy to see that the Poisson bracket is bilinear (linear in both entries) and antisymmetric ($\{f,g\} = -\{g,f\}$) from the definition (7.18). The brackets also posses a Leibniz rule,

$$\{fg, h\} = f\{g,h\} + g\{f,h\}\,, \tag{7.21}$$

which can be seen by using the product rule for fg and expanding. They also behave nicely with the coordinates and momenta:

$$\{q_i, q_k\} = 0, \qquad \{p_i, p_k\} = 0, \qquad \{p_i, q_k\} = \delta_{ik}. \tag{7.22}$$

Also, if $f(p,q,t)$ is a function as before and $g(p,q,t) \equiv G(f(q,p,t))$, then their bracket

$$\{f, G(f)\} = \sum_{k=1}^{d} \left(\frac{\partial f}{\partial p_k} G'(f) \frac{\partial f}{\partial q_k} - \frac{\partial f}{\partial q_k} G'(f) \frac{\partial f}{\partial p_k} \right) = 0 \tag{7.23}$$

vanishes by the chain rule.

The Poisson bracket also satisfies the **Jacobi identity**:

$$\{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\} = 0, \tag{7.24}$$

which we will now verify. As $\{f,g\}$ is a linear function where each term contains a first derivative of f and g, then $\{h, \{f,g\}\}$ (for example) is a linear function in which each term contains a second derivative. Let $D_g(\phi) = \{g, \phi\}$ and $D_h(\phi) = \{h, \phi\}$. Note that the first term of (7.24) does not contain any second derivatives of f, and so all of the second derivatives of f are contained within

$$\begin{aligned}\{g, \{h,f\}\} + \{h, \{f,g\}\} &= \{g, \{h,f\}\} - \{h, \{g,f\}\} \\ &= D_g(D_h(f)) - D_h(D_g(f)) = (D_g D_h - D_h D_g) f\end{aligned}$$

To simplify notation and observe some cancellation, we write the operators D_g and D_h as

$$D_g = \sum_j \left(\frac{\partial g}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial g}{\partial q_j}\frac{\partial}{\partial p_j} \right) = \sum_k \xi_k \frac{\partial}{\partial x_k}$$
$$D_h = \sum_j \left(\frac{\partial h}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial h}{\partial q_j}\frac{\partial}{\partial p_j} \right) = \sum_k \eta_k \frac{\partial}{\partial x_k}$$

where $x = (q, p)$, $\xi = (\frac{\partial g}{\partial p}, -\frac{\partial g}{\partial q})$, and $\eta = (\frac{\partial h}{\partial p}, -\frac{\partial h}{\partial q})$. Therefore

$$D_g D_h = \sum_k \xi_k \frac{\partial}{\partial x_k}\left(\sum_l \eta_l \frac{\partial}{\partial x_l} \right) = \sum_{k,l} \left(\xi_k \frac{\partial \eta_l}{\partial x_k}\frac{\partial}{\partial x_l} + \xi_k \eta_l \frac{\partial^2}{\partial x_k \partial x_l} \right)$$
$$D_h D_g = \sum_k \eta_k \frac{\partial}{\partial x_k}\left(\sum_l \xi_l \frac{\partial}{\partial x_l} \right) = \sum_{k,l} \left(\eta_k \frac{\partial \xi_l}{\partial x_k}\frac{\partial}{\partial x_l} + \eta_k \xi_l \frac{\partial^2}{\partial x_k \partial x_l} \right)$$

In taking the difference of these, we see that the second terms in these last equalities cancel and we are left with

$$D_g D_h - D_h D_g = \sum_{k,l} \left(\xi_k \frac{\partial \eta_l}{\partial x_k} - \eta_k \frac{\partial \xi_l}{\partial x_k} \right) \frac{\partial}{\partial x_l}$$

That is, all of the second derivatives of f cancel, leaving only first derivatives. Symmetrically, this must also be true for g and h, and since every term of (7.24) contains only terms containing exactly one second derivative, then all the terms must cancel.

We will record one final property of the Poisson bracket.

Proposition 7.6. *(Poisson's theorem) If $f(q,p)$ and $g(q,p)$ are constants of motion for a Hamiltonian system, then their Poisson bracket $\{f, g\}$ is also conserved.*

Although powerful, it should be noted that the new quantity $\{f, g\}$ may be trivial, especially considering that a system with n degrees of freedom can only have up to $2n - 1$ conserved quantities.

Proof. As f is conserved and time-independent we have

$$0 = \frac{\mathrm{d}f}{\mathrm{d}t} = \{H, f\},$$

and similarly for g. The Jacobi identity (7.24) yields

$$\begin{aligned} 0 &= \{H, \{f, g\}\} + \{f, \{g, H\}\} + \{g, \{H, f\}\} \\ &= \{H, \{f, g\}\} - \{f, \{H, g\}\} + \{g, \{H, f\}\} \\ &= \{H, \{f, g\}\}. \end{aligned}$$

Rearranging, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\{f,g\} = \{H,\{f,g\}\} = 0,$$

and so $\{f,g\}$ is an integral of the motion. □

Example 7.7. If $L_1 = x_2p_3 - x_3p_2$ and $L_2 = x_3p_1 - x_1p_3$ are the first and second components of the angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ of a particle $\mathbf{x} \in \mathbb{R}^3$, then their bracket

$$\{L_1, L_2\} = \{r_2p_3, r_3p_1\} + \{r_3p_2, r_1p_3\} = r_2p_1 - p_2r_1 = -L_3$$

is the remaining component of the angular momentum.

7.5 Canonical transformations

For a system with n degrees of freedom, any choice of coordinates $q = (q_1, q_2, \ldots, q_n)$ is of course far from unique. In fact, we may choose any set of n independent quantities $Q_i(q,t)$, $i = 1, \ldots, n$ that describe the system's state and Lagrange's equations must still hold in terms of the new coordinates. For the Hamiltonian approach, we would like to consider the more general transformation of (q,p) to the $2n$ independent quantities $Q_i(q,p,t)$, $P_i(q,p,t)$, and $T(q,p,t)$ for $i = 1, \ldots, n$ so that on the extended phase space $T^*M \times I$, $I \subset \mathbb{R}$ the one-forms

$$p\,\mathrm{d}q - H\,\mathrm{d}t = P\,\mathrm{d}Q - K\,\mathrm{d}T + \mathrm{d}S \tag{7.25}$$

are equal, for some smooth functions $K(Q,P,T)$ and $S(Q,P,T)$. A transformation satisfying (7.25) is called **canonical**.

The definition (7.25) is made so that Hamilton's equations

$$\frac{\mathrm{d}Q_i}{\mathrm{d}T} = \frac{\partial K}{\partial P_i}, \qquad \frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{\partial K}{\partial Q_i}$$

hold for the new variables and the new Hamiltonian $K(Q,P,t)$ (sometimes referred to as the **Kamiltonian**); in particular, if the canonical transformation is time-independent then we have $K(Q,P) = H(q,p)$. As remarked at the end of section 4.1, this is a consequence of the Lagrangians differing by a total time derivative:

$$p \cdot \dot{q} - H(q,p,t) = P \cdot \dot{Q} - K(Q,P,t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[F(q(t),p(t),Q(t),P(t),t)\right].$$

In physics the function F is called the **generating function** for the transformation.

Remark. Some authors define canonical transformations as any diffeomorphism which obeys Hamilton's equations (7.5), and even falsely claim the equivalence of the two definitions (e.g. [LL76, Sec. 45]). The condition that a transformation preserves Hamilton's equations (7.5) is much weaker—for example $P = 2p$, $Q = q$ satisfies (7.5) but not (7.25)—but our definition has deeper geometric significance.

The first type of generating function we will consider is $F \equiv F(q, Q, t)$. Applying the chain rule and rearranging (7.25) yields

$$p \cdot \dot{q} - P \cdot \dot{Q} + K(Q, P, t) - H(q, p, t) = \frac{\partial F}{\partial q} \cdot \dot{q} + \frac{\partial F}{\partial Q} \cdot \dot{Q} + \frac{\partial F}{\partial t}.$$

Matching terms we find

$$p_i = \frac{\partial F}{\partial q_i}, \qquad P_i = -\frac{\partial F}{\partial Q_i}, \qquad K = H + \frac{\partial F}{\partial t}. \tag{7.26}$$

In this way, the generating function does indeed characterize the transformation.

Now consider a generating function of the form $\Phi \equiv \Phi(q, P, t)$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{d} P_i Q_i\right) = \sum_{i=1}^{d} \dot{P}_i Q_i + \sum_{i=1}^{d} P_i \dot{Q}_i.$$

We add this to (7.25) to solve for the remaining variables p and Q:

$$\begin{aligned} p \cdot \dot{q} - \cancel{P \cdot \dot{Q}} + \dot{P} \cdot Q + \cancel{P \cdot \dot{Q}} + K(Q, P, t) - H(q, p, t) \\ = \frac{\mathrm{d}}{\mathrm{d}t}\left[P \cdot Q + F(q, P, t)\right] = \frac{\partial \Phi}{\partial q} \cdot \dot{q} + \frac{\partial \Phi}{\partial P} \cdot \dot{P} + \frac{\partial \Phi}{\partial t} \end{aligned}$$

which requires that our generating function be of the form $\Phi(q, P, t) = P \cdot Q + F(q, P, t)$ and

$$p_i = \frac{\partial \Phi}{\partial q_i}, \qquad Q_i = \frac{\partial \Phi}{\partial P_i}, \qquad K = H + \frac{\partial \Phi}{\partial t}. \tag{7.27}$$

Similar procedures yield the relations corresponding to generating functions of p, Q or of p, P. Notice that in matching terms we need the $2n$ generating function coordinates to be independent. For example, if the new coordinates $Q_i \equiv Q_i(q, t)$ do not depend on the old momenta, then the coordinates Q and q are not independent, and so a generating function $F(q, Q, t)$ would not work. Canonical transformations highlight the point that the Hamiltonian perspective regards both q and p equally as independent coordinates. In particular, the transformation $Q_i = p_i$, $P_i = -q_i$ leaves the Hamiltonian formulation unchanged, which would not be legal in the Lagrangian framework.

Lastly, we can characterize time-independent canonical transformations with Poisson brackets. Statement (c) is often how a transformation is verified as canonical in practice.

Proposition 7.8. *For a time-independent transformation $(q, p) \to (Q, P)$ the following conditions are equivalent.*

(a) The transformation is canonical, i.e. $p\,dq = P\,\mathrm{d}Q + \mathrm{d}S$.

(b) The transformation preserves the Poisson bracket:

$$\{f,g\}_{p,q} = \{f,g\}_{P,Q}$$

for all smooth functions f and g on phase space, where the subscripts denote the respective variables of differentiation.

(c) The new variables satisfy the coordinate relations with respect to the old Poisson brackets:

$$\{Q_i,Q_j\}_{p,q} = 0, \qquad \{P_i,P_j\}_{p,q} = 0, \qquad \{P_i,Q_j\}_{p,q} = \delta_{ij}.$$

Proof. (b) implies (c): This is immediate, since we can replace the differentiation variables q,p with Q,P by assumption—for example,

$$\{Q_i,Q_j\}_{p,q} = \{Q_i,Q_j\}_{P,Q} = 0.$$

(c) implies (b): By premise, we compute

$$\begin{aligned}\begin{pmatrix} \frac{\partial P}{\partial p} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial Q}{\partial q} \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q} \end{pmatrix} &= \begin{pmatrix} \frac{\partial P}{\partial p} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial Q}{\partial q} \end{pmatrix} \begin{pmatrix} -\frac{\partial P}{\partial q} & -\frac{\partial Q}{\partial q} \\ \frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \end{pmatrix} \\ &= -\begin{pmatrix} \left[\{P_i,P_j\}_{p,q}\right]_{i,j} & \left[\{P_i,Q_j\}_{p,q}\right]_{i,j} \\ \left[\{Q_i,P_j\}_{p,q}\right]_{i,j} & \left[\{Q_i,Q_j\}_{p,q}\right]_{i,j} \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\end{aligned}$$

Plugging this in, we have

$$\begin{aligned}\{f,g\}_{p,q} &= -\begin{pmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial g}{\partial p} \\ \frac{\partial g}{\partial q} \end{pmatrix} \\ &= -\begin{pmatrix} \frac{\partial f}{\partial P} & \frac{\partial f}{\partial Q} \end{pmatrix} \begin{pmatrix} \frac{\partial P}{\partial p} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial Q}{\partial q} \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q} \end{pmatrix} \begin{pmatrix} \frac{\partial g}{\partial P} \\ \frac{\partial g}{\partial Q} \end{pmatrix} \\ &= -\begin{pmatrix} \frac{\partial f}{\partial P} & \frac{\partial f}{\partial Q} \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial g}{\partial P} \\ \frac{\partial g}{\partial Q} \end{pmatrix} = \{f,g\}_{P,Q}\end{aligned}$$

(a) implies (c): The time-independence requires that the new Hamiltonian $K(Q,P,t) = H(q,p,t)$ is the same as the old Hamiltonian. On the one hand, the chain rule and Hamilton's equations insist

$$\dot{Q}_i = \frac{\partial Q_i}{\partial q}\cdot\dot{q} + \frac{\partial Q_i}{\partial p}\cdot\dot{p} = \frac{\partial Q_i}{\partial q}\cdot\frac{\partial H}{\partial p} - \frac{\partial Q_i}{\partial p}\cdot\frac{\partial H}{\partial q} = \{H,Q_i\}_{p,q}.$$

On the other hand, the new Hamilton's equations (7.5) tell us that this should be equal to

$$\frac{\partial H}{\partial P_i} = \frac{\partial H}{\partial q}\cdot\frac{\partial q}{\partial P_i} + \frac{\partial H}{\partial p}\cdot\frac{\partial p}{\partial P_i}.$$

Equating these, we have

$$\frac{\partial Q_i}{\partial p_j} = -\frac{\partial q_j}{\partial P_i}, \qquad \frac{\partial Q_i}{\partial q_j} = \frac{\partial p_j}{\partial P_i}.$$

The analogous computation for $\dot{P}_i$ lead to the relations

$$\frac{\partial P_i}{\partial p_j} = \frac{\partial q_j}{\partial Q_i}, \qquad \frac{\partial P_i}{\partial q_j} = -\frac{\partial p_j}{\partial Q_i}.$$

Together these are exactly the conditions of (c).

(c) implies (a): The same computation from the previous implication shows that $\mathrm{d}p \wedge \mathrm{d}q = \mathrm{d}P \wedge \mathrm{d}Q$. Using Stokes' theorem to pass to the boundary, we deduce that for any closed contour γ contractible to a point we have

$$\oint_\gamma p\,\mathrm{d}q = \oint_\gamma P\,\mathrm{d}Q.$$

Therefore we conclude that the difference form $p\,\mathrm{d}q - P\,\mathrm{d}Q$ is closed since its antiderivative is well-defined. □

7.6 Infinitesimal canonical transformations

A Hamiltonian flow can be thought of as a continuous canonical transformation. Consider a Hamiltonian system undergoing a canonical transformation

$$Q_i = q_i + \delta q_i, \qquad P_i = p_i + \delta p_i, \qquad i = 1, \ldots, d, \tag{7.28}$$

which is infinitesimal in the sense that we will be sending δq_i and δp_i to zero later to obtain a statement about derivatives. We expect the corresponding generating function to be a perturbation of that which generates the identity transformation. It is the generating function

$$\Phi_0(q, P, t) = \sum_{i=1}^{n} q_i P_i$$

which yields the identity transformation, since according to (7.27) we have

$$p_i = \frac{\partial \Phi_0}{\partial q_i} = P_i, \qquad Q_i = \frac{\partial \Phi_0}{\partial P_i} = q_i, \qquad K(Q, P) = H(q, p).$$

Consequently, we seek a generating function for the transformation (7.28) of the form

$$\Phi(q, P, t) = \sum_{i=1}^{n} q_i P_i + \epsilon G(q, P) \tag{7.29}$$

for some small parameter ϵ. We will be considering the first order behavior as $\epsilon \to 0$, and δq_i and δp_i will depend linearly on ϵ. By (7.27), the function Φ generates the transformation

$$p_i = \frac{\partial \Phi}{\partial q_i} = P_i + \epsilon \frac{\partial G}{\partial q_i}, \qquad Q_i = \frac{\partial \Phi}{\partial P_i} = q_i + \epsilon \frac{\partial G}{\partial P_i} = q_i + \epsilon \frac{\partial G}{\partial p_i}\frac{\partial p_i}{\partial P_i}.$$

The leading term of the derivative $\frac{\partial p_i}{\partial P_i} = \frac{\partial}{\partial P_i}(P_i - \delta p_i) = 1 + \mathcal{O}(\delta p_i)$ is one, and with the factor of ϵ in front this derivative does not make a first order contribution. We made this change of variables so that we may take the function $G = G(q, p)$ to be dependent only on the old coordinates and momenta q, p. Solving for δq_i and δp_i in their definition (7.28), we get

$$\delta p_i = P_i - p_i = -\epsilon \frac{\partial G}{\partial q_i} + \mathcal{O}(\epsilon^2), \qquad \delta q_i = Q_i - q_i = \epsilon \frac{\partial G}{\partial p_i} + \mathcal{O}(\epsilon^2). \tag{7.30}$$

If we take the function $G = H(q, p)$ to be the system's Hamiltonian and let the parameter ϵ be an infinitesimal time interval δt, this becomes

$$\delta q_i = \delta t \frac{\partial H}{\partial p_i} + \mathcal{O}(\delta t^2), \qquad \delta p_i = -\delta t \frac{\partial H}{\partial q_i} + \mathcal{O}(\delta t^2). \tag{7.31}$$

Taking $\delta t \to 0$ we recover Hamilton's equations, which indicates that **the Hamiltonian is the instantaneous generator of the system's motion with time**.

