Some applications in classical mechanics of the double and the dual numbers

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Abstract. We give some examples of the application in classical mechanics of the double and the dual numbers, which are analogous to the complex numbers.

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1. Introduction

The complex numbers are employed in classical physics as a useful tool, usually taking advantage of their basic algebraic properties, but not as an essential ingredient; the objects of physical interest are real numbers or real-valued functions. (By contrast, in quantum mechanics, the complex numbers have a more fundamental role.) Despite the existence of many deep results in the theory of complex variables, most applications of the complex numbers in classical mechanics, electrodynamics, and special or general relativity rely on the fact that the complex numbers, apart from having all the algebraic properties of the real numbers (such as associativity and commutativity of the sum and the product), possess an imaginary unit, i, characterized by the property $\mathrm{i}^2 = -1$.

There exist other sets of "hypercomplex" numbers that in many respects imitate the complex numbers, and are characterized by the presence of certain units, j and ε, with the defining properties $\mathrm{j}^2 = 1$ (but $\mathrm{j} \neq \pm 1$) and $\varepsilon^2 = 0$ (but $\varepsilon \neq 0$). The numbers of the form $a + \mathrm{j}b$, with $a, b \in \mathbb{R}$, are called double (or split-complex) numbers, and the numbers of the form $a + \varepsilon b$, with $a, b \in \mathbb{R}$, are called dual numbers. Unlike the real and the complex numbers, the double and the dual numbers are not fields, but for many applications this is not a problem at all (see, e.g., [1, 2, 3]).

The aim of this paper is to give some elementary applications of the double and the dual numbers in the solution of the differential equations that appear in analytical mechanics. In section 2 we give two examples of mechanical systems with one degree of freedom, showing that in each case the Hamilton equations are equivalent to a single first-order ordinary differential equation. In section 3, following [4], we consider a family of Hamiltonians which includes those of the two-dimensional isotropic harmonic oscillator, the repulsive oscillator, and the free particle, and we find in a unified way constants of motion and symmetry groups.

2. Two simple applications in classical mechanics

In this section we give two very simple examples related with classical mechanics, illustrating the application of the double and the dual numbers separately.

2.1. A bead in a rotating straight wire

The standard Hamiltonian for a bead of mass m in a straight wire which rotates about the z-axis with a constant angular velocity ω, forming a constant angle θ_0 with the z-axis, is given by

$$H = \frac{p^2}{2m} - \frac{m}{2}\omega^2 r^2 \sin^2\theta_0 + mgr\cos\theta_0,$$

assuming that the z-axis points upwards. Here, r is the distance from the origin to the bead, p is the momentum canonically conjugate to r, and g is the acceleration of gravity. The Hamilton equations yield the coupled equations

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{p}{m}, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = m\omega^2 r\sin^2\theta_0 - mg\cos\theta_0$$

and, making use of the hypercomplex unit j, we can write these equations as the single equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(p + \mathrm{j}\, m\omega r\sin\theta_0) = \mathrm{j}\,\omega\sin\theta_0(p + \mathrm{j}\, m\omega r\sin\theta_0) - mg\cos\theta_0,$$

which, using the fact that $\mathrm{j}^2 = 1$ and that $mg\cot\theta_0/\omega$ is constant, can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p + \mathrm{j}\, m\omega r\sin\theta_0 - \mathrm{j}\frac{mg\cos\theta_0}{\omega\sin\theta_0}\right) = \mathrm{j}\,\omega\sin\theta_0\left(p + \mathrm{j}\, m\omega r\sin\theta_0 - \mathrm{j}\frac{mg\cos\theta_0}{\omega\sin\theta_0}\right).$$