We will now consider how the Hamiltonian is affected by this transformation. Let $u(q, p)$ be a smooth function, which is perturbed as the result of the transformation (7.28) by

$$u(q_i + \delta q_i, p_i + \delta p_i) = u(q_i, p_i) + \delta u \tag{7.32}$$

Taylor expanding, we have

$$\begin{aligned} \delta u &= \sum_{i=1}^{d} \left(\frac{\partial u}{\partial q_i} \delta q_i + \frac{\partial u}{\partial p_i} \delta p_i \right) + \mathcal{O}(\delta q_i^2 + \delta p_i^2) \\ &= \sum_i \left(\frac{\partial u}{\partial q_i} \epsilon \frac{\partial G}{\partial p_i} - \frac{\partial u}{\partial p_i} \epsilon \frac{\partial G}{\partial q_i} \right) + \mathcal{O}(\epsilon^2) = -\epsilon \{u, G\} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{7.33}$$

after plugging in for δq_i and δp_i using (7.30). If we take the function $u(q, p) = H(q, p)$ to be the Hamiltonian, then

$$\left. \frac{\mathrm{d}H}{\mathrm{d}\epsilon} \right|_{\epsilon=0} = -\{H, G\}. \tag{7.34}$$

As G is time-independent, then this quantity vanishes if and only if G is a constant of the motion, and so **constants of the motion are instantaneous generators of those transformations which leave the Hamiltonian invariant**. This is another appearance of Noether's theorem (Proposition 4.12).

7.7 Canonical variables

For any system with bounded periodic motion there is always a change of variables so that the motion is dictated by a linearly increasing angular variable and an energy parameter. In this section we will consider a conservative system with one degree of freedom and bounded motion. As in section 6.3 this analysis also applies coordinate-wise to a system with periodic motion in each variable (but not necessarily jointly periodic motion).

As the motion is bounded and necessarily periodic, we may consider the action variable

$$I = \frac{1}{2\pi} \oint p \, \mathrm{d}q \tag{7.35}$$

which was first glimpsed in the period integral (2.8) and we later identified in (6.20). The integral is evaluated over one period, around a loop in the phase plane around which q returns to an extreme value once. We may also write

$$I = \frac{1}{2\pi} \iint \mathrm{d}p \, \mathrm{d}q \tag{7.36}$$

where the double integral is evaluated over the region enclosed by the trajectory $(q(t), p(t))$ in phase space.

We would like to treat I as a momentum variable in order to obtain the Hamiltonian and Hamilton's equations in terms of this geometric quantity. Consider the generating function

$$\Phi = \int p(q, E) \, \mathrm{d}q \tag{7.37}$$

where $E \equiv H(q(t), p(t))$ is the constant energy along a trajectory and $p(q, E)$ is implicitly determined. For our conservative system $I \equiv I(E)$ is strictly a function of the energy, and so the generating function $\Phi \equiv \Phi(q, I)$ may be considered a function of I rather than E. The first equation of (7.27) requires that

$$p = \frac{\partial \Phi}{\partial q} \tag{7.38}$$

which agrees with the fundamental theorem of calculus applied to the definition (7.37). The new coordinate—which we will denote by w—is dictated by the second equation of (7.27):

$$w = \frac{\partial \Phi}{\partial I} \tag{7.39}$$

Together, w and I are called the **canonical variables** of the system, and individually I is the **action variable** and w is the **angle variable**.

As the generating function $\Phi(q, I, \lambda)$ does not explicitly depend on time, the last equation of (7.27) tells us that the Hamiltonian is unaffected:

$$K = H + \frac{\partial \cancel{\Phi}}{\cancel{\partial t}} = H, \tag{7.40}$$

and is just the old one in terms of the new variables. However, since the action variable $I \equiv I(E)$ is determined by the energy then in the new variables $H \equiv E \equiv E(I)$. Hamilton's equations are now

$$\dot{I} = -\frac{\partial H}{\partial w} = 0, \qquad \dot{w} = \frac{\partial H}{\partial I} = \frac{\mathrm{d}E}{\mathrm{d}I}. \tag{7.41}$$

The first equation is expected, since I is given by the system'senergy which is constant for a conservative system. The latter can be easily integrated to obtain

$$w(t) = \frac{\mathrm{d}E}{\mathrm{d}I}t + \text{constant} = \omega(I)t + \text{constant}, \tag{7.42}$$

and so w is the phase of the oscillations.

From their definitions (7.35) and (7.37) we can see that after each oscillation the generating function Φ increments by $2\pi I$. In turn, the equation of motion (7.42) tells us that the angle variable w increases by 2π with each oscillation, much like an actual angle coordinate. Conversely, the original variables q and p are unchanged after an integer number of periods.

7.8 Exercises

7.1 (Harmonic oscillator)**.** The one-dimensional harmonic oscillator has the kinetic and potential energies

$$T = \tfrac{1}{2}mv^2, \qquad V = \tfrac{1}{2}kx^2.$$

As this system is conservative, the Hamiltonian $H = E$ is given by the total energy

$$H = T + V = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2$$

where $\omega^2 = k/m$.

(a) Directly apply Hamilton's equations to obtain $\ddot{x} + \omega^2 x = 0$, which has solution

$$x(t) = x_0 \cos\left[\omega(t - t_0)\right]$$

as in Example 2.2. Hamilton's equations may not make solving the motion easier than using Lagrange's or Newton's equations, but rather provide a new interpretation of the system's motion.

(b) By considering the shape of trajectories in phase space, find the angle variable I and show that the energy as a function of the angle variable I is $E(I) = \omega I$.

(c) Write down (but do not evaluate) the integral for the generating function Φ, and show that the angle variable w is

$$w = \tan^{-1}\left(\frac{x}{\sqrt{\frac{2I}{m\omega} - x^2}}\right) + \text{constant}.$$

(d) By design, the angle variable w is cyclic for the new Hamiltonian $E(I) = \omega I$. From the equality

$$w(t) = \frac{\mathrm{d}E}{\mathrm{d}I}t + \text{constant} = \omega t + \text{constant},$$

invert this equation to conclude

$$x(t) = x_0 \cos\left[\omega(t - t_0)\right]$$

as in part (a). This method is unwieldy in application, and is intended for the purpose of physical understanding rather than solving for the equations of motion.

7.2 (Charged particle in an electromagnetic field)**.** In Exercise 4.7, we found the Lagrangian for a charged particle in $\mathbb{R}^3$ with charge q and mass m moving through an electromagnetic field to be

$$\mathscr{L} = \tfrac{1}{2}mv^2 - q\phi + \frac{q}{c}v \cdot A$$

where ϕ and A are the scalar and vector potentials for the electric and magnetic fields:

$$B = \nabla \times A, \qquad E = -\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t}.$$

(a) Show that the Hamiltonian for this system is

$$H = \frac{1}{2m}\left(p - \frac{q}{c}A\right)^2 + q\phi.$$

(b) Although the electric and magnetic fields are uniquely expressed, the scalar and vector potentials that ϕ and A are not unique and they appear explicitly in the Hamiltonian. In fact, substituting $A' = A + \nabla f$ for any smooth function $f(x, y, z, t)$ leaves B unchanged since the curl of a gradient is also zero. Show that for E to remain unchanged we need to also substitute

$$\phi' = \phi - \frac{1}{c}\frac{\partial f}{\partial t}.$$

Together, replacing $(A, \phi) \to (A', \phi')$ is called a **gauge transformation**, and a specific pair (A, ϕ) is called a choice of **gauge**.

(c) As the electric and magnetic fields are unaffected by the choice of gauge, then any physical laws in terms of the potentials should also be invariant. Show that under a gauge transformation the Hamiltonian becomes

$$H' = H - \frac{q}{c}\frac{\partial f}{\partial t},$$

and that Hamilton's equations still hold in the new variables.

7.3 (Young's inequality)**.** Show that for any Legendre transform pair $f(\mathbf{x})$ and $f^*(\boldsymbol{\xi})$ we have

$$\mathbf{x} \cdot \boldsymbol{\xi} \leq f(\mathbf{x}) + f^*(\boldsymbol{\xi}).$$

Apply this inequality to the function $f(\mathbf{x})$ of Example 7.1.

7.4 (Properties of the Legendre transform)**.** In this exercise, we will assume $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and not identically $+\infty$.

(a) Define the sub-differential

$$\partial f(\mathbf{x}) = \{\mathbf{v} \in \mathbb{R}^n : f(\mathbf{y}) - f(\mathbf{x}) \geq \mathbf{v} \cdot (\mathbf{y} - \mathbf{x}) \text{ for all } \mathbf{y} \in \mathbb{R}^n\}.$$

Show that if f is convex, then $\partial f(\mathbf{x})$ is nonempty for all $\mathbf{x} \in \mathbb{R}^n$. Show that the Legendre transform $f^*(\boldsymbol{\xi})$ (defined by (7.6)) is equal to $\mathbf{x}\cdot\boldsymbol{\xi} - f(\mathbf{x})$ if and only if $\boldsymbol{\xi} \in \partial f(\mathbf{x})$.

(b) Show that $f^*(\boldsymbol{\xi})$ is a lower semicontinuous and convex function. Moreover, if f is convex, show that $f^{**} = f$. In general, show that f^{**} is the largest lower semicontinuous convex function that is less than or equal to f.

(c) Show that if f is C^1, strictly convex, and superlinear (i.e. $f(\mathbf{x})/|\mathbf{x}| \to +\infty$ as $|\mathbf{x}| \to \infty$), then f^* is C^1.

7.5 (Example of Poisson's theorem)**.** Show that for the angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ of a particle $\mathbf{x} \in \mathbb{R}^3$ we have

$$\{\mathbf{L}, \mathbf{L} \cdot \mathbf{n}\} = \mathbf{L} \times \mathbf{n}$$

for any unit vector $\mathbf{n}$. Do so by picking a choice of Cartesian coordinates with respect to $\mathbf{n}$, writing down the infinitesimal canonical transformation (7.28) given by rotation about the $\mathbf{n}$ axis by an angle $\mathrm{d}\theta$, and apply the calculation (7.33) with u replaced by $\mathbf{L}$. Conclude that $\{L_i, L_j\} = -L_k$ for $i, j, k \in \{1, 2, 3\}$ in cyclic order.

7.6. Hamilton's equations also arise from a variational principle, but the functional is unbounded and hence less useful. Given a smooth Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ that is strictly convex and satisfies $H(0) = 0$ and an energy level $\alpha \in \mathbb{R}$, consider the functional

$$E(\mathbf{x}(t)) = \tfrac{1}{2} \int_0^1 \mathbf{x}(t) \cdot J\dot{\mathbf{x}}(t)\, \mathrm{d}t$$

with the domain

$$M_\alpha = \left\{ \mathbf{x} \in C^1(\mathbb{R}; \mathbb{R}^{2n}) : \mathbf{x}(t+1) = \mathbf{x}(t),\ \int_0^1 H(\mathbf{x}(t))\, \mathrm{d}t = \alpha \right\}.$$

Here, J is the $2n \times 2n$ matrix (8.5). Show that if $\mathbf{x}(t)$ is a critical point of E on M_α, then $\mathbf{x}(t) = (\mathbf{q}(t), \mathbf{p}(t))$ is a periodic solution of Hamilton's equations (7.4). Show that E is not bounded below on $\{x \in C^1(\mathbb{R}; \mathbb{R}^{2n}) : \mathbf{x}(t+1) = \mathbf{x}(t)\}$.

Chapter 8

Symplectic geometry

Symplectic geometry is the differential geometric generalization of the time-independent Hamiltonian structure of phase space to general manifolds, and the physical results for conservative systems from the previous chapter can be lifted to this perspective. The material for this section is based on [Arn89, Ch. 8] chapter 8 and [Lee13, Ch. 22], and we will try to meld the physics and mathematical notations.

8.1 Symplectic structure

Let M be a smooth even-dimensional manifold of dimension $2n$. A **symplectic form** or **symplectic structure** on M is a two-form ω on M that is closed ($\mathrm{d}\omega = 0$) and **nondegenerate**: for each $x \in M$ the map $T_xM \to T_x^*M$ which takes $\xi \mapsto \xi \lrcorner \omega = \omega(\xi, \cdot)$ is invertible. The pair (M, ω) is called a **symplectic manifold**. There are other more easily verifiable criteria for a two-form ω (cf. Exercise 8.1), but conceptually this is the classification we will rely upon. Heuristically, the symplectic form ω is an identification of the tangent and cotangent spaces at each point in M, which bears resemblance to Riemannian structure but as we shall see exhibits drastically different behavior.

Example 8.1. Consider the cotangent bundle T^*N of a n-dimensional manifold N, or simply the Euclidean space $T^*\mathbb{R}^n = \mathbb{R}^n_q \times \mathbb{R}^n_p$. Then the two-form

$$\omega = \mathrm{d}p \wedge \mathrm{d}q = \sum_{i=1}^{n} \mathrm{d}p^i \wedge \mathrm{d}q^i \tag{8.1}$$

is a symplectic form on T^*N, and is called the **canonical symplectic form** on the cotangent bundle; we will later see how this form naturally encapsulates Hamilton's equations. This form is closed, since it is the exterior derivative of the one-form

$$p\,\mathrm{d}q = \sum_{i=1}^{n} p^i\,\mathrm{d}q^i, \tag{8.2}$$

which is in turn called the **tautological one-form** on T^*M; we first saw this form in the period integral (2.8) and then again later in defining the action variable (6.20). To see that ω is nondegenerate, fix $(q,p) \in T^*N$ and let $(v_1,\dots,v_n,a_1,\dots,a_n) \in T_{(q,p)}(T^*N)$ denote the dual basis to $(\mathrm{d}q,\mathrm{d}p)$ so that

$$\mathrm{d}q^i(v_j) = \delta_{ij}, \qquad \mathrm{d}p^i(a_j) = \delta_{ij}, \qquad \mathrm{d}q^i(a_j) = \mathrm{d}p^i(v_j) = 0$$

for $i,j = 1,\dots,n$. Often in differential topology we write ∂_{q_i} for the dual vector to $\mathrm{d}q^i$, but physically we like to think of the tangent vector to a point in phase space as the velocity and (mass-times-)acceleration. The action of ω on these basis vectors is

$$\omega(v_i,v_j) = \omega(a_i,a_j) = 0, \qquad \omega(v_i,a_j) = -\omega(a_j,v_i) = \delta_{ij}$$

for $i,j = 1,\dots,n$. To check nondegeneracy, fix $\xi \in T_{(q,p)}(T^*N)$ and suppose we have $\omega(\xi,\eta) = 0$ for all $\eta \in T_{(q,p)}(T^*N)$. Expanding $\xi = \sum_{i=1}^n (b^i v_i + c^i a_i)$ for constants $b^i, c^i \in \mathbb{R}$, we see that

$$0 = \omega(\xi, v_i) = -c^i, \qquad 0 = \omega(\xi, a_i) = b^i$$

and so $\xi = 0$ as desired.

In fact, the form of Example 8.1 is the fundamental example in the following sense.

Theorem 8.2 (Darboux's theorem)**.** *Let (M,ω) be a $2n$-dimensional symplectic manifold. For any $\xi \in M$ there exists local coordinates (q,p) centered at ξ with respect to which ω has the representation* (8.1)*.*

For a proof, see [Lee13, Th. 22.13].

Remark. Symplectic manifolds are automatically orientable because ω^n is a nonvanishing $2n$ form on M. Indeed, if we fix $x \in M$ and write $\omega = \sum_i \mathrm{d}p^i \wedge \mathrm{d}q^i$ at x, then the n-fold wedge product

$$\omega^n = \sum_I \mathrm{d}p^{i_1} \wedge \mathrm{d}q^{i_1} \wedge \cdots \wedge \mathrm{d}p^{i_d} \wedge \mathrm{d}q^{i_d}$$

where the sum ranges over all multi-indices $I = (i_1,\dots,i_n)$ of length n. Any term where I contains a repeated index is zero because $\mathrm{d}q^i \wedge \mathrm{d}q^i = 0$, and since 2-forms commute under the wedge product then we can rewrite this as

$$\omega^d = d!\,\mathrm{d}p^1 \wedge \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}p^d \wedge \mathrm{d}q^d.$$

This is nonvanishing at x by definition of the nondegeneracy of ω.

8.2 Hamiltonian vector fields

Next let us see how to use the symplectic structure to obtain the familiar dynamics from a Hamiltonian. Let (M,ω) be a symplectic manifold and $H : M \to \mathbb{R}$

be a smooth function. Then $\mathrm{d}H$ is a differential one-form which associates a covector to each point in M. On the other hand, for Hamilton's equations we need to specify a vector field for the right-hand side of the differential equation. By definition, for each $x \in M$ the map $T_xM \to T_x^*M$ which takes $\xi \mapsto \xi \lrcorner \omega$ is invertible, and so let $J : T_x^*M \to T_xM$ denote its inverse. Then $J\,\mathrm{d}H$ is a vector field on M, called the **Hamiltonian vector field** associated to the Hamiltonian H. The induced flow $e^{tJ\,\mathrm{d}H} : M \to M$, $x_0 \mapsto x(t)$ defined to solve the ODE

$$\dot{x}(t) = J\,\mathrm{d}H(x(t)), \qquad x(0) = x_0 \tag{8.3}$$

is called a **Hamiltonian flow** on M. In differential geometry the notation $\widehat{\omega} : T_xM \to T_x^*M$ is used for the map $\xi \mapsto \xi \lrcorner \omega$, and so in place of J the notations $\widehat{\omega}^{-1}$ and

$$X_H := \widehat{\omega}^{-1}(\mathrm{d}H) \qquad \Longleftrightarrow \qquad X_H \lrcorner \omega = \mathrm{d}H \tag{8.4}$$

are often used.

Example 8.3. Consider the canonical symplectic form (8.1) on Euclidean phase space $T^*\mathbb{R}^n = \mathbb{R}^n_q \times \mathbb{R}^n_p$. On Euclidean space we already have a natural identification of vectors and covectors via the dot product, and we can express J as a linear transformation in terms of this identification. Given a vector field

$$X = \sum_{i=1}^{n} \left(b^i \frac{\partial}{\partial q^i} + c^i \frac{\partial}{\partial p^i} \right)$$

for some smooth coefficients b^i and c^i, we compute

$$X \lrcorner \omega = \sum_{i=1}^{n} \left(c^i\,\mathrm{d}q^i - b^i\,\mathrm{d}p^i \right) = \begin{pmatrix} c \\ -b \end{pmatrix} = J \begin{pmatrix} b \\ c \end{pmatrix}$$

and hence

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{8.5}$$

for I the identity matrix. Writing

$$\mathrm{d}H = \frac{\partial H}{\partial q}\,\mathrm{d}q + \frac{\partial H}{\partial p}\,\mathrm{d}p = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q^i}\,\mathrm{d}q^i + \frac{\partial H}{\partial p^i}\,\mathrm{d}p^i \right) = \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix},$$

we have

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J\,\mathrm{d}H = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}$$

which agrees with Hamilton's equations (7.4).

Symplectic geometry is the generalization of time-independent Hamiltonian dynamics since the Hamiltonian function H is automatically conserved under its Hamiltonian flow.

Proposition 8.4 (Conservation of energy)**.** *Let (M, ω) be a symplectic manifold and let H be a smooth function on M. Then H is constant along the integral curves of $J\,\mathrm{d}H$, and when $\mathrm{d}H \neq 0$ the vector field $\mathrm{d}H$ is tangent to the level sets of H.*

Proof. Both are a consequence of

$$(J\,\mathrm{d}H)(H) = \mathrm{d}H(J\,\mathrm{d}H) = \omega(J\,\mathrm{d}H, J\,\mathrm{d}H) = 0,$$

which holds since ω is alternating. □

8.3 Integral invariants

Consider a smooth diffeomorphism $g : M \to M$. A k-form η is said to be an integral invariant for g if

$$\int_{g(N)} \eta = \int_N \eta \tag{8.6}$$

for all orientable k-dimensional submanifolds N with boundary.

Theorem 8.5. *A Hamiltonian flow $g_t = e^{tJ\,\mathrm{d}H}$ preserves the symplectic structure: $g_t^*\omega = \omega$. In other words, the symplectic form ω is an integral invariant of g_t.*

Proof. The map g_t is smoothly homotopic to the identity via the family of diffeomorphisms g_s, $s \in [0, t]$, in the sense that at time $s = 0$ the map $g_0 : M \to M$ is the identity, and at time $s = t$ the map $g_t : M \to M$ is what we are given. Fix N a smooth orientable 2-dimensional submanifold, and let $\Omega_N := \{g_s(N) : s \in [0, t]\}$ denote the image of N under the homotopy. We can think of ω_N as an orientable 3-manifold in $[0, t] \times M$ or as being immersed in M. With this choice of orientation we have

$$\partial\Omega_N = g_t(N) \cup (-N) \cup (-\Omega_{\partial N}). \tag{8.7}$$

We claim that for any smooth curve γ in M we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\gamma} \omega = \int_{g^t(\gamma)} \mathrm{d}H, \tag{8.8}$$

where H is the Hamiltonian for the flow g^t. Let $\phi : [a, b] \to M$ be a parametrization of γ. Then Ω_γ is parameterized by $\Phi(s, x) := g^s(\phi(x))$ and we have

$$\int_{\Omega_\gamma} \omega = \int_0^t \int_a^b \omega\left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial s}\right) \mathrm{d}x\,\mathrm{d}s$$

where $\frac{\partial\Phi}{\partial s}(s, x)$ is in $T_{\Phi(s,x)}M$. By definition of the Hamiltonian phase flow $g_t = e^{tJ\,\mathrm{d}H}$ the tangent vector $\frac{\partial\Phi}{\partial s}$ points in the direction of $J\,\mathrm{d}H$, and so we have $\omega(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial s}) = \mathrm{d}H(\frac{\partial\Phi}{\partial x})$. Therefore

$$\int_{\Omega_\gamma} \omega = \int_0^t \int_a^b \mathrm{d}H\left(\frac{\partial\Phi}{\partial x}\right) \mathrm{d}x\,\mathrm{d}s = \int_0^t \left(\int_{g^s(\gamma)} \mathrm{d}H\right) \mathrm{d}s,$$

and the identity (8.8) follows from the fundamental theorem of calculus.

For a closed curve like ∂N we note that

$$\int_{g^t(\partial N)} \mathrm{d}H = \int_{g^t(\emptyset)} H = 0,$$

and so the identity (8.8) implies

$$\int_{\Omega_{\partial N}} \omega = 0. \tag{8.9}$$

As ω is closed by definition, then by Stokes' theorem we have

$$0 = \int_{\Omega_N} \mathrm{d}\omega = \int_{\partial\Omega_N} \omega.$$

Decomposing the boundary $\partial\Omega_N$ according to (8.7) we obtain

$$0 = \int_{g_t(N)} \omega - \int_N \omega - \int_{\Omega_{\partial N}} \omega.$$

From (8.9) we know the last integral vanishes, and so we conclude that ω is an integral invariant of g_t. □

In the previous section we saw that ω^n defines a volume form on M and so we immediately obtain an analog of Theorem 7.4.

Corollary 8.6 (Liouville's theorem)**.** *Each of the forms ω^2, ω^4, ... is preserved by a Hamiltonian flow. In particular, every Hamiltonian flow preserves the volume form ω^n.*

Recall from section 7.5 that on a cotangent bundle $M = T^*Q$ with canonical symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}q$, a (time-independent) canonical transformation $g : T^*Q \to T^*Q$ satisfies

$$g^*(p\,\mathrm{d}q) = p\,\mathrm{d}q + \mathrm{d}S. \tag{8.10}$$

This definition does not make $p\,\mathrm{d}q$ an integral invariant for a canonical transformation, since the condition (8.6) only holds for closed curves N. Instead, we employ the following useful observation.

Proposition 8.7. *Let $g : M \to M$ be a smooth diffeomorphism. If η is a k-form such that* (8.6) *holds for only closed orientable k-submanifolds N, then $\mathrm{d}\eta$ is an integral invariant.*

Proof. Let N be an orientable $(k+1)$-submanifold. Then by Stokes' theorem we have

$$\int_N \mathrm{d}\eta = \int_{\partial N} \eta, \qquad \int_{g(N)} \mathrm{d}\eta = \int_{\partial g(N)} \eta = \int_{g(\partial N)} \eta.$$

As ∂N is closed then these two right-hand sides are equal by premise. Therefore we conclude that (8.6) holds for the form $\mathrm{d}\eta$, and hence $\mathrm{d}\eta$ is an integral invariant. □

Noting that $\mathrm{d}(p\,\mathrm{d}q) = \mathrm{d}p \wedge \mathrm{d}q$, we conclude:

Corollary 8.8. *Canonical transformations preserve the symplectic form ω and the volume form ω^n.*

Remark. In section 7.6 we saw that for Euclidean phase space $T^*\mathbb{R}^n$ Hamiltonian flows are canonical transformations. However, the converse of Proposition 8.7 is not true in general, and so Theorem 8.5 does not imply the identity (8.10). We must assume that M is simply connected in order for Hamiltonian flows are canonical transformations, since then Theorem 8.5 implies (8.10).

8.4 Poisson bracket

In section 7.4 we saw how to phrase Hamiltonian dynamics in terms of the Poisson bracket, and in this section we will see how this notion manifests in terms of symplectic structure. For smooth functions $f, g \in C^\infty(M)$ on a symplectic manifold (M, ω) we define the **Poisson bracket** of f and g to be

$$\{f, g\} = \omega(J\,\mathrm{d}f, J\,\mathrm{d}g) = \mathrm{d}g(J\,\mathrm{d}f) = (J\,\mathrm{d}f)(g). \tag{8.11}$$

As in (7.19) the Poisson bracket $\{H, f\}$ is also the evolution of the quantity f under the Hamiltonian flow H, since the Lie derivative of the function f along $J\,\mathrm{d}H$ is given by

$$\mathcal{L}_{J\,\mathrm{d}H} f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ e^{tJ\,\mathrm{d}H} = (J\,\mathrm{d}H)(f) = \{H, f\} \tag{8.12}$$

according to the definition (8.3) of the Hamiltonian flow. In particular, we again have that f is conserved by the Hamiltonian flow of H if and only if $\{H, f\} = 0$.

Example 8.9. Let us check that the new definition (8.11) agrees with the phase space definition (7.18) on Euclidean space $T^*\mathbb{R}^n = \mathbb{R}^n_q \times \mathbb{R}^n_p$. Using the calculation of J from Example 8.3 we have

$$J\,\mathrm{d}f = J\left(\frac{\partial f}{\partial q}\,\mathrm{d}q + \frac{\partial f}{\partial p}\,\mathrm{d}p\right) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial q} \\ \frac{\partial f}{\partial p} \end{pmatrix} = \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p}.$$

Consequently, the definition (8.11) yields

$$\{f, g\} = (J\,\mathrm{d}f)(g) = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p},$$

which agrees with the first definition (7.18).

The properties listed in section 7.4 resemble those of the commutator because the Poisson bracket at the level of functions corresponds to the commutator of the respective Hamiltonian vector fields.

Proposition 8.10. *If (M, ω) is a symplectic manifold, then the Poisson bracket is the Hamiltonian of the commutator of the corresponding Hamiltonian vector fields:*

$$J\,\mathrm{d}\{f, g\} = [J\,\mathrm{d}f, J\,\mathrm{d}g]. \tag{8.13}$$

In particular, the vector space $C^\infty(M)$ is a Lie algebra under the Poisson bracket: the Poisson is bilinear, antisymmetric, and satisfies the Jacobi identity.

Proof. In fact, it suffices to prove that the Poisson bracket satisfies the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \tag{8.14}$$

for arbitrary smooth functions $f, g, h \in C^\infty(M)$. This is sufficient because we can use the antisymmetry of ω to write

$$\{\{f, g\}, h\} = \{f, \{g, h\}\} - \{g, \{f, h\}\},$$

and recognizing the Poisson bracket as a Lie derivative via (8.12) we conclude

$$\mathcal{L}_{J\,\mathrm{d}\{f,g\}}h = \mathcal{L}_{J\,\mathrm{d}f}\mathcal{L}_{J\,\mathrm{d}g}h - \mathcal{L}_{J\,\mathrm{d}g}\mathcal{L}_{J\,\mathrm{d}f}h = \mathcal{L}_{[J\,\mathrm{d}f, J\,\mathrm{d}g]}h.$$

As $h \in C^\infty(M)$ is arbitrary, the identity (8.13) must hold.

It remains to demonstrate the Jacobi identity (8.14). The left-hand side of (8.14) is a combination of terms which each have at least one second order derivative. Isolating the terms containing second order derivatives of f and recognize the Poisson bracket as a Lie derivative via (8.12), we have

$$\begin{aligned}\{\{f, g\}, h\} + \{\{h, f\}, g\} &= \{h, \{g, f\}\} - \{g, \{h, f\}\} \\ &= \mathcal{L}_{J\,\mathrm{d}h}\mathcal{L}_{J\,\mathrm{d}g}f - \mathcal{L}_{J\,\mathrm{d}g}\mathcal{L}_{J\,\mathrm{d}h}f.\end{aligned}$$

However, we know $\mathcal{L}_{J\,\mathrm{d}h}\mathcal{L}_{J\,\mathrm{d}g} - \mathcal{L}_{J\,\mathrm{d}g}\mathcal{L}_{J\,\mathrm{d}h} = \mathcal{L}_{[J\,\mathrm{d}h, J\,\mathrm{d}g]}$ is only a first order differential operator, and so we conclude that the terms with second order derivatives of f vanish. Lastly, since the left-hand side of (8.14) is symmetric in f, g, and h, we conclude that there can be no second order derivatives and hence must vanish. □

8.5 Time-dependent systems

Recall that Hamilton's equations (7.4) remain valid for time-dependent Hamiltonian systems, while so far we have only allowed H to be a smooth function on phase space M. This section is dedicated to extending the symplectic geometry developed thus far to time-dependent Hamiltonian systems.

Let (M, ω) be a $2n$-dimensional symplectic manifold, and define the **extended phase space** $M \times \mathbb{R}$. Given a possibly time-dependent Hamiltonian H, we define the **Poincaré–Cartan one-form** locally in terms of the canonical coordinates (q, p) on M guaranteed by Theorem 8.2 as

$$\tau = p\,\mathrm{d}q - H\,\mathrm{d}t = \sum_{i=1}^{n} p^i\,\mathrm{d}q^i - H\,\mathrm{d}t. \tag{8.15}$$

Note that the first term above is the tautological one-form $p\,\mathrm{d}q$ on M, the differential of which yields the symplectic form ω. This is the form that we insisted be preserved by a canonical transformation in (7.25), and the same notion of canonical transformations holds on a general extended phase space $M \times \mathbb{R}$ in terms of the canonical coordinates (q, p, t).

Remark. The extended phase space $M \times \mathbb{R}$ together with the Poincaré–Cartan one-form τ do indeed define a contact manifold, but as we will see in the next chapter it is not the natural contact extension of M since τ depends on the system's Hamiltonian H.

On $M \times \mathbb{R}$ we define the **extended Hamiltonian vector field**

$$Y_H = X_H + \frac{\partial}{\partial t}, \tag{8.16}$$

where $X_H(t)$ is the Hamiltonian vector field on $M \times \{t\}$ defined by (8.3). In analogy with the second condition of (8.4), the vector field (8.16) is the unique solution to

$$Y_H \lrcorner \mathrm{d}\tau = 0. \tag{8.17}$$

The flow of Y_H is given by

$$\dot{q}^i = \frac{\partial H}{\partial p^i}, \qquad \dot{p}^i = -\frac{\partial H}{\partial q^i}, \qquad \dot{t} = 1, \tag{8.18}$$

which is just Hamilton's equations (7.4) joined with the trivial equation $\dot{t} = 1$. It follows that any smooth time-dependent function f on $M \times \mathbb{R}$ evolves according to

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{H, f\}_{p,q} + \frac{\partial f}{\partial t},$$

as was the case in section 7.4. In particular, a time-dependent Hamiltonian H is no longer conserved under its own flow.

Example 8.11. [Cal41, Kan48] showed that in addition to naturally occurring time-dependent Hamiltonian systems, this framework can also be applied to some systems which are time-independent and dissipative by introducing artificial time dependence into the Hamiltonian. Consider the one-dimensional Hamiltonian system

$$H(Q, P, t) := e^{-\gamma t}\frac{P^2}{2m} + e^{\gamma t}V(Q) \tag{8.19}$$

on the extended phase space $\mathbb{R}_Q \times \mathbb{R}_P \times \mathbb{R}_t$, where $\gamma \geq 0$ is a constant and the coordinates Q, P are related to the physical coordinates q, p by the non-canonical transformation

$$P = e^{\gamma t}p, \qquad Q = q.$$

Then the equations of motion (8.18) yield

$$m\ddot{q} + m\gamma\dot{q} + V'(q) = 0.$$

This represents a Newtonian system with a friction force that depends linearly on the velocity, like the damped harmonic oscillator of Example 2.8.

8.6 Locally Hamiltonian vector fields

In section 8.1 we called a vector field V on a symplectic manifold (M, ω) Hamiltonian if it is equal to $J\,\mathrm{d}H$ for some smooth function $H \in C^\infty(M)$, and in Theorem 8.5 we saw that ω is necessarily invariant under the flow of $J\,\mathrm{d}H$. In fact, any smooth vector field V on M whose flow leaves ω invariant (i.e. $(e^{tV})^*\omega = \omega$) is called **symplectic**. However, a vector field being Hamiltonian is a global condition—in the sense that the corresponding Hamiltonian H must extend smoothly to all of M—while being symplectic is a pointwise condition and is hence only local.

Consequently, a smooth vector field V on M is called **locally Hamiltonian** (as opposed to **globally** Hamiltonian) if for each point p there exists a neighborhood on which V is Hamiltonian. As an extension of Theorem 8.5, locally Hamiltonian vector fields are exactly those which are symplectic.

Proposition 8.12. *Let (M, ω) be a symplectic manifold. A smooth vector field V on M is symplectic if and only if it is locally Hamiltonian. If M is also simply connected, then every locally Hamiltonian vector field is globally Hamiltonian.*

Proof. Differentiating the condition $(e^{tV})^*\omega = \omega$ with respect to t at $t = 0$, we see that a vector field V is symplectic if and only if $\mathcal{L}_V\omega = 0$. From Cartan's magic formula we have

$$\mathcal{L}_V\omega = \mathrm{d}(V \lrcorner \omega) + V \lrcorner \mathrm{d}\omega = \mathrm{d}(V \lrcorner \omega)$$

since ω is closed. Therefore V is symplectic if and only if $V \lrcorner \omega$ is closed.

If V is locally Hamiltonian, then in a neighborhood of any point there exists a function f so that $V = J\,\mathrm{d}f$, and hence $V \lrcorner \omega = \mathrm{d}f$ which is certainly closed. Conversely, if V is a symplectic vector field, then $V \lrcorner \omega$ is closed and hence exact in a neighborhood of any point, and writing $V \lrcorner \omega = \mathrm{d}f$ we deduce that $V = J\,\mathrm{d}f$ on a neighborhood as desired.