The solution of this equation is given by

$$p + \mathrm{j}\, m\omega r\sin\theta_0 - \mathrm{j}\frac{mg\cos\theta_0}{\omega\sin\theta_0} = \exp(\mathrm{j}\,\omega t\sin\theta_0)\left(p_0 + \mathrm{j}\, m\omega r_0\sin\theta_0 - \mathrm{j}\frac{mg\cos\theta_0}{\omega\sin\theta_0}\right),$$

where p_0 and r_0 denote the values of p and r, respectively, at $t = 0$. Making use of the relation $\exp\mathrm{j}\,x = \cosh x + \mathrm{j}\,\sinh x$ (which follows from the series expansion of the exponential), and separating the "real" and "imaginary" parts of the preceding equation we obtain

$$p = p_0\cosh(\omega t\sin\theta_0) + \left(m\omega r_0\sin\theta_0 - \frac{mg\cos\theta_0}{\omega\sin\theta_0}\right)\sinh(\omega t\sin\theta_0)$$

and

$$m\omega r\sin\theta_0 - \frac{mg\cos\theta_0}{\omega\sin\theta_0} = \left(m\omega r_0\sin\theta_0 - \frac{mg\cos\theta_0}{\omega\sin\theta_0}\right)\cosh(\omega t\sin\theta_0) + p_0\sinh(\omega t\sin\theta_0)$$

which constitute the solution of the equations of motion.

2.2. A point mass in a uniform gravitational field

The standard Hamiltonian for a particle of mass m in a uniform gravitational field is

$$H = \frac{p^2}{2m} + mgx$$

and the corresponding Hamilton equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p}{m}, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -mg.$$

With the aid of the unit ε (which satisfies $\varepsilon^2 = 0$), these two equations can be merged into the linear first-order equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(p + \varepsilon Kx) = \frac{\varepsilon K}{m}(p + \varepsilon Kx) - mg,$$

where K is a constant with dimensions of momentum/length.

The standard formula for the solution of a linear equation given in the elementary textbooks on differential equations yields

$$p + \varepsilon Kx = \mathrm{e}^{\varepsilon Kt/m}\left(-\int mg\mathrm{e}^{-\varepsilon Kt/m}\mathrm{d}t + \text{const.}\right)$$

and, using the fact that $\exp\varepsilon x = 1 + \varepsilon x$, we have

$$\begin{aligned} p + \varepsilon Kx &= \left(1 + \frac{\varepsilon Kt}{m}\right)\left(-\int mg\left(1 - \frac{\varepsilon Kt}{m}\right)\mathrm{d}t + c_1 + \varepsilon c_2\right) \\ &= \left(1 + \frac{\varepsilon Kt}{m}\right)\left(-mgt + \frac{\varepsilon Kgt^2}{2} + c_1 + \varepsilon c_2\right) \\ &= -mgt + c_1 + \varepsilon\left(c_2 - \frac{Kgt^2}{2} + \frac{Kc_1t}{m}\right), \end{aligned}$$

where c_1 and c_2 are real constants. This last equation now leads to the two separated expressions

$$p = -mgt + c_1, \qquad x = \frac{c_2}{K} + \frac{c_1t}{m} - \frac{gt^2}{2}.$$

The constant terms c_1 and c_2/K represent the initial values of p and x, respectively.

3. Symmetries of the isotropic harmonic oscillator and related problems

We shall consider a mechanical system with two degrees of freedom, with Hamiltonian

$$H = \frac{1}{2m}({p_x}^2 + {p_y}^2) - \frac{\mathrm{h}^2}{2}m\omega^2(x^2 + y^2), \tag{1}$$

where m and ω are constants, and h may be the imaginary unit, i (in which case H corresponds to the usual two-dimensional isotropic harmonic oscillator); the hypercomplex unit j (corresponding to a repulsive force); or the hypercomplex unit ε (corresponding to a free particle). (That is, h^2 is equal to -1, $+1$, or 0, respectively, and in all cases H is real.) This Hamiltonian can be expressed in the form

$$H = \frac{1}{2m}\Psi^\dagger\Psi, \tag{2}$$

where Ψ is the two-component vector [4]

$$\Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv \begin{pmatrix} p_x + \mathrm{h} m\omega x \\ p_y + \mathrm{h} m\omega y \end{pmatrix} \tag{3}$$

and the Hermitian adjoint of a matrix is defined as the transpose of the conjugate matrix (in all cases, under conjugation, $\overline{\mathrm{h}} \equiv -\mathrm{h}$).