Now suppose M is also simply connected. Then every closed one form is exact, and so $V \lrcorner \omega$ is closed if and only if $V \lrcorner \omega = \mathrm{d}f$ and $V = J\,\mathrm{d}f$ for a smooth function f defined on all of M. □

A smooth vector field V is called an **infinitesimal symmetry** of the Hamiltonian H if both ω and H are invariant under the flow e^{tV} of V. The second condition $(e^{tV})^*H = H$ can be recast as $V(H) = 0$, as can be seen by differentiating $H \circ e^{tV} = H$ with respect to t at $t = 0$. With this notion, we can prove the following analogue of Proposition 4.12.

Proposition 8.13 (Noether's theorem)**.** *Let (M, ω) be a symplectic manifold and H a fixed Hamiltonian. If f is a conserved quantity, then the Hamiltonian vector field $J\,\mathrm{d}f$ is an infinitesimal symmetry. Conversely, if M is also simply connected then each infinitesimal symmetry is the Hamiltonian vector field of a conserved quantity, and the quantity is unique up to the addition of a function that is constant on each component of M.*

Proof. First suppose f is a conserved quantity. Then from the identity (8.12) we know that $\{H, f\} = 0$. However, since the Poisson bracket is antisymmetric, we deduce that $0 = \{f, H\} = (J\,\mathrm{d}f)(H)$ as well. This demonstrates that H is conserved by $J\,\mathrm{d}f$ and from Theorem 8.5 we know that ω is also conserved by $J\,\mathrm{d}f$, and hence $J\,\mathrm{d}f$ is an infinitesimal symmetry.

Now suppose that M is simply connected and V is an infinitesimal symmetry. Then V is symplectic by definition, and by Proposition 8.12 we know V is globally Hamiltonian. Writing $V = J\,\mathrm{d}f$, since H is conserved by f then we have $0 = (J\,\mathrm{d}f)(H) = \{f, H\} = -\{H, f\}$, and therefore f is a conserved quantity. If g is another function with $J\,\mathrm{d}g = V = J\,\mathrm{d}f$, then by definition of J we have $\mathrm{d}(f - g) = (J\,\mathrm{d}f - J\,\mathrm{d}g) \lrcorner \omega = 0$ and hence $f - g$ is constant on the components of M. □

8.7 Exercises

8.1. Show that the following criteria for a nondegenerate two-form ω are equivalent.

(a) For each $x \in M$ the map $T_xM \to T_x^*M$ which takes $\xi \mapsto \omega(\xi, \cdot)$ is invertible.

(b) For each $x \in M$ and $\xi \in T_xM$ nonzero there exists $\eta \in T_xM$ such that $\omega(\xi, \eta) \neq 0$.

(c) The local matrix representation of ω in terms of some (hence every) basis is invertible.

8.2. (a) Show that rotation on the two-dimensional sphere S^2 is a Hamiltonian flow.

(b) Show that translations $g_t(q, p) = (q + t, p)$ on the torus $\mathbb{R}^2/\mathbb{Z}^2$ is a locally Hamiltonian flow, but is not globally Hamiltonian.

8.3. Let (M, ω) be a $2n$-dimensional compact symplectic manifold.

(a) Show that the n-fold wedge product ω^n is not exact.

(b) Show that the de-Rham cohomology groups $H^{2k}_{\mathrm{dR}}(M)$ are nontrivial for $k = 1, \ldots, n$.

(c) Conclude that the two-dimensional sphere S^2 is the only sphere that admits a symplectic structure.

8.4. (Real symplectic matrices) A linear transformation $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called **symplectic** if it preserves the canonical symplectic form (8.1). The set of all symplectic matrices forms a group with matrix multiplication and is often denoted by $Sp(2n, \mathbb{R})$.

(a) Prove that the determinant of any symplectic matrix is equal to 1.

(b) Show that a matrix A is symplectic if and only if $A^T JA = J$ for the matrix J defined in (8.5).

(c) Show that the Jacobian of a canonical transformation (8.10) is symplectic. (Note that this calculation implicitly appears in the proof of Proposition 7.8.)

8.5. (Symplectic and complex structure) Identify Euclidean phase space $\mathbb{R}^{2n} = \mathbb{R}^n_q \times \mathbb{R}^n_p$ with the complex space $\mathbb{C}^n$ via $z_j := q_j + ip_j$.

(a) Show that matrix multiplication by matrix J (8.5) on $\mathbb{R}^{2n}$ corresponds to multiplication by $-i$ on $\mathbb{C}^n$. This is why the matrix (8.5) is called J, and in fact some authors (e.g. [Arn89]) choose the opposite sign for the canonical symplectic form (8.1) and use the notation I in place of J.

(b) Show that the Hermitian inner product $(z, w) \mapsto \sum z_j \overline{w}_j$ on $\mathbb{C}^n$ corresponds to $(\xi, \eta) \mapsto \xi \cdot \eta + i\,\xi \cdot J\eta$ on $\mathbb{R}^{2n}$. In other words, the scalar product and symplectic product are the real and imaginary parts of the Hermitian inner product, respectively.

8.6. (Hamiltonian PDE [Gar71]) The Hamiltonian mechanics we have developed thus far originates from the time derivative of trajectories in phase space and hence provides a class of ODEs on a finite dimensional manifold; however, this can be extended to PDE when we consider the time derivative of trajectories in an infinite dimensional function space. Given a smooth functional F of, say, smooth real-valued periodic functions $C^\infty(\mathbb{R}/\mathbb{Z})$ we denote the Fréchet derivative kernel by $\delta F/\delta q(x)$, so that

$$\mathrm{d}F\big|_q(f) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} F(q + sf) = \int \frac{\delta F}{\delta q}(x) f(x)\,\mathrm{d}x.$$

The Fréchet space $C^\infty(\mathbb{R}/\mathbb{Z}; \mathbb{R})$ will now take the role of the manifold M. Specifically, for two smooth functionals $F(q)$ and $G(q)$ on $C^\infty(\mathbb{R}/\mathbb{Z})$ we define the Poisson bracket

$$\{F, G\} = \int \left(\frac{\delta F}{\delta q}\right)'(x) \frac{\delta G}{\delta q}(x)\,\mathrm{d}x.$$

In other words, we have replaced the dot product on the Euclidean phase space $\mathbb{R}^{2n}$ and the matrix J (8.5) with the L^2 real inner product and the skew-symmetric operator $J = \frac{\partial}{\partial x}$. Given a smooth functional $H(q)$ on $C^\infty(\mathbb{R}/\mathbb{Z})$, we define the associated Hamiltonian PDE

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial q}\right)$$

in the spirit of (8.3) and (8.12).

It turns out that many of the results of finite dimensional Hamiltonian mechanics have analogs in this infinite dimensional setting. For example, the global

minimum of a classical Hamiltonian is a (Liapunov) stable equilibrium, and the global minimum of a Hamiltonian functional is orbitally stable (i.e. concentration compactness).

(a) Check that this Poisson bracket is bilinear, antisymmetric, and satisfies the Jacobi identity.

(b) Given a Hamiltonian functional $H(q)$, show that a smooth functional $F(q)$ is constant for solutions q to the PDE associated to H if and only if $\{H, F\} = 0$ as a functional on $C^\infty(\mathbb{R}/\mathbb{Z})$.

(c) Show that for any Hamiltonian functional $H(q)$ both the Hamiltonian H and the **mass functional**

$$M(q) := \int_0^1 q(x)\,\mathrm{d}x$$

are automatically conserved for solutions q to the PDE associated to H.

(d) Show that the **momentum functional**

$$P(q) := \tfrac{1}{2}\int_0^1 q(x)^2\,\mathrm{d}x$$

does indeed generate translations, in the sense that the solution to the associated PDE with initial data $q(0, x) = f(x)$ is $q(t, x) = f(x + t)$.

(e) The Korteweg–de Vries (KdV) equation is the PDE associated to the Hamiltonian

$$H_{\mathrm{KdV}}(q) := \int_0^1 \left(\tfrac{1}{2}q'(x)^2 - q(x)^3\right)\mathrm{d}x,$$

and arises as the long-wavelength and shallow-water limit for unidirectional water waves of height q from the undisturbed water level $q = 0$. Show that the mass $M(q)$, momentum $P(q)$, and energy $H_{\mathrm{KdV}}(q)$ are all constant for solutions q to KdV. In fact, these are the first three of an infinite hierarchy of conserved quantities for KdV.

Chapter 9

Contact geometry

Just as symplectic geometry extends the structure of conservative Hamiltonian dynamics on phase space, contact geometry is the natural generalization of nonconservative dynamics on the product of phase space with the time axis. The material for this chapter is based on [BCT17] and [Lee13, Ch. 22].

9.1 Contact structure

A **contact manifold** is a smooth $(2n+1)$-dimensional manifold M paired with a contact form η. A **contact form** η is a one-form required to satisfy the following nondegeneracy condition: for each $x \in M$ the restriction of $\mathrm{d}\eta_x$ to the subspace $\ker(\eta_x) \subset T_x M$ is nondegenerate (i.e. $\mathrm{d}\eta_x$ is a symplectic tensor for all $x \in M$). The rank-$2n$ distribution $N \subset TM$ satisfying $N_x = \ker(\eta_x)$ for each $x \in M$ is called a **contact structure** on M; it plays a fundamental role and is sometimes taken in the literature to be the defining geometric concept instead of η.

Similar to how the symplectic form nondegeneracy condition is equivalent to the nonvanishing of the n-fold wedge product $(\omega)^n$, this nondegeneracy condition turns out to be equivalent to

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0; \tag{9.1}$$

the proof of this equivalence is Exercise 9.1. Consequently, $\eta \wedge (\mathrm{d}\eta)^n$ defines a volume form on M, and so in particular M must be orientable. Although the condition (9.1) is sometimes easily verifiable in practice, we will conceptually be relying on the first condition.

Example 9.1. On the Euclidean contact space $\mathbb{R}_S \times T^*\mathbb{R}^n = \mathbb{R}_S \times \mathbb{R}^n_q \times \mathbb{R}^n_p$ we have the **canonical contact form**

$$\eta = \mathrm{d}S - p\,\mathrm{d}q = \mathrm{d}S - \sum_{i=1}^{n} p^i\,\mathrm{d}q^i. \tag{9.2}$$

Note that this is the combination of $\mathrm{d}S$ and the tautological one-form (8.2) on $T^*\mathbb{R}^n$. A straightforward computation shows that $\eta \wedge (\mathrm{d}\eta)^n = \mathrm{d}S \wedge \mathrm{d}q \wedge \mathrm{d}p$ is the Euclidean volume form on $\mathbb{R}_t \times T^*\mathbb{R}^n$, but we can also check that $\mathrm{d}\eta$ is a symplectic tensor. Note that $\mathrm{d}\eta = -\mathrm{d}p \wedge \mathrm{d}q$, and so the rank-$2n$ distribution $N \subset T\mathbb{R}^{2n+1}$ annihilated by η is spanned by the vector fields

$$X_i = \frac{\partial}{\partial q^i} + p^i \frac{\partial}{\partial S}, \qquad Y_i = \frac{\partial}{\partial p^i}$$

for $i = 1, \dots, n$. Moreover, we have

$$\mathrm{d}\eta(X_i, X_j) = 0, \qquad \mathrm{d}\eta(Y_i, Y_j) = 0, \qquad \mathrm{d}\eta(X_i, Y_j) = \delta_{ij}$$

for $i, j = 1, \dots, n$, and it follows that $\mathrm{d}\eta|_N$ is nondegenerate as in Example 8.1.

This example is again fundamental in the following sense.

Theorem 9.2 (Contact Darboux theorem)**.** *If (M, η) is a $(2n+1)$-dimensional contact manifold, then for each $x \in M$ there exist local coordinates (q, p, S) centered at x with respect to which η has the representation* (9.2)*.*

For a proof, see [Lee13, Th. 22.31].

The contact structure automatically induces an associated vector field called the Reeb field, which heuristically points orthogonally to the distribution N and plays the role of S-axis.

Proposition 9.3 (The Reeb field)**.** *If (M, η) is a contact manifold, then there exists a unique smooth vector field ξ on M called the* Reeb field *satisfying*

$$\xi \lrcorner \mathrm{d}\eta = 0, \qquad \eta(\xi) = 1. \tag{9.3}$$

Proof. The map Φ which takes $X \mapsto X \lrcorner \mathrm{d}\eta$ defines a smooth bundle homomorphism $\Phi : TM \to T^*M$, and for each $x \in M$ it reduces to a linear map $\Phi_x : T_xM \to T_x^*M$. As $\mathrm{d}\eta_x$ restricted to the subspace N_x is nondegenerate by definition, then $\Phi_x|_{N_x}$ is injective and hence Φ_x has rank at least $2n$. On the other hand, we know that Φ_x cannot have rank $2n+1$ because then $\mathrm{d}\eta_x$ would be nondegenerate and contradict that T_xM is odd-dimensional. Therefore, we conclude that $\ker \Phi_x$ is one-dimensional. Moreover, since $\ker(\Phi_x)$ is not contained in $N_x = \ker(\eta_x)$ by definition, we know there exists a unique $\xi \in \ker(\Phi_x)$ with $\eta_x(\xi_x) = 1$; these correspond to the two conditions (9.3) respectively.

The smoothness of ξ follows from the smoothness of η. Note that $\ker \Phi \subset TM$ is a smooth rank-one subbundle, and so around any $x \in M$ we can pick a smooth nonvanishing section X of $\ker \Phi$ near x. As $\eta(X) \neq 0$, then we can write $\xi = \eta(X)^{-1} X$ as a composition of smooth maps near x. □

Example 9.4. For the Euclidean contact space of Example 9.1, we see that Reeb field is

$$\xi = \frac{\partial}{\partial S}$$

as the two conditions (9.3) are easily verified.

9.2 Hamiltonian vector fields

Given a smooth function H on a contact manifold (M, η), the associated **contact Hamiltonian vector field** X_H is uniquely determined by the two conditions

$$\eta(X_H) = H, \qquad (X_H \lrcorner \mathrm{d}\eta)|_N = -\mathrm{d}H|_N. \tag{9.4}$$

As $\mathrm{d}\eta|_N$ is nondegenerate by definition, there is a unique vector field Y on N satisfying the second condition of (9.4). The vector field $X_H := Y + H\xi$ is then the unique solution of the conditions (9.4).

In comparison to symplectic Hamiltonian vector fields, the first condition of (9.4) looks like the primitive of $\omega(X_H) = \mathrm{d}H$ (which we wrote as $X_H = J\,\mathrm{d}H$) and the second condition is like $X_H \lrcorner \omega = \mathrm{d}H$; in the symplectic case these conditions were redundant (cf. (8.4)), but now we need both in order to determine X_H on and off of the kernel N_x of η_x.

Example 9.5. Let us see what the contact Hamiltonian vector field X_H looks like for the Euclidean contact space of Example 9.1 (and hence also the expression of X_H in the local coordinates guaranteed by Theorem 9.2). Given a smooth function $H(s, q, p)$ it is easily verified that

$$X_H = \sum_{i=1}^{n} \left(p^i \frac{\partial H}{\partial p^i} - H \right) \frac{\partial}{\partial S} + \frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p^i \frac{\partial H}{\partial S} \right) \frac{\partial}{\partial p^i} \tag{9.5}$$

satisfies the two conditions (9.4), from which we obtain the differential equation system

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{i=1}^{n} p^i \frac{\partial H}{\partial p^i} - H, \qquad \frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p^i}, \qquad \frac{\mathrm{d}p^i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} - p^i \frac{\partial H}{\partial S}. \tag{9.6}$$

When H is independent of s, the second two sets of equations reduce to Hamilton's equations of motion (7.4). We also recognize the quantity S as the action $S(q, t)$; indeed, if we use the formula (6.1) for the momentum and use the Hamilton–Jacobi equation (6.4) we see that the time derivative of the action is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial S}{\partial q^i} \frac{\mathrm{d}q^i}{\mathrm{d}t} + \frac{\partial S}{\partial t} = \sum_{i=1}^{n} p^i \frac{\mathrm{d}q^i}{\mathrm{d}t} - H.$$

To conclude the section, let us consider a one-dimensional physical example for to illustrate how contact geometry encapsulates time-independent nonconservative dynamics.

Example 9.6. Take $n = 1$ in the Euclidean contact space of Example 9.1, and consider the Hamiltonian

$$H(q, p, S) = \frac{p^2}{2m} + V(q) + \gamma S \tag{9.7}$$

where γ is a real constant. The contact Hamilton's equations (9.6) read

$$\dot{q} = \frac{p}{m}, \qquad \dot{p} = -V'(q) - \gamma p, \qquad \dot{S} = \frac{p^2}{2m} - V(q) - \gamma S.$$

This represents a Newtonian system with a friction force that depends linearly on the velocity, like the damped harmonic oscillator of Example 2.8. Note that as opposed to Example 8.11, in this approach the momentum coordinate still coincides with the physical momentum defined via the velocity.

9.3 Dynamics

Using the expression (9.5) of a contact Hamiltonian vector field X_H in terms of the local coordinates (q, p, S) of Theorem 9.2, we see that a smooth function $F(q, p, S)$ on a contact manifold (M, η) evolves according to

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}t} &= X_H(F) \\ &= -H\frac{\partial F}{\partial S} + \sum_{i=1}^{n} p^i \left[\frac{\partial H}{\partial p^i}\frac{\partial F}{\partial S} - \frac{\partial H}{\partial S}\frac{\partial F}{\partial p^i}\right] + \sum_{i=1}^{n}\left[\frac{\partial H}{\partial p^i}\frac{\partial F}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial F}{\partial p^i}\right] \\ &= -H\frac{\partial F}{\partial S} + \sum_{i=1}^{n} p^i \{H, F\}_{p^i, S} + \{H, F\}_{p,q}. \end{aligned} \tag{9.8}$$

The last term above we recognize from (7.19) and the first two terms are corrections introduced by the contact structure. The notation $\{H, F\}_{p^i,S}$ should be interpreted as a convenient shorthand defined via the above formula with no deeper meaning.

Taking $F = H(q, p, S)$ to be the Hamiltonian for the vector field X_H in the formula (9.8) above, we see that the Hamiltonian evolves according to

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -H\frac{\partial H}{\partial S}. \tag{9.9}$$

From this we see that H is a conserved quantity if and only if H is independent of S or $H = 0$. In particular, for a conservative Hamiltonian $H \equiv H(q, p)$ we recover the conservation of the Hamiltonian. However, in general the rate of decrease of the Hamiltonian is proportional to the system's energy H and its dissipation $\partial H/\partial S$.

Specifically, let us consider a Hamiltonian of the form

$$H(q, p, S) = H_0(q, p) + f(S) \tag{9.10}$$

where $H_0(q, p)$ is the mechanical energy of the system (e.g. $H_0(q, p) = p^2/2m + V(q)$), then according to the formula (9.8) the mechanical energy obeys

$$\frac{\mathrm{d}H_0}{\mathrm{d}t} = -\sum_{i=1}^{n} p^i \frac{\partial H_0}{\partial p^i} f'(S). \tag{9.11}$$

We interpret this as saying that $f(S)$ is the potential for the system's dissipative force. Moreover, the evolution (9.9) of the Hamiltonian can be integrated to obtain

$$H(t) = H(0) \exp\left[- \int_0^t f'(S) \right].$$

Plugging in the Hamiltonian (9.10), we obtain an implicit equation which in principle determines the action $S = S(q, p, t)$; thus the equations of motion (9.6) reduce to the $2n$ equations for the positions q^i and momenta p^i.

Example 9.7. If we take $d = 1$ and the dissipative potential $f(S) = \gamma S$ to be linear as in Example 9.6 then equation (9.11) becomes

$$\dot{H}_0 = -m\gamma \dot{q}^2$$

which agrees with what we found for the damped harmonic oscillator of Example 2.8. The energy of this system decays exponentially to zero, according to

$$H(t) = H(0) e^{-\gamma t}.$$

Solving for the action S we obtain

$$S(q, p, t) = \frac{1}{\gamma} \left[H(0) e^{-\gamma t} - \frac{p^2}{2m} - V(q) \right].$$

9.4 Contact transformations

A **contact transformation** is a smooth transformation which leaves the contact structure invariant. As opposed to canonical transformations, we allow the contact form $f\eta$ in the new coordinates to differ by a smooth nonvanishing factor $f \in C^\infty(M)$ since we are interested in the contact structure N. In terms of the canonical coordinates (q, p, S) guaranteed by Theorem 9.2, the new coordinates $(\tilde{q}, \tilde{p}, \tilde{S})$ must satisfy

$$\tilde{\eta} = \mathrm{d}\tilde{S} - \tilde{p}\, \mathrm{d}\tilde{q} = f(\mathrm{d}S - p\, \mathrm{d}q) = f\eta. \tag{9.12}$$

Writing $(\tilde{S}, \tilde{q}, \tilde{p})$ as functions of (q, p, S), this is equivalent to the conditions

$$f = \frac{\partial \tilde{S}}{\partial S} - \sum_{j=1}^n \tilde{p}^j \frac{\partial \tilde{q}^j}{\partial S}, \qquad -f p^i = \frac{\partial \tilde{S}}{\partial q^i} - \sum_{j=1}^n \tilde{p}^j \frac{\partial \tilde{q}^j}{\partial q^i}, \qquad 0 = \frac{\partial \tilde{S}}{\partial p^i} - \sum_{j=1}^n \tilde{p}^j \frac{\partial \tilde{q}^j}{\partial p^i}$$

for $i = 1, \dots, d$. Note that canonical transformations are independent of $S, \tilde{S}$ and are defined by the condition (9.12) with $f \equiv 1$.

As we allow for a conformal factor f in the definition (9.12), the volume form is also rescaled. Indeed, if $\tilde{\eta} = f\eta$ then $\mathrm{d}\tilde{\eta} = \mathrm{d}f \wedge \eta + f\, \mathrm{d}\eta$, and so

$$\tilde{\eta} \wedge (\mathrm{d}\tilde{\eta})^n = f^{d+1} \eta \wedge (\mathrm{d}\eta)^n$$

is the new volume form. In the case of canonical transformations we have $f \equiv 1$, and hence we recover volume preservation.

As with canonical transformations in section 7.5, we may consider a contact transformation as being generated by a generating function. For an example, assume that the coordinates $(q, \tilde{q}, S)$ are independent. Then the differential of the generating function $\tilde{S} = \tilde{S}(q, \tilde{q}, S)$ may be written

$$\mathrm{d}\tilde{S} = \frac{\partial \tilde{S}}{\partial S}\,\mathrm{d}S + \sum_{i=1}^{n} \frac{\partial \tilde{S}}{\partial q^i}\,\mathrm{d}q^i + \sum_{i=1}^{n} \frac{\partial \tilde{S}}{\partial \tilde{q}^i}\,\mathrm{d}\tilde{q}^i.$$

Plugging this into (9.12), we see that the remaining coordinates are determined in terms of $\tilde{S}$ by

$$f = \frac{\partial \tilde{S}}{\partial S}, \qquad f p^i = -\frac{\partial \tilde{S}}{\partial q^i}, \qquad \tilde{p}^i = \frac{\partial \tilde{S}}{\partial \tilde{q}^i}.$$

Taking $f \equiv 1$ and comparing to (7.26), we see that $\tilde{S}$ is related to the canonical transformation generating function $F(q, \tilde{q})$ via

$$\tilde{S} = S - F(q, \tilde{q}).$$

In particular, we conclude that all physicist's time-independent canonical transformations (cf. the remark of section 7.5) are a special case of contact transformations.

Recall that for symplectic structures the Hamiltonian dynamics generate instantaneous canonical transformations; this was first seen in section 7.6 computationally, and then abstractly as an equivalence in Proposition 8.12. In analogy with symplectic vector fields, we will call a smooth vector field V on M if a **contact vector field** if its flow e^{tV} preserves the contact structure N, in that

$$\mathrm{d}\big(e^{tV}\big)_x (N_x) = N_{e^{tV}x} \tag{9.13}$$

for all $t \in \mathbb{R}$ and $x \in M$ in the domain of definition of the flow e^{tV}.

Proposition 9.8. *Let (M, η) be a contact manifold. A smooth vector field V on M is a contact vector field if and only if it is a contact Hamiltonian vector field.*

Proof. From the condition (9.13) and the definition of the Lie derivative, we see that V is a contact vector field if and only if $\mathcal{L}_V \eta = 0$ on N.

First assume that V is a contact Hamiltonian vector field, and write $V = X_H$ for a Hamiltonian H. Then from Cartan's magic formula and the first condition (9.4) defining X_H we have

$$\mathcal{L}_{X_H}\eta = \mathrm{d}\left[\eta(X_H)\right] + X_H \lrcorner \mathrm{d}\eta = \mathrm{d}H + X_H \lrcorner \mathrm{d}\eta.$$

From the second condition of (9.4) we know that $X_H \lrcorner \mathrm{d}\eta$ is equal to $-\mathrm{d}H$ on N. By the definition (9.3) of the Reeb field it then follows that

$$\mathcal{L}_{X_H}\eta = -\xi(H)\eta = -\frac{\partial H}{\partial S}\eta, \tag{9.14}$$

where the last equality is merely the expression in terms of the local canonical coordinates. Comparing this to the definition (9.12) of a contact transformation, we see that the flow is a contact transformation with $f = -\xi(H) = -\partial H/\partial S$. In particular, if we restrict to the contact structure N, we have $\eta = 0$ and hence $\mathcal{L}_{X_H}\eta = 0$ as desired.

Conversely, assume that V is contact vector field. Then Cartan's magic formula reads

$$0 = (\mathcal{L}_V\eta)|_N = \mathrm{d}^{\eta(X_H)}|_N + (V \lrcorner \mathrm{d}\eta)|_N.$$

Consider the smooth function $H = \eta(V)$, defined so that the first condition of the definition (9.4) holds. Then we obtain the second condition of (9.4) from the above equality, and so we conclude that $V = X_H$ is the contact Hamiltonian vector field for H. □

Heuristically, we do not expect the volume form to be preserved by a general contact Hamiltonian flow since contact dynamics includes dissipative systems. Using (9.14) we see that the volume form evolves according to

$$\begin{aligned}\mathcal{L}_{X_H}[\eta \wedge (\mathrm{d}\eta)^n] &= (\mathcal{L}_{X_H}\eta) \wedge (\mathrm{d}\eta)^n + \sum_{i=0}^{n-1} \eta \wedge (\mathrm{d}\eta)^i \wedge [\mathrm{d}(\mathcal{L}_{X_H}\eta)] \wedge (\mathrm{d}\eta)^{n-1-i} \\ &= -(n+1)\frac{\partial H}{\partial S}\, \eta \wedge (\mathrm{d}\eta)^n.\end{aligned}$$

This illustrates the connection between the Hamiltonian's S-dependence to the system's dissipation, and consequently systems for which $\frac{\partial H}{\partial S}$ is nonvanishing are called **dissipative**.

Instead, we have a variant of Liouville's theorem due to [**?**], in which a rescaled volume form is preserved away from the zero set $H^{-1}(0)$.

Proposition 9.9 (Canonical measure for dissipative contact systems)**.** *Let (M, η) be a $(2n+1)$-dimensional contact manifold and H a smooth function on M. Then the volume form*

$$|H|^{-(n+1)}\, \eta \wedge (\mathrm{d}\eta)^n$$

is an invariant measure for the contact Hamiltonian flow for H along orbits outside of $H^{-1}(0)$. Moreover, up to scalar multiplication it is the unique such measure whose density with respect to the standard volume form depends only on H.

Proof. For a smooth function ρ on M, a computation using (9.14) shows that

$$\begin{aligned}\mathcal{L}_{X_H}[\rho\, \eta \wedge (\mathrm{d}\eta)^n] &= (\mathcal{L}_{X_H}\rho)\, \eta \wedge (\mathrm{d}\eta)^n - (n+1)\frac{\partial H}{\partial S}\rho\, \eta \wedge (\mathrm{d}\eta)^n \\ &= \left[X_H(\rho) - (n+1)\frac{\partial H}{\partial S}\rho\right]\, \eta \wedge (\mathrm{d}\eta)^n.\end{aligned}$$

If we assume $\rho = \rho(H)$ then

$$X_H(\rho) = -H\rho'(H)\frac{\partial H}{\partial S},$$

and so the vanishing of $\mathcal{L}_{X_H}[\rho\,\eta \wedge (\mathrm{d}\eta)^n]$ occurs exactly when ρ solves

$$\rho'(H) = -(n+1)H^{-1}\rho.$$

This equation has the solution $\rho(H) = |H|^{-(n+1)}$ and it is unique up to scalar multiplication. □

9.5 Time-dependent systems

Thus far we have allowed H to be a function on the contact manifold M, and hence have only considered time-independent dissipative systems. In this section, we present the extension introduced in [BCT17] of contact Hamiltonian systems to include time-dependence.

For (M, η) a $(2n+1)$-dimensional contact manifold, we define the **extended manifold** $M \times \mathbb{R}$. In analogy with the Poincaré–Cartan one-form (8.15), given a possibly time-dependent Hamiltonian H we extend the contact form to

$$\theta = \mathrm{d}S - p\,\mathrm{d}q + H\,\mathrm{d}t = \mathrm{d}S - \sum_{i=1}^{n} p^i\,\mathrm{d}q^i + H\,\mathrm{d}t \tag{9.15}$$

in terms of the canonical coordinates (q, p, S) on M guaranteed by Theorem 9.2.

On $M \times \mathbb{R}$ we define the **extended contact Hamiltonian vector field**

$$Y_H = X_H + \frac{\partial}{\partial t}.$$

In place of the conditions (9.4), it can be checked that this vector field is uniquely determined by

$$\theta(Y_H) = 0, \qquad Y_H \lrcorner \mathrm{d}\theta = -\frac{\partial H}{\partial S}\theta. \tag{9.16}$$

Here, the first condition is analogous to how (8.17) replaced (8.4) for time-dependent symplectic systems, and the second condition is the analog of (9.14) (which serves as a rephrasing for the second condition of (9.4) that does not involve N).

The flow of Y_H is given by

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{i=1}^{n} p^i\frac{\partial H}{\partial p^i} - H, \qquad \frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p^i}, \qquad \frac{\mathrm{d}p^i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} - p^i\frac{\partial H}{\partial S}, \qquad \dot{t} = 1,$$

which are the old equations of motion (9.6) joined with the trivial equation $\dot{t} = 1$. It follows that any smooth time-dependent function F on $M \times \mathbb{R}$ evolves according to

$$\frac{\mathrm{d}F}{\mathrm{d}t} = -H\frac{\partial F}{\partial S} + \sum_{i=1}^{n} p^i\{H, F\}_{p^i,S} + \{H, F\}_{p,q} + \frac{\partial F}{\partial t}, \tag{9.17}$$

using the notation of eq. (9.8). In particular, we see that under its own flow the Hamiltonian now changes according to both its dissipation $\partial H/\partial S$ and its time-dependence.

Lastly, let us extend the notion of contact transformations to our extended manifold $M\times\mathbb{R}$. In terms of canonical coordinates, a **time-dependent contact transformation** $(q,p,S,t)\mapsto(\tilde{q},\tilde{p},\tilde{S},\tilde{t})$ must satisfy

$$\tilde{\theta}=\mathrm{d}\tilde{S}-\tilde{p}\,\mathrm{d}\tilde{q}+K\,\mathrm{d}\tilde{t}=f(\mathrm{d}S-p\,\mathrm{d}q+H\,\mathrm{d}t)=f\theta \tag{9.18}$$

for a smooth nonvanishing factor $f\in C^\infty(M\times\mathbb{R})$ and a new Hamiltonian $K\in C^\infty(M\times\mathbb{R})$. Expanding $\mathrm{d}\tilde{q}$ and $\mathrm{d}\tilde{S}$, we see that the new Hamiltonian must satisfy

$$fH=\frac{\partial\tilde{S}}{\partial t}-\sum_{i=1}^{n}\tilde{p}^i\frac{\partial\tilde{q}^i}{\partial t}+K.$$

As before, we may consider a contact transformation as being generated by a generating function. For example, let us assume that the coordinates $(q,\tilde{q},S,t)$ are independent. After substituting the differential of the generating function $\tilde{S}=\tilde{S}(q,\tilde{q},S,t)$ into (9.18), we see that the remaining coordinates are determined in terms of $\tilde{S}$ by

$$f=\frac{\partial\tilde{S}}{\partial S},\qquad fp^i=-\frac{\partial\tilde{S}}{\partial q^i},\qquad \tilde{p}^i=\frac{\partial\tilde{S}}{\partial\tilde{q}^i},\qquad fH=\frac{\partial\tilde{S}}{\partial t}+K. \tag{9.19}$$

The first three conditions are unchanged and the last condition defines the new Hamiltonian $K=K(q,\tilde{q},S,t)$. Taking $f\equiv 1$ and comparing to (7.26), we see that $\tilde{S}$ is related to the canonical transformation generating function $F(q,\tilde{q},t)$ via

$$\tilde{S}=S-F(q,\tilde{q},t).$$

However, now there is an additional constraint on $\tilde{S}$ imposed by the invariance of the second condition of (9.16). After the transformation we must have

$$Y_H\lrcorner\mathrm{d}\theta=-\frac{\partial K}{\partial\tilde{S}}\tilde{\theta}.$$

Using $\tilde{\theta}=f\theta$, $\tilde{Y}_H=Y_H$, and the extended contact Hamiltonian vector field conditions (9.16), this yields

$$f\frac{\partial K}{\partial\tilde{S}}=f\frac{\partial H}{\partial S}+\mathrm{d}f(Y_H).$$

In the special case $f\equiv 1$ we note that if H is independent of S then $K=0$ is a solution, in which case the last condition of (9.19) becomes the familiar Hamilton–Jacobi equation (6.4). However, in general f may be S-dependent and so the notion of contact transformations is strictly more general than even the physicist's notion of canonical transformations (cf. the remark of section 7.5).

Example 9.10. In Example 8.11 we introduced a non-canonical coordinate transformation $(q,p) \mapsto (q, e^{\gamma t})$ to describe a dissipative system using time-dependent Hamiltonian dynamics. However, it is easily checked that the transformation

$$\tilde{q} = q, \qquad \tilde{p} = e^{\gamma t} p, \qquad \tilde{S} = e^{\gamma t} S, \qquad \tilde{t} = t$$

satisfies eq. (9.15) and is hence a time-dependent contact transformation. Here, the conformal factor is $f = e^{\gamma t}$, and the Hamiltonians are expressed in their respective coordinates: H is given by eq. (9.7) and K is given by eq. (8.19).

9.6 Exercises

9.1. Show that a smooth one-form η on a $(2n+1)$-dimensional manifold M satisfies the nondegeneracy condition of a contact form if and only if it satisfies the nonvanishing top form condition (9.1).

9.2 (Contact structure on S^{2n+1})**.** On the Euclidean space $\mathbb{R}^{2n+2}$ consider the coordinates $(x^1, \dots, x^{n+1}, y^1, \dots, y^{n+1})$ and define the one-form

$$\theta := \sum_{i=1}^{n+1} \left(x^i \, \mathrm{d}y^i - y^i \, \mathrm{d}x^i \right).$$

The **standard contact form** on the sphere S^{2n+1} is $\eta := \iota^* \theta$, where $\iota := : S^{2n+1} \hookrightarrow \mathbb{R}^{2n+2}$ is the inclusion map.