Making use of the Hamilton equations we find that, for example,

$$\frac{\mathrm{d}(p_x + \mathrm{h} m\omega x)}{\mathrm{d}t} = -\frac{\partial H}{\partial x} + \mathrm{h} m\omega \frac{\partial H}{\partial p_x} = \mathrm{h}^2 m\omega^2 x + \mathrm{h}\omega p_x = \mathrm{h}\omega(p_x + \mathrm{h} m\omega x),$$

with a similar result for the time derivative of ψ_2, thus

$$\frac{\mathrm{d}\psi_A}{\mathrm{d}t} = \mathrm{h}\omega\psi_A \qquad (A = 1, 2).$$

The solution of these equations of motion is given by

$$\psi_A = (\psi_A)_0 \exp \mathrm{h}\omega t, \tag{4}$$

where $(\psi_A)_0$ is the initial value of ψ_A and

$$\exp \mathrm{h}\theta = \begin{cases} \cos\theta + \mathrm{i}\sin\theta & \text{if } \mathrm{h} = \mathrm{i}, \\ \cosh\theta + \mathrm{j}\,\sinh\theta & \text{if } \mathrm{h} = \mathrm{j}, \\ 1 + \varepsilon\theta & \text{if } \mathrm{h} = \varepsilon. \end{cases} \tag{5}$$

Equations (5) are obtained by means of the series expansion of the exponential.

Since we have already the solution of the equations of motion, we can find all the constants of motion. In particular, from equation (4) we see that the products of the form $\overline{\psi_A}\psi_B$ are constants of motion. In this way we obtain four (real) constants of motion; two of them are

$$\overline{\psi_1}\psi_1 = {p_x}^2 - \mathrm{h}^2 m^2\omega^2 x^2, \tag{6}$$
$$\overline{\psi_2}\psi_2 = {p_y}^2 - \mathrm{h}^2 m^2\omega^2 y^2, \tag{7}$$

and from $\overline{\psi_1}\psi_2 = (p_x - \mathrm{h} m\omega x)(p_y + \mathrm{h} m\omega y) = p_x p_y - \mathrm{h}^2 m^2\omega^2 xy - \mathrm{h} m\omega(xp_y - yp_x)$, separating the "real" and "imaginary" parts, we obtain the two additional constants of motion

$$p_x p_y - \mathrm{h}^2 m^2\omega^2 xy, \qquad xp_y - yp_x. \tag{8}$$

These four constants of motion cannot be functionally independent since, for a system with two degrees of freedom, there exist three time-independent functionally independent constants of motion only. In fact, the are related by

$$(p_x p_y - \mathrm{h}^2 m^2\omega^2 xy)^2 = ({p_x}^2 - \mathrm{h}^2 m^2\omega^2 x^2)({p_y}^2 - \mathrm{h}^2 m^2\omega^2 y^2) + \mathrm{h}^2 m^2\omega^2 (xp_y - yp_x)^2.$$

It may be noticed that the sum of the constants of motion (6) and (7), divided by $2m$, is the Hamiltonian (1).

3.1. Finite symmetries of the Hamiltonian

The functions (6), (7) and (8), defined above, being constants of motion, must be the infinitesimal generators of symmetries of the Hamiltonian. However, the corresponding finite transformations can be found directly noting that the expression (2) is invariant under unitary transformations $\Psi \mapsto U\Psi$. (As usual, a matrix U is unitary if $U^\dagger = U^{-1}$.) In fact, under the transformation $\Psi \mapsto U\Psi$, we have $\Psi^\dagger\Psi \mapsto (U\Psi)^\dagger(U\Psi) = \Psi^\dagger U^\dagger U\Psi = \Psi^\dagger\Psi$. Furthermore, the unitary transformation $\Psi \mapsto U\Psi$ corresponds to a canonical transformation.