(a) Show that the vector fields

$$V = \sum_{i=1}^{n+1} \left(x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} \right), \qquad W = \sum_{i=1}^{n+1} \left(x^i \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \right)$$

satisfy $V \lrcorner \mathrm{d}\theta = 2\theta$ and $W \lrcorner \mathrm{d}\theta = -\,\mathrm{d}\big(x^2 + y^2\big)$.

(b) Let $S \subset T(\mathbb{R}^{2n+1} \setminus \{0\})$ denote the subbundle spanned by V and W, and let

$$S^\perp = \bigcup_{p \in S^{2n+1}} \big\{ X \in T_p \mathbb{R}^{2n+2} : \mathrm{d}\theta_p(V_p, X_p) = \mathrm{d}\theta_p(W_p, X_p) = 0 \big\}$$

denote its symplectic complement. Show that θ is indeed a contact form with respect to the contact structure $S^\perp$.

(c) Show that the corresponding Reeb field is given by W restricted S^{2n+1}.

9.3. (Solving the damped parametric oscillator via expanding coordinates) Consider the one-dimensional damped parametric oscillator

$$H = \frac{p^2}{2m} + \frac{m}{2} \omega^2(t) q^2 + \gamma S$$

with time-dependent frequency $\omega(t)$ and damping parameter γ. Show that the new **expanding coordinates**

$$\tilde{q} = e^{\gamma t/2} q, \qquad \tilde{p} = e^{\gamma t/2}(p + \tfrac{1}{2} m\gamma q), \qquad \tilde{S} = e^{\gamma t}(S + \tfrac{1}{4} m\gamma q^2), \qquad \tilde{t} = t$$

define a contact transformation, with respect to which the Hamiltonian takes the form

$$K = \frac{\tilde{p}^2}{2m} + \frac{m}{2}\left(\omega^2(t) - \tfrac{1}{4}\gamma^2\right)\tilde{q}^2.$$

This new Hamiltonian K now corresponds to an undamped parametric oscillator with the new frequency $\sqrt{\omega^2(t) - \frac{1}{4}\gamma^2}$. The undamped oscillator has been extensively studied and solutions for the equations of motion can be obtained. In the harmonic oscillator case $\omega(t) \equiv \omega_0$ the Hamiltonian K is a conserved quantity, and the coordinates expand exponentially in time so that the trajectories form closed orbits at a slower frequency.

9.4. (Solving the damped parametric oscillator via conserved quantities) Consider the one-dimensional damped parametric oscillator

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2(t) q^2 + \gamma S$$

with time-dependent frequency $\omega(t)$ and damping parameter γ.

(a) We seek a solution F to eq. (9.17) with vanishing left-hand side. Substituting the quadratic ansatz

$$F(q, p, S, t) := \beta(t) p^2 - 2\xi(t) qp + \eta(t) q^2 + \zeta(t) S,$$

obtain a system of first-order equations for β, η, ξ, and ζ, where the $\dot{\zeta}$ has solution $\zeta(t) = \zeta_0 e^{\gamma t}$.

(b) Using the substitution $\beta(t) := \frac{1}{2m} e^{\gamma t} \alpha^2(t)$, show that the β equation is solved if and only if α satisfies the Ermakov equation

$$\ddot{\alpha} + [\omega^2(t) - \tfrac{1}{4}\gamma^2]\alpha = \alpha^{-3},$$

and the remaining equations become

$$\eta(t) = \tfrac{1}{2} m e^{\gamma t}\{[\dot{\alpha} - \tfrac{1}{2}\gamma\alpha]^2 + \alpha^{-2}\}, \qquad \xi(t) = \tfrac{1}{2} e^{\gamma t}[\dot{\alpha} - \tfrac{1}{2}\gamma\alpha] + \tfrac{1}{4}.$$

(c) Conclude that the quantity $F(q, p, S, t) = I(q, p, t) + \zeta_0 G(q, p, S, t)$ is conserved, with

$$I = \frac{m}{2} e^{\gamma t}\left\{\left[\frac{p}{m}\alpha - \left(\dot{\alpha} - \frac{\gamma}{2}\alpha\right) q\right]^2 + \left(\frac{q}{\alpha}\right)^2\right\}, \qquad G = e^{\gamma t}(S - \tfrac{1}{2} qp),$$

and $\alpha(t)$ solves the Ermakov equation. Moreover, since F is invariant for all initial conditions and ζ_0 is determined solely by initial conditions, then I and G must be separately conserved.

(d) Show that the new coordinates

$$\tilde{q} = \arctan\left[(\dot{\alpha} - \tfrac{1}{2}\gamma\alpha)\alpha - \alpha^2 \frac{p}{mq}\right], \quad \tilde{p} = I(q,p,t), \quad \tilde{S} = G(q,p,S,t),$$

(and $\tilde{t} = t$) define a contact transformation, with respect to which the new Hamiltonian is simply $K = I\alpha^{-2}$. Solve the new equations of motion for $\tilde{q}, \tilde{p}, \tilde{S}$ and obtain the solution

$$q(t) = \sqrt{\frac{2I}{m}} e^{\gamma t}\alpha(t)\cos\phi(t), \qquad S(t) = Ge^{-\gamma t} + \tfrac{1}{2}q(t)p(t),$$
$$p(t) = \sqrt{2mI}e^{\gamma t}\left[\left(\dot{\alpha} - \frac{\gamma}{2}\alpha\right)\cos\phi(t) - \frac{1}{\alpha}\sin\phi(t)\right], \qquad \phi(t) = \int_{t_0}^{t} \frac{\mathrm{d}\tau}{\alpha^2(\tau)}.$$

Here, $\alpha(t)$ solves the Ermakov equation and the conserved quantities I and G are determined by the initial conditions.

Appendix A

Fundamentals of ODE theory

We collect some facts from introductory ODE theory in this chapter for easy reference. The material is based on [CL55].

A.1 Picard iteration

Throughout this chapter, we will study the **initial value problem** (IVP)

$$\dot{x} = f(t, x(t)), \qquad x(0) = x_0, \tag{A.1}$$

where $\dot{g} = \frac{\mathrm{d}g}{\mathrm{d}t}$ denotes a time derivative. The equation (A.1) describes the evolution of a point x, which we will take to lie in a Banach space $(X, |\cdot|)$. Although we will primarily be concerned with the case $X = \mathbb{R}^d$ in these notes, this level of generality is useful as it includes some systems with infinite degrees of freedom (e.g. an infinite chain of harmonic oscillators). However, this is not broad enough to really include PDE, except for some boring examples.

Our first step will be to recast the IVP (A.1) as an integral equation:

Lemma A.1. *If $f : \mathbb{R} \times X \to X$ is continuous, then the following are equivalent:*

(a) (Classical solution) $x : (-T, T) \to X$ is C^1 and solves the IVP (A.1)*.*

(b) (Strong solution) $x : (-T, T) \to X$ is C^0 and solves the integral equation

$$x(t) = x(0) + \int_0^t f(s, x(s))\,\mathrm{d}s. \tag{A.2}$$

Proof. Both directions easily follow from the fundamental theorem of calculus.

First assume that (a) holds. Then both sides of the IVP (A.1) are continuous, and so integration yields (A.2) by the fundamental theorem of calculus.

Now assume that (b) holds. Then $t \mapsto f(t, x(t))$ is continuous, and so by the fundamental theorem of calculus the integral equation (A.2) says that x is differentiable with derivative $f(t, x(t))$. □

In the case $X = \mathbb{R}^d$, we can give a more general measure-theoretic version of Lemma A.1 via the following statement of the fundamental theorem of calculus: given $g : (-T, T) \to \mathbb{R}^d$ in L^1, the function $x : (-T, T) \to \mathbb{R}^d$ is absolutely continuous and solves $\dot{x}(t) = g(t)$ almost everywhere iff $x(t) = x(0) + \int_0^t g(s)\,\mathrm{d}s$. The theory of absolute continuity for Banach-valued functions exists, but is quite involved.

Lemma A.1 does not apply to many PDE, not even to the transport equation $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$, because spatial differentiation $u \mapsto \frac{\partial u}{\partial x}$ is not continuous in most Banach spaces. (There are some exceptions to this though, like the space of holomorphic functions on a strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < c\}$ containing the real axis.) Moreover, the conclusion of Lemma A.1 does not hold, and (a) and (b) yield distinct notions of solutions. Another common notion is that of weak solutions: an L^∞ function $x : (-T, T) \to X$ that solves

$$\int_{-T}^{T} x(t)\psi(t)\,\mathrm{d}t = x(0)\int_{-T}^{T} \psi(t)\,\mathrm{d}t + \int_{-T}^{T} \psi(t)\int_0^t f(s, x(s))\,\mathrm{d}s\,\mathrm{d}t$$

for all $\psi \in C_0^\infty([-T, T])$. For ODEs, it turns out that this is also equivalent provided that f is continuous, but we will not need this fact.

In trying to argue that solutions to the IVP (A.1) exist, the formulation (b) is better than that of (a). This is because integrals are stable while derivatives are highly unstable. For example, consider the Fourier series $g(t) = \sum_{k \neq 0} c_k e^{2\pi i k t}$. If the coefficients c_k are absolutely summable, then this defines a periodic function $g(t)$. Roughly speaking, the rate of decay of c_k corresponds to the smoothness of $g(t)$, because the character $e^{2\pi i k t}$ is rapidly oscillating for large frequencies k. Integration suppresses high frequencies, because it replaces the coefficients of $g(t)$ by more rapidly decaying sequence $\frac{c_k}{2\pi i k}$. Conversely, differentiation amplifies high frequencies, because it replaces the coefficients of $g(t)$ by more slowly decaying sequence $2\pi i k c_k$.

In addition to differential equations, the theory of integral equations also exists. The integral equation (A.2) is special in that the integration is over $[0, t]$, and such equations are said to be of Volterra type (in analogy with triangular matrices). The more general case

$$x(t) = b(t) + \int_0^1 K(t, s)x(s)\,\mathrm{d}s$$

is called a Fredholm integral equation, of which the linear case is shown above.

The following theorem is a fundamental result on the existence of solutions:

Theorem A.2 (Picard–Lindelöf)**.** *If $f : \mathbb{R} \times X \to X$ is continuous and is Lipschitz in x:*

$$|f(t, x) - f(t, y)| \leq C|x - y| \quad \textit{for all } x, y \in X \tag{A.3}$$

for some constant $C > 0$, then for each $x_0 \in X$ there exists a unique continuous function $x : \mathbb{R} \to X$ so that the integral equation (A.2) *holds. Moreover, $x \in C([-T, T] \to X)$ depends continuously upon $x_0 \in X$ for all $T > 0$.*

Theorem A.2 says that the ODE (A.1) is (globally-in-time) **well-posed** (in the sense of Hadamard): solutions exist, solutions are unique, and solutions depend continuously upon the initial data. The statement of continuous dependence can take various forms; for example, it follows that given a convergent sequence of initial data, the corresponding sequence of solutions converges uniformly on compact time intervals. Note that continuous dependence upon initial data is not extremely restrictive; indeed, chaotic systems, where trajectories exhibit complex geometric structure and are sensitive to small changes in initial conditions, can still be well-posed.

We will only present one proof of Theorem A.2, but there are multiple arguments that apply. Having multiple methods is particularly useful in the study of PDEs, because different ODE proofs yield distinct PDE statements.

Proof. We will argue by Picard iteration. Recursively define the sequence

$$x_0(t) \equiv x_0, \qquad x_{n+1}(t) = x_0 + \int_0^t f(s, x_n(s))\,\mathrm{d}s$$

of successive approximate solutions. Ultimately we will show $\{x_n\}$ is a Cauchy sequence, and we will take its limit to be our solution.

We want to show that the difference between successive approximations is shrinking. Using the Lipschitz condition (A.3), we estimate

$$\begin{aligned} |x_{n+1}(t) - x_n(t)| &\le \int_0^t |f(s, x_n(s)) - f(s, x_{n-1}(s))|\,\mathrm{d}s \\ &\le C \int_0^t |x_n(s_n) - x_{n-1}(s_n)|\,\mathrm{d}s_n. \end{aligned}$$

We now have the difference between the previous two approximations on the RHS. Applying this estimate iteratively to the RHS, we obtain

$$\begin{aligned} &\le C^2 \int_0^t \int_0^{s_n} |x_{n-1}(s_{n-1}) - x_{n-2}(s_{n-1})|\,\mathrm{d}s_{n-1}\,\mathrm{d}s_n \\ &\vdots \\ &\le C^n \int_{0<s_1<s_2<\cdots<s_n<t} |x_1(s_1) - x_0(s_1)|\,\mathrm{d}s_1\,\mathrm{d}s_2 \ldots \mathrm{d}s_n. \end{aligned}$$

For the first two functions in our sequence, we can bound

$$|x_1(s_1) - x_0(s_1)| = \left| \int_0^{s_1} f(s_0, x_0)\,\mathrm{d}s_0 \right| \le s_1 \sup_{s \in (0,t)} |f(s, x_0)|.$$

Together, this yields

$$|x_{n+1}(t) - x_n(t)| \le \sup_{s \in (0,t)} |f(s, x_0)|\ \frac{C^n t^{n+1}}{(n+1)!}.$$

The RHS is summable in n, and so we conclude that $\{x_n(t)\}$ is a Cauchy sequence in $C([-T,T] \to X)$.

Define x to be the limit of x_n in $C([-T,T] \to X)$, which exists because X is complete. As x_n converges to x in this space, we have

$$\sup_{t\in[-T,T]} |x_n(t) - x(t)| \to 0 \quad \text{as } n \to \infty$$

for all $T > 0$. To see that $x(t)$ solves the IVP (A.1), we take $n \to \infty$ in the definition

$$x_{n+1}(t) = x_0 + \int_0^t f(s, x_n(s))\,\mathrm{d}s.$$

We have $f(s, x_n(s)) \to f(s, x(s))$ by continuity, so the integrals converge, and hence we obtain

$$x(t) = x_0 + \int_0^t f(s, x(s))\,\mathrm{d}s.$$

Therefore $x(t)$ is a solution to the integral equation (A.2), and consequently the IVP (A.1) by Lemma A.1.

Next, we claim that the solution is unique. Suppose $x(t)$ and $\tilde{x}(t)$ are both solutions to the IVP (A.1) that are in $C([-T,T] \to X)$. Arguing as before, we estimate

$$\begin{aligned}
|x(t) - \tilde{x}(t)| &\le \int_0^t |f(s, x(s)) - f(s, \tilde{x}(s))|\,\mathrm{d}s \\
&\le C \int_0^t |x(s_n) - \tilde{x}(s_n)|\,\mathrm{d}s_n \\
&\le C^2 \int_0^t \int_0^{s_n} |x(s_{n-1}) - \tilde{x}(s_{n-1})|\,\mathrm{d}s_{n-1}\,\mathrm{d}s_n \\
&\vdots \\
&\le C^n \int_{0<s_1<s_2<\cdots<s_n<t} |x(s_1) - \tilde{x}(s_1)|\,\mathrm{d}s_1\,\mathrm{d}s_2 \ldots \mathrm{d}s_n \\
&\le \frac{C^n}{n!} \sup_{s\in(-T,T)} |x(s) - \tilde{x}(s)|.
\end{aligned}$$

Taking a supremum over $t \in (-T,T)$, we obtain

$$\sup_{t\in(-T,T)} |x(t) - \tilde{x}(t)| \le \frac{C^n}{n!} \sup_{s\in(-T,T)} |x(s) - \tilde{x}(s)|$$

for all n. As the LHS and RHS are finite since x and $\tilde{x}$ are in $C([-T,T] \to X)$, we send $n \to \infty$ to conclude that $x(t) \equiv \tilde{x}(t)$.

Lastly, we claim that the solution depends continuously upon the initial data. Suppose $x(t)$ and $\tilde{x}(t)$ are both solutions to (A.1) in $C([-T,T] \to X)$

with different initial data. Arguing as before, we estimate

$$\begin{aligned}|x(t)-\tilde{x}(t)| &\leq |x(0)-\tilde{x}(0)| + C\int_0^t |x(s_{n-1})-\tilde{x}(s_{n-1})|\,\mathrm{d}s_{n-1}\\ &\vdots\\ &\leq |x(0)-\tilde{x}(0)|\left[1+Ct+\frac{C^2t^2}{2!}+\cdots+\frac{C^nt^n}{n!}\right]\\ &\quad+\frac{C^{n+1}t^{n+1}}{(n+1)!}\sup_{s\in(-T,T)}|x(s)-\tilde{x}(s)|.\end{aligned}$$

The supremum is finite and independent of n since x and $\tilde{x}$ are in $C([-T,T]\to X)$. Therefore the last term on the RHS converges to zero, and so sending $n\to\infty$ yields

$$|x(t)-\tilde{x}(t)| \leq |x(0)-\tilde{x}(0)|e^{C|t|}.$$

Taking a supremum over $t\in[-T,T]$, we see that the map $x_0\mapsto x(t)$ is Lipschitz continuous on any bounded time interval. □

Mimicking the proof of continuous dependence, we can also prove the following useful fact:

Lemma A.3 (Grönwall's inequality)**.** *If $f:[0,T]\to[0,\infty)$ is continuous, $a:[0,T]\to[0,\infty)$ is L^1, and*

$$f(t)\leq A+\int_0^t a(s)f(s)\,\mathrm{d}s,$$

then

$$f(t)\leq A\exp\left\{\int_0^t a(s)\,\mathrm{d}s\right\}.$$

Many authors choose to prove Lemma A.3 first, and then cite it in the proof of Theorem A.2.

Notice that the proof actually shows that the **data-to-solution map** $x_0\mapsto x(t)$ is a Lipschitz function from X into $C([-T,T]\to X)$, and that the Lipschitz constant is bounded by e^{CT}. In fact, it follows that for fixed t the map $x_0\mapsto x(t)$ is a bi-Lipschitz homeomorphism, because we can reconstruct $x(0)$ from $x(t)$ by solving the ODE backwards in time and citing uniqueness. The Lipschitz continuity here matches the Lipschitz continuity of f, and in general we cannot do better.

Another common proof of Theorem A.2 is based on contraction mapping. However, this only proves the result for $T>0$ sufficiently small. The full statement of Theorem A.2 requires our iteration argument.

Next, we would like to extend our existence result to include equations where f is not globally Lipschitz, but instead is smooth. Note that f being smooth does not imply that there are global solutions:

Example A.4. The equation

$$\dot{x} = x^2, \qquad x(0) = 1$$

has solution

$$x(t) = \frac{1}{1-t},$$

which blows up at $t = 1$.

Smoothness does guarantee local solutions however:

Theorem A.5. *If $f : \mathbb{R} \times X \to X$ is continuous and is C^1 in x, then given $x_0 \in X$ there exists $T > 0$ and a unique solution $x : (-T, T) \to X$ in $C([-T, T] \to X)$ to the IVP* (A.1) *that depends continuously upon the initial data.*

Proof. As f' is continuous, there exists $\delta > 0$ and $A > 0$ such that $|t|+|x-x_0| < \delta$ implies $\|f'(t, x)\| \leq A$. In particular, f is Lipschitz on the set $|t|+|x-x_0| < \delta$.

Let $\psi : [0, \infty) \to \mathbb{R}$ be a smooth cutoff function so that $\psi(r) \equiv 1$ for $r \in [0, \frac{\delta}{2}]$ and $\psi(r) \equiv 0$ for $r \geq \delta$. We can now apply the Picard–Lindelöf theorem (Theorem A.2) to

$$\dot{x} = f(t, x)\psi(t)\psi(|x - x_0|), \qquad x(0) = x_0.$$

Lastly, we choose $T > 0$ small enough to stop the solution from noticing the change in the RHS. □

Lastly, we note that solutions $x(t)$ should always be defined on open intervals, since given one defined on a closed interval we can always extend it a bit further.

Corollary A.6 (Blowup criterion)**.** *Suppose $f : \mathbb{R} \times X \to X$ is continuous and is C^1 in x, and fix $x_0 \in X$. Then there exists a maximal interval of existence (T_-, T_+) for some $-\infty \leq T_- < 0 < T_+ \leq \infty$ and a unique solution $x : (-T_-, T_+) \to X$ to the IVP* (A.1)*. Moreover, if T_+ is finite then $|x(t)| \to \infty$ as $t \uparrow T_+$, and similarly for T_-.*

Proof. We define the maximal interval of existence to be the union of all open intervals containing t_0 on which a solution $x(t)$ exists. By Theorem A.5, this is an open and connected set and hence is indeed an interval (T_-, T_+). We may then glue all of these solutions together by uniqueness to obtain a solution $u : (T_-, T_+) \to X$.

Suppose for a contradiction that $T_+ < +\infty$ and $|x(t)| \not\to \infty$ as $t \uparrow T_+$. Then there exists an increasing sequence t_n converging to T_+ on which $|x(t)|$ is bounded. Together with $t = T_+$, this is a bounded set on which f' is continuous and hence bounded. Arguing as in Theorem A.5, we may apply Theorem A.2 to construct a solution defined for a short time after $t = T_+$—but this contradicts the maximality of T_+. □

A.2 Alternative approaches to well-posedness

In this section, we display some important arguments which should be included in a study of ODEs. However, they are not strictly necessary for these notes, and so this section can be skipped if desired.

Existence. In the Picard–Lindelöf theorem (Theorem A.2), the data-to-solution map is automatically Lipschitz. This is convenient for nice problems, but it limits the applicability to ODEs where this is true. The following is an alternative method for proving existence based on compactness, and hence only works for $X = \mathbb{R}^d$.

Theorem A.7 (Cauchy–Peano)**.** *Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ continuous. Then for every $x_0 \in \mathbb{R}^d$ the IVP* (A.1) *has at least one solution $x : (-T, T) \to \mathbb{R}^d$.*

Proof. We may assume $f(t, x)$ is continuous and bounded, after replacing f by a truncation as in the proof of Theorem A.5. Let $\phi(t, x)$ be a smooth nonnegative function with $\int \phi = 1$, and define $\phi_\epsilon(t, x) := \epsilon^{-d-1}\phi(\frac{t}{\epsilon}, \frac{x}{\epsilon})$. We replace f by the convolution

$$f_\epsilon(t, x) := \iint \phi_\epsilon(s, y) f(t - s, x - y) \, \mathrm{d}y \, \mathrm{d}s.$$

Then f_ϵ is bounded pointwise independently of ϵ, and is Lipschitz for fixed ϵ.

By the Picard–Lindelöf theorem (Theorem A.2), there exists a unique solution $x_\epsilon(t)$ to the modified equation

$$\dot{x} = f_\epsilon(t, x), \qquad x(0) = x_0.$$

Notice that

$$x_\epsilon(t) = x_0 + \int_0^t f_\epsilon(s, x_\epsilon(s)) \, \mathrm{d}s \tag{A.4}$$

is Lipschitz uniformly in ϵ, since

$$|x_\epsilon(t) - x_\epsilon(s)| \le \int_s^t |f_\epsilon(\tau, x_\epsilon(\tau))| \, \mathrm{d}\tau \le |t - s| \|f\|_{L^\infty}.$$

Therefore $x_\epsilon : [-1, 1] \to \mathbb{R}^d$ for $\epsilon \in (0, 1]$ form a bounded (take $s = 0$ above) and equicontinuous set of functions. By the Arzelà–Ascoli theorem, there exists a sequence $\epsilon_n \to 0$ such that $x_{\epsilon_n}(t)$ converges uniformly on $[-1, 1]$ to some $x(t)$. (This step requires Euclidean space $\mathbb{R}^d$ instead of a general Banach space X, but still works if the image of f is precompact.) Sending $\epsilon_n \to 0$ in the integral equation (A.4), we get

$$x(t) = x_0 + \int_0^t f(s, x(s)) \, \mathrm{d}s.$$

Therefore $x(t)$ solves the IVP by Lemma A.1. We then pick $T > 0$ small so that $x(t)$ solves the original IVP. □

Uniqueness. Note that the statement of Theorem A.7 provides existence, but not uniqueness. This is good for problems where solutions are not unique. For example, the IVP

$$\dot{x} = 2\sqrt{|x|}, \qquad x(0) = 0$$

has solutions $x(t) \equiv 0$ and $x(t) = t^2$.

However, for all other IVPs we would like to prove that solutions are unique. We will now present various arguments for uniqueness.

Our first tool is Grönwall's inequality:

Proposition A.8 (Uniqueness by Grönwall)**.** *Suppose $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz in x. Then for every $x_0 \in X$ the IVP* (A.1) *has at most one solution.*

Proof. Suppose $x(t)$ and $\tilde{x}(t)$ both solve the IVP (A.1). As f is Lipschitz,

$$|x(t) - \tilde{x}(t)| \leq \int_0^t C|x(s) - \tilde{x}(s)| \, \mathrm{d}s,$$

and so by Grönwall's inequality (Lemma A.3) we have $x(t) \equiv \tilde{x}(t)$. □

Second, we have a monotonicity argument:

Proposition A.9 (Uniqueness by monotonicity)**.** *Suppose $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies*

$$(x - y) \cdot [f(t, x) - f(t, y)] \leq 0. \tag{A.5}$$

Then for every $x_0 \in X$ the IVP (A.1) *has at most one solution.*

When $d = 1$, the condition (A.5) says that f is decreasing.

Proof. Given two solutions $x(t)$ and $\tilde{x}(t)$ with the same initial data, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - \tilde{x}(t)]^2 = 2[x(t) - \tilde{x}(t)] \cdot [f(t, x(t)) - f(t, \tilde{x}(t))] \leq 0.$$

So by Grönwall's inequality (Lemma A.3) we have $x(t) \equiv \tilde{x}(t)$. □

Note that the proof still applies even if we only have

$$(x - y) \cdot [f(t, x) - f(t, y)] \leq C|x - y|^2.$$

Third, we have a barrier argument:

Proposition A.10 (Uniqueness by barrier)**.** *Suppose $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies*

$$|f(t, x) - f(t, y)| \leq |x - y| \log \tfrac{1}{|x-y|} \quad \text{for } |x - y| \leq \tfrac{1}{2}. \tag{A.6}$$

Then for every $x_0 \in X$ the IVP (A.1) *has at most one solution.*

Note that the assumption (A.6) on f is slightly weaker than Lipschitz continuity.

Proof. Fix two solutions $x(t)$ and $\tilde{x}(t)$ with the same initial data. We may assume that $|x(t) - \tilde{x}(t)| \leq \frac{1}{2}$ by first restricting our attention to sufficiently small time intervals, and then patching these intervals together. This allows us to estimate

$$|x(t) - \tilde{x}(t)| \leq \int_0^t |x(s) - \tilde{x}(s)| \log \tfrac{1}{|x(s)-\tilde{x}(s)|} \, \mathrm{d}s.$$

Fix $A > 0$, and define the "barrier" $b(t) = \exp\{-Ae^{-t}\}$, which solves the equation

$$\dot{b} = b \log \tfrac{1}{b}, \qquad b(0) = e^{-A}. \tag{A.7}$$

We claim that $|x(t) - \tilde{x}(t)| \leq b(t)$. Suppose this is false. It is true at $t = 0$, and so by continuity there exists a minimal time $t_0 > 0$ where it fails, i.e.

$$|x(t) - \tilde{x}(t)| < b(t) \quad \text{for all } t \in [0, t_0), \qquad |x(t_0) - \tilde{x}(t_0)| = b(t_0).$$

Then by continuity,

$$|x(t_0) - \tilde{x}(t_0)| \leq \int_0^{t_0} |x(s) - \tilde{x}(s)| \log \tfrac{1}{|x(s)-\tilde{x}(s)|} \, \mathrm{d}s < \int_0^{t_0} b \log \tfrac{1}{b} \, \mathrm{d}s = b(t_0),$$

which contracts the choice of t_0.

Together, we have shown that for any $A > 0$ we have $|x(t) - \tilde{x}(t)| \leq b(t)$. Sending $A \to \infty$, we conclude that $|x(t) - \tilde{x}(t)| \leq 0$ and hence $x(t) \equiv \tilde{x}(t)$. □

The assumption (A.6) we must place on f is dictated by the differential equation (A.7) for the barrier b. All we need to make this argument work though is that the differential equation $\dot{b} = g(b)$ satisfies $\int_0^\epsilon \frac{1}{g(b)} \, \mathrm{d}b = \infty$.

Fourth, we have Carleman's approach:

Proposition A.11 (Uniqueness by Carleman estimate)**.** *Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and*

$$|f(t, x) - f(t, y)| \leq C|x - y| \quad \textit{for all } x, y \in \mathbb{R}.$$

Then for every $x_0 \in X$ the IVP (A.1) *has at most one solution.*

We begin with an inequality:

Lemma A.12 (A Carleman estimate)**.** *If $w : \mathbb{R} \to \mathbb{R}$ is C^1 and compactly supported, then*

$$\int e^{2\lambda t} \dot{w}^2 \, dt \geq \int \lambda^2 e^{2\lambda t} w^2 \, dt \quad \textit{for all } \lambda \in \mathbb{R}. \tag{A.8}$$

Proof. We write

$$\begin{aligned} \int e^{2\lambda t} \dot{w}^2 \, \mathrm{d}t &= \int \left[\frac{\mathrm{d}}{\mathrm{d}t}(e^{\lambda t} w) - \lambda e^{\lambda t} w \right]^2 \mathrm{d}t \\ &= \int \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}(e^{\lambda t} w) \right)^2 - 2\lambda (e^{\lambda t} w) \frac{\mathrm{d}}{\mathrm{d}t}(e^{\lambda t} w) + \lambda^2 e^{2\lambda t} w^2 \right] \mathrm{d}t. \end{aligned}$$

The first term on the RHS is nonnegative, and so we can drop it to obtain an inequality. The second term is equal to $-\lambda\frac{\mathrm{d}}{\mathrm{d}t}[(e^{\lambda t}w)^2]$, which integrates to zero since w has compact support. The inequality (A.8) follows. □

Proof of Proposition A.11. Let $x(t)$ and $\tilde{x}(t)$ be two solutions to the IVP (A.1) with the same initial data. We may assume that x and $\tilde{x}$ disagree in the future after substituting $t \mapsto -t$ if necessary. We may also modify $\tilde{x}$ so that $\tilde{x}(t) \equiv x(t)$ for all $t \leq 0$.

Fix $\delta > 0$, and let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function so that $\chi(t) \equiv 1$ for $t \leq \delta$ and $\chi(t) \equiv 0$ for $t \geq 2\delta$. Then the weight $w(t) = \chi(t)[\tilde{x}(t) - x(t)]$ is C^1 and compactly supported, and so the Carleman estimate (A.8) yields

$$\begin{aligned}
&\lambda^2 \int e^{2\lambda t}\chi(t)^2[\tilde{x}(t) - x(t)]^2 \,\mathrm{d}t \\
&\leq \int e^{2\lambda t}\big[\dot{\chi}(\tilde{x} - x) + \chi(f(t,\tilde{x}(t)) - f(t,x(t)))\big]^2 \,\mathrm{d}t \\
&\leq C^2 \int e^{2\lambda t}\chi(t)^2|\tilde{x}(t) - x(t)|^2 \,\mathrm{d}t \\
&\quad + \int e^{2\lambda t}\dot{\chi}(t)^2|\tilde{x}(t) - x(t)|^2 \,\mathrm{d}t + 2C \int e^{2\lambda t}|\dot{\chi}(t)\chi(t)||\tilde{x}(t) - x(t)|^2 \,\mathrm{d}t.
\end{aligned}$$

Let $A = \sup\{|\dot{\chi}(t)| : t \in \mathbb{R}\}$, and note that $A \geq c\delta^{-1}$ for some $c > 0$ since χ decreases by 1 over an interval of length δ. Then

$$\begin{aligned}
&\lambda^2 \int e^{2\lambda t}\chi(t)^2[\tilde{x}(t) - x(t)]^2 \,\mathrm{d}t \\
&\leq C^2 \int e^{2\lambda t}\chi(t)^2|\tilde{x}(t) - x(t)|^2 \,\mathrm{d}t + (A^2 + 2AC)\int_\delta^{2\delta} e^{2\lambda t}2(|\tilde{x}|^2 + |x|^2)\,\mathrm{d}t.
\end{aligned}$$

The second term on the RHS is bounded by a constant times $e^{2\delta\lambda}$. Therefore, for all $\lambda \leq -1$ sufficiently large we have

$$\frac{\lambda^2}{4}\int e^{2\lambda t}\chi(t)^2[\tilde{x}(t) - x(t)]^2 \,\mathrm{d}t \leq C'e^{-2\delta|\lambda|}$$

for some constant C'. This implies that the LHS is zero. Indeed, if $|\tilde{x}(t) - x(t)| \not\equiv 0$ on $(0,\delta)$, then there is a time $t_0 \in (0,\delta)$ such that

$$\frac{\lambda^2}{4}\int e^{2\lambda t}\chi(t)^2[\tilde{x}(t) - x(t)]^2 \,\mathrm{d}t \geq c\lambda^2 e^{-2t_0\lambda}$$

for some $c > 0$, and the RHS cannot bounded by $e^{-2\delta|\lambda|}$ for all $\lambda \leq -1$ large.

As $\delta > 0$ was arbitrary, we conclude that $x(t) \equiv \tilde{x}(t)$. □

In other applications, the parameter δ often needs to be fixed small for some other reason. To accommodate this, we can pick $t = 0$ to be the first time after which $x(t)$ and $\tilde{x}(t)$ disagree.

One application of Carleman's argument is Laplace's equation $\Delta u = 0$ in $\mathbb{R}^d$. For $d = 2$, uniqueness can be easily proved by methods of complex analysis. These methods do not carry over at all to $d \geq 3$ however, but Carleman's argument does.

Continuous dependence. Once we have proved existence and uniqueness, we can sometimes recover continuous dependence via a compactness argument.

Proposition A.13. *Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous, and suppose all solutions to the IVP* (A.1) *exist, are bounded, and are unique on some interval $[0, T]$. Then the IVP* (A.1) *is well-posed.*

Proof. It only remains to show continuous dependence. Consider a convergent sequence of initial data $\xi_n \to \xi$. We want to show that the corresponding solutions $x_n(t) := x(t; \xi_n)$ converge uniformly to $x(t; \xi)$.

Observe that the sequence $\{x_n(t)\}$ is equicontinuous on $[0, T]$, since

$$|\dot{x}_n(t)| \leq |f(t, x_n(t))|;$$

the input $x_n(t)$ is bounded uniformly in $n \in \mathbb{N}$ and $t \in [0, T]$ and f is continuous, and so the RHS is also uniformly bounded. Fix an arbitrary subsequence of $\{x_n(t)\}$. By the Arzelà–Ascoli theorem, there exists a further subsequence which converges uniformly. The uniform convergence implies that the limit $y(t)$ will solve the integral equation

$$y(t) = \xi + \int_0^t f(s, y(s)) \, \mathrm{d}s,$$

and hence will solve the IVP (A.1) with initial data ξ by Lemma A.1. Solutions are unique by premise, and so the limit function must be $x(t; \xi)$.