With the Poisson bracket defined in such a way that $\{x, p_x\} = 1$, from (3) one readily finds that

$$\{\psi_A, \psi_B\} = 0 = \{\overline{\psi_A}, \overline{\psi_B}\}, \qquad \{\overline{\psi_A}, \psi_B\} = -2\mathrm{h}m\omega\delta_{AB}. \tag{9}$$

Then, in order to show that the mapping $\Psi \mapsto U\Psi$ corresponds to a canonical transformation, we only have to prove that the Poisson brackets (9) are invariant under this mapping. For instance, if $U = (U_{AB})$ then

$$\begin{aligned}\{\overline{\psi_A}, \psi_B\} \mapsto \{\sum_{C=1}^{2} \overline{U_{AC}\psi_C}, \sum_{D=1}^{2} U_{BD}\psi_D\} &= \sum_{C,D=1}^{2} \overline{U_{AC}}\, U_{BD}\, \{\overline{\psi_C}, \psi_D\} \\ &= \sum_{C,D=1}^{2} \overline{U_{AC}}\, U_{BD}(-2\mathrm{h}m\omega\delta_{CD}) = -2\mathrm{h}m\omega \sum_{C=1}^{2} \overline{U_{AC}}\, U_{BC} = -2\mathrm{h}m\omega\delta_{AB} \\ &= \{\overline{\psi_A}, \psi_B\}.\end{aligned}$$

Recalling that the SU(2) matrices are of the form $\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$, with $\alpha, \beta \in \mathbb{C}$, such that $|\alpha|^2 + |\beta|^2 = 1$, we consider matrices of the form

$$\begin{pmatrix} a + \mathrm{h}b & c + \mathrm{h}d \\ -c + \mathrm{h}d & a - \mathrm{h}b \end{pmatrix}, \tag{10}$$

with $a, b, c, d \in \mathbb{R}$, such that $a^2 + c^2 - \mathrm{h}^2b^2 - \mathrm{h}^2d^2 = 1$ and we find that the product of this matrix by its conjugate transpose is given by

$$\begin{pmatrix} a + \mathrm{h}b & c + \mathrm{h}d \\ -c + \mathrm{h}d & a - \mathrm{h}b \end{pmatrix} \begin{pmatrix} a - \mathrm{h}b & -c - \mathrm{h}d \\ c - \mathrm{h}d & a + \mathrm{h}b \end{pmatrix} = (a^2 + c^2 - \mathrm{h}^2b^2 - \mathrm{h}^2d^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which means that the 2×2 matrix (10) is unitary (and special, in the sense that its determinant is equal to 1). One can verify that the matrices of the form (10) form a group with the usual matrix multiplication and, therefore, we have a group of canonical transformations that leave invariant the Hamiltonian (1). This group possesses three (real) parameters owing to the condition $a^2 + c^2 - \mathrm{h}^2b^2 - \mathrm{h}^2d^2 = 1$. The generators of this group are the two constants of motion (8) together with the difference between the functions (6) and (7). The Hamiltonian, which is the sum of (6) and (7) divided by $2m$, generates the time evolution (4).

As pointed out above, the matrices of the form (10) with $\mathrm{h} = \mathrm{i}$ and $a^2+b^2+c^2+d^2 = 1$, form the group usually denoted by SU(2), which is homomorphic to the rotation group SO(3). As is well known, SU(2) is a symmetry group for the two-dimensional isotropic harmonic oscillator (see [4] and the references cited therein).

When $\mathrm{h} = \mathrm{j}$, the matrices of the form (10) with and $a^2+c^2-b^2-d^2 = 1$, form a group isomorphic to the group $\mathrm{SL}(2, \mathbb{R})$, formed by the 2×2 real matrices with determinant equal to 1. An isomorphism between these two groups is given by

$$\begin{aligned}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{j} \\ -\mathrm{j} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{j} \\ \mathrm{j} & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} a+d+\mathrm{j}(b+c) & b-c+\mathrm{j}(d-a) \\ -b+c+\mathrm{j}(d-a) & a+d-\mathrm{j}(b+c) \end{pmatrix}.\end{aligned}$$

This expression shows that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\mathrm{SL}(2, \mathbb{R})$, then $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is of the form (10). The group $\mathrm{SL}(2, \mathbb{R})$ is homomorphic to the group $\mathrm{SO}(2, 1)$ of the Lorentz transformations in a space-time with two spatial directions.

Finally, when $\mathrm{h} = \varepsilon$, the matrices of the form (10) with and $a^2 + c^2 = 1$, form a group homomorphic to the group of rigid motions of the Euclidean plane.

The Hamiltonians (1) are also interesting because of their relation with the Kepler problem. This connection is obtained by expressing the standard Hamiltonian for the Kepler problem in two-dimensions,

$$H_{\mathrm{Kepler}} = \frac{{p_x}^2 + {p_y}^2}{2m} - \frac{k}{\sqrt{x^2+y^2}},$$

where k is a constant, in terms of the parabolic coordinates (u, v), which can be defined by $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$. The result is

$$H_{\mathrm{Kepler}} = \frac{{p_u}^2 + {p_v}^2}{2m(u^2+v^2)} - \frac{2k}{u^2+v^2},$$

and therefore the condition $H_{\mathrm{Kepler}} = E$, where E is the value of the energy, is equivalent to

$$\frac{{p_u}^2 + {p_v}^2}{2m} - E(u^2+v^2) = 2k. \tag{11}$$

The left-hand side of (11) has the form (1), with E in place of $\frac{1}{2}\mathrm{h}^2 m\omega^2$. Since the energy, E, can be positive, negative, or zero, these three cases correspond to h equal to j, i, or ε, respectively. Among other things, this implies that the Kepler problem with E positive, negative, or zero admits a symmetry group homomorphic to $\mathrm{SL}(2, \mathbb{R})$, SU(2), or the group of rigid motions of the Euclidean plane, respectively.

4. Concluding remarks

In one of the standard procedures employed in the solution of systems of coupled ordinary differential equations of the form found in sections 2 and 3, as a first step, one obtains a decoupled equation by raising the order of the equations. The advantage of using the complex, double, or dual numbers is that, in those special cases where they are useful, one is able to reduce the number of equations to solve or to obtain decoupled equations, without raising the order of the equations.

In the case of the Hamiltonians (1), thanks to the use of complex, double and dual numbers, one is able to find, by inspection, a three-parameter symmetry group.

In the examples presented above, the differential equations are linear and can be easily solved directly, but the complex, double and dual numbers are also useful in other cases. For instance, the nonlinear equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2a_1 x - a_2 + a_3(x^2 + \mathrm{h}^2 y^2), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -2a_1 y + 2a_3 xy, \tag{12}$$

where a_1, a_2, a_3 are real constants and $\mathrm{h}^2 = \pm 1$, arise in the search for symmetries of certain two-dimensional Riemannian manifolds. Equations (12) also make sense when $\mathrm{h}^2 = 0$ and, with the definition $z \equiv x + \mathrm{h}y$, the system (12) is given by the single equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = a_3 z^2 - 2a_1 z - a_2,$$

which can be readily solved. In fact, in the three cases $\mathrm{h} = \mathrm{i}, \mathrm{j}, \varepsilon$, its solution is given by

$$z(t) = \frac{\alpha z(0) + \beta}{\gamma z(0) + \delta}, \tag{13}$$

where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$. It turns out that, taking $\mathrm{h} = \mathrm{i}, \mathrm{j}, \varepsilon$, equation (13) represents all the three nonequivalent actions of $\mathrm{SL}(2, \mathbb{R})$ on the xy-plane [2].

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