As the initial subsequence was arbitrary, we conclude that the entire sequence $x_n(t)$ converges to $x(t; \xi)$. □

A.3 Smooth dependence upon initial data

If f is C^1 in x, then the following proposition shows that the data-to-solution map is also C^1. This will be useful because we will want to be able to compute volumes in phase space.

Proposition A.14. *Let $f : \mathbb{R} \times X \to X$ be continuous, and be Lipschitz and C^1 in x, and let $x(t; x_0)$ denote the unique solution to the IVP* (A.1). *Then $x(t; x_0)$ is a C^1 function of x_0, and the derivative $V(t; x_0) = \partial_{x_0} x(t)$ is the unique solution to*

$$\dot{V}(t) = f'(t, x(t))V(t), \qquad V(0) = Id_X.$$

Proof. The Picard–Lindelöf theorem (Theorem A.2) applies to V equation, because the RHS is continuous in times and linear in V.

Let $E_h(t)$ denote the quantity

$$\sup_{|\eta|\le h}\left|\frac{f(t,x(t;x_0+\eta))-f(t,x(t;x_0))}{h}-f'(t,x(t;x_0))\frac{x(t;x_0+\eta)-x(t;x_0)}{h}\right|.$$

Observe that $|E_h(t)|\lesssim 1$ uniformly on $[0,T]$, because f and $x_0\mapsto x(t;x_0)$ are both Lipschitz. We also know that $E_h(t)\to 0$ as $h\downarrow 0$ for each t, because $|x(t;x_0+\eta)-x(t;\eta)|\lesssim e^{C|t|}|\eta|$ and f is differentiable at x_0.

Now, for $\eta\neq 0$ we have

$$\begin{aligned}&\left|x(t;x_0+\eta)-x(t;x_0)-\left(\eta+\int_0^t f'(s,x(s))[x(t;x_0+\eta)-x(t;x_0)]\,\mathrm{d}s\right)\right|\\&\le|\eta|\int_0^T E_{|\eta|}(s)\,\mathrm{d}s.\end{aligned}$$

This is the integral equation for

$$V(t)\eta=\eta+\int_0^t f'(s,x(s))V(s)\eta\,\mathrm{d}s.$$

Thus

$$\begin{aligned}&|x(t;x_0+\eta)-x(t;x_0)-V(t)\eta|\\&\le|\eta|\int_0^T E_{|\eta|}(s)\,\mathrm{d}s+\int_0^T\|f'(s,x(s))\|_{\mathrm{op}}|x(s;x_0+\eta)-x(s;x_0)-V(s)\eta|\,\mathrm{d}s.\end{aligned}$$

Note that $\|f'(s,x(s))\|_{\mathrm{op}}$ is bounded uniformly for $s\in[0,T]$. So by Grönwall's inequality (Lemma A.3),

$$|x(t;x_0+\eta)-x(t;x_0)-V(t)\eta|\le|\eta|e^{\sup_s\|f'\|_{\mathrm{op}}}\int_0^T E_h(s)\,\mathrm{d}s.$$

Taking a supremum in η, we obtain

$$\sup_{|\eta|\le h}|x(t;x_0+\eta)-x(t;x_0)-V(t)\eta|\le he^{\sup_s\|f'\|_{\mathrm{op}}}\int_0^T E_h(s)\,\mathrm{d}s.$$

The integral on the RHS converges to zero as $h\downarrow 0$ by the dominated convergence theorem. (For infinite dimensional X, these are Riemann integrals, and so we replace the dominated convergence theorem by the Arzelà convergence theorem—a fact surprisingly more difficult to prove within the context of Riemann integration.) Therefore $\frac{\partial x(t)}{\partial x_0}$ exists and is equal to $V(t)$.

It only remains to show that $x_0 \mapsto V(t; x_0)$ is continuous. We estimate

$$\begin{aligned}
&\|V(t, x_0 + \eta) - V(t; x_0)\| \\
&\leq \int_0^t \|f'(s, x(s; x_0 + \eta))V(s; x_0 + \eta) - f'(s, x(s; x_0))V(s; x_0)\| \, \mathrm{d}s \\
&\leq \int_0^t \|f'(s, x(s))\|_{\mathrm{op}} \|V(s; x_0 + \eta) - V(s; x_0)\| \, \mathrm{d}s \\
&\quad + \int_0^T \|f'(s, x(s; x_0 + \eta)) - f'(s, x(s; x_0))\| \|V(s; x_0 + \eta)\| \, \mathrm{d}s.
\end{aligned}$$

For the second integral on the RHS, we have

$$\|f'(s, x(s; x_0 + \eta)) - f'(s, x(s; x_0))\| \to 0$$

for each s, since

$$|x(s; x_0 + \eta) - x(s; x_0)| \leq |\eta| \exp\left\{\|f\|_{\mathrm{Lip}} T\right\}.$$

Also, the factor $\|V(s; x_0 + \eta)\|$ is bounded, since by the differential equation we have

$$\|V(s; x_0 + \eta)\| \leq 1 \cdot \exp\left\{\|f\|_{\mathrm{Lip}} T\right\}.$$

Therefore, an application of Grönwall's inequality (Lemma A.3) finishes the proof. □

We can extend this to include both higher degrees of regularity and dependence on parameters:

Corollary A.15. *If $f : \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^k_\mu \to \mathbb{R}^d$ is C^r in (x, μ) and Lipschitz in x, then the solution to*

$$\dot{x} = f(t, x(t), \mu), \qquad x(0) = \xi \tag{A.9}$$

is C^r in (ξ, μ).

Proof. We iterate the previous theorem. For example, consider the augmented system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x(t) \\ \frac{\partial x}{\partial \xi}(t) \\ \mu(t) \end{pmatrix} = \begin{pmatrix} f(t, x(t), \mu) \\ \frac{\partial f}{\partial x}(t, x(t), \mu) \frac{\partial x}{\partial \xi}(t) \\ 0 \end{pmatrix} \tag{A.10}$$

with initial data (ξ, Id, μ). Assume that f is C^2 in x and μ. Then the RHS obeys the hypotheses of Proposition A.14, and so we conclude that $x(t)$, $\frac{\partial x}{\partial \xi}(t)$, and $\mu(t)$ are C^1 functions of ξ and μ. □

The augmented system (A.10) is not only useful in proofs, but is also commonly used in numerical integration of the system (A.9) with parameter μ. The time t can also be included in the augmented system, but this yields a weaker smoothness result.

A.4 Vector fields and flows

In differential geometry, vector fields are conflated with first-order differential operators:

$$X(x) = (X^1(x), \dots, X^d(x)) \qquad \longleftrightarrow \qquad X \cdot \nabla = \sum_{i=1}^{d} X^i(x) \frac{\partial}{\partial x^i},$$

where $x = (x^1, \dots, x^d)$. (We intentionally use superscript indices here, because they are useful for bookkeeping. In fact, some authors use Einstein summation notation, where we omit the summation symbol and automatically sum over repeated indices provided that one is superscript and the other is subscript.) This way, first-order differential operators are characterized by being a linear operator that satisfies the product rule.

We define the **commutator** of the vector fields X and Y to be the differential operator

$$[X, Y]f = (XY - YX)f.$$

At first this appears to be a second-order differential operator; however, a computation shows that it is a first order differential operator with coefficients

$$[X, Y]^j = \sum_i \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right).$$

The commutator satisfies the Jacobi identity

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$$

for all vector fields X, Y, and Z. This can be verified directly, but is not particularly illuminating. Ultimately this is true because operators form an associative algebra, which in turn is true because function composition is always associative. (The Jacobi identity does not exclusively arise from associative algebras however; the cross product also obeys the Jacobi identity, but is not associative.)

A vector field X also has an associated first-order differential equation $\dot{x} = X(x)$. Given a vector field X, we define the **flow** $\Phi_X(t) := x(t; \cdot)$ which maps the initial data $\xi \in \mathbb{R}^d$ to the solution $x(t; \xi) \in \mathbb{R}^d$ at time t for the differential equation $\dot{x} = X(x)$.

To leading order, the commutator $[X, Y]$ measures the failure of the flows Φ_X and Φ_Y to commute:

Lemma A.16. *Let X and Y be smooth vector fields on $\mathbb{R}^d$. Then*

$$\big[\Phi(t) \circ \Phi_Y(s) - \Phi_Y(s) \circ \Phi_X(t)\big](\xi) = -st[X, Y](\xi) + \mathcal{O}(s^3 + t^3)$$

as $s, t \to 0$.

Proof. We will Taylor expand the LHS. This is valid because flows are always smooth, and consequently we may also differentiate in any order. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_X(t)\circ\Psi_Y(s) = X\circ\Phi_X(t)\circ\Phi_Y(s),$$

and so

$$\big[\Phi_X(t)\circ\Phi_Y(0)\big](\xi) = \xi + tX(\xi) + \tfrac{1}{2}t^2(X\cdot\nabla)X(\xi) + \mathcal{O}(t^3).$$

As $\Phi_Y(0) = Id$, we get the same expression for $\Phi_Y(0)\circ\Phi_X(t)$. In this way, there are no terms involving only t or only s in the Taylor expansion.

Therefore the leading order term in the Taylor expansion is quadratic, with only the st term not necessarily vanishing. We have

$$\left.\frac{\partial^2}{\partial s\,\partial t}\right|_{s,t=0}\Phi_X(t)\circ\Phi_Y(s) = (Y\cdot\nabla)X,$$
$$\left.\frac{\partial^2}{\partial s\,\partial t}\right|_{s,t=0}\Phi_Y(s)\circ\Phi_X(t) = (X\cdot\nabla)Y.$$

Together, we compute the coefficient of the st term to be

$$\begin{aligned}\left.\frac{\partial^2}{\partial s\,\partial t}\right|_{s,t=0}\big[\Phi_X(t)\circ\Phi_Y(s) - \Phi_Y(s)\circ\Phi_X(t)\big](\xi) &= \big\{(Y\cdot\nabla)X - (X\cdot\nabla)Y\big\}(\xi)\\ &= -[X,Y](\xi)\end{aligned}$$

as desired. □

Integrating in time, we obtain the following important fact:

Theorem A.17. *Let X and Y be smooth vector fields. Then $[X,Y]=0$ if and only if the flows commute:*

$$\Phi_X(t)\circ\Phi_Y(s) = \Phi_Y(s)\circ\Phi_X(t).$$

Proof. We follow the analytic argument from [Arn89, §39.E].

$\Longleftarrow$: This follows immediately from the previous lemma.

$\Longrightarrow$: Fix $s,t\neq 0$. We may assume $s,t>0$ after reversing time if necessary. Fix a positive integer N, and divide the rectangle $[0,t]\times[0,s]$ into an $N\times N$ grid.

Each path from $(0,0)$ to (s,t) along this grid corresponds to a composition of N copies of $\Phi_X(\frac{t}{N})$ and N copies of $\Phi_Y(\frac{s}{N})$ in some order. In particular,

$$\Phi_X(t)\circ\Phi_Y(s) = \Phi_X(\tfrac{t}{N})\circ\cdots\circ\Phi_X(\tfrac{t}{N})\circ\Phi_Y(\tfrac{s}{N})\circ\cdots\circ\Phi_Y(\tfrac{s}{N})$$

and similarly for $\Phi_Y(s)\circ\Phi_X(t)$.

We write the difference $\Phi_X(t)\circ\Phi_Y(s) - \Phi_Y(s)\circ\Phi_X(t)$ as a telescoping sum of N^2 terms, where the summand consists of the difference of two paths that

differ around one grid square. Within the summand, there is a difference of two flows that agree up to one point, differ around a $N^{-1} \times N^{-1}$ box, and then continue as the flows of two possibly different points. Applying the lemma with $[X, Y] = 0$, the difference around one box is

$$\Phi_X(\tfrac{t}{N}) \circ \Phi_Y(\tfrac{s}{N}) - \Phi_Y(\tfrac{s}{N}) \circ \Phi_X(\tfrac{t}{N}) = 0 + \mathcal{O}(\tfrac{|s|^3+|t|^3}{N^3}).$$

After this, the flows can then deviate at most exponentially. Altogether, we estimate the whole sum as

$$\Phi_X(t) \circ \Phi_Y(s) - \Phi_Y(s) \circ \Phi_X(t) = N^2 e^{C(|t|+|s|)} \mathcal{O}(\tfrac{|s|^3+|t|^3}{N^3}).$$

Sending $N \to \infty$, the RHS vanishes. As s, t were arbitrary, we conclude that $\Phi_X(t) \circ \Phi_Y(s) - \Phi_Y(s) \circ \Phi_X(t) \equiv 0$. □

A.5 Behavior away from fixed points

We are ultimately interested in the qualitative characterization of all Hamiltonian flows. First, we will need a general fact about the behavior of solutions away from fixed points.

We say that a diffeomorphism $x = \Psi(y)$ **conjugates** the flow $\dot{x} = X(x)$ into the flow $\dot{y} = Y(y)$ if

$$Y(y) = (\Psi'(y))^{-1}(X \circ \Psi)(y). \tag{A.11}$$

Indeed, under the change of variables $x = \Psi(y)$ we have

$$\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\Psi(y) = \Psi'(y)\dot{y} = \Psi'(y)Y(y).$$

On the other hand,

$$\dot{x} = X(x) = (X \circ \Psi)(y),$$

and so rearranging yields the condition (A.11). This is the notion with which we can describe the qualitative behavior of solutions.

Proposition A.18. *If $X(x_0) \neq 0$, then there exists a local diffeomorphism conjugating*

$$\dot{x} = X(x) \qquad \textit{to} \qquad \dot{y} = \boldsymbol{e}_1 = (1, 0, \ldots, 0).$$

Proof. We may assume $x_0 = 0$ after translating. First we rotate coordinates so that $X(0) = c\mathbf{e}_1$ with $c > 0$. Let $\Phi_X(t; x) := x(t; \xi)$ denote the flow of the initial data ξ by time t. Define

$$\Psi(y) = \Phi_X(y_1; (0, y_2, \ldots, y_d)).$$

That is, we flow the initial data $(0, y_2, \ldots, y_d)$ by a "time" y_1.

We check that this choice does indeed work. We have

$$\Psi'(y) = \begin{pmatrix} \frac{\partial \Psi_1}{\partial y_1} & \cdots & \frac{\partial \Psi_1}{\partial y_d} \\ \vdots & & \vdots \\ \frac{\partial \Psi_d}{\partial y_1} & \cdots & \frac{\partial \Psi_d}{\partial y_d} \end{pmatrix} = \begin{pmatrix} | & * & \cdots & * \\ X \circ \Psi(y) & \vdots & \ddots & \vdots \\ | & * & \cdots & * \end{pmatrix}, \tag{A.12}$$

where "$*$" denotes an unspecified nonzero entry. In particular,

$$\Psi'(0) = \begin{pmatrix} c & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

has determinant $c > 0$ and hence is nonsingular. (The lower right submatrix is the identity because $\Phi_X(0; (0, y_2, \dots, y_d)) = (0, y_2, \dots, y_d)$.) Therefore Ψ is a local diffeomorphism by the inverse function theorem. Also, from (A.12) we see that

$$\Psi'(y)\mathbf{e}_1 = (X \circ \Psi)(y),$$

and so

$$\dot{y} = (\Psi'(y))^{-1}(X \circ \Psi)(y) \equiv \mathbf{e}_1$$

as desired. □

To paraphrase, a nonvanishing vector field X is locally a coordinate vector field $\frac{\partial}{\partial x_1}$ for some choice of coordinates. Unlike general collections of vector fields, coordinate vector fields commute with each other.

Proposition A.19. *If $X_1, \dots, X_n$ smooth vector fields on $\mathbb{R}^d$ that commute and $X_1(x_0), \dots, X_n(x_0) \in \mathbb{R}^d$ are linearly independent, then there exists a local diffeomorphism that conjugates $X_1, \dots, X_n$ into $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$.*

Proof. We may assume $x_0 = 0$ after translating. First, we make a linear change of variables so that $X_1(x_0) = \mathbf{e}_1, \dots, X_n(x_0) = \mathbf{e}_n$, which can be done by linear independence. Let $[\Phi_{X_i}(t)](x) := x_i(t; \xi)$ denote the flow of the initial data ξ under the equation $\dot{x}_i = X_i$ by time t. Define

$$\Psi(y) = \big[\Phi_{X_1}(y_1) \circ \Phi_{X_2}(y_2) \circ \cdots \circ \Phi_{X_n}(y_n)\big](0, \dots, 0, y_{n+1}, \dots, y_d).$$

We have $\Psi'(0) = Id$ as before. As the vector fields commute, so do their flows. Therefore, in computing the ith derivative of Ψ, we may pull the ith flow map Φ_{X_i} out to the front:

$$\frac{\partial \Psi}{\partial y_i} = \frac{\partial}{\partial y_i} \Psi_1 X_i(y_i) \circ [*] = X_i \circ \Psi(y) \circ [*].$$

The rest of the calculation is the same as in Proposition A.18. □

A.6 Exercises

A.1. (a) Fix $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ that is C^1 and consider the initial value problem

$$\dot{x} = f(t, x) \quad \text{with} \quad x(0) = \xi.$$

Suppose that for some ξ_0 this problem admits a (necessarily unique) solution on the interval $[0, T)$. Show that for each $\epsilon > 0$ there is a $\delta > 0$ so that if $|\xi - \xi_0| < \delta$, then the initial value problem admits a solution on the interval $[0, T - \epsilon)$.

(b) Compute the maximal (forward) existence time as a function of the initial data for the problem

$$\dot{x} = \frac{x^2}{1 + y^2 x^2}, \quad \dot{y} = 0.$$

Deduce that the existence time may fail to be continuous (in the natural topology on $[0, \infty]$). By part (a), however, it is always lower semi-continuous.

